Title: QFT with minimum length

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Abstract:

Quantum Field Theory with a Minimal Length

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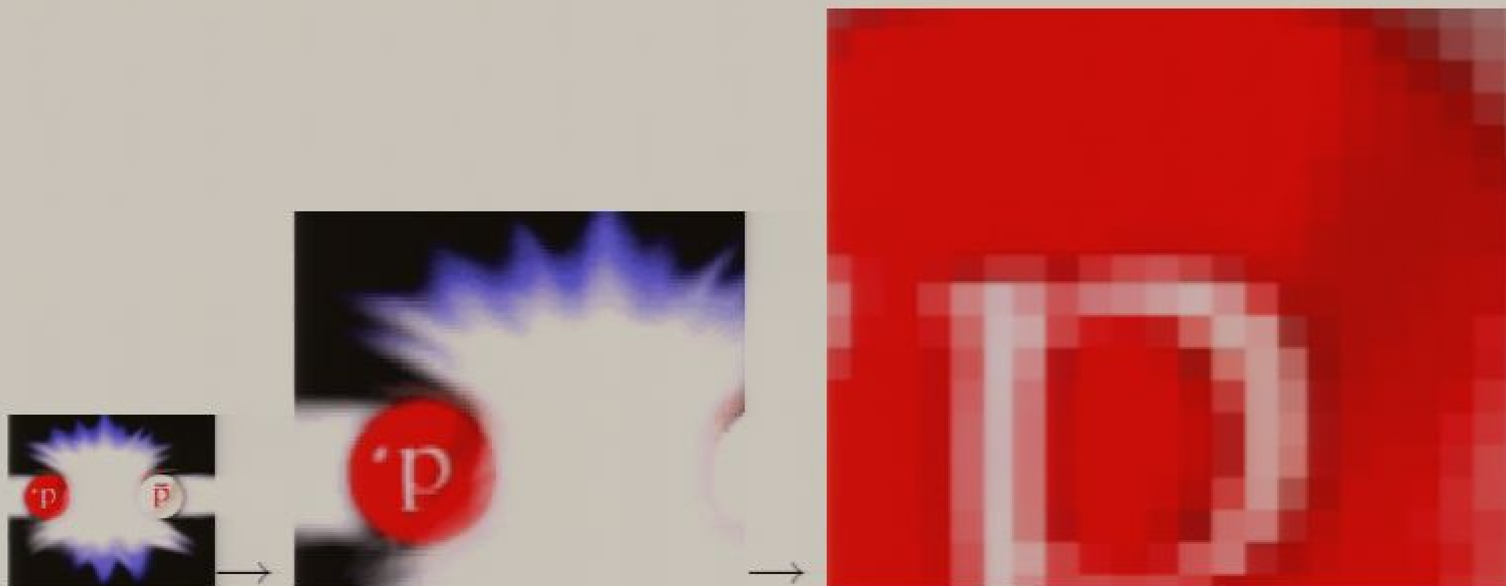
Based on

Phys. Lett. B **575**, 85 (2003) [arXiv:hep-th/0305262]
Class. Quantum Grav. 23 (2006) 1815 [arXiv:hep-th/0510245]
Phys. Rev. D **73** 105013 (2006) [arXiv:hep-th/0603032]

The Minimal Length Scale



- Very general expectation for quantum gravity: fluctuations of spacetime itself disable resolution of small distances
- Can be found e.g. in String Theory, Loop Gravity, NCG, etc.
- Minimal length scale acts as UV cutoff

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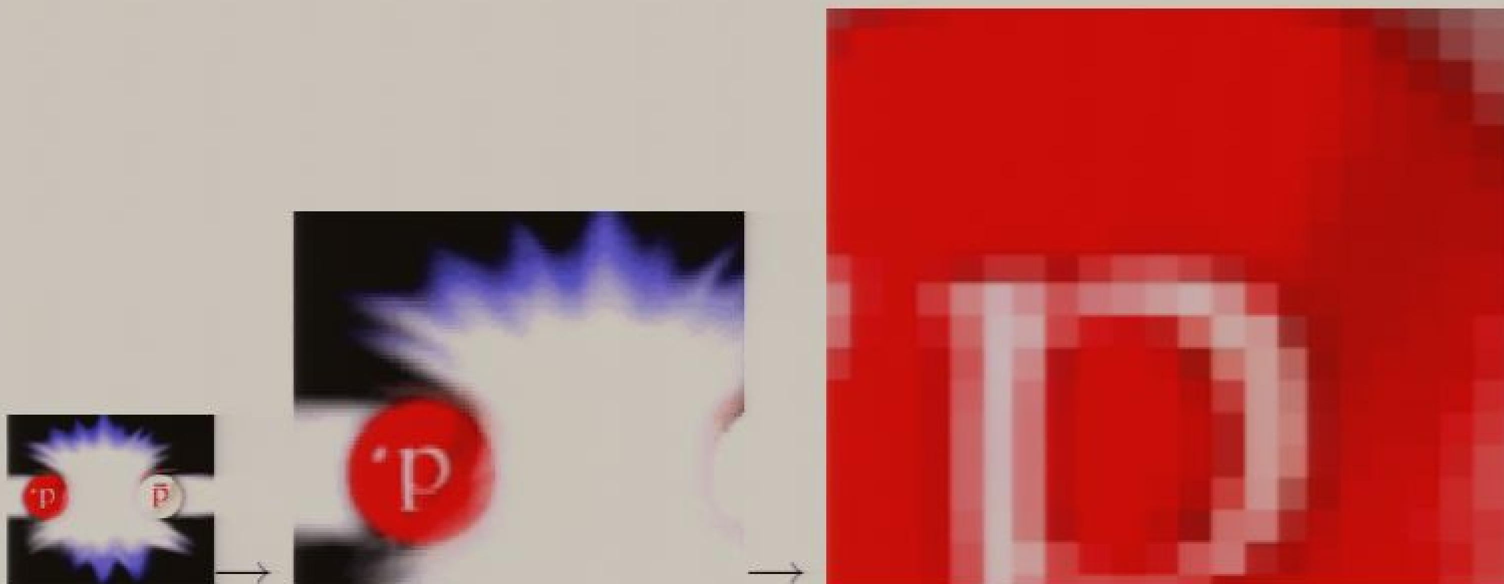

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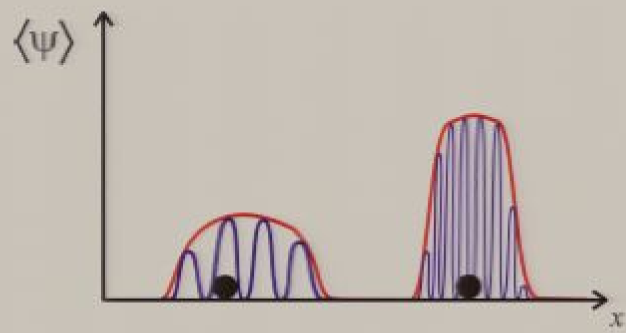
With Large Extra Dimensions

- Lowering the Planck mass means raising the Planck length!
- Effects of a finite resolution become important at the same energy scale as other new signatures.
- Relevance for high energy and high precicision experiments.

$$m_{\mathrm{p}^2} = M_{\mathrm{f}}^{d+2} R^d$$

Finite Resolution of Structures



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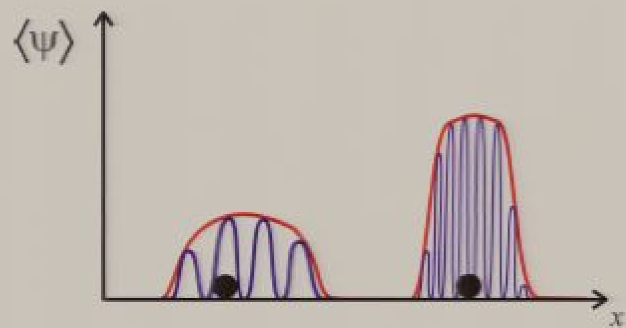
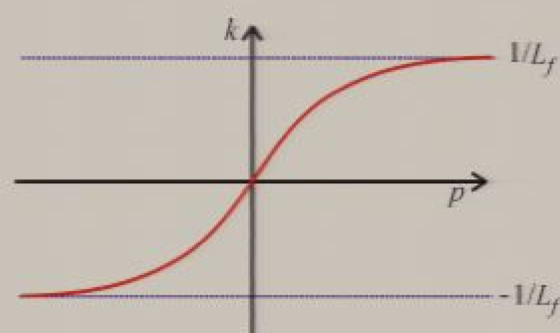


- For large momenta, p, Compton-wavelength $\lambda = 1/k$ can not get arbitrarily small $\lambda > L_{\mathrm{min}} = 1/M_{\mathrm{f}}$



A Model for the Minimal Length

- Modify wave-vector k and commutation relations $k = k(p) = \hbar p + a_1 p^3 + a_2 p^5 \ldots \Rightarrow [p_i, x_j] = i \partial p_i / \partial k_j$
- Results in a generalized uncertainty principle

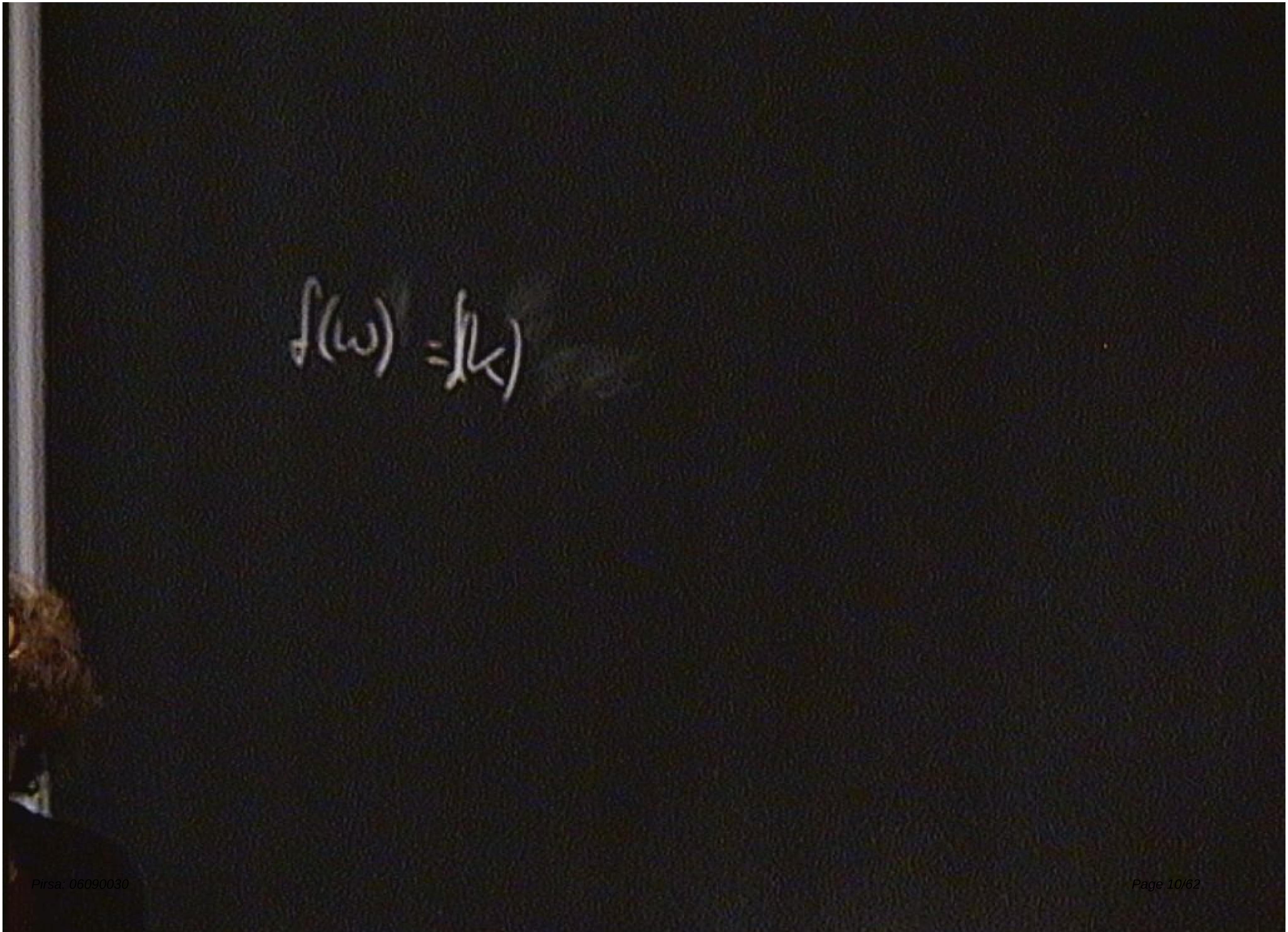
$$\Delta x \Delta p \geq \frac{1}{2} \hbar \left(1 + b_1 L_{\rm min}^2 \langle p^2 \rangle \right)$$

- And a squeezed phase space at high energies

$$\langle p | p' \rangle = \frac{\partial p}{\partial k} \delta(p - p') \Rightarrow dk \rightarrow \hbar \; dp \; \frac{\partial k}{\partial p} = \hbar \; dp \; e^{-L_{\rm min}^2 p^2}$$

- Can but need not have a varying speed of light $d\omega / dk \neq 1$.

$$\omega^2 = k^2$$


f(ω) = f(k)

$$f(\omega) = f(k)$$

$$\omega^2 - f(\omega)k^2 = 0$$

Two Approximations

1. First order expansion

$$k \approx p + \frac{\beta}{3} p \left(\frac{p}{M_f} \right)^2 \; , \; \frac{\partial k}{\partial p} \approx 1 + \beta \left(\frac{p}{M_f} \right)^2$$

2. High energy limit

$$k \approx M_{\mathrm{f}} \, \mathrm{Erf} \left(- \frac{p}{M_{\mathrm{f}}} \right) \; , \; \frac{\partial k}{\partial p} \approx \exp \left(- \frac{p^2}{M_{\mathrm{f}}^2} \right)$$

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Quantisation with a Minimal Length

Parametrization in $p_\nu = f_\nu(k)$ – input from underlying theory.

- Quantize via $k \to -\mathrm{i}\partial$, $p \to f(-\mathrm{i}\partial) := F(\partial)$

$$p \quad = \quad k + \sum_{n\geq 1} a_n k^{2n+1} \ , \ F(\partial) = -\mathrm{i}\partial + \sum_{n\geq 1} a_n (-\mathrm{i})^{2n+1} \partial^{2n+1}$$

- The Klein-Gordon equation, alias modified dispersion relation

$$E^2 - p^2 = m^2 \quad \Rightarrow \quad F^\nu(\partial) F_\nu(\partial)\psi = m^2 \psi$$

- The Dirac equation

$$(\not{F}(\partial) - m)\psi \quad = \quad 0$$

- (Anti)-commutation relations

$$\left[\hat{a}^\dagger(p), \hat{a}(p')\right]_\pm \quad = \quad \delta(\vec{p} - \vec{p}\,') \left| \frac{\partial p}{\partial k} \right|$$

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Field Expansion

- Express everything in momentum-space

$$\phi = \sum\!\!\!\!\!\!\int dk \left(a_k v_k + a_k^\dagger v_k^* \right)$$

$$\phi = \sum\!\!\!\!\!\!\int dp \left| \frac{\partial k}{\partial p} \right| \left(a_p v_p + a_p^\dagger v_p^* \right)$$

- Momenta are observables, have usual properties, and final results are expressed in terms of momentum variables
- In intermediate steps (strong curvature regime), the relation between wave-vector and momentum is modified

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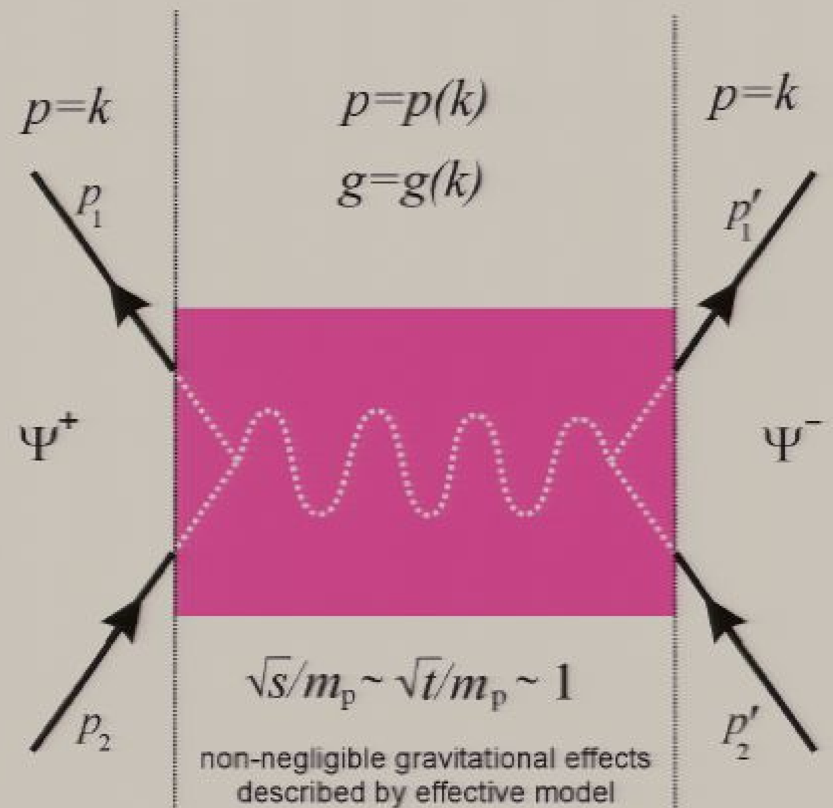
The Collision Region



Alternative Description with Energy Dependent Metric*

- Even on a classical level, a particle's energy influences the metric it propagates in

$$R_{\kappa\nu} - \frac{1}{2} g_{\kappa\nu} = \frac{1}{m_{\rm p}^2} T_{\kappa\nu}$$

- On a quantum level, expect this dependence to be dominant
- Re-write modified dispersion relation in collision region as

$$g^{\kappa\nu}(k) k_\kappa k_\nu = \mu^2$$

- At small energies $g^{\kappa\nu} \sim \eta^{\kappa\nu}$

*Kimberly, Magueijo and Medeiros, Phys. Rev. D **70**, 084007 (2004).

Quantisation with a Minimal Length

- Lagrangian for free fermions

$$\mathcal{L}_f = \mathrm{i}\overline{\Psi}(\not{F}(k) - m)\Psi \qquad \mathcal{L}_f = \mathrm{i}\overline{\Psi}(g^{\nu\kappa}(k)\gamma_\nu k_\nu - m)\Psi$$

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- To first order one finds the usual $\mathcal{L} = \mathcal{L}_f - e\overline{\Psi}\eta^{\kappa\nu}\gamma_\kappa A_\nu\Psi + \mathcal{O}(eL_{\mathrm{min}}^2)$ and the dominant modification comes from the propagators

$$(\not{F}(k) - m)^{-1} \qquad (g^{\nu\kappa}(k)\gamma_\nu k_\kappa - m)^{-1}$$
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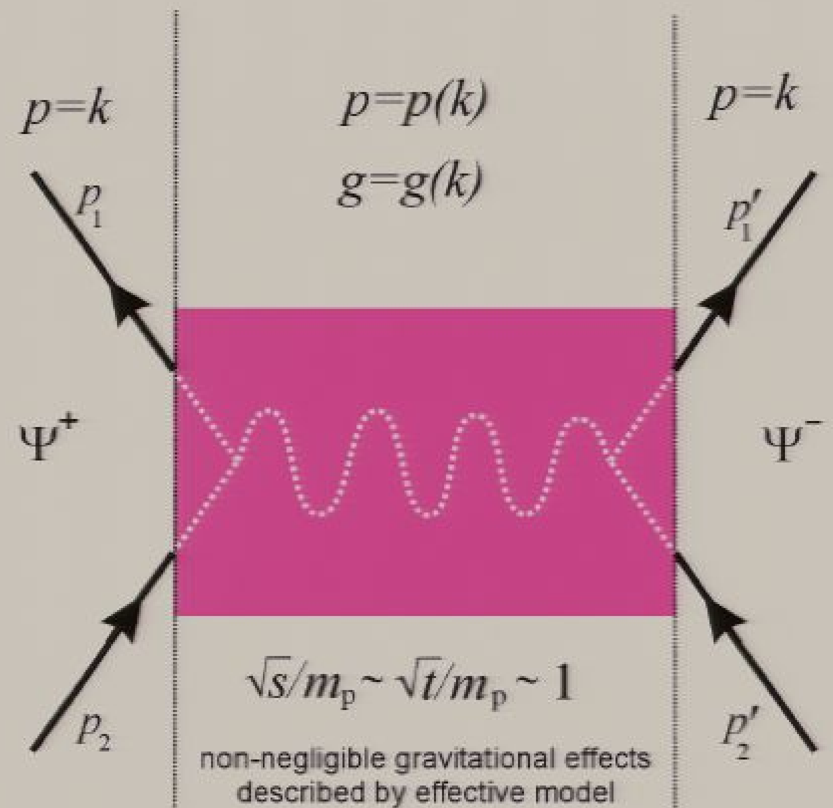

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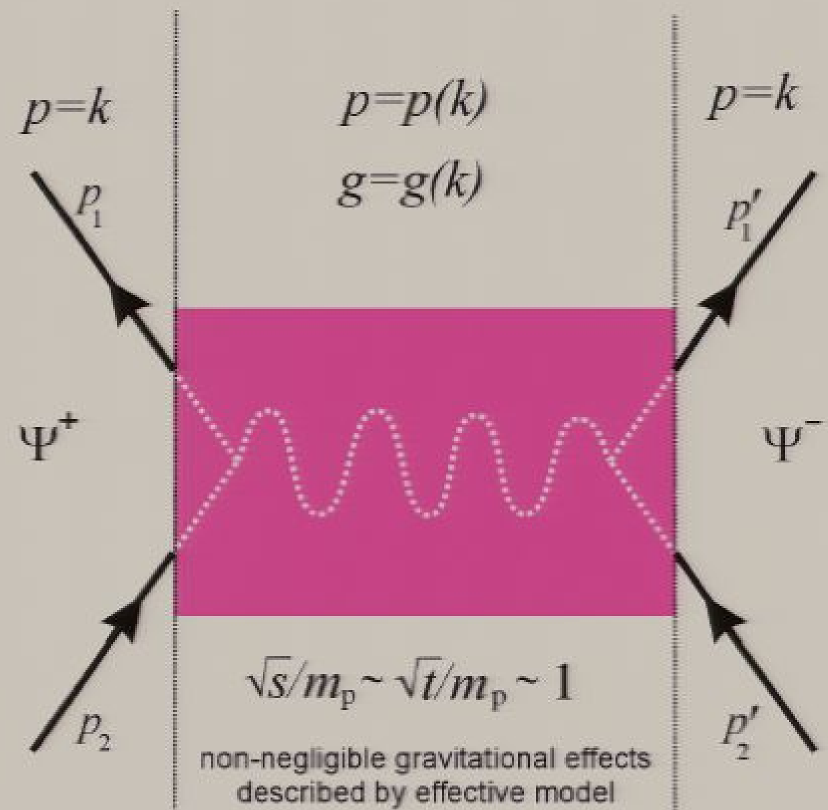

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- Note: $\delta = F(\partial)$ has higher order derivatives $\longrightarrow$ messy Euler-Lagrange formalism
- Make sure that $\mathcal{L} = \delta^\mu\phi\delta_\mu\phi$ indeed gives desired equations of motion
- Requires shifting of derivatives
- It can be shown that $\phi_\mu(\delta^\mu\psi) = -(\delta^\mu\phi_\mu)\psi +$ total divergence and everything works with $\delta\phi$ as conjugated field
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Feynman rules

- Replace every

$$dp \quad \text{with} \quad dp \left| \frac{\partial k}{\partial p} \right|$$

- Keep vertices
- Make sure to express energy/momentum conservation in terms of momenta

$$\delta(\mathbf{p}_\mathrm{i} - \mathbf{p}_\mathrm{f})$$

- Keep in mind: for asymptotic in/out-going fields applies the usual translation invariance

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The Locality Bound*

From the commutator

$$[a_p, a^\dagger_{p'}] = \delta(p - p') \left| \frac{\partial p}{\partial k} \right|$$

And the field expansion

$$\phi(x) = \int \mathrm{d}^3 p \left| \frac{\partial k}{\partial p} \right| \left[v_p(x) a_p + v^*_p(x) a^\dagger_p \right]$$

One finds the equal time commutator for $x = (\mathbf{x}, t), y = (\mathbf{y}, t)$.

$$[\phi(x), \pi(y)] = \mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left| \frac{\partial k}{\partial p} \right| \mathrm{e}^{\mathrm{i}k(x-y)} \to \mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{\mathrm{i}k(x-y) - \varepsilon p^2}$$

where $\varepsilon \sim L^2_{\mathrm{min}}$.

*Giddings and Lippert, Phys. Rev. D **65**, 024006 (2002), Phys. Rev. D **69**, 124019 (2004).

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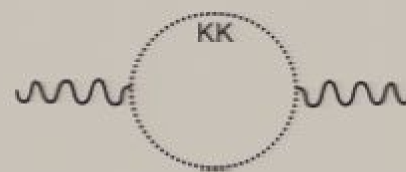
The Minimal Length as UV-Regulator

- Minimal length regularizes loop-integrals

$$\int d^4p \frac{1}{p^2(p-q)^2} \sim \ln\frac{\Lambda}{\mu_0} \quad \text{for} \quad d=0$$

$$\int d^{4+d}p \frac{1}{p^2(p-q)^2} \sim \left(\frac{\Lambda}{\mu_0}\right)^d \quad \text{for} \quad d>1?$$

$$\rightarrow \int d^{4+d}p \left|\frac{\partial k}{\partial p}\right| \frac{1}{p^2(p-q)^2} < \infty$$



- Regulararizes also higher-dimensional QFT's.
- Natural interpretation of regulator.

SH, Phys. Rev. D 70 (2004) 105003.

The Minimal Length as UV-Regulator

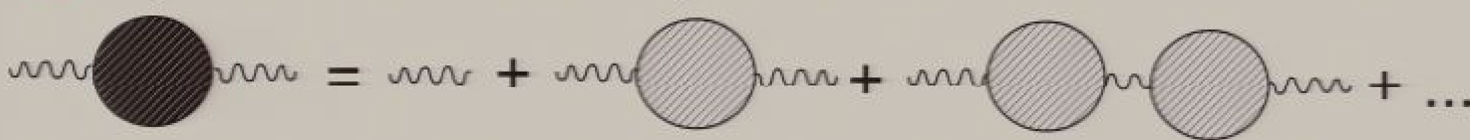
Example: Running Coupling

- Looking close, the propagator exhibits complex structures

$$D^{\mu\nu}(q) = -g^{\mu\nu}/q^2$$

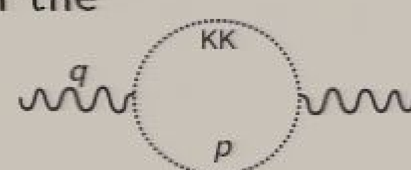
$$\to \widetilde{D}^{\mu\nu}$$

- Expansion in series of one-particle irreducible contributions



$$i\widetilde{D}^{\mu\nu} = iD^{\mu\nu} + i\widetilde{D}^{\mu\lambda}[ie^2\Pi_{\lambda\sigma}]i\widetilde{D}^{\sigma\nu} + i\widetilde{D}^{\mu\lambda}[ie^2\Pi_{\lambda\sigma}]i\widetilde{D}^{\sigma\kappa}[ie^2\Pi_{\kappa\rho}]i\widetilde{D}^{\rho\nu} + \dots$$

- Can be summarized in $[q^2 g_{\mu\nu} - e^2\Pi_{\mu\nu}]\widetilde{D}^{\nu}_{\lambda} = -g_{\mu\lambda}$
- Loop-integral is dampened at high energies from squeezing of the momentum space measure $\longrightarrow$ becomes finite



The Minimal Length as UV-Regulator

Example: Running Coupling

- Feynman rules yield

$$\Pi_{\mu\nu}(q,d) = e^2 \frac{(2\pi)^d}{\Omega_d} \varepsilon^{d/2} \int \frac{d^{4+d}p}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma_\mu}{\not p}\right]\left[\frac{\gamma_\nu}{\not p - \not q}\right] e^{-\varepsilon p^2} \quad ,$$

- After Wick-rotation

$$\pi(q,d) = 3b_i \frac{\alpha_i}{2\pi} \frac{(2\pi)^d}{\Omega_d} (\pi\varepsilon)^{d/2} \left[\int_0^1 \mathrm{d}x x (1-x)^{1+d/2} \int_\varepsilon^\infty \mathrm{d}z \, e^{-zxq^2} z^{-1-d/2} + \frac{1}{q^2} \frac{(d+4)}{2(d+3)} \varepsilon^{-1-d/2} \int_0^1 \mathrm{d}x (1-x)^{1+d/2} \left(e^{-\varepsilon x q^2} - 1 \right) \right]$$

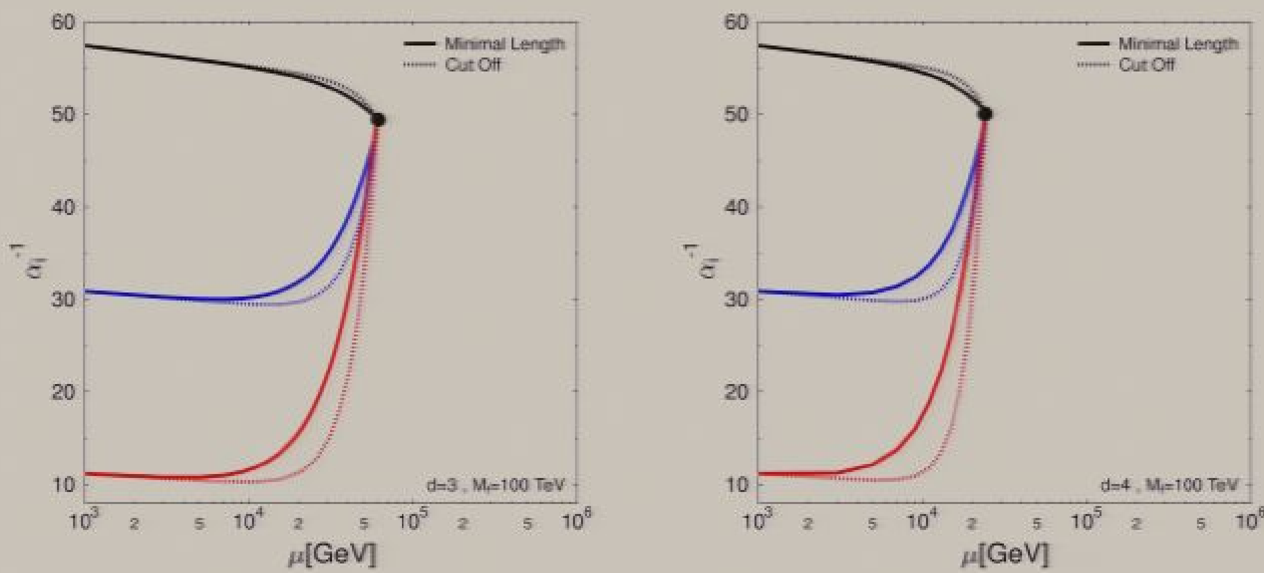
with $\Pi_{\mu\nu}(q,d) = \pi(q,d)(q_\mu q_\nu - g_{\mu\nu} q^2)$ and $\varepsilon \sim L_{\rm min}^2$.

- In 4-dim (d=0) Schwinger parametrization, results in std. log-running.

The Minimal Length as UV-Regulator

Example: Running Coupling

- Interpretation: finite resolution of vacuum structure.
- In extra dimensions, power-law running accelerates unification



Dienes, Dudas and Gherghetta, Phys. Lett. B **436**, 55 (1998), SH, Phys. Rev. D **70** (2004) 105003.

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$$\Pi_{\mu\nu}(q,d) = e^2 \frac{(2\pi)^d}{\Omega_d} \varepsilon^{d/2} \int \frac{d^{4+d}p}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma_\mu}{\not p}\right]\left[\frac{\gamma_\nu}{\not p - \not q}\right] e^{-\varepsilon p^2} \quad ,$$

- After Wick-rotation

$$\begin{aligned} \pi(q,d) &= 3b_i \frac{\alpha_i}{2\pi} \frac{(2\pi)^d}{\Omega_d} (\pi\varepsilon)^{d/2} \left[\int_0^1 \mathrm{d}x x (1-x)^{1+d/2} \int_\varepsilon^\infty \mathrm{d}z \, e^{-zxq^2} z^{-1-d/2} \right. \\ &+ \left. \frac{1}{q^2} \frac{(d+4)}{2(d+3)} \varepsilon^{-1-d/2} \int_0^1 \mathrm{d}x (1-x)^{1+d/2} \left(e^{-\varepsilon x q^2} - 1 \right) \right] \end{aligned}$$

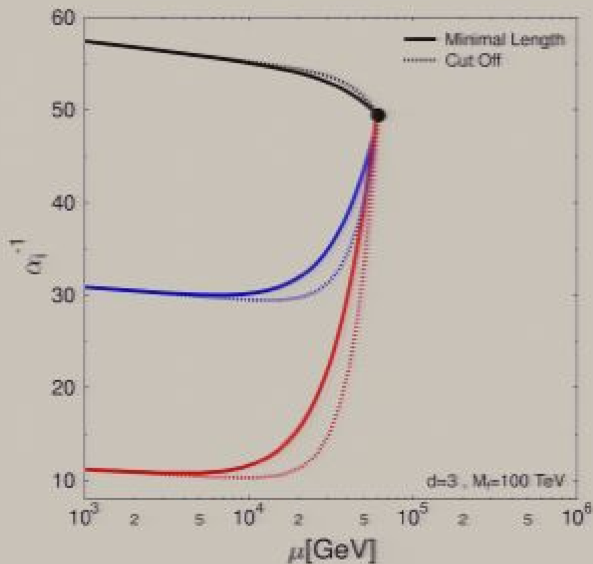
with $\Pi_{\mu\nu}(q,d) = \pi(q,d)(q_\mu q_\nu - g_{\mu\nu} q^2)$ and $\varepsilon \sim L_{\mathrm{min}}^2$.

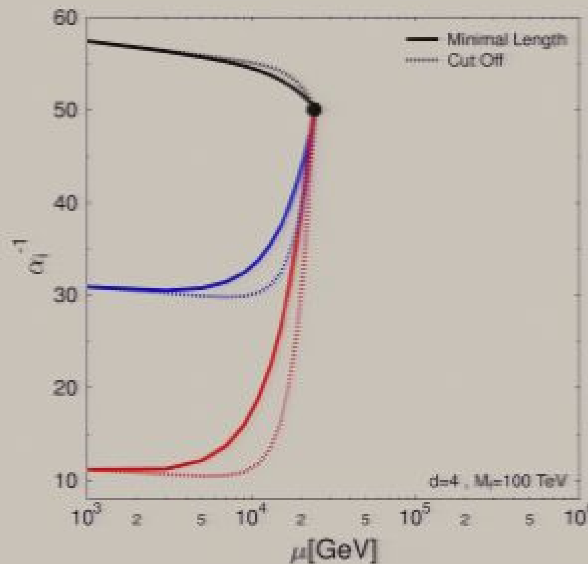
- In 4-dim (d=0) Schwinger parametrization, results in std. log-running.

The Minimal Length as UV-Regulator

Example: Running Coupling

- Interpretation: finite resolution of vacuum structure.
- In extra dimensions, power-law running accelerates unification





Dienes, Dudas and Gherghetta, Phys. Lett. B **436**, 55 (1998), SH, Phys. Rev. D **70** (2004) 105003.

Tree-Level Amplitudes

- S-Matrix elements

$$\widetilde{S}_{\rm fi} = (2\pi)^4 \widetilde{M}_{\rm fi} \delta(\underline{\mathbf{p}}_{\rm i} - \underline{\mathbf{p}}_{\rm f}) \left| \frac{\partial p}{\partial k} \right|_{\underline{\mathbf{p}}_{\rm i} = \underline{\mathbf{p}}_{\rm f}}$$

where the amplitude is unmodified.

- This yields ($2 \to 2$ process in lab frame) the diff. cross-sections

$$d\tilde{\sigma}(\mathrm{i} \to \mathrm{f}) = \hbar^2 (2\pi)^4 |\tilde{M}_{\rm fi}|^2 \frac{E_{\rm f1} E_{\rm f2}^3}{m E_{\rm i2}} \left| \frac{\partial k}{\partial p} \right| d\Omega \quad .$$

- Ratio of total cross-section relative to std. result

$$\frac{d\tilde{\sigma}}{d\sigma} = \prod_n \frac{E_n}{\omega_n} \left| \frac{\partial k}{\partial p} \right|_{\underline{\mathbf{p}}_{\rm i} = \underline{\mathbf{p}}_{\rm f}}$$

Deformed Special Relativity

- Minimal length $L_{\min}$ requires new Lorentz-transformations
- New transformations have 2 invariants: c and $L_{\min}$
- Generalized Uncertainty $\Longleftrightarrow$ Deformed Special Relativity
  - * When relation $k(p)$ is known and p's (usual) transformation, then also the transformation of k is known.
  - * When the new transformation on k is known, then one gets $k(p)$ by boosting in and out of the restframe where $k = p$.

SH, Class. Quantum Grav. 23 (2006) 1815.

Deformed, Non-linear Action on Momentum Space

- Lorentz-algebra remains unmodified

$$[J^i, K^j] = \varepsilon^{ijk} K_k \ , \ [K^i, K^j] = \varepsilon^{ijk} K_k \ , \ [J^i, J^j] = \varepsilon^{ijk} J_k$$

- But it acts non-linearly on momentum space, e.g.*

$$e^{-iL_{ab}\omega^{ab}} \to U^{-1}(p_0) e^{-iL_{ab}\omega^{ab}} U(p_0) \quad \text{with} \quad U(p_0) = e^{L_{\min} p_0 p_a \partial p^a}$$

- Leads to Lorentz-boost (z-direction)

$$p_0' = \frac{\gamma(p_0 - v p_z)}{1 + L_{\min}(\gamma - 1) p_0 - L_{\min} \gamma v p_z}$$

$$p_z' = \frac{\gamma(p_z - v p_0)}{1 + L_{\min}(\gamma - 1) p_0 - L_{\min} \gamma v p_z}$$

which transforms $(1/L_{\min}, 1/L_{\min}) \to (1/L_{\min}, 1/L_{\min})$

*Magueijo and Smolin, Phys. Rev. Lett. 88, 190403 (2002).

Interpretation of an Invariant Minimal Length

Besides c there is a second invariant $L_{\rm min}$ for all observers

- Standard DSR approach:
  - * DSR applies for each observer to agree on minimal-ness ... ?
  - * Therefore deformed transformation applies to free particles
  - * If caused by quantum gravity effects what sets the scale?
- Here:
  - * Two observers can not compare lengths without interaction
  - * The strength of gravitational effects sets the scale for the importance of quantum gravity
  - * Free particles do not experience any quantum gravity or DSR
  - * Effects apply for particles in the interaction region only

$\longrightarrow$ Propagator of exchange particles is modified

Problems of DSR

The Soccer-Ball-Problem

* Usual DSR: A particle has an upper bound on its energy. There better were no such bound for composite objects. How to get the correct multi-particle limit for DSR-trafos?
* Here: Composite objects don't experience anything funny as long as the gravitational interaction among the components is weak.

The Conservation-Law Problem

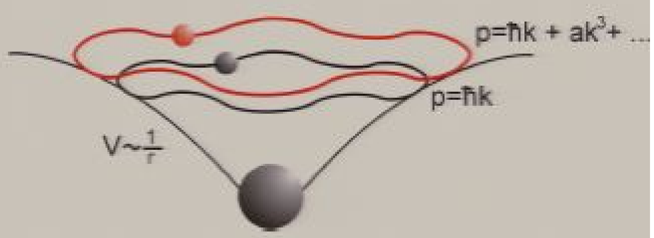
* Usual DSR: the physical momentum p is the one that experiences DSR. But then p is not additive and $f(p_1 + p_2) \neq f(p_1) + f(p_2)$. Which quantity is to be conserved in multi-particle interactions*?
* Here: the physical momenta p_1, p_2 are the asymptotic momenta, they transform, are added, and are conserved in the usual way†.

---

* Judes and Visser, Phys. Rev. D **68**, 045001 (2003) [arXiv:gr-qc/0205067].
† SH, Phys. Rev. D **15** ?????? (2006) [arXiv:hep-th/0603032]

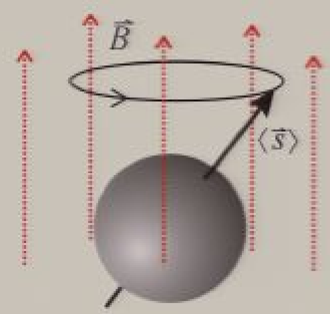
Observables of a Minimal Length - High Precision

- Hydrogen atom: shift of energy levels



SH, M. Bleicher, S. Hofmann, S. Scherer, J. Ruppert and H. Stöcker, Phys. Lett. B **598** (2004) 92-98

- g-2



U. Harbach, SH, M. Bleicher and H. Stöcker Phys. Lett. B **584** (2004) 109-113

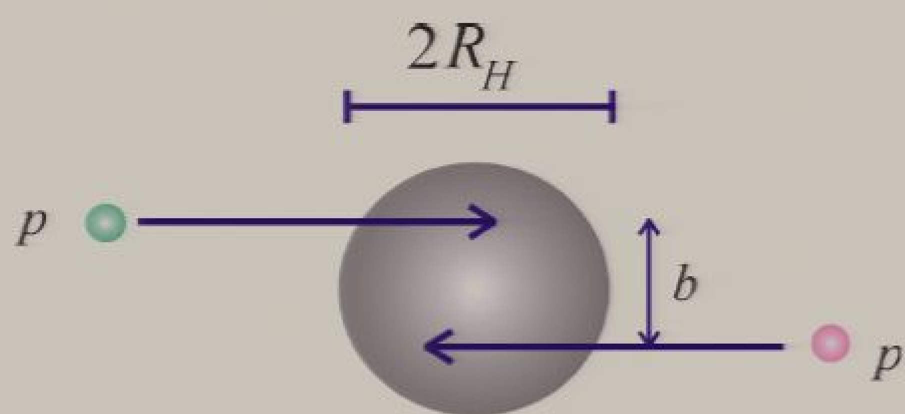
- Varying speed of light – energy dependent time of flight

Amelino-Camelia, Phys. Rev. D **64** (2001) 036005
Magueijo and Smolin, Phys. Rev. Lett. **88** (2002) 190403
Judes and Visser, Phys. Rev. D **68** (2003) 045001

Advertisement Break

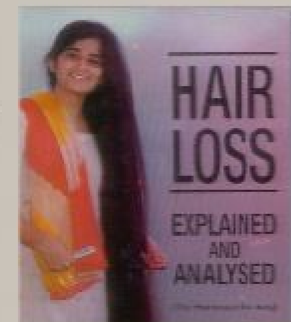
Large Extra Dimensions

- More volume: the volume factor makes gravity appear weak at large distances $\gg$ radius
- True strenght: at small distances (~ 1TeV) gravity becomes stronger
- 2 in 1: graviton- and black hole production at the LHC becomes possible*



* Restrictions apply

Evaporation proceeds in 3 stages:



1. Balding phase: hair loss – the black holes radiates off angular momentum and multipole moments
2. Hawking phase: thermal radiation into all particles of the standard model as well as gravitons
3. Final decay (remaining black hole relic?)

Black hole thermodynamics: $T = \kappa/2\pi \sim 200$ GeV

Numerical investigation: black hole event generator CHARYBDIS

TEILCHENPHYSIK

Angst vor dem großen Knall

Physiker wollen bei New York den Anfang des Universums erforschen und lösen Endzeitstimmung aus

VOR DEM ERSTEN STOSS Seit Juli flitzen Goldatome durch den unterirdischen Ringtunnel. Ab Herbst gehen sie auf Kollisionskurs

In der „Unendlichen Geschichte" von Michael Ende breitet sich das Nichts unaufhaltsam aus. Es reißt Tiere und Pflanzen fort, verschlingt Berge und Seen – und lässt von ganz Phantásien nicht mehr als ein Sandkorn übrig.

Solch ein Schicksal steht vielleicht der Erde bevor, fürchten jetzt viele Amerikaner, wenn ein neuer Teilchenbeschleuniger bei New York ab Herbst schwere Atome aufeinander hetzt. Der Relativistische Schwerionen-Collider (RHIC) in Brookhaven lässt die Teilchen so heftig zusammenkrachen, dass sie 10 000-mal heißer als die Sonne werden. Damit wollen die Physiker Bedingungen schaffen, wie sie direkt nach dem Urknall herrschten.

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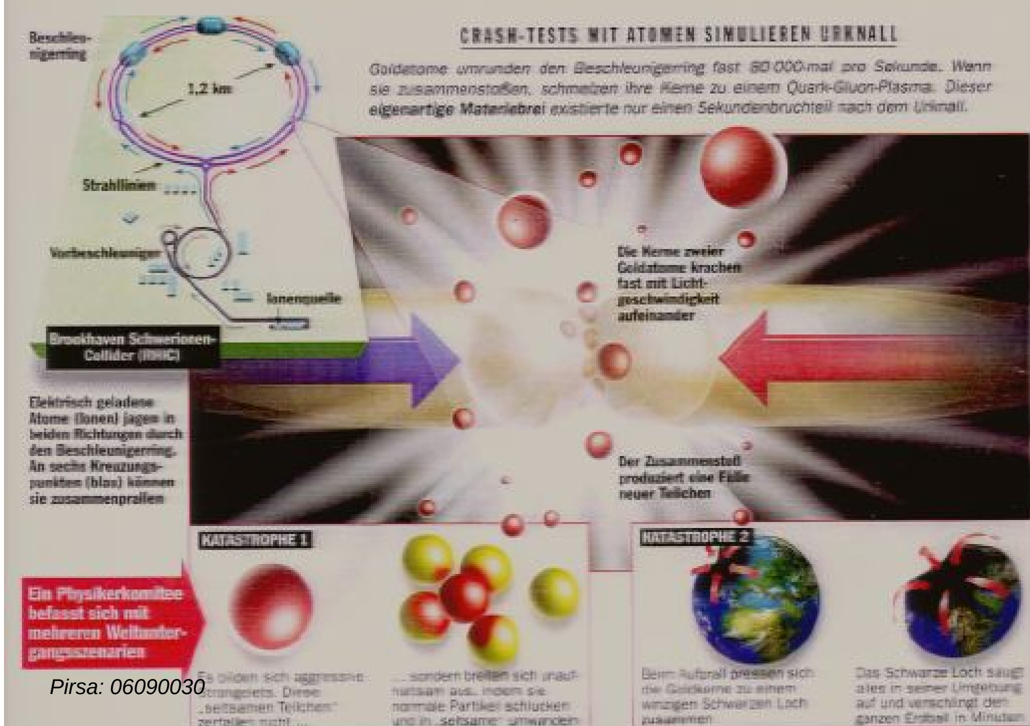
CRASH-TESTS MIT ATOMEN SIMULIEREN URKNALL

Goldatome umrunden den Beschleunigerring fast 80 000-mal pro Sekunde. Wenn sie zusammenstoßen, schmelzen ihre Kerne zu einem Quark-Gluon-Plasma. Dieser ***eigenartige Materiebrei*** *existierte nur einen Sekundenbruchteil nach dem Urknall.*



Big Bang Machine: Will it destroy Earth?

The London Times July 18, 1999

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FORSCHUNG & TECHNIK

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Beim Aufprall pressen sich die Goldkerne zu einem winzigen Schwarzen Loch zusammen

Das Schwarze Loch saugt alles in seiner Umgebung auf und verschlingt den ganzen Erdball in Minuten

...back to the main feature

Observables of a Minimal Length - High Energy

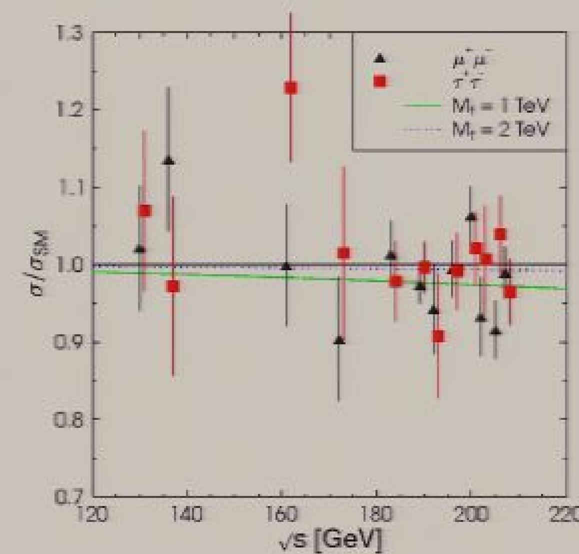
- Suppression of cross-section from $\sqrt{s} \sim M_f$ on.

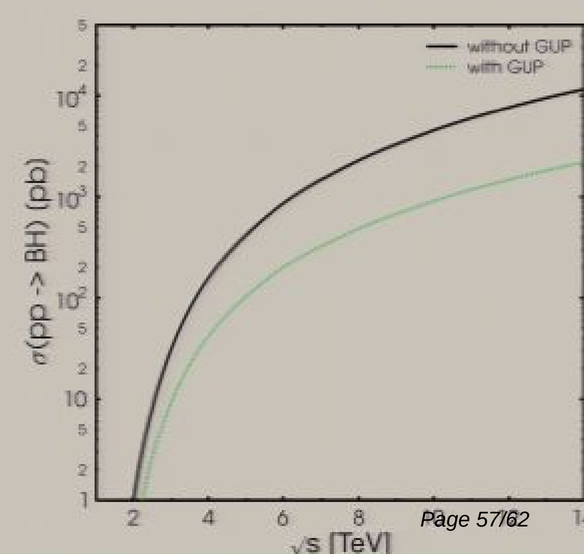
  SH, M. Bleicher, S. Hofmann, S. Scherer, J. Ruppert and H. Stöcker, Phys. Lett. B598 (2004) 92-98

- Black hole production becomes more difficult

  SH, Phys. Lett. B598 (2004) 92-98

- Cosmic ray puzzle? (modified treshold)





Outlook

- Thermodynamics with minimal length
  * Finite minimal density?
  * Modified equation of state?
- MDR from mean field approximation
  * Integrating out the graviton soup
  * Non-local effects compatible with stringy limits?

Detour: Thermoduality*

- Quantum Field Theory at finite temperature:

  Compactify timelike coordinate to radius $R = 1/2\pi T$.
- Stringy T-duality:

  R has a dual radius $R \longrightarrow \frac{R_{\rm d}^2}{R}$
- Together: T has a dual temperature

$$T \longrightarrow \frac{T_{\rm d}^2}{T}$$

* Dienes and Lennek, Phys. Rev. D **70**, 126005 (2004) [arXiv:hep-th/0312216]
Dienes and Lennek, Phys. Rev. D **70**, 126006 (2004) [arXiv:hep-th/0312217].

Detour: Thermoduality*

BUT

Detour: Thermoduality*

Standard Thermodynamics is not invariant under this duality

- Even if some quantities (free energy) exhibit symmetries, others (entropy, specific heat) don't.
- Reason is that d/dT does not respect the symmetry
- Thermoduality requires a "covariant" derivative D_T which leads to modifications
- Low $T \ll T_\mathrm{d}$ limit reproduces the familiar thermodynamics

$\longrightarrow$ Early universe? Equation of state?

Summary

- Effective theory that allows to examine phenomenology
- Related to DSR but different interpretation of free particles
- Input $p(k)$, rspt. $\sqrt{g(p)}$, should eventually be provided by the underlying theory
- With more investigation, it should be possible to further classify and constrain the model