Title: Towards Understanding String Spectrum in AdS5 x S5

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Abstract:

Towards understanding string spectrum in $AdS_5 \times S^5$

J.A. Minahan, A. Tirziu and A.A.T.

"Infinite spin limit of semiclassical string states," hep-th/0606145

M. Kruczenski, J. Russo and A.A.T.

"Spiky strings and giant magnons on S5," hep-th/0607044

work in progress with R. Roiban

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

Parameters:

$\lambda = g^2_{_{YM}} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - global charges of $SO(2,4) \times SO(6)$:

spins $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve susy 4-d CFT = string in R-R background:

compute $E = \Delta$ for any λ (and J,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

"Constructive" approach:

use perturbative results on both sides and other properties (integrability, susy,+?) to guess exact answer (Bethe ansatz,...)

Remarkable recent progress:

– "semiclassical" states with large quantum numbers

dual to "long" gauge operators

$E = \Delta$ – same dependence on $J, m, ...$

coefficients = interpolating functions of λ

– connection to spectrum of integrable spin chains

– advances in uncovering underlying Bethe ansatz

String Theory in $AdS_5 \times S^5$

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar\theta(D + F_5)\theta\partial x$$

$$+ \bar\theta\theta\bar\theta\theta\partial x\partial x + ... \Big]$$

$T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$ (Metsaev, AT 98)

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability (Bena, Polchinski, Roiban 02)

Progress in detailed understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert et al 05; Dorey, Vacedo 06,...)

Explicit computation of 1-loop quantum superstring $(1/T)$ corrections to classical string energies (Frolov, AT 02-4, ...)

Near-geodesic expansion (Parnachev, Ryzhov; Callan, Lee, McLoughlin, Schwarz, Swanson, Wu 03; ...)

1-loop S-matrix? Beyond 1-loop? Quantum integrability?

$\mathcal{N}=4$ Conformal Gauge Theory

Dimensions of operators: eigenvalues of dilatation operator

e.g., operators built out of SYM scalars (dual to strings in S^5)

$SU(2)$ sector: $\mathrm{Tr}(\Phi^{J_1}\Phi^{J_2}) + \ldots, \quad J = J_1 + J_2$

$\Phi_1 = \phi_1 + i\phi_2, \quad \Phi_2 = \phi_3 + i\phi_4$

planar 1-loop dilatation operator of $\mathcal{N}=4$ SYM:

= Hamiltonian of ferromagnetic Heisenberg $\mathrm{XXX}_{1/2}$ spin chain (Minahan, Zarembo 02):

$$H_1 = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1})$$

Higher orders (Beisert, Kristjansen, Staudacher 03; Beisert 04; Eden, Jarczak, Sokatchev 04):

$H_2 = \frac{\lambda^2}{(4\pi)^4} \sum_{l=1}^{J} (-3 + 4\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - \vec{\sigma}_l \cdot \vec{\sigma}_{l+2})$

H_3 contains $\vec{\sigma}_l \cdot \vec{\sigma}_{l+3}$ but also $(\vec{\sigma}_l \cdot \vec{\sigma}_{l+1})(\vec{\sigma}_{l+2} \cdot \vec{\sigma}_{l+3})$, etc.

operator dimensions = eigenvalues of "long-range" ferromagnetic spin chain H with "multi-spin" interactions

H_{eff} for Hubbard model (at least to 3 loop order) (Rej, Serban, Staudacher 05)

Spectrum? Compare to string theory?

Integrability (!) $\longrightarrow$ Bethe ansatz $\longrightarrow$ Spectrum

1-loop: Heisenberg model $\longrightarrow$ Bethe ansatz equations:

$$e^{ip_k J} = \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i},$$

$u_j = \frac{1}{2}\cot\frac{p_j}{2}, \qquad J = J_1 + J_2, \qquad M = J_2$

$E = J + \frac{\lambda}{\pi^2}\sum_{j=1}^{M}\sin^2\frac{p_j}{2}, \quad \sum_{j=1}^{M} p_j = 2\pi m$

Indications of integrability of both string ($\lambda \gg 1$) and gauge ($\lambda \ll 1$) theory: expect Bethe ansatz description for any λ (Beisert, Dippel, Staudacher 04)

$$e^{ip_k J} = \prod_{j\neq k}^{M} S(p_k, p_j; \lambda), \qquad S = S_1\, e^{i\theta}$$

$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad \theta = \theta(p_k, p_j; \lambda)$

$u_j(p_j, \lambda) = \frac{1}{2}\cot\frac{p_j}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}}$

p_j for bound states with $\quad \sum_{k=1}^{M} p_k = 2\pi m$

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(Arutyunov, Frolov, Staudacher 04)

θ – common to all sectors, structure fixed by symmetries

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Compute $c_{rs}(\lambda)$ from "first principles"

– from quantum $AdS_5 \times S^5$ superstring

String 1-loop corrections to string energies (Frolov, AT 03; Park, Tirziu, AT 05) imply $a_{rs} \neq 0$ (Beisert, AT 05)

1-loop string results translate into (Hernandez, Lopez 06)
$a_{rs} = \frac{2}{\pi}[1-(-1)^{r+s}]\frac{(r-1)(s-1)}{(r-1)^2-(s-1)^2}$

Consistent (Arutyunov, Frolov 06; Beisert 06) with crossing condition (Janik 06)

Beyond 1-loop order? Which are additional constraints?

Various Attempts:

• compute S-matrix directly from superstring theory

Important conceptual role played by non-relativistic "Landau-Lifshitz" type effective action for positive energy magnons (Kruczenski 03)

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• String sigma model (in conformal gauge): suggests interpret S as "effective" scattering matrix of integrable Lorentz-invariant 2d field theory whose effective excitations correspond to spin chain magnons (Polchinski, Mann 05; Gromov, Kazakov, Sakai, Viera 06; Gromov, Kazakov 06)

• detailed study of spectrum in various limits on gauge and string sides $\longrightarrow$ extra constraints on S-matrix

Key assumption:

Expect spectrum to have qualitatively same structure at any λ (at least for large J)

smooth change with λ: no transition on the way from small to large λ

Indeed, remarkable evidence (qualitative and quantitative) of correspondence between string and gauge states

sometimes works better than one could expect (susy: non-renormalization of some coefficients, ...)

Plan:

compare weak-coupling spin chain spectrum with semiclassical string spectrum

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$$q_{r+1}(p) = \frac{2}{r} \sin\frac{rp}{2}\left(\frac{\sqrt{1+4\bar{\lambda}\sin^2\frac{p}{2}}-1}{\bar{\lambda}\sin\frac{p}{2}}\right)^r, \qquad \bar{\lambda} \equiv \frac{\lambda}{(2\pi)^2}$$

Matching to classical string:

$(c_{rs})_{\lambda\to\infty} \ \to \ \lambda^{\frac{r+s-1}{2}}\delta_{r,s-1}$ (AFS)

at large λ expect from string theory

$c_{rs}(\lambda \gg 1) = \bar{\lambda}^{\frac{r+s-1}{2}}[\delta_{r,s-1} + \frac{1}{\sqrt{\bar{\lambda}}}a_{rs} + \frac{1}{(\sqrt{\bar{\lambda}})^2}b_{rs} + ...]$

$c_{rs}(\lambda \ll 1) \ \to \ 0$?

Spectrum? Compare to string theory?

Integrability (!) $\longrightarrow$ Bethe ansatz $\longrightarrow$ Spectrum

1-loop: Heisenberg model $\longrightarrow$ Bethe ansatz equations:

$$e^{ip_k J} = \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i},$$

$u_j = \frac{1}{2} \cot \frac{p_j}{2}, \qquad J = J_1 + J_2, \qquad M = J_2$

$E = J + \frac{\lambda}{\pi^2} \sum_{j=1}^{M} \sin^2 \frac{p_j}{2}, \qquad \sum_{j=1}^{M} p_j = 2\pi m$

Indications of integrability of both string ($\lambda \gg 1$) and gauge ($\lambda \ll 1$) theory: expect Bethe ansatz description for any λ (Beisert, Dippel, Staudacher 04)

$$e^{ip_k J} = \prod_{j \neq k}^{M} S(p_k, p_j; \lambda), \qquad S = S_1 \, e^{i\theta}$$

$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad \theta = \theta(p_k, p_j; \lambda)$

$u_j(p_j, \lambda) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}$

p_j for bound states with $\sum_{k=1}^{M} p_k = 2\pi m$

$E = J + \sum_{j=1}^{M} \left(\sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} - 1 \right)$

$S =$ phase shift due to magnon scattering (Staudacher 05)

What about θ ?

Perturbative gauge theory: "Asymptotic" BDS ansatz

$J \to \infty$, up to λ^J order: $\quad S = S_1, \;\; \theta = 0$

But to match semiclassical string theory need $\theta \neq 0$

Perturbative string theory: "String" AFS ansatz

(Arutyunov, Frolov, Staudacher 04)

θ – common to all sectors, structure fixed by symmetries

(Beisert 05)

$$\theta(p,p';\lambda) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{rs}(\lambda)[q_s(p')q_r(p) - q_s(p)q_r(p')]$$

$$q_{r+1}(p) = \frac{2}{r}\sin\frac{rp}{2}\left(\frac{\sqrt{1+4\bar\lambda\sin^2\frac{p}{2}}-1}{\bar\lambda\sin\frac{p}{2}}\right)^r, \quad \bar\lambda \equiv \frac{\lambda}{(2\pi)^2}$$

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$c_{rs}(\lambda \ll 1) \;\to\; 0 \;\;?$

Compute $c_{rs}(\lambda)$ from "first principles"

– from quantum $AdS_5 \times S^5$ superstring

String 1-loop corrections to string energies (Frolov, AT 03; Park, Tirziu, AT 05) imply $a_{rs} \neq 0$ (Beisert, AT 05)

1-loop string results translate into (Hernandez, Lopez 06)

$a_{rs} = \frac{2}{\pi}[1 - (-1)^{r+s}]\frac{(r-1)(s-1)}{(r-1)^2-(s-1)^2}$

Consistent (Arutyunov, Frolov 06; Beisert 06) with crossing condition (Janik 06)

Beyond 1-loop order? Which are additional constraints?

Various Attempts:

- compute S-matrix directly from superstring theory

Important conceptual role played by non-relativistic "Landau-Lifshitz" type effective action for positive energy magnons (Kruczenski 03)

S-matrix of magnons with "non-relativistic" dispersion relation (Klose, Zarembo 06)

S= effective string theory S-matrix of "positive-energy" branch of BMN-type string modes: "integrate out" negative-energy branch (Roiban, Tirziu, AT 06)

• String sigma model (in conformal gauge): suggests interpret S as "effective" scattering matrix of integrable Lorentz-invariant 2d field theory whose effective excitations correspond to spin chain magnons (Polchinski, Mann 05; Gromov, Kazakov, Sakai, Viera 06; Gromov, Kazakov 06)

• detailed study of spectrum in various limits on gauge and string sides $\rightarrow$ extra constraints on S-matrix

Key assumption:

Expect spectrum to have qualitatively same structure at any λ (at least for large J)

smooth change with λ: no transition on the way from small to large λ

Indeed, remarkable evidence (qualitative and quantitative) of correspondence between string and gauge states

sometimes works better than one could expect (susy: non-renormalization of some coefficients, ...)

Plan:

compare weak-coupling spin chain spectrum with semiclassical string spectrum

Gauge theory spectrum at $\lambda \ll 1$ and $J \gg 1$

1-loop: $XXX_{1/2}$ Heisenberg, length $J = J_1 + J_2$, solve BA

energy $E - J = \lambda E_1[1 + O(\frac{1}{J})] + O(\lambda^2)$

- $E_1 = 0$: ferromagnetic vacuum (BPS operator Tr Φ^J)
- $E_1 = \frac{a}{J^2}$: $J_2 = 2$, magnons

$p = \frac{2\pi n}{J}$, $w \sim p^2$: BMN operators

$\sum e^{ipJ} \mathrm{Tr}([\Phi_1....\Phi_1]\Phi_2[\Phi_1....\Phi_1]\Phi_2...)$

- $E_1 = \frac{b}{J}$: $J_1 \sim J_2 \gg 1$, low-energy spin waves

"Thermodynamic" limit: bound states of

large number ($J_2 \sim J \gg 1$) of magnons, $b = b(\frac{J_2}{J})$

"Bloch walls" or "macroscopic Bethe strings" (Sutherland 95; Dhar, Shastry 00; Beisert, Minahan, Staudacher, Zarembo 03); "locally BPS" operators

Tr$([\Phi_1....\Phi_1][\Phi_2...\Phi_2][\Phi_1....\Phi_1][\Phi_2...\Phi_2]...)$

- $E_1 = c$: bound states of finite no. of magnons

"Bethe strings" (Bethe 31), $c \sim \frac{1}{J_2}$

- $E_1 = kJ$: antiferromagnetic ($J_1 = J_2 \gg 1$) state

$k = \frac{\ln 2}{4\pi^2}$ (Huelthen 38)

LL action beyond leading order

effective actions from gauge-theory spin chain and string theory: $S = \int dt \int_0^J dx\, L\,, \qquad \bar\lambda = \frac{\lambda}{(2\pi)^2}\,, \quad x = J\sigma$

$$L = \vec C(n)\cdot\partial_t\vec n - \frac{1}{4}\vec n\,(\sqrt{1-\bar\lambda\partial_x^2}-1)\,\vec n\; - \frac{3\bar\lambda^2}{128}(\partial_x\vec n)^4$$

$$+\frac{\bar\lambda^3}{64}\Big[\frac{7}{4}(\partial_x\vec n)^2(\partial_x^2\vec n)^2 \;-\; \mathrm{b}(\lambda)\,(\partial_x\vec n\partial_x^2\vec n)^2 \;-\; \mathrm{c}(\lambda)\,(\partial_x\vec n)^6\Big]+\ldots$$

quadratic part is exact: reproduces the BMN dispersion relation for small ("magnon") fluctuations near $\vec n = (0,0,1)$

$$\partial_t^2 - \partial_x^2 + m^2 \to (i\partial_t - \sqrt{m^2-\partial_x^2})(-i\partial_t - \sqrt{m^2-\partial_x^2})$$

and $\sqrt{1+4\bar\lambda\sin^2\frac{p}{2}} \;\to\; \sqrt{1+\bar\lambda p^2}, \quad p = \frac{2\pi n}{J}\to 0$

Orders $\bar\lambda$ and $\bar\lambda^2$: direct agreement

"3-loop" coefficients are interpolating functions:

$$\lambda \gg 1\ :\qquad \mathrm{b} = -\frac{25}{2} + O(\frac{1}{\sqrt\lambda})\,, \qquad \mathrm{c} = \frac{13}{16} + O(\frac{1}{\sqrt\lambda})$$

$$\lambda \ll 1\ :\qquad \mathrm{b} = -\frac{23}{2} + O(\lambda)\,, \qquad \mathrm{c} = \frac{12}{16} + O(\lambda)$$

implied by non-analytic terms in 1-loop string correction: $J\bar\lambda^3\frac{1}{\sqrt\lambda} = \frac{\lambda^{5/2}}{J^5}$ (Beisert, AT; Schafer-Nameki, Zamaklar 05)

How to solve string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

complicated solitonic spectrum

Light-cone gauge: analog of

$x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0?$

Two natural options:

(i) null geodesic parallel to the boundary in Poincare patch – explicit action/Hamiltonian quartic in fermions

(Metsaev, Thorn, AT, 01)

(ii)null geodesic wrapping S^5: complicated action but hidden $su(2|2) \times su(2|2)$ symmetry (Callan et al, 03; Arutyunov, Frolov, Plefka, Zamaklar, 05-06)

Problem: lack of 2d Lorentz symmetry, hard to apply known integrable 2d field theory methods (S-matrix does not depend only on difference of rapidities, constraints on it are unclear, etc.)

Some conclusions

- Correspondence between gauge and string spectra near and far from BPS limit
- Presence of non-trivial interpolation functions in Bethe ansatz phase, string energies, effective LL action
- Special large J limit: simplicity, non-renormalizability, relation to S-matrix
- Additional constraints on S-matrix?

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An alternative approach based on Pohlmeyer map ("P-map"):

impose only conformal gauge, solve Virasoro conditions classically using a (non-local) field redefinition

map string theory into some 2d Lorentz invariant integrable massive local field theory ("P-theory")

classical equivalence: same solitonic solutions

integrable theories: expect/hope that semiclassical solitonic spectrum determines quantum spectrum

Relation between quantum string theory and quantum version of "P-theory" ? Not obvious: non-local map, different Poisson structures (Mikhailov 06)

But close relation may be true in quantum conformally invariant case: quantum P-theory corresponding to string theory should be a CFT (?)

the two solitonic S-matrices may be closely related

Which is integrable P-theory for $AdS_5 \times S^5$ string ?

...supercoset generalization of non-abelian Toda theory...

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...supercoset generalization of non-abelian Toda theory...

very special model with 4+4 dimensional bosonic target...

quantum conformal invariance? 2d world-sheet susy?

Prototypical example:

S^2 sigma model $\longrightarrow$ sin-Gordon model (Pohlmeyer, 76)

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)$$

$$\partial_+ \partial_- X^m + \Lambda X^m = 1\,, \qquad \Lambda = \partial_+ X^m \partial_- X^m$$

$$T_{+-} = 0\,,\ \partial_+ T_{--} = 0,\ \partial_- T_{++} = 0,\ T_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$$

then $T_{++} = f(\sigma_+),\ T_{--} = h(\sigma_-)$ and can do conformal transformation to make

$$\partial_+ X^m \partial_+ X^m = \kappa^2, \qquad \partial_- X^m \partial_- X^m = \kappa^2$$

Same as Virasoro constraints for classical string on $R_t \times S^2$ in conformal gauge and $t = \kappa\tau$ fixing residual conformal transformation freedom.

define new field ϕ by

$$\kappa^2 \cos\phi = \partial_+ X^m \partial_- X^m, \quad X^m X^m = 1\,, \quad m = 1,2,3$$

then eqs. for X^m and constraints are equivalent to

$$\partial_+ \partial_- \phi + \kappa^2 \sin\phi = 0$$

"conformal reduction" of S^2 sigma model = sin-Gordon theory, i.e. SG theory =P-theory for string on $R_t \times S^2$

$$L_P = \frac{1}{2}\partial_+\phi\partial_-\phi - \kappa^2\cos\phi$$

Classical solutions and integrable structures (Lax pair, Backlund transformations, etc) are directly related

2d Lorentz invariant; manifest $SO(3)$ symmetry hidden; infinite sets of conserved charges are in correspondence

[sin-Gordon soliton on an infinite line mapped into rotating open string on S^2 with $J=\infty$: "giant magnon" (Hofman, Maldacena 06)]

Quantum equivalence of S^2 sigma model and sin-Gordon model? No – no quantum conformal invariance, P-map does not formally apply... (yet there is a relation between solitonic S-matrices, Zamolodchikov^2)

But what if S^2 model is embedded into conformal sigma model describing string theory?

Another example:

S^3 sigma model $\longrightarrow$ "complex sin-Gordon" model

(Pohlmeyer; Lund, Regge 76)

$$L = \frac{1}{2}(\partial_+\phi\partial_-\phi + \tan^2\frac{\phi}{2}\,\partial_+\chi\partial_-\chi) - \kappa^2\cos\phi$$

$\kappa^2\cos\phi = \partial_+ X^m \partial_- X^m,$

$X^m X^m = 1, \ \ m = 1,2,3,4$

$\partial_\pm\chi \ \sin^2\frac{\phi}{2} = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial^2_\pm X^l$

non-local map producing 2d relativistic P-model for string on $R_t \times S^3$ in conformal gauge

an example of non-abelian Toda model (Leznov, Saveliev): massive integrable perturbation of a coset WZW model (here $SO(3)/SO(2)$) (Hollowood, Miramontes, Park 94)

related to massive Thirring model via 1st order form of classical eqs: $\psi = (u, v)$

$i\partial_+ u + v - |u|^2 v = 0, \ i\partial_- v + u - |v|^2 u = 0,$

quantum S-matrix is known (Dorey, Hollowood)

General bosonic case:

sigma model on symmetric space F/G reduces to "symmetric space SG" model with Lagrangian formulation in terms of G/H gauged WZW model with a potential (Bakas, Park, Shin 95)

potential determined by choice of 2 elements T_+, T_- in k:
$f = g + k$ and H corresponds to h=centralizer of $T_\pm$ in g

$$S = I_{gwzw}(g, A) - \kappa^2 \int \mathrm{tr}(T_- g^{-1} T_+ g)$$

$g \in G,\ A_\pm \in h$; gauging $g' = UgU^{-1},\ U \in H$

$[A_\pm, T_\pm] = 0$ implies existence of Lax pair: $[L_+, L_-] = 0$

$L_+ = \partial_- + g^{-1}\partial_- g + g^{-1}A_- g + sT_-,$
$L_- = \partial_+ + A_+ + s^{-1}\kappa^2 g^{-1} T_+ g$

gauge $A_\pm = 0$, $(g^{-1}\partial_- g)_h = 0,\ (\partial_+ g g^{-1})_h = 0$

$$\partial_+(g^{-1}\partial_- g) - \kappa^2[T_-, g^{-1}T_+ g] = 0$$

$F/G = S^n = SO(n+1)/SO(n)$:

$SO(n)/SO(n-1)$ gWZW model + potential

$(T_+)_{12} = (T_-)_{12} = 1$, rest =0: $H = SO(n-1)$

$(g^{-1}T_- g)_{1,k} = (0, V_0, V_1, ..., V_{n-1})$

$V_0^2 + V_i V_i = 1, \quad i = 1, ..., n-1$

$\partial_+ \left(\frac{\partial_- V_i}{\sqrt{1 - V_k V_k}}\right) - \kappa^2 V_i = 0$

e.g. $S^5 = SO(6)/SO(5)$ reduces to $SO(5)/SO(4)$:

4d target space coordinates are $SO(6)$ invariants constructed out of $X_m, \partial_\pm X_m, \partial_\pm^2 X_m, \partial_\pm^3 X_m$ $(X_m X_m = 1, m = 1, ..., 6)$ using δ_{mn} and ϵ_{mnklpq}

Similar story for non-compact case:

$AdS_n = SO(2, n-1)/SO(1, n-1)$

constraints $Y^p Y_p = -1, \quad \partial_\pm Y^p \partial_\pm Y_p = -\kappa^2$

e.g. $AdS_2 \longrightarrow$ sinh-Gordon theory

$AdS_n \times S^n$: $\quad \partial_\pm Y^p \partial_\pm Y_p + \partial_\pm X^m \partial_\pm X^m = 0$

P-theory: solves classical eqs. and conformal gauge constraints for string on $AdS_n \times S^n$

Quantum theory: need to embed into a conformal theory

Include fermions: "super P-map"

reduction of GS string on $AdS_5 \times S^5$? hidden 2d susy ?

curios observations:

1. $N = 2$ supersymmetric extension of sin-Gordon has

bosonic part ($z = \phi + i\varphi$)

$$L = \frac{1}{2}(\partial_+\phi\partial_-\phi + \partial_+\varphi\partial_-\varphi) - \kappa^2(\cos\phi - \cosh\varphi)$$

same as P-theory for $AdS_2 \times S^2$ string

GS string in $AdS_2 \times S^2$ reduces to $N = 2$ SG ?

2. reduced theory for standard (non-unitary) supercoset $SU(2|1)/S(U(1) \times U(1))$ is same as $N = 1$ susy complex SG model (Napolitano, Sciuto 81)

complex SG: $ds^2 = d\phi^2 + \tan^2\frac{\phi}{2}d\chi^2$ is Kahler – has actually $N = 2$ susy

[$N = 4$ extension is related to reduction of $AdS_3 \times S^3$? Not naively: 4d target is direct product and is not hyperKahler...]

GS string on $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$:

superalgebra $psu(2,2|4)$ admits Z_4 grading

$$J = g^{-1}dg = H + Q_1 + P + Q_2$$

H represents $so(1,4) + so(5)$

Action (cf. $\int \mathrm{tr}[(g^{-1}\partial g)_{G/H}]^2$):

$$S = \int \mathrm{str}(P \wedge *P + Q_1 \wedge Q_2)$$

bosonic part of P is direct sum of AdS_5 (P^q) and S^5 (P^m)

Eqs. of motion and constraints ($P = P_+ d\sigma^+ + P_- d\sigma^-$):

$$\partial_+ P_- - [H_+, P_-] + [Q_{2-}, Q_{2+}] = 0$$

$$\partial_- P_+ - [H_-, P_+] + [Q_{1-}, Q_{1+}] = 0$$

$$[P_+, Q_{1-}] = 0, \quad [P_-, Q_{2+}] = 0$$

$$\partial_+ H_- - \partial_- H_+ =$$
$$[H_+, H_-] + [P_+, P_-] + [Q_{1+}, Q_{2-}] + [Q_{2+}, Q_{1-}]$$

$$\partial_+ Q_{1-} - \partial_- Q_{1+} =$$
$$[H_+, Q_{1-}] - [H_-, Q_{1+}] + [P_+, Q_{2-}] - [P_-, Q_{2+}]$$

$$\partial_+ Q_{2-} - \partial_- Q_{2+} =$$
$$[H_+, Q_{2-}] - [H_-, Q_{2+}] + [P_+, Q_{1-}] - [P_-, Q_{1+}]$$

$$\mathrm{Tr}[(P^m_\pm)^2] = \kappa^2, \qquad \mathrm{Tr}[(P^q_\pm)^2] = \kappa^2$$

Remarkably, Virasoro conditions can be split as in the bosonic case (cf. Bena, Polchinski, Roiban)

above eqs. implicitly define analog of sin-Gordon model in the case of string on $R_t \times S^2$ (Mikhailov 06) cf. P-map for string on a coset: going from eqs for coordinates parametrizing g to eqs for new fields parametrizing $J = g^{-1} dg$

to construct super P-map:

choose parametrization for J solving constraints, fix kappa-symmetry gauge, find action for new independent (4+4 bosonic and ? fermionic) fields

simplest example: $AdS_2 \times S^2$ with RR flux

GS theory for $PSU(1,1|2)/[U(1) \times U(1)]$

bosonic part: sin-Gordon + sinh-Gordon

fermionic eqs look similar to those of $N = 2$ extension of sin-Gordon... (in progress with R. Roiban)

Many questions:

P-action also has supercoset structure? interpretation in terms of non-abelian Toda for some supercoset? hidden 2d susy? quantum theory? conformal invariance? solitonic S-matrix? spectrum of its poles? ...