Title: On measures of distance between quantum observables and a new joint measurement uncertainty relation

Date: Oct 18, 2006 04:00 PM

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Abstract: I will discuss various different ways of quantifying the differences between two quantum observables (POVMs). Each of these approaches gives rise to a notion of approximately measuring one observable by means of measuring some other observable. This will be illustrated in the case of position and momentum by studying the question which POVMs on phase space can reasonably be said to represent a joint approximate determination of these observables. A new, universally valid trade-off relation for the associated inaccuracies will be rigorously formulated. I will sketch the proof which is an adaptation of some interesting techniques and properties of covariant phase space observables used recently by R Werner in a related project.

Recommended reading (optional):
quant-ph/0405184 (R Werner), quant-ph/0609185 (PB et al),
and also for further background information quant-ph/0309091 (M Hall), quant-ph/0310070 (M Ozawa), quant-ph/9803051 (DM Appleby).

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Universal joint-measurement uncertainty relation for error bars

October 18, 2006

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We formulate and prove a novel form of uncertainty relation, valid for any joint measurements of position and momentum that satisfy a condition of finite error bars. The measurement inaccuracies, defined as minimal lengths of confidence intervals, satisfy a trade-off relation of the form $\delta q \cdot \delta p \geq C\hbar$, with a positive constant C.

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1 Introduction

In his seminal paper of 1927 [?], Heisenberg formulated not one but in fact three distinct variants of uncertainty relations of the general form

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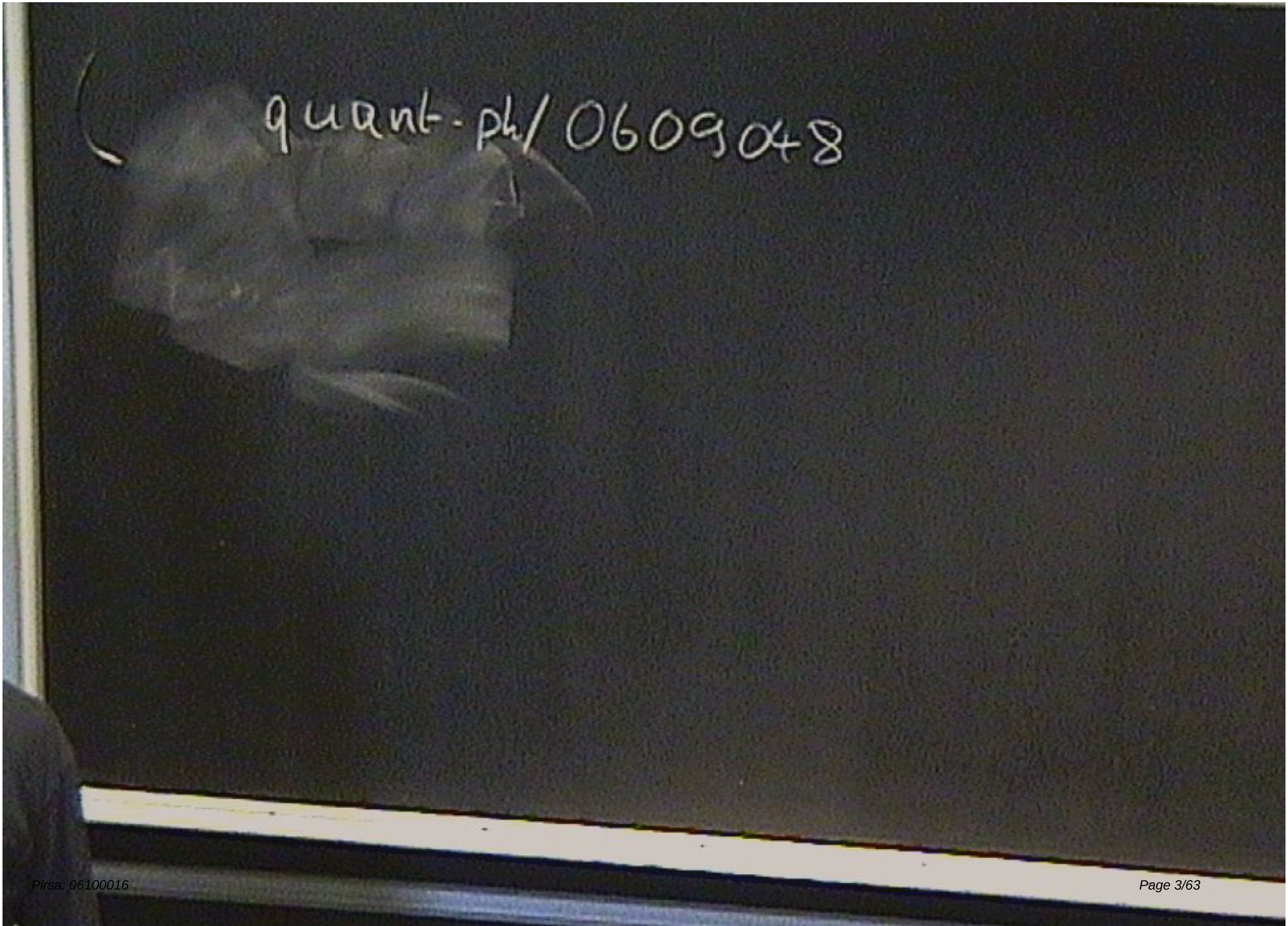
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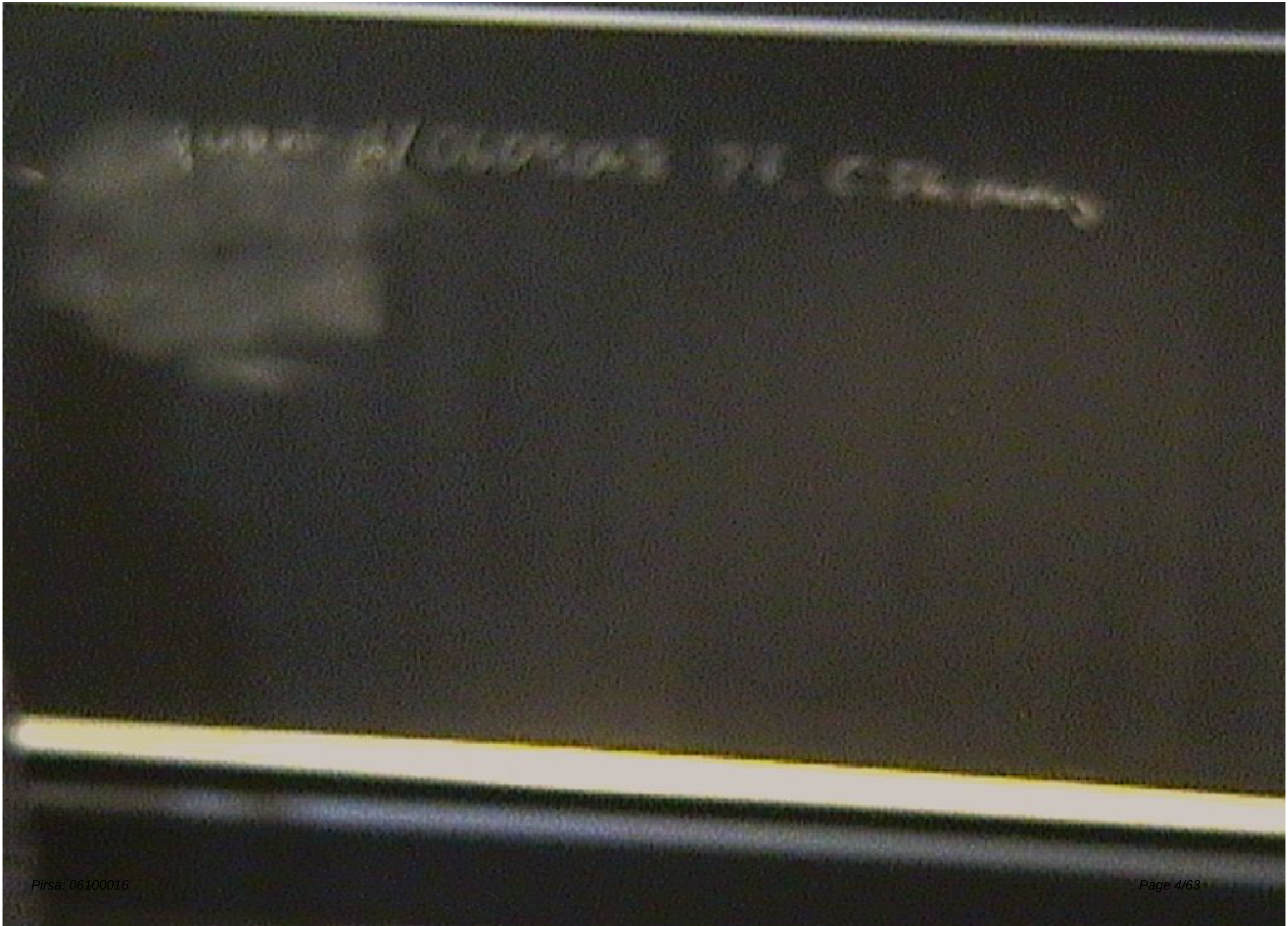

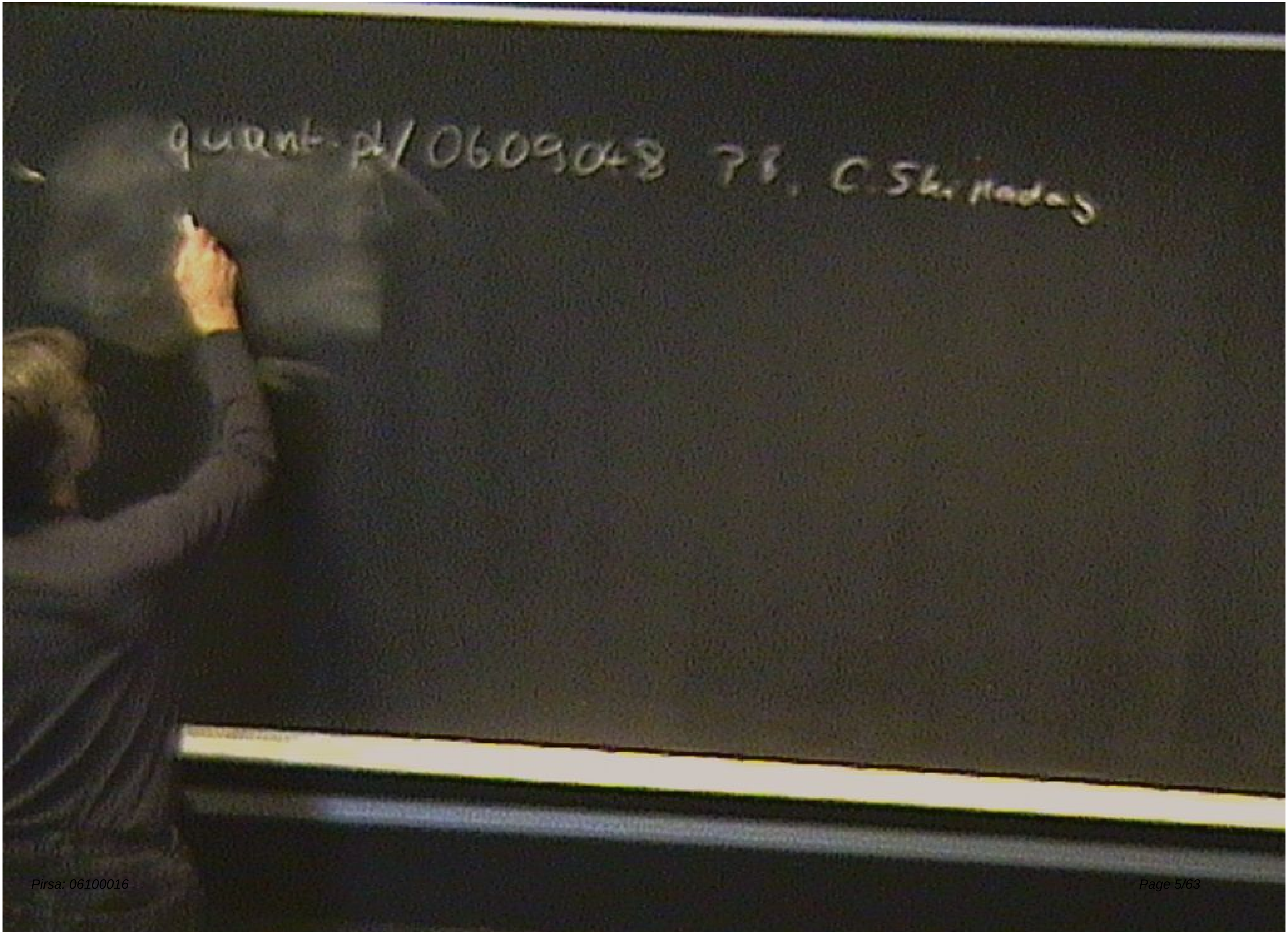
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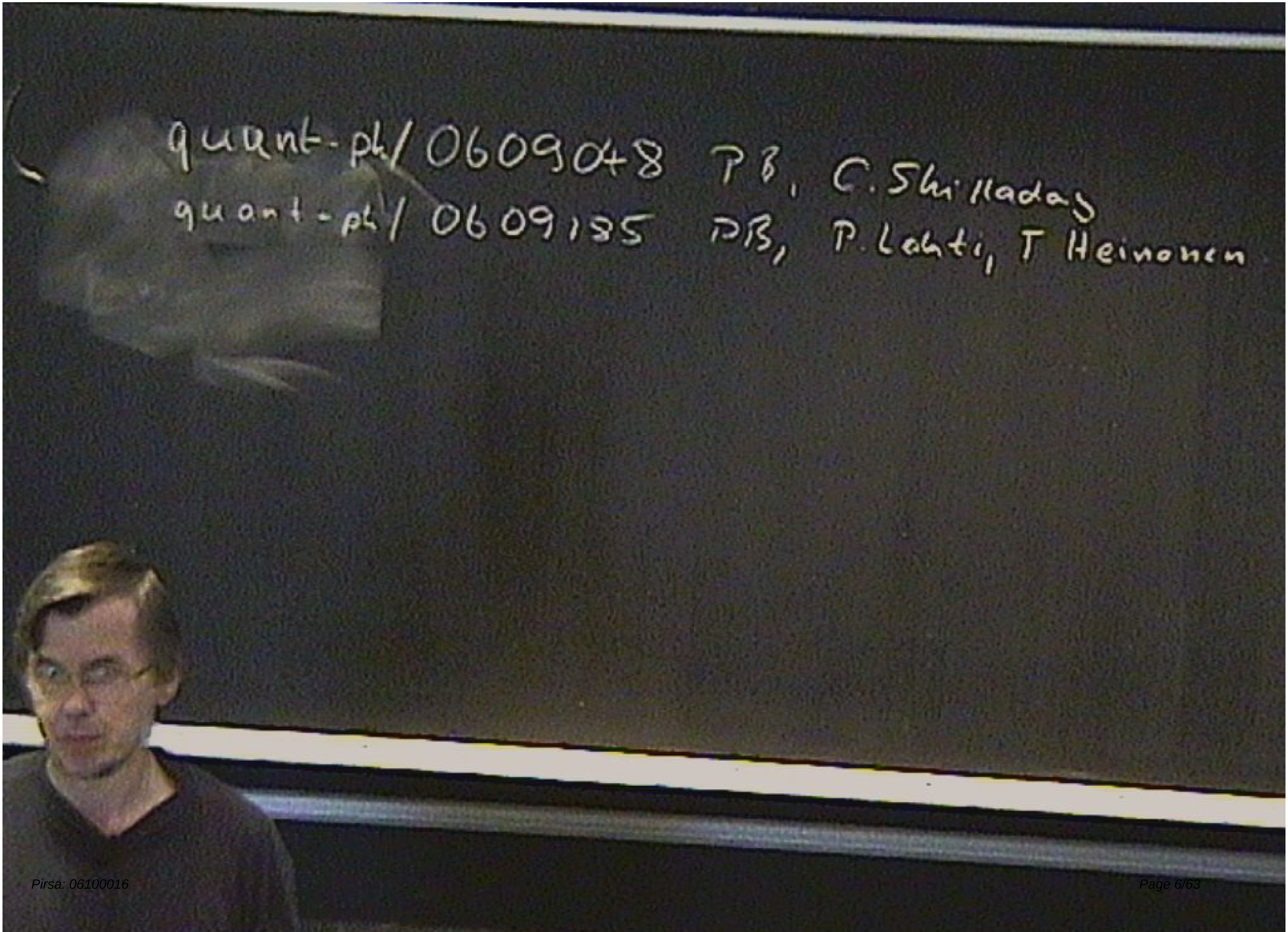
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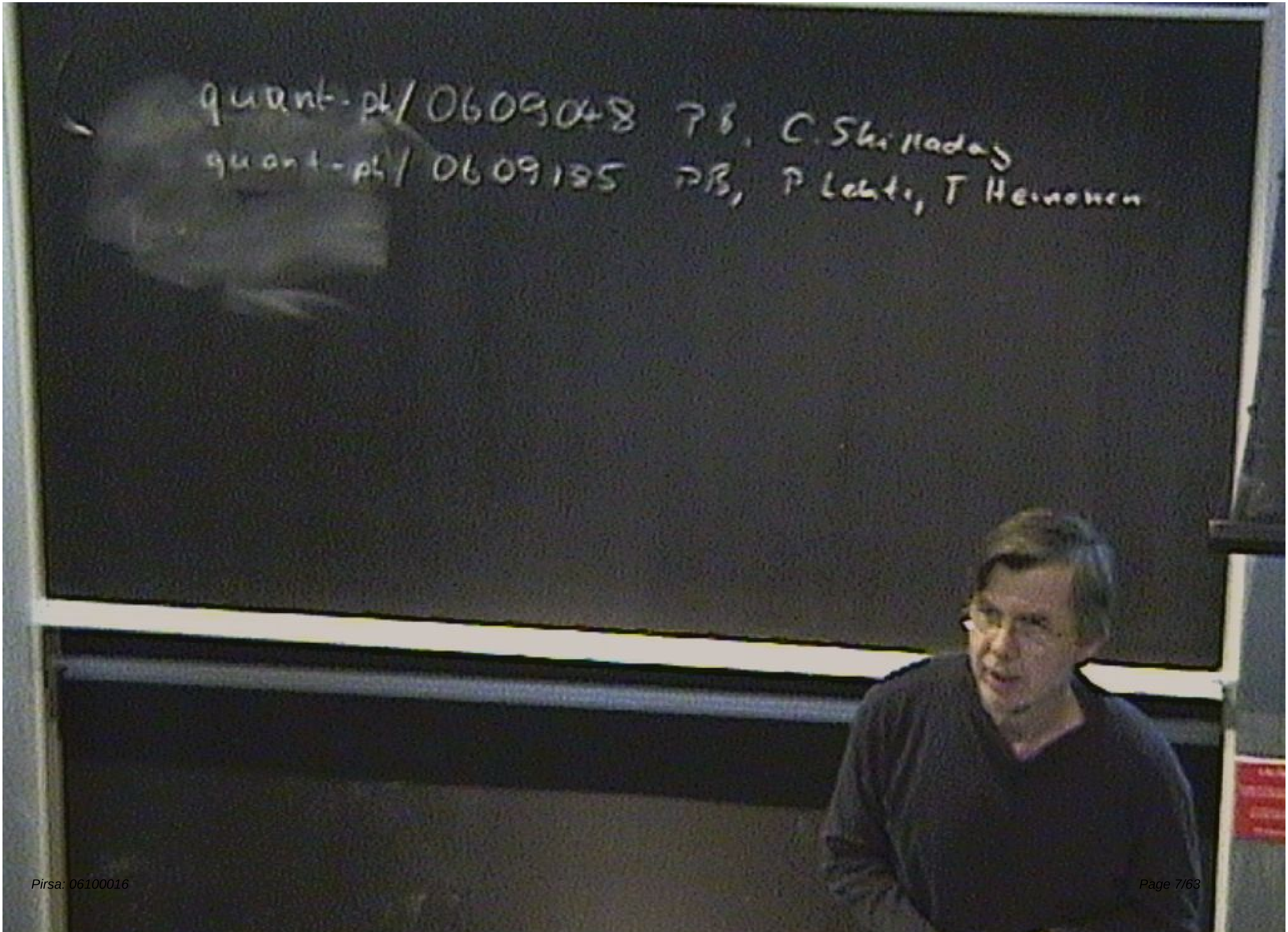
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1 Introduction

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which together comprise the full content of the uncertainty principle [?] but which were not received with equal approval. The uncontroversial form of uncertainty relation is the one which characterizes possibilities of *preparations*; it was made precise in terms of standard deviations, first for position Q and momentum P and any vector state φ,

$$\Delta(Q,\varphi) \cdot \Delta(P,\varphi) \geq \frac{\hbar}{2}, \tag{2}$$

and generalized soon afterwards to apply to all pairs of observables and all quantum states [?, ?, ?, ?]. This

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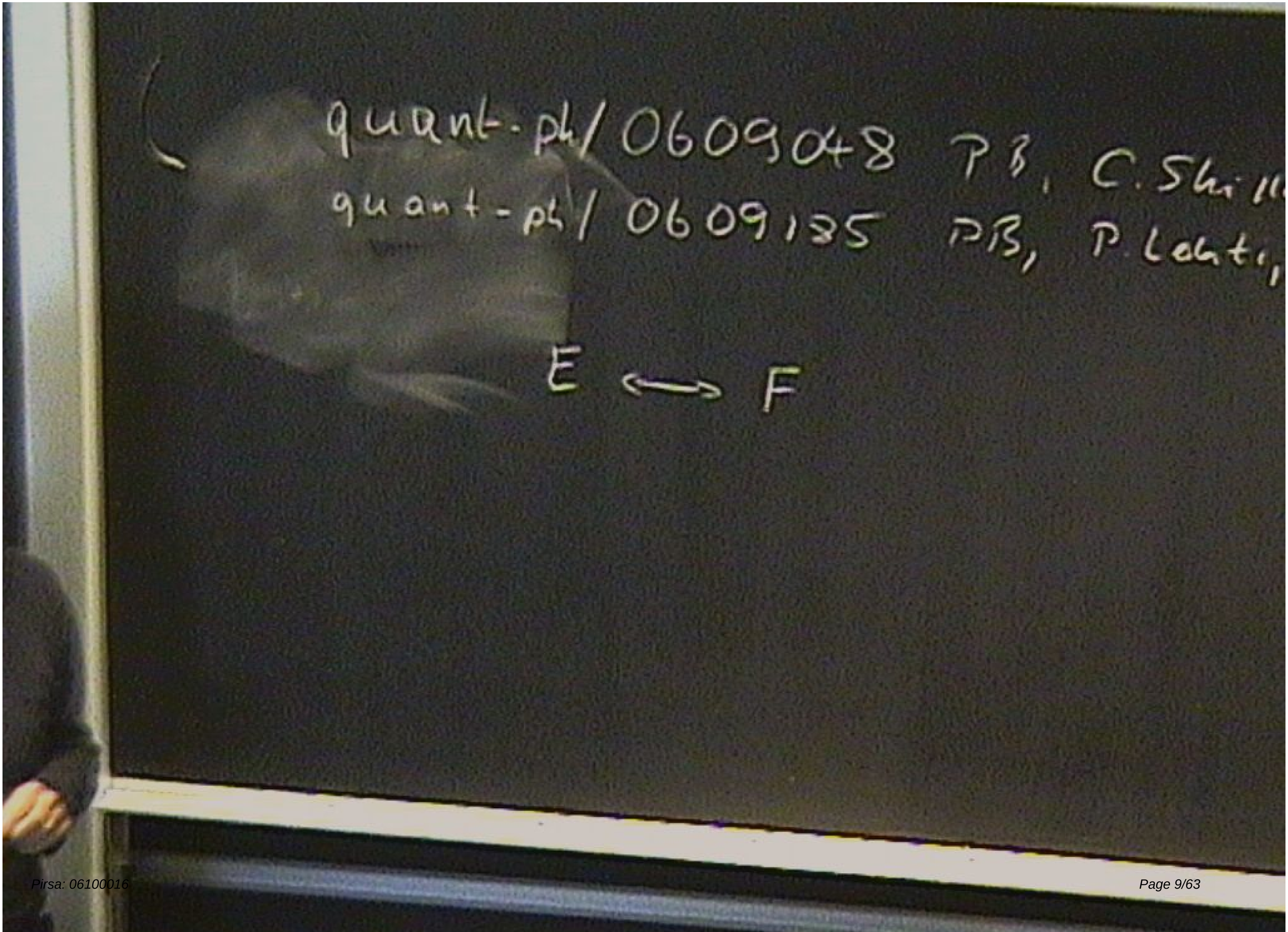
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quant-ph/0609135 P.B., P. Lahti,
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significant, Heisenberg uncertainty relation for error bars in joint measurements of position and momentum. Here an observable on phase space is accepted as representing a joint measurement if it has finite errors, defined operationally as the widths of confidence intervals obtained in the calibration of the joint measurement.

2 Covariant phase space observables

For the rest of this paper we consider a quantum particle in one spatial dimension, with Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and canonical position and momentum operators Q, P. By Q and P we denote the spectral measures of Q and P, respectively, and $W(q,p) = \exp(\frac{i}{\hbar}(Pq - Qp)$ are the Weyl operators which comprise an irreducible unitary projective representation of the translations on phase space $\mathbb{R}^2$. States are represented as positive operators ρ of trace 1, the convex set of all states being denoted S. Occasionally we use unit vectors $\varphi \in \mathcal{H}$ to represent pure states. Observables are represented as normalized ($E(\Omega) = I$) positive operator measures (POMs) on a measurable space (Ω, Σ), which in the present context will be either $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. We write ρ^E for the probability measure induced by a state ρ and an observable E via the formula $\rho^E(X) = \mathrm{tr}\,[\rho E(X)]$, $X \in \Sigma$. Finally we use the notation $E[k]$ for the moment operators $\int x^k E(dx)$ of an observable E.

The earliest version of a quantum mechanical phase space probability distributions was discovered in 1939 by Husimi [?]. This is of the form (written for any vector state φ)

$$(q,p) \mapsto |\langle \varphi | \eta_{qp} \rangle|^2, \tag{3}$$

where $\eta_{qp} = W(q,p)^* \eta$ denotes the family of Gaussian coherent states. The totality of distributions (3) determine a unique covariant phase space observable G_η via

$$\mathrm{tr}\,[\rho G_\eta(Z)] = \frac{1}{2\pi\hbar} \int_Z \langle \eta_{qp} | \rho \eta_{qp} \rangle \, dq dp, \quad \rho \in S,\ Z \in \mathcal{B}(\mathbb{R}^2). \tag{4}$$

A more general form of covariant phase space observable is obtained if the above integrand is replaced with $|\langle W(q,p)^* \xi | \rho W(q,p)^* \xi \rangle$ [?]; finally one can take convex combinations of such phase space POVMs to obtain a covariant phase space observable

$$\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto G(Z) = \frac{1}{2\pi\hbar} \int_Z W(q,p)^* \mathbf{m} W(q,p) \, dq dp, \tag{5}$$

where the integral is defined weakly and the operator density is generated by an arbitrary fixed positive

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For later reference we write out the covariance property. For $x = (q,p) \in \mathbb{R}^2$, let τ_x be the shift map on the space $C(\mathbb{R}^2)$ of bounded continuous functions f, so that $\tau_x f(y) = f(y-x)$. We can extend τ_x to act on indicator functions or the associated Borel sets in the obvious way. An observable G on phase space $\mathbb{R}^2$ is *covariant* if for all $Z \in \mathcal{B}(\mathbb{R}^2)$,

$$G(\tau_{(q,p)}Z) = W(q,p)^* G(Z) W(q,p). \tag{6}$$

It is in fact the case that *every* covariant phase space observable is of the form (5). This important result is implied by results of [?] using methods of quantum harmonic analysis and has been made explicit in[?] using Mackey's machine of induced representations and in [?] using the theory of integration with respect to operator measures.

The marginal observables of G are convolutions of the spectral measures Q, P with the probability measures $\mathbf{m}_\Pi^{\mathsf{Q}}, \mathbf{m}_\Pi^{\mathsf{P}}$, that is,

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Here $\mathbf{m}_\Pi = \Pi \mathbf{m} \Pi^*$ is the operator obtained from $\mathbf{m}$ under the action of the parity transformation Π ($\Pi\varphi(x) = \varphi(-x)$). Taking the standard deviations as measures of inaccuracy,

$$\delta(\mathsf{Q}, G) = \Delta(Q, \mathbf{m}), \quad \delta(\mathsf{P}, G) = \Delta(P, \mathbf{m}), \tag{8}$$

one has the general measurement uncertainty relation for covariant phase space observables:

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The operational meaning of (say) the distribution $\mathbf{m}^{\mathsf{Q}}$ becomes fully transparent if one considers a sequence of states ρ_n such that the distributions ρ_n^Q are increasingly sharply peaked, approaching the Dirac distribution concentrated at a point q_0. In this case the output distributions $\rho_n^{G_1}$ approach $(\mathbf{m}_\Pi)_{q_0}^{\mathsf{Q}}$, that is, the translate of the distribution $\mathbf{m}_\Pi^{\mathsf{Q}}$ by q_0. Thus the width of that distribution characterizes the likely range of the prepared values of position if it is known that a fairly sharp distribution was prepared and an outcome q_0 is recorded.

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(NOTE: I think it must be possible to come up with tighter lower bounds since for very small $\varepsilon_1, \varepsilon_2$, one

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(NOTE: I think it must be possible to come up with tighter lower bounds since for very small $\varepsilon_1, \varepsilon_2$, one must allow rather large intervals. Check the Hilgevoord-Uffink relation for overall width and mean peak width, from which they derive another lower bound for the above product.)

Following [?], we define the resolution $\gamma(E; 1 - \varepsilon)$ (at confidence level $1 - \varepsilon$) of a POM E on $\mathcal{B}(\mathbb{R})$ as follows. then

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$$G_1 = \mathsf{Q} * \mathbf{m}_\Pi, \quad G_2 = \mathsf{P} * \mathbf{m}_\Pi. \tag{7}$$

Here $\mathbf{m}_\Pi = \Pi \mathbf{m} \Pi^*$ is the operator obtained from $\mathbf{m}$ under the action of the parity transformation Π ($\Pi\varphi(x) = \varphi(-x)$). Taking the standard deviations as measures of inaccuracy,

$$\delta(\mathsf{Q}, G) = \Delta(Q, \mathbf{m}), \quad \delta(\mathsf{P}, G) = \Delta(P, \mathbf{m}), \tag{8}$$

one has the general measurement uncertainty relation for covariant phase space observables:

$$\delta(\mathsf{Q}, G) \cdot \delta(\mathsf{P}, G) \geq \frac{\hbar}{4}. \tag{9}$$

The operational meaning of (say) the distribution $\mathbf{m}^{\mathsf{Q}}$ becomes fully transparent if one considers a sequence of states ρ_n such that the distributions ρ_n^{Q} are increasingly sharply peaked, approaching the Dirac distribution concentrated at a point q_0. In this case the output distributions $\rho_n^{G_1}$ approach $(\mathbf{m}_\Pi)_{q_0}^{\mathsf{Q}}$, that is, the translate of the distribution $\mathbf{m}_\Pi^{\mathsf{Q}}$ by q_0. Thus the width of that distribution characterizes the likely range of the prepared values of position if it is known that a fairly sharp distribution was prepared and an outcome q_0 is recorded.

Since measures of the width of a distribution in terms of (second or other) moments are of limited use, we note, for later reference, an alternative formulation of uncertainty relation for joint measurement inaccuracies, valid for covariant phase space observables.

For $q \in \mathbb{R}$, $\delta > 0$, let $I_{q;\delta}$ denote the interval $[q - \delta/2, q + \delta/2)$. (We will occasionally use the same symbol to denote the indicator function of this set.) Then, for given confidence level $1 - \varepsilon$, $\varepsilon > 0$, we define the *overall width* of a probability distribution p on $\mathbb{R}$ as the smallest length of all intervals for which the probability exceeds $1 - \varepsilon$:

$$W(\mathsf{p}, \varepsilon) := \inf\{d \,:\, \mathsf{p}(I_{x;d}) > 1 - \varepsilon, \text{ for some } x \in \mathbb{R}\}. \tag{10}$$

The following relation was shown in [?] to hold for any state ρ and positive $\varepsilon_1, \varepsilon_2 > 0$ for which $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} \leq 1$; we write it here for $\rho = \mathbf{m}$:

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$$\gamma(E; 1 - \varepsilon) = \inf\{d > 0 \,:\, \rho^E(I_{x;d}) > 1 - \varepsilon \text{ for all } x \in \mathbb{R}\}. \tag{12}$$

It follows readily (see also [?]) that for the marginals G_1, G_2 of a covariant phase space observable G with generating operator $\mathbf{m}$, one has

$$\gamma(G_1; 1 - \varepsilon_1) = W(\mathbf{m}^{\mathsf{Q}}, \varepsilon_1), \quad \gamma(G_2; 1 - \varepsilon_2) = W(\mathbf{m}^{\mathsf{P}}, \varepsilon_2). \tag{13}$$

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3 Uncertainty relations for "classical" error measures

If one considers the idea that possibly *any* observable M on phase space could provide information about position and momentum, such that the marginals M_1, M_2 represent approximate measurements of Q, P, respectively, then it is of interest to study the necessary minimal inaccuracies that must be present due to the noncommutativity of Q and P. Classical intuition suggests that a suitable error measure could be given by the average deviation of the value of an indicator observable of the measuring apparatus from the value of the observable to be measured approximately, where these observables are represented as selfadjoint operators Z and A, respectively, that is, the root mean square $\epsilon(Z, A) = \langle (Z - A)^2 \rangle^{1/2}$. This measure has indeed been widely used in the physical literature, and has in recent years been studied in the present foundational context (e.g., [?, ?, ?] ; these papers also give good surveys of the properties and applications of this error measure). As shown, for instance, in [?], the above definition can be recast in terms of the POM E that represents the given measurement and the observable A to be measured approximately; thus a *classical error* measure can be defined as follows:

$$\epsilon(E, A; \rho)^2 = \mathrm{tr}\left[\rho(E[1] - A)^2\right] + \mathrm{tr}\left[\rho(E[2] - E[1]^2)\right]. \tag{14}$$

We note that $\epsilon(E, A : \rho) = 0$ for all ρ exactly when $E = A$. The first term measures the difference between the first moments of the observable to be measured, represented by the selfadjoint operator A, and the observable E that represents the measurement. Note that this term cannot, in general, be determined from the statistics of measurements of E and A alone.

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The classical error measure $\epsilon(E, A; \rho)$ is thus seen to violate what would appear to be a natural requirement of *operational significance*: the requirement that the measure can be determined by the measurement statistics of the observables to be compared (E, A). This requirement can only be satisfied if $(E[1]) - A)^2$ and $E[1]^2$ are functions of A or of E. This happens, in particular, if $E[1] - A = cI$ for some constant $c \in \mathbb{R}$. If $E[1] = A$, the measurement is called *globally unbiased*. This property, applied to the marginals of an observable on phase space, is taken as a defining condition for a joint measurement of position and momentum by some authors (e.g., [?, ?]).

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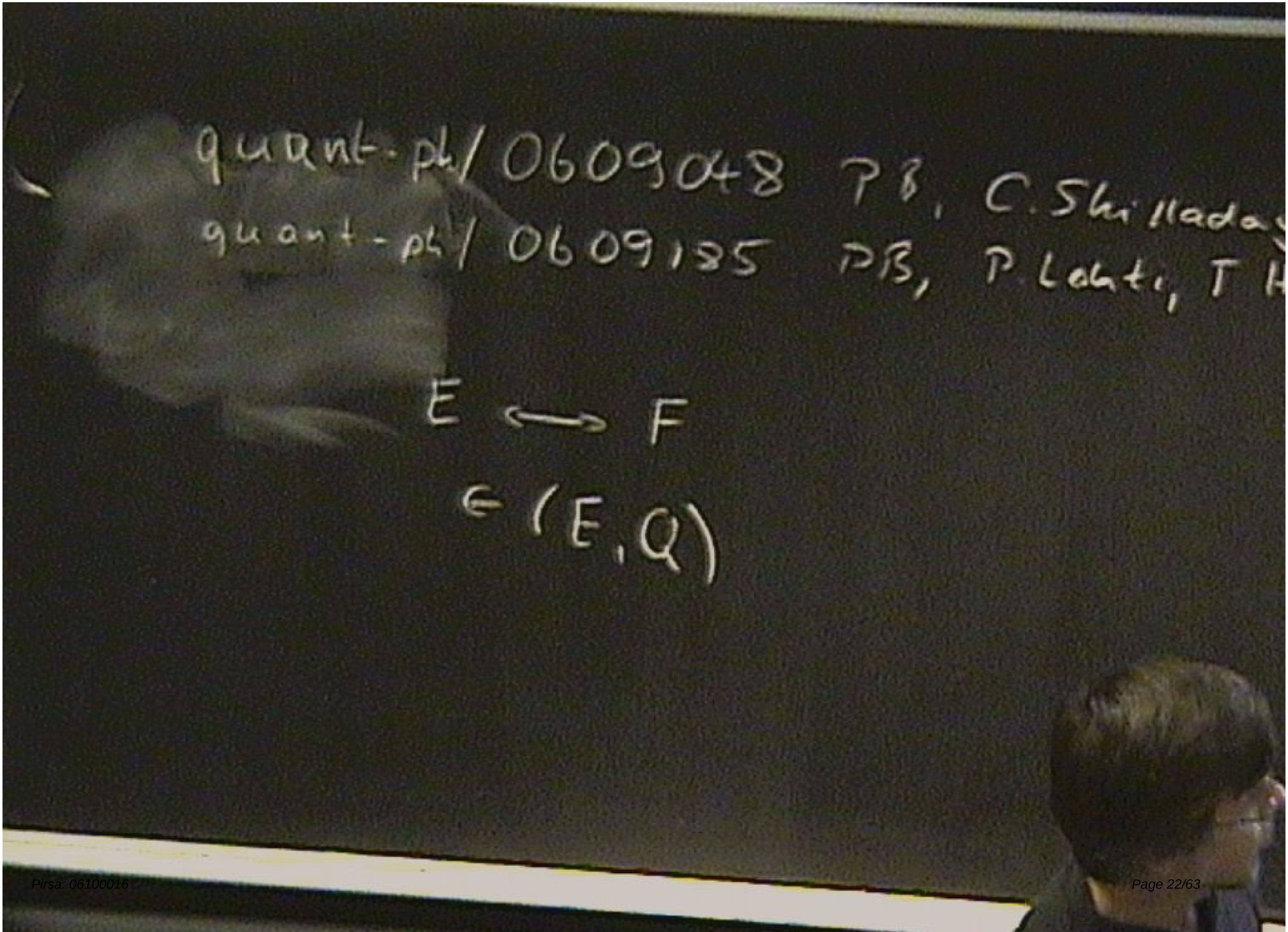

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quant-ph/0609185 PB, P. Lahti, T H
E ⟷ F
∈ (E, Q)

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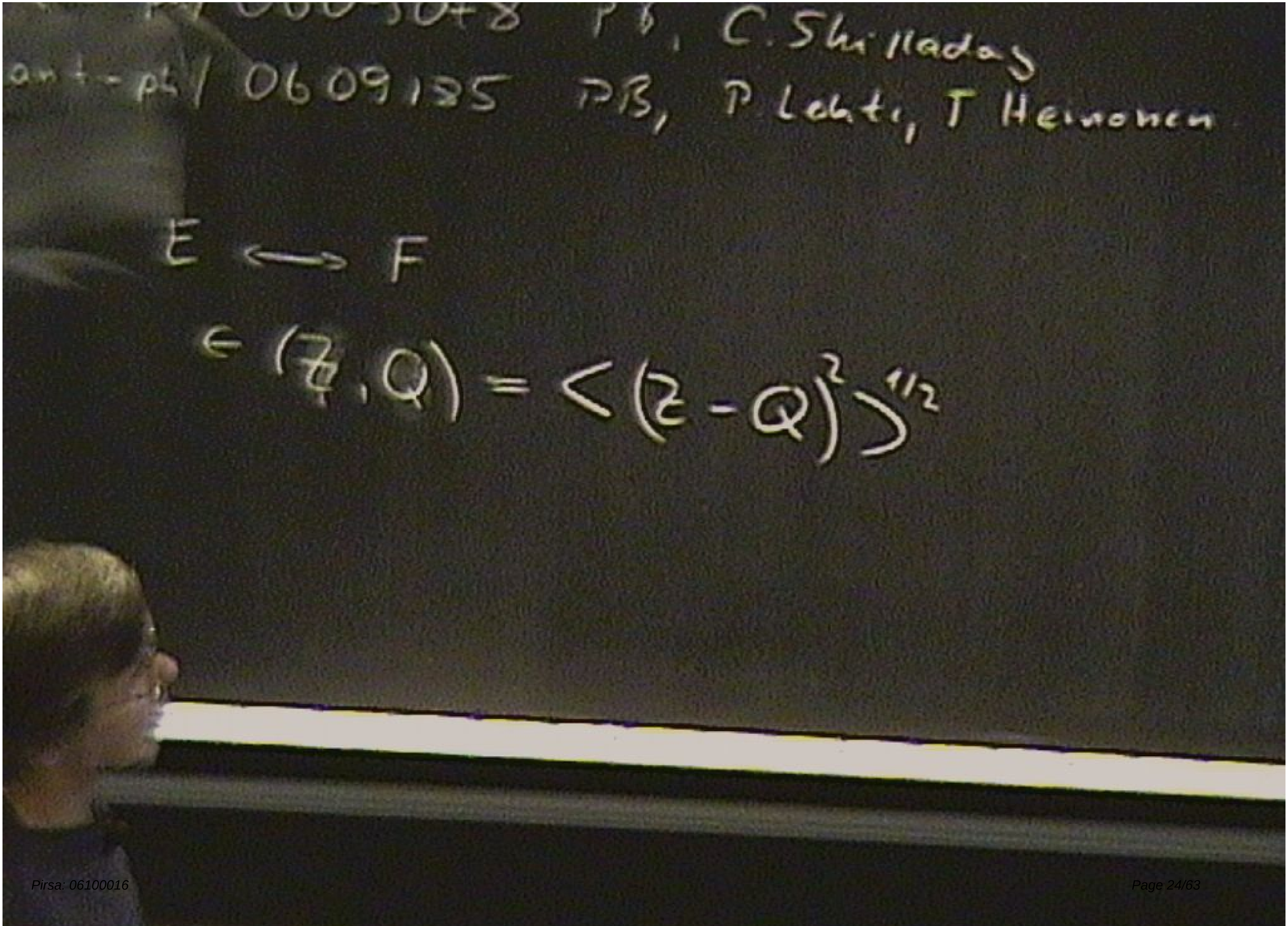
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ant-ph/ 0609185 PB, P. Lahti, T Heinonen
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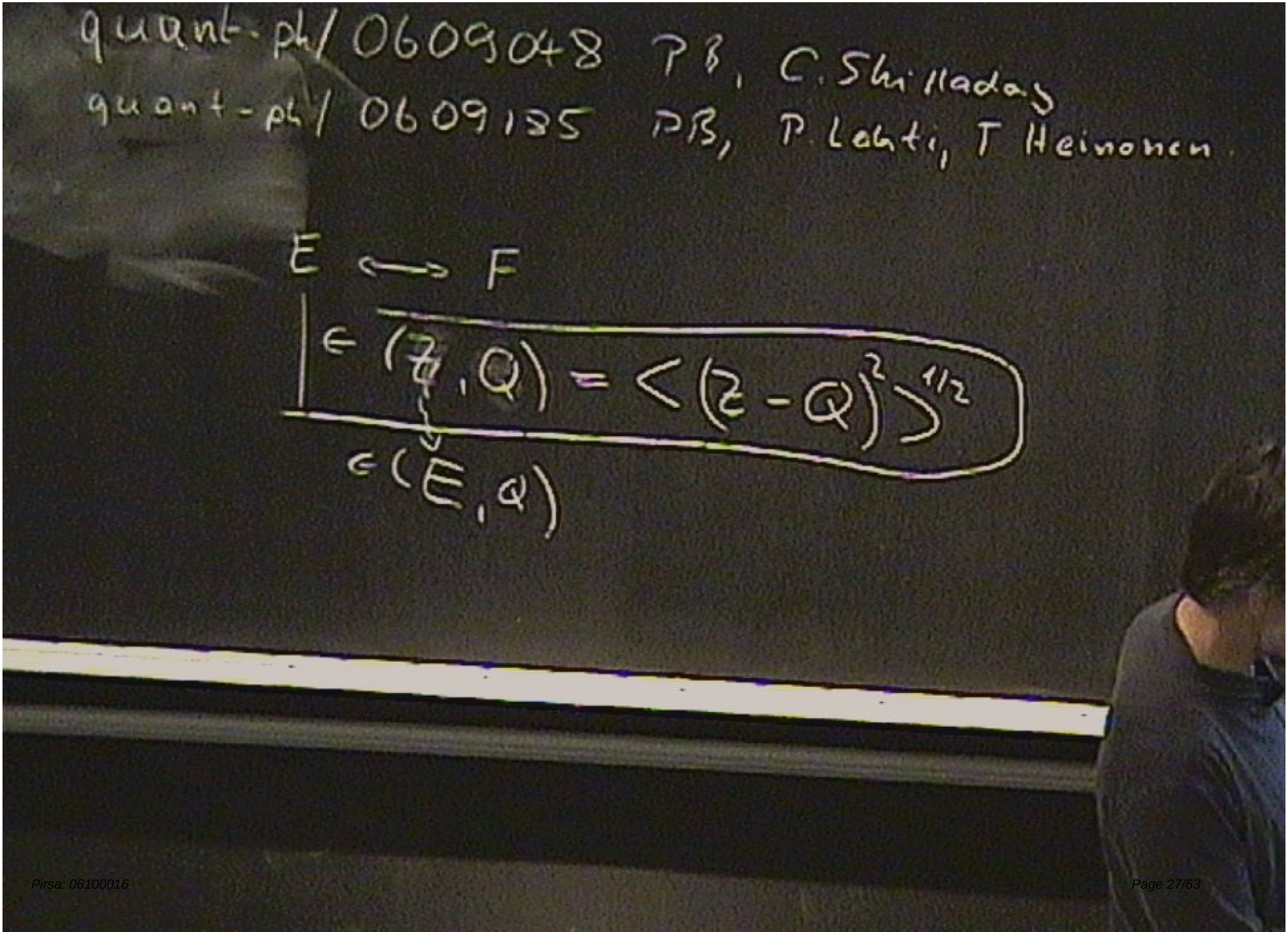
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This is the surplus of the standard deviation of the distribution ρ^E over that of the distribution ρ^A.

It has been shown [?, ?, ?] that the classical error measures for the marginals of an observable M on phase space do not, in general satisfy the standard uncertainty relation with a state-independent lower bound for the error products. Instead, a rather weaker relation holds [?]:

$$\epsilon(M_1,\mathsf{Q};\rho)\cdot\epsilon(M_2,\mathsf{P};\rho) + \epsilon(M_1,\mathsf{Q};\rho)\cdot\Delta(\mathsf{P},\rho) + \Delta(\mathsf{Q},\rho)\cdot\epsilon(M_2,\mathsf{P};\rho) \geq \frac{\hbar}{2}. \quad (16)$$

However, for unbiased measurements, where the classical errors are operationally significant, an uncertainty relation for the error products stronger than (16) has been shown to hold (for a proof and a survey of further relevant work, see [?]):

$$\epsilon(M_1,\mathsf{Q};\rho)\cdot\epsilon(M_2,\mathsf{P};\rho) \geq \frac{\hbar}{2} \quad (M_1[1]=Q+c_1I,\ M_2[1]=P+c_2). \quad (17)$$

(QUESTION: is the unbiasedness condition strong enough to ensure covariance of the marginals? I suspect not but have no proof.)

The classical error measures can be calculated in the case of a covariant phase space observable G [?,

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The classical error measures can be calculated in the case of a covariant phase space observable G [?, Theorem 4]; one obtains

$$G_1 = Q + \mathrm{tr}[\mathrm{m}Q]\, I, \quad G_2 = P + \mathrm{tr}[\mathrm{m}P]\, I, \tag{18}$$

and so

$$\epsilon(G_1, \mathsf{Q}; \rho) = \Delta(Q, \mathrm{m}), \quad \epsilon(G_2, \mathsf{P}; \rho) = \Delta(P, \mathrm{m}). \tag{19}$$

In this case the uncertainty relation (17) holds and coincides in fact with (9).

The following example illustrates that the classically motivated error measure seems indeed misleading.

Example 1 *Let M be an observable on $\mathcal{B}(\mathbb{R}^2)$ such that the first marginal is $M_1 = \mathsf{Q}$ and the second is a* trivial observable, *defined as $M_2 = \mu\, I$, where μ is a fixed probability measure on $\mathcal{B}(\mathbb{R})$. Thus, for $X, Y \in \mathcal{B}(\mathbb{R})$, one has $M(X \times Y) = \mathsf{Q}(X)\mu(Y)$. [REF on bimeasure extension?] We assume that μ has finite first and second moments $\mu[1], \mu[2]$. It follows that*

$$\epsilon(M_1, \mathsf{Q}; \rho) = 0, \quad \epsilon(M_2, \mathsf{P}; \rho)^2 = \Delta(P, \rho)^2 + \big(\mathrm{tr}\big[\rho P\big] - \mu[1]\big)^2 + \big(\mu[2] - \mu[1]^2\big)\,. \tag{20}$$

This is finite for many ρ (in fact for all ρ whose momentum distribution ρ^{P} have finite first and second moments), and can be made arbitrarily small by choosing the variances of μ and ρ^{P} sufficiently small and $\mathrm{tr}\big[\rho P\big] = \mu[1]$. *On the other hand, it is clear that $\epsilon(M_2, \mathsf{P}; \rho)$ is not bounded from above, reflecting the fact that as a trivial observable, M_2 is the worst possible approximate measurement of P, providing no information at all about P. In any case it is evident that the generalized uncertainty relation (16) is fulfilled:*

$$\Delta(\mathsf{Q}, \rho) \cdot \epsilon(M_2, \mathsf{P}; \rho) \geq \Delta(\mathsf{P}, \rho) \cdot \Delta(\mathsf{P}, \rho) \geq \frac{\hbar}{2}. \tag{21}$$

It is the occurrence of the momentum standard deviation $\Delta(\mathsf{P}, \rho)$ in $\epsilon(M_2, \mathsf{P}; \rho)$ that enforces the validity of this uncertainty relation; so this relation now coincides with the standard uncertainty relation for preparations. Moreover, the finiteness of the quantity $\epsilon(M_2, \mathsf{P}; \rho)$ should not be interpreted as reflecting finite error margins: the output distribution being the same fixed μ for all *input states ρ, it follows that in a calibration scenario any momentum-localized state could have been the input given the output distribution; in other words, the measurement should be considered to have infinite inaccuracy.*

Thus it is seen that giving up unbiasedness as a criterion for a good approximate measurement leads to a very "liberal" notion of joint measurement, where a vanishing error product occurs even when one marginal provides no information at all.

Examples like this one suggest that a reliable measure of error should reflect the overall quality of the approximate determination of an observable in terms of a given measurement scheme. The above classically motivated error measure is state dependent and does not reflect adequately the notion of *calibration*; that is the idea that the systematic and random errors (bias and width) of a given measurement procedure are tested by applying it to a sufficiently large family of input states in which the observable one wishes to measure with this setup has fairly sharp values.

$$\epsilon(M_1, \mathsf{Q}; \rho) = 0, \quad \epsilon(M_2, \mathsf{P}; \rho)^2 = \Delta(P, \rho)^2 + \left(\mathrm{tr}[\rho P] - \mu[1]\right)^2 + \left(\mu[2] - \mu[1]^2\right). \tag{20}$$

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We note that Hall [?] uses somewhat different definitions of inaccuracy, but arrives at similar conclusions, including also the situation where in estimating the simultaneous values of two observables, one uses prior information about the state. The above critical comments apply to this approach, as well as to an inaccuracy-disturbance trade-off relation studied by Ozawa [?] who uses a measure of disturbance similar to the above classical error measure.

4 Werner's joint-measurement uncertainty relation

The Werner distance between two observables E, F on $\mathbb{R}$ is defined as

$$\epsilon(M_1, \mathsf{Q}; \rho) = 0, \quad \epsilon(M_2, \mathsf{P}; \rho)^2 = \Delta(P, \rho)^2 + \left(\mathrm{tr}[\rho P] - \mu[1]\right)^2 + \left(\mu[2] - \mu[1]^2\right). \tag{20}$$

This is finite for many ρ (in fact for all ρ whose momentum distribution ρ^{P} have finite first and second moments), and can be made arbitrarily small by choosing the variances of μ and ρ^{P} sufficiently small and $\mathrm{tr}[\rho P] = \mu[1]$. *On the other hand, it is clear that $\epsilon(M_2, \mathsf{P}; \rho)$ is not bounded from above, reflecting the fact that as a trivial observable, M_2 is the worst possible approximate measurement of P, providing no information at all about P. In any case it is evident that the generalized uncertainty relation (16) is fulfilled:*

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Examples like this one suggest that a reliable measure of error should reflect the overall quality of the approximate determination of an observable in terms of a given measurement scheme. The above classically motivated error measure is state dependent and does not reflect adequately the notion of *calibration*; that is the idea that the systematic and random errors (bias and width) of a given measurement procedure are tested by applying it to a sufficiently large family of input states in which the observable one wishes to measure with this setup has fairly sharp values.

We note that Hall [?] uses somewhat different definitions of inaccuracy, but arrives at similar conclusions, including also the situation where in estimating the simultaneous values of two observables, one uses prior information about the state. The above critical comments apply to this approach, as well as to an inaccuracy-disturbance trade-off relation studied by Ozawa [?] who uses a measure of disturbance similar to the above classical error measure.

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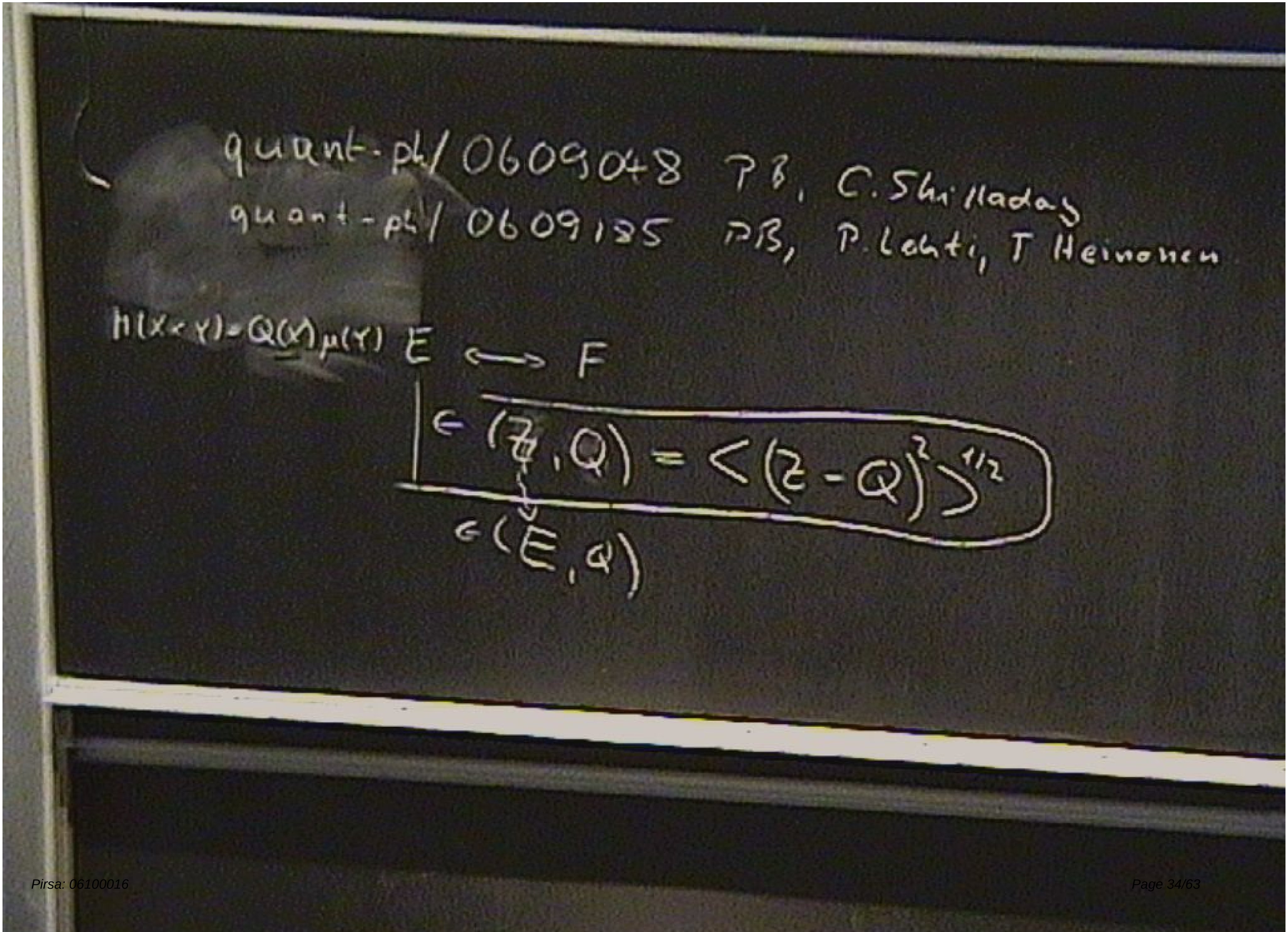
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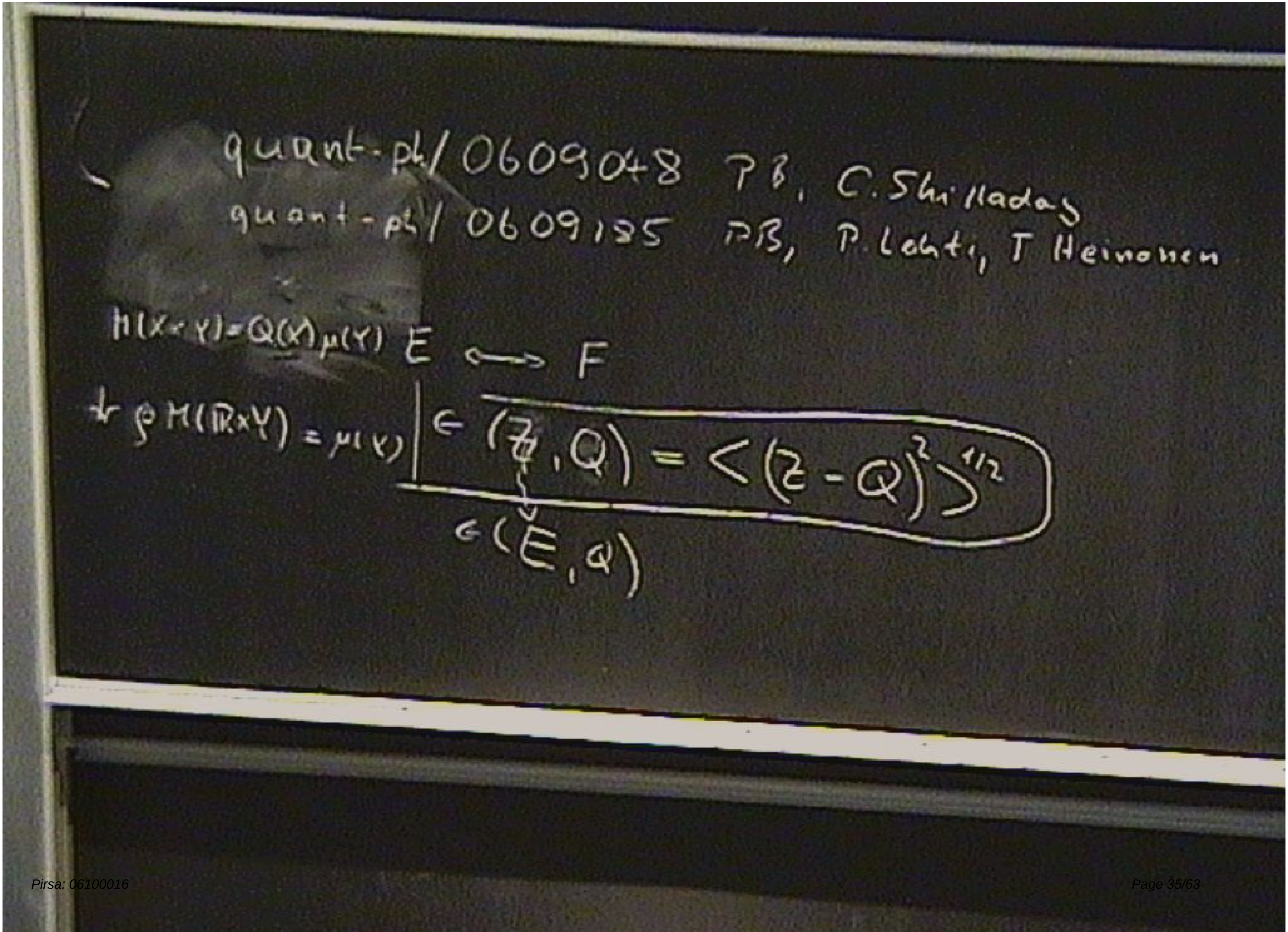
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$$d(G_1, \mathsf{Q}) = \int |q|\mathbf{m}^{\mathsf{Q}}(dq), \ \ d(G_2, \mathsf{P}) = \int |p|\mathbf{m}^{\mathsf{P}}(dp). \tag{24}$$

The relation (23) states that for any attempted joint measurement in which one marginal serves to estimate position and the other marginal serves to estimate momentum, the marginal observables cannot both be arbitrarily close to position and momentum, respectively.

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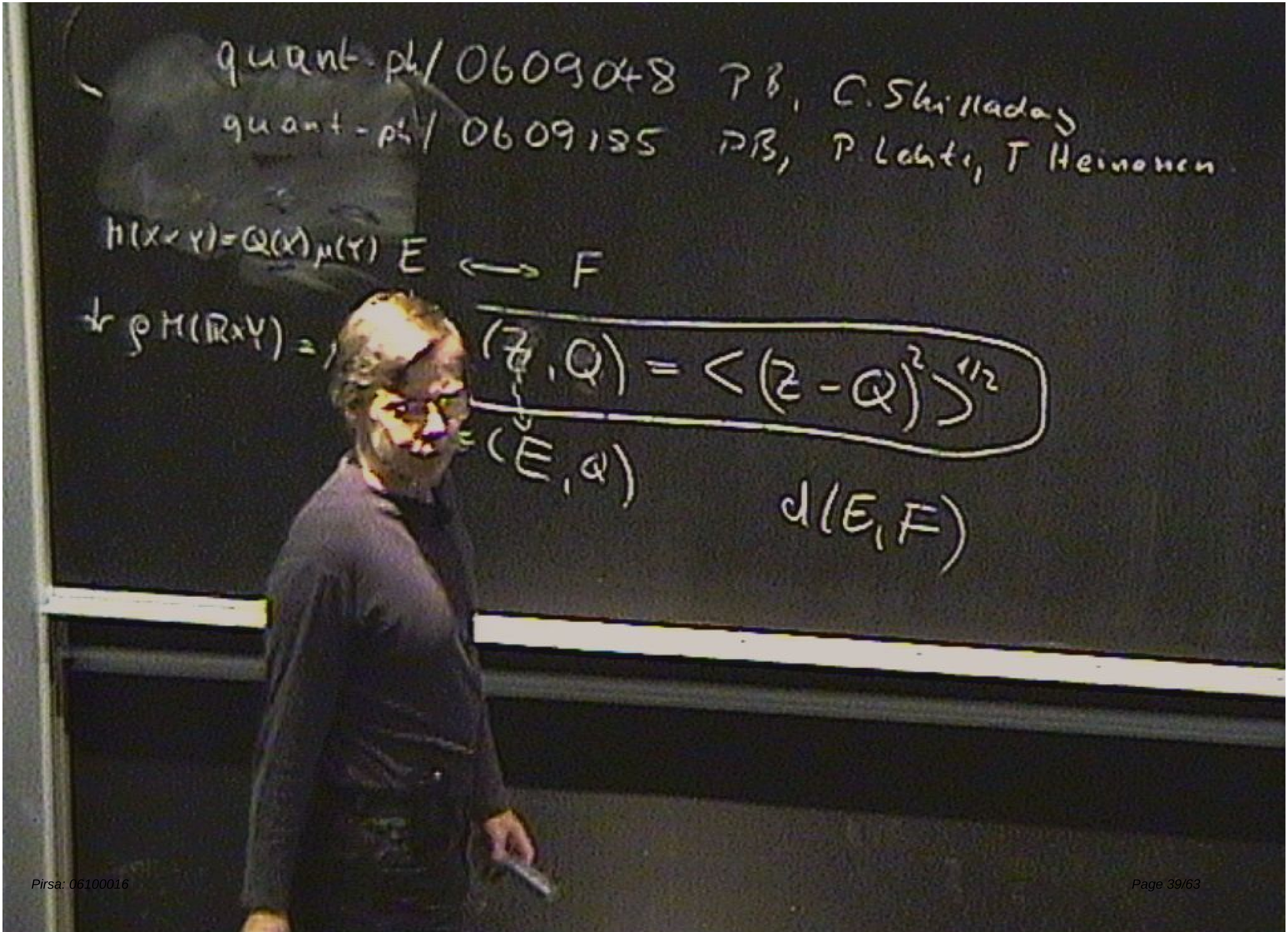
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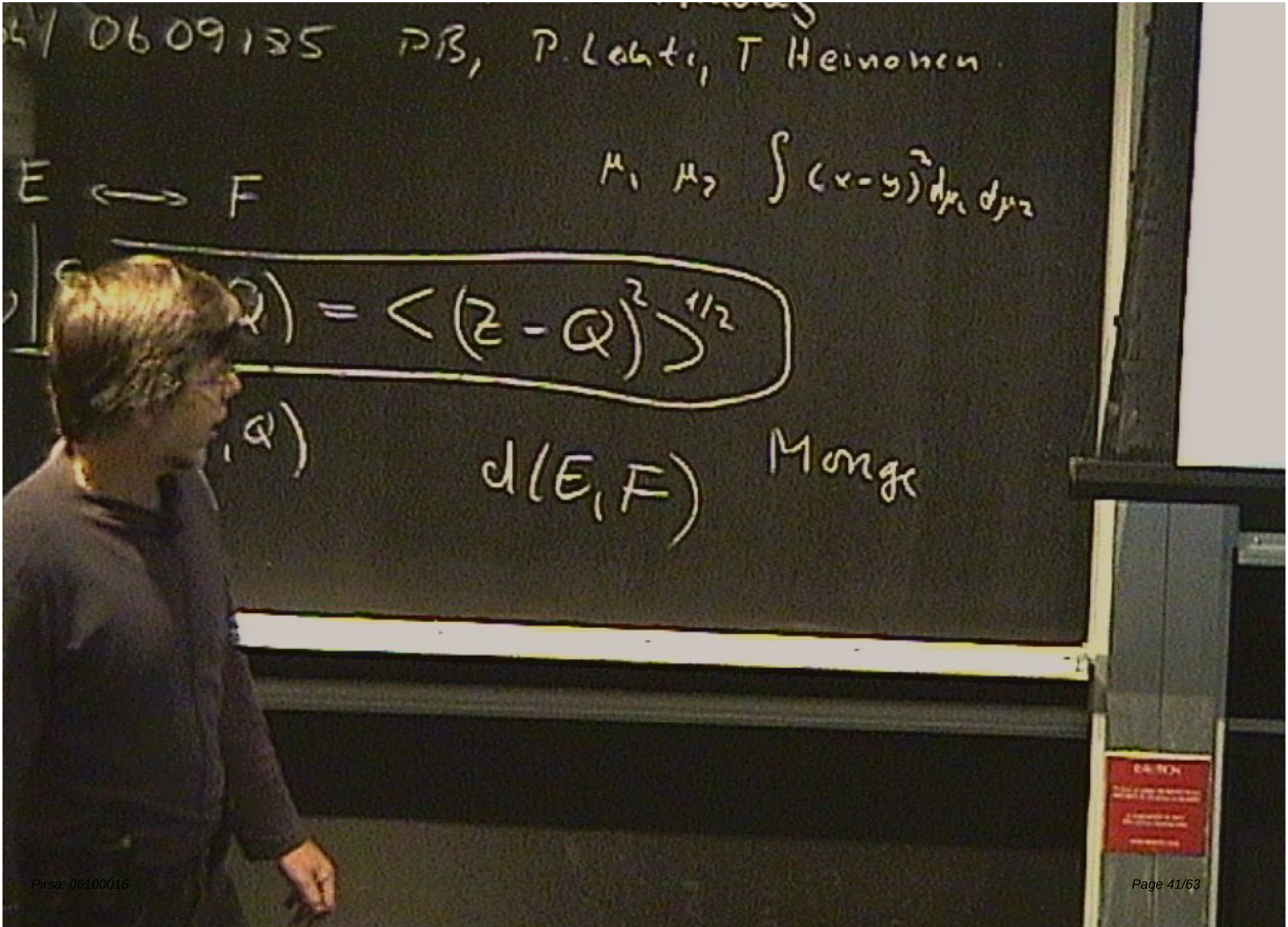
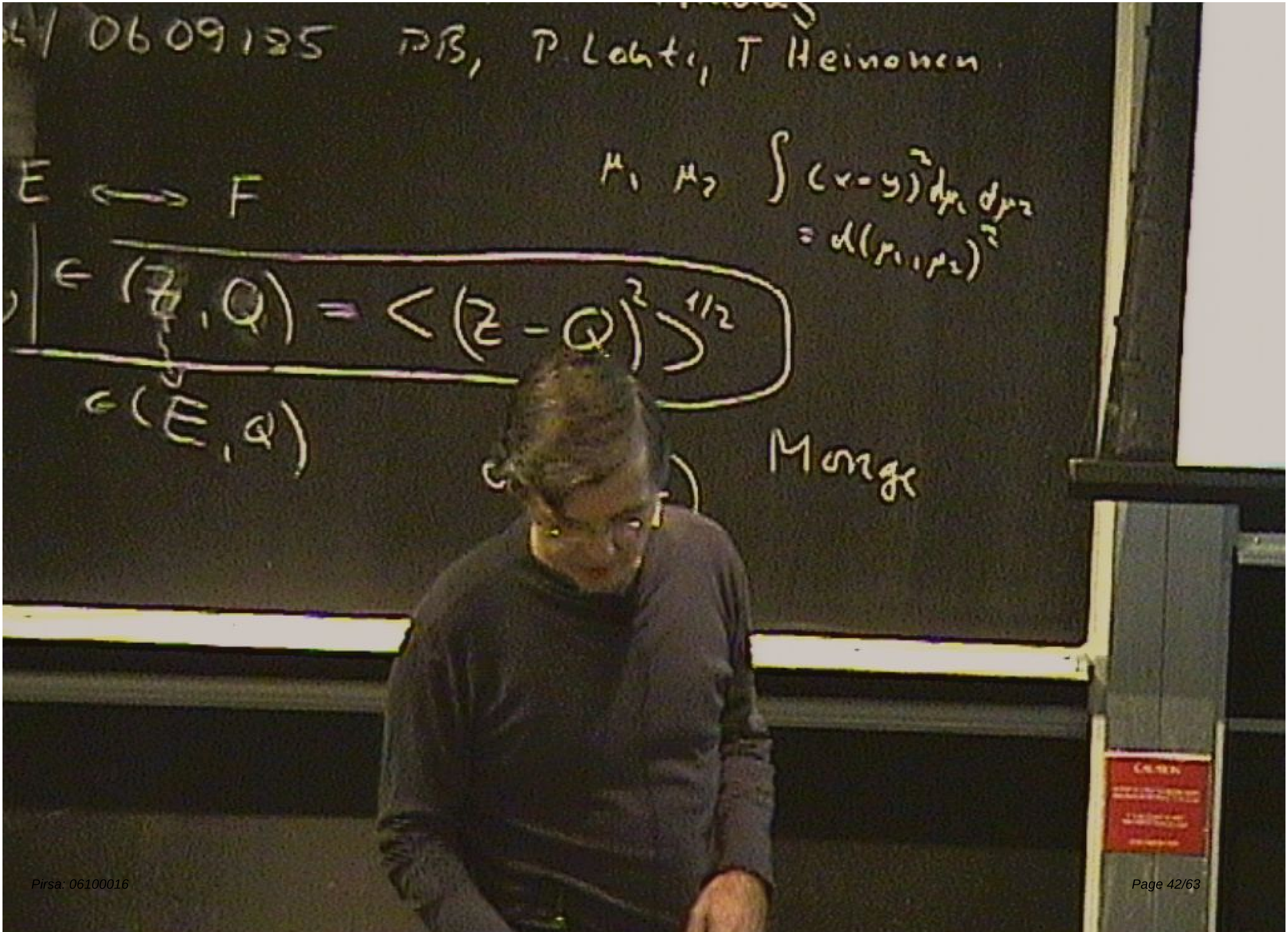

0609135 PB, P. Lahti, T Heinonen
μ1 μ2 ∫(x-y)² dμ1 dμ2
E ⟷ F
Q) = <(Z - Q)²>^1/2
d(E, F)
Monge


0609185 DB, P Lahti, T Heinonen
E ⟷ F
μ1 μ2 ∫ (x-y)² dμ1 dμ2
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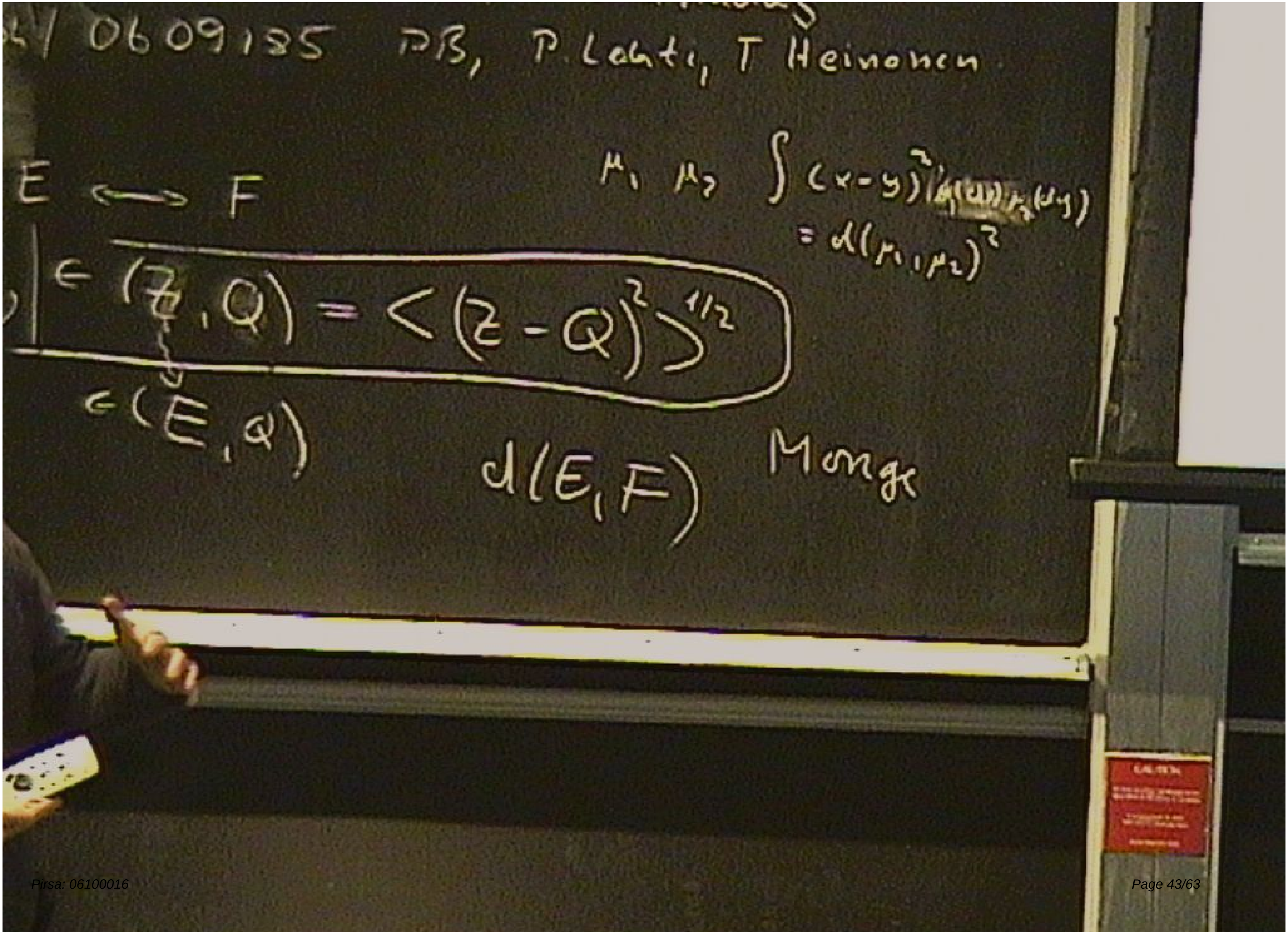
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The Werner distance between two observables E, F on $\mathbb{R}$ is defined as

$$d(E,F) := \sup_{h\in\lambda,\rho\in S} |\mathrm{tr}\,[\rho E(h)] - \mathrm{tr}\,[\rho F(h)] = \sup_{h\in\Lambda} \|E(h) - F(h)\| = \sup_{\rho\in S} d(\rho^E, \rho^F), \tag{22}$$

where $\Lambda := \{h : \mathbb{R} \to \mathbb{R}\,|\, h \text{ bounded}, \ |h(x) - h(y)| \le |x-y| \ \forall x, y \in \mathbb{R}\}$, and $E(h) = \int h(x) E(dx)$. The function d is the so-called Monge metric between probability distributions on $\mathbb{R}$.

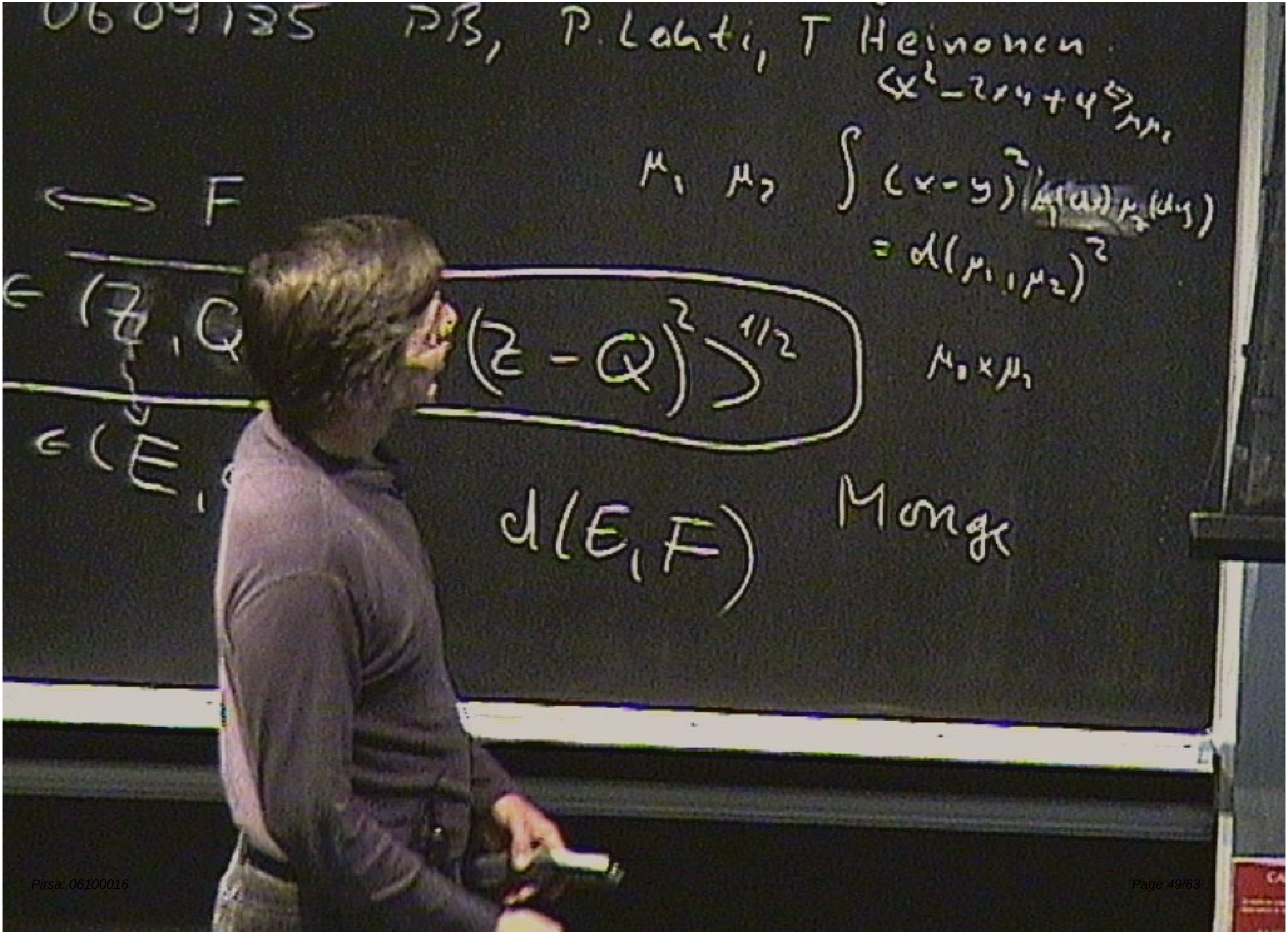
Werner's joint-measurement uncertainty relation then holds for any observable M on phase space, with marginals M_1, M_2:

$$d(M_1, \mathsf{Q}) \cdot d(M_2, \mathsf{P}) \ge C\hbar. \tag{23}$$

A crucial step of the proof consists in showing that for any M there is a covariant phase space observable G whose associated distances $d(G_1, \mathsf{Q})$ and $d(G_2, \mathsf{P})$ are not greater than those of M. The tightest lower bound for the product of distances can thus be determined within the class of covariant phase space observables and has a value of approximately 0.3047. We note that for a covariant G, with generating $\mathbf{m}$, one has

$$d(G_1, \mathsf{Q}) = \int |q| \mathbf{m}^{\mathsf{Q}}(dq), \ \ d(G_2, \mathsf{P}) = \int |p| \mathbf{m}^{\mathsf{P}}(dp). \tag{24}$$

The relation (23) states that for any attempted joint measurement in which one marginal serves to estimate position and the other marginal serves to estimate momentum, the marginal observables cannot both be arbitrarily close to position and momentum, respectively.


0609135 PB, P. Lahti, T Heinonen
⟨x² − 2xy + y²⟩
μ1 μ2 ∫ (x−y)² μ1(dx) μ2(dy)
= d(μ1, μ2)²
F
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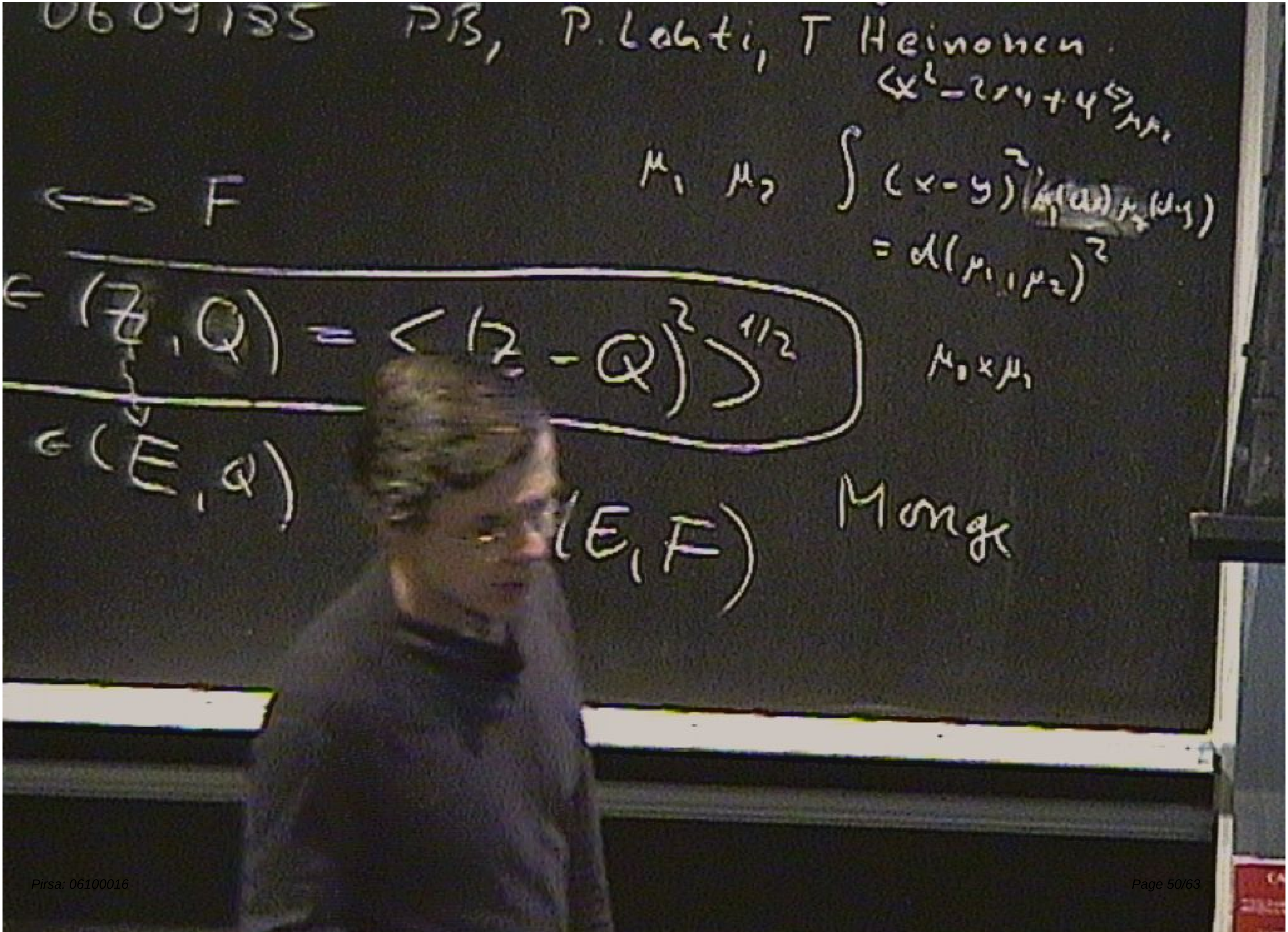
0609135 PB, P. Lahti, T Heinonen
μ1 μ2
= d(μ1, μ2)^2
F
(E, Q)
(E, F)
μ0 × μ1
Monge

It follows that the distance $d(\mathsf{P}, E) = \infty$.□

For the observables M on phase space discussed in Example 1, with marginals $M_1 = \mathsf{Q}$, $M_2 = \mu I$..... This occurs in certain models of sequential measurements where first Q is measured, followed by a measurement of P. The sharp position measurement thus distorts the subsequent momentum measurement, and Example 2 demonstrates that Werner's uncertainty relation is recovered in that $d(M_1 \mathsf{Q}) = 0$ goes along with $d(M_2, \mathsf{P}) = \infty$. The next example exhaustively generalizes this conclusion.

Example 3 *Let M be an observable on phase space $\mathbb{R}^2$ whose first marginal M_1 is sharp position Q. Then the second marginal M_2 has infinite Werner distance from sharp momentum P. (This case was not covered by Werner's proof, which required the distances to be finite.)*

Proof. *We note first that all positive operators (effects) $M_2(X)$ in the range of M_2 commute with Q and are thus functions of Q (see, e.g., [?] (quant-ph/0610122). Thus one can write $M_2(X) = \int \mathsf{Q}(dq) m(q, X)$, where the functions $m(\cdot, X)$ are defined almost everywhere for all (Borel) subsets X of $\mathbb{R}$, and $X \mapsto m(q, X)$ is then a probability measure. We consider states ρ with the same fixed position distribution, $\rho^{\mathsf{Q}} = \mathsf{p}$, and compute*

$$\mathrm{tr}\left[\rho M_2(h)\right] = \int h(x) m_{\mathsf{p}}(dx), \;\; m_{\mathsf{p}}(X) = \int \mathsf{p}(dq) m(q, X).$$

We will let h run through a family $h_n \in \Lambda$ and ρ through a family $\rho_n \in S_q$ such that $\rho_n^{\mathsf{Q}} = \mathsf{p}$ and $\mathrm{tr}\left[\rho_n M_2(h_n)\right] \to 0$, while $\mathrm{tr}\left[\rho_n \mathsf{P}(h_n)\right] \to \infty$. This shows that $d(M_2, \mathsf{P}) = \infty$.

Choose h_n as in Example 2, where we have now $\mu = m_{\mathsf{p}}$. This gives $\mathrm{tr}\left[\rho_n M_2(h_n)\right] \to 0$ for any ρ_n (yet to be specified) with $\rho_n^{\mathsf{Q}} = \mathsf{p}$.

Let $\rho_n = W(0, c_n + n - (c_1 + 1)) \rho_1 W(0, c_n + n - (c_1 + 1))^*$, with ρ_1 a state whose momentum distribution is centered symmetrically at $c_1 + 1$, the peak location of h_1. Then the momentum distribution of ρ_n is centered at the peak location $c_n + n$ of h_n. Also note that $\rho_n^{\mathsf{Q}} = \rho_1^{\mathsf{Q}} =: \mathsf{p}$. Specifically we take ρ_n such that the densities $\rho_n^{\mathsf{P}}(p) = \chi_{I_n}(p)$, $I_n = [c_n + n - 1/2, c_n + n + 1/2]$. Then we have $\mathrm{tr}\left[\rho_n \mathsf{P}(h_n)\right] = n - 1/4 \to \infty$ as $n \to \infty$. □

Example 4 *Let g be a bounded measurable function on $\mathbb{R}$. Then the Werner distance of Q and Q^g is $d(\mathsf{Q}, \mathsf{Q}^g) = \infty$.*

Proof. *We will show that $||\mathsf{Q}(h_n) - \mathsf{Q}^g(h_n)|| \to \infty$ as $n \to \infty$ for a suitable sequence of functions $h_n \in \Lambda$. To this end we use the inequality $||\mathsf{Q}(h_n) - \mathsf{Q}^g(h_n)|| \geq \big| ||\mathsf{Q}(h_n)|| - ||\mathsf{Q}^g(h_n)|| \big|$, and choose h_n such that $||\mathsf{Q}(h_n)|| \to \infty$ as $n \to \infty$, while $||\mathsf{Q}^g(h_n)||$ will remain bounded.*

Let $|g(x)| \leq g_0$. Choose $h_n(x) = n - |x - n|$ if $|x - n| \leq n$ and $h_n(x) = 0$ otherwise. Then we have $h_n \in \Lambda$. Further, $\mathsf{Q}(h_n) = \int h_n(x) \mathsf{Q}(dx)$, so $||\mathsf{Q}(h_n)|| = n \to \infty$ as $n \to \infty$. Next, we see that ur-talk(161006).pdf

the second marginal M_2 has infinite Werner distance from sharp momentum P. *(This case was not covered by Werner's proof, which required the distances to be finite.)*

Proof. *We note first that all positive operators (effects) $M_2(X)$ in the range of M_2 commute with Q and are thus functions of Q (see, e.g., [?] (quant-ph/0610122). Thus one can write $M_2(X) = \int \mathsf{Q}(dq)m(q,X)$, where the functions $m(\cdot,X)$ are defined almost everywhere for all (Borel) subsets X of $\mathbb{R}$, and $X \mapsto m(q,X)$ is then a probability measure. We consider states ρ with the same fixed position distribution, $\rho^{\mathsf{Q}} = \mathsf{p}$, and compute*

$$\mathrm{tr}\,[\rho M_2(h)] = \int h(x) m_{\mathsf{p}}(dx), \;\; m_{\mathsf{p}}(X) = \int \mathsf{p}(dq) m(q,X).$$

We will let h run through a family $h_n \in \Lambda$ and ρ through a family $\rho_n \in S_q$ such that $\rho_n^{\mathsf{Q}} = \mathsf{p}$ and $\mathrm{tr}\,[\rho_n M_2(h_n)] \to 0$, while $\mathrm{tr}\,[\rho_n \mathsf{P}(h_n)] \to \infty$. This shows that $d(M_2, \mathsf{P}) = \infty$.

Choose h_n as in Example 2, where we have now $\mu = m_{\mathsf{p}}$. This gives $\mathrm{tr}\,[\rho_n M_2(h_n)] \to 0$ for any ρ_n (yet to be specified) with $\rho_n^{\mathsf{Q}} = \mathsf{p}$.

Let $\rho_n = W(0, c_n + n - (c_1+1))\rho_1 W(0, c_n + n - (c_1+1))^*$, with ρ_1 a state whose momentum distribution is centered symmetrically at $c_1 + 1$, the peak location of h_1. Then the momentum distribution of ρ_n is centered at the peak location $c_n + n$ of h_n. Also note that $\rho_n^{\mathsf{Q}} = \rho_1^{\mathsf{Q}} =: \mathsf{p}$. Specifically we take ρ_n such that the densities $\rho_n^{\mathsf{P}}(p) = \chi_{I_n}(p)$, $I_n = [c_n + n - 1/2, c_n + n + 1/2]$. Then we have $\mathrm{tr}\,[\rho_n \mathsf{P}(h_n)] = n - 1/4 \to \infty$ as $n \to \infty$. $\square$

Example 4 *Let g be a bounded measurable function on $\mathbb{R}$. Then the Werner distance of Q and Q^g is $d(\mathsf{Q}, \mathsf{Q}^g) = \infty$.*

Proof. *We will show that $||\mathsf{Q}(h_n) - \mathsf{Q}^g(h_n)|| \to \infty$ as $n \to \infty$ for a suitable sequence of functions $h_n \in \Lambda$. To this end we use the inequality $||\mathsf{Q}(h_n) - \mathsf{Q}^g(h_n)|| \geq \big| ||\mathsf{Q}(h_n)|| - ||\mathsf{Q}^g(h_n)|| \big|$, and choose h_n such that $||\mathsf{Q}(h_n)|| \to \infty$ as $n \to \infty$, while $||\mathsf{Q}^g(h_n)||$ will remain bounded.*

Let $|g(x)| \leq g_0$. Choose $h_n(x) = n - |x - n|$ if $|x - n| \leq n$ and $h_n(x) = 0$ otherwise. Then we have $h_n \in \Lambda$. Further, $\mathsf{Q}(h_n) = \int h_n(x)\mathsf{Q}(dx)$, so $||\mathsf{Q}(h_n)|| = n \to \infty$ as $n \to \infty$. Next, we see that

$$\mathsf{Q}^g(h_n) = \int h_n(t)\mathsf{Q}^g(dt) = \int h_n(g(x))\mathsf{Q}(dx)$$

is a bounded operator since $|h_n(g(x))| \leq g_0$ for some positive constant $g_0 \in \mathbb{R}$, and so $||\mathsf{Q}^g(h_n)|| \leq g_0$. $\square$

This example can be taken as an indication that the Werner metric is somewhat coarse: it assigns infinite distance to the position observable from any of its bounded functions, although the latter might in certain circumstances be considered as reasonable approximations to the former.

5 Joint-measurement uncertainty relation for error bars

We will say that a state ρ is (Q-)localized in an interval $I_{q;\delta}$ if the position distribution ρ^{Q} vanishes outside $I_{q;\delta}$; similarly, ρ is localized in the momentum interval $I_{p;\delta}$ if the momentum distribution ρ^{P} vanishes outside $I_{p;\delta}$.

Let $\varepsilon \in (0,1), \delta > 0$. We say that an observable M_1 on$\mathbb{R}$ is an (ε,δ)-approximation to Q if there is a positive number $w < \infty$ such that for all $q \in \mathbb{R}$ and all states ρ localized in $I_{q;\delta}$, one has $\rho^{M_1}(I_{q;w}) > 1-\varepsilon$. The infimum of all such w is called the *inaccuracy* (of M_1 with respect to Q) and will be denoted $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q})$. Finally, the *resolution* (of M_1 with respect to Q is defined as the smallest inaccuracy across all $\delta > 0$: $\Delta_\varepsilon(M_1,\mathsf{Q}) := \inf_\delta \Delta_{\varepsilon,\delta}(M_1,\mathsf{Q})$.

The inaccuracies describe the range of values within which the input values can be inferred from the output distributions, with confidence level ε, given initial localizations within δ.

We propose to accept M_1 as representing an *approximate measurement* of Q (in short: M_1 is an *approximation* to Q) if M_1 has *finite accuracies*, that is, $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q}) < \infty$, for all $\varepsilon,\delta > 0$. An approximation M_1 of Q is said to have *finite resolution* if $\Delta_\varepsilon(M_1,\mathsf{Q}) > 0$ for all $\varepsilon > 0$.

Similar definition apply to approximations M_2 of momentum P, with ensuing inaccuracy $\Delta_{\varepsilon,\delta}(M_2,\mathsf{P})$ and resolution $\Delta_\varepsilon(M_2,\mathsf{P})$.

Using this notion of approximation, we say that an observable on phase space M is an (ε,δ)-*approximate* joint observable of position and momentum if the marginals M_1, M_2 are (ε,δ)-approximations to Q and P, respectively. For later use we state this condition explicitly:
there are positive numbers $w, w' < \infty$ such that the following conditions hold:

(I) for all $q \in \mathbb{R}$ and all ρ localized in $I_{q;\delta}$, $\mathrm{tr}\,[\rho M_1(I_{q;w})] > 1-\varepsilon$;

(II) for all $p \in \mathbb{R}$ and all ρ localized in $I_{q;\delta}$, $\mathrm{tr}\,[\rho M_2(I_{p;w'})] > 1-\varepsilon$.

Thus, a joint measurement of position and momentum is required to yield pointer distributions which are concentrated around the values of position and momentum if these are prepared to lie within intervals $I_{q;\delta}$ and $I_{p;\delta}$, respectively; and it is required that for given confidence level ε, the associated bulk widths of the output distributions, the inaccuracies $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q})$, $\Delta_{\varepsilon,\delta}(M_2,\mathsf{P})$ are finite.

We propose to accept as an *approximate joint observable* of Q, P any observable M on phase space that has *finite* accuracies $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q}) < \infty$, $\Delta_{\varepsilon,\delta}(M_2,\mathsf{P}) < \infty$ for all $\varepsilon > 0, \delta > 0$.

(It is straightforward to adapt this definition and the subsequent considerations to the case where the ur-talk(161006).pdf

Finally, the *resolution* (of M_1 with respect to Q is defined as the smallest inaccuracy across all $\delta > 0$: $\Delta_\varepsilon(M_1, \mathsf{Q}) := \inf_\delta \Delta_{\varepsilon,\delta}(M_1, \mathsf{Q})$.

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We propose to accept M_1 as representing an *approximate measurement* of Q (in short: M_1 is an *approximation* to Q) if M_1 has *finite accuracies*, that is, $\Delta_{\varepsilon,\delta}(M_1, \mathsf{Q}) < \infty$, for all $\varepsilon, \delta > 0$. An approximation M_1 of Q is said to have *finite resolution* if $\Delta_\varepsilon(M_1, \mathsf{Q}) > 0$ for all $\varepsilon > 0$.

Similar definition apply to approximations M_2 of momentum P, with ensuing inaccuracy $\Delta_{\varepsilon,\delta}(M_2, \mathsf{P})$ and resolution $\Delta_\varepsilon(M_2, \mathsf{P})$.

Using this notion of approximation, we say that an observable on phase space M is an (ε, δ)*-approximate* joint observable of position and momentum if the marginals M_1, M_2 are (ε, δ)-approximations to Q and P, respectively. For later use we state this condition explicitly:
there are positive numbers $w, w' < \infty$ such that the following conditions hold:

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(It is straightforward to adapt this definition and the subsequent considerations to the case where the conditions for M_1 and M_2 formulated with different values $\varepsilon, \varepsilon'$ and δ, δ' of the confidence parameters and localization widths.)

It is possible to characterize the case of a *sharp* measurement of Q. First note

$$\Delta_{\varepsilon,\delta}(\mathsf{Q}, \mathsf{Q}) = \delta \quad \text{and} \quad \Delta_\varepsilon(\mathsf{Q}, \mathsf{Q}) = 0. \tag{25}$$

Proposition 1 *Let M_1 be an approximation to Q. Then the following are equivalent:*

(a) $\Delta_{\varepsilon,\delta}(M_1, \mathsf{Q}) = \delta$ *for all* $\varepsilon \in (0, 1), \delta > 0$;

(b) $M_1 = \mathsf{Q}$.

Proof. Assume (b) holds. Let $\varepsilon \in (0, 1)$, $\delta > 0$. Certainly one can choose $w = \delta$ so that for any $q \in \mathbb{R}$ and

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Using this notion of approximation, we say that an observable on phase space M is an (ε,δ)-*approximate* joint observable of position and momentum if the marginals M_1, M_2 are (ε,δ)-approximations to Q and P, respectively. For later use we state this condition explicitly:
there are positive numbers $w, w' < \infty$ such that the following conditions hold:

(I) for all $q \in \mathbb{R}$ and all ρ localized in $I_{q;\delta}$, $\mathrm{tr}\,[\rho M_1(I_{q;w})] > 1-\varepsilon$;

(II) for all $p \in \mathbb{R}$ and all ρ localized in $I_{q;\delta}$, $\mathrm{tr}\,[\rho M_2(I_{p;w'})] > 1-\varepsilon$.

Thus, a joint measurement of position and momentum is required to yield pointer distributions which are concentrated around the values of position and momentum if these are prepared to lie within intervals $I_{q;\delta}$ and $I_{p;\delta}$, respectively; and it is required that for given confidence level ε, the associated bulk widths of the output distributions, the inaccuracies $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q})$, $\Delta_{\varepsilon,\delta}(M_2,\mathsf{P})$ are finite.

We propose to accept as an *approximate joint observable* of Q,P any observable M on phase space that has *finite* accuracies $\Delta_{\varepsilon,\delta}(M_1,\mathsf{Q}) < \infty$, $\Delta_{\varepsilon,\delta}(M_2,\mathsf{P}) < \infty$ for all $\varepsilon > 0, \delta > 0$.

(It is straightforward to adapt this definition and the subsequent considerations to the case where the conditions for M_1 and M_2 formulated with different values ε,ε' and δ,δ' of the confidence parameters and localization widths.)

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(b) $M_1 = \mathsf{Q}$.

Proof. Assume (b) holds. Let $\varepsilon \in (0,1)$, $\delta > 0$. Certainly one can choose $w = \delta$ so that for any $q \in \mathbb{R}$ and any state ρ with $\rho^{\mathsf{Q}}(I_{q;\delta}) = 1$, we also have $\rho^{\mathsf{Q}}(I_{q;\delta}) > 1-\varepsilon$. This shows that $\Delta_{\varepsilon,\delta}(\mathsf{Q},\mathsf{Q}) \le \delta$. If $\Delta_{\varepsilon,\delta}(\mathsf{Q},\mathsf{Q})$ were smaller than δ, one could choose $w < \delta$ such that still $\rho^{\mathsf{Q}}(I_{q;w}) > 1-\varepsilon$ for all ρ localized in $I_{q;\delta}$. But this is violated by any ρ localized in $I_{q;\delta} \setminus I_{q;w}$, for which $\rho^{\mathsf{Q}}(I_{q;w}) = 0$. (Here we are using the continuity of the (spectrum) of Q.) Hence $\Delta_{\varepsilon,\delta}(\mathsf{Q},\mathsf{Q}) = \delta$.

Conversely, assume that (a) holds. Consider any $\varepsilon \in (0,1), \delta > 0$. For $w = \Delta_{\varepsilon,\delta}(M_1,\mathsf{Q}) = \delta$, we have, for all $q \in \mathbb{R}$ and all ρ with $\rho^{\mathsf{Q}}(I_{q;\delta}) = 1$, that $\rho^{M_1}(I_{q;w}) \ge 1-\varepsilon$. This entails for any vector state φ for which $\mathsf{Q}(I_{q;\delta})\varphi = \varphi$ that $\langle\varphi|M_1(I_{q;\delta})\varphi\rangle \ge 1-\varepsilon$. As this holds for any $\varepsilon \in (0,1)$, it follows that $\langle\varphi|M_1(I_{q;\delta})\varphi\rangle = 1$. This entails that $\mathsf{Q}(I_{q;\delta}) \le M_1(I_{q;\delta})$. Since $q \in \mathbb{R}$ and $\delta > 0$ are arbitrary, this ordering holds for any interval $J = [a,b)$. Let J be a member of a partition J_n of $\mathbb{R}$, then $I = \sum_n \mathsf{Q}(J_n) \le \sum_n M_1(J_n) = I$. This ensures

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Proposition 2 *Let M be an observable on phase space whose first marginal coincides with sharp position, $M_1 = \mathsf{Q}$, so that $\Delta_{\varepsilon,\delta}(M_1, \mathsf{Q}) = 0$. Then the second marginal M_2 cannot be an (ε,δ)-approximation to P for any $\varepsilon > 0, \delta > 0$, that is, $\Delta_{\varepsilon,\delta}(M_2, \mathsf{P}) = \infty$. In particular, M cannot be an approximate joint observable to* Q, P.

Proof. Let ϵ $(0 < \varepsilon < 1)$ and $\delta > 0$ be given and $w' > 0$ be arbitrary. We have to show that there is an interval $I_{p;\delta}$ and a state ρ localized in $I_{p;\delta}$ so that $\mathrm{tr}\,[\rho M_2(I_{p;w'})] < 1 - \varepsilon$.

As noted in Example 3, since $M_1 = \mathsf{Q}$, the observable M is commutative, and every effect in its range is a function of Q, and we can write: $M_2(X) = \int m(q, X)\,\mathsf{Q}(dx)$. Consider a partition of $\mathbb{R}$ into disjoint intervals $I_{p_n;w'}$. Since $I = M_2(\mathbb{R}) = \sum_n M_2(I_{p_n;w'})$ (ultraweakly), it follows that for every state ρ,

$$\mathrm{tr}\,[\rho M_2(I_{p_n;w'})] = \int \rho^{\mathsf{Q}}(dq) m(q, I_{p_n;w'}) \to 0 \quad \text{as } |p_n| \to \infty.$$

Let ρ_0 be localized in $I_{p_0;\delta}$, that is, the distribution ρ_0^{P} vanishes outside that interval. Then $\rho_n := W(0, p_n)\rho_0$ is localized in $I_{p_n;\delta}$, while the position distribution is unchanged, $\rho_n^{\mathsf{Q}} = \rho_0^{\mathsf{Q}}$.

For the given $\varepsilon \in (0, 1)$, there is an $n \in \mathbb{N}$ such that for the fixed state ρ_0, $\mathrm{tr}\,[\rho_0 M_2(I_{p_n;w'}))] < 1 - \varepsilon$. Then, since $\rho_0^{\mathsf{Q}} = \rho_n^{\mathsf{Q}}$, we also have $\mathrm{tr}\,[\rho_n M_2(I_{p_n;w'})] < 1 - \varepsilon$, whereas ρ_n is localized in $I_{p_n;\delta}$. □

This result reproduces, in particular, the well-known fact that there is no observable on phase space whose marginals are sharp position and sharp momentum.

As an observable M on phase space with $M_1 = \mathsf{Q}$ cannot be regarded as an approximate joint measurement of position and momentum, it is appropriate to consider (ε, δ)-approximate measurements whose inaccuracies are bounded away from 0 across all δ. Thus we focus on the class of approximate joint measurements of Q, P with *finite resolutions*, that is, $\Delta_\varepsilon(M_1, \mathsf{Q}) > 0$, $\Delta_\varepsilon(M_2, \mathsf{P}) > 0$. We will see shortly that this class is not empty. In particular, all covariant phase space observables belong to it. Our main result is the following.

Theorem 1 *Let M be an approximate joint observable for Q, P. Then for all ε, δ with $0 < \varepsilon < 1/2$, $\delta > 0$, the inaccuracies and resolutions of M_1 and M_2 satisfy the uncertainty relation*

$$\Delta_{\varepsilon,\delta}(M_1, \mathsf{Q}) \cdot \Delta_{\varepsilon,\delta}(M_2, \mathsf{P}) \geq \Delta_\varepsilon(M_1, \mathsf{Q}) \cdot \Delta_\varepsilon(M_2, \mathsf{P}) \geq C(\varepsilon)\hbar. \tag{26}$$

For the proof of Theorem 1, we set out to show that for each member M in this class of approximate joint measurements, there is a covariant phase space observable G whose resolutions are not greater than those of M, that is, $\Delta_\varepsilon(G_i, \mathsf{Q}) \leq \Delta_\varepsilon(M_i, \mathsf{Q})$, $i = 1, 2$. We then use the results reviewed in Section 2 to prove

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$W(0,p_n)\rho_0$ is localized in $I_{p_n;\delta}$, while the position distribution is unchanged, $\rho_n^Q = \rho_0^Q$.

For the given $\varepsilon \in (0,1)$, there is an $n \in \mathbb{N}$ such that for the fixed state ρ_0, $\mathrm{tr}\,[\rho_0 M_2(I_{p_n;w'})] < 1-\varepsilon$. Then, since $\rho_0^Q = \rho_n^Q$, we also have $\mathrm{tr}\,[\rho_n M_2(I_{p_n;w'})] < 1-\varepsilon$, whereas ρ_n is localized in $I_{p_n;\delta}$. □

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For the proof of Theorem 1, we set out to show that for each member M in this class of approximate joint measurements, there is a covariant phase space observable G whose resolutions are not greater than those of M, that is, $\Delta_\varepsilon(G_i,\mathsf{Q}) \leq \Delta_\varepsilon(M_i,\mathsf{Q})$, $i=1,2$. We then use the results reviewed in Section 2 to prove the uncertainty relation (26) for G.

Following Werner, we make use of the concept of the invariant mean on the group of phase space translations to introduce a covariant phase space observable M^{av} associated with any observable M on phase space. The invariant mean is a linear functional η on $C(\mathbb{R}^2)$ which is linear, positive (it sends nonnegative functions to nonnegative numbers), and has the invariance property $\eta(\tau_x f) = \eta(f)$. This extends the operation of integrating f over an interval, dividing by the interval length, and letting that length go to infinity. While this operation only works for a very limited class of functions, the existence of η is guaranteed by the axiom of choice.

Any observable M on phase space can be viewed as a functional on the space $C_{uc}(\mathbb{R}^2)$ of bounded uniformly continuous functions via $M(f) = \int f(q,p)dM(q,p)$. For any $f \in C_{uc}(\mathbb{R}^2)$, the operator $M^{\mathrm{av}}(f)$ is defined via the following equations, required to hold for all $\rho \in S$:

$$\mathrm{tr}\,[\rho M^{\mathrm{av}}(f)] = \eta(u(\rho,f)), \quad u(\rho,f)(q,p) = \mathrm{tr}\left[W(q,p)^*\rho W(q,p)M(\tau_{(q,p)}f)\right]. \tag{27}$$

M^{av} is defined first only as a positive, normalized, linear functional $f \mapsto M^{\mathrm{av}}(f)$ on $C_{uc}(\mathbb{R})$. The covariance of M^{av} is an immediate consequence of the invariance of η. We will show that under the assumptions of

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It seems that the definition of approximate measurement based on finite inaccuracy and finite resolution is somewhat more restrictive than the definition based on finite Werner distances $d(M_1, \mathsf{Q})$, $d(M_2, \mathsf{P})$, although the latter condition need not be satisfied for a phase space observable M to be an approximate joint measurement of Q, P.

Following the reasoning of Werner [1] these results can be adapted as follows to cover the second form of measurement uncertainty relation for accuracy and disturbance. Assume a (sharp or approximate) position measurement is made. This will in general lead to a change of the state and hence a change of the momentum distribution, which can be tested by a subsequent momentum measurement. This sequence of measurements induces an observable M on phase space whose first marginal is the (approximate) position observable measured first, while the second marginal represents the momentum which is distorted by the position measurement. Since our measures of error quantify the differences between the marginal distributions M_1, M_2 and their corresponding sharp counterparts Q, P, the measurement resolution for the first marginal describes the accuracy of the first measurement, while the resolution for the second marginal represents a measure of disturbance of the momentum distribution.

The joint measurement uncertainty relation is thus seen to entail the accuracy-disturbance uncertainty relation. Finally, we note that the joint measurement uncertainty relation was obtained by reduction to the covariant case, for which the standard type of uncertainty relations for quantum states was utilized to deduce the joint measurement uncertainty relation. In this way, all three forms of uncertainty relations are ultimately seen to be formally equivalent within quantum mechanics.

References

[1] R.F. Werner, quant-ph/0405184.

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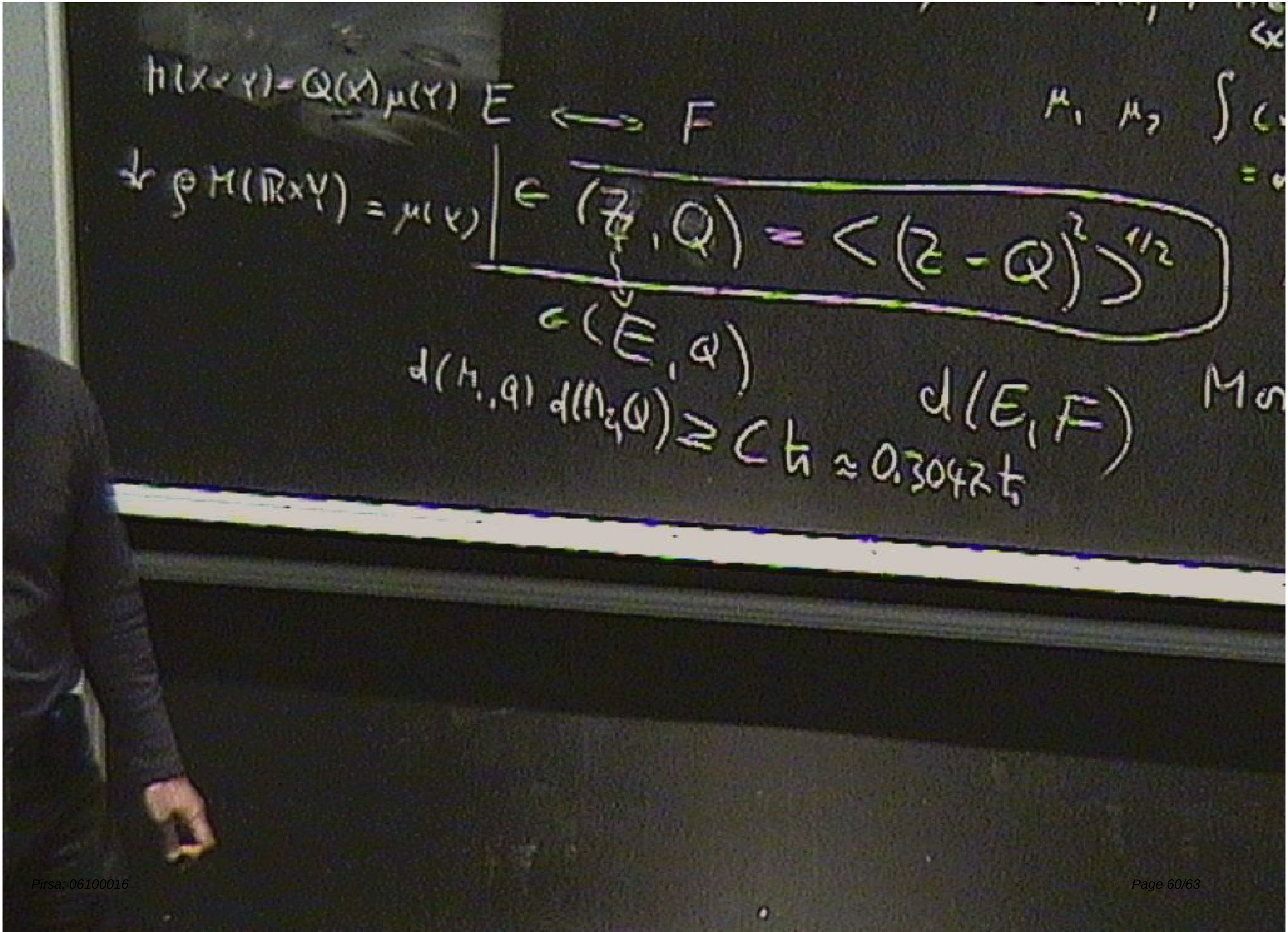
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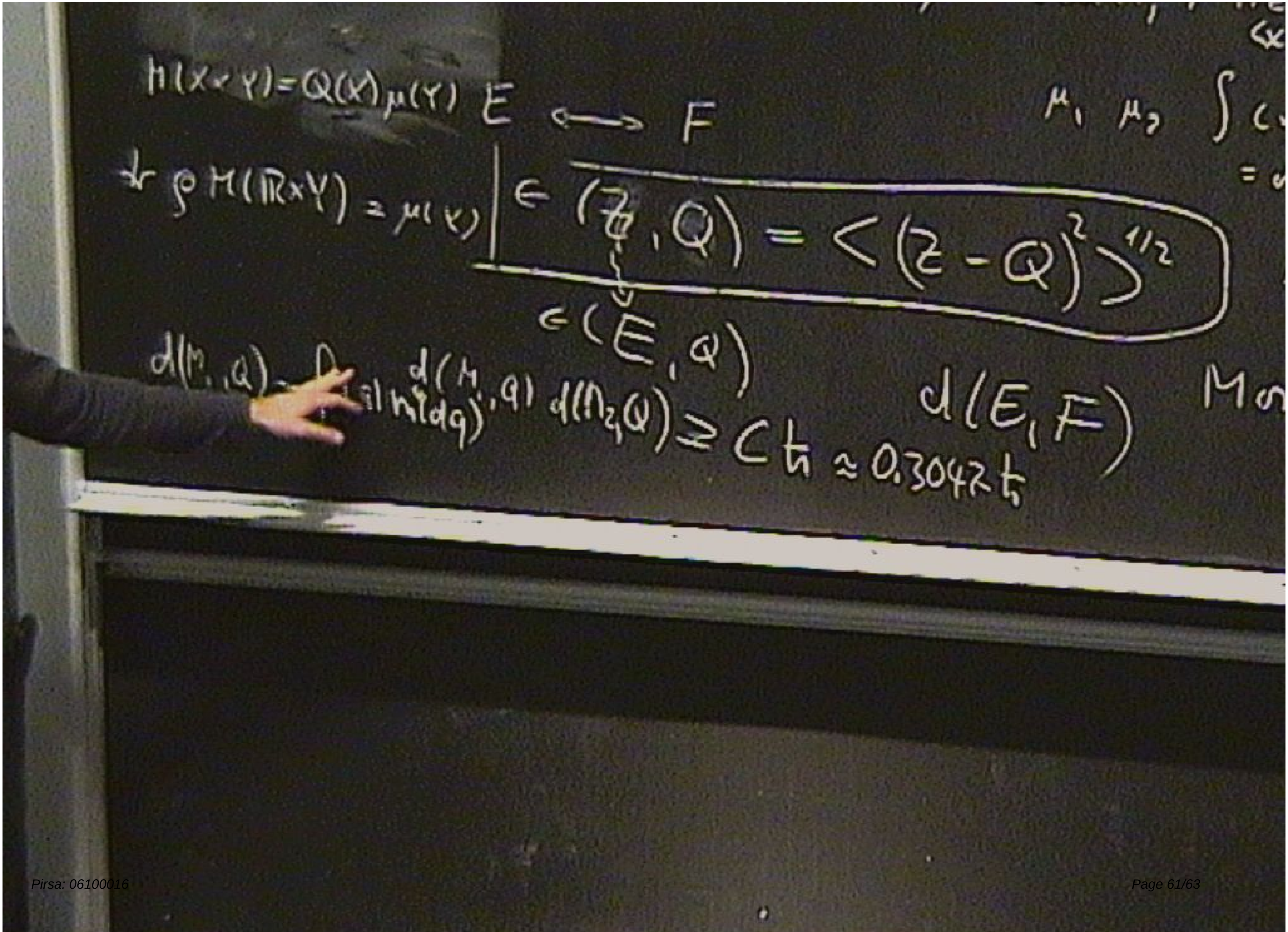

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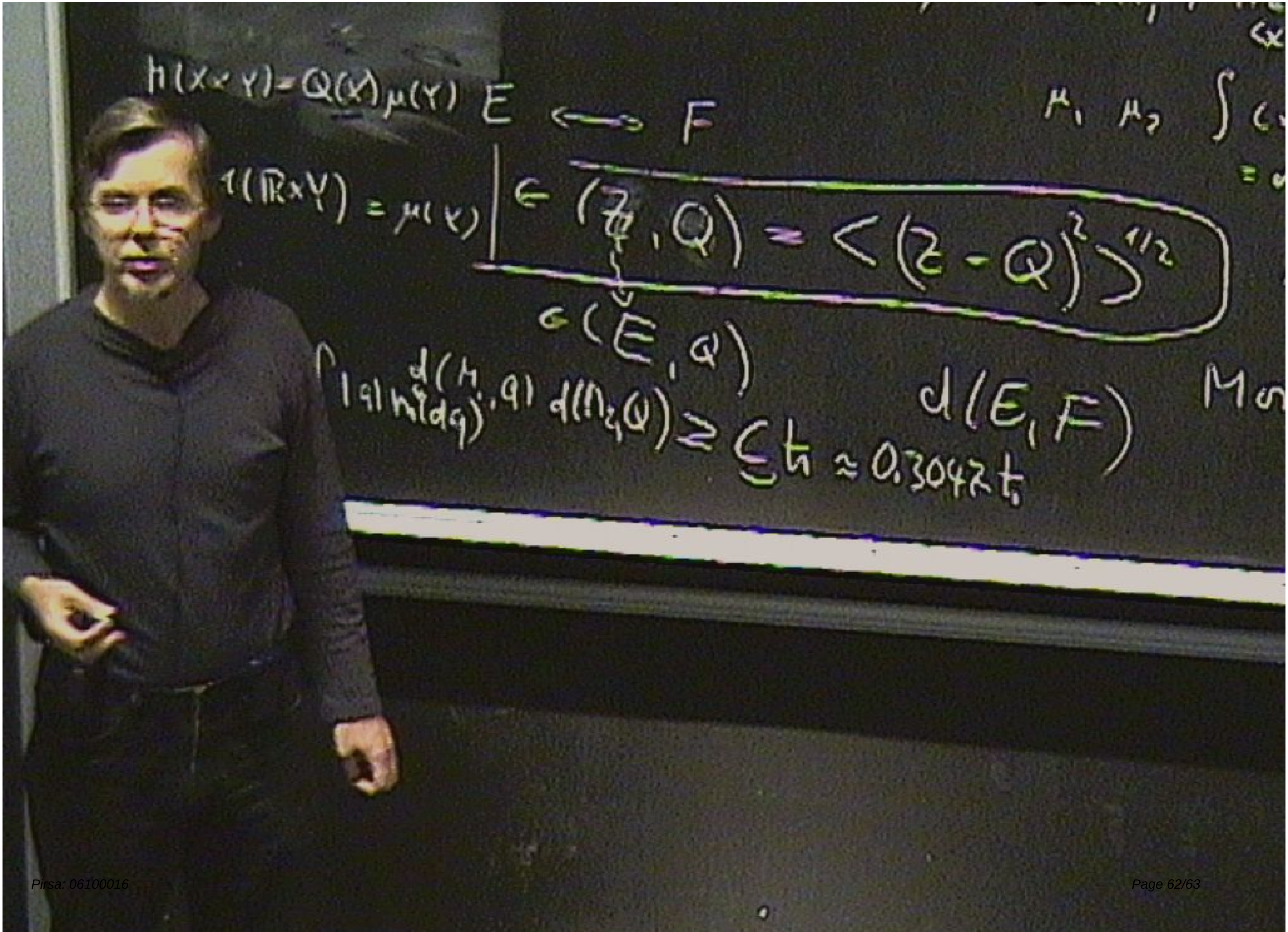
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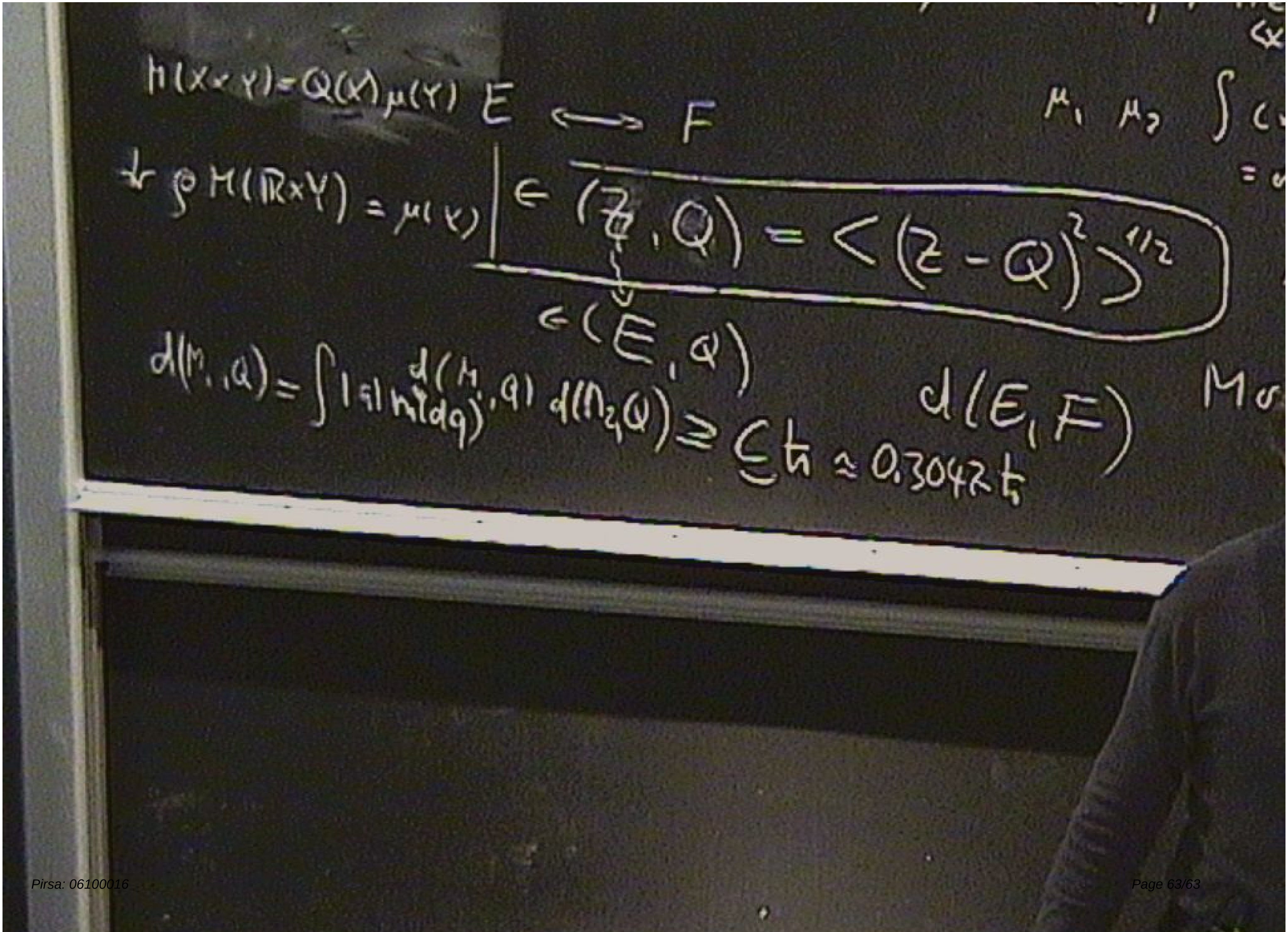
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