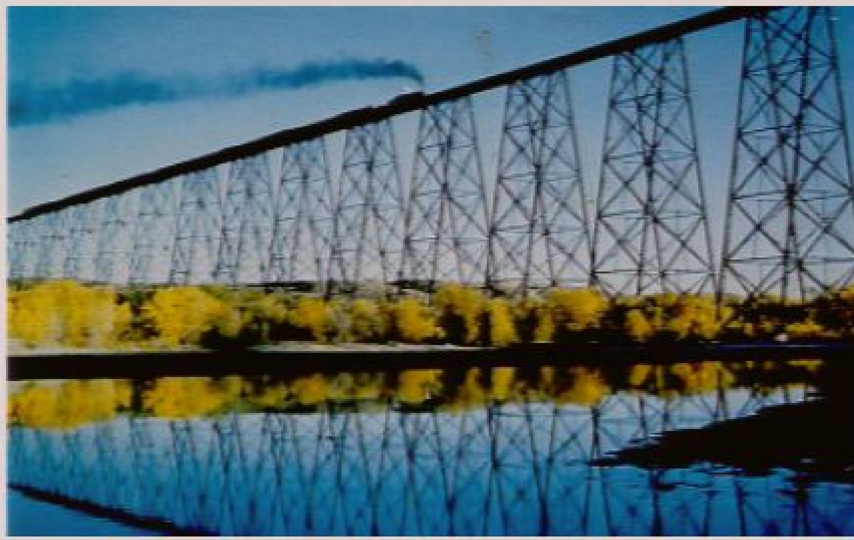
Title: 1/4 BPS Loops in N=4 Super-Yang-Mills: Bridges between weak and strong coupling

Date: Oct 10, 2006 11:00 AM

URL: http://pirsa.org/06100009

Abstract: TBA

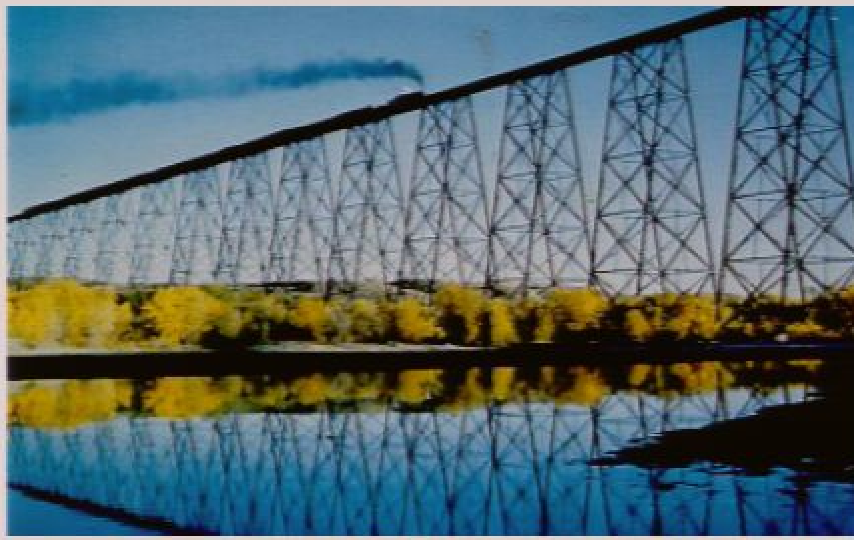
1/4 BPS Loops in $\mathcal{N} = 4$ SYM: Bridges Between Weak and Strong Coupling



Donovan Young
University of British Columbia

Perimeter Institute, October 10, 2006

1/4 BPS Loops in $\mathcal{N} = 4$ SYM: Bridges Between Weak and Strong Coupling



Donovan Young
University of British Columbia

Perimeter Institute, October 10, 2006

Outline

1. A cross-section of results
2. A new 1/4 BPS circular loop
3. Recent work (hep-th/0609158) with Gordon Semenoff on two point functions with chiral primary operators

Introduction

The Wilson loop of $\mathcal{N} = 4$ SYM is given by

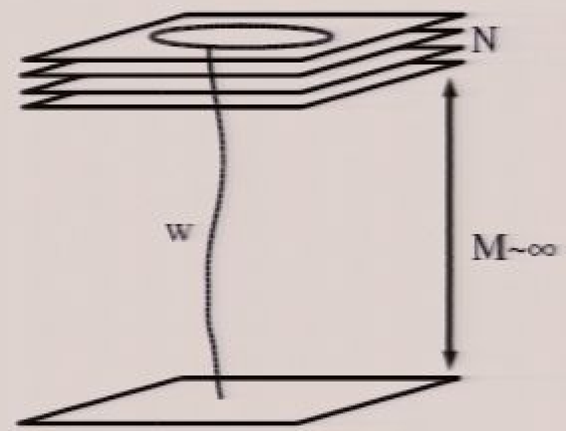
$$W = \frac{1}{N} \operatorname{Tr} P \exp \left[\oint_C d\tau \left(i \dot{x}_\mu(\tau)\, A_\mu(x) + |\dot{x}(\tau)| \theta^I(\tau)\, \Phi^I(x) \right) \right]$$

with $\theta^I \theta^I = 1$. The construction for $\theta^I =$ const. may be understood as the holonomy of a heavy, fundamental W-boson:

The six scalars of the $SU(N+1)$ theory are given a VEV:

$$\hat{\Phi}^I = \begin{pmatrix} \Phi^I & w^I \\ w^{I\dagger} & M\theta^I \end{pmatrix}$$

this gives the w^I (fundamental rep.) a mass M.



One can then show that

$$\int dy \langle w(x)\, w^\dagger(x)\, w(y)\, w^\dagger(y) \rangle \sim \int \mathcal{D}x_\mu \int \mathcal{D}A_\mu \mathcal{D}\Phi^I e^{-S_{SU(N)} - ML(x_\mu)} W(x_\mu)$$

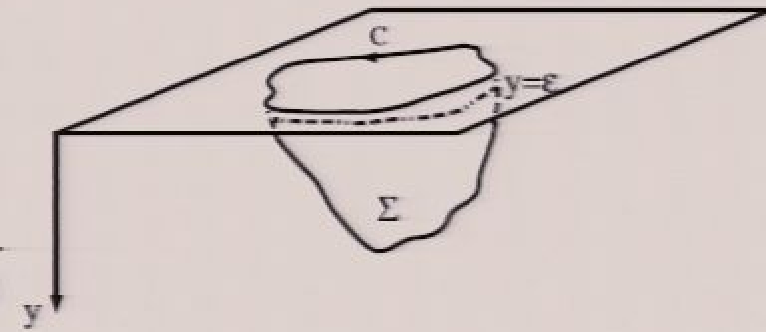
Wilson loop at strong coupling

At strong coupling the Wilson loop is given by the semi-classical partition function for a string in $AdS_5 \times S^5$:

$$Z = \int \mathcal{D}X^\mu \, \mathcal{D}Y^I \, \mathcal{D}h_{ab} \mathcal{D}\vartheta^\alpha \exp\left(-\frac{\sqrt{\lambda}}{4\pi} \int_\Sigma d^2\sigma \sqrt{h}\, h_{ab} \frac{\partial_a X^\mu \partial_b X^\mu + \partial_a Y^I \partial_b Y^I}{Y^2} + \text{fermions} \right)$$

$$X^\mu|_{\partial\Sigma} = x_\mu(\tau), \qquad Y^I|_{\partial\Sigma} = \theta^I(\tau) Y|_{\partial\Sigma}, \qquad Y|_{\partial\Sigma} = 0$$

The saddle-point is obtained when the string worldsheet Σ describes a surface of minimal area $\mathcal{A}$



$$\frac{\sqrt{\lambda}}{2\pi} \int_\Sigma d^2\sigma \frac{1}{Y^2} \sqrt{\det\left(\partial_a X^\mu \partial_b X^\mu + \partial_a Y^I \partial_b Y^I\right)}$$

$$= \mathcal{A}_{\text{reg.}} + \frac{L(C)}{\varepsilon}$$

And so $1/\varepsilon$ corresponds to M, the mass of the heavy boson. We then have,

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi} \mathcal{A}_{\text{reg.}}}$$

Introduction

The Wilson loop of $\mathcal{N}=4$ SYM is given by

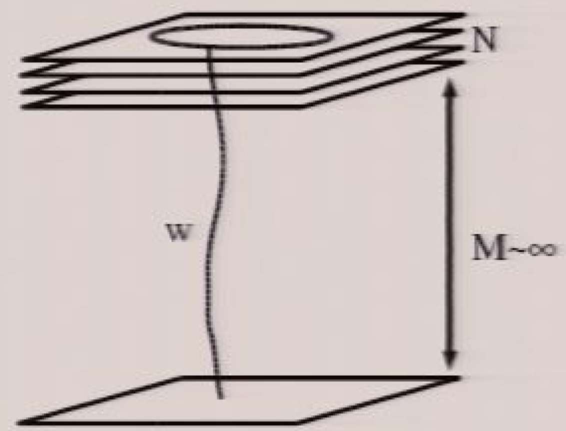
$$W = \frac{1}{N}\,\mathrm{Tr}\,P\exp\left[\oint_C d\tau\left(i\dot{x}_\mu(\tau)\,A_\mu(x) + |\dot{x}(\tau)|\theta^I(\tau)\,\Phi^I(x)\right)\right]$$

with $\theta^I\theta^I = 1$. The construction for $\theta^I = \text{const.}$ may be understood as the holonomy of a heavy, fundamental W-boson:

The six scalars of the $SU(N+1)$ theory are given a VEV:

$$\hat{\Phi}^I = \begin{pmatrix} \Phi^I & w^I \\ w^{I\dagger} & M\theta^I \end{pmatrix}$$

this gives the w^I (fundamental rep.) a mass M.



One can then show that

$$\int dy \langle w(x)\,w^\dagger(x)\,w(y)\,w^\dagger(y)\rangle \sim \int \mathcal{D}x_\mu \int \mathcal{D}A_\mu \mathcal{D}\Phi^I e^{-S_{SU(N)}-ML(x_\mu)} W(x_\mu)$$

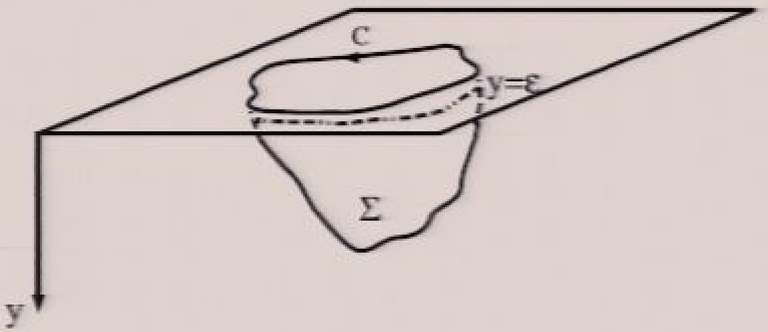
Wilson loop at strong coupling

At strong coupling the Wilson loop is given by the semi-classical partition function for a string in $AdS_5 \times S^5$:

$$Z = \int \mathcal{D}X^\mu \mathcal{D}Y^I \mathcal{D}h_{ab}\mathcal{D}\vartheta^\alpha \exp\left(-\frac{\sqrt{\lambda}}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}\, h_{ab}\frac{\partial_a X^\mu \partial_b X^\mu + \partial_a Y^I \partial_b Y^I}{Y^2} + \text{fermions}\right)$$

$$X^\mu|_{\partial\Sigma} = x_\mu(\tau), \qquad Y^I|_{\partial\Sigma} = \theta^I(\tau) Y|_{\partial\Sigma}, \qquad Y|_{\partial\Sigma} = 0$$

The saddle-point is obtained when the string worldsheet Σ describes a surface of minimal area $\mathcal{A}$



$$\frac{\sqrt{\lambda}}{2\pi}\int_\Sigma d^2\sigma \frac{1}{Y^2}\sqrt{\det\left(\partial_a X^\mu \partial_b X^\mu + \partial_a Y^I \partial_b Y^I\right)}$$

$$= \mathcal{A}_{\text{reg.}} + \frac{L(C)}{\varepsilon}$$

And so $1/\varepsilon$ corresponds to M, the mass of the heavy boson. We then have,

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}\mathcal{A}_{\text{reg.}}}$$

A class of special loops

The straight line: $x_\mu(\tau) = (\tau, 0, 0, 0) \qquad \theta^I = \text{const.}$

$$\langle W \rangle = 1 + \frac{1}{N}\left\langle \mathrm{Tr} \int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \, (iA_0 + \theta \cdot \Phi)(\tau_1)\,(iA_0 + \theta \cdot \Phi)(\tau_2) + \dots \right\rangle$$

$$= 1 + \frac{-1 + \theta \cdot \theta}{4\pi^2 \left(x(\tau_1) - x(\tau_2)\right)^2} \leftarrow \text{ZERO!} + \dots$$

So the loop-to-loop propagator vanishes for the straight line.

In fact this is a special case of a class of **supersymmetric Wilson loops**, due to Zarembo, which have

$$\theta^I(\tau) = \frac{\dot{x}_\mu(\tau)}{|\dot{x}(\tau)|} M^I_\mu, \qquad \text{where} \qquad M^I_\mu M^I_\nu = \delta_{\mu\nu}$$

For these loops, the loop-to-loop propagator always vanishes,

$$\left\langle (i\dot{x}_\mu A_\mu + |\dot{x}|\theta \cdot \Phi)(\tau_1)\,(i\dot{x}_\mu A_\mu + |\dot{x}|\theta \cdot \Phi)(\tau_2) \right\rangle = \frac{-\dot{x}(\tau_1)\cdot\dot{x}(\tau_2) + M^I_\mu M^I_\nu \dot{x}_\mu(\tau_1)\dot{x}_\nu(\tau_2)}{4\pi^2 \left(x(\tau_1) - x(\tau_2)\right)^2}$$

$$= 0$$

Supersymmetry of SUSY loops

$$\delta_\epsilon W = \frac{1}{N} \mathrm{Tr}\, P \int d\tau\, \bar{\psi}\, (i\dot{x}_\mu \gamma^\mu + |\dot{x}|\theta \cdot \Gamma)\, \epsilon \exp\left(\int d\tau' \, (i\dot{x}_\mu A_\mu + |\dot{x}|\theta \cdot \Phi) \right)$$

So if $(i\dot{x}_\mu \gamma^\mu + |\dot{x}|\theta \cdot \Gamma)\epsilon = 0$ for some constant ϵ, we'll have some SUSY. In fact this operator is nilpotent, indicating a halving of the supersymmetry,

$$(i\dot{x}_\mu \gamma^\mu + |\dot{x}|\theta \cdot \Gamma)^2 = -\dot{x}^2 + \dot{x}^2 \theta \cdot \theta = 0$$

but, in general solutions will require $\epsilon(\tau)$ which is **local** SUSY - not a symmetry of $\mathcal{N} = 4$.

In the SUSY loop case, the path dependence factorizes,

$$\dot{x}_\mu(\tau) \left(i\gamma^\mu + M_\mu^I \Gamma^I \right) \left(\epsilon_0 + x_\nu(\tau) \gamma^\nu \epsilon_1 \right) = 0$$

which gives one halving for each non zero component of $\dot{x}_\mu(\tau)$, and which acts independently on the Poincaré (ϵ_0) and superconformal (ϵ_1) supersymmetries.

$$\text{d-dimensional loop} = \left(\frac{1}{2} \right)^d \text{ BPS}$$

Protection of SUSY loops

$$\langle W_{\rm SUSY} \rangle = 1 \qquad \text{(independent of contour)}$$

This has been proven for $1/2 \to 1/8$ BPS loops by Guralnik & Kulik using the fact that $A_{0,1,2} + 3$ of the Φ's form a chiral superfield $\widetilde{\Phi}$ of an $\mathcal{N} = 2$ subalgebra of SYM's superalgebra:

$$W_{\rm SUSY} \sim \exp\left(\int \widetilde{\Phi} \cdot d\vec{y} \right) \in \text{ chiral ring of subalgebra}$$

Explicit cancellation was shown up to λ^2 order for all SUSY loops in original Zarembo work.

1/16 BPS loop protection remains conjectured as far as gauge theory is concerned.

Protection from AdS/CFT: Simple Example

Here we expect $\mathcal{A}_{\text{reg.}} = 0$, since $\langle W \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}\mathcal{A}_{\text{reg.}}}$. Simple to see for the 1/2 BPS straight line $x_\mu = (\sigma, 0, 0, 0)$, θ^I =const. Here the string worldsheet sits at a point on the S^5, so we just need the AdS_5 piece

$$\mathcal{A} = \int d\tau d\sigma \frac{1}{Y^2}\sqrt{(X'^2 + Y'^2)\left(\dot{X}^2 + \dot{Y}^2\right) - \left(X' \cdot \dot{X} + Y'\dot{Y}\right)}$$

If we set $X^{1,2,3} = 0$, we have two embedding functions to worry about $X^0(\sigma,\tau)$ and $Y(\sigma,\tau)$. But we also have this much gauge invariance. Actually the choice

$$X^0(\sigma,\tau) = \sigma \qquad Y(\sigma,\tau) = \tau \tag{1}$$

solves the equations of motion trivially and obeys the B.C.'s

$$X^\mu(\sigma,0) = x_\mu = (\sigma,0,0,0) \qquad Y(\sigma,0) = 0 \tag{2}$$

so then we have

$$\mathcal{A} = \int d\sigma \int_\varepsilon^\infty \frac{1}{\tau^2} = \frac{\int d\sigma}{\varepsilon} = \frac{L(C)}{\varepsilon} \tag{3}$$

Protection from AdS/CFT

Zarembo: found the minimal surface for the 1/4 BPS circle $x_\mu = R(\cos\sigma, \sin\sigma, 0, 0)$ and found $\mathcal{A}_{\text{reg.}} = 0$. In fact the AdS_5 contribution is always negative after subtracting the divergence. The S^5 contribution, suffering no divergence, is of course positive.

Dymarsky, Gubser, Guralnik, and Maldacena: proved $\mathcal{A}_{\text{reg.}} = 0$ for **all** $1/2, \ldots, 1/16$ BPS loops using a calibration. One can decompose the $AdS_5 \times S^5$ metric as follows

$$ds^2 = \frac{Y^2 + U^2}{R^2} dX^\mu dX^\mu + \frac{R^2}{Y^2 + U^2}\Big(dY^m dY^m + dU^i dU^i\Big)$$

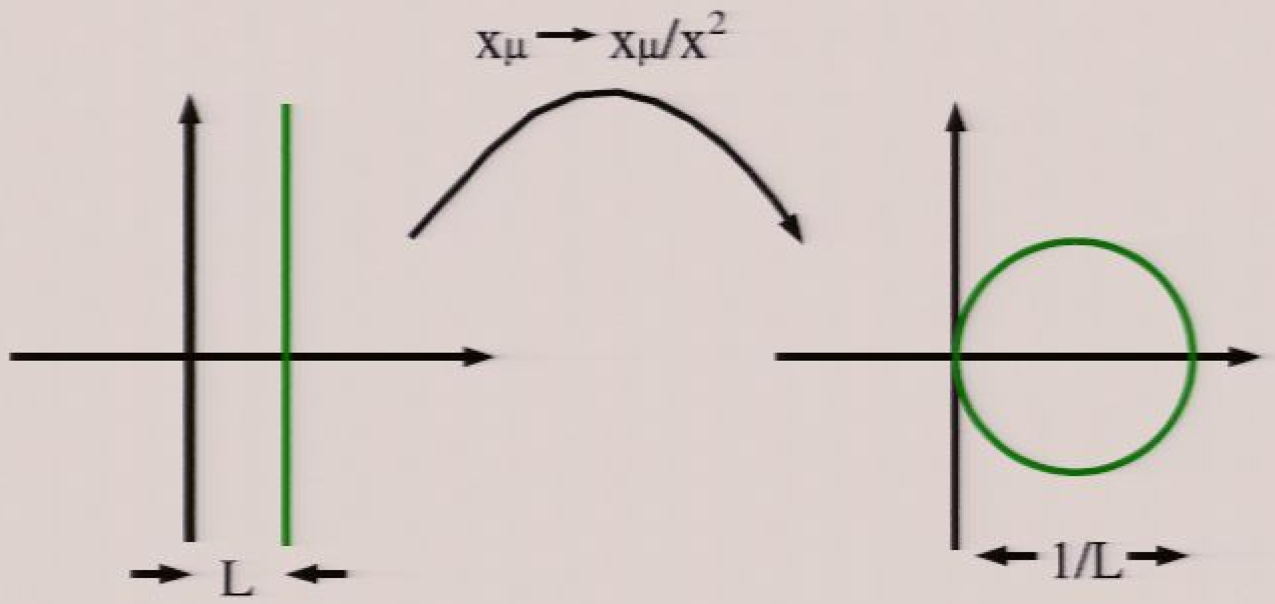
where $\mu = 0, \ldots, 3$, $m = 4, \ldots, 7$, $i = 8, 9$. They showed that

$$\text{area} \geq \int_\Sigma J \qquad \text{where} \qquad J = J_{AB}\, d\mathbb{X}^A \wedge d\mathbb{X}^B$$

where $\mathbb{X}^A = (X^\mu, Y^m, U^i)$ and J_{AB} is an *almost complex* structure obeying $J_A^B J_B^C = -\delta_A^\mu \delta_\mu^C - \delta_A^m \delta_m^C$. A minimal surface saturates the bound, then

$$\int_\Sigma J = \frac{L(C)}{\varepsilon}$$

The 1/2 BPS Circle: The straight line's conformal half-brother



$\langle W \rangle = 1$ — 1/2 BPS, ϵ_0, ϵ_1 independent

$\langle W \rangle \neq 1$ — 1/2 BPS because ϵ_0 related to ϵ_1

So the 1/2 BPS circle is given by $x_\mu = R\,(\cos\tau, \sin\tau, 0, 0)$ and θ^I =const. If we analyze the supersymmetry,

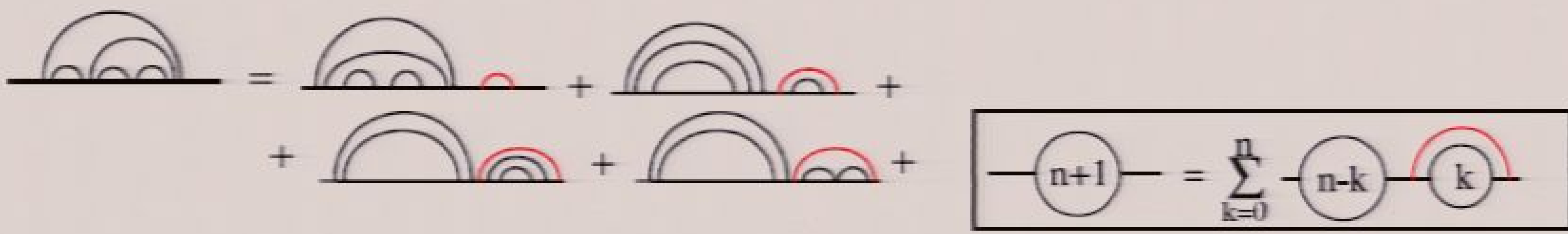
$$(i\dot{x}_\mu\gamma^\mu + \theta\cdot\Gamma)\,(\epsilon_0 + x_\mu\gamma^\mu\epsilon_1) = 0 \rightarrow \boxed{i\gamma^1\gamma^0\epsilon_1 = -\theta\cdot\Gamma\epsilon_0}$$

Erickson, Semenoff, Zarembo

The loop-to-loop propagator on the circle is a constant:

$$\left\langle (i\dot{x}_\mu A_\mu + \theta\cdot\Phi)(i\dot{x}_\mu A_\mu + \theta\cdot\Phi)\right\rangle = \frac{1-\cos\tau_1\cos\tau_2-\sin\tau_1\sin\tau_2}{4\pi^2\left[(\cos\tau_1-\cos\tau_2)^2+(\sin\tau_1-\sin\tau_2)^2\right]} = \frac{1}{8\pi^2}$$

So summing planar ladder diagrams becomes a counting exercise:



This gives a recursion relation which may be solved

$$N_{n+1} = \sum_{k=0} N_{n-k}N_k \qquad \to \qquad N_n = \frac{(2n)!}{(n+1)!n!}$$

Taking care of factors from the path-ordered integration, one has

$$\langle W\rangle_{\text{ladders}} = \sum_{n=0}^{\infty}\frac{(\lambda/4)^n}{(n+1)!n!} = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})$$

Drukker and Gross

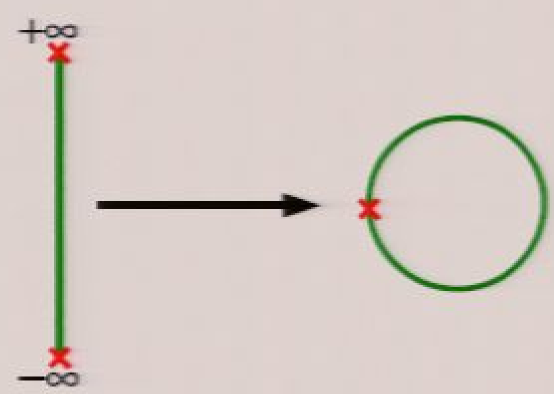
The diagrams with one internal vertex were shown to cancel by ESZ, further, they noticed that the counting of planar ladders can equally be accomplished through a large-N limit of a Hermitian one-matrix model

$$\langle W_{\rm circle}\rangle = \frac{1}{Z}\int DM \frac{1}{N}\,{\rm Tr}\exp M \exp\left(-\frac{2N}{\lambda}\,{\rm Tr}\, M^2\right)$$

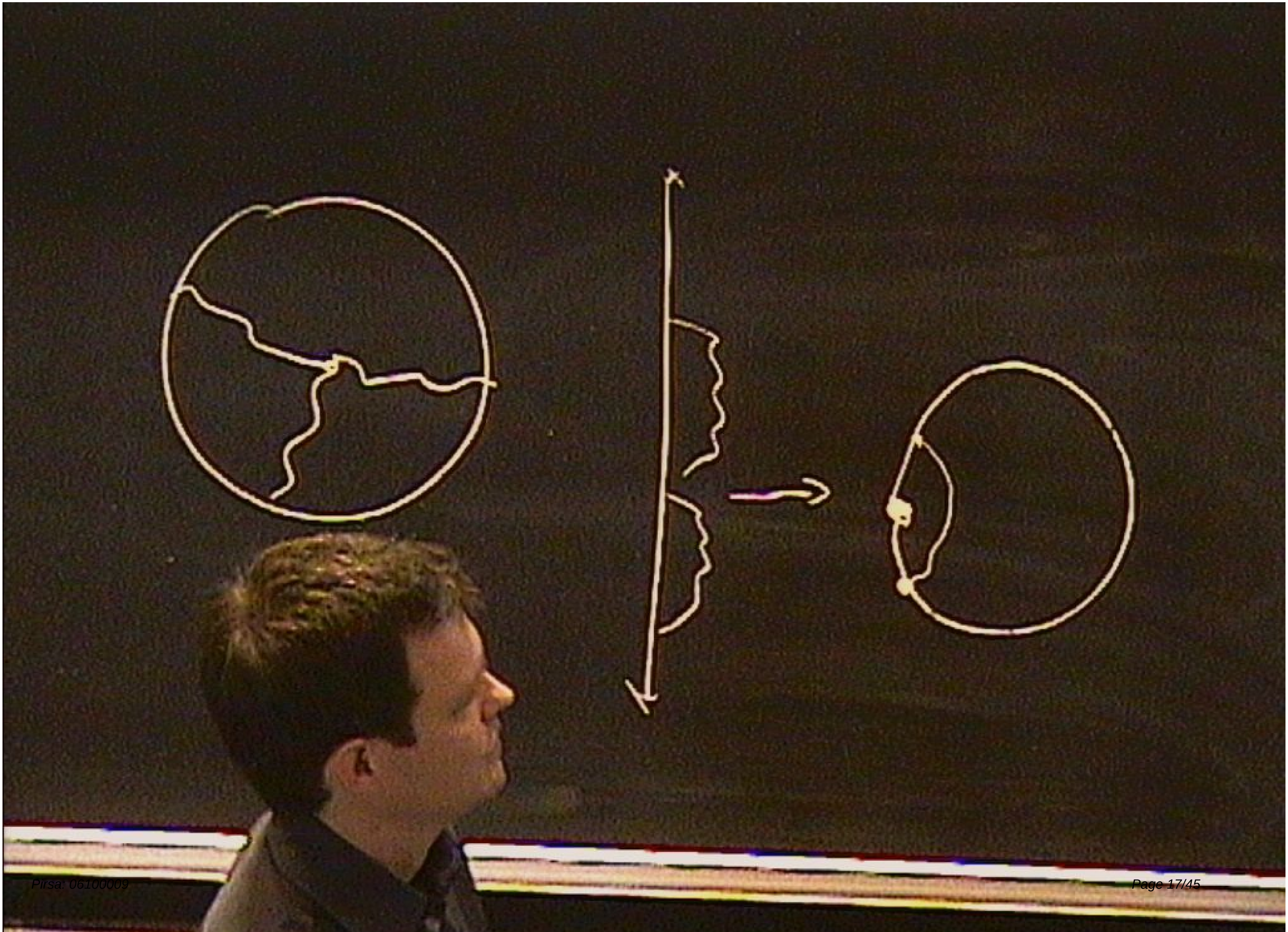
Drukker & Gross went further with the matrix model and solved also for arbitrary N,

$$\langle W_{\rm circle}\rangle = \frac{1}{N}L^1_{N-1}\left(-\sqrt{\lambda}/4N\right)e^{-\lambda/8N} = \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2}I_2(\sqrt{\lambda}) + \dots$$

They also understood that the inversion $x_\mu \rightarrow x_\mu/x^2$ is a singular one, which gives a sort of *conformal anomaly*. The dynamics are captured by a 0-dimensional theory at the point mapped from infinity, voilà the matrix model. In fact, the result is general



$$\langle W_{\rm closed}\rangle = F(\lambda, N)\langle W_{\rm open}\rangle$$



Drukker and Gross

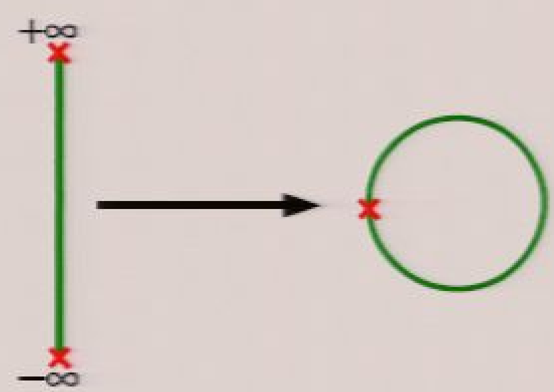
The diagrams with one internal vertex were shown to cancel by ESZ, further, they noticed that the counting of planar ladders can equally be accomplished through a large-N limit of a Hermitian one-matrix model

$$\langle W_{\rm circle}\rangle = \frac{1}{Z}\int DM \frac{1}{N}\,{\rm Tr}\exp M \exp\left(-\frac{2N}{\lambda}\,{\rm Tr}\, M^2\right)$$

Drukker & Gross went further with the matrix model and solved also for arbitrary N,

$$\langle W_{\rm circle}\rangle = \frac{1}{N} L^1_{N-1}\left(-\sqrt{\lambda}/4N\right) e^{-\lambda/8N} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \ldots$$

They also understood that the inversion $x_\mu \to x_\mu/x^2$ is a singular one, which gives a sort of *conformal anomaly*. The dynamics are captured by a 0-dimensional theory at the point mapped from infinity, voilà the matrix model. In fact, the result is general



$$\langle W_{\rm closed}\rangle = F(\lambda, N)\langle W_{\rm open}\rangle$$

Circle from AdS/CFT

The minimal surface has been found, and the regulated area is -2π giving $\langle W \rangle = e^{\sqrt{\lambda}}$. The large λ and N limit of the gauge theory result is

$$\langle W \rangle_{\text{gauge}} \simeq \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}$$

The discrepancy is resolved because of the path integral over zero modes associated with the disk amplitude. There are 3 constraints, each gives a factor of $\lambda^{-1/4}$.

Although coefficients C_p are unknown, DG argued that the disk may be decorated by degenerate handles, which gives

$$\langle W \rangle = \sum_p \frac{C_p}{N^{2p}} \frac{\lambda^{(6p-3)/4}}{p!} e^{\sqrt{\lambda}} \left(1 + \mathcal{O}(1/\sqrt{\lambda})\right)$$

In fact, a large λ expansion of their matrix model result gives exactly this, with

$$C_p = \sqrt{\frac{2}{\pi}} \frac{1}{96^p}$$

Circle from AdS/CFT

The minimal surface has been found, and the regulated area is -2π giving $\langle W \rangle = e^{\sqrt{\lambda}}$. The large λ and N limit of the gauge theory result is

$$\langle W \rangle_{\text{gauge}} \simeq \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}$$

The discrepancy is resolved because of the path integral over zero modes associated with the disk amplitude. There are 3 constraints, each gives a factor of $\lambda^{-1/4}$.

Although coefficients C_p are unknown, DG argued that the disk may be decorated by degenerate handles, which gives

$$\langle W \rangle = \sum_p \frac{C_p}{N^{2p}} \frac{\lambda^{(6p-3)/4}}{p!} e^{\sqrt{\lambda}} \left(1 + \mathcal{O}(1/\sqrt{\lambda})\right)$$

In fact, a large λ expansion of their matrix model result gives exactly this, with

$$C_p = \sqrt{\frac{2}{\pi}} \frac{1}{96^p}$$

1/4 BPS Circles

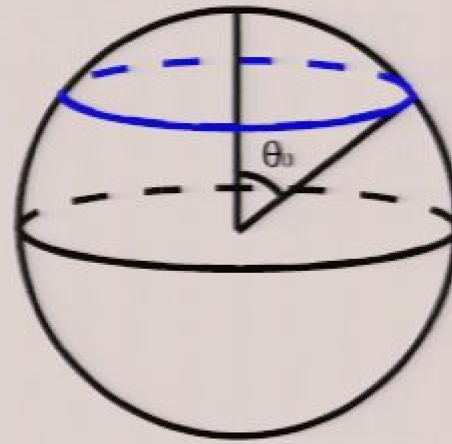
Recently, Drukker presented the following Wilson loop

$$x_\mu = R\,(\cos\tau, \sin\tau, 0, 0) \qquad \theta^I = (\sin\theta_0\cos\tau, \sin\theta_0\sin\tau, \cos\theta_0, 0, 0, 0)$$

If $\theta_0 = \pi/2 \rightarrow 1/4$ BPS SUSY circle of Zarembo. If $\theta_0 = 0 \rightarrow 1/2$ BPS circle.

For general θ_0, there is one condition each of ϵ_0 and ϵ_1, **and** one more condition relating them.

Also, the loop-to-loop propagator is constant: $\cos^2\theta_0/8\pi^2$. This is just $\cos^2\theta_0$ by the 1/2 BPS circle propagator.



Further, leading internal vertex diagrams cancel in the calculation of $\langle W \rangle$ by the same mechanism as for the 1/2 BPS circle. The only difference is that $\lambda \rightarrow \lambda' = \cos^2\theta_0\lambda$. On the string side, the minimal surface for the loop was found and $\langle W \rangle = \exp(\sqrt{\lambda'})$. Thus

$$\langle W_{1/4} \rangle = \langle W_{1/2} \rangle (\lambda \rightarrow \lambda')$$

1/4 BPS Loop - Chiral Primary Correlator

DY, Semenoff hep-th/0609158

We are interested in calculating a connected correlator $\langle W_{1/4}\mathcal{O}\rangle / \langle W_{1/4}\rangle$ of the loop with a chiral primary operator $\mathcal{O}$.

In particular we are interested in the limit of large separation between the two. When viewed from a large distance, the Wilson loop should look like an assembly of local operators

$$W[C] = \langle W[C]\rangle \left(1 + \sum_{\Delta_i > 0} \mathcal{O}_{\Delta_i}(0)\, L[C]^{\Delta_i} \xi_{\Delta_i}[C] + \ldots \right)$$

The leading behaviour of the correlator is given by the operators of smallest conformal dimension - the chiral primaries, which we normalize as

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(0)\rangle = \frac{\delta_{\Delta\Delta'}}{(4\pi^2 x^2)^\Delta}$$

We then expect

$$\frac{\langle W[C]\; \mathcal{O}_\Delta(x)\rangle}{\langle W[C]\rangle} = \frac{L[C]^\Delta}{(4\pi^2 |x|^2)^\Delta}\xi_\Delta + \ldots$$

1/4 BPS Circles

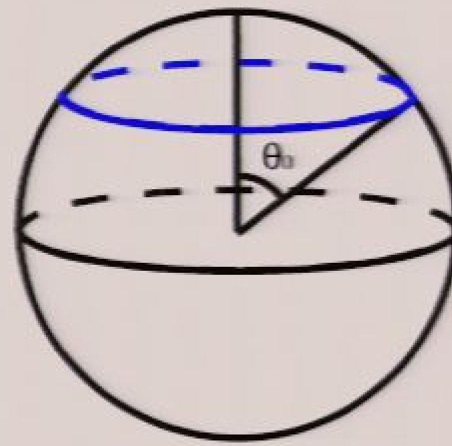
Recently, Drukker presented the following Wilson loop

$$x_\mu = R(\cos\tau, \sin\tau, 0, 0) \qquad \theta^I = (\sin\theta_0\cos\tau, \sin\theta_0\sin\tau, \cos\theta_0, 0, 0, 0)$$

If $\theta_0 = \pi/2 \to 1/4$ BPS SUSY circle of Zarembo. If $\theta_0 = 0 \to 1/2$ BPS circle.

For general θ_0, there is one condition each of ϵ_0 and ϵ_1, **and** one more condition relating them.

Also, the loop-to-loop propagator is constant: $\cos^2\theta_0/8\pi^2$. This is just $\cos^2\theta_0$ by the 1/2 BPS circle propagator.



Further, leading internal vertex diagrams cancel in the calculation of $\langle W\rangle$ by the same mechanism as for the 1/2 BPS circle. The only difference is that $\lambda \to \lambda' = \cos^2\theta_0\lambda$. On the string side, the minimal surface for the loop was found and $\langle W\rangle = \exp(\sqrt{\lambda'})$. Thus

$$\langle W_{1/4}\rangle = \langle W_{1/2}\rangle(\lambda \to \lambda')$$

1/4 BPS Loop - Chiral Primary Correlator

DY, Semenoff hep-th/0609158

We are interested in calculating a connected correlator $\langle W_{1/4}\mathcal{O}\rangle/\langle W_{1/4}\rangle$ of the loop with a chiral primary operator $\mathcal{O}$.

In particular we are interested in the limit of large separation between the two. When viewed from a large distance, the Wilson loop should look like an assembly of local operators

$$W[C] = \langle W[C]\rangle\left(1 + \sum_{\Delta_i>0} \mathcal{O}_{\Delta_i}(0)\ L[C]^{\Delta_i}\xi_{\Delta_i}[C] + \ldots\right)$$

The leading behaviour of the correlator is given by the operators of smallest conformal dimension - the chiral primaries, which we normalize as

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(0)\rangle = \frac{\delta_{\Delta\Delta'}}{(4\pi^2x^2)^\Delta}$$

We then expect

$$\frac{\langle W[C]\ \mathcal{O}_\Delta(x)\rangle}{\langle W[C]\rangle} = \frac{L[C]^\Delta}{(4\pi^2|x|^2)^\Delta}\xi_\Delta + \ldots$$

Example: 1/2 BPS circle

Take the 1/2 BPS circle with $\theta^I = (1, 0, ..., 0)$.

$$W = \langle W \rangle \left(\sum_k (2\pi R)^k \frac{1}{N} \frac{1}{k!\,2^k} \operatorname{Tr}\left(Z(0) + \bar{Z}(0)\right)^k + \dots \right)$$

$$= \langle W \rangle \left(1 + \sum_{J \geq 2} \mathcal{O}_J(0) (2\pi R)^J \xi_J + \dots \right)$$

where $\mathcal{O}_J(x) = \frac{1}{\sqrt{\lambda^J J}} \operatorname{Tr} Z^J$, $Z = \Phi_1 + i\Phi_2$. We then have

$$\frac{\langle \mathcal{O}_J(x) W(0) \rangle}{\langle W(0) \rangle} = \left(\frac{2\pi R}{4\pi^2 x^2} \right)^J \xi_J \qquad \xi_J = \frac{1}{N} \frac{1}{2^J J!} \sqrt{J \lambda^J}$$

In fact, all planar (loop-to-loop) ladders can also be summed for the calculation of ξ_J. Leading corrections also vanish, the result is (Semenoff, Zarembo)

$$\xi_J = \frac{1}{N} \frac{1}{2} \sqrt{\lambda J} \frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \qquad \text{confirmed with AdS/CFT - more to come}$$

Comparing SUSY's: CPO and 1/4 BPS Loop

Comparing supersymmetries:

1/4 BPS loop:

$$\begin{aligned} \sin\theta_0(\gamma^1\Gamma^2+\gamma^2\Gamma^1)\epsilon_0 &= 0 \\ \sin\theta_0(\gamma^1\Gamma^2+\gamma^2\Gamma^1)\epsilon_1 &= 0 \\ \cos\theta_0\epsilon_0 &= R(-i\gamma^1+\sin\theta_0\Gamma^2)\Gamma^3\gamma^2\epsilon_1 \end{aligned}$$

General CPO $\mathrm{Tr}(u\cdot\Phi)^J$
where $\vec{u}\in\mathbb{C}^6$, $u^2=0$:

$$u\cdot\Gamma\epsilon_0=0, \qquad \epsilon_1 \text{ free}$$

For a 1/2 BPS circle $\theta_0=0$ and there is one relation between ϵ_0 and ϵ_1. The CPO condition will further halve this, so that $\mathcal{O}\,W_{1/2}$ respects 1/4 of the SUSY.

This is likely responsible for the protection of this quantity. What about for $W_{1/4}$?

Comparing SUSY's: CPO and 1/4 BPS Loop

Comparing supersymmetries:

1/4 BPS loop:

$$\begin{aligned} \sin\theta_0(\gamma^1\Gamma^2+\gamma^2\Gamma^1)\epsilon_0 &= 0 \\ \sin\theta_0(\gamma^1\Gamma^2+\gamma^2\Gamma^1)\epsilon_1 &= 0 \\ \cos\theta_0\epsilon_0 &= R(-i\gamma^1+\sin\theta_0\Gamma^2)\Gamma^3\gamma^2\epsilon_1 \end{aligned}$$

General CPO $\mathrm{Tr}(u\cdot\Phi)^J$
where $\vec{u}\in\mathbb{C}^6$, $u^2=0$:

$$u\cdot\Gamma\epsilon_0=0, \qquad \epsilon_1 \text{ free}$$

For a 1/2 BPS circle $\theta_0=0$ and there is one relation between ϵ_0 and ϵ_1. The CPO condition will further halve this, so that $\mathcal{O}\,W_{1/2}$ respects 1/4 of the SUSY.

This is likely responsible for the protection of this quantity. What about for $W_{1/4}$?

Comparing SUSY's: CPO and 1/4 BPS Loop

For $\theta_0 \neq 0$, we find shared supersymmetry for 3 cases (where $\vec{u} = (u_1, \ldots, u_6)$):

- $\underline{u_1 = u_2 = 0}$: The loop commutes with an $SO(6)$ R-symmetry rotation which we act on (u_4, u_5, u_6) to set $u_5 = u_6 = 0$. Then only solution is for $u_3 = \pm i u_4 \rightarrow 1/8$ SUSY's shared.

- $\underline{u_3 = u_4 = 0}$: ditto but with $u_1 = \pm i u_2$, $1/8$ shared SUSY.

- $\underline{u_1 = \pm i u_2}$: Only solution is for $u_3 = \pm i u_4$, and only $1/16$ of SUSY's are in common.

Thus the least supersymmetric choice is $\mathrm{Tr}\left(\chi(\Phi_1 + i\Phi_2) + (\Phi_3 + i\Phi_4)\right)^J$.

Anything more general we do not expect to be protected.

R-symmetry pares down CPO further

In fact, all $(\Phi_1 + i\Phi_2)$ terms in the two-point function will vanish by R-symmetry.

This is because a spatial rotation in the x_0-x_1 plane can be un-done by shifting the loop parameter τ, and then by applying a compensating R-symmetry rotation in the θ^1-θ^2 plane. Let $\widetilde{\mathcal{O}}_J = \mathrm{Tr}(\Phi_1 + i\Phi_2)^J$:

$$\langle \widetilde{\mathcal{O}}_J(x)\, W_{1/4} \rangle = \langle U \widetilde{\mathcal{O}}_J(x)\, W_{1/4} U^\dagger \rangle = \langle \widetilde{\mathcal{O}}_J(Ux)\, R W_{1/4} R^\dagger \rangle$$
$$= \langle R \widetilde{\mathcal{O}}_J(Ux) R^\dagger\, W_{1/4} \rangle = e^{iJ\phi} \langle \widetilde{\mathcal{O}}_J(Ux)\, W_{1/4} \rangle = e^{iJ\phi} \langle \widetilde{\mathcal{O}}_J(x)\, W_{1/4} \rangle$$

So we've proven $\langle \widetilde{\mathcal{O}}_J(x)\, W_{1/4} \rangle = 0$. Therefore we can take

$$\mathcal{O}_J = \frac{1}{\sqrt{\lambda^J J}} \mathrm{Tr}\left(\Phi_3 + i\Phi_4\right)^J$$

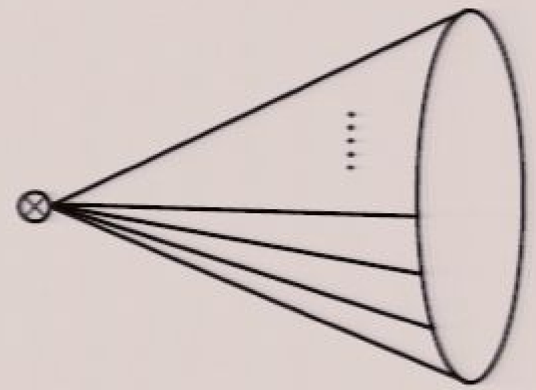
without loss of generality.

Gauge Theory Computation

We want to evaluate

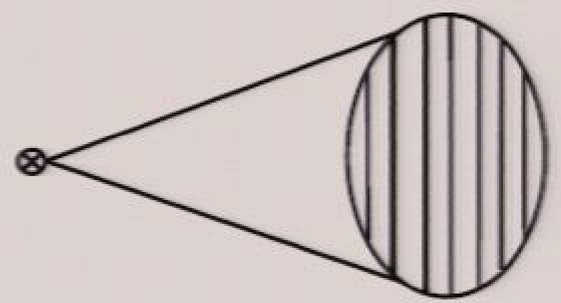
$$\frac{\langle \frac{1}{\sqrt{\lambda}^J J} \mathrm{Tr}\,(\Phi_3 + i\Phi_4)^J W_{1/4} \rangle}{\langle W_{1/4} \rangle}$$

in perturbation theory.



Recall that $\theta^3(\tau) = \cos\theta_0$, and $\theta^4(\tau) = 0$. Thus the J scalar lines simply provide J powers of $\cos\theta_0$ over the 1/2 BPS circle case. There are J powers of λ, and a factor of $\lambda^{-J/2}$ from the CPO normalization. Thus $\lambda \to \lambda' = \cos^2\theta_0 \lambda$.

Next: Ladders. Recall that the loop-to-loop propagator is constant. The 1/2 BPS result comes over again, with the modified coupling.

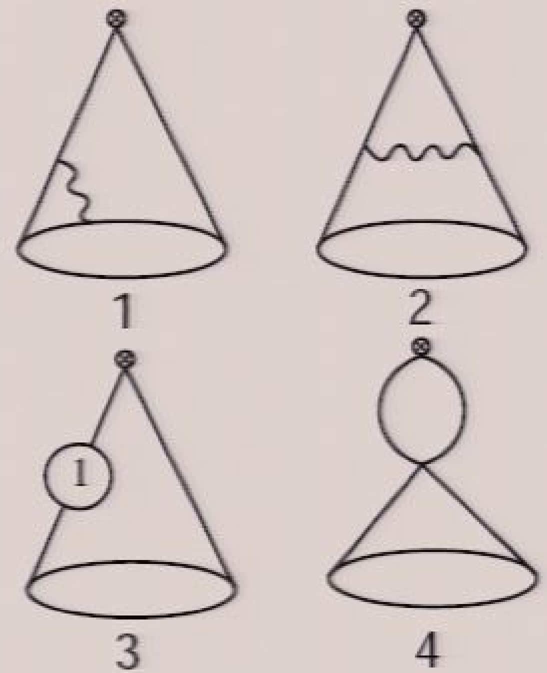


$$\boxed{\xi_J = \frac{1}{N}\frac{1}{2}\sqrt{\lambda' J}\frac{I_J(\sqrt{\lambda'})}{I_1(\sqrt{\lambda'})}}$$

Leading Corrections

Leading ladders decorated with internal vertices vanish as we have seen. The other leading corrections are as follows:

We were able to show that these diagrams add to zero for the 1/4 BPS loop, as they do in the 1/2 BPS case.

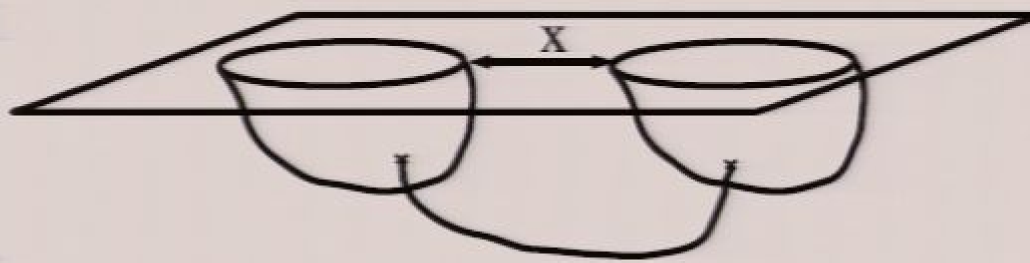


Conjecture: Higher-order analogues all vanish.

Strong Coupling Calculation from AdS/CFT

The chiral primaries are dual to supergravitons propagating in $AdS_5 \times S^5$. The large distance correlator may be thought of as an exchange of such a mode, between the loop's worldsheet and the boundary

The easier way is to extract ξ_J from a loop-loop correlator. BCFM showed that



$$\frac{\langle W(x)W(0)\rangle}{\langle W(x)\rangle\langle W(0)\rangle} = \sum_J \xi_J^2 \left(\frac{R}{x}\right)^{2J} + \ldots$$

That is, the bulk-to-bulk exchange is dominated at large distance by the chiral primaries.

1/4 BPS Loop: Minimal Surface

The embedding for our 1/4 BPS loop was given in the original reference. We chose coordinates on $AdS_5 \times S^5$ as follows

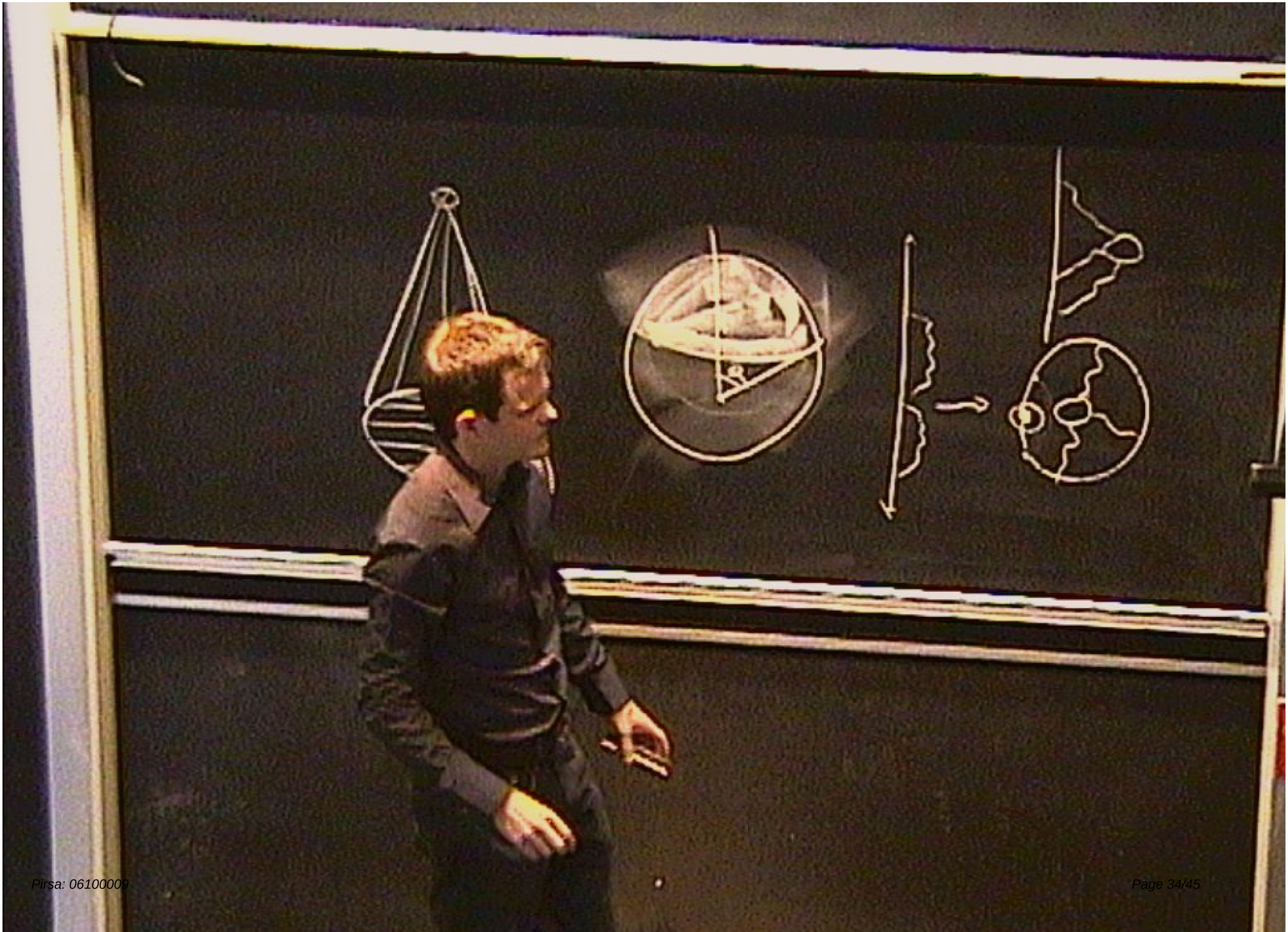
$$
\begin{aligned}
ds^2 &= \sqrt{\lambda}\Big(\frac{dy^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2}{y^2} \\
&+ d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta\left(d\rho^2 + \sin^2\rho\, d\hat{\phi}^2 + \cos^2\rho\, d\tilde{\phi}^2\right)\Big)
\end{aligned}
$$

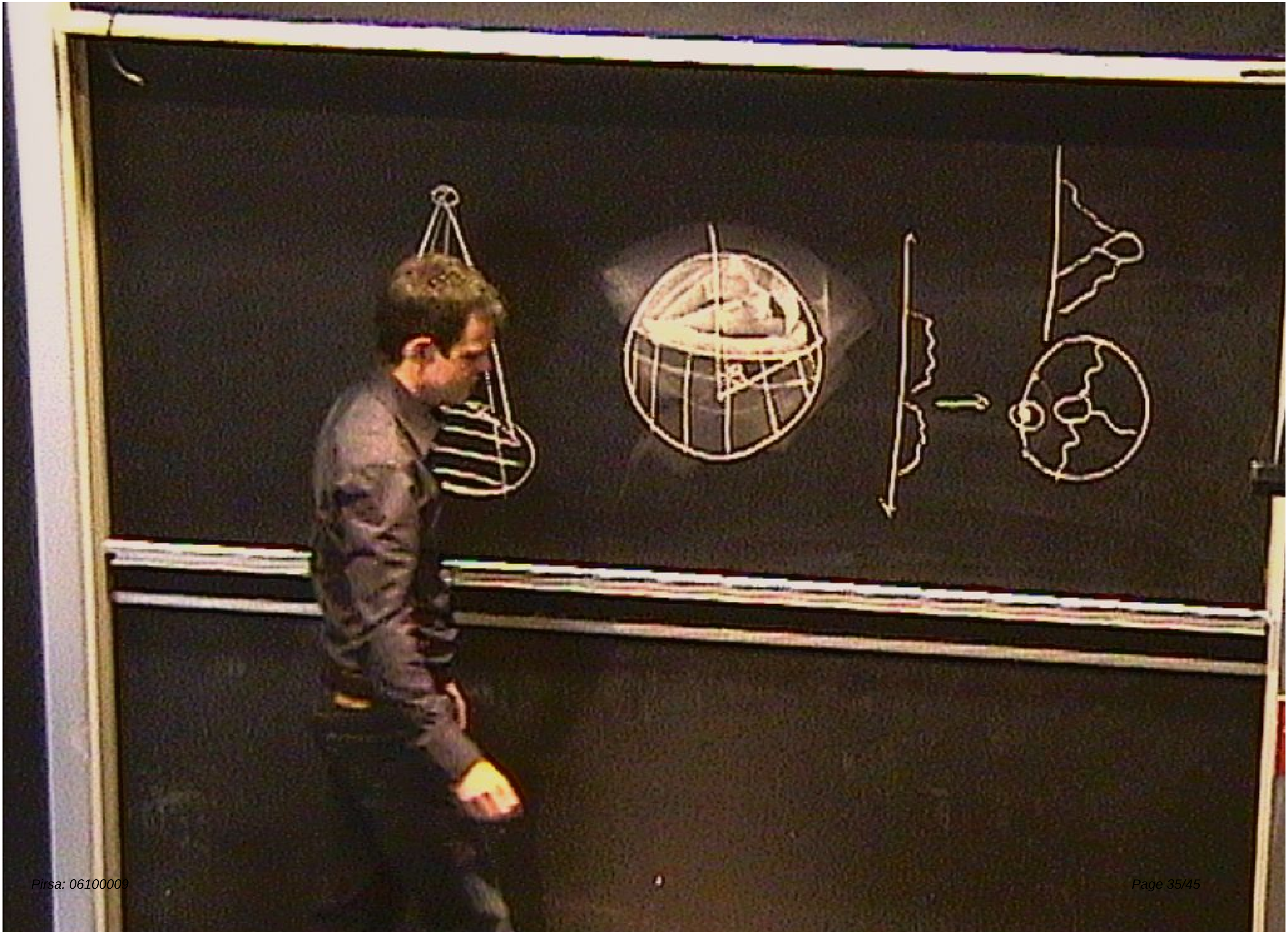
The string world-sheet is then embedded as follows,

$$
y = R\tanh\sigma \qquad r_1 = \frac{R}{\cosh\sigma} \qquad \phi_1 = \tau \qquad r_2 = 0 \qquad \phi_2 = \text{const.}
$$

$$
\sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \qquad \phi = \tau \qquad \rho = \frac{\pi}{2} \qquad \hat{\phi} = 0 \qquad \tilde{\phi} = \text{const.}
$$

where $\sigma \in [0, \infty]$ and $\tau \in [0, 2\pi]$ are the world-sheet coordinates. The $\pm$ in the θ embedding reflects the fact that the string action has two saddle points. They give $\langle W \rangle = \exp(\pm\sqrt{\lambda'})$. Obviously the minimal saddle-point is the dominant one - but it turns out to be interesting to consider both.





1/4 BPS Loop: Minimal Surface

The embedding for our 1/4 BPS loop was given in the original reference. We chose coordinates on $AdS_5 \times S^5$ as follows

$$ds^2 = \sqrt{\lambda}\left(\frac{dy^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2}{y^2} + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta\left(d\rho^2 + \sin^2\rho\, d\hat{\phi}^2 + \cos^2\rho\, d\tilde{\phi}^2\right)\right)$$

The string world-sheet is then embedded as follows,

$$y = R\tanh\sigma \qquad r_1 = \frac{R}{\cosh\sigma} \qquad \phi_1 = \tau \qquad r_2 = 0 \qquad \phi_2 = \text{const.}$$

$$\sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \qquad \phi = \tau \qquad \rho = \frac{\pi}{2} \qquad \hat{\phi} = 0 \qquad \tilde{\phi} = \text{const.}$$

where $\sigma \in [0, \infty]$ and $\tau \in [0, 2\pi]$ are the world-sheet coordinates. The $\pm$ in the θ embedding reflects the fact that the string action has two saddle points. They give $\langle W \rangle = \exp(\pm\sqrt{\lambda'})$. Obviously the minimal saddle-point is the dominant one - but it turns out to be interesting to consider both.

Relevant Supergravity Modes

The supergravity modes dual to CPO's are the lightest scalars $s^J(x)$ with $m^2 = J(J-4)$, $J \geq 2$. These are responsible for metric fluctuations as follows

$$\begin{aligned} \delta g_{\alpha\beta} &= \left[-\frac{6\,J}{5} g_{\alpha\beta} + \frac{4}{J+1} D_{(\alpha} D_{\beta)} \right] s^J(X)\, Y_J(\Omega) \\ \delta g_{IK} &= 2\,k\, g_{IK}\, s^J(X)\, Y_J(\Omega) \end{aligned}$$

The orientation of the Φ fields in the CPO correspond to the choice of $Y_J(\Omega)$. Specifically

$$\mathrm{Tr}(u \cdot \Phi)^J \leftrightarrow Y_J(\Omega) = \mathcal{N}_J(u)(u \cdot \mathbb{X})^J$$

where $\mathbb{X}^I$ are the embedding functions of the S^5 in $\mathbb{R}^6$.

Our embedding of the string worldsheet sets three of the $\mathbb{X}^I$'s to zero, so we have

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) \Big[u_1 \sin\theta \cos\phi + u_2 \sin\theta \sin\phi + u_3 \cos\theta \Big]^J$$

1/4 BPS Loop: Minimal Surface

The embedding for our 1/4 BPS loop was given in the original reference. We chose coordinates on $AdS_5 \times S^5$ as follows

$$
\begin{aligned}
ds^2 &= \sqrt{\lambda}\Bigg(\frac{dy^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2}{y^2} \\
&+ d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta\left(d\rho^2 + \sin^2\rho\, d\hat{\phi}^2 + \cos^2\rho\, d\tilde{\phi}^2\right)\Bigg)
\end{aligned}
$$

The string world-sheet is then embedded as follows,

$$
y = R\tanh\sigma \qquad r_1 = \frac{R}{\cosh\sigma} \qquad \phi_1 = \tau \qquad r_2 = 0 \qquad \phi_2 = \text{const.}
$$

$$
\sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \qquad \phi = \tau \qquad \rho = \frac{\pi}{2} \qquad \hat{\phi} = 0 \qquad \tilde{\phi} = \text{const.}
$$

where $\sigma \in [0, \infty]$ and $\tau \in [0, 2\pi]$ are the world-sheet coordinates. The $\pm$ in the θ embedding reflects the fact that the string action has two saddle points. They give $\langle W \rangle = \exp(\pm\sqrt{\lambda'})$. Obviously the minimal saddle-point is the dominant one - but it turns out to be interesting to consider both.

Relevant Supergravity Modes

The supergravity modes dual to CPO's are the lightest scalars $s^J(x)$ with $m^2 = J(J-4)$, $J \geq 2$. These are responsible for metric fluctuations as follows

$$\begin{aligned} \delta g_{\alpha\beta} &= \left[-\frac{6\,J}{5} g_{\alpha\beta} + \frac{4}{J+1} D_{(\alpha} D_{\beta)} \right] s^J(X)\, Y_J(\Omega) \\ \delta g_{IK} &= 2\,k\, g_{IK}\, s^J(X)\, Y_J(\Omega) \end{aligned}$$

The orientation of the Φ fields in the CPO correspond to the choice of $Y_J(\Omega)$. Specifically

$$\mathrm{Tr}(u \cdot \Phi)^J \leftrightarrow Y_J(\Omega) = \mathcal{N}_J(u)(u \cdot \mathbb{X})^J$$

where $\mathbb{X}^I$ are the embedding functions of the S^5 in $\mathbb{R}^6$.

Our embedding of the string worldsheet sets three of the $\mathbb{X}^I$'s to zero, so we have

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) \Big[u_1 \sin\theta \, \cos\phi + u_2 \sin\theta \, \sin\phi + u_3 \cos\theta \Big]^J$$

Supergravity Mode Exchange

The loop-loop correlator is then computed like this

$$\frac{\langle W(x)W(0)\rangle}{\langle W(x)\rangle\langle W(0)\rangle} = \int_\Sigma \int_{\bar{\Sigma}} \partial_a X^M \partial^a X^N \, \delta g_{MN} \, P(X,\bar{X}) \, \delta \bar{g}_{\bar{M}\bar{N}} \, \partial_{\bar{a}} X^{\bar{M}} \partial^{\bar{a}} X^{\bar{N}}$$

where $M, N = 1, ..., 10$, and $P(X, \bar{X}) = \langle s^J(X) s^J(\bar{X})\rangle \sim y^J \bar{y}^J / x^{2J}$, and we've used the gauge-fixed Polyakov action.

What we find has the following structure

$$\left[\int d\tau \int d\sigma \left[(y'^2 + r_1'^2 + r_1^2) y^{J-2} + (\theta'^2 + \sin^2\theta) y^J\right] Y_J(\theta,\phi)\right]^2$$

But τ appears only in $Y_J(\theta, \phi)$, since $\phi = \tau$.

$$\int_0^{2\pi} d\tau \, Y_J(\theta,\phi) = \mathcal{N}_J(u) \int_0^{2\pi} d\tau \left[u_1 \sin\theta \cos\tau + u_2 \sin\theta \sin\tau + u_3 \cos\theta\right]^J$$

Now, if we take $u_1 = \pm i u_2$, so that we have the minimum SUSY, this integral evaluates to $2\pi \mathcal{N}_J(u)(u_3 \cos\theta)^J$, i.e. dependence on u_1 and u_2 vanish.

Results

In the SUSY case our results are

$$\frac{\langle W(x)W(0)\rangle}{\langle W(x)\rangle\langle W(0)\rangle} = \frac{J(J+1)^2}{N^2}\left(\frac{R}{x}\right)^{2J}\frac{\lambda}{4}\left[\left\{\int_{-1}^{\mp\cos\theta_0} dz - \int_0^1 dz\right\}\left(\frac{\pm z+\cos\theta_0}{1\pm z\cos\theta_0}\right)^J z^J\right]^2$$

$$= \left(\frac{R}{x}\right)^{2J}\frac{J\lambda}{N^2}\frac{1}{4}\left[-(\pm)^{J+1}\cos\theta_0\right]^2$$

This gives the result

$$\boxed{\xi_J = \sqrt{J\lambda\cos^2\theta_0}\frac{1}{2N}}$$

exactly the large λ limit of the gauge theory result, and the 1/2 BPS circle result with $\lambda \to \cos^2\theta_0\lambda$.

What if we choose $u_1 \neq \pm i u_2$, so that there is no SUSY? Well, for example for $J=2$, we get things like this:

$$\frac{\langle W(x)W(0)\rangle}{\langle W(x)\rangle\langle W(0)\rangle} \propto \ln(1+\cos\theta_0)\left(\frac{2(1-\cos^2\theta_0)}{\cos^3\theta_0}\right) + \frac{4-\cos\theta_0}{3} + \frac{\cos\theta_0 - 2}{\cos^2\theta_0}$$

indicating a lack of protection.

Double Saddle Points

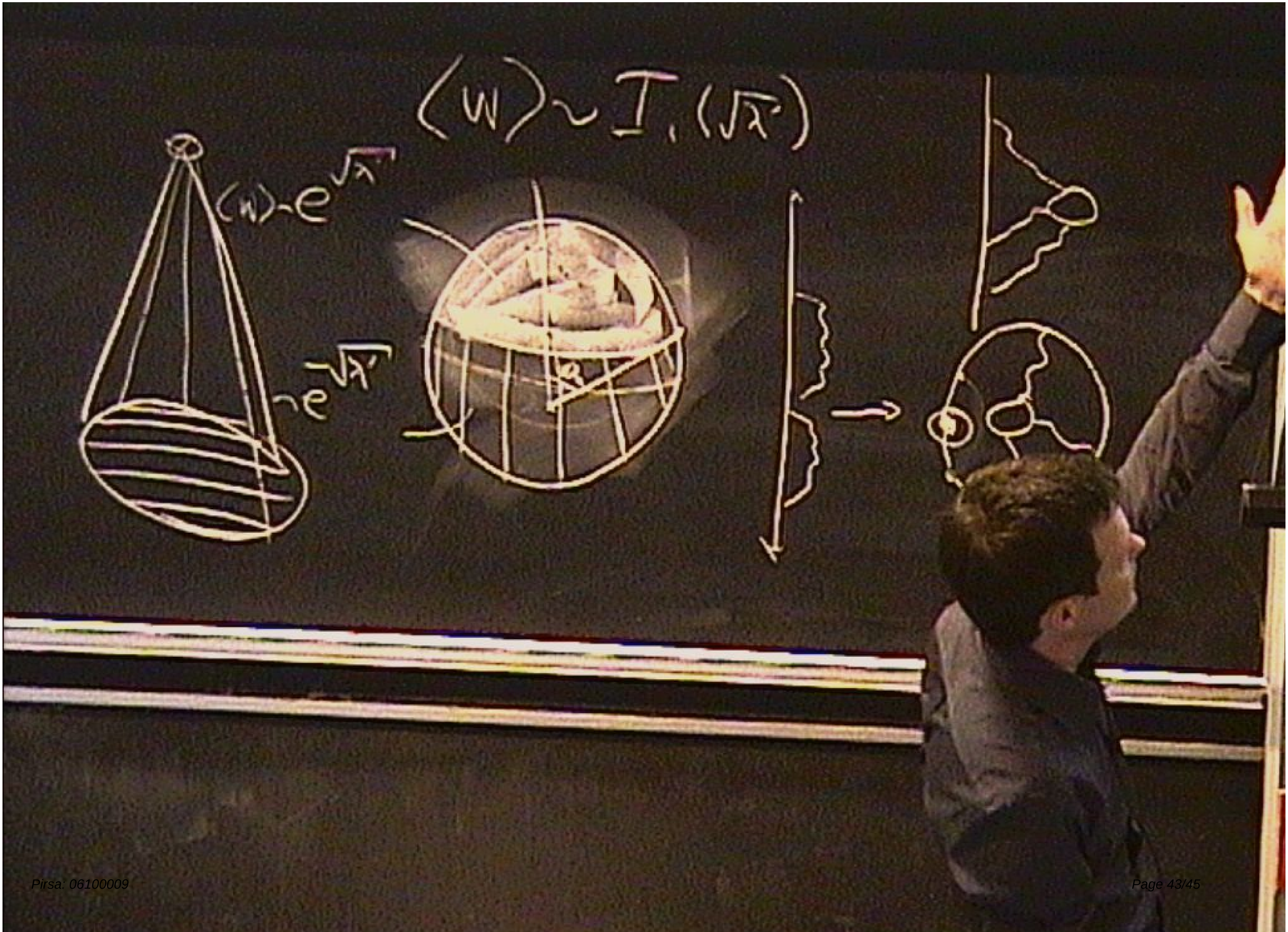
I mentioned that the string action had two saddle points earlier. One then expects the result for $\langle W \rangle$ to be a sum of two terms; one proportional to $\exp(\sqrt{\lambda'})$ and the other to $\exp(-\sqrt{\lambda'})$. In fact the asymptotic λ' expansion of the Bessel function (large N result) gives

$$\frac{e^{\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\,\frac{e^{-\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)},$$

The factor of i was associated with the fluctuation determinant of tachyonic modes associated with the worldsheet slipping off the unstable pole of the five-sphere.

Due to the sign structure in our result we expect a sum of a term proportional to $\exp(\sqrt{\lambda'})$ and of another proportional to $(-1)^{J+1}\exp(-\sqrt{\lambda'})$. Our asymptotic expansions for the the gauge theory result give

$$\frac{e^{\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)} \mp i\,(-1)^J\,e^{-\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)}}{e^{\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\,e^{-\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)}}. \tag{4}$$


$\langle W \rangle \sim I_1(\sqrt{\lambda})$
$\langle W \rangle \sim e^{\sqrt{\lambda}}$
$\sim e^{\sqrt{\lambda}}$

Double Saddle Points

I mentioned that the string action had two saddle points earlier. One then expects the result for $\langle W \rangle$ to be a sum of two terms; one proportional to $\exp(\sqrt{\lambda'})$ and the other to $\exp(-\sqrt{\lambda'})$. In fact the asymptotic λ' expansion of the Bessel function (large N result) gives

$$\frac{e^{\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\,\frac{e^{-\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)},$$

The factor of i was associated with the fluctuation determinant of tachyonic modes associated with the worldsheet slipping off the unstable pole of the five-sphere.

Due to the sign structure in our result we expect a sum of a term proportional to $\exp(\sqrt{\lambda'})$ and of another proportional to $(-1)^{J+1}\exp(-\sqrt{\lambda'})$. Our asymptotic expansions for the the gauge theory result give

$$\frac{e^{\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)} \mp i\,(-1)^J e^{-\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k!\,\Gamma(J-k+1/2)}}{e^{\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)} \pm i\,e^{-\sqrt{\lambda'}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k!\,\Gamma(3/2-k)}}. \tag{4}$$

Perspectives

- I didn't touch upon a very beautiful picture which has emerged describing "giant loops". These are loops with very large representations - not the fundamental representation used here.

- They correspond to D-branes expanding into either the AdS_5 or S^5, much like the giant gravitons.

- Can we find a D-brane description of the loop presented here?

- Can we find overlaps with CPO's, giant gravitons, other giant loops?

Thank-You