Title: Large N q-deformed Yang-Mills Theory and Hecke Algebras

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Abstract:

Large N Expansion of q-deformed two-dimensional Yang-Mills theory and Hecke algebras

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Based on – mainly hep-th/0603056
by S. de Haro, S. Ramgoolam,A. Torrielli (HRT-2006)

1 Introduction and Outline

- 2D Yang-Mills theory, with gauge group $SU(N)$, on a Riemann surface of genus G and area A

$$Z = \int DA e^{-\frac{1}{2g^2_{YM}} \int_{\Sigma_G} d^2x (trF^2)}$$

  is exactly solvable.

$$Z(G,A) = \sum_R (DimR)^{2-2G} e^{-\frac{g^2_{YM}}{2} AC_2(R)}$$

  The sum is over all irreducible representations of the $SU(N)$ gauge group.

- Gross (1992) initiated a program of investigating **gauge-string duality** in this simple model. Gross-Taylor (1993) found the large N expansion of the partition function in terms of symmetric groups. The symmetric group data counts branched covers.

- The large N expansion takes the form

$$Z \sim \sum Z^+ Z^-$$

- This was developed further in Cordes-Moore-Ramgoolam (1994) to show that the large N expansion computes Euler characters of the moduli space of maps

$$Z^+(G, A=0) = \sum_h N^{2-2h} \chi(\mathcal{M}_{hol}(\Sigma_h \rightarrow \Sigma_G))$$

  $1/N$ can be identified with the string coupling constant.

Worldsheet

Base space of YM2

- In this talk we will focus Z_+ which is defined by setting

$$\sum_R \to \sum_{n=0}^{\infty} \sum_{R \in Y_n}$$

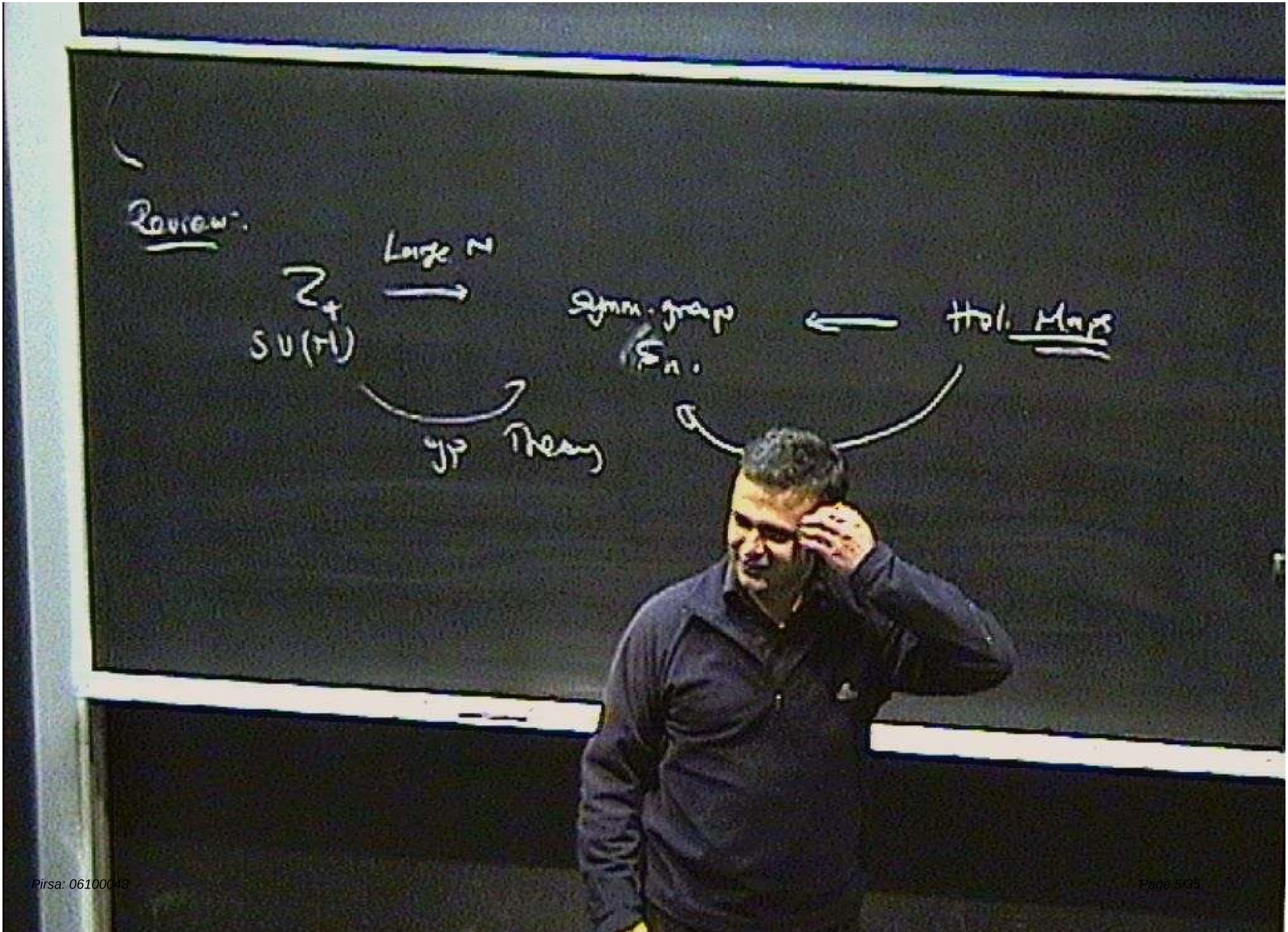
and treating n as a small number compared to N in the summand.

- This is not the complete large N expansion, since there are also representations which conjugates of small repesentations and which have comparable Casimirs and dimensions .. Hence the Z_-.

- The paper HRT-2006, we describe the **deformation of the symmetric group data** of the chiral large N expansion to Hecke algebra data. At $q = 1$ the symmetric group data is directly related to holomorphic maps (branched covers) and hence to the string theory on Σ_G target.

$Z_+ \to$	symmetric group data $\to$	Holomorphic maps	$\to$	String action

$Z_+(q) \to$	Hecke algebra data	$\to$	?? $\to$	String on Calabi-Yau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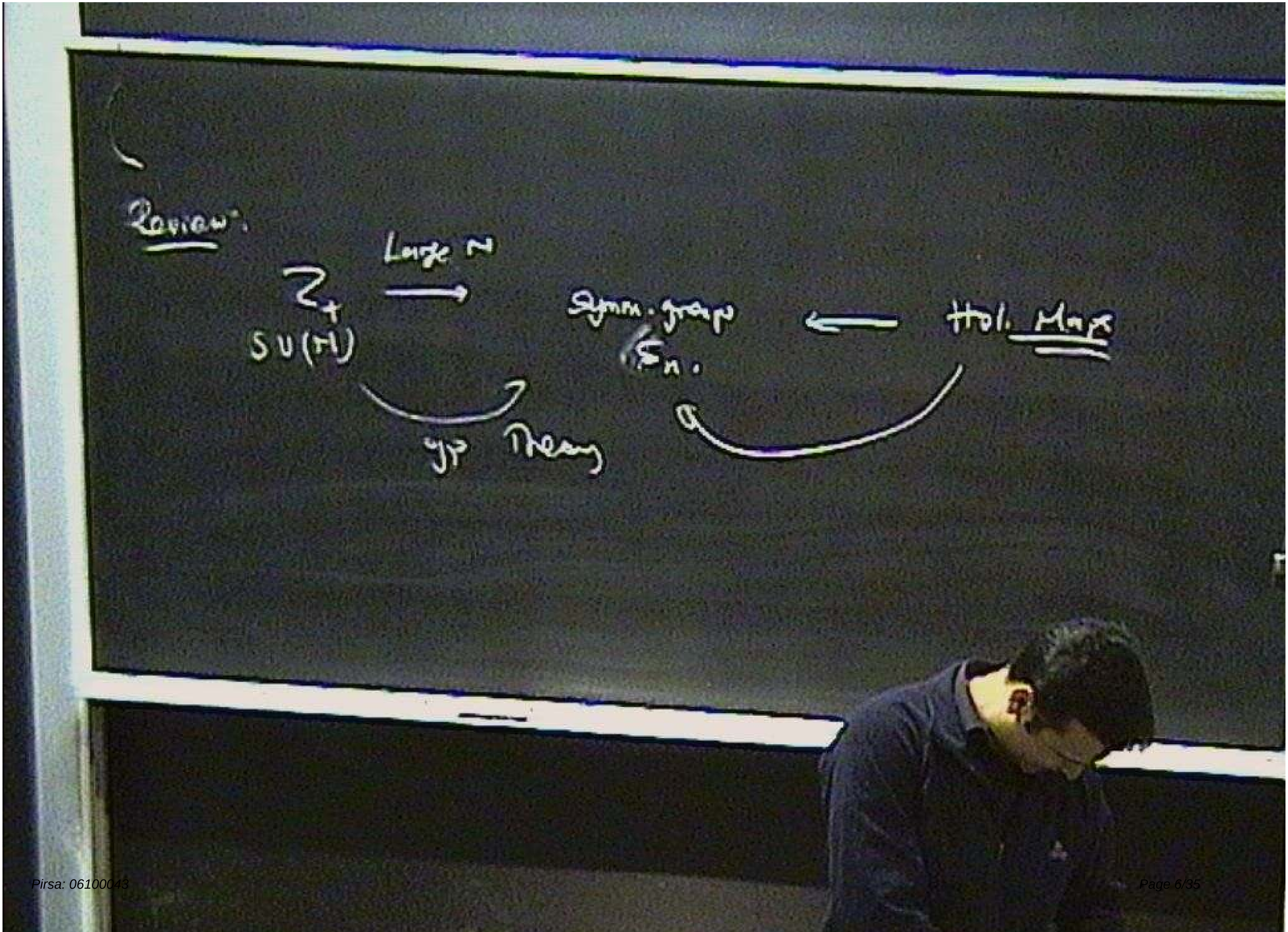
Large N
SU(N)
Symm. groups
Hol. Maps
Top Theory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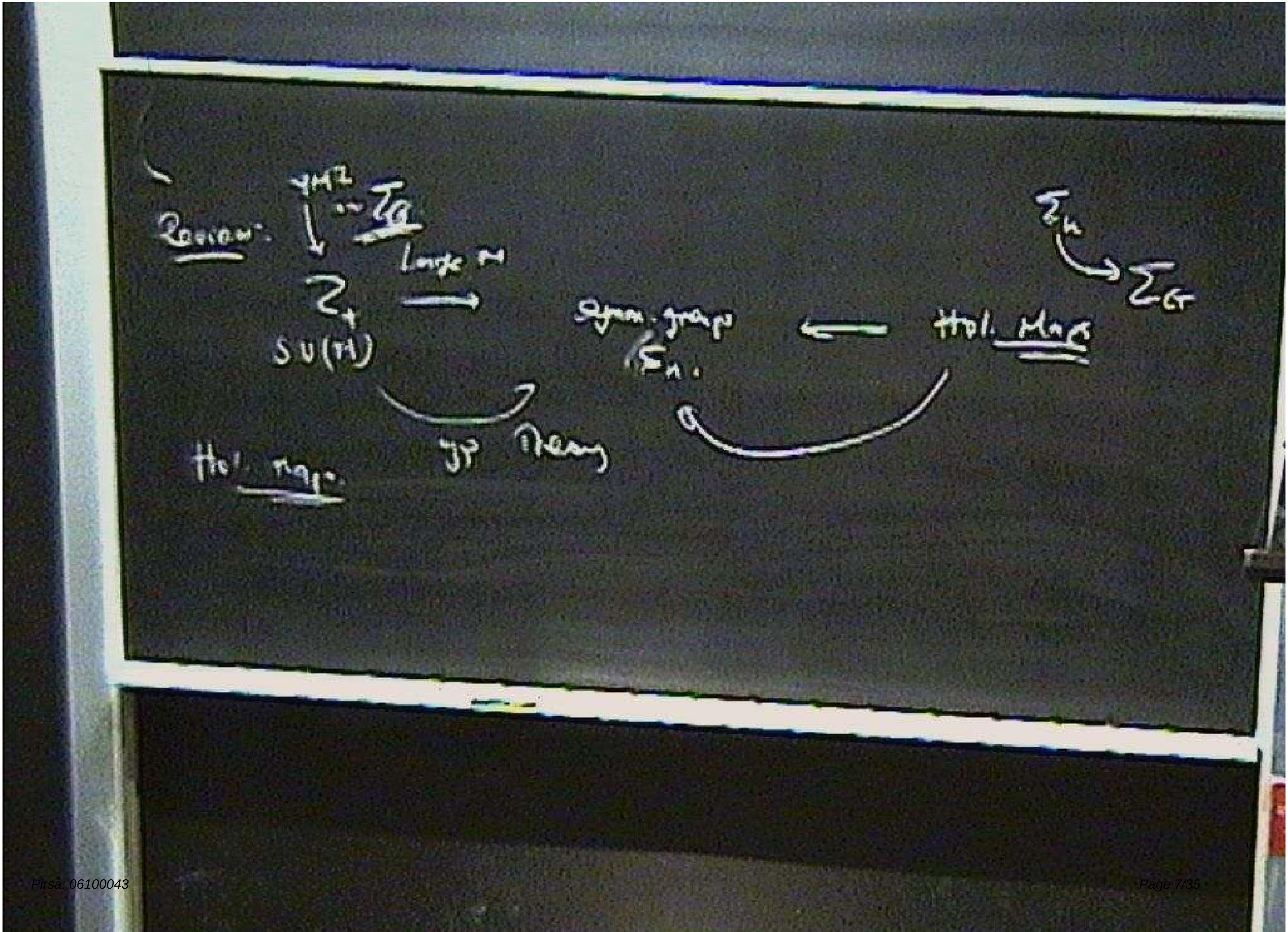

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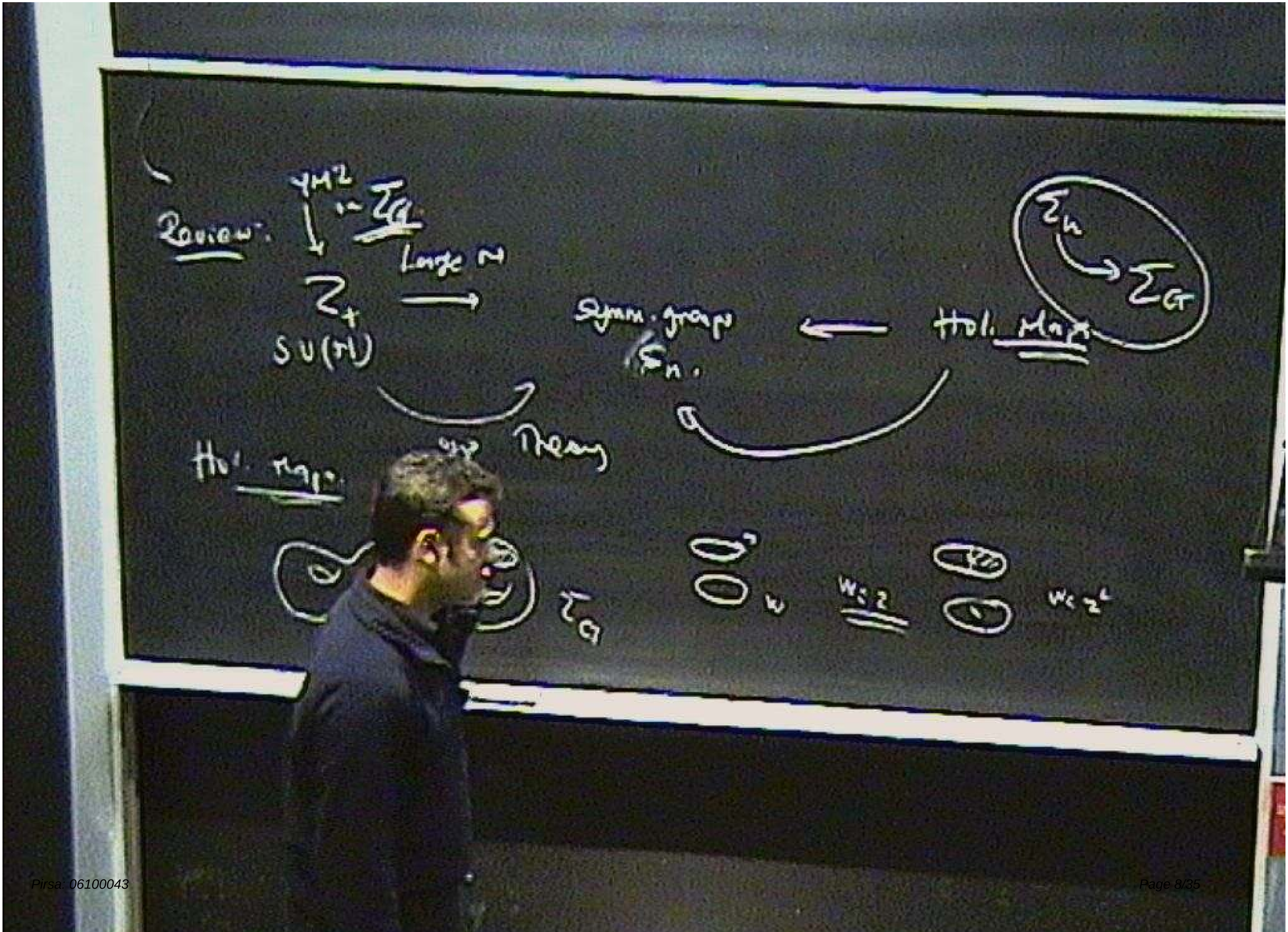
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Hol. Maps
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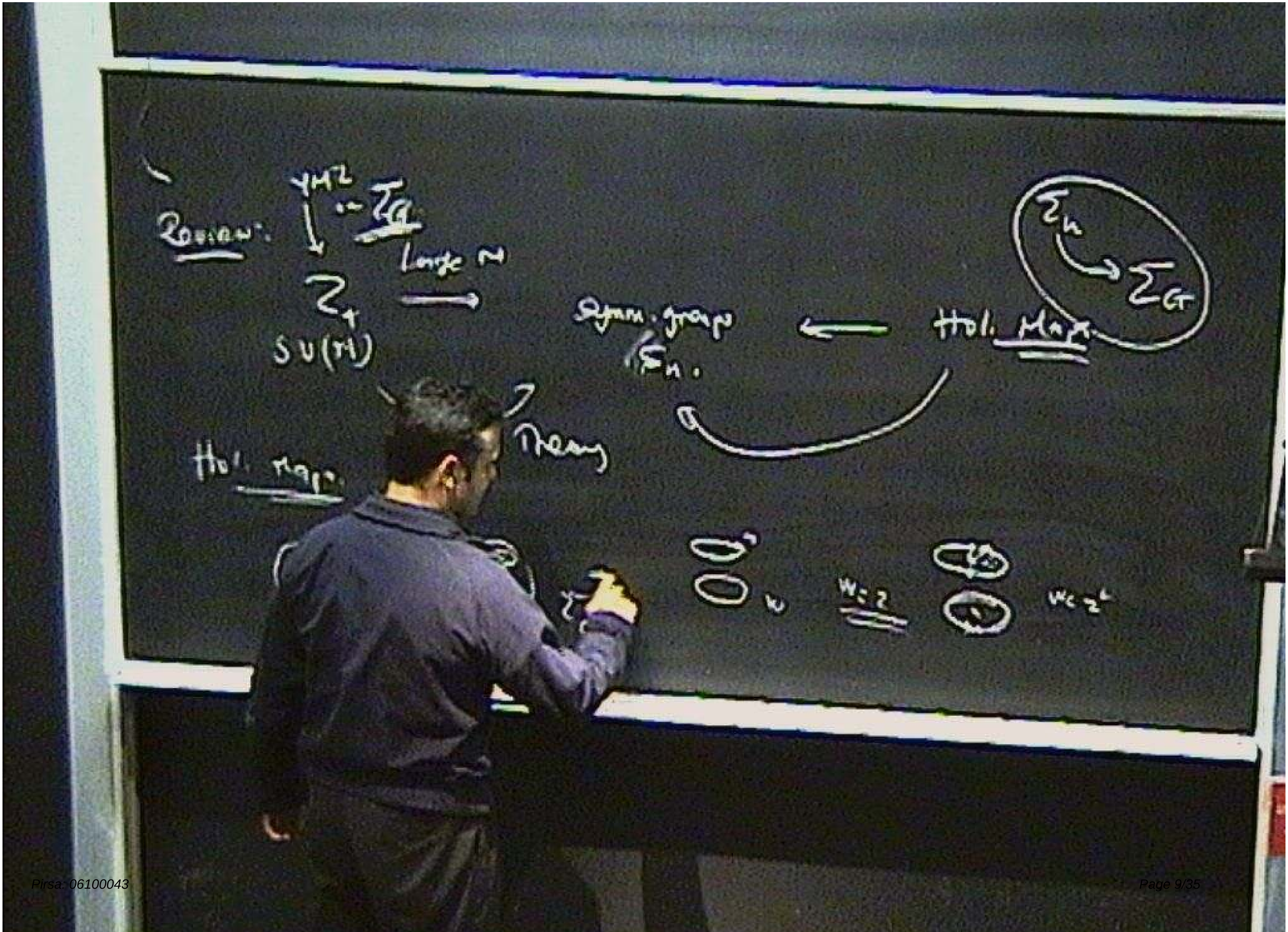
w=z
w=z^2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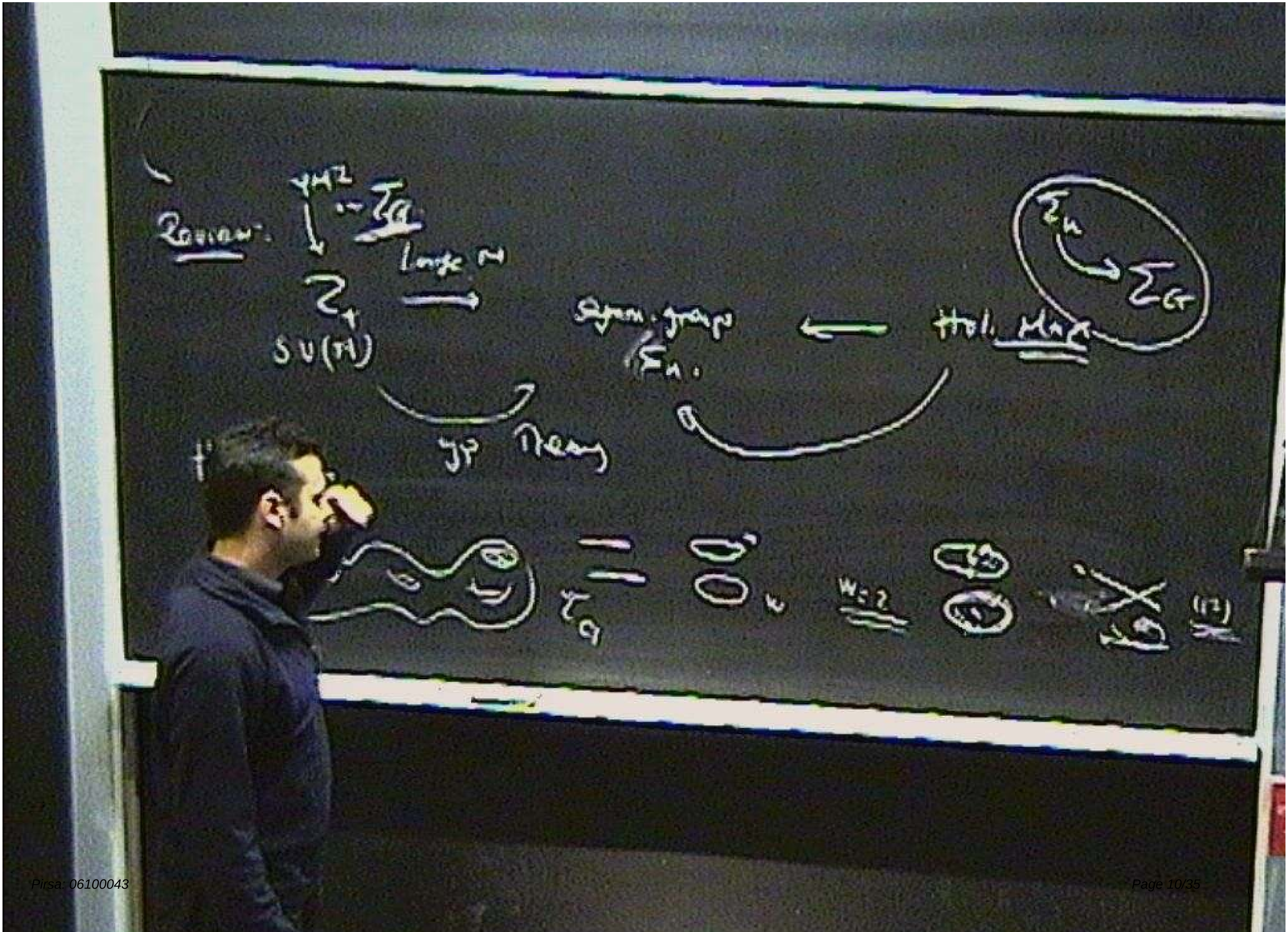
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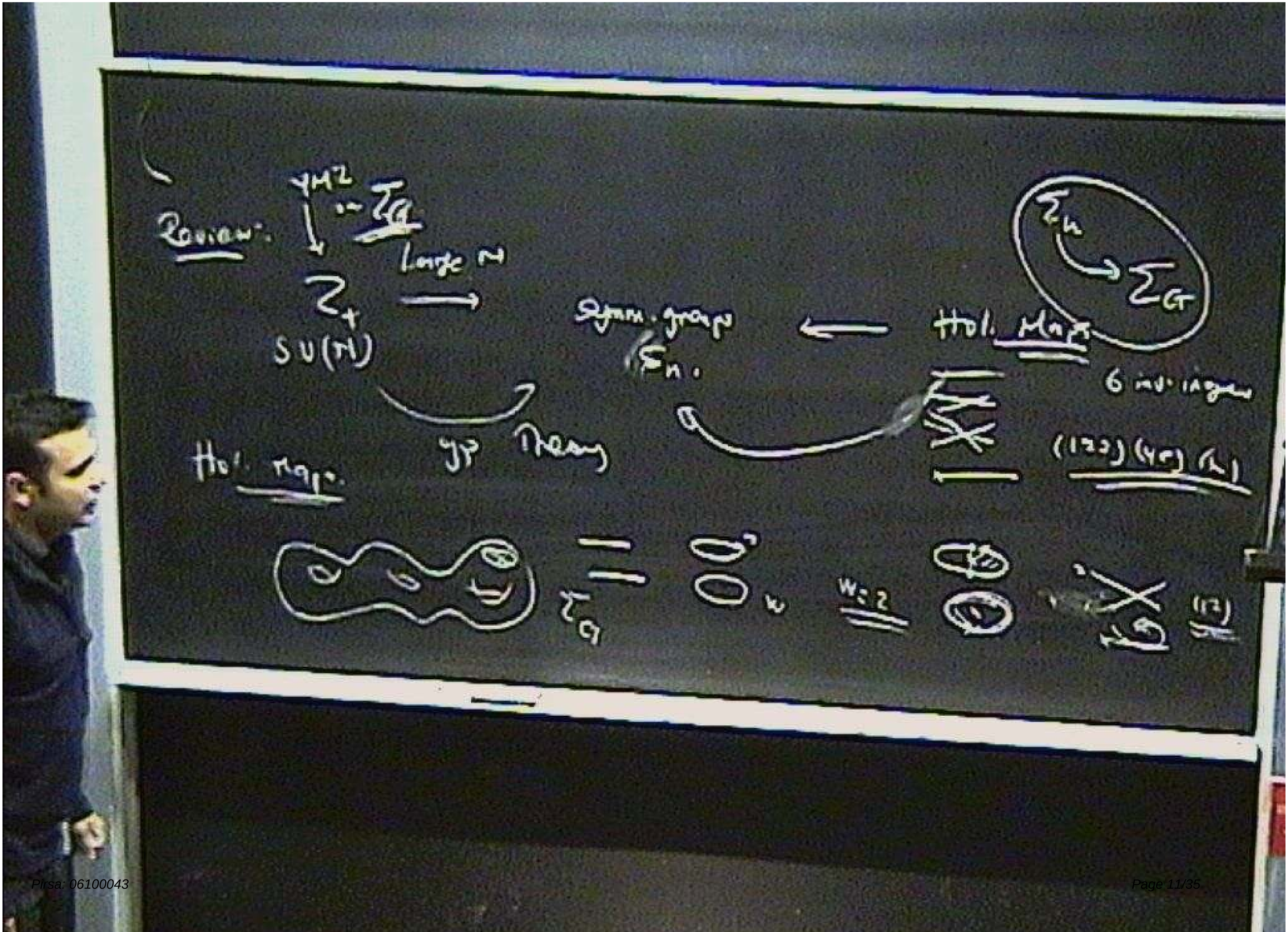
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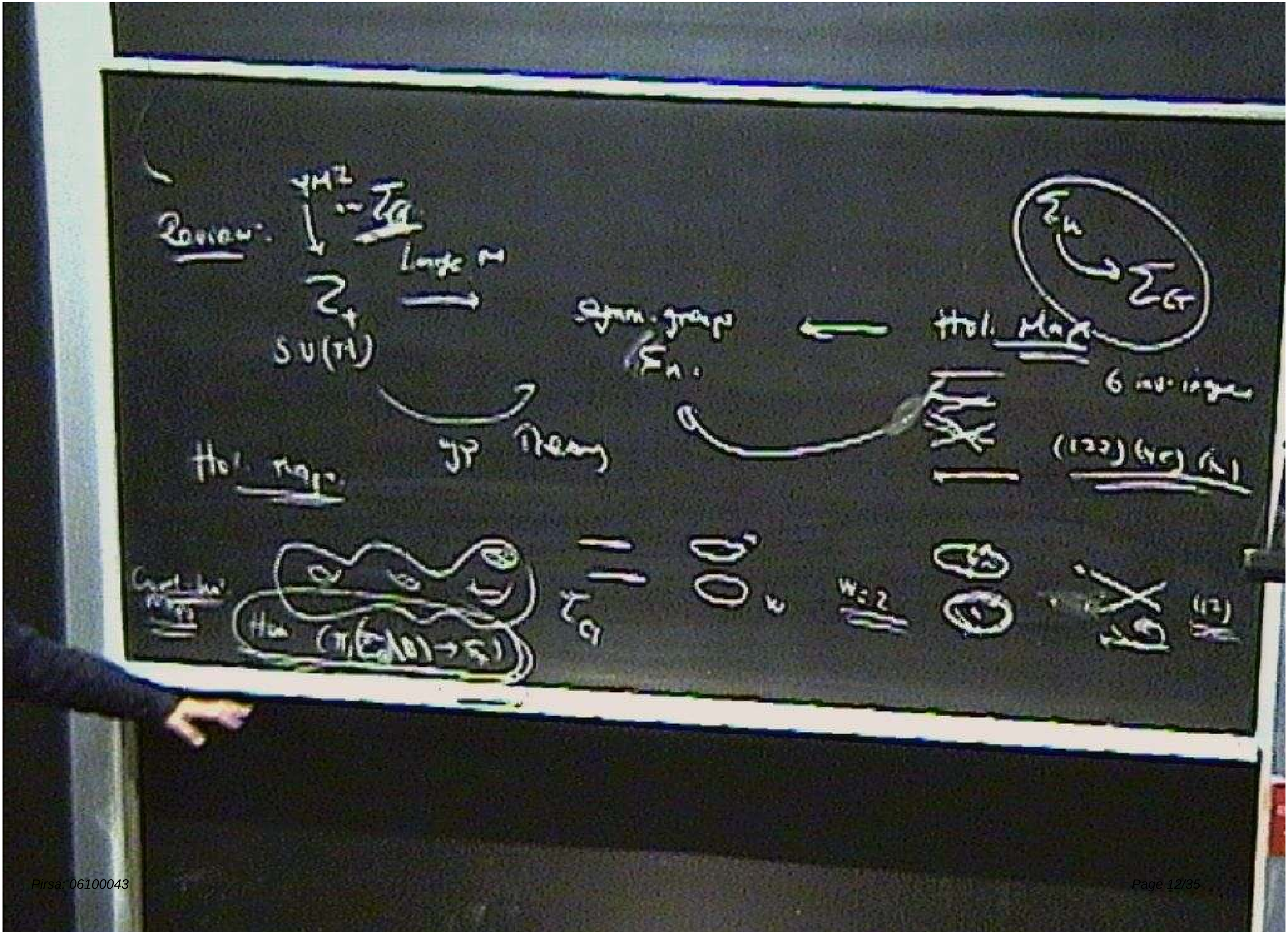
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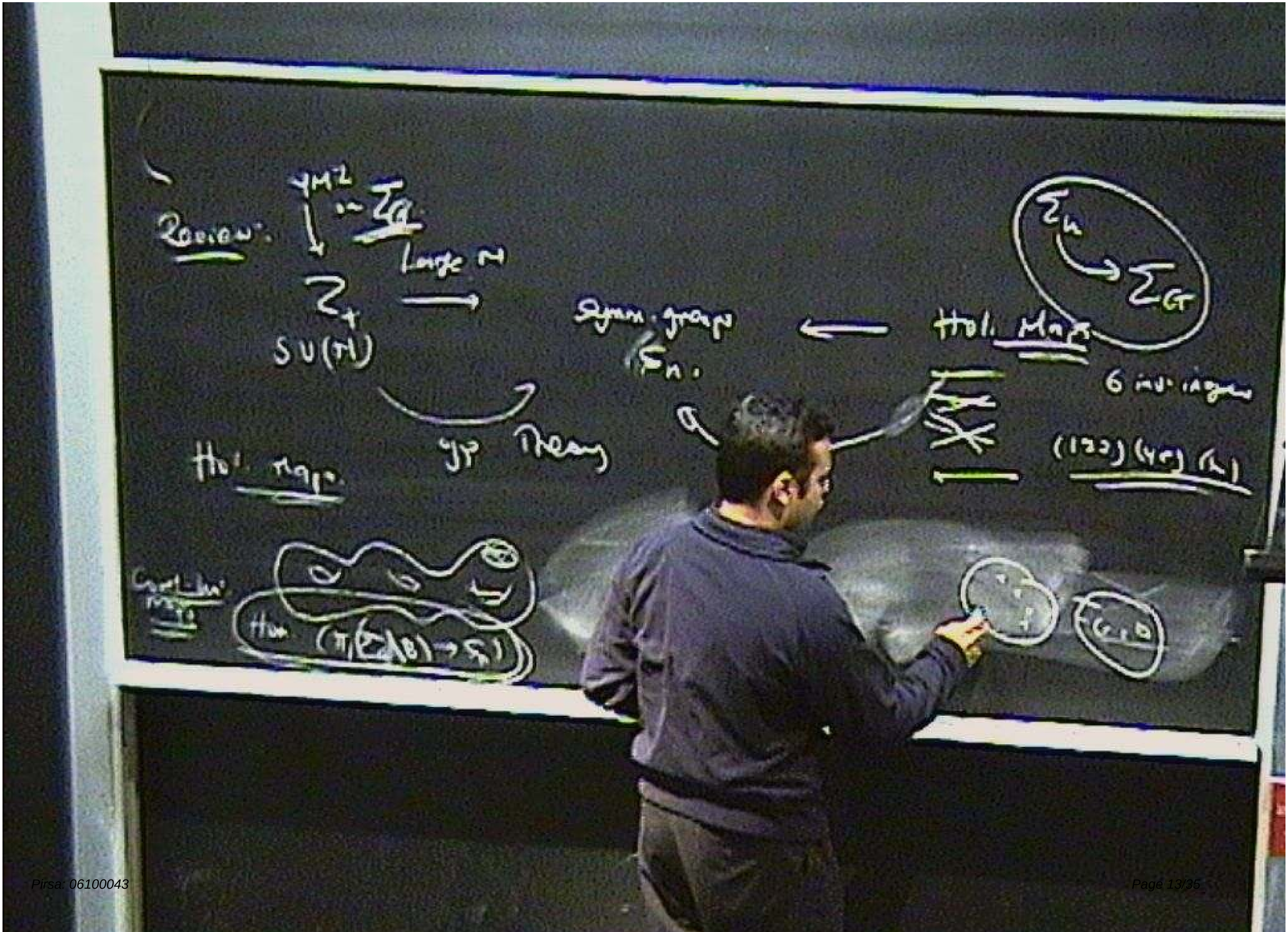

Review:
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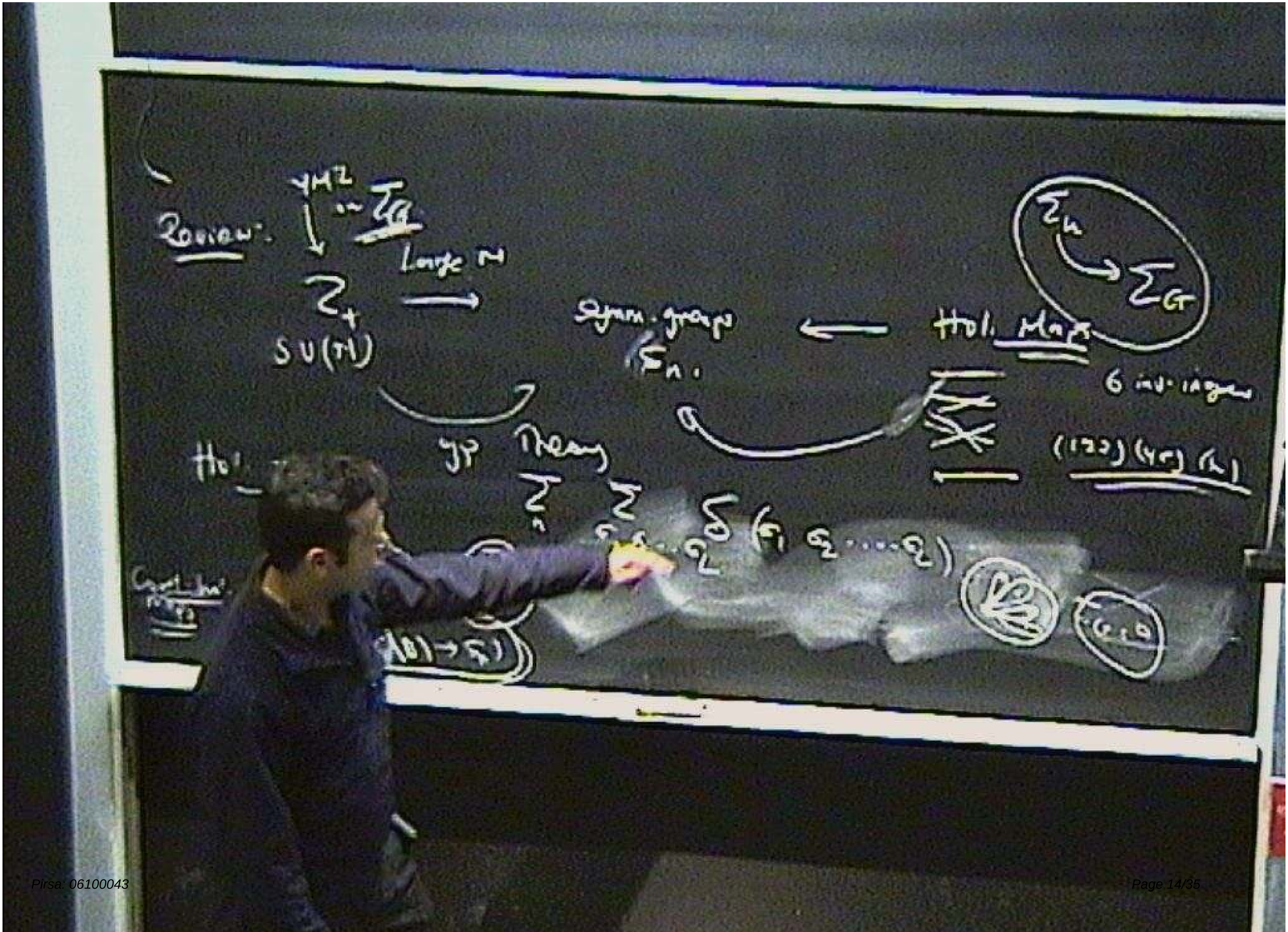
6 inv. images
Theory
Hol. maps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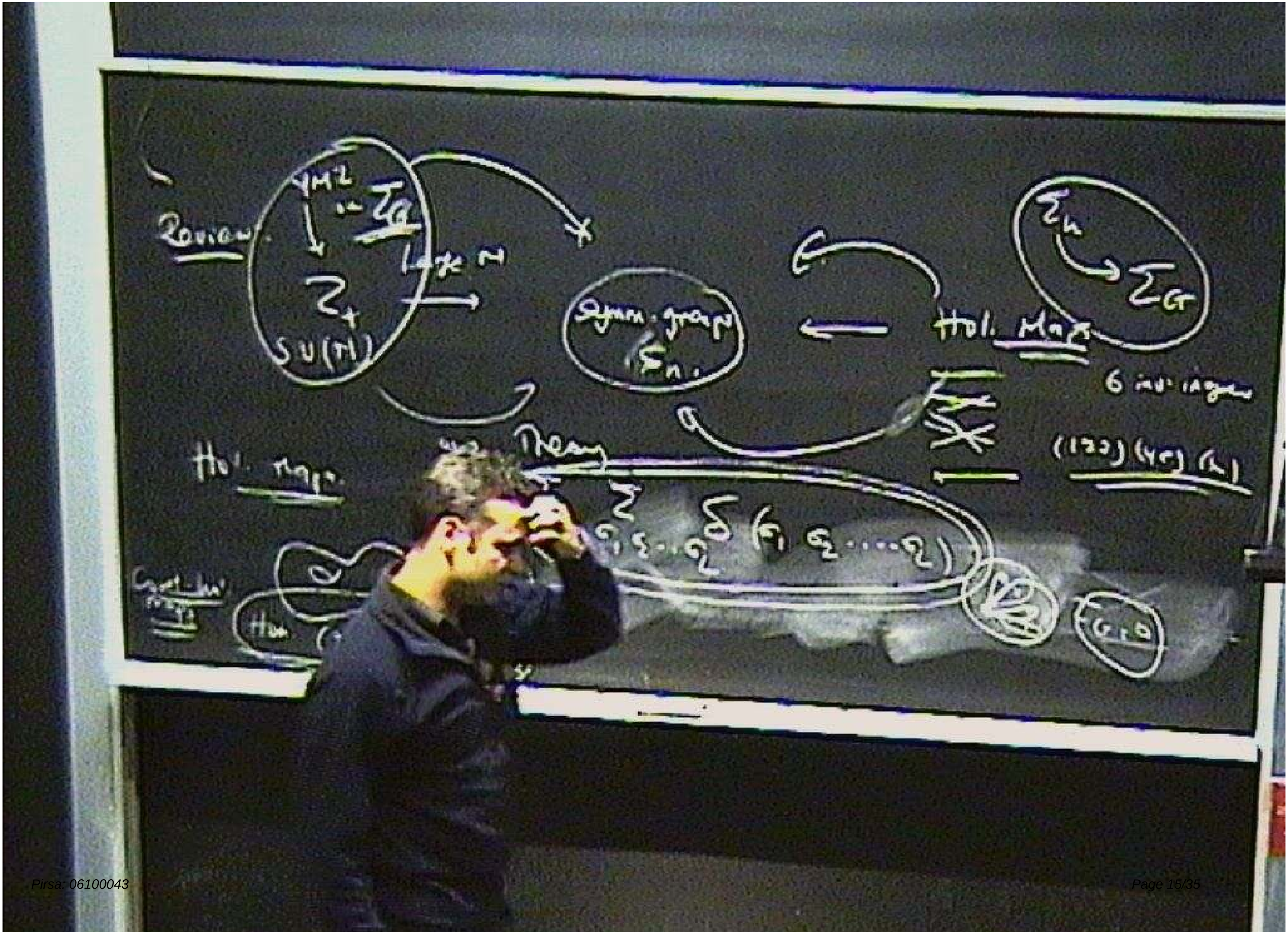
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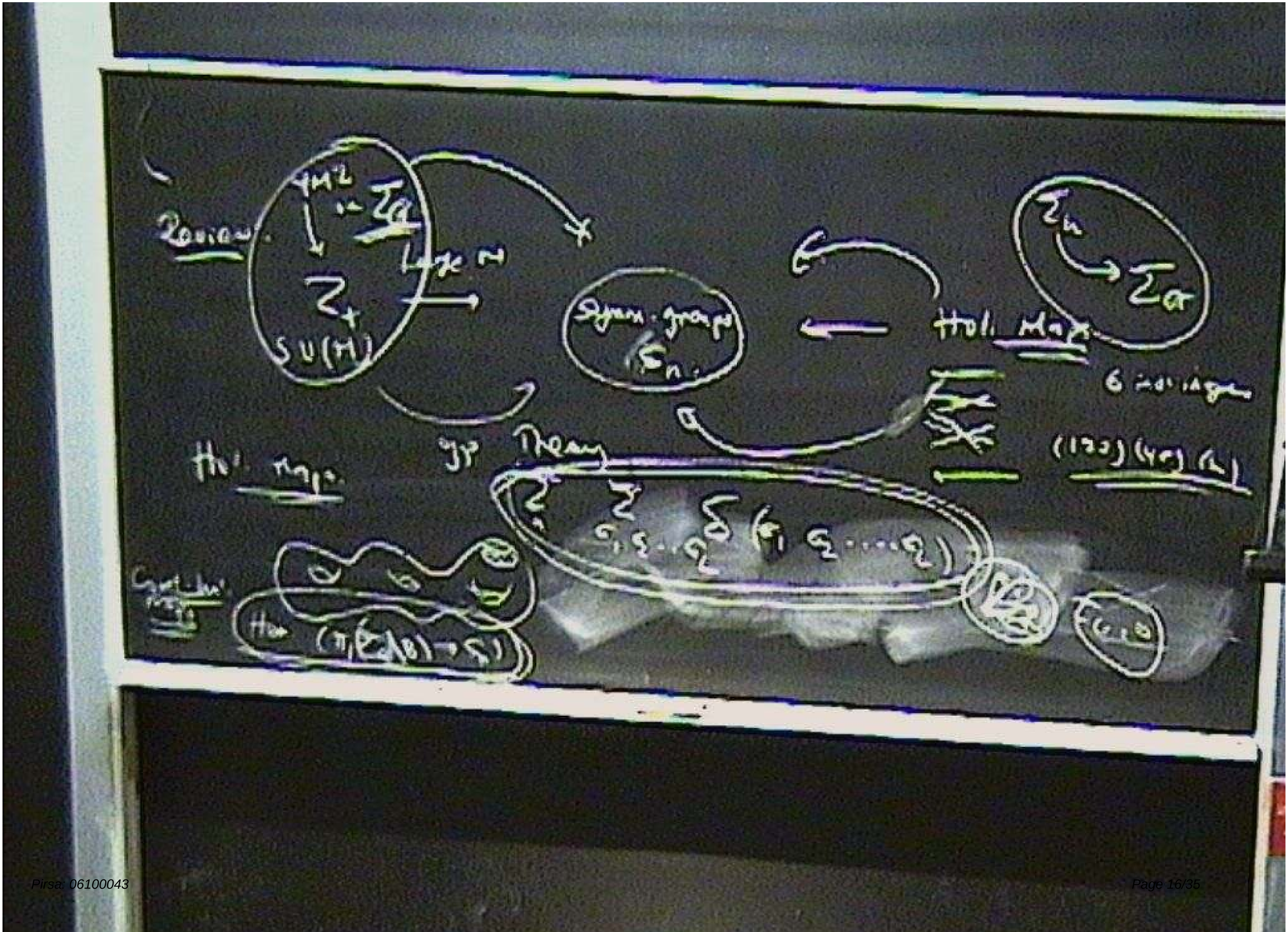
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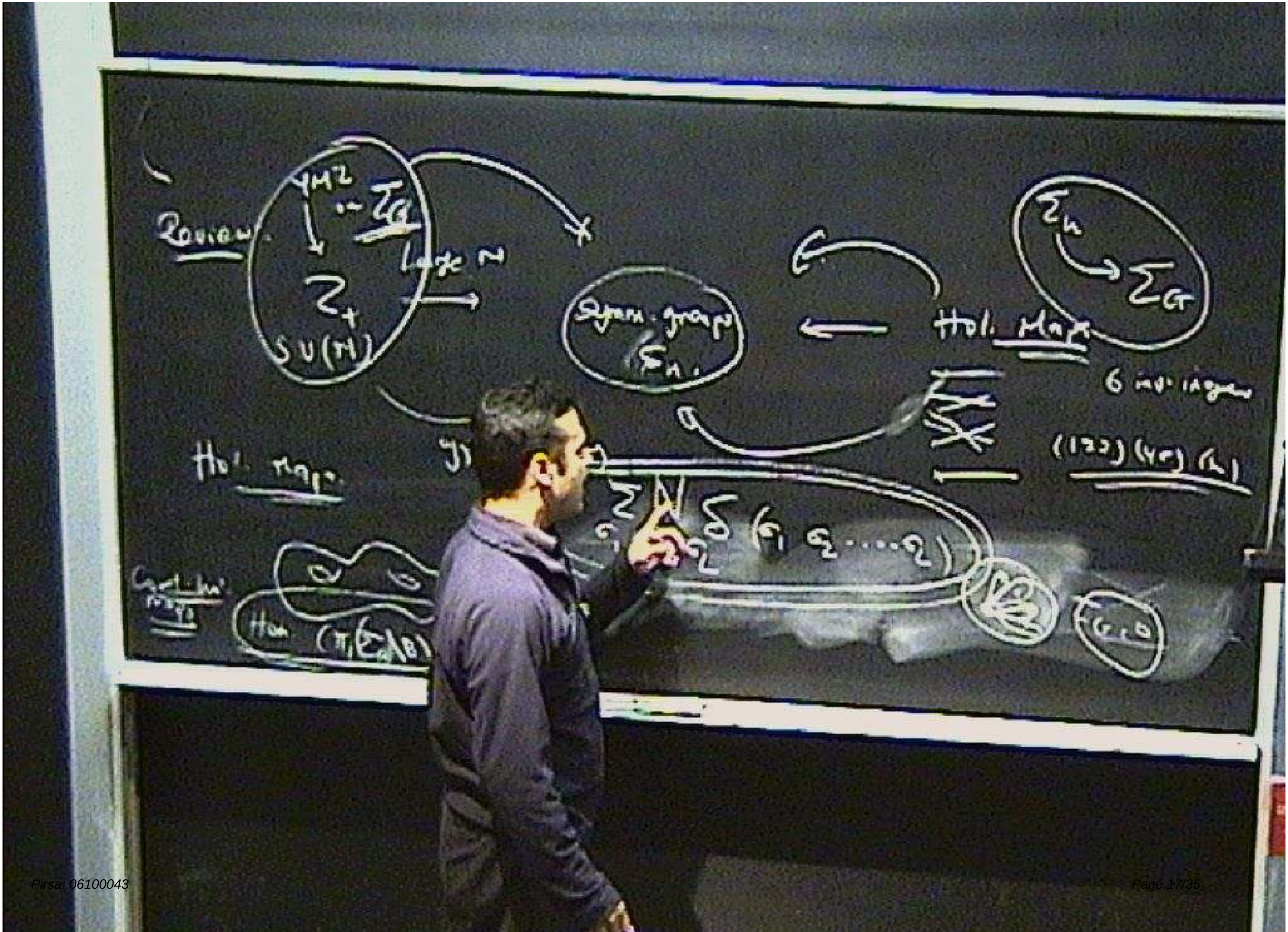
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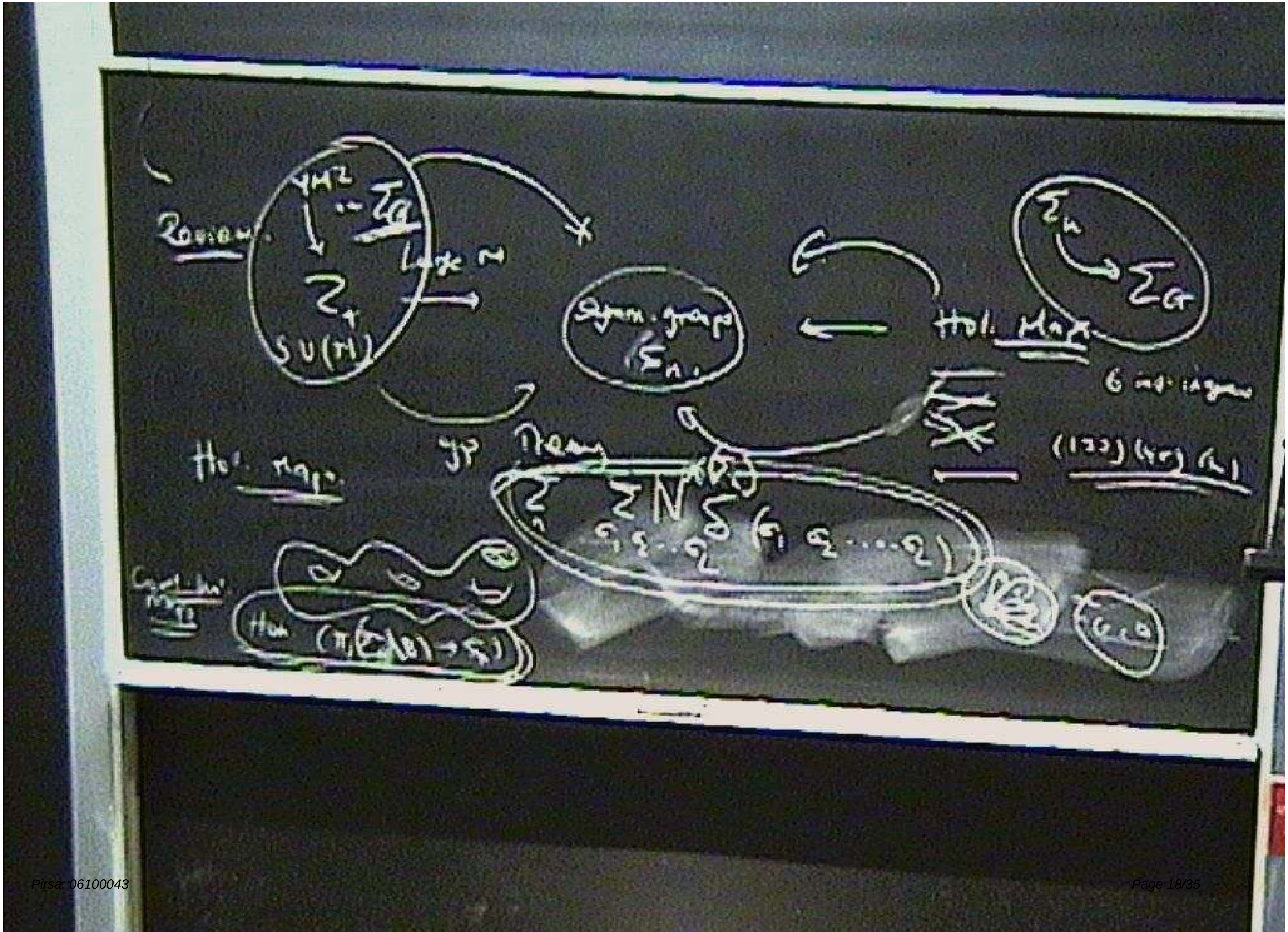
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Part I : Large N expansion of $Z_+(q=1)$

- The large N expansion of

$$Z_+ = \sum_{n=0}^{\infty} \sum_{R \in Y_n} (DimR)^{2-2G}$$

can be given in terms of symmetric groups as follows :

$$Z_+ = \sum_{n=0}^{\infty} \sum_{s_i, t_i} \frac{N^{n(2-2G)}}{n!} \delta(\Omega^{2-2G} \prod_{i=1}^{G} s_i t_i s_i^{-1} t_i^{-1})$$

- $s_1, t_1 ... s_G, t_G$ are summed over the symmetric group S_n. Ω is an element in the group algebra $\mathbb{C}(S_n)$.

$$\begin{aligned} \Omega &= \sum_{\sigma} N^{K_\sigma - n} \sigma \quad = \left(1 + \sum_{\sigma}{}' N^{K_\sigma - n} \sigma\right) \\ &= 1 + \frac{1}{N} T_2 + \cdots \end{aligned}$$

It is related to $Dim\ R$ by

$$Dim\ R = \frac{N^n}{n!} \chi_R(\Omega)$$

- **Note :** Ω is a sum of central elements.

$$\begin{aligned} \Omega &= \sum_{T} (\sum_{\sigma \in T} \sigma) N^{K_T - n} \\ &= \sum_{T} C_T N^{K_T - n} \end{aligned}$$

- $\Omega^{-1} = 1 - \sum_{\sigma} N^{K_\sigma - n} \sigma + \frac{(-1)(-2)}{1 \cdot 2} \sum_{\sigma_1, \sigma_2} N^{(K_{\sigma_1} - n) + (K_{\sigma_2} - n)} \sigma_1 \sigma_2 \cdots$

Ω^{2-2G} ∞

- This formula can be derived from **Schur-Weyl duality**

$$V^{\otimes n} = \oplus_{R \in Y_n} R^{U(N)} \otimes R^{S_n}$$

which leads to

$$tr(\sigma U) = \sum_R \chi_R(\sigma)\chi_R(U)$$

Then use orthogonality for S_n characters to obtain

$$\chi_R(U) = \frac{1}{n!}\sum_\sigma \chi_R(\sigma) tr(\sigma U)$$

Setting $U = 1$ gives the dimension formula in terms of Ω.

- Can also use this to derive the projector

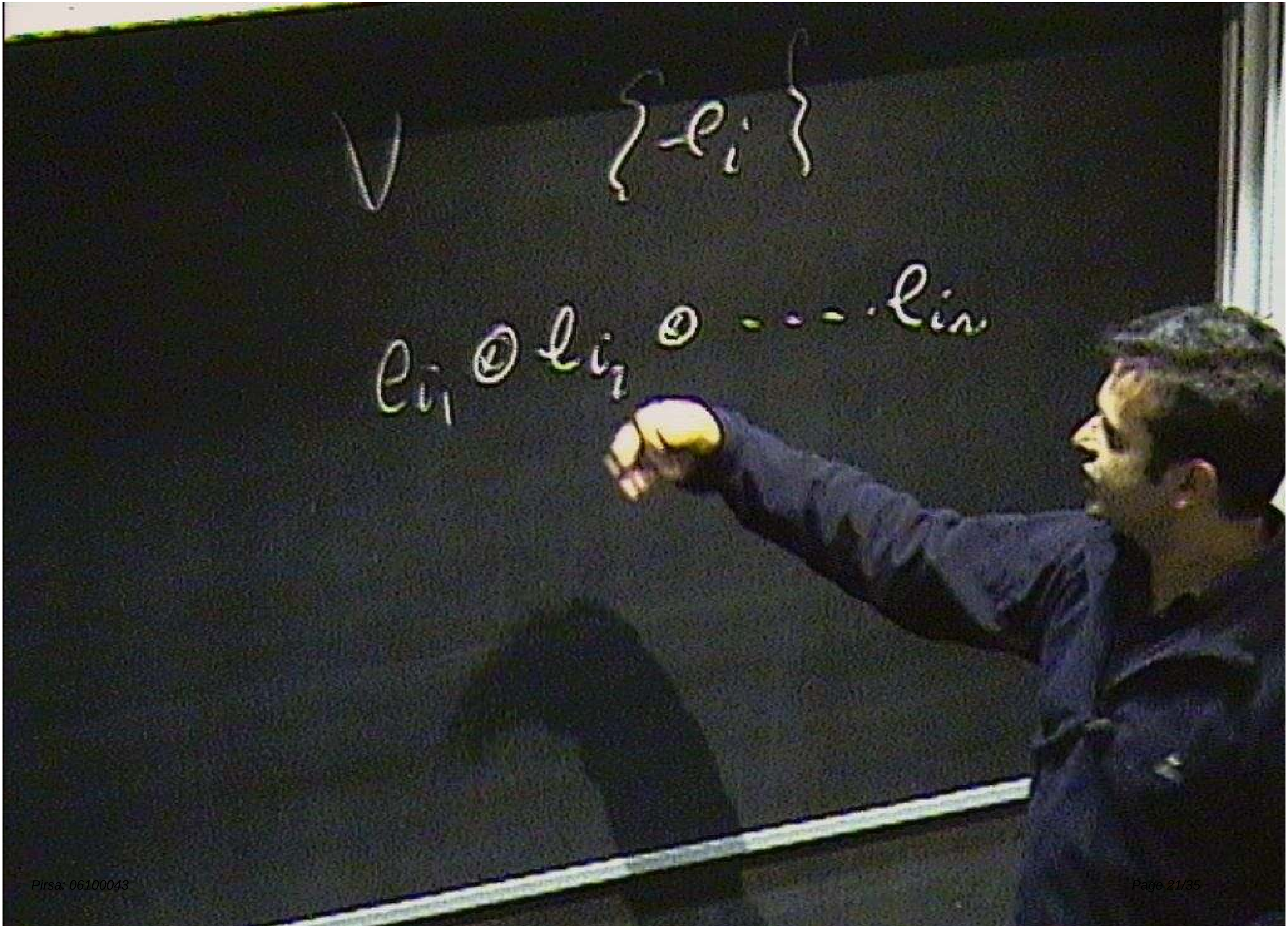
$$P_R = \frac{d_R}{n!}\sum_\sigma \chi_R(\sigma^{-1})\sigma$$

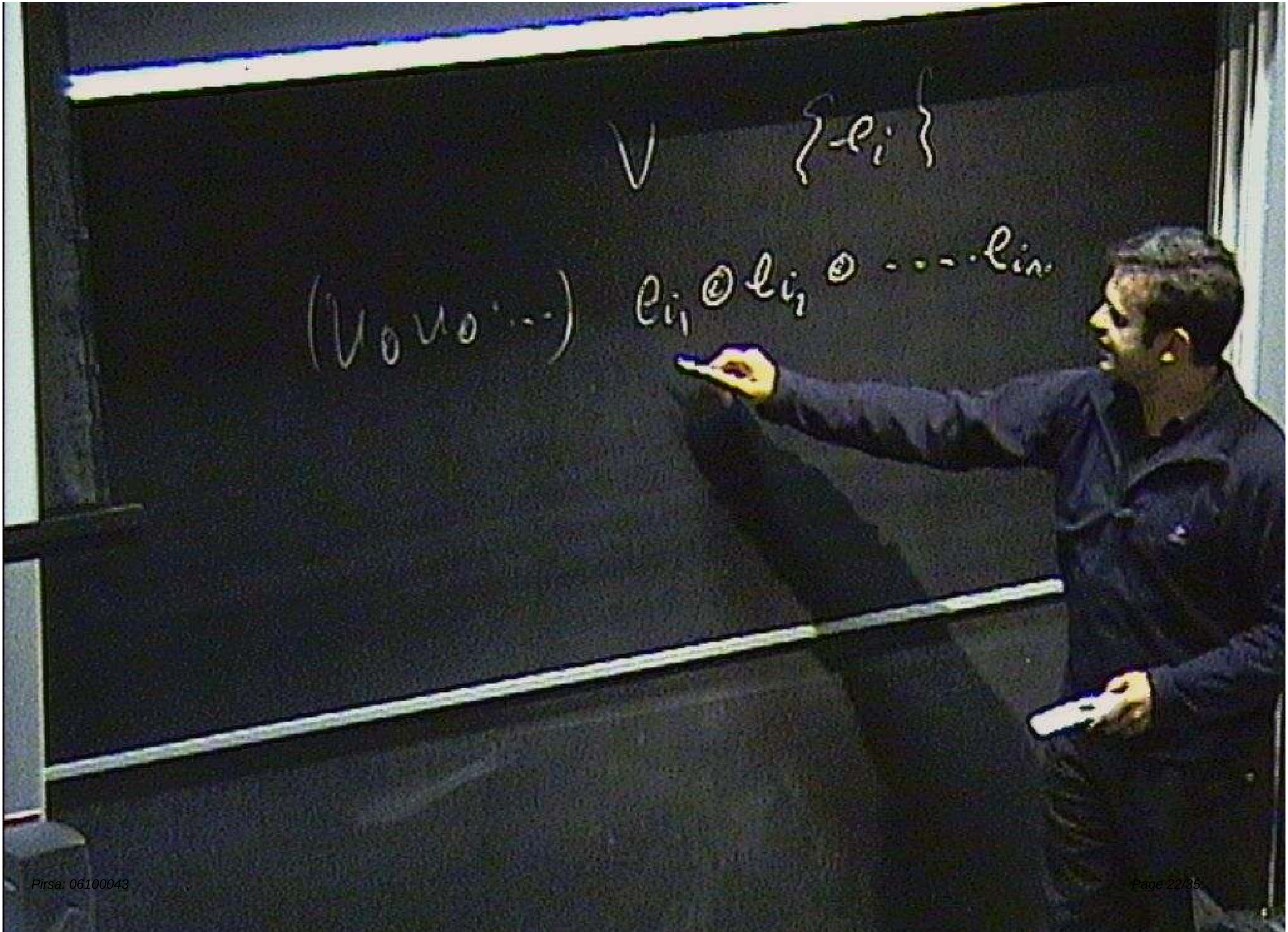
- The delta function on the symmetric grop

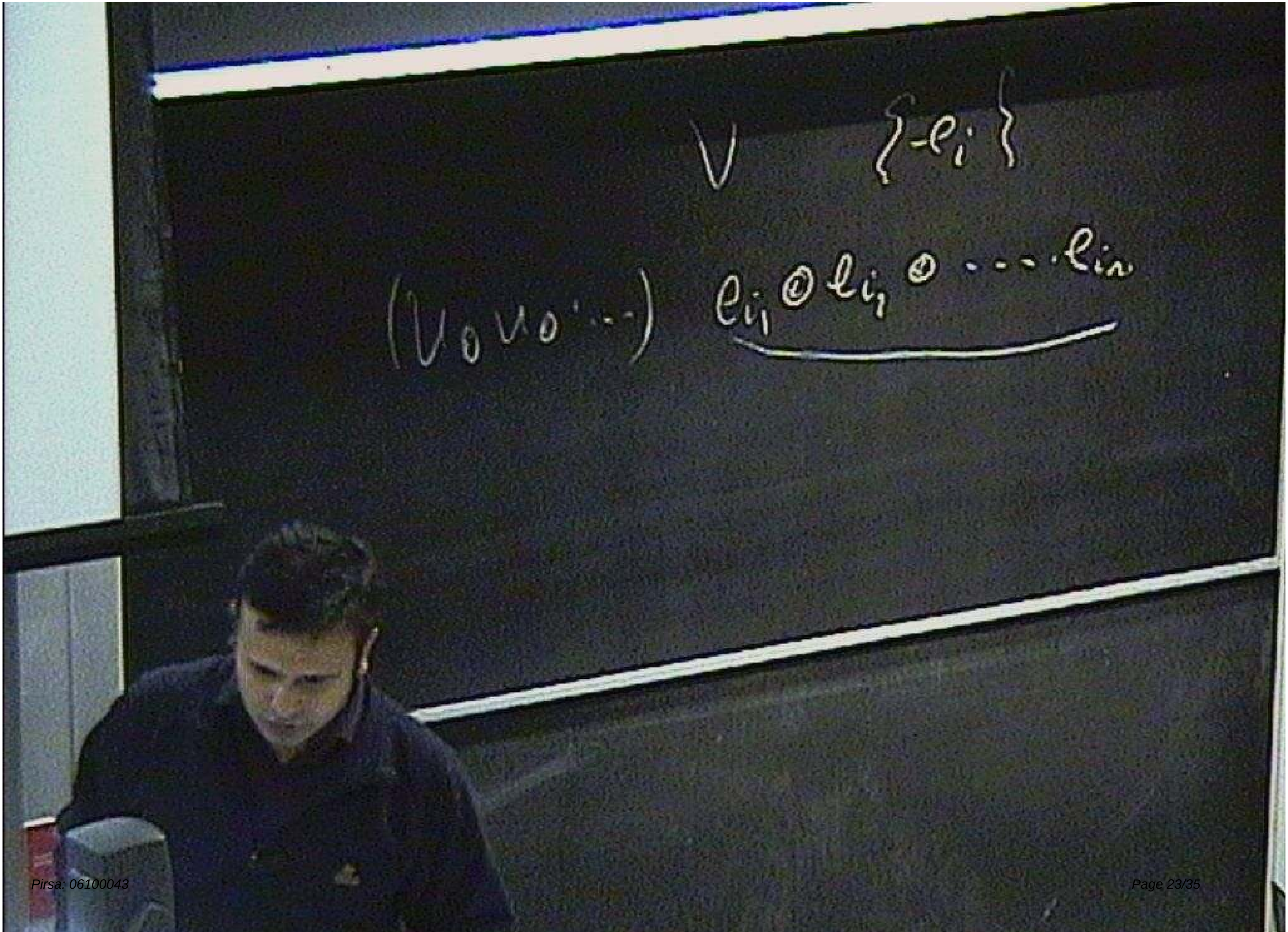
$$\begin{aligned}\delta(\sigma) &= 1 \text{ if } \sigma = 1 \\ &= 0 \text{ otherwise}\end{aligned}$$

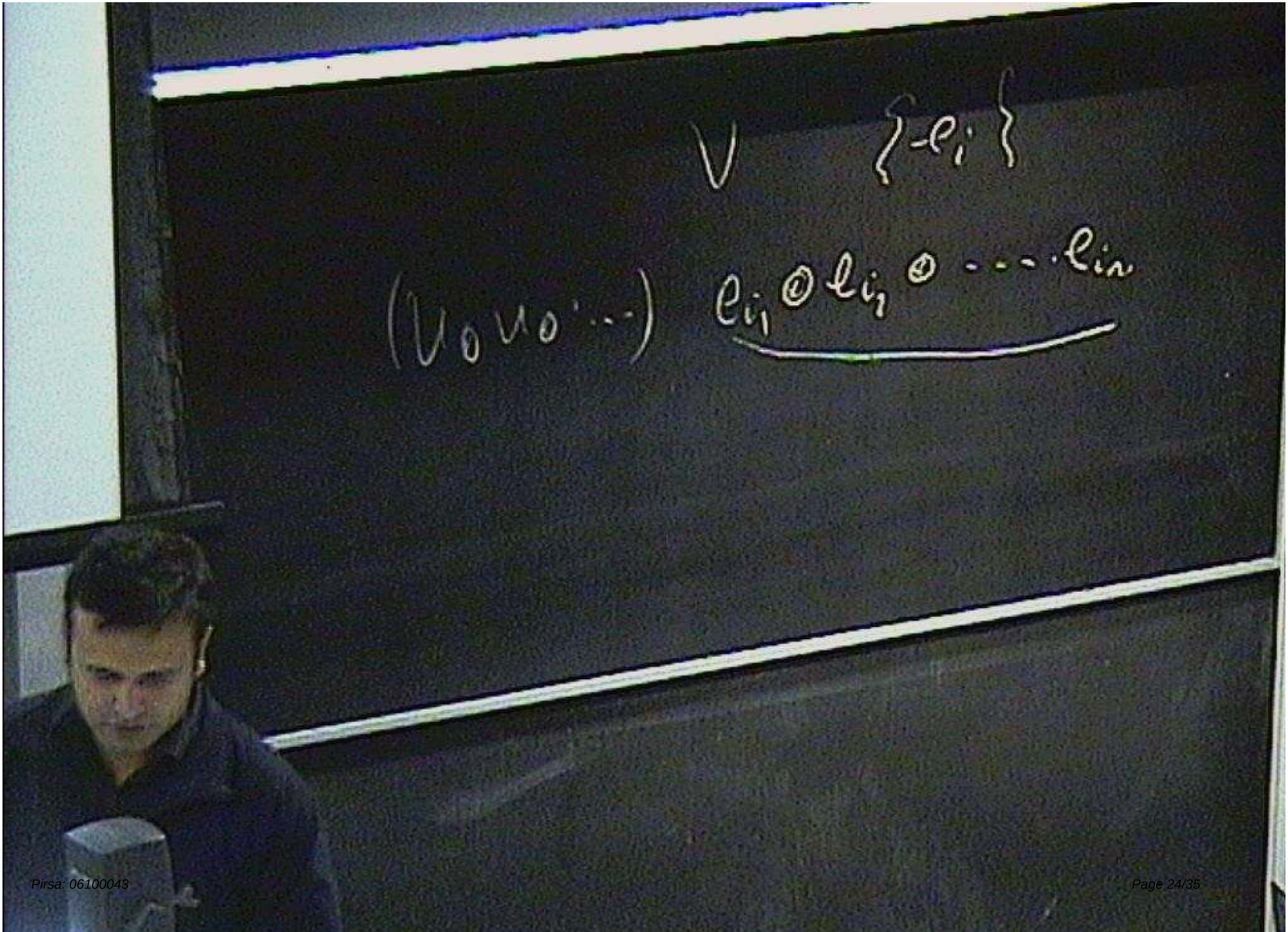
It has a character expansion

$$\delta(\sigma) = \sum_R \frac{d_R}{n!}\chi_R(\sigma)$$


V
{e_i}
e_{i_1} ⊗ e_{i_2} ⊗ ... e_{i_n}


V
{e_i}
(V⊗V⊗...)
e_{i_1} ⊗ e_{i_2} ⊗ .... e_{i_n}


V
$\{e_i\}$
$(V \otimes V \otimes \cdots)$
$e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n}$


V $\{e_i\}$
$(v \otimes v \otimes \cdots)$ $e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n}$

PART II : The q-deformed chiral partition function

- The key theorem leading to the chiral expansion of Z_+ was **Schur-Weyl duality** between representations of $SU(N)$ (or equivalently of the universal enveloping algebra $U(su(N))$) in $V^{\otimes n}$ and its commutant $\mathbb{C}(S_n)$.

- The Schur-Weyl duality can be extended to $q \neq 1$ and relates $U_q(su(N))$ and $H_q(n)$. (Jimbo 1986).

- For concreteness recall $U_q(su(2))$ is the deformation of $U(su(2))$

$$
\begin{aligned}
[H, X_+] &= 2X_+ \\
[H, X_-] &= -2X_- \\
[X_-, X_+] &= \frac{(q^{H/2} - q^{-H/2})}{q^{1/2} - q^{-1/2}} \longrightarrow H
\end{aligned}
$$

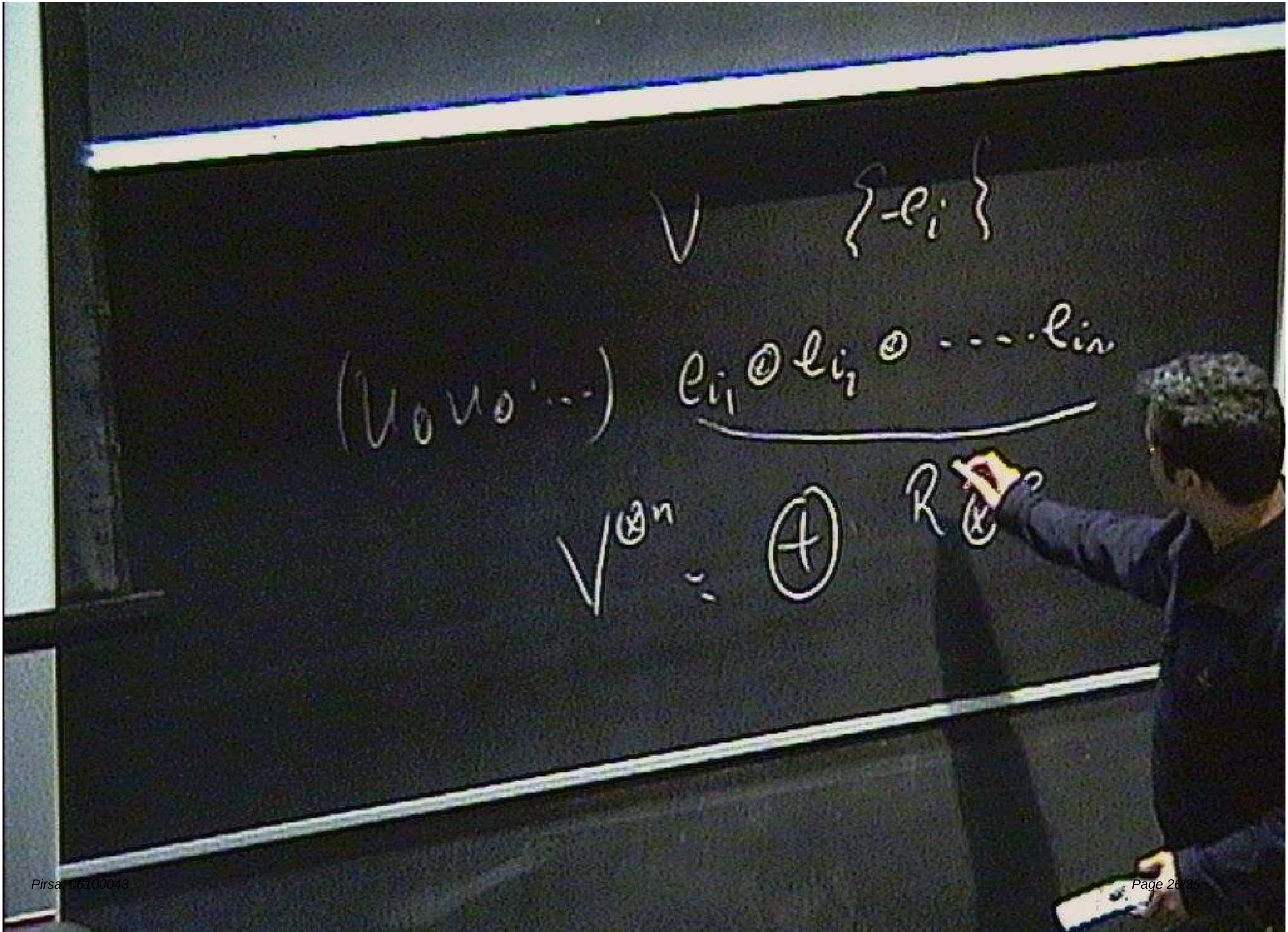
The action of $X_\pm$ on $V \otimes V$ is given by

$$\Delta(X_\pm) = X_\pm \otimes q^{H/4} + q^{-H/4} \otimes X_\pm$$

This is not symmetric under exchange of the two factors, hence the symmetric group is no longer the commutant. Rather, it is the Hecke algebra, which is constructed from the action of the R-matrix in $V \otimes V$.

$$\Delta'(X_\pm) = q^{H/4} \otimes X_\pm + X_\pm \otimes q^{-H/4}$$

$$\Delta' = \text{"} \ \underline{\underline{R}} \, \Delta \, \underline{\underline{R}}^{-1}$$


$V \quad \{e_i\}$
$(v \otimes v \otimes \cdots) \quad e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n}$
$V^{\otimes n} = \bigoplus \quad R \otimes$

$V \qquad \{e_i\}$

$(V \otimes V \otimes \cdots) \qquad \underbrace{e_{i_1} \otimes e_{i_2} \otimes \cdots \cdots e_{i_n}}_{U(n)}$

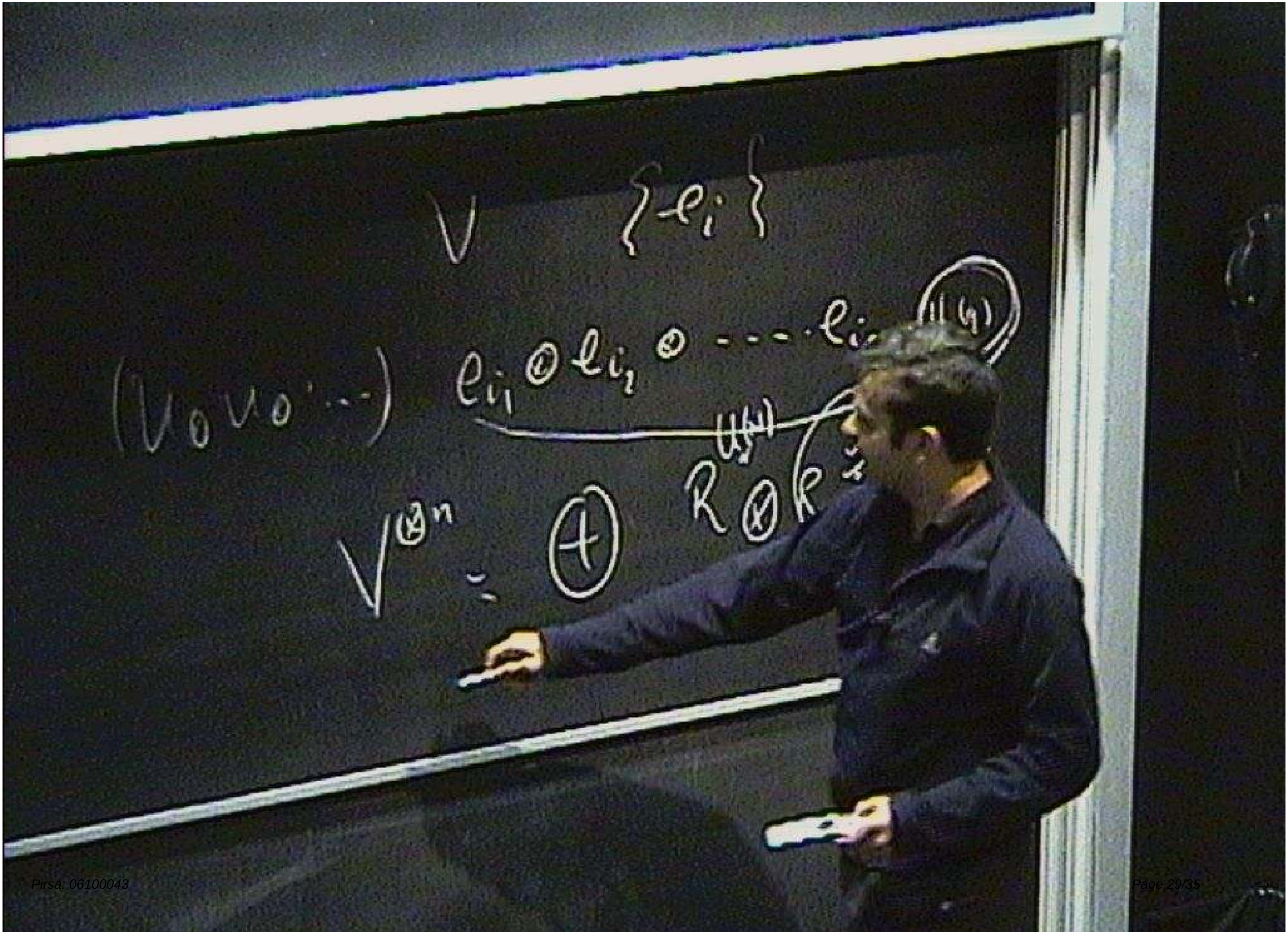
$V^{\otimes n} = \bigoplus \; R \otimes R$

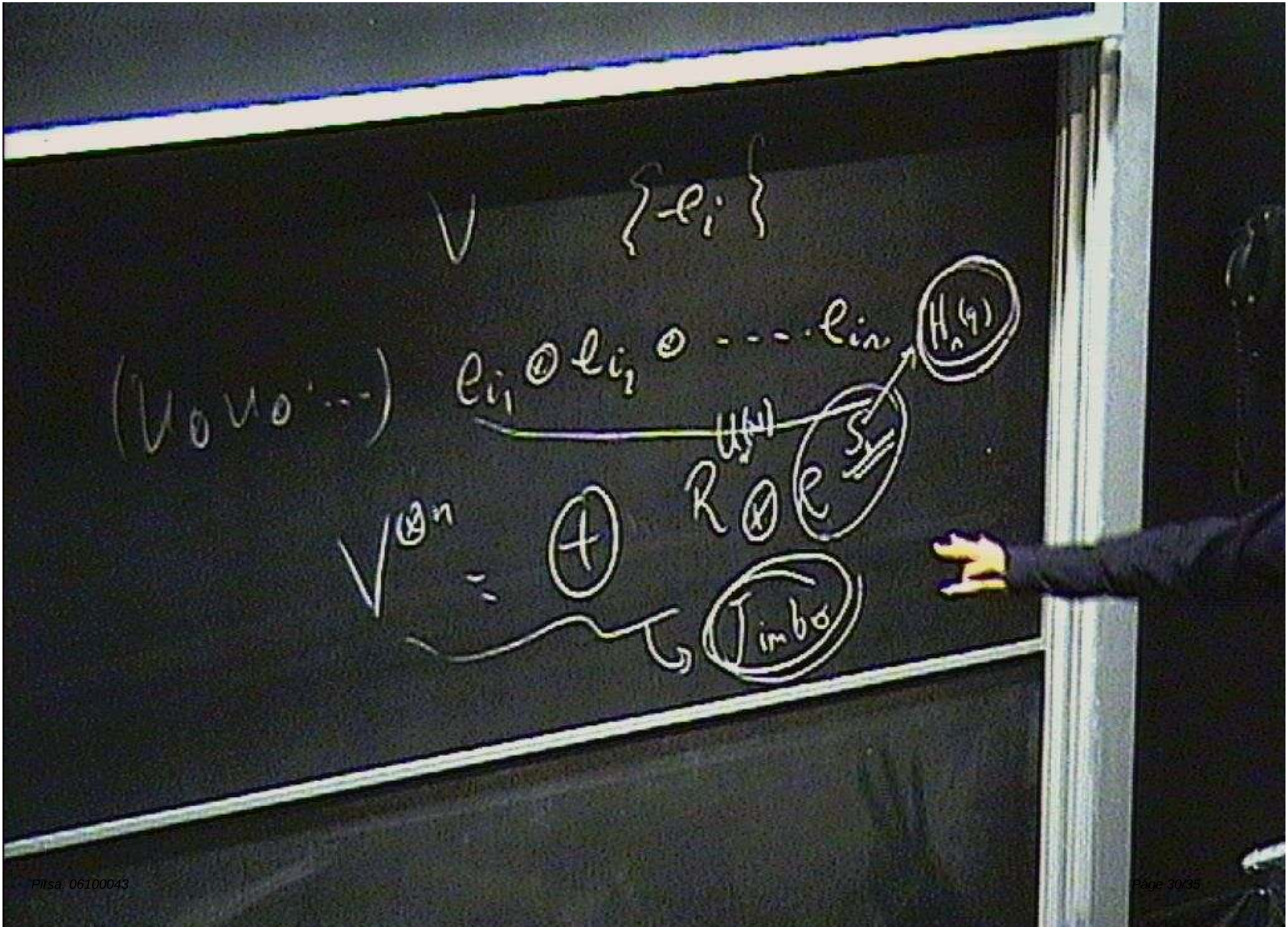
$V \quad \{e_i\}$

$(V \otimes V \otimes \cdots)$ $e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n}$ $H_n^{(q)}$

$U_n^{(q)}$

$V^{\otimes n} = \bigoplus R \otimes e^{S_n}$


V {e_i}
(V⊗V⊗...)
e_{i1} ⊗ e_{i2} ⊗ ....e_{in}
V^{⊗n} = ⊕ R⊗R


V
$\{e_i\}$
$(V \otimes V \otimes \cdots)$
$e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n}$
$H_n(q)$
$U_q(sl_2)$
S_n
$R \otimes R$
$V^{\otimes n} = \bigoplus$
Jimbo

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The action of $X_\pm$ on $V \otimes V$ is given by

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This is not symmetric under exchange of the two factors, hence the symmetric group is no longer the commutant. Rather, it is the Hecke algebra, which is constructed from the action of the R-matrix in $V \otimes V$.

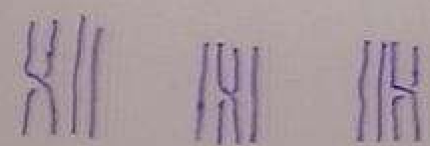
$$\Delta'(X_\pm) = q^{H/4} \otimes X_\pm + X_\pm \otimes q^{-H/4}$$

$$\Delta' = \underline{\underline{R}} \, \Delta \, \underline{\underline{R}}^{-1}$$

- The symmetric group S_n can be described in terms of generators $s_1, \cdots, s_{n-1}$ obeying relations :

$$
\begin{aligned}
s_i^2 &= 1 \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \\
s_i s_j &= s_j s_i \qquad \text{for} |i-j| \geq 2
\end{aligned}
$$

FIGURE



s_1 s_2 s_3

s_1, s_2, s_3 generate S_4.

- The **Hecke algebra generators** obey the deformed relations

$$
\begin{aligned}
g_i^2 &= (q-1)g_i + q \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \\
g_i g_j &= g_j g_i \qquad \text{for} |i-j| \geq 2
\end{aligned}
$$

- There is a linear basis for $H_q(n)$ given by $h(\sigma)$ for $\sigma \in S_n$. h is defined by expressing σ as a word in the s_i of minimal length (i.e use a reduced decomposition) and then replace $s_i \to g_i$.

- Using these ingredients we can write the q-deformed chiral expansion

$$Z^{+} = \sum_{n=0}^{\infty} \sum_{s_i,t_i} \frac{1}{g} [N]^{(2-2G)n} \delta\left(D\,\Omega^{2-2G} \prod_{i=1}^{G} q^{-l(s_i)-l(t_i)} h(s_i)h(t_i)h(s_i^{-1})h(t_i^{-1}) \right).$$

(9)

- $[N]$ is the q-number

$$[N] = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}$$

- The element D defined by

$$\chi_R(D) = d_R(q)$$

It goes to 1 as $q \to 1$.

We can give an expicit q-expansion as follows. Define

$$\begin{aligned} E &= \textstyle\sum_s q^{-l(s)} h(s^{-1})h(s) \\ &= 1 + \textstyle\sum_s' q^{-l(s)} h(s^{-1})h(s) \end{aligned}$$

We can show that

$$D = gE^{-1}$$

Summary and Outlook

- We described the deformation of the symmetric group data of the chiral large N expansion to Hecke algebra data. At $q = 1$ the symmetric group data is directly related to holomorphic maps (branched covers) and hence to the string theory on Σ_G target.

$$Z_+(q=1) \rightarrow \begin{array}{c}\text{symmetric}\\ \text{group}\\ \text{data}\end{array} \rightarrow \begin{array}{c}\text{Holomorphic}\\ \text{maps}\end{array} \rightarrow \begin{array}{c}\text{String}\\ \text{action}\end{array}$$

$$Z_+(q) \rightarrow \begin{array}{c}\text{Hecke}\\ \text{algebra}\\ \text{data}\end{array} \rightarrow\ ??\ \rightarrow \begin{array}{c}\text{String}\\ \text{on}\\ \text{Calabi-Yau}\end{array}$$

- The first question raised by this picture is : What does the Hecke algebra have to do with maps from Σ_h to the Calabi-Yau.

$$\begin{array}{l} X \leftarrow L_1 \oplus L_2 \\ \downarrow \\ \Sigma_G \end{array}$$

- The large N expansion of YM2 on Σ_G, expressed in terms of symmetric groups, has been interpreted in terms of string theory with target space Σ_G. The key connection between algebraic data and geometry was the correpondence between

$$\begin{array}{c} Hom(\pi_1(\Sigma) \rightarrow (\Sigma_G \setminus B)) \\ \downarrow \\ \text{Branched covers of } \Sigma_G \end{array} \qquad (2)$$

- In the q-deformed case, the algebraic data has been q-deformed and is expressed in terms of Hecke algebras. A second conjecture/guess is that the Hecke data corresponds to maps between **q-deformed Riemann surfaces**. For example they could be surfaces which "locally" look like the q-plane $xy = qyx$. Even a demonstration in the case of the q-plane, of such a connection between holomorphic functions and Hecke algebras, would be an interesting step.

- Another possible geometric interpretation comes from the fact that the same Hecke algebra $H_q(n)$ has another origin, not obviously related to quantum groups.

$H_n(q)$ is an **algebra of double cosets**

$$B_n(F_q) \setminus GL_n(F_q)/B_n(F_q)$$

Here F_q is the finite field with q elements, where q is a power of a prime p.

$GL_n(F_q)$ is the group of $n \times n$ matrices with entries in F_q. $B_n(F_q)$ is the subgroup of the upper triangular matrices. This generalises the fact that S_n appears from double cosets

$$B_n(\mathbb{C}) \setminus GL_n(\mathbb{C})/B_n(\mathbb{C})$$

Hence the deformation of $\mathbb{C}S_n$ to the Hecke algebra $H_n(q)$ corresponds to going from $\mathbb{C}$ to F_q.

This suggests that, at least for q equal to a power of a prime, the Hecke-q-deformed Hurwitz counting problem, might be related to Riemann surfaces over F_q.