Title: Non-Abelian Gauge Dynamics in 2d N=(2,2) Theories

Date: Oct 05, 2006 11:00 AM

URL: http://pirsa.org/06100006

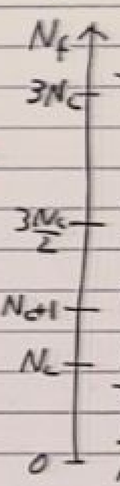
Abstract:

Motivations

① Field Theory (4 supercharges)

$4d$ $\mathcal{N}=1$

Seiberg 1993~94



free

SCFT [E-M duality]

free magnetic theory

WZ model M^i_j, B^c $\tilde{B}_c$

NLSM (quantum deformed Moduli space)

No stable vacuum (Affleck-Dine-Seiberg)

N_c massive vacua $\mathbb{Z}_{2N_c} \xrightarrow{\chi SB} \mathbb{Z}_2$

$SU(N_c)$ SQCD
N_f flavors

$3d$ $\mathcal{N}=2$ -- vacuum structure understood.

1997 Berkeley, New Jersey.

$2d$ $(2,2)$ Abelian ✓ Witten 1993
H. Vafa 2000

Non-Abelian ?

② String theory

Compactifications to 4d $\mathcal{N}=2$ ($\overset{\text{or}}{\mathcal{N}}=1$) theories

e.g. Type II on $\mathbb{R}^{3+1} \times \underbrace{M}_{\uparrow}$ (with Orientifold + D-branes)

2d (2,2) SCFT with $\hat{c}=3$

(e.g. CY manifold of $\dim_{\mathbb{C}}=3$)

Need to know Variety of M's

Moduli space of (2,2) SCFT's

Past : M ~ Complete intersections in toric manifold

~ IR fixed points of Abelian gauge theories

IR fixed points of **Non-Abelian** gauge theories

are not well-studied.

Witten index of $SU(k)$ SQCD

IR dynamics of $SU(k)$ gauge theories

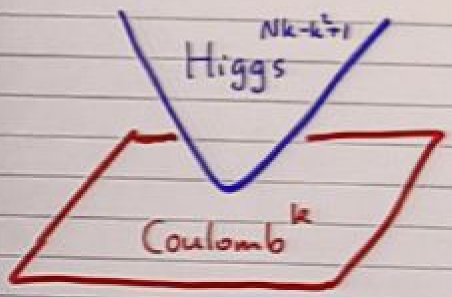
$U(k)$ linear sigma models

Glop transition

Derivation

Witten index of SU(k) SQCD

with N fundamentals (No antifundamentals)



← Non-compact space of classical vacua

$\mathrm{Tr}(-1)^F$ --- tricky / ill-defined

Hanany-H.

But **Twisted Masses** for the fundamentals lift both Higgs and Coulomb branches

$$\boxed{\mathrm{Tr}(-1)^F = n(k,N) := \#\left\{(\omega_1,\cdots,\omega_k) \;\middle|\; \begin{array}{l} \omega_a^N = 1, \quad \omega_a \neq \omega_b \\ \omega_1 + \cdots + \omega_k \neq 0 \end{array}\right\} \Big/ \begin{array}{l} \text{scaling} \\ \text{permutation} \end{array}}$$

Scaling $(\omega_1,\cdots,\omega_k) \mapsto (\omega\omega_1,\cdots,\omega\omega_k) \quad \omega^N=1$

permutation $(\omega_1,\cdots,\omega_k) \mapsto (\omega_{\sigma(1)},\cdots,\omega_{\sigma(k)})$

Witten index of SU(k) SQCD

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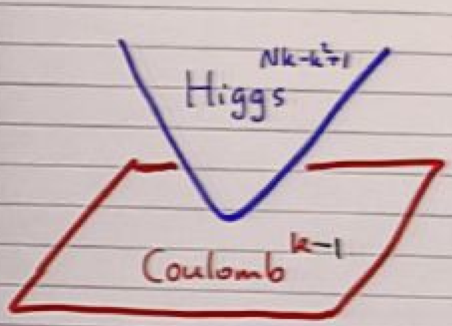

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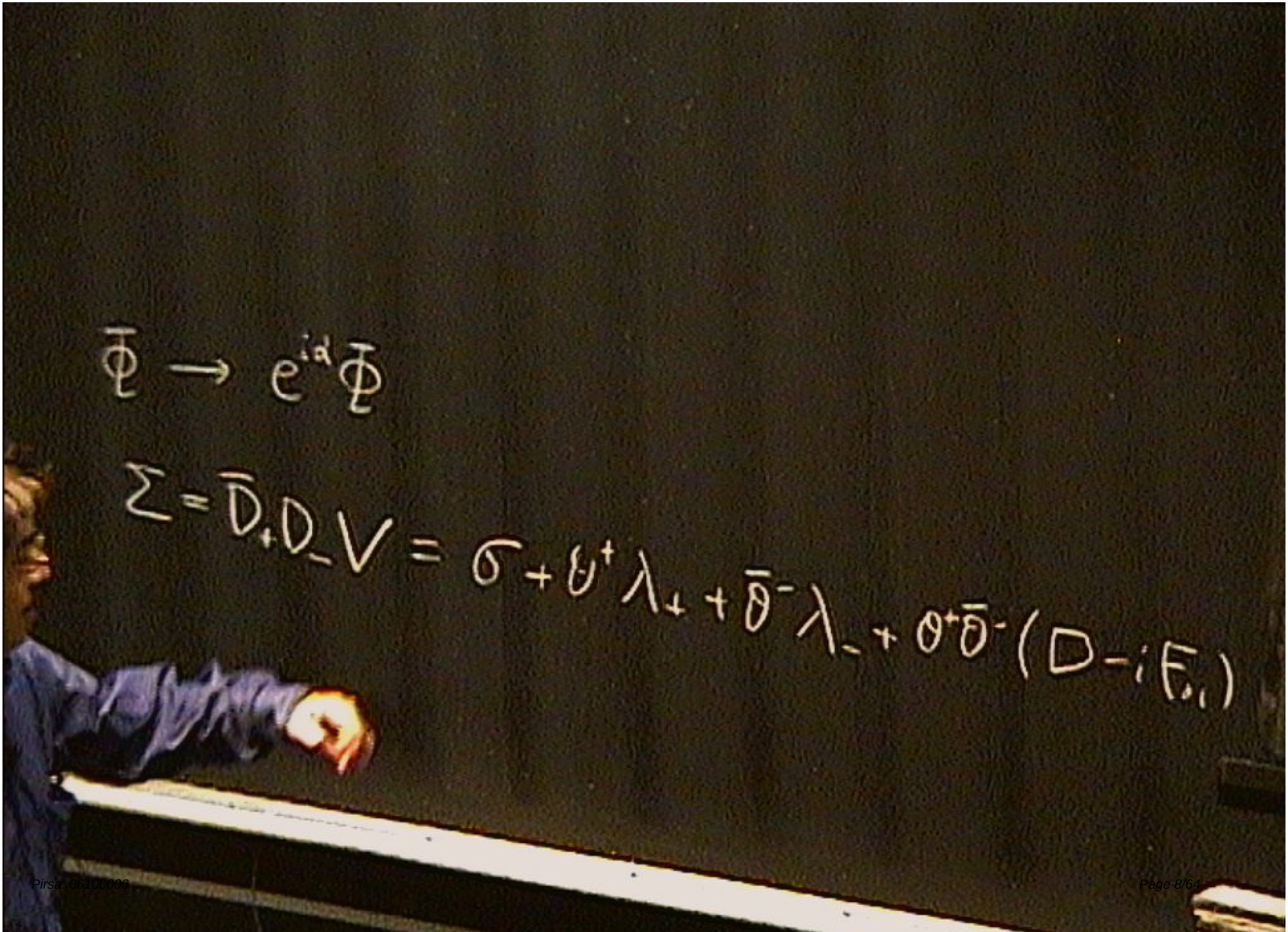
$$\bar{\Phi} \rightarrow e^{i\lambda}\bar{\Phi}$$



$$\Phi \longrightarrow e^{i\alpha}\Phi$$

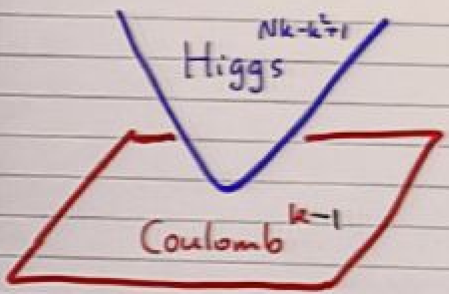
$$\Sigma = \bar{D}_+ D_- V = \sigma + \theta^+ \lambda_+ + \bar{\theta}^- \lambda_- + \theta^+ \bar{\theta}^- (D - iF_{01})$$

$$\Phi \longrightarrow e^{i\alpha}\Phi$$

$$\Sigma = \bar{D}_+ D_- V = \underbrace{\sigma} + \theta^+\lambda_+ + \bar{\theta}^-\lambda_- + \theta^+\bar{\theta}^-(D - iF_{01})$$

Witten index of SU(k) SQCD

with N fundamentals (No antifundamentals)



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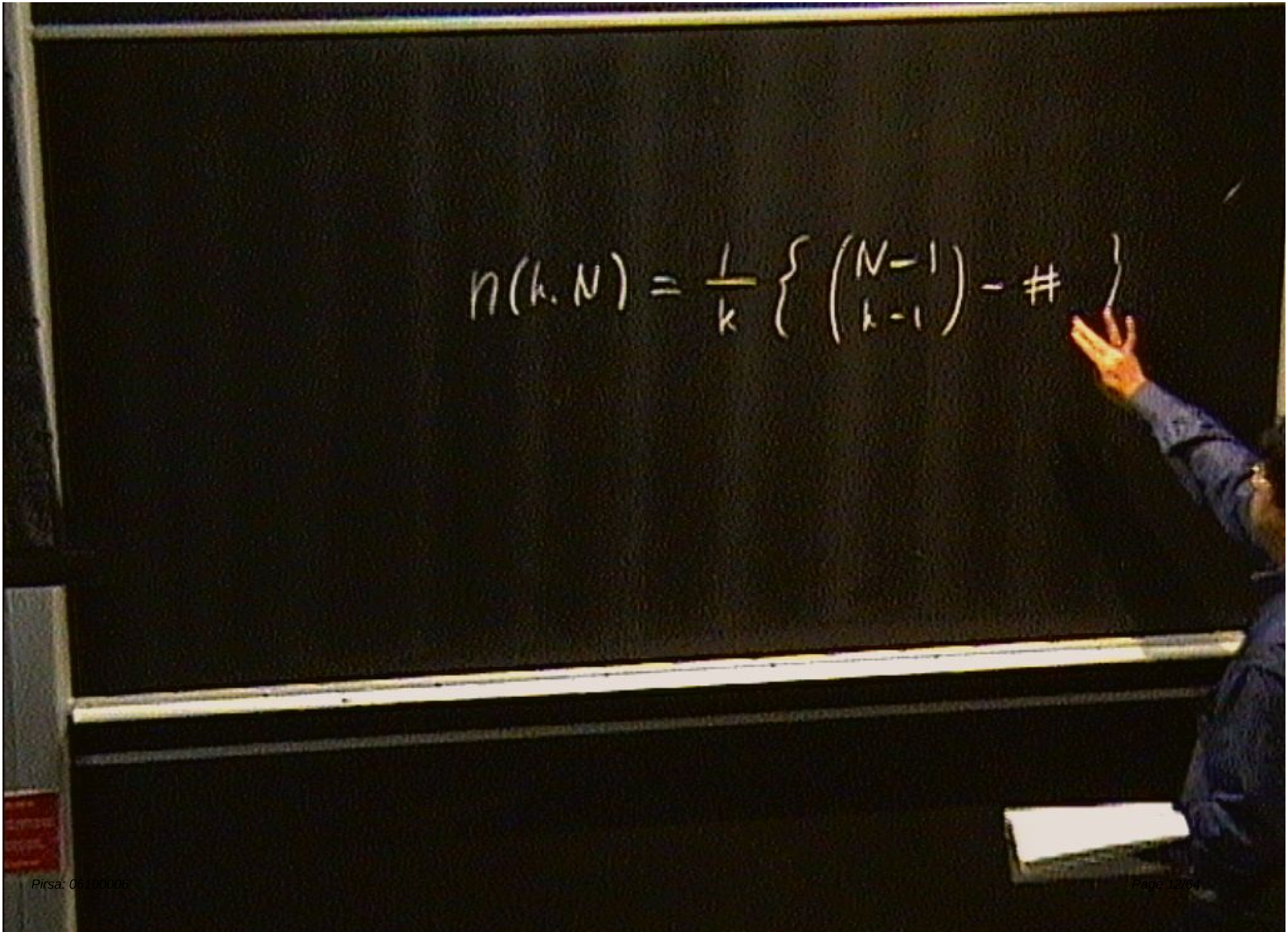
permutation $(\omega_1, \dots, \omega_k) \mapsto (\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)})$

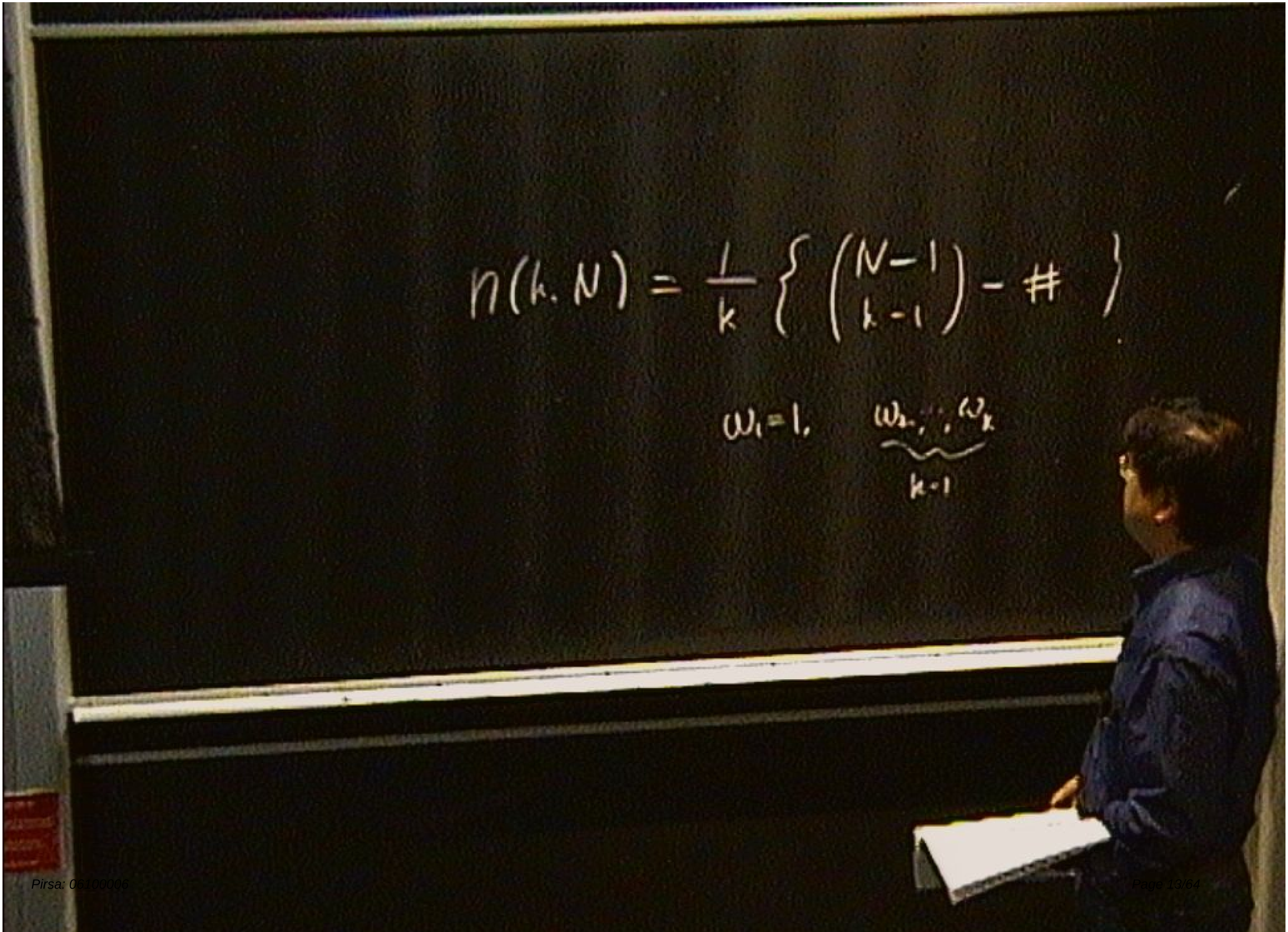
e.g. $\mathrm{Tr}(-1)^F = 0$ if $1 \leq N \leq k$

$\mathrm{Tr}(-1)^F = 1$ if $N = k+1$

$k \backslash N$	1	2	3	4	5	6	7	8	9	10	···
2	0	0	1	1	2	2	3	3	4	4	
3	0	0	0	1	2	3	5	7	9	12	
4	0	0	0	0	1	2	5	8	14	20	
5	0	0	0	0	0	1	3	7	14	25	
6	0	0	0	0	0	0	1	3	9	20	
7	0	0	0	0	0	0	0	1	4	12	
⋮											



$$n(k,N) = \frac{1}{k}\left\{ \binom{N-1}{k-1} - \# \right\}$$



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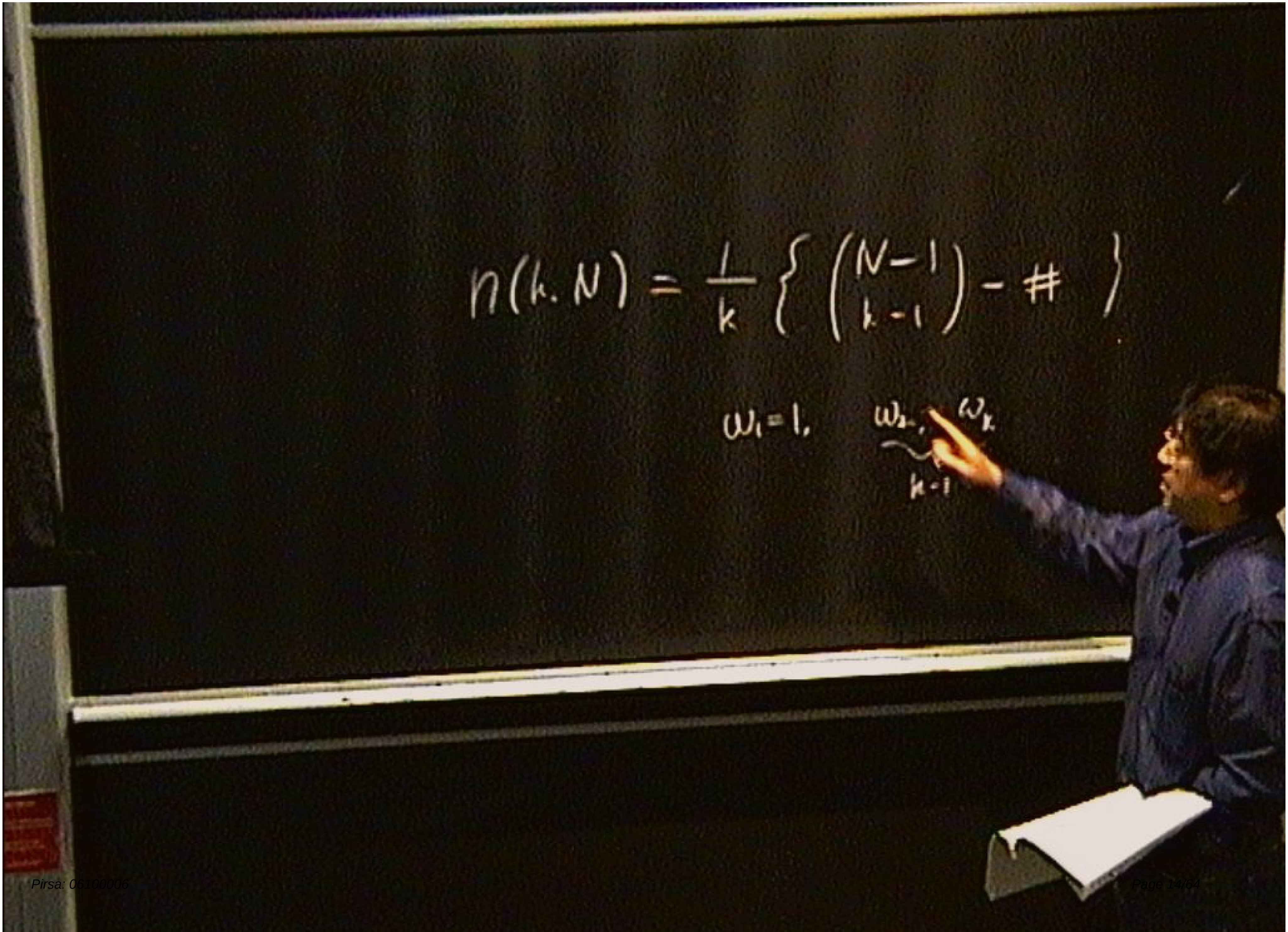
$$\omega_1 = 1, \quad \underbrace{\omega_2, \ldots, \omega_k}_{k-1}$$



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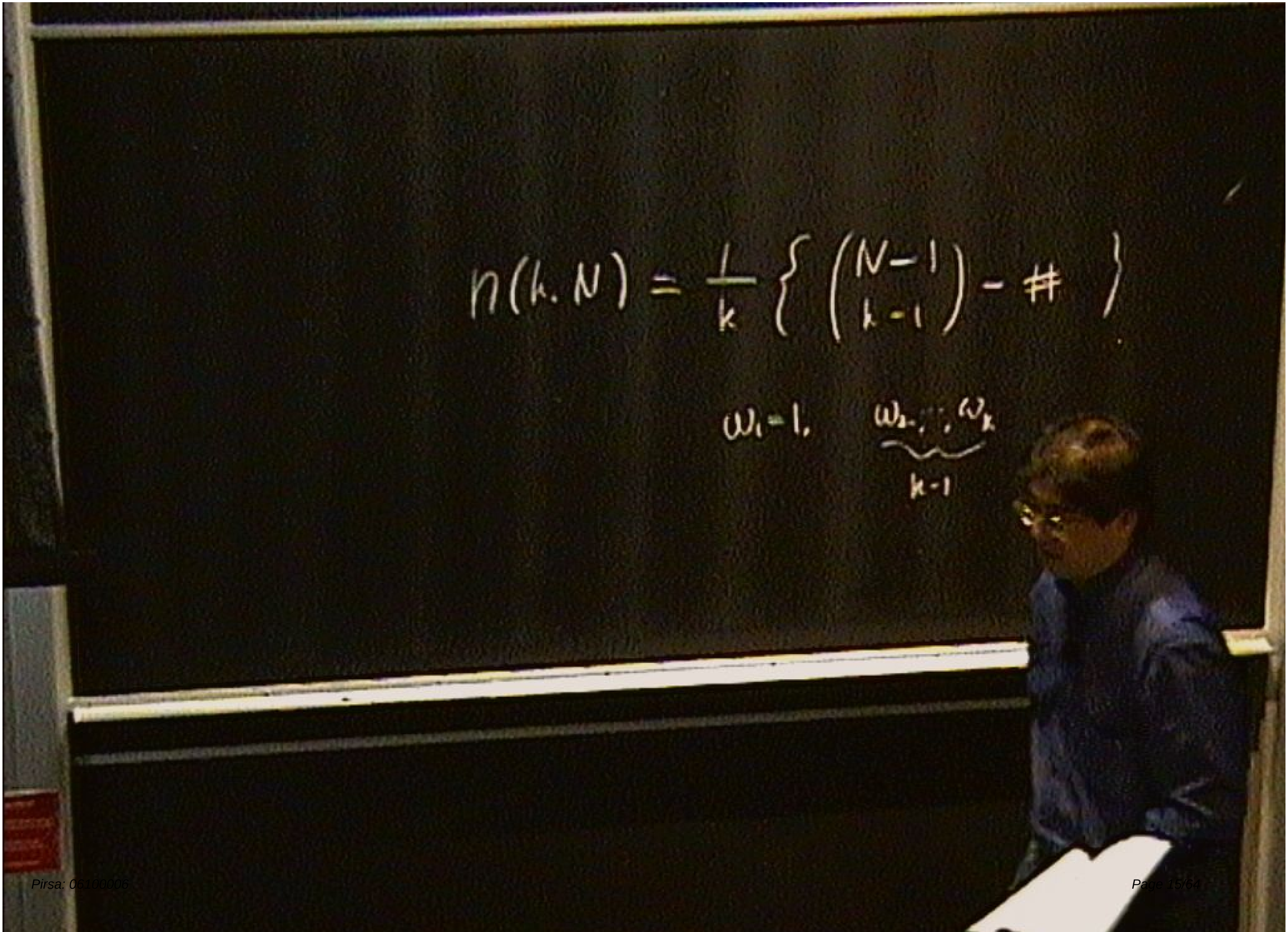
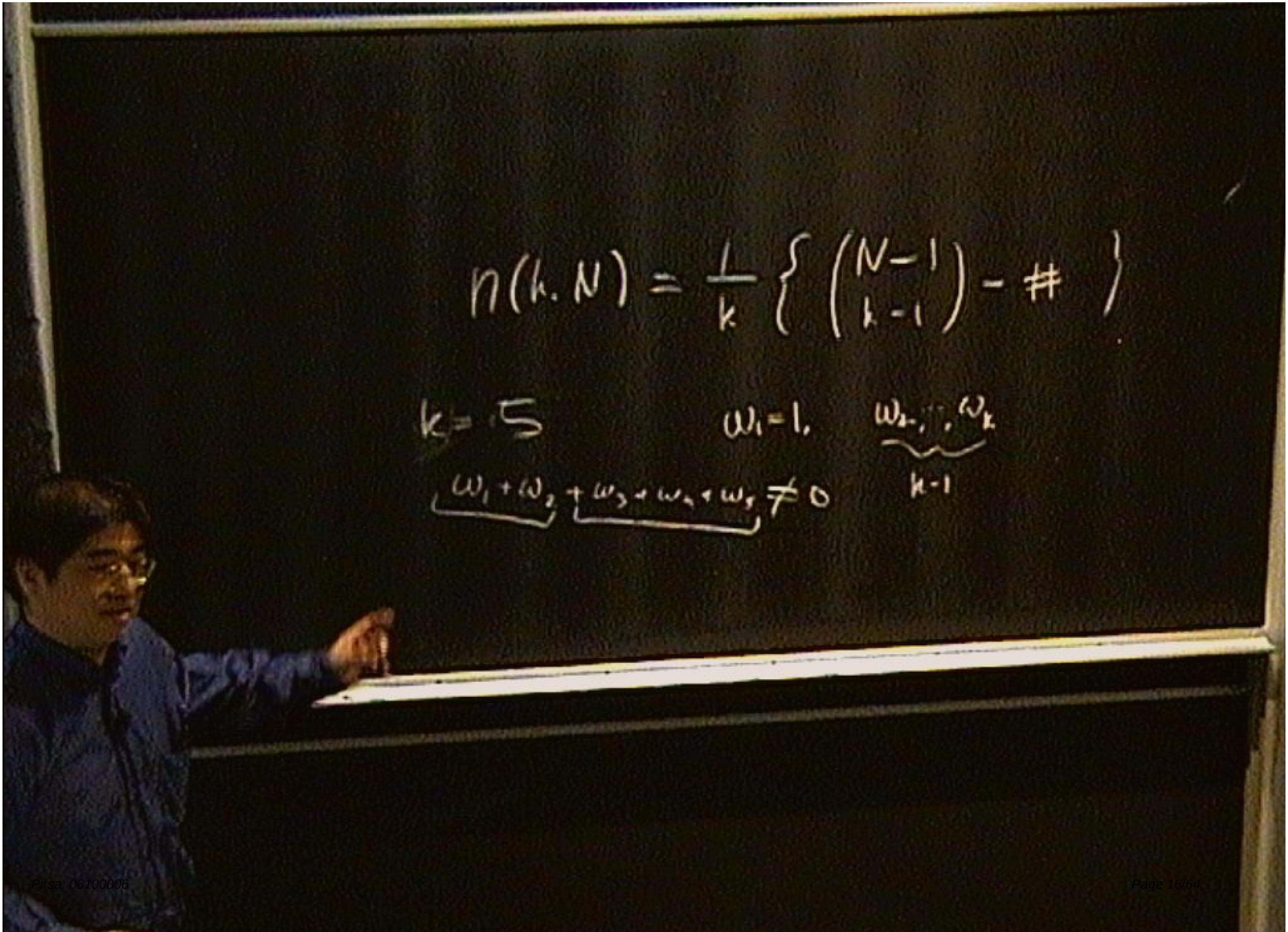


$$n(k.N) = \frac{1}{k}\left\{ \binom{N-1}{k-1} - \# \quad \right\}$$

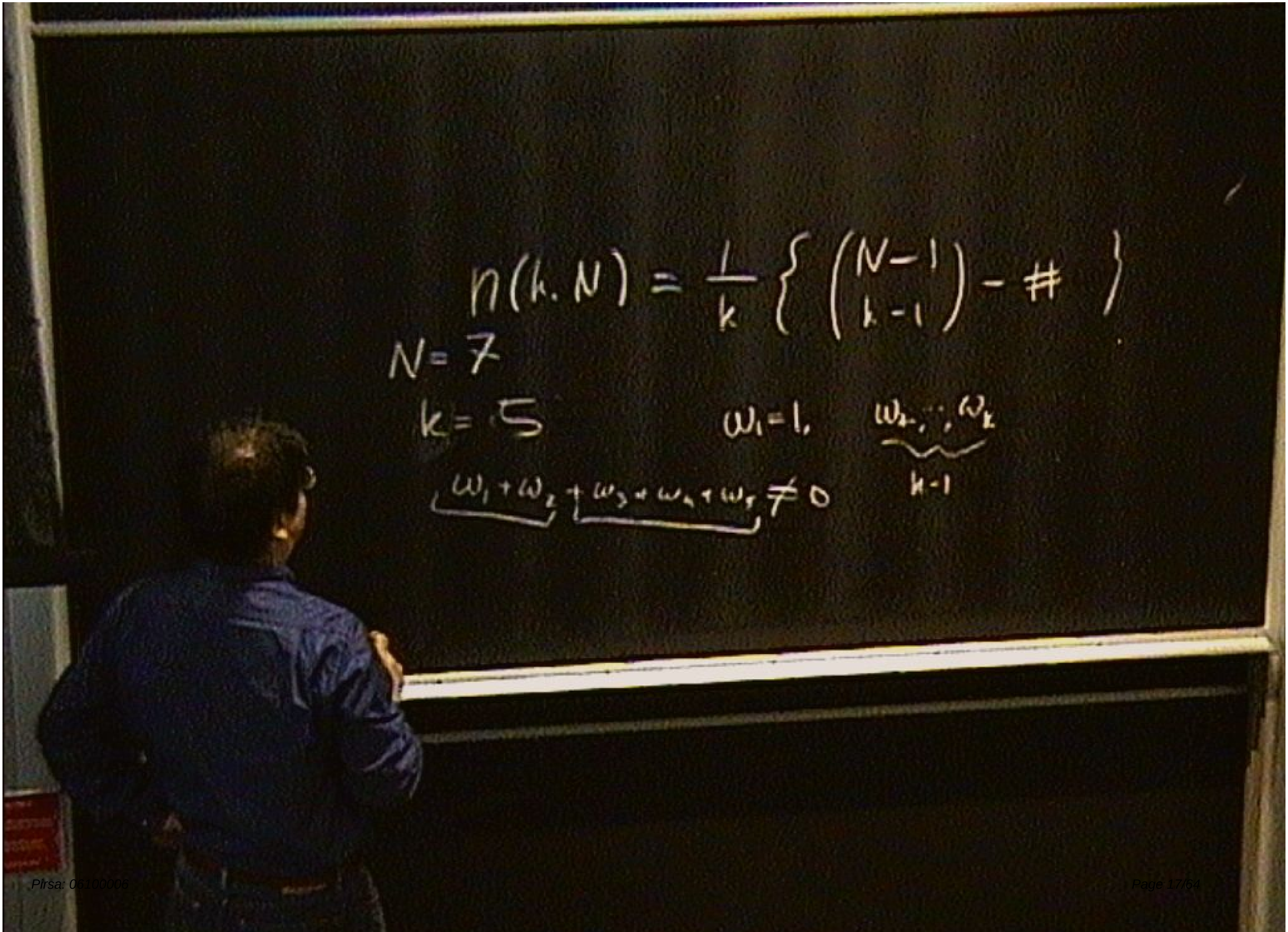
$$\omega_1 = 1, \quad \underbrace{\omega_2, \cdots, \omega_k}_{k-1}$$



$$n(k,N) = \frac{1}{k}\left\{ \binom{N-1}{k-1} - \# \right\}$$

$k = 5$ $\qquad$ $\omega_1 = 1,$ $\quad \underbrace{\omega_2, \ldots, \omega_k}_{k-1}$

$$\underbrace{\omega_1 + \omega_2} + \underbrace{\omega_3 + \omega_4 + \omega_5} \neq 0$$



$$n(k.N) = \frac{1}{k}\left\{ \binom{N-1}{k-1} - \# \right\}$$

$N = 7$

$k = 5$ $\quad \omega_1 = 1, \quad \underbrace{\omega_2, \dots, \omega_k}_{k-1}$

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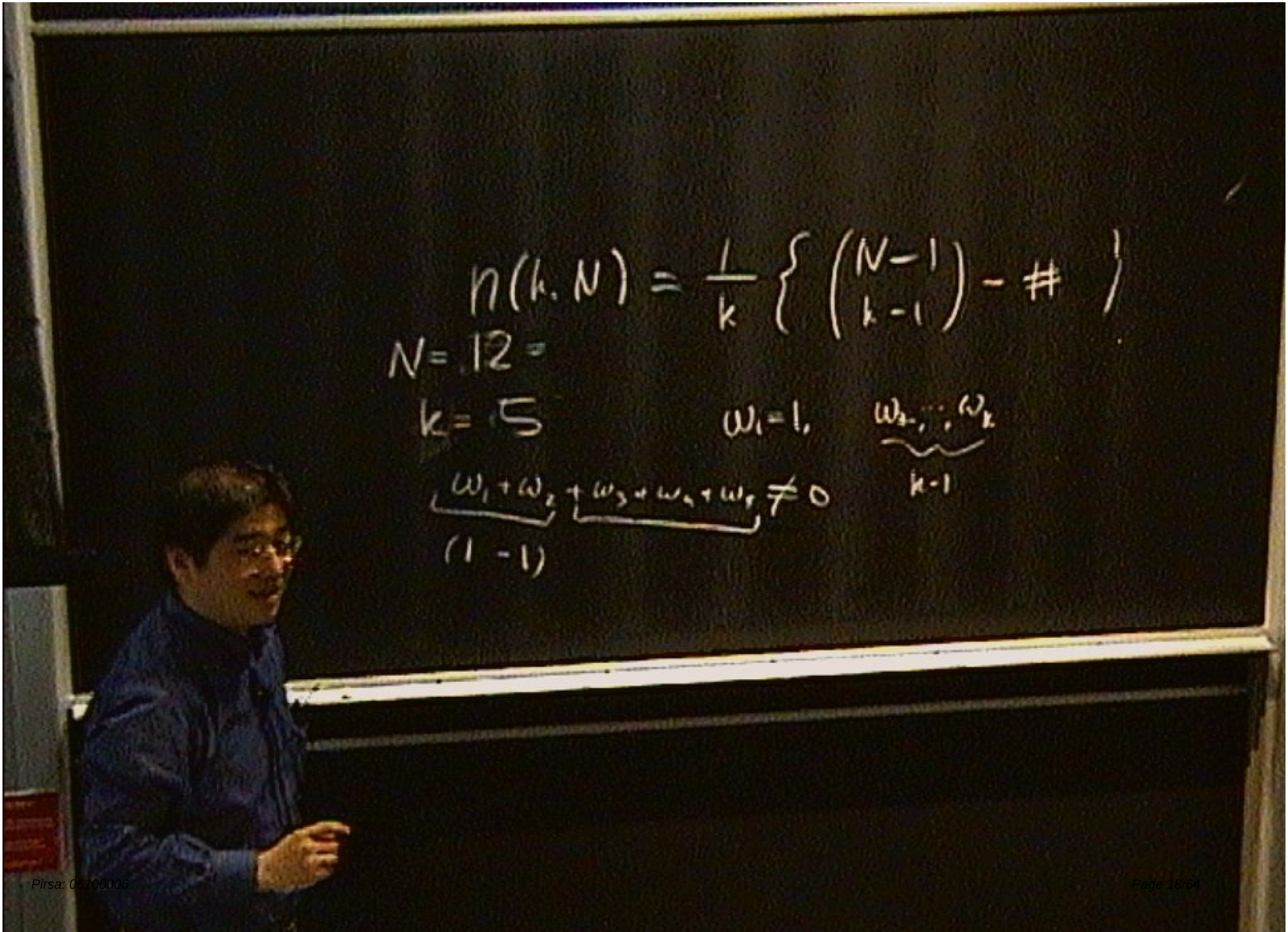

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$N = 12 =$

$k = 5$

$\omega_1 = 1,$ $\underbrace{\omega_2, \cdots, \omega_k}_{k-1}$

$$\underbrace{\omega_1 + \omega_2}_{(1\ \ -1)} + \underbrace{\omega_3 + \omega_4 + \omega_5} \neq 0$$

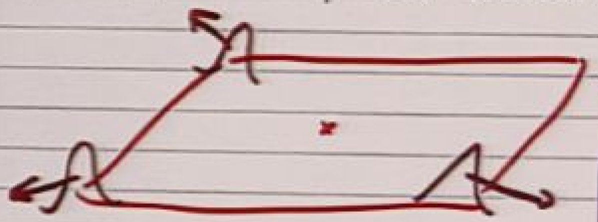
The vacua are localized at points on the Coulomb branch

location depends on the twisted masses $\widetilde{m}_1, \dots, \widetilde{m}_N$

e.g. Equal mass case : $\widetilde{m}_1 = \dots = \widetilde{m}_N = \widetilde{m}$

$$\sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} \qquad \sigma_a = \widetilde{m} - \frac{\omega_a}{\omega_1 + \dots + \omega_k} k \widetilde{m}$$

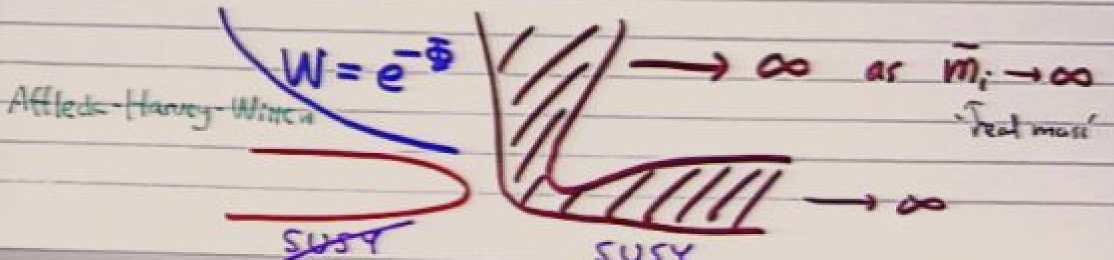
As $\widetilde{m}_i \xrightarrow{\text{all}} \infty$, all vacua run away to ∞



Suggests:

$\nexists$ SUSY ground states in pure SU(k) SYM
2d (2,2)

Similar to **3d $\mathcal{N}$=2 SQCD** de Boer et al / Aharony et al



IR dynamics of SU(k) gauge theories

With N massless fundamentals

SU(k) simple $\Rightarrow$ $\boxed{\mathrm{Tr}_R F_{01} = 0}$ $\forall$ finite-dim repr. R

$\Rightarrow$ axial rotation is anomaly free

$\rightsquigarrow$ IR SCFT may be studied by chiral ring

e.g. $\boxed{\hat{c} = \sum_{i:\ \text{Dirac fermion}} (-q_i)}$ ← $U(1)_V$ R-charge

$\psi_{i\pm}$ $U(1)_A$ charge ∓ 1

For example, $W = G_d(B)$ degree d polynomial

in baryon operators $B_{i_1 \cdots i_k} = \epsilon_{a_1 \cdots a_k} \bar{\Phi}^{a_1}_{i_1} \cdots \bar{\Phi}^{a_k}_{i_k}$

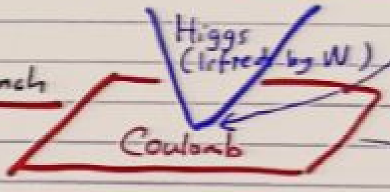
$$R[W] = 2 \Rightarrow R[B] = \frac{2}{d} \Rightarrow R[\bar{\Phi}] = \frac{2}{dk}$$

$$\underline{\hat{c} = \underset{\substack{\uparrow \\ \psi_{i\pm}}}{Nk\left(1 - \frac{2}{dk}\right)} - \underset{\substack{\uparrow \\ \text{from } \lambda_\pm}}{(k^2 - 1)}}$$

call the IR fixed pt $\underline{C_{k,N}(G_d)}$

But: The theory may be Singular (at $\bar{\Phi}=0$) by

∃ non-compact Coulomb branch



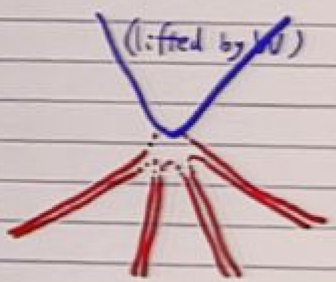
→ dim = k classically

If $\exists(\omega_1,\cdots,\omega_k)$ $\omega_a^N=1$, $\omega_a\neq\omega_b$, $\omega_1+\cdots+\omega_k=0$,

then ∃ one-dimensional Coulomb branch in the direction

$$\sigma=\begin{pmatrix}\omega_1 s & & \\ & \ddots & \\ & & \omega_k s\end{pmatrix} \qquad s\in\mathbb{C}$$

Singular ✓



If ∃ several such $(\omega_1,\cdots,\omega_k)$

∃ several (one-dim) Coulomb branches **Singular ✓**

However, if ∄ such $(\omega_1,\cdots,\omega_k)$, Coulomb branch is lifted and **the theory is non-singular** (at $\bar{\Phi}=0$).

property of (k,N)

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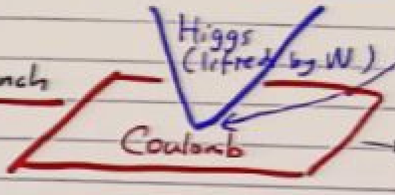

→ dim $=(k-1)$ classically

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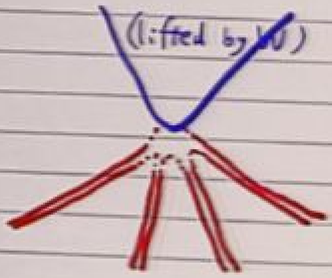

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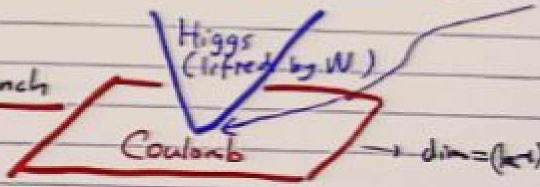

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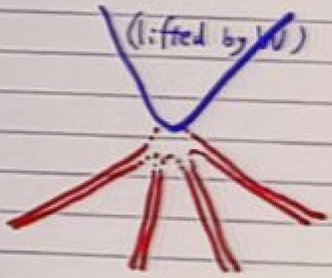

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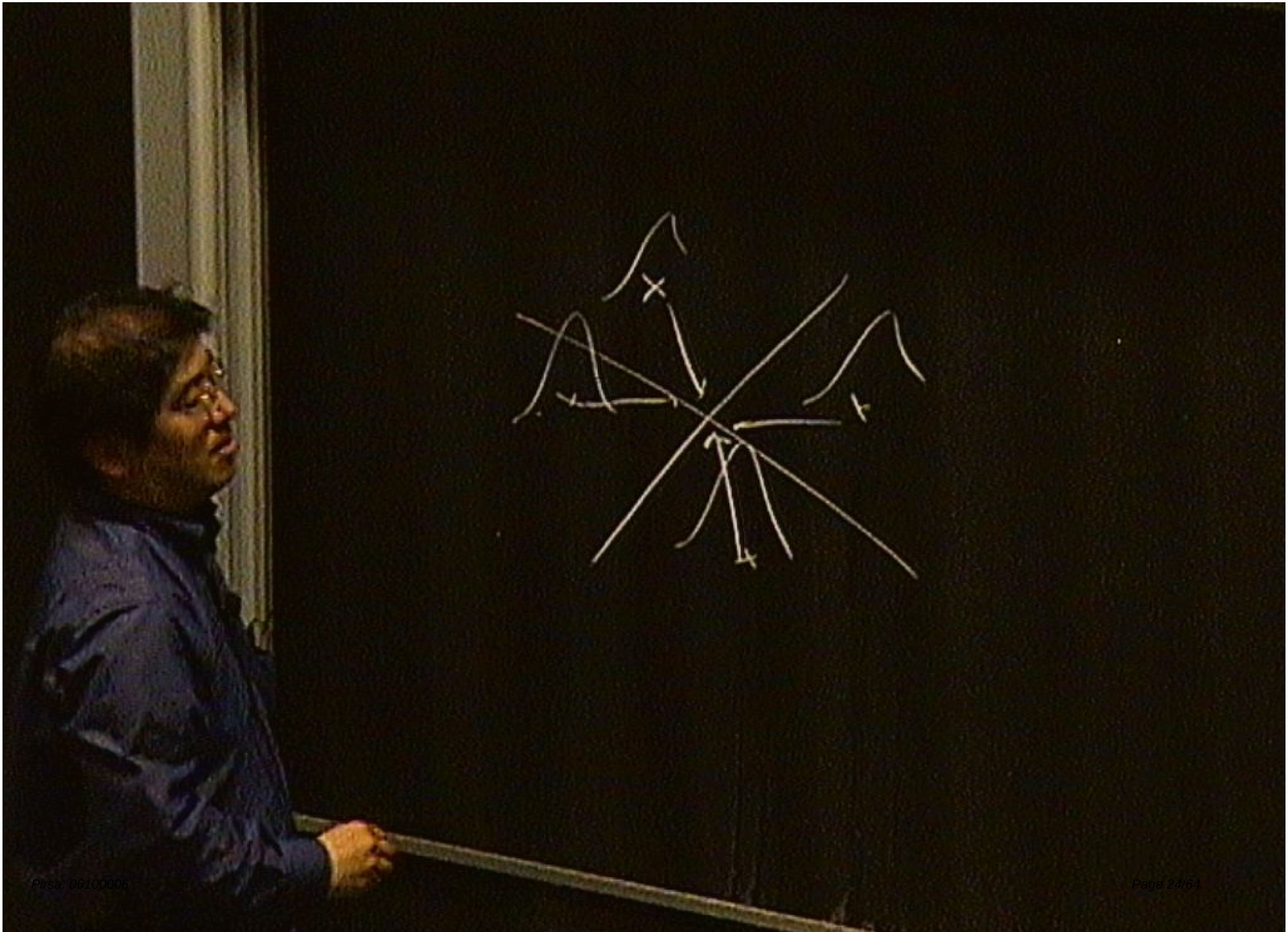
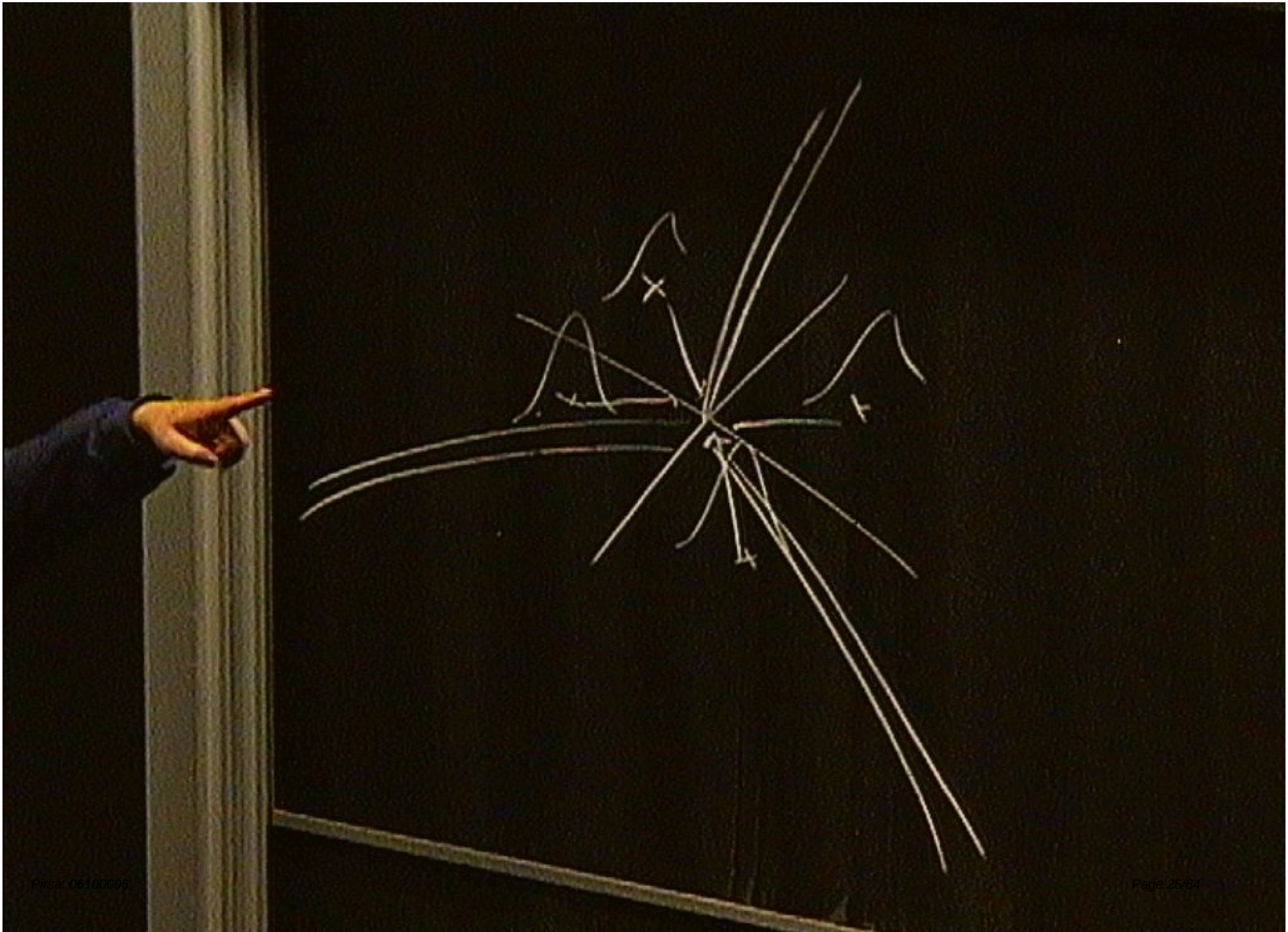
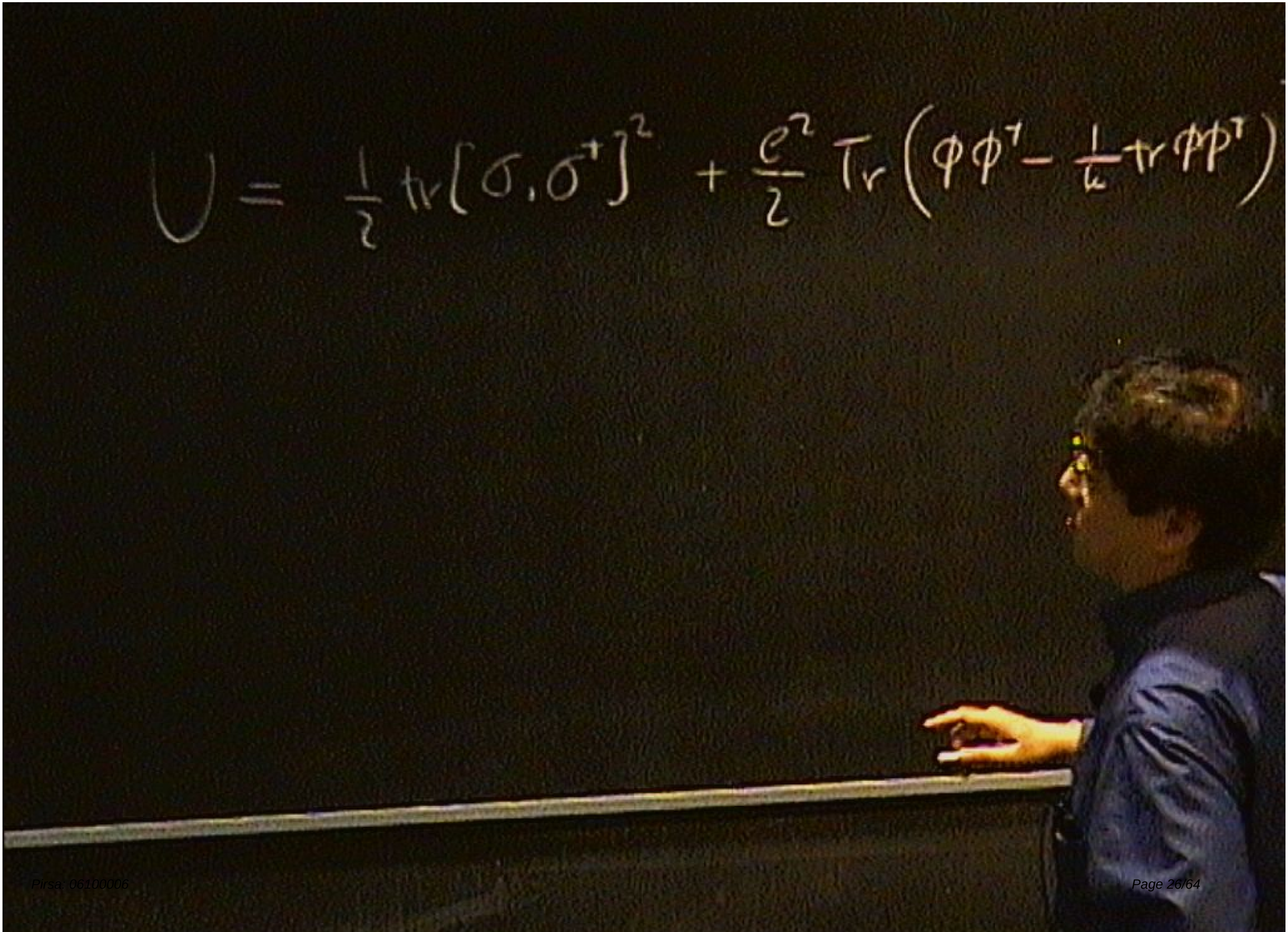
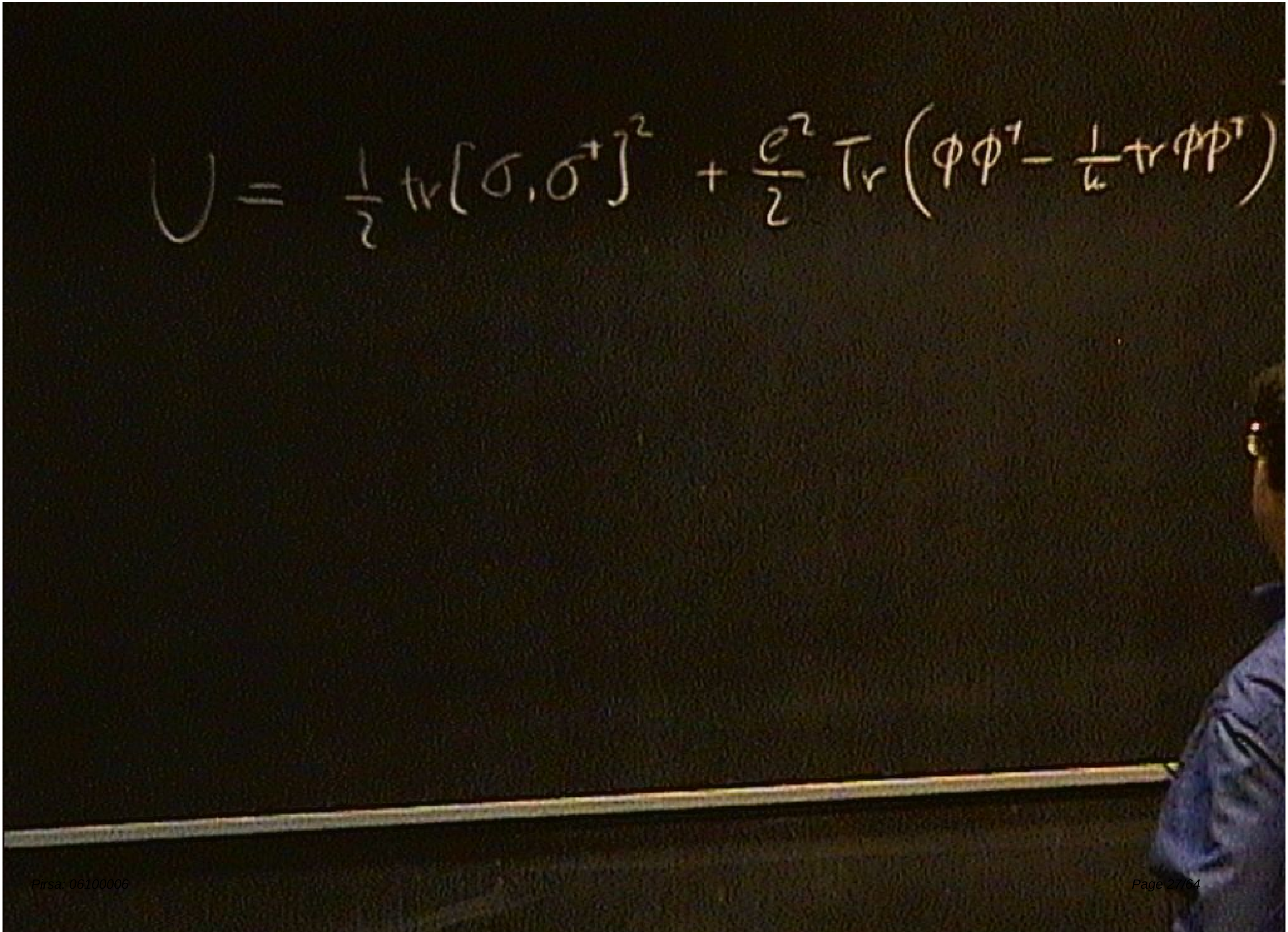






$$U = \frac{1}{2}\,\mathrm{tr}[\sigma, \sigma^\dagger]^2 + \frac{e^2}{2}\,\mathrm{Tr}\left(\phi\phi^\dagger - \frac{1}{k}\,\mathrm{tr}\,\phi\phi^\dagger\right)$$



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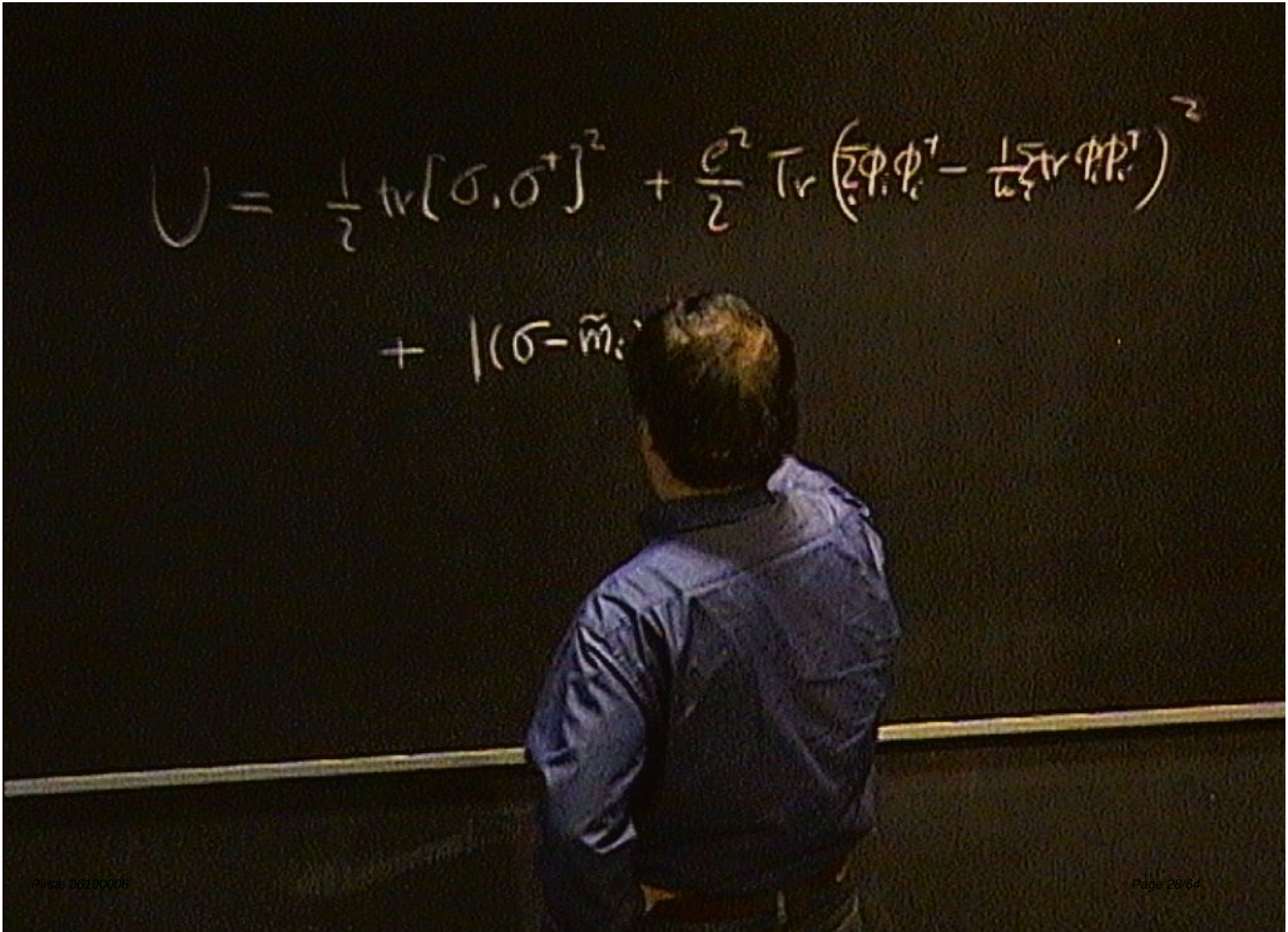

$$U = \frac{1}{2}\mathrm{tr}[\sigma, \sigma^{\dagger}]^2 + \frac{e^2}{2}\mathrm{Tr}\left(\sum_i \phi_i \phi_i^{\dagger} - \frac{1}{N}\sum_i \mathrm{tr}\,\phi_i \phi_i^{\dagger}\right)^2$$

$$+ |(\sigma - \tilde{m}_i$$

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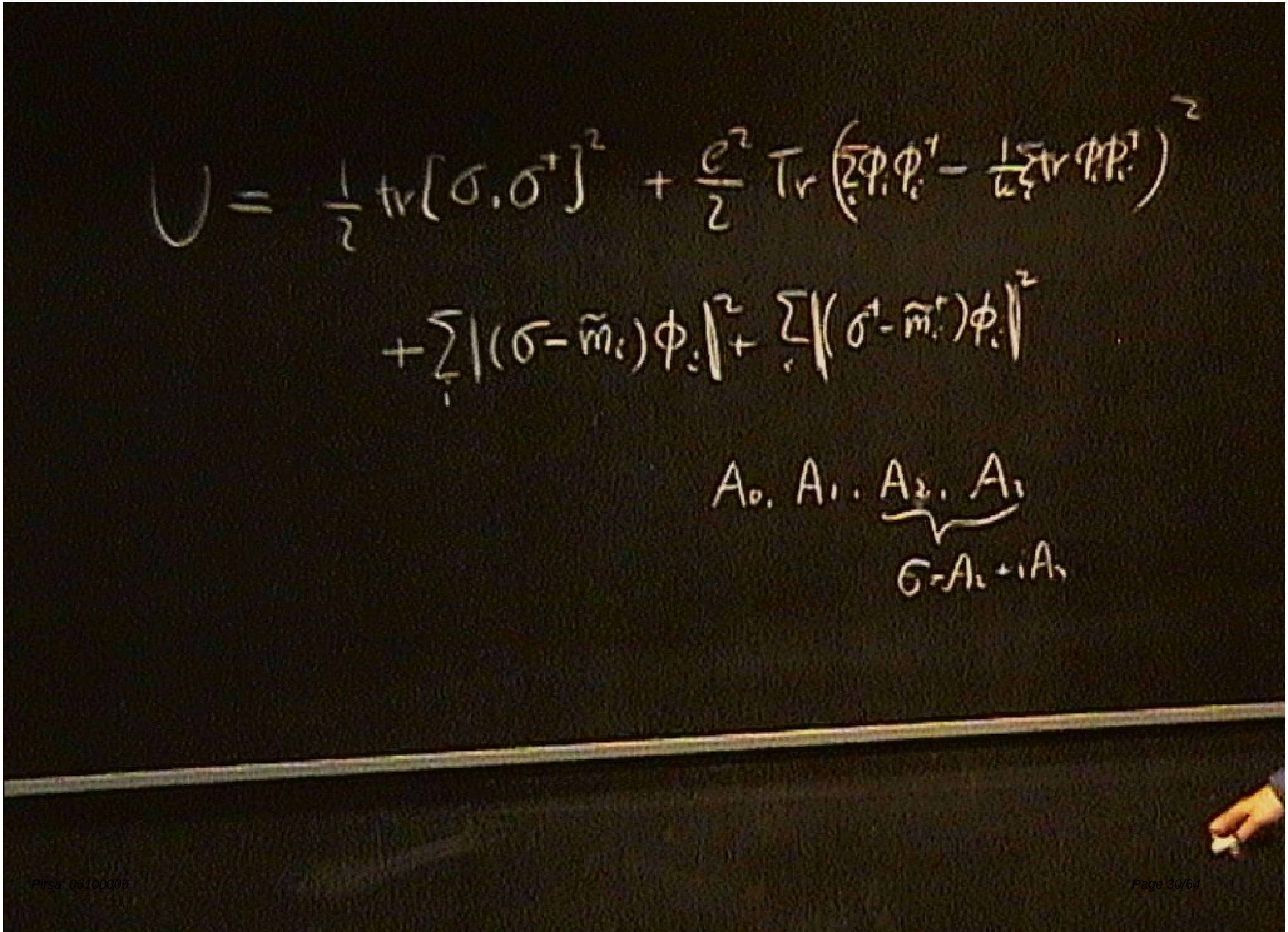
$$+ \sum_i \left| (\sigma - \tilde{m}_i) \phi_i \right|^2 + \sum_i \left| (\sigma^\dagger - \tilde{m}_i^\dagger) \phi_i \right|^2$$



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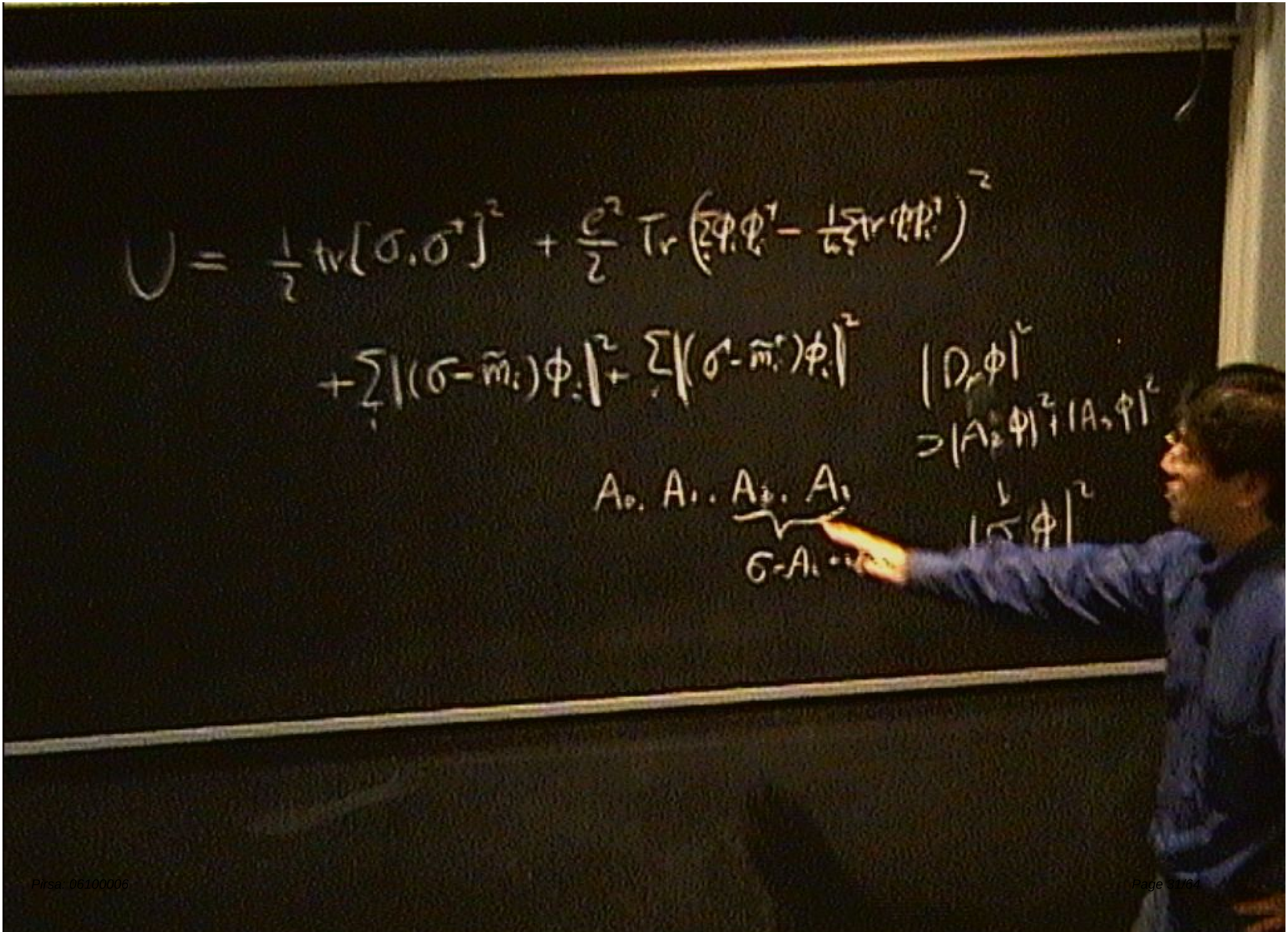
$$+\sum_i |(\sigma-\tilde{m}_i)\phi_i|^2 + \sum_i |(\sigma^\dagger-\tilde{m}_i^\dagger)\phi_i|^2$$

$$A_0,\ A_1,\ \underbrace{A_2,\ A_3}_{\sigma = A_2 + iA_3}$$



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$$+\sum_i |(\sigma-\tilde{m}_i)\phi_i|^2 + \sum_i |(\sigma-\tilde{m}_i')\phi_i|^2$$

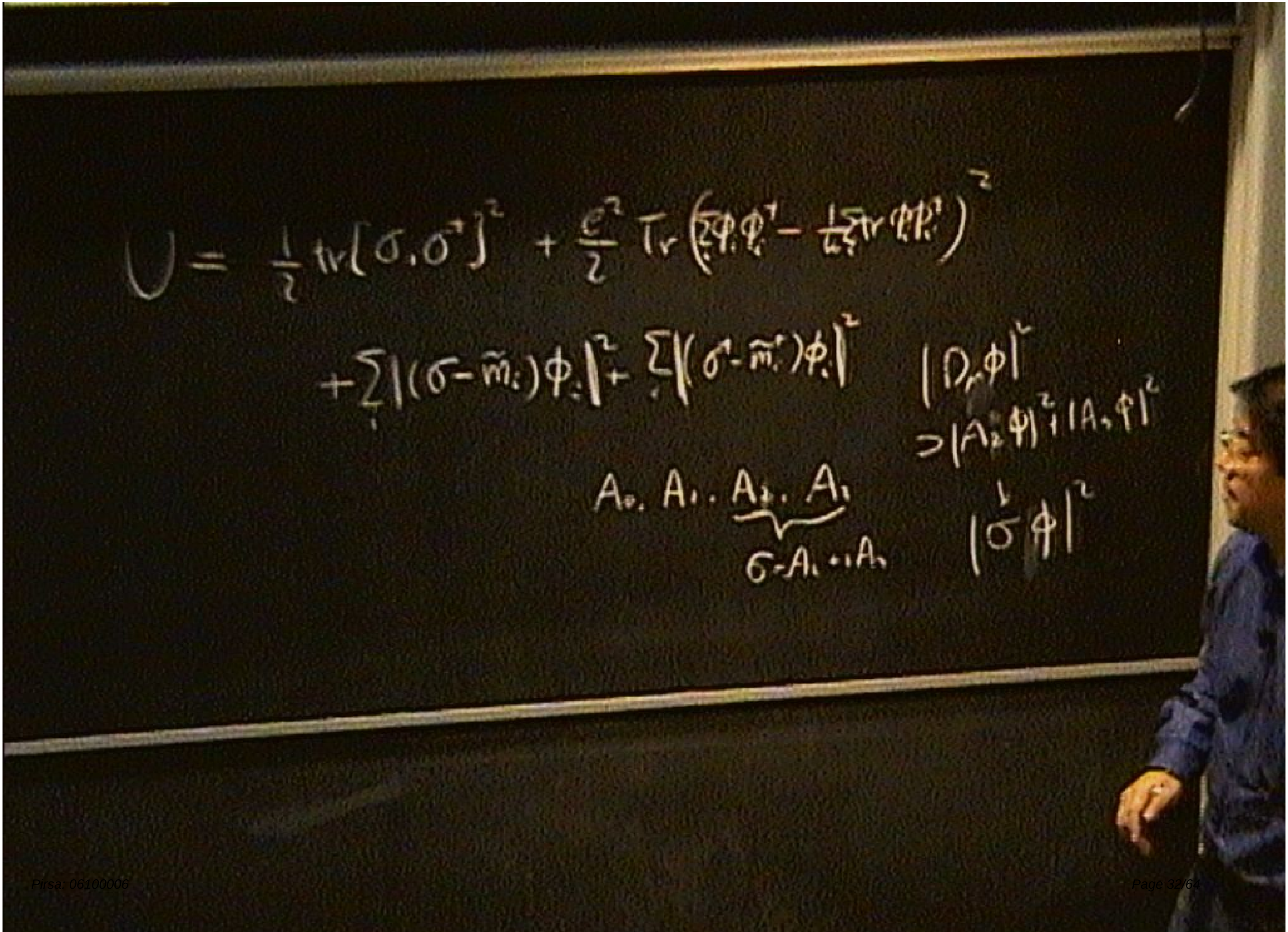
$$|D_\mu\phi|^2$$

$$\supset |A_2\phi|^2 + |A_3\phi|^2$$

$$\downarrow$$

$$|\sigma\phi|^2$$

$$A_0,\ A_1,\ \underbrace{A_2,\ A_3}_{\sigma - A_2 \cdot}$$



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$$|\overset{\dagger}{\sigma}\phi|^2$$

$$U = \frac{1}{2}\mathrm{tr}[\sigma, \sigma^\dagger]^2 + \frac{e^2}{2}\mathrm{Tr}\Big(\sum_i \phi_i\phi_i^\dagger - \frac{1}{N_c}\sum_i \mathrm{tr}(\phi_i\phi_i^\dagger)\Big)^2$$

$$+ \sum_i |(\sigma - \tilde{m}_i)\phi_i|^2 + \sum_i |(\sigma^\dagger - \tilde{m}_i^*)\phi_i|^2$$

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$$|\sigma \phi|^2$$

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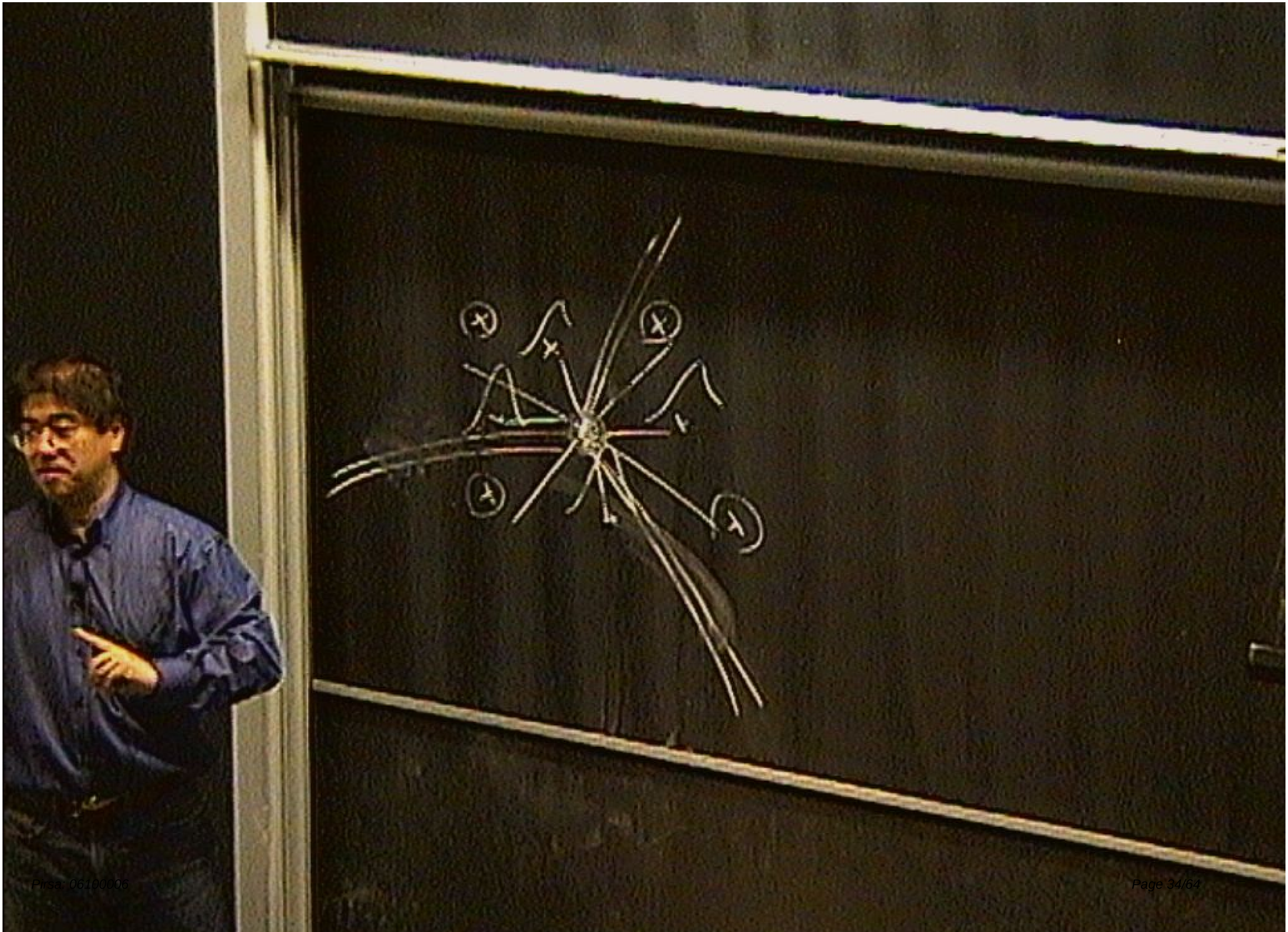


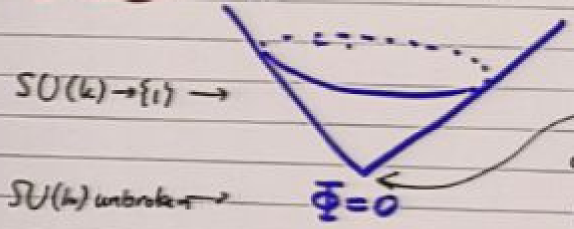
N=k+1 $\nexists(\omega_1,\cdots,\omega_k)$ $\omega_a \neq \omega_b$ $\omega_a^N = 1$

$\omega_1 + \cdots + \omega_k = 0$

W=0 (SQCD): Higgs branch $\overset{\text{cplx}}{\approx} \mathbb{C}^N$ spanned by

$B^i = \epsilon^{i j_1 \cdots j_k} B_{j_1 \cdots j_k}$ $i=1,\cdots,N.$

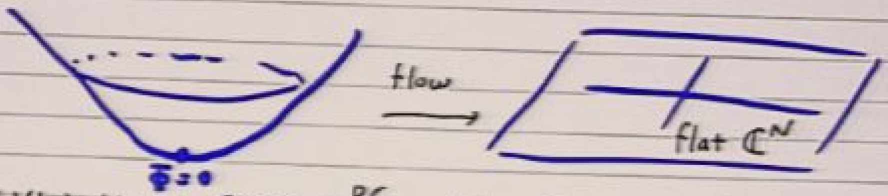
Classical Higgs branch is a cone, geometrically



Singularity from elimination of SU(k) gauge multiplet that becomes massless at $\Phi=0$.

Quantum: Coulomb lifted ⇒ SU(k) gauge multiplet is massive

⇒ No singularity at $\Phi=0$



SU(k) N=k+1 SQCD $\xrightarrow{RG}$ Free SCFT $\hat{c}=N$

$W=G_d(B)$ $\xrightarrow{RG}$ LG model of N indep. variables $B^1,\cdots,B^N$ with $W=G_d(B)$

U(k) Linear Sigma Models

$\Phi_1, \dots, \Phi_N$: fundamentals

P : $\det^{-d}$ (charge $-dk$ under center $U(1)$)

$W = P G_d(B)$ degree d poly in $B_{i_1 \dots i_k}$'s.

Center $= U(1)$: one Fayet-Iliopoulos-Theta $t = r - i\theta$

$r \gg 0$ Φ has max rank $(=k)$: $U(k) \to \{1\}$

$\xrightarrow{\text{Low}}$ Non-linear σ-model on $X_{k,N}(G_d) :=$

$$\{ \phi = (\phi_i^a) \mid \phi\phi^\dagger = r\mathbb{1}_k ,\ G_d(B) = 0 \} / U(k)$$

$=$ hypersurface $G_d(B) = 0$ in Grassmannian $G(k,N)$

$r \ll 0$ $P \neq 0$: $U(k) \to \{ g \in U(k) \mid \det^d g = 1 \} = G$

$$1 \to SU(k) \to G \to \mathbb{Z}_d \to 1$$

$r \to -\infty$ $C_{k,N}(G_d)/\mathbb{Z}_d$ orbifold CFT

$$U = \frac{1}{2}\mathrm{tr}[\sigma,\sigma^\dagger]^2 + \frac{e^2}{2}\mathrm{Tr}\left(\sum_i \phi_i\phi_i^\dagger - |\tilde{\phi}|^2 - r\right)^2 + \left|\tilde{\phi} + \sigma\, p\right|^2$$

$$+\sum_i \left|(\sigma - \ )\phi_i\right|^2 + \sum_i \left|(\sigma^\dagger \ )\phi_i\right|^2$$

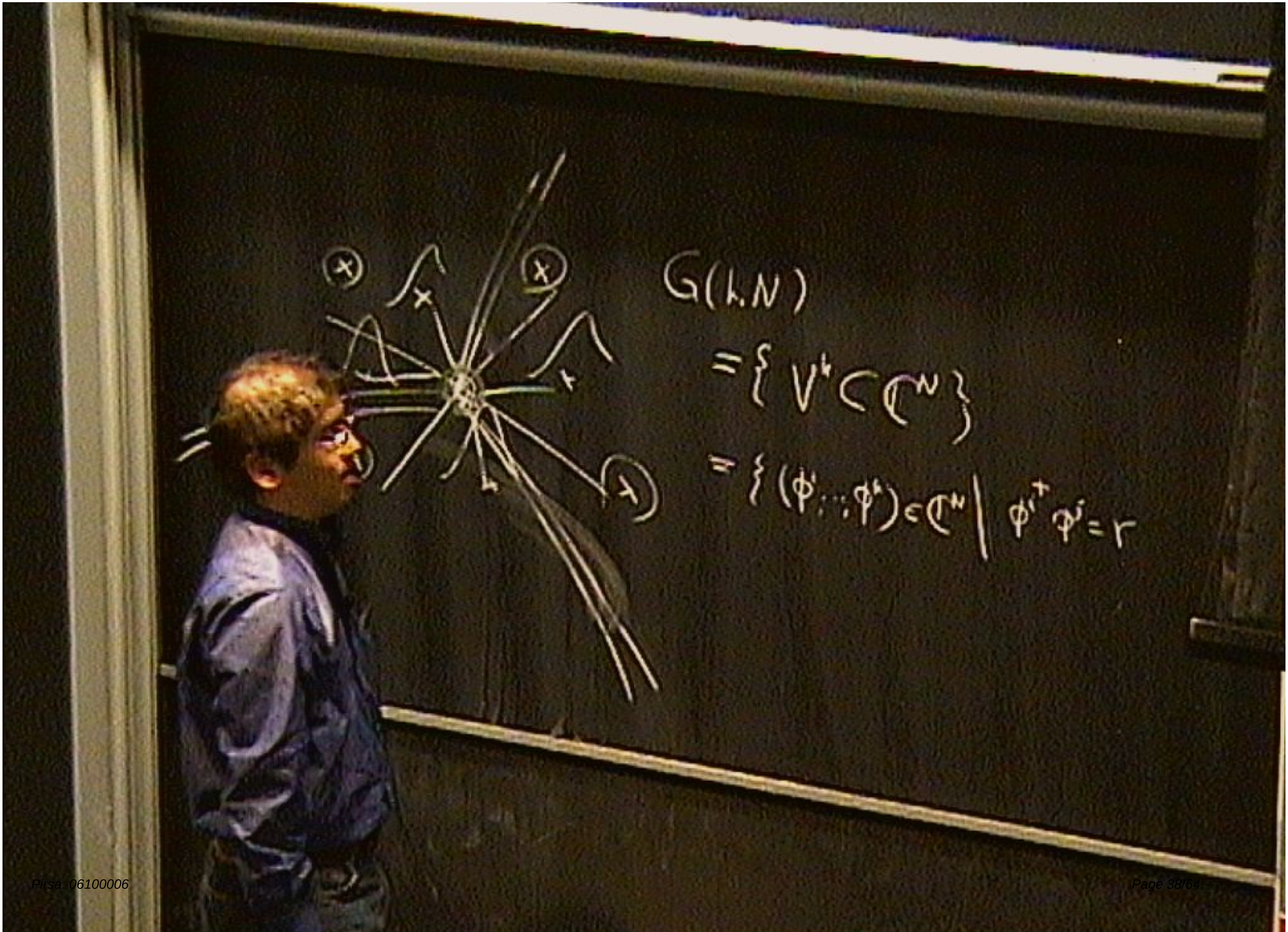
$$|D_\mu \phi|^2$$

$$\supset |A_2\phi|^2 + |A_3\phi|^2$$

$$A_0,\ A_1,\ \underbrace{A_2,\ A_3}_{\sigma = A_2 + iA_3}$$

$$\downarrow$$

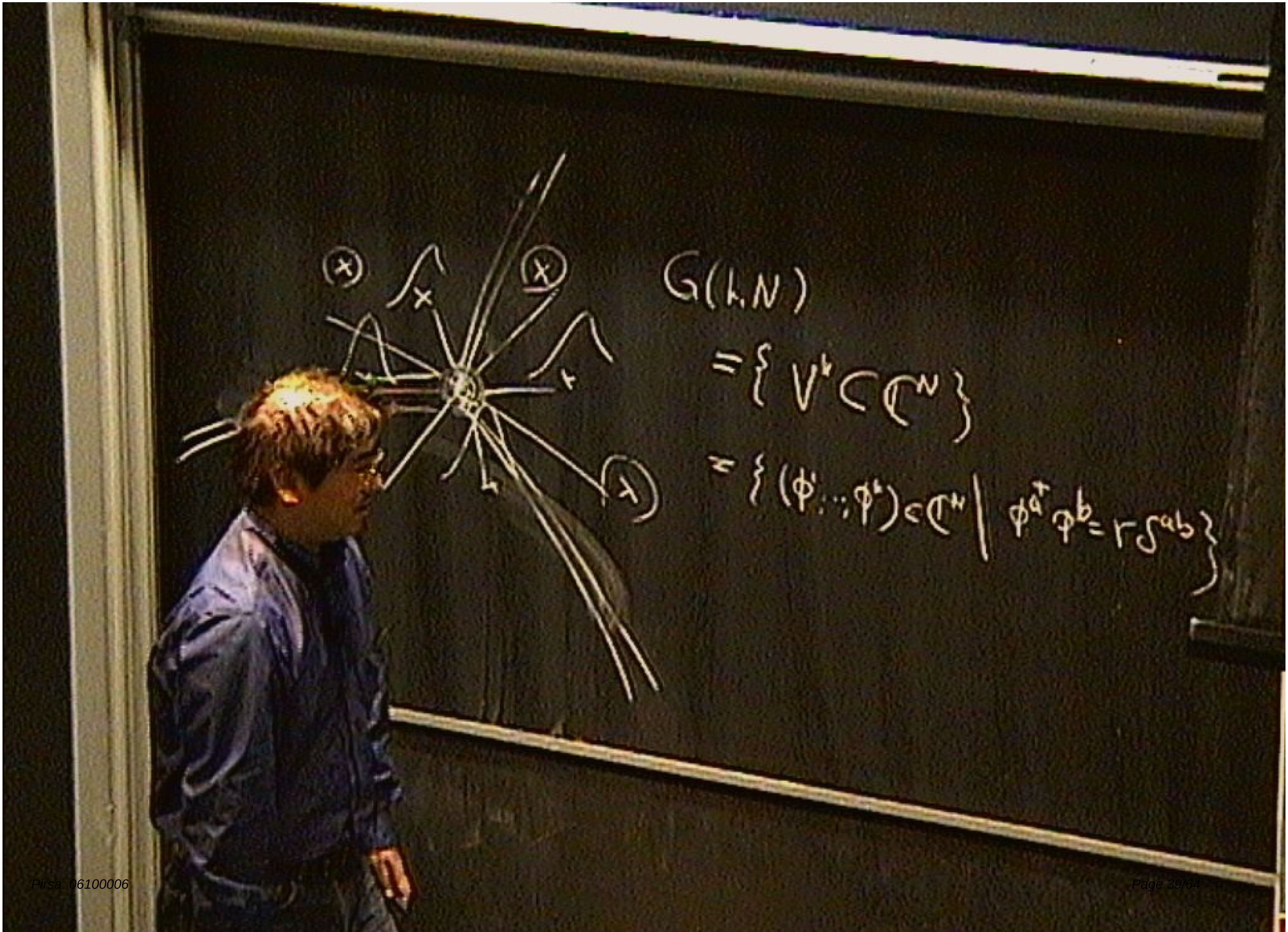
$$|\sigma\phi|^2$$



$$G(k,N)$$

$$= \{ V^k \subset \mathbb{C}^N \}$$

$$= \{ (\phi^1, \cdots, \phi^k) \in \mathbb{C}^N \mid \phi^{i\dagger} \phi^j = r$$



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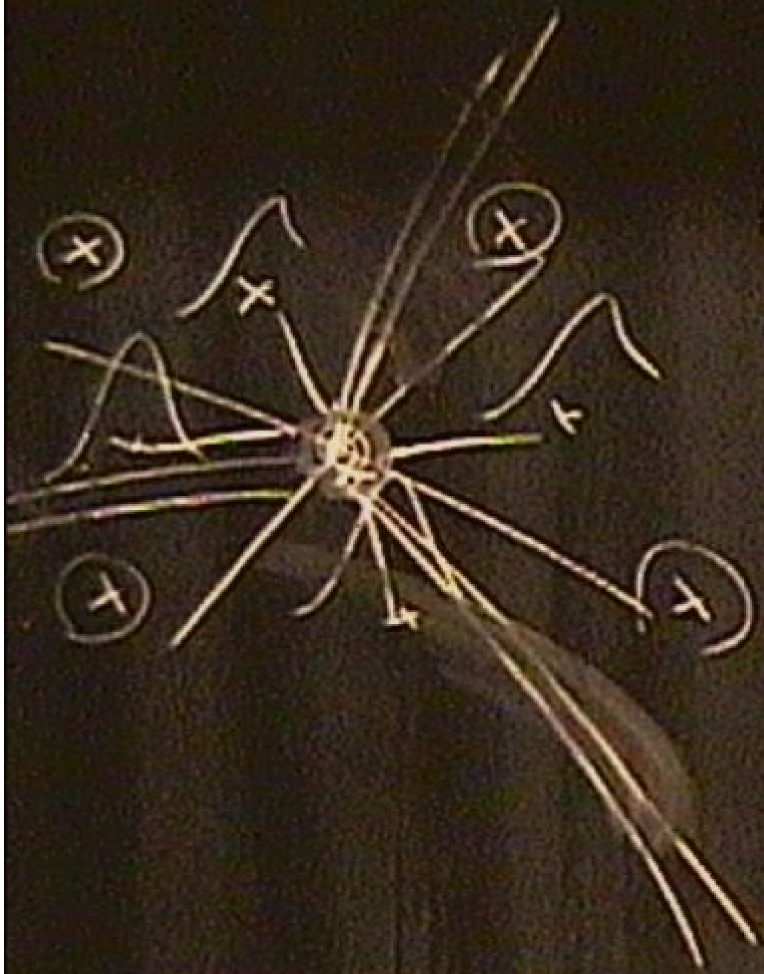

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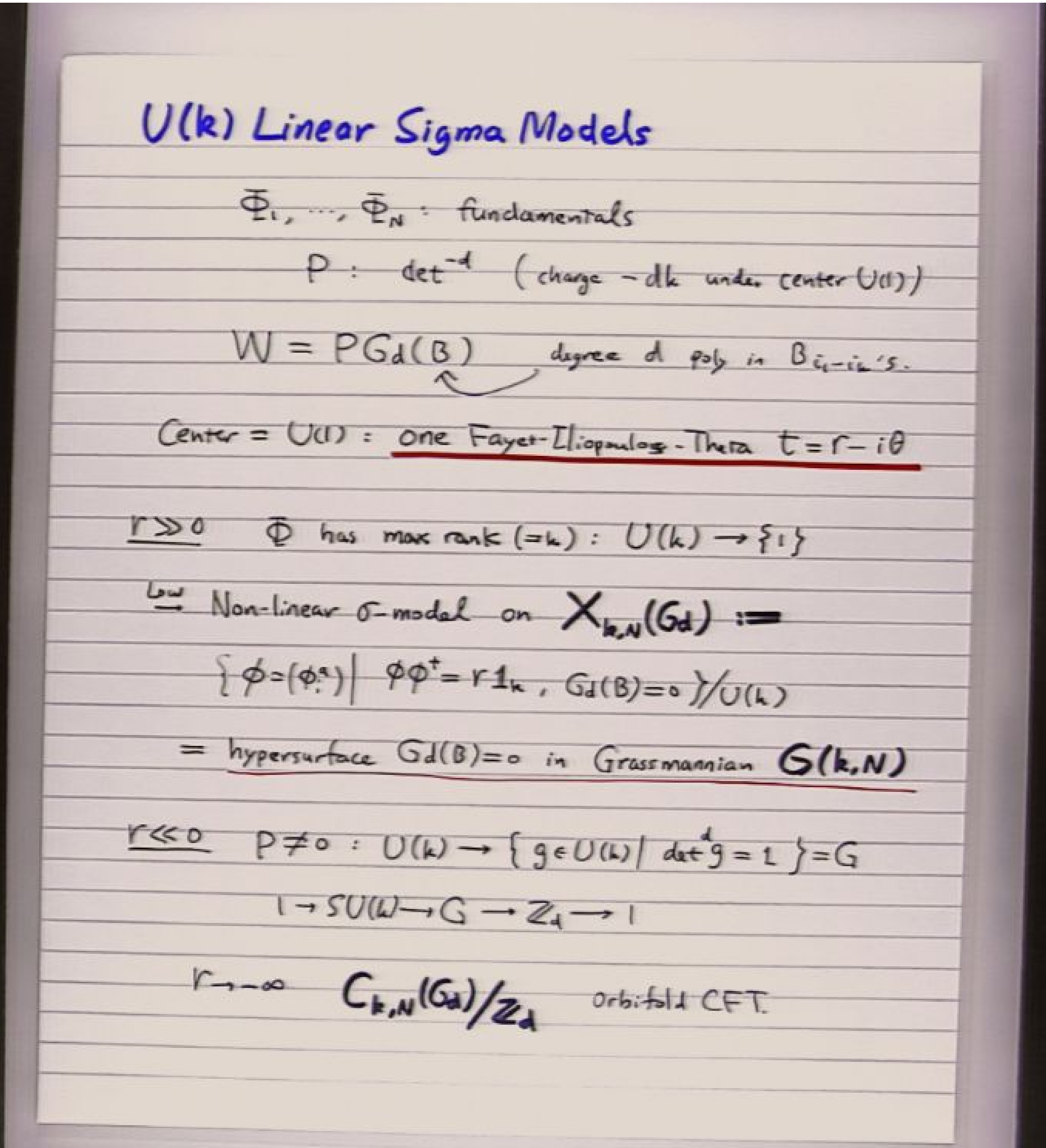

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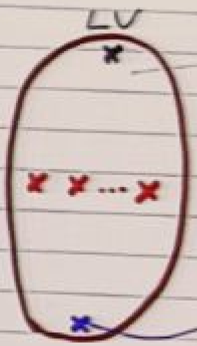
$r(\mu) = (N-d)\log(M/\mu') + r(\mu')$

d=N



NLSM on $X_{k,N}(G_d)$ ← Calabi-Yau

← $n(k,N)$ singular points

$C_{k,N}(G_d)/\mathbb{Z}_d$ (singular or non-singular depending on (k,N))

d<N

× NLSM on $X_{k,N}(G_d)$ ← Fano

$$\Big\downarrow \text{RG}$$

× $C_{k,N}(G_d)/\mathbb{Z}_d$ + × $(N-d)\cdot n(k,N)$ massive vacua on Coulomb branch

d>N

× $C_{k,N}(G_d)/\mathbb{Z}_d$

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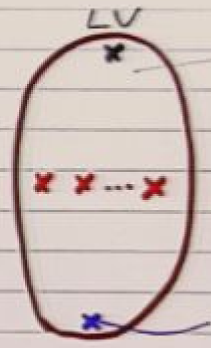

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check: $\hat{c}\big|$ $= (N$

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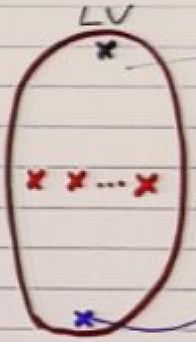

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d>N × $C_{k,N}(G_d)/\mathbb{Z}_d$

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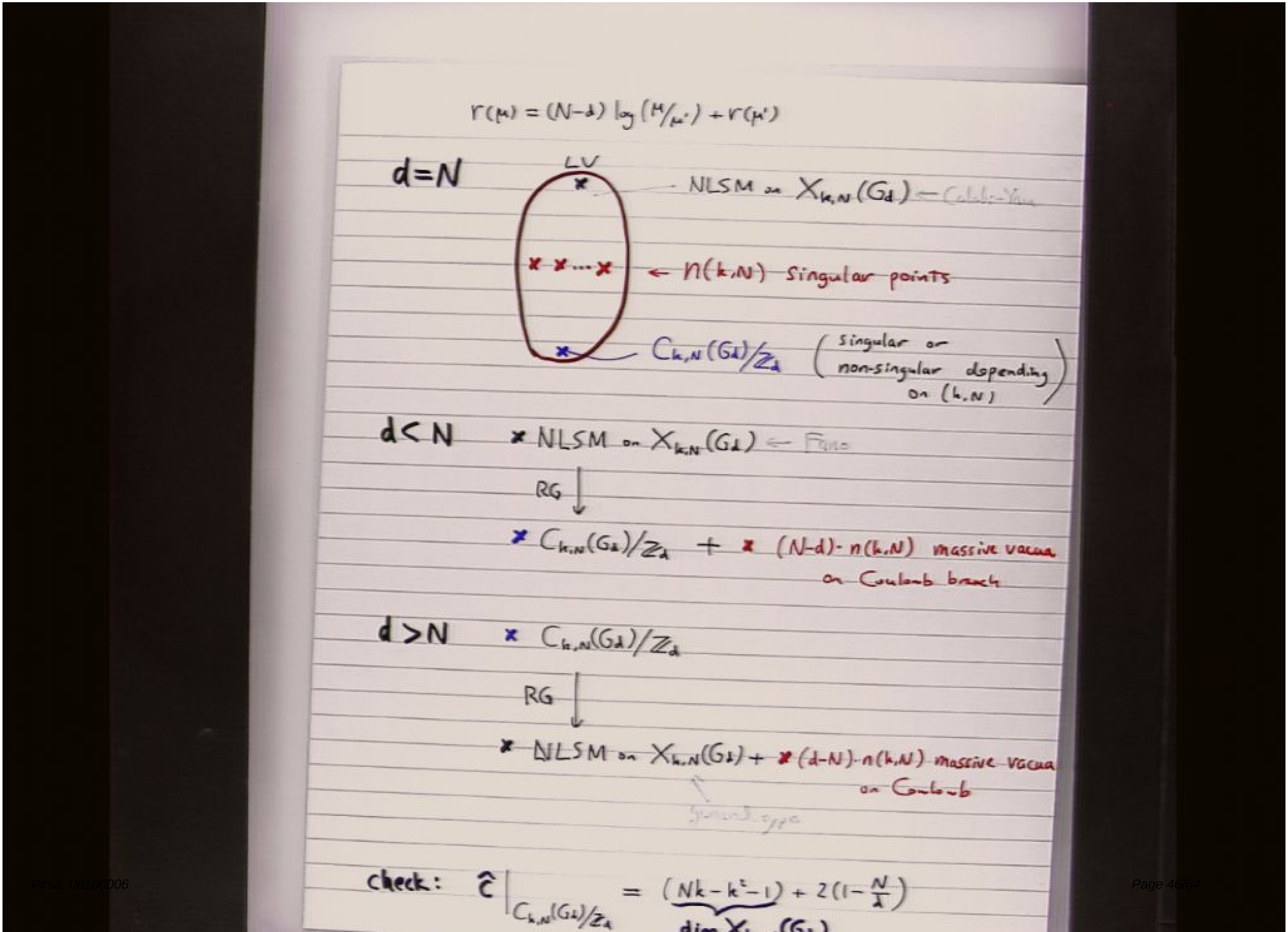
× NLSM on $X_{k,N}(G_d)$ + × $(d-N)\cdot n(k,N)$ massive vacua on Coulomb

? special type

check: $\hat{c}\big| \quad = (Nk-k^2-1)+2(1-\frac{N}{\ })$



$$r(\mu) = (N-d)\log(\mu/\mu') + r(\mu')$$

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× × ... × ← $n(k,N)$ singular points

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d>N × $C_{k,N}(G_d)/\mathbb{Z}_d$

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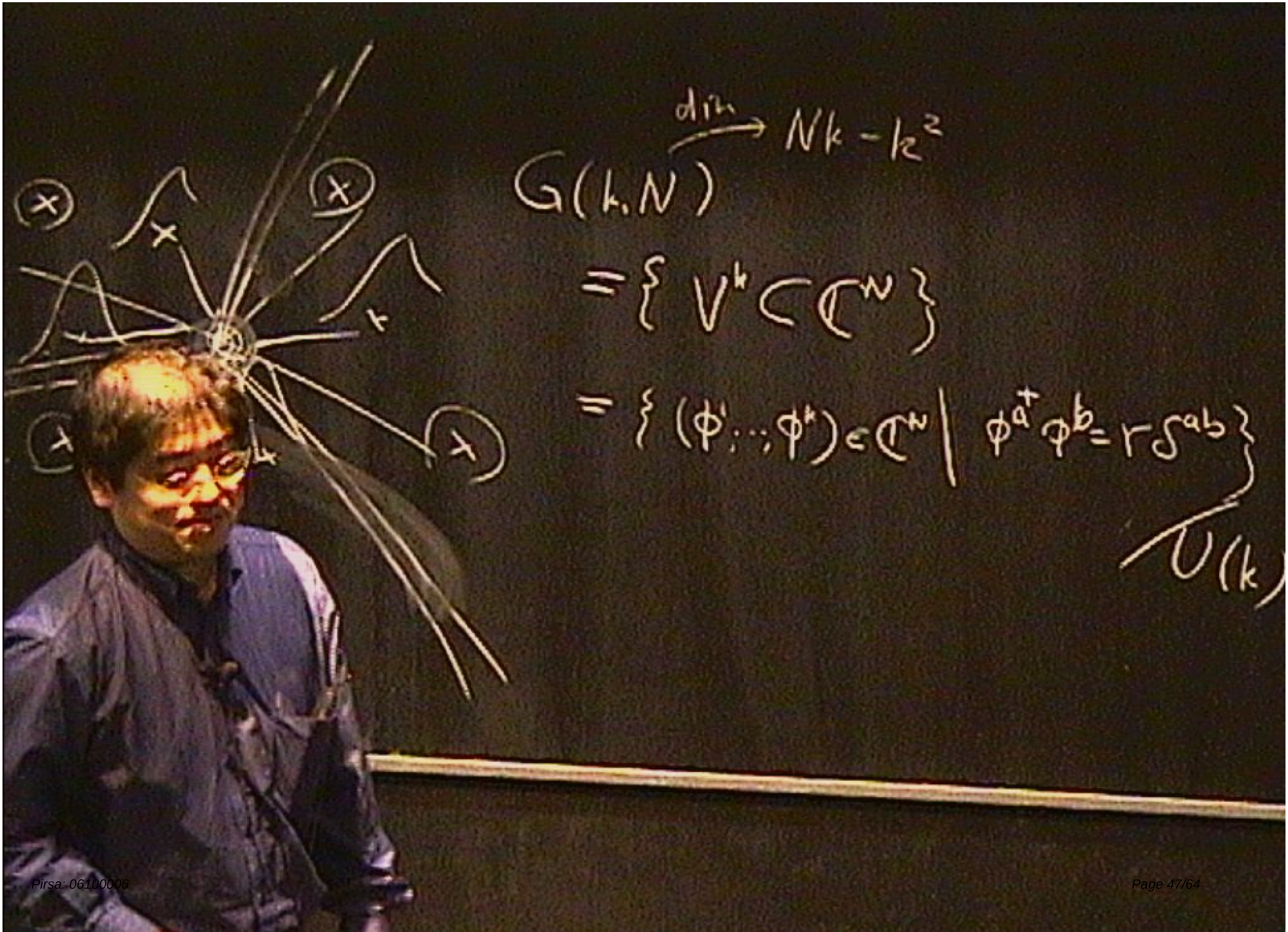
× NLSM on $X_{k,N}(G_d)$ + × $(d-N)\cdot n(k,N)$ massive vacua on Coulomb

↑ Gepner-type

check: $$\hat{c}\Big|_{C_{k,N}(G_d)/\mathbb{Z}_d} = (\underbrace{Nk-k^2-1}_{\dim X_{k,N}(G_d)}) + 2\left(1-\frac{N}{d}\right)$$



$$G(k,N) \xrightarrow{\dim} Nk - k^2$$

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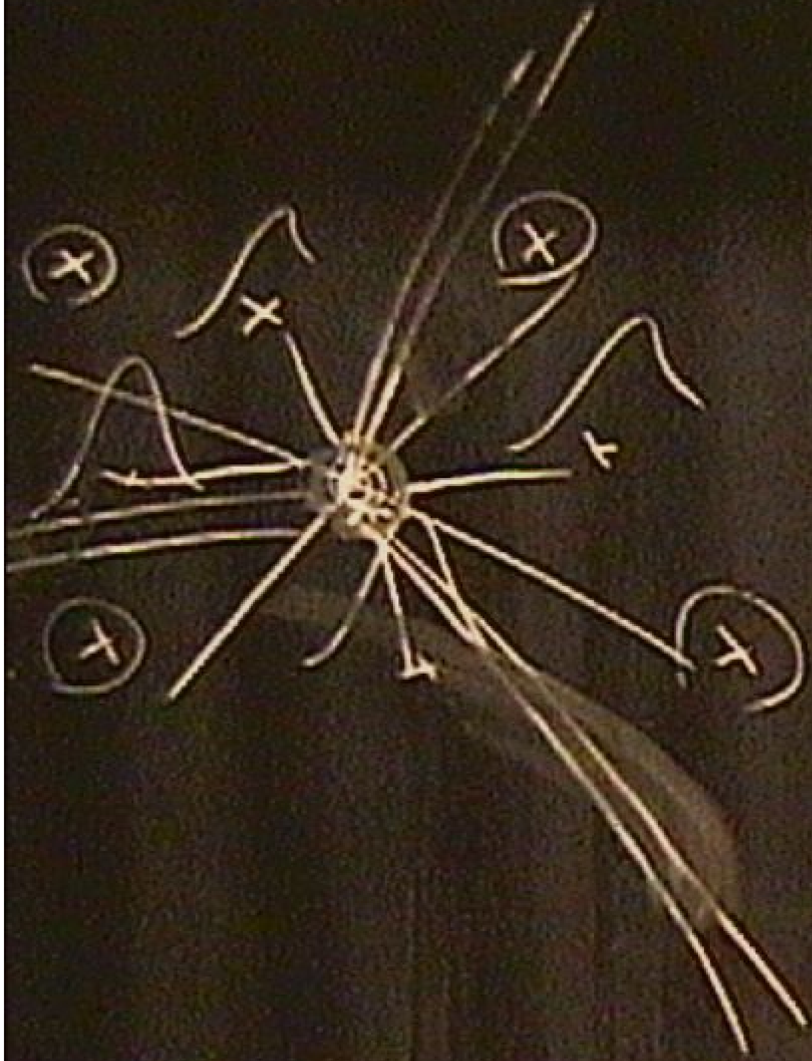

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$$Nk\left(1 - \frac{2}{dk}\right) - k^2 \underset{-1 \; +2}{\underline{+1}}$$

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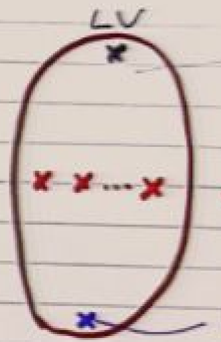
d=N



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d<N × NLSM on $X_{k,N}(G_d)$ ← Fano

$$\Big\downarrow \text{RG}$$

× $C_{k,N}(G_d)/\mathbb{Z}_d$ + × $(N-d)\cdot n(k,N)$ massive vacua on Coulomb branch

d>N × $C_{k,N}(G_d)/\mathbb{Z}_d$

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× NLSM on $X_{k,N}(G_d)$ + × $(d-N)\cdot n(k,N)$ massive vacua on Coulomb

general type

check: $$\hat{c}\Big|_{C_{k,N}(G_d)/\mathbb{Z}_d} = \underbrace{(Nk-k^2-1)}_{\dim X_{k,N}(G_d)} + 2\left(1-\frac{N}{d}\right)$$

Duality

$$\begin{tikzcd} G(k,N) \arrow[r, phantom, "\cong"] \arrow[d, equal] & G(N-k,N) \arrow[d, equal] \\ \{V^k \subset \mathbb{C}^N\} & \{(V^k)^\perp \subset (\mathbb{C}^N)^*\} \end{tikzcd}$$

$$B_{i_1 \cdots i_k} \longleftrightarrow \epsilon_{i_1 \cdots i_k j_1 \cdots j_{N-k}} B'^{j_1 \cdots j_{N-k}}$$

suppose $G_d(B) = G'_d(B')$, then

$$\begin{tikzcd} X_{k,N}(G_d) \arrow[r, phantom, "\cong"] \arrow[d, "RG"'] & X_{N-k,N}(G'_d) \arrow[d, "RG"] \\ C_{k,N}(G_d)/\mathbb{Z}_d & C_{N-k,N}(G'_d)/\mathbb{Z}_d \end{tikzcd}$$

$\underline{d<N}$

$$\rightsquigarrow \quad \underline{C_{k,N}(G_d) \cong C_{N-k,N}(G'_d)}$$

Glop Transition

$U(2)$ $\Phi_1, \cdots, \Phi_7$ fundamentals

$P^1, \cdots, P^7$ $\det^{-1}$'s

$$W = \frac{1}{2}\sum_{i,j,k=1}^{7} A_i^{jk} P^i \Phi_j^a \Phi_k^b \epsilon_{ab} = A(P)^{jk} \Phi_j^1 \Phi_k^2$$

$r \gg 0$ NLSM on $\underset{\substack{\uparrow \\ \text{CY 3-fold.}}}{X} = \{ \underset{\substack{| \\ \text{7 eqn}}}{A_i^{jk} \Phi_j^1 \Phi_k^2 = 0 \quad i=1\cdots 7} \} \subset \underset{\substack{| \\ \text{10-dim}}}{G(2,7)}$

$r \ll 0$ $P \neq 0$: $U(2) \rightarrow SU(2)$

$\hookrightarrow$ span $\{ p \mid \sum_{i=1}^{7} |p_i|^2 = -r \} / U(1) = \mathbb{CP}^6$

$$\boxed{\text{SU(2) gauge theory with 7 fundamentals } / \mathbb{CP}^6 \text{ with varying mass matrix } A(P).}$$

$$\text{rank } \underbrace{A(P)}_{\substack{7\times 7 \\ \text{antisymmetric}}} = \underset{\substack{\uparrow \\ N_{\text{eff}}=1}}{6}, \ \underset{\substack{\uparrow \\ N_{\text{eff}}=3}}{4}, \ \underset{\text{absent if } A_i^{jk} \text{ generic}}{\cancel{2}, \ \cancel{0}}$$

Witten index $\rightsquigarrow$ $SU(2)$ with $N_{eff} = 1$ has no SUSY vacuum

$\therefore$ low energy dynamics localizes on rank $A(p) = 4$ locus in $\mathbb{CP}^6$ $=$ Pfaffian Calabi-Yau Y

Born-Oppenheimer approx:

Slow variables: $p = p_* \in Y$, rank $A(p^*) = 4$

fast variables: everything else, $\bar{\Phi}_i^a$, $\underset{\substack{\uparrow \\ \text{Transverse to } Y}}{\delta P_{i=1,2,3}}$, $SU(2)$

Φ's that acquire mass from $A(p^*)$ decouple

$\longrightarrow$ rename: $\bar{\Phi}_{i=1,2,3}$ are massless. Then

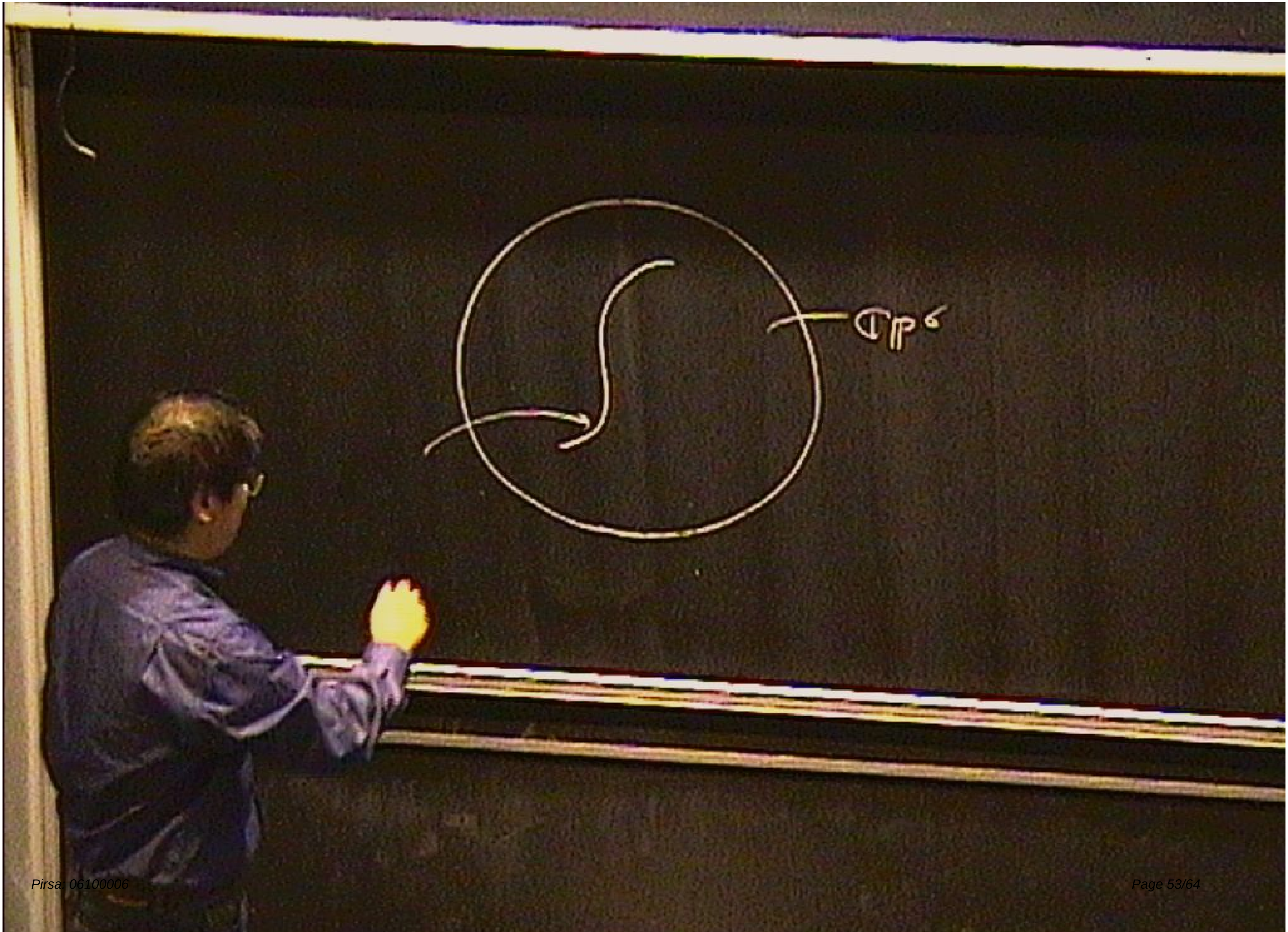
$$W_{fast} = \sum_{i,j=1}^{3} A(\delta P)^{ij} \bar{\Phi}_i^1 \bar{\Phi}_j^2$$

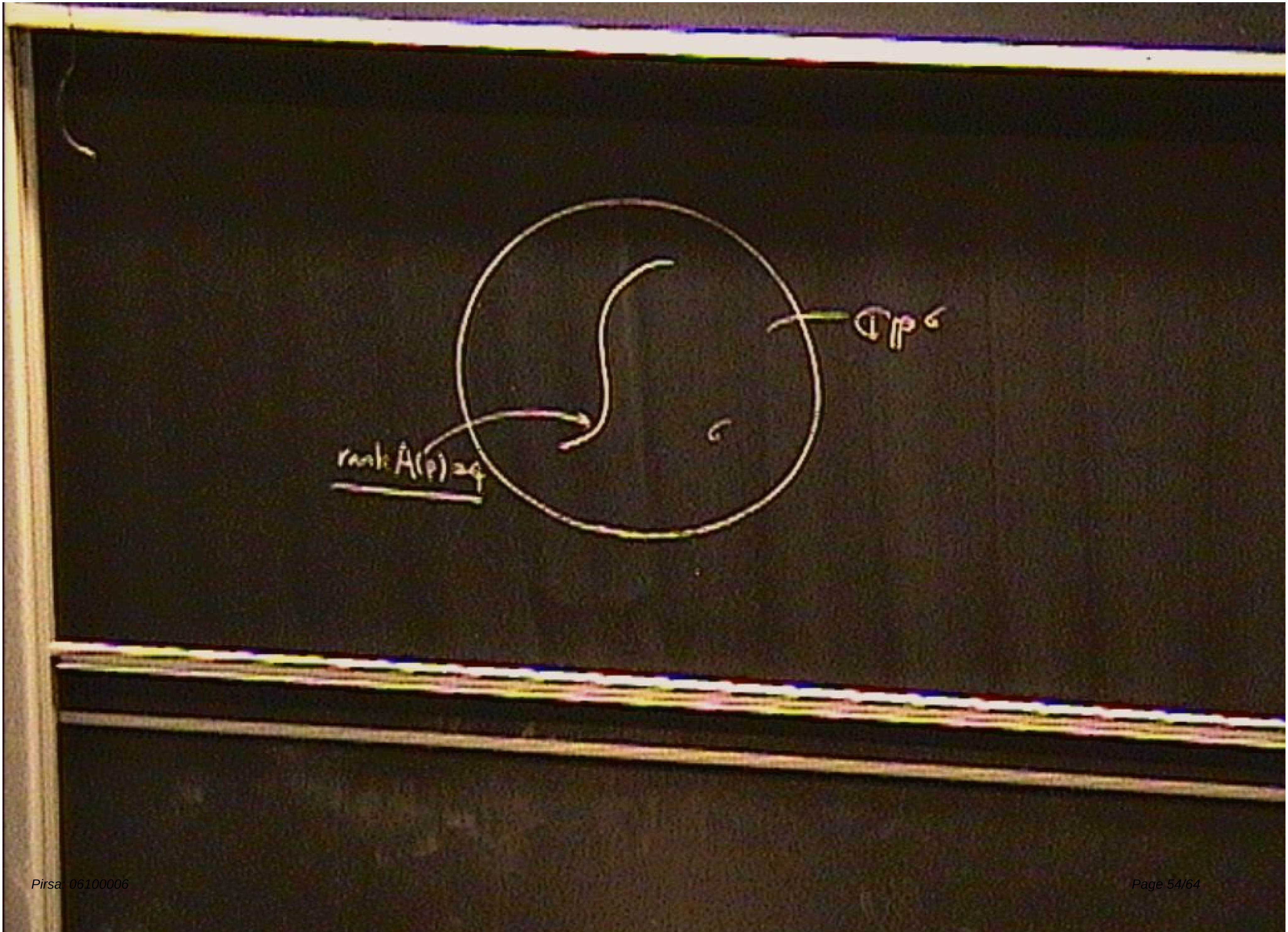
$\xleftrightarrow{\text{IR dual}}$ LG model with 6 variables δP_i, B_{ij}

$$W_{dual} = \sum_{i,j=1}^{3} A(\delta P)^{ij} B_{ij} \underset{\substack{\uparrow \\ \text{def of } \delta P_i}}{=} \delta P_1 B_{23} + \delta P_2 B_{31} + \delta P_3 B_{12}$$

$\rightarrow$ $\exists!$ **SUSY ground state with mass gap.**

B.O. approx valid.


$\mathbb{CP}^6$


$\mathbb{CP}^6$
rank $A(p) \geq 4$
6

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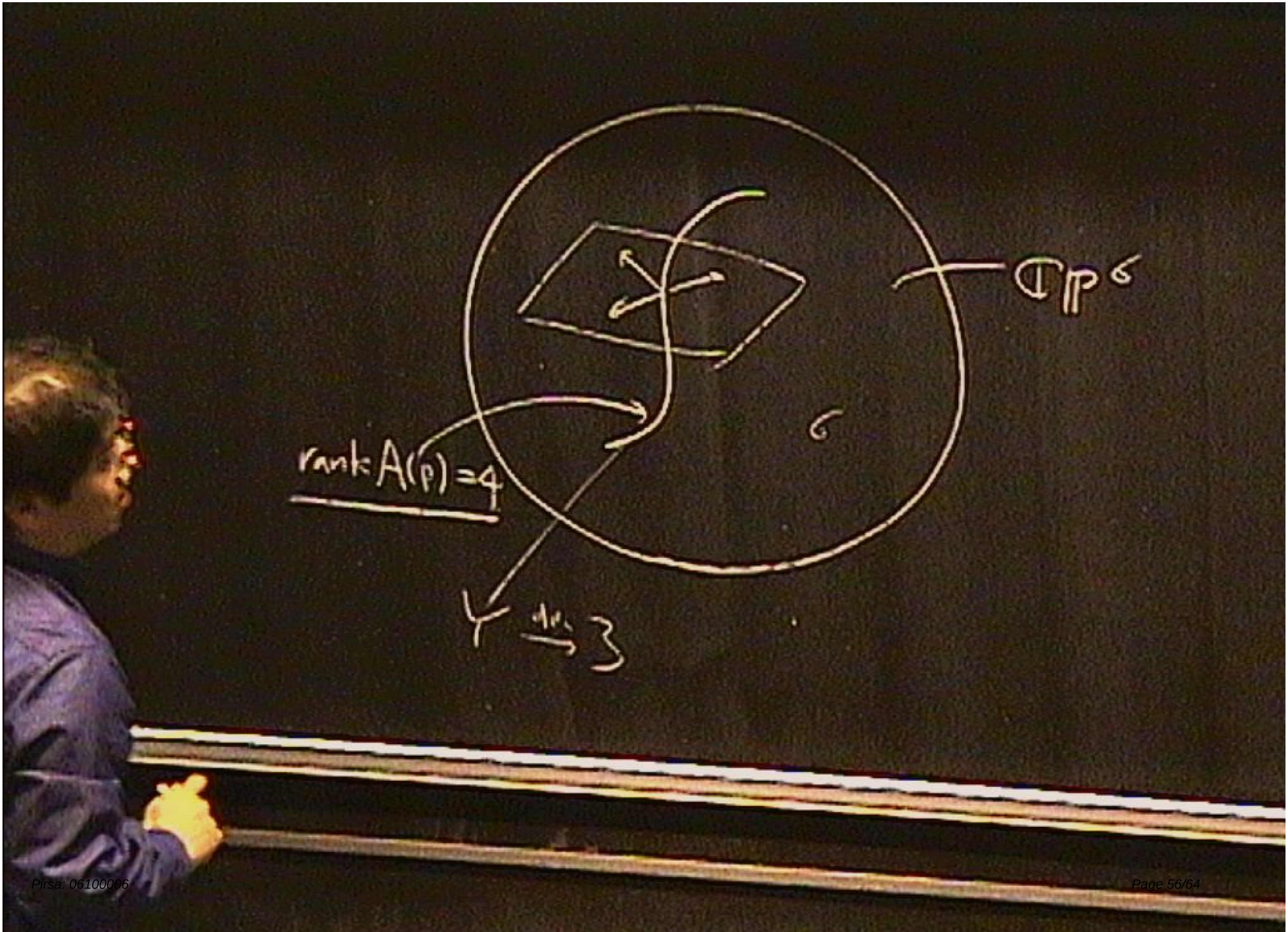
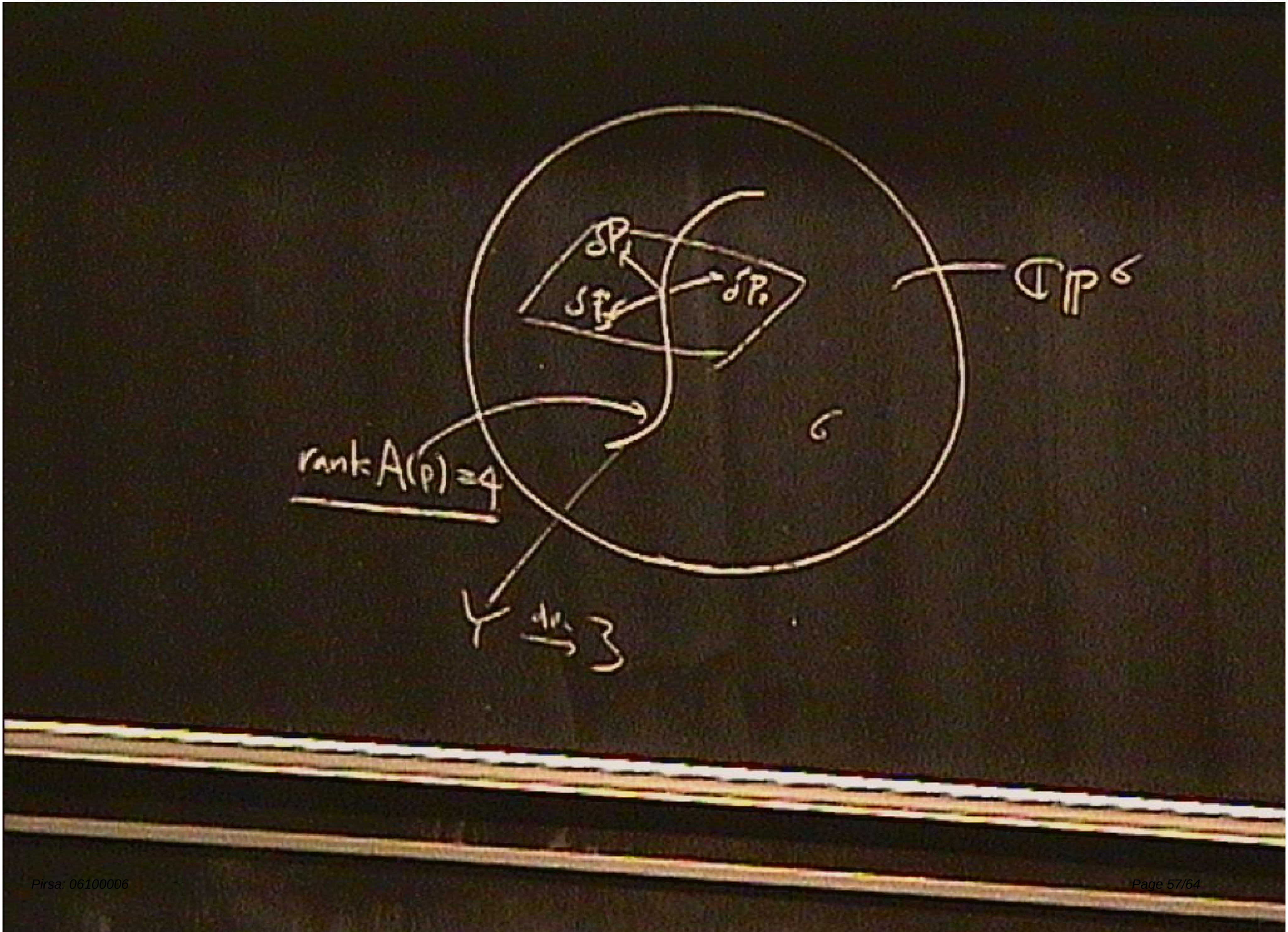

$\mathbb{CP}^6$
σ
rank A(p) = 4
$Y \xrightarrow{\dim} 3$


δP_1
δP_3
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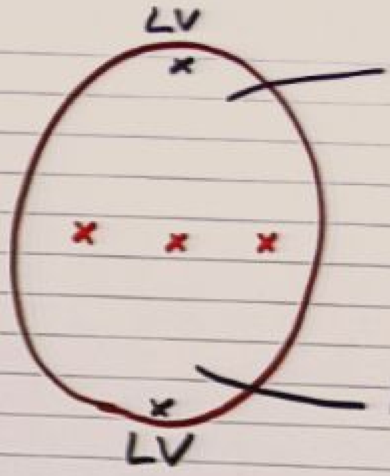


NLSM on X (CY in $G(2,7)$)

3 singular points

NLSM on Y (Pfaffian CY)

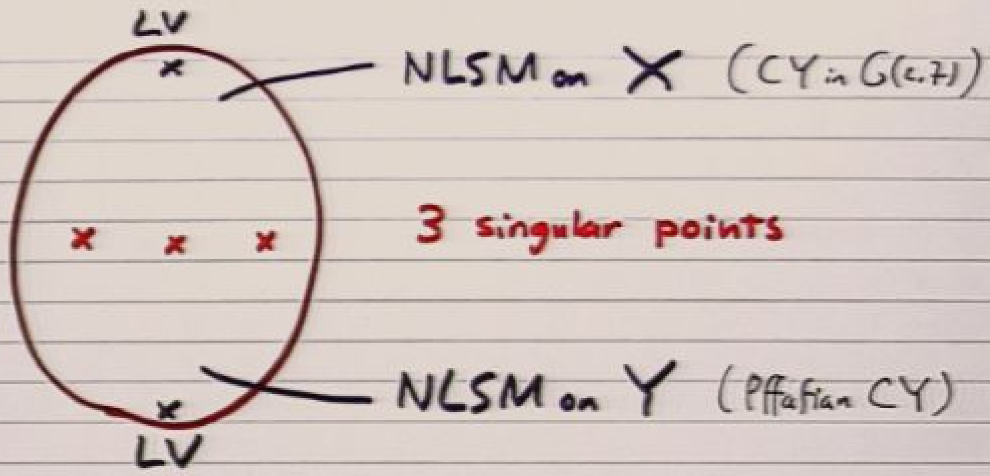
Looks like a flop transition, but $X \rightleftarrows Y$ is not a local operation. (X and Y are not birationally equivalent).

— G(rassmannian) lop

✱ Consequence: The categories of B-branes on X and Y must be equivalent

$$D^b(\mathrm{Coh}\,X) \underset{\text{equiv}}{\approx} D^b(\mathrm{Coh}\,Y)$$

Surprize since $X \underset{\text{birationally}}{\not\simeq} Y$. Proved by Borisov-Caldararu



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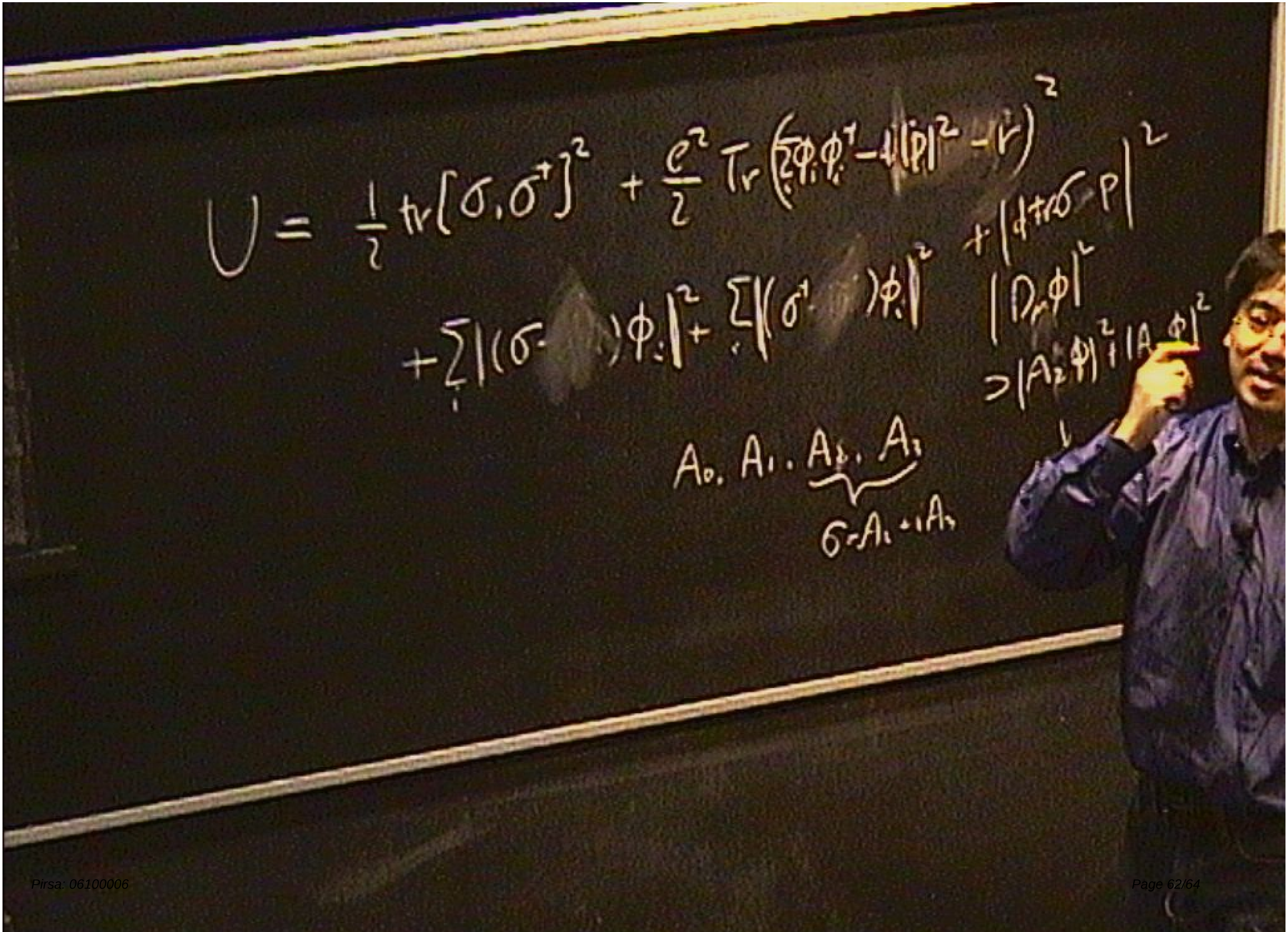


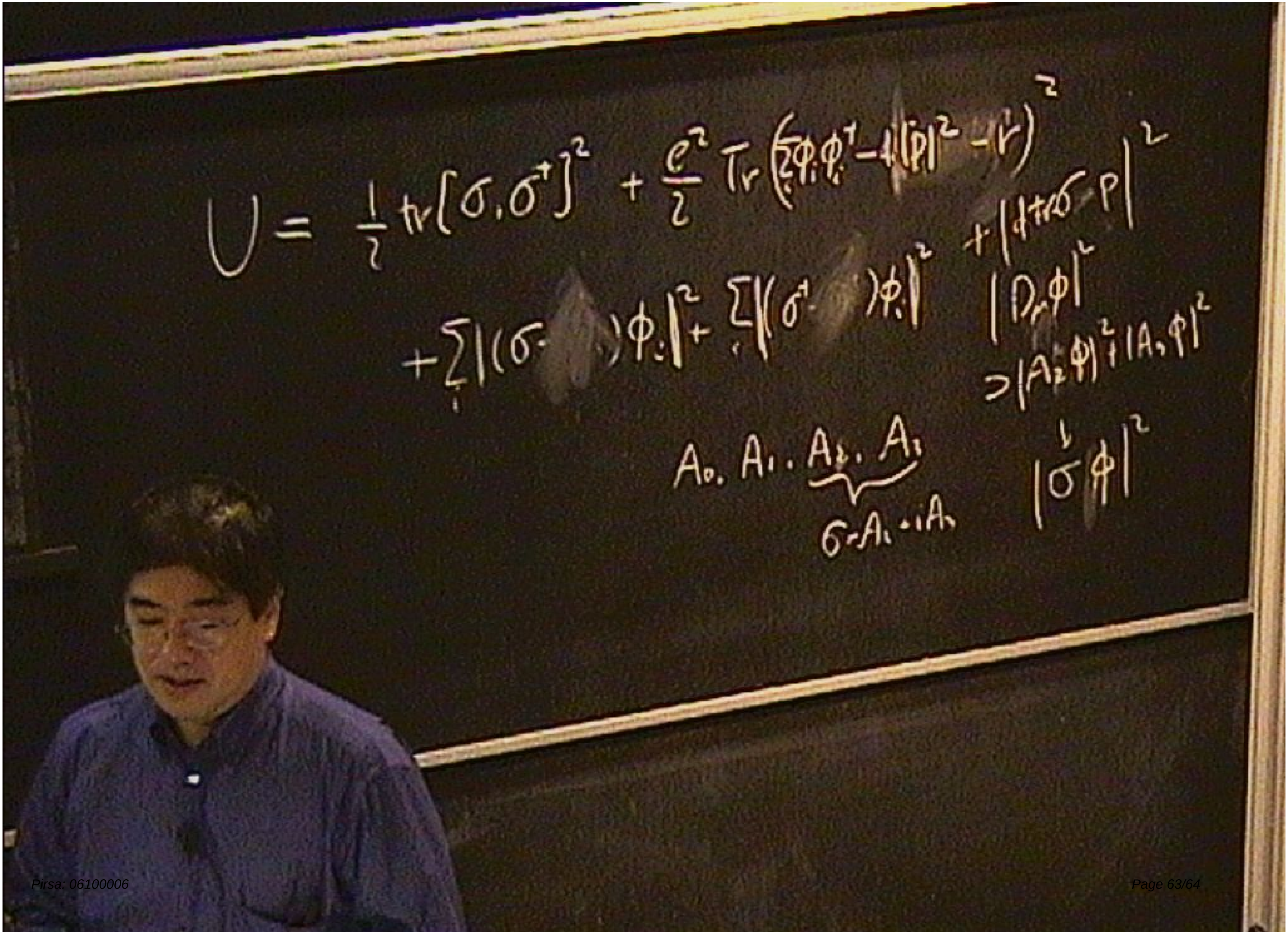
$$U = \frac{1}{2}\mathrm{tr}[\sigma,\sigma^\dagger]^2 + \frac{e^2}{2}\mathrm{Tr}\left(\sum_i \phi_i\phi_i^\dagger - |\tilde{p}|^2 - r\right)^2 + |(d + \sigma)p|^2$$

$$+ \sum_i |(\sigma - \quad)\phi_i|^2 + \sum_i |(\sigma^\dagger - \quad)\phi_i|^2$$

$$|D_\mu\phi|^2$$

$$\supset |A_2\phi|^2 + |A_3\phi|^2$$

$$A_0,\ A_1,\ \underbrace{A_2,\ A_3}_{\sigma = A_2 + iA_3}$$



$$U = \frac{1}{2}\mathrm{tr}[\sigma,\sigma^\dagger]^2 + \frac{e^2}{2}\mathrm{Tr}\left(\sum_i \phi_i\phi_i^\dagger - |\tilde{p}|^2 - r\right)^2 + |(\ \ \ )\,p|^2$$

$$+\sum_i |(\sigma - \ \ )\phi_i|^2 + \sum_i |(\sigma^\dagger - \ \ )\phi_i|^2$$

$$|D_\mu\phi|^2$$

$$\supset |A_2\phi|^2 + |A_3\phi|^2$$

$$A_0,\ A_1,\ \underbrace{A_2,\ A_3}_{\sigma = A_2 + iA_3}$$

$$|\overset{\downarrow}{\sigma}\phi|^2$$

NLSM on X (CY in G(2,7))

3 singular points

NLSM on Y (Pfaffian CY)

Looks like a flop transition, but $X \rightleftarrows Y$ is <u>not</u> a local operation (X and Y are not birationally equivalent).

— G(rassmannian) flop

Consequence: The categories of B-branes on X and Y must be equivalent

$$D^b(\mathrm{Coh}\,X) \underset{\text{equiv}}{\approx} D^b(\mathrm{Coh}\,Y)$$

surprize since $X \underset{\text{birationally}}{\not\cong} Y$. Proved by Borisov-Caldararu, Kuznetsov