Title: Black holes in tidal environments

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Abstract: A nonrotating black hole placed in a tidal environment (that is, subjected to the gravitational interactions produced by other nearby bodies) is not described by the Schwarzschild solution to the Einstein field equations. Instead, its metric is given by a perturbed version of this exact solution, and the spacetime is no longer stationary nor spherically symmetric. After reviewing the situation in Newtonian theory, I shall describe how the metric of a tidally distorted black hole is calculated. Special attention will be placed on the general description of the tidal environment, the choice of a good coordinate system to describe the perturbed black hole, and the consequences on the structure of the event horizon

Black holes in tidal environments

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Outline

- Goals and motivation
- Separation of scales
- Newtonian problem
- Relativistic problem
- Binary black holes and gravitational waves
- Conclusions

[Phys. Rev. D **70**, 08044 (2004); Phys. Rev. Lett. **94**, 161103 (2005)]

Goals and motivation

- To calculate the metric of a nonrotating black hole moving in an external universe and subjected to a tidal gravitational field.
- To calculate (and better understand) the tidal heating of a black hole.

In the context of a binary system, the increase of the black-hole mass by tidal heating impacts the energy balance between radiated energy and orbital energy.

This affects the phasing of the gravitational waves, and has measurable consequences.

[Poisson and Sasaki (1995); Tagoshi et al (1997); Alvi, gr-qc/0107080; Poisson, gr-qc/0407050]

Separation of scales

The problem involves two length scales:

- the black-hole mass M
- the tidal radius (local radius of curvature) $\mathcal{R}$

For a two-body system (external mass $M_{\rm ext}$, separation d),

$$\mathcal{R}^{-2} \sim \frac{M_{\rm ext}}{d^3} = \frac{M_{\rm ext} V^6}{(M + M_{\rm ext})^3}$$

$$V = \sqrt{\frac{M + M_{\rm ext}}{d}} \sim \text{orbital velocity}$$

Separation of scales

We assume that the **black hole is well isolated**, so that

$$M/\mathcal{R} \ll 1$$

We work in the **black-hole zone**, so that

$$M < r \ll \mathcal{R}$$

Separation of scales

For a two-body system,

$$\frac{M}{\mathcal{R}} \sim \frac{(M/M_{\rm ext})V^3}{(1+M/M_{\rm ext})^{3/2}} \ll 1$$

This is small whenever $M/M_{\rm ext} \ll 1$ **(small-hole approximation)**, in which case V is arbitrary.

This is also small whenever $V \ll 1$ **(slow-motion approximation)**, in which case $M/M_{\rm ext}$ is arbitrary.

Newtonian problem

A Newtonian body of mass M is assumed to be spherical in complete isolation; it is nonrotating.

Its unperturbed gravitational potential is $\Phi_{\rm body} = -M/r$.

It is placed in a tidal environment described by an external potential $\Phi_{\rm ext}$.

What is the response of the body?

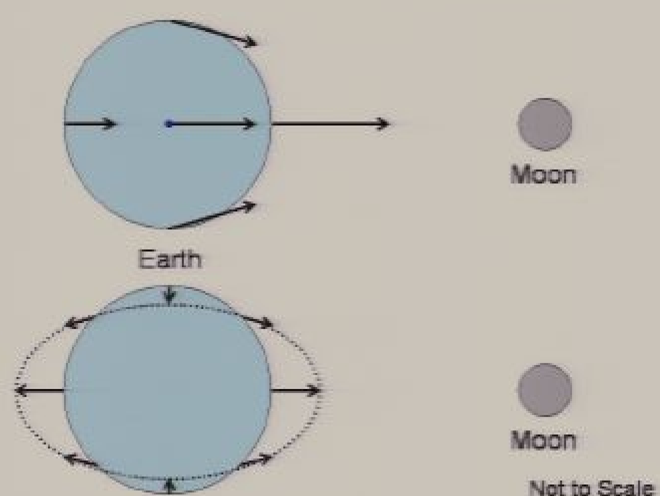
What is its perturbed gravitational potential?

What are the physical consequences of the tidal interaction?

Newtonian problem — Tidal environment

Using the separation of scales, the external potential can be expressed as a Taylor expansion about the body's centre of mass.

$$
\begin{aligned}
\Phi_{\rm ext}(t,\boldsymbol{r}) &= \Phi_{\rm ext}\Big|_{\boldsymbol{r}=\boldsymbol{0}} + \frac{\partial \Phi_{\rm ext}}{\partial x^a}\Big|_{\boldsymbol{r}=\boldsymbol{0}} x^a + \frac{1}{2}\frac{\partial^2 \Phi_{\rm ext}}{\partial x^a \partial x^b}\Big|_{\boldsymbol{r}=\boldsymbol{0}} x^a x^b + \cdots \\
&= \underbrace{\Phi_{\rm ext}(t,\boldsymbol{0})}_{\text{constant}} - \underbrace{g_a(t)}_{\text{CM acceleration}} x^a + \frac{1}{2}\underbrace{\mathcal{E}_{ab}(t)}_{\text{tidal field}} x^a x^b + \cdots
\end{aligned}
$$



[http://www-astronomy.mps.ohio-state.edu/ pogge/ Ast161/Unit4/tides.html]

Newtonian problem — Tidal environment

The centre-of-mass acceleration $g_a(t)$ is removed by working in the body's moving frame.

The description of the tidal environment is contained in the **tidal gravitational field**

$$\mathcal{E}_{ab}(t) := \left.\frac{\partial^2 \Phi_{\rm ext}}{\partial x^a \partial x^b}\right|_{\boldsymbol{r}=\boldsymbol{0}} \sim \frac{1}{\mathcal{R}^2}$$

The external potential is a **quadrupolar field**, and it is small:

$$\Phi_{\rm ext}(t, \boldsymbol{r}) = \frac{1}{2}\mathcal{E}_{ab}(t) x^a x^b + \cdots \sim (r/\mathcal{R})^2 \ll 1$$

Newtonian problem — Body response

The response of the body to the tidal perturbation depends on its internal composition.

Here the body is modeled as a viscous, incompressible fluid.

The response is calculated by solving the Navier-Stokes equation for the fluid's velocity field.

The details will not be shown. . .

Newtonian problem — Body response

The tidal distortion is measured by the body's **quadrupole moment tensor** [$Q^{ab} = \int \rho(x^a x^b - \frac{1}{3}\delta^{ab} r^2)\, dV$], which is given by

$$Q_{ab}(t) = -\frac{1}{2} a^5 \mathcal{E}_{ab}(t - \tau)$$

$a =$ body's radius

$\tau = \frac{19}{2}\left(\frac{a}{M}\right)\nu =$ viscous delay

$\nu =$ kinematic viscosity

The body's perturbed gravitational potential is

$$\Phi_{\text{body}} = -\frac{M}{r} - \frac{3}{2}\frac{Q_{ab}x^a x^b}{r^5} + \cdots$$

Newtonian problem — Physical consequences

The viscosity-induced **phase mismatch** between the applied force and the body's deformation allows the tidal forces to do (net) work on the body.

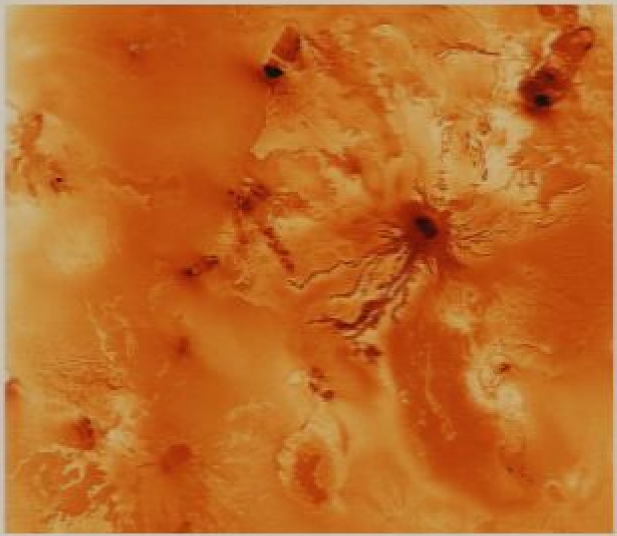
The rate of work done is

$$\frac{dW}{dt} = \frac{1}{4} a^5 \tau \dot{\mathcal{E}}_{ab} \dot{\mathcal{E}}^{ab}$$

This work is dissipated into heat by the fluid's viscosity.

This **tidal heating** is responsible for Io's spectacular volcanic activity. [Peale, Cassen, and Reynolds, Science **203**, 892 (1979)]

Newtonian problem — Physical consequences



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[http://www.solarviews.com/eng/io.htm]

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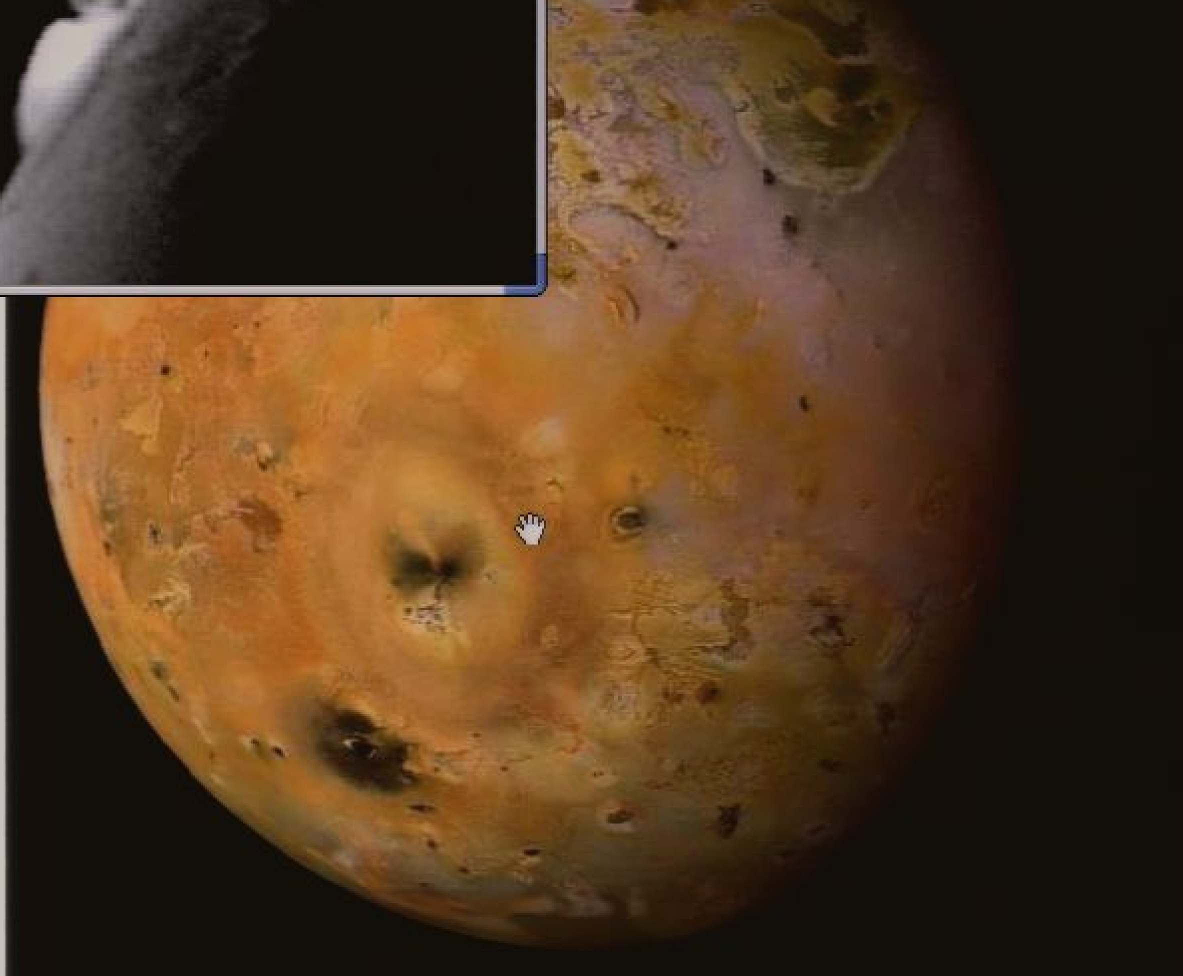

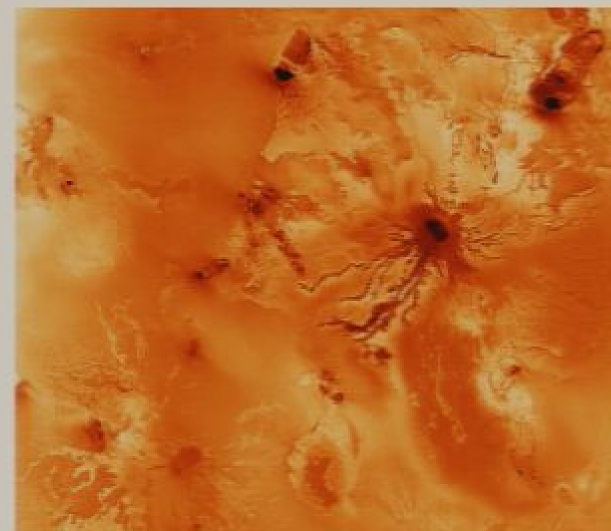

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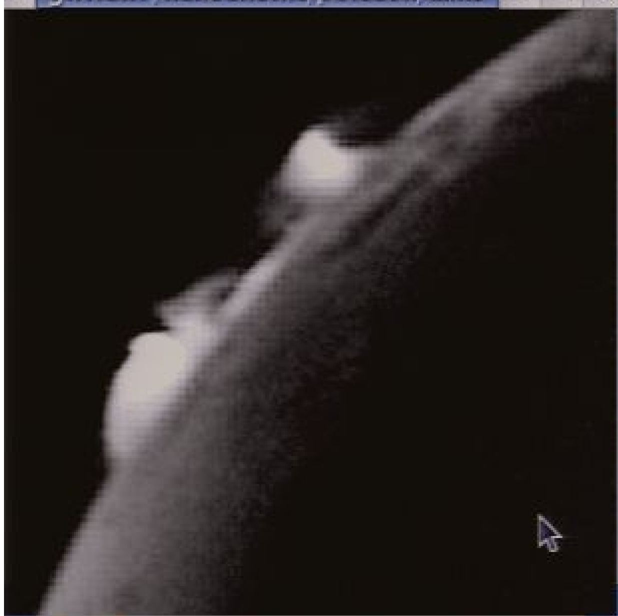

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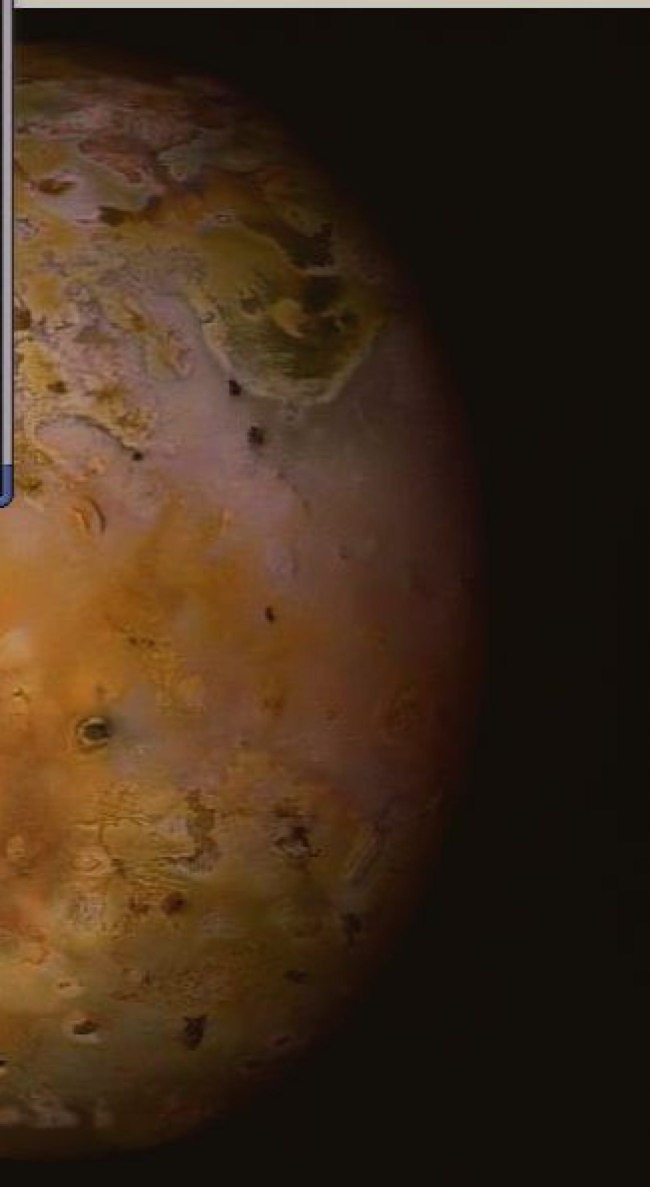

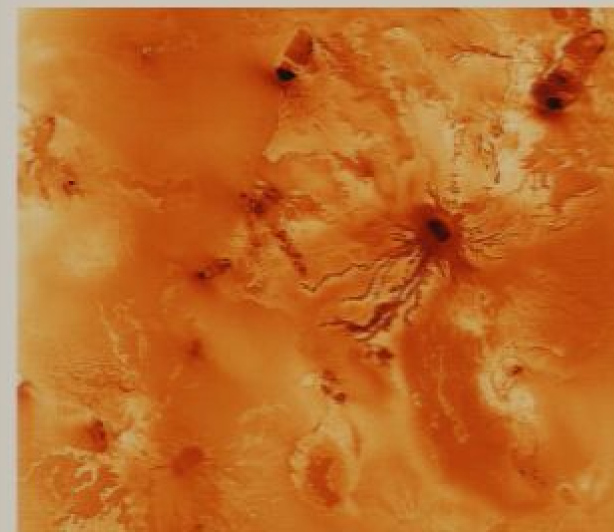

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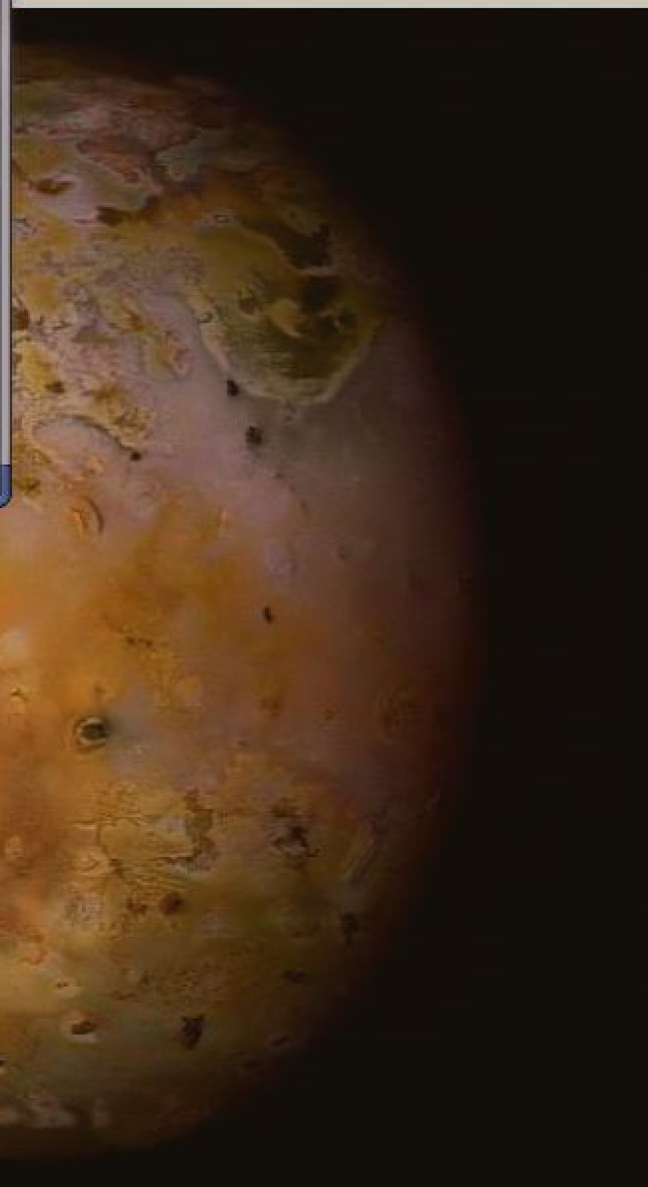

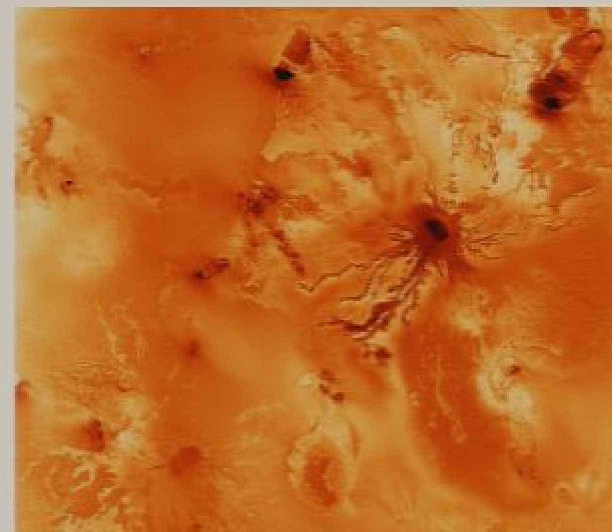

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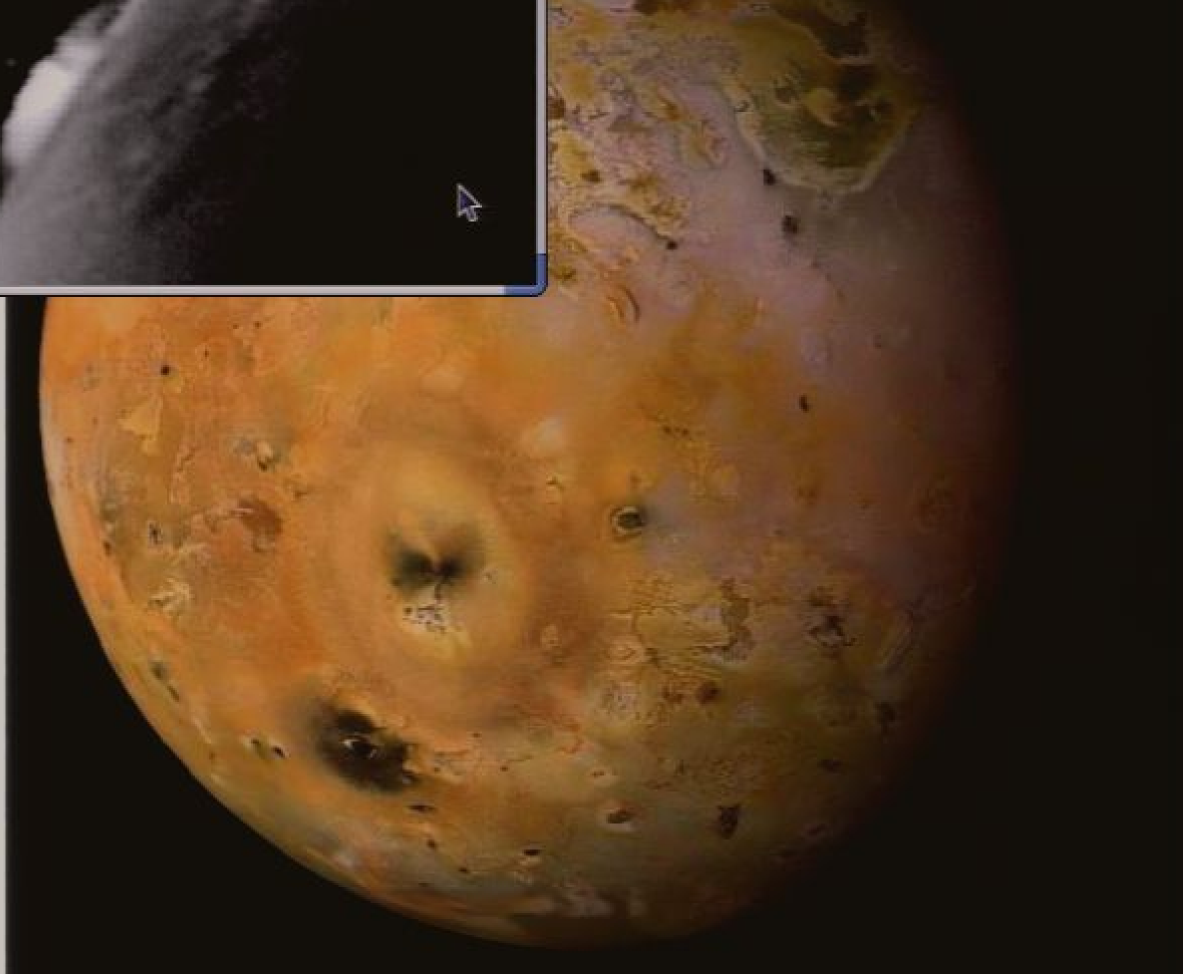

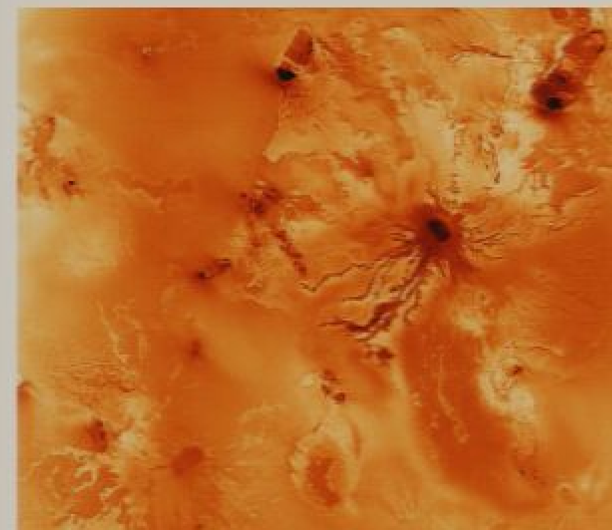

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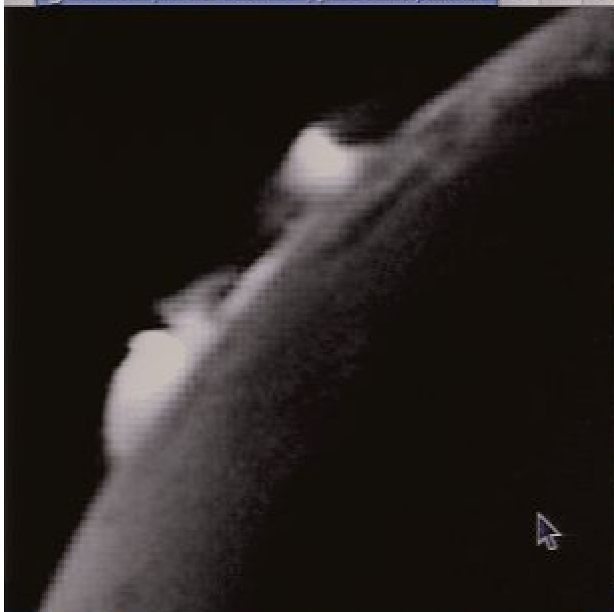

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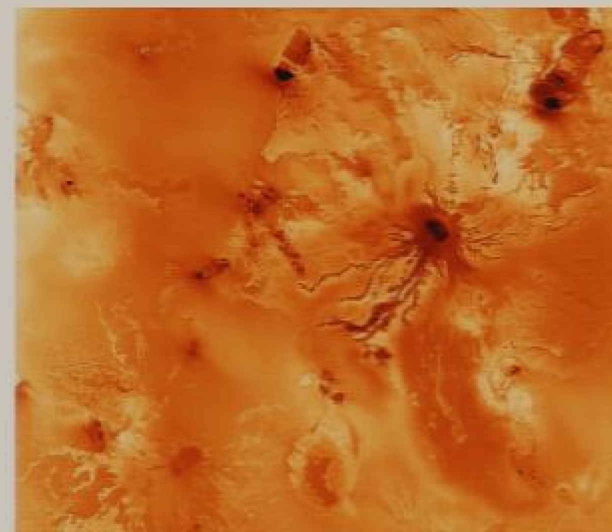

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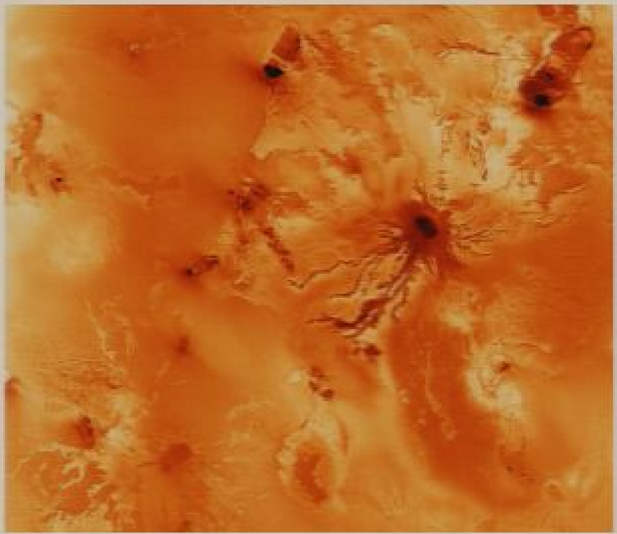
Newtonian problem — Physical consequences



[animation]

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Relativistic problem

We want to calculate the metric of a tidally distorted, nonrotating black hole by integrating the vacuum equations of black-hole perturbation theory.

We want to express the metric in geometrically meaningful coordinates that penetrate the event horizon.

We want to parameterize the perturbation with tidal fields defined in terms of the spacetime's Weyl tensor in the asymptotic region $r \gg M$.

We want to calculate the deformation of the event horizon.

We want to extract physical consequences.

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Relativistic problem — Unperturbed BH

The unperturbed spacetime contains a nonrotating black hole in complete isolation.

The metric is given by Schwarzschild's solution,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The metric is expressed in incoming Eddington-Filkenstein coordinates, which possess a nice geometric interpretation:

- v labels incoming light cones that converge toward $r = 0$
- (θ, ϕ) labels the null generators of each light cone
- r is an affine parameter on each generator
- r is an areal radius

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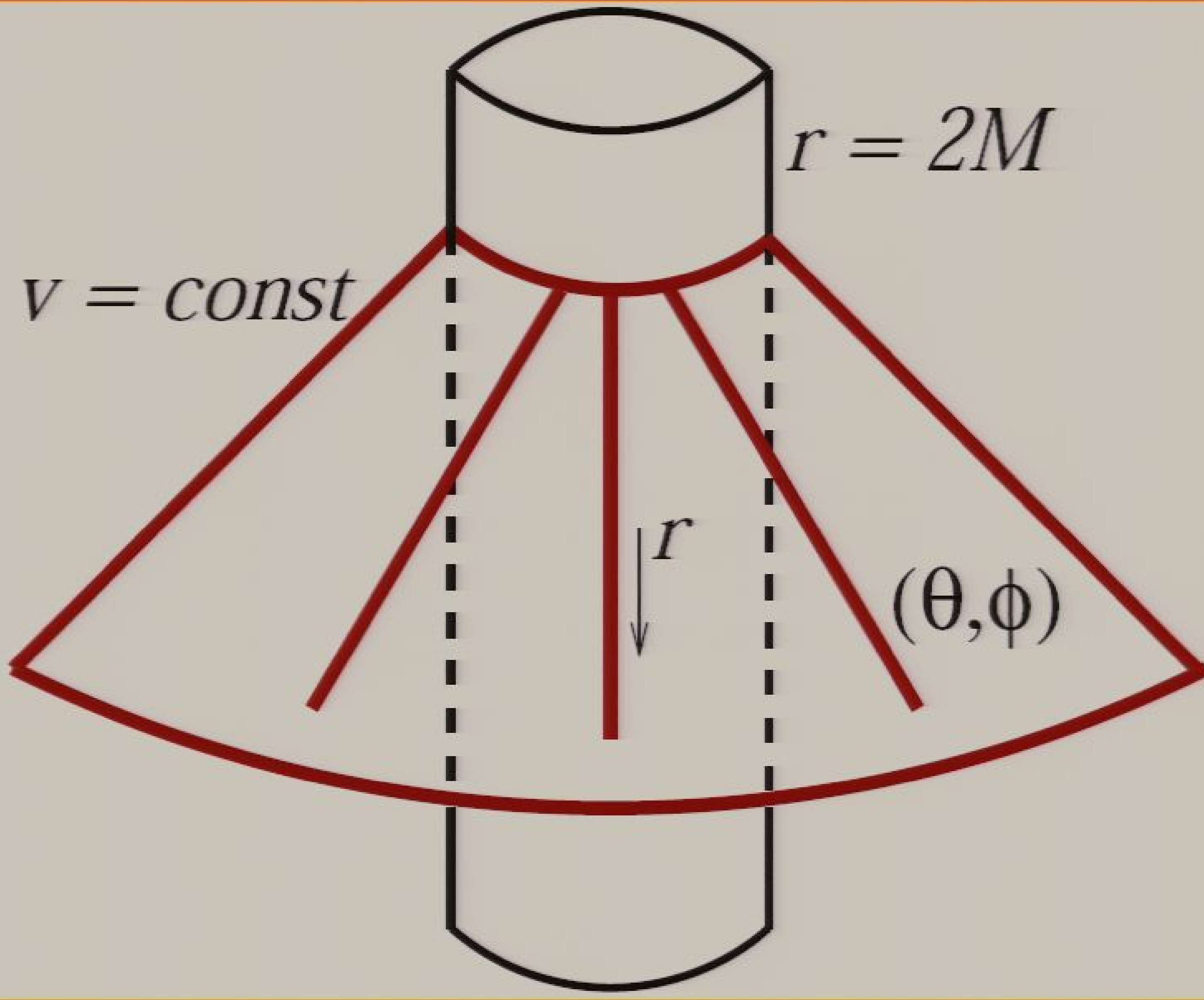

Relativistic problem — Tidal environment

Let the black hole move on a geodesic within a tidal environment created by other bodies.

When $r \gg M$ the gravitational field measured by an observer comoving with the black hole is dominated by the other bodies; the black hole is negligible.

Assuming that $r \ll \mathcal{R}$, the metric can be expressed in light-cone coordinates

$$v, \qquad x^a = [r\sin\theta\cos\phi, r\sin\theta\cos\phi, r\cos\theta]$$

centered on the black hole.

These are a variant of Fermi normal coordinates.

Relativistic problem — Tidal environment

In the region $M \ll r \ll \mathcal{R}$, the metric is given by

$$g_{vv} = -1 - \mathcal{E}_{ab}(v)x^a x^b + O(M/r, r^3/\mathcal{R}^3)$$

$$\boxed{\mathcal{E}_{ab}(v) = C_{avbv} = O(\mathcal{R}^{-2}) = \text{asymptotic Weyl tensor}}$$

The quantities $\mathcal{E}_{ab}$ (and other similar quantities) describe the hole's tidal environment.

The driving term $-\mathcal{E}_{ab}(v)x^a x^b$ provides boundary conditions for the metric perturbation.

At this order of approximation, the perturbation is entirely quadrupolar ($\ell = 2$).

Relativistic problem — Coordinates

We wish to integrate the equations of black-hole perturbation theory for

$$g_{\alpha\beta} = g_{\alpha\beta}^{\rm Schwarzschild} + h_{\alpha\beta}$$

This requires a choice of coordinate (gauge) conditions.

The **light-cone gauge** preserves the geometrical meaning of the background Eddington-Finkelstein coordinates:

- v continues to label incoming light cones
- (θ, ϕ) are still constant on generators
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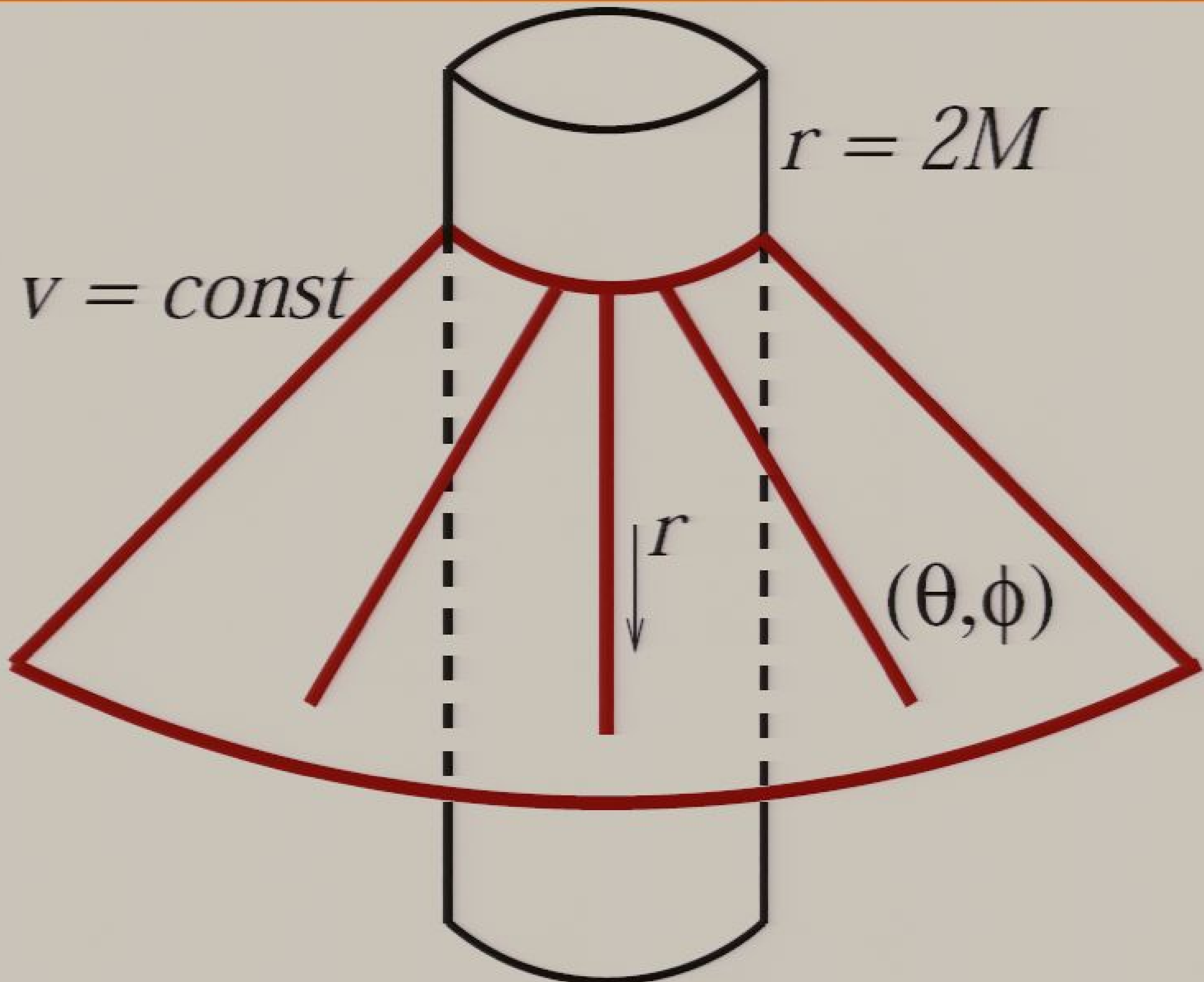
- the event horizon is still described by $r = 2M$

Relativistic problem — Unperturbed BH



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Relativistic problem — Perturbed metric

Integrating the equations of black-hole perturbation theory returns

$$g_{vv} = -\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^2 \mathcal{E}_{ab}(v) x^a x^b + O(r^3/\mathcal{R}^3)$$

A more accurate calculation gives

$$\begin{aligned} g_{vv} &= -f - f^2 \mathcal{E}_{ab}(v) x^a x^b \\ &\quad + \frac{1}{3} r f \left[1 + \frac{1}{2}\frac{M}{r}\left(5 + 12 \ln \frac{r}{2M}\right) - 3\frac{M^2}{r^2}\left(9 + 4 \ln \frac{r}{2M}\right) \right. \\ &\qquad \left. + 14\frac{M^3}{r^3} + 12\frac{M^4}{r^4}\right] \dot{\mathcal{E}}_{ab}(v) x^a x^b \\ &\quad - \frac{1}{3} f^2 \left(1 - \frac{M}{r}\right) \mathcal{E}_{abc}(v) x^a x^b x^c + O(r^4/\mathcal{R}^4) \end{aligned}$$

where $f = 1 - 2M/r$.

Relativistic problem — Perturbed horizon

The coordinate description of the event horizon is preserved by the perturbation,

$$r = 2M\left[1 + O(M^4/\mathcal{R}^4)\right]$$

But the intrinsic geometry of the horizon is nonspherical. For example, the surface gravity is given by

$$\kappa = \frac{1}{4M}\left[1 + \frac{4}{3}M\dot{\mathcal{E}}_{ab}(v)x^a x^b + \cdots\right]$$

At this level of approximation the event horizon is an apparent horizon; the expansion of the horizon's null generators is

$$\Theta = O(M^4/\mathcal{R}^5)$$

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Relativistic problem — Tidal heating

The slow growth of the event horizon is not revealed by a direct examination of the horizon's perturbed geometry, but it can be calculated by integrating the optical-scalar equations:

$$\frac{dM}{dv} = \frac{16}{45} M^6 \dot{\mathcal{E}}_{ab}(v)\dot{\mathcal{E}}^{ab}(v) + \text{higher-order terms}$$

similar to the Newtonian expression for a fluid body,

$$\frac{dW}{dt} = \frac{1}{4} a^5 \tau \dot{\mathcal{E}}_{ab}(t)\dot{\mathcal{E}}^{ab}(t), \qquad \tau = \frac{19}{2}\left(\frac{a}{M}\right)\nu$$

An approximate match is obtained by setting $a = 2M$ and $\tau \sim M$, which sets the kinematic viscosity to $\nu \sim M$.

[Hartle, Phys. Rev. D **9**, 2749 (1974)]

Relativistic problem — Tidal heating

For a nonrelativistic orbit of radius b around an external body of mass $M_{\rm ext}$, this becomes

$$\frac{dM}{dv} \simeq \frac{32}{5} \frac{M^6 M_{\rm ext}^2}{(M + M_{\rm ext})^8} V^{18} \qquad \textbf{(slow-motion)}$$

where $V = \sqrt{(M + M_{\rm ext})/b}$ is the orbital velocity.

For a relativistic orbit around an external Schwarzschild black hole, this becomes

$$\frac{dM}{dv} \simeq \frac{32}{5} \left(\frac{M}{M_{\rm ext}}\right)^6 V^{18} \frac{(1 - V^2)(1 - 2V^2)}{(1 - 3V^2)^2} \qquad \textbf{(small-hole)}$$

where $V = \sqrt{M_{\rm ext}/b}$ is the orbital velocity.

Conclusions

- The metric of a nonrotating black hole in a tidal environment is expressed as an expansion in powers of $\mathcal{R}^{-1}$, the inverse tidal radius.
- The metric is presented in light-cone coordinates; these possess a clear geometrical meaning.
- The tidal interaction produces a growth of the horizon.
- The result is similar to the rate at which tidal forces do work on fluid bodies; a viscosity can be assigned to the event horizon.
- Tidal heating of black holes has observational consequences for gravitational-wave astronomy.
- The tidal heating of a rotating black hole can also be calculated (with the Teukolsky equation).

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