Title: Towards a generally covariant cutoff using sampling theory

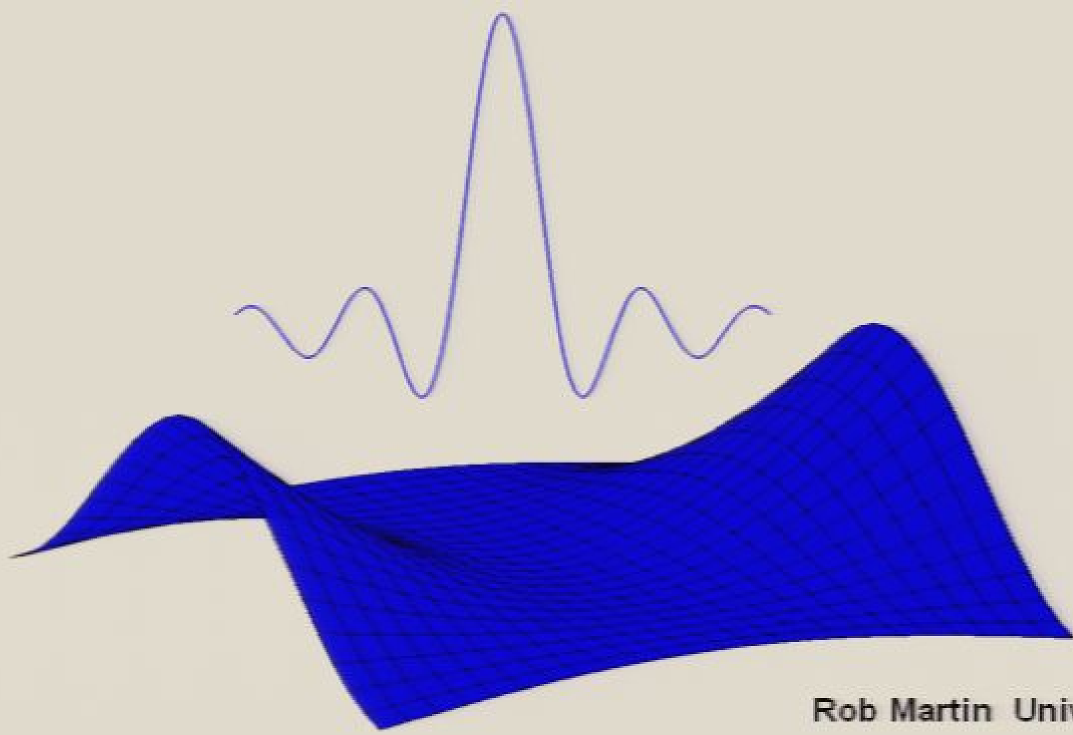
Date: Sep 06, 2006 04:30 PM

URL: http://pirsa.org/06090028

Abstract:

A covariant UV cutoff:

Sampling Theory on curved spacetime



Rob Martin University of Waterloo 06/09/06

Outline

1. **Motivation**: generally covariant UV cutoff using Sampling Theory

2. **Introduction to Sampling Theory:** Bandlimited functions

3. **Sampling Theory on Space-time:**

   -covariantly bandlimited functions

   -Minkowski spacetime

   -preliminary results on Sampling Theory in De Sitter spacetime

4. **Summary & Outlook**

1.

Motivation: the problem

- Quantum gravity arguments → Expect an UV cutoff
- UV divergencies → QFT ill-defined on continuous curved spacetime
- QFT becomes well defined if spacetime is discrete
- GR lives on a smooth, continuous manifold, discreteness of spacetime violates manifold symmetries like Lorentz invariance
- Q: How to satisfy both?

1.

Motivation: A proposed solution

- **Idea:** Spacetime continuous but physical fields possess a finite spatial density of degrees of freedom.
  → path integral is over a countable number of spatial degrees of freedom

- The field values $\{\phi(t, x_n)\}_{t \in R}$ on any `sufficiently dense' discrete set of points $\Lambda = \{x_n\}_{n \in Z}$ determine ϕ uniquely & can reconstruct $\phi(t, x)$ perfectly from this info.

→ Any physical theory can be defined on the smooth manifold, or on any lattice of sufficient density. No preferred lattice → spacetime symmetries preserved

Q: Is there a theory that describes classes of fields/functions with these special properties?

A: Yes, its called Sampling Theory

2.

Introduction to Sampling Theory

Sampling theory: study of classes of functions, reconstructible from their values taken on certain discrete sets of points

- Example: Ω-Bandlimited functions
  - Fourier transforms of elements of $L^2[-\Omega, \Omega]$
  - Ω called the bandlimit
- Resources: ∃ large body of literature:
  - Mathematics
  - communication engineering
  - Information theory

Bandlimited functions

Example: B(Ω) := Hilbert space of functions bandlimited by Ω:

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \qquad F \in L^2(-\Omega, \Omega)$$

- Resolution of Identity: $1 = \sum_n (\cdot, b_{x_n}) b_{x_n}$

$$b_{x_n} := \frac{1}{\sqrt{2\Omega}} e^{ix_n w} \qquad w \in [-\Omega, \Omega] \qquad x_n := \frac{n\pi}{\Omega}$$

- The Shannon Sampling Formula:

$$F = \sum_n (F, b_{x_n}) b_{x_n} \qquad (F, b_{x_n}) = \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} F(w) e^{iwx_n} dw = \frac{1}{\sqrt{2\Omega}} f(x_n)$$

$$f = \mathcal{F}^{-1} F = \sum_n (F, b_{x_n}) \mathcal{F}^{-1} b_{x_n} \qquad \mathcal{F}^{-1} b_{x_n} = 2\Omega \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)}$$

$$\Rightarrow \boxed{f(x) = \sum_n f(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)}}$$

Sampling Lattices for B(Ω)

- Sampling Lattice: a set of points $\Lambda = \{y_n\}$, $|y_n - y_m| \geq \epsilon > 0$ s.t. there is a linear map $R: B(\Omega) \to l^2(Z)$, $Rf = \{f(y_n)\}_{n \in Z}$ which is bounded above and below.
  - E.g. $\Lambda = \{x_n = \frac{n\pi}{\Omega}\}$

- **Any lattice of sufficient density will do:**

$n^-(r) :=$ min number of y_n in any interval of length r

$$D^-(\Lambda) := \lim_{r \to \infty} \frac{n^-(r)}{r}$$

Theorem:

(a) if Λ is a set of sampling $D^-(\Lambda) \geq \frac{\Omega}{\pi}$ (Beurling)

(b) Λ is a set of sampling if $D^-(\Lambda) > \frac{\Omega}{\pi}$ (Duffin & Shaeffer / Beurling)

Bandlimited functions

Example: $B(\Omega)$:= Hilbert space of functions bandlimited by Ω:

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \qquad F \in L^2(-\Omega, \Omega)$$

- Resolution of Identity: $1 = \sum_n (\cdot, b_{x_n}) b_{x_n}$

$$b_{x_n} := \frac{1}{\sqrt{2\Omega}} e^{ix_n w} \qquad w \in [-\Omega, \Omega] \qquad x_n := \frac{n\pi}{\Omega}$$

- The Shannon Sampling Formula:

$$F = \sum_n (F, b_{x_n}) b_{x_n} \qquad (F, b_{x_n}) = \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} F(w) e^{iwx_n} dw = \frac{1}{\sqrt{2\Omega}} f(x_n)$$

$$f = \mathcal{F}^{-1} F = \sum_n (F, b_{x_n}) \mathcal{F}^{-1} b_{x_n} \qquad \mathcal{F}^{-1} b_{x_n} = 2\Omega \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)}$$

$$\Rightarrow \boxed{f(x) = \sum_n f(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)}}$$

Sampling Lattices for B(Ω)

- Sampling Lattice: a set of points $\Lambda = \{y_n\}$, $|y_n - y_m| \geq \epsilon > 0$ s.t. there is a linear map $R: B(\Omega) \to l^2(Z)$, $Rf = \{f(y_n)\}_{n \in Z}$ which is bounded above and below.
  - E.g. $\Lambda = \{x_n = \frac{n\pi}{\Omega}\}$

- **Any lattice of sufficient density will do:**

$n^-(r) :=$ min number of y_n in any interval of length r

$$D^-(\Lambda) := \lim_{r \to \infty} \frac{n^-(r)}{r}$$

Theorem:

(a) if Λ is a set of sampling $D^-(\Lambda) \geq \frac{\Omega}{\pi}$ (Beurling)

(b) Λ is a set of sampling if $D^-(\Lambda) > \frac{\Omega}{\pi}$ (Duffin & Shaeffer / Beurling)

Sampling Lattices for B(S)

- B(S) := Functions frequency limited by compact $S \subset \mathbb{R}^n$

$$f(x) = \int_S F(w) e^{iwx} dw \qquad F \in L^2(S)$$

- Theorem: (*Landau*)
  If Λ is a set of sampling for B(S) then $D^-(\Lambda) \geq \frac{\mu(S)}{(2\pi)^n}$

minimum possible sample density $\propto$ bandwidth volume

Sufficiency conditions not known in general

Sampling Lattices for B(Ω)

- **Sampling Lattice:** a set of points $\Lambda = \{y_n\}$, $|y_n - y_m| \geq \epsilon > 0$ s.t. there is a linear map $R: B(\Omega) \to l^2(Z)$, $Rf = \{f(y_n)\}_{n \in Z}$ which is bounded above and below.
  - E.g. $\Lambda = \{x_n = \frac{n\pi}{\Omega}\}$

- **Any lattice of sufficient density will do:**

$n^-(r)$:= min number of y_n in any interval of length r

$$D^-(\Lambda) := \lim_{r \to \infty} \frac{n^-(r)}{r}$$

Theorem:

(a) if Λ is a set of sampling $D^-(\Lambda) \geq \frac{\Omega}{\pi}$ (Beurling)

(b) Λ is a set of sampling if $D^-(\Lambda) > \frac{\Omega}{\pi}$ (Duffin & Shaeffer / Beurling)

Sampling Lattices for B(S)

- B(S) := Functions frequency limited by compact $S \subset \mathbb{R}^n$

$$f(x) = \int_S F(w) e^{iwx} dw \qquad F \in L^2(S)$$

- Theorem: (*Landau*)
  If Λ is a set of sampling for B(S) then $D^{-}(\Lambda) \geq \frac{\mu(S)}{(2\pi)^n}$

minimum possible sample density $\propto$ bandwidth volume

Sufficiency conditions not known in general

A generally covariant bandlimit

- A bandlimit is a cutoff on the spectrum of the Laplacian in $L^2(M)$ on the manifold M= $\mathbb{R}$.

$$\Delta := \frac{d^2}{dx^2} \qquad \frac{d^2}{dx^2}e^{\pm iwx} = w^2 e^{\pm iwx} \ \forall\ w \in \mathbb{R}$$

- B(Ω):= subspace spanned by $e^{\pm iwx}$ for $|w| \leq \Omega^2$

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \qquad \Leftrightarrow \ f \in B(\Omega)$$

- **Definition:** B(M, Ω) := subspace of $L^2(M)$ spanned by eigenfunctions to Δ (or □) whose eigenvalues λ obey $|\lambda| \leq \Omega^2$

(Ω := the bandlimit)

Pesenson (2000)

A generally covariant UV cutoff

- Idea: Impose this generally covariant bandlimit / UV cutoff on physical fields on spacetime.
- i.e. Given spacetime M, we restrict the space of allowed physical fields to be B(M, Ω).
- This is a restriction on the set of fields that the path integral runs over
- Investigate: as in the simple case M= $\mathbb{R}$, does the subspace of fields B(M,Ω) have a countable (or even finite?) # of spatial degrees of freedom

Sampling in Minkowski spacetime

- D'Alembertian: $-\frac{\partial^2}{\partial t^2} + \Delta$ Eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$

Spectral cutoff: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$ Fix $p^0 = \Omega b$

$$\Omega\sqrt{1+b^2} \geq |\mathbf{p}| \geq \Omega\sqrt{b^2-1}$$

min. possible sample density $\propto$ bandwidth volume (Landau's thm)

	$b \leq 1$	$b > 1$
1D	$2\Omega\sqrt{1+b^2}$	$2\Omega\left(\sqrt{1+b^2} - \sqrt{b^2-1}\right)$
2D	$\pi\Omega^2(1+b^2)$	$2\pi\Omega^2$
3D	$\frac{4\pi}{3}\Omega^3(1+b^2)^{\frac{3}{2}}$	$\frac{4\pi}{3}\Omega^3\left((1+b^2)^{\frac{3}{2}} - (b^2-1)^{\frac{3}{2}}\right)$ $\xrightarrow{b\to\infty} \infty$

Reconstruction in flat space-time

- Note: in 1+1, 1+2, 1+3 D, each fixed temporal mode obeys a spatial sampling theorem and vice versa
- E.g. $\phi \in B(M,\Omega)$ M:= 1+1 D Minkowski spacetime.
- Φ := temporal Fourier transform of ϕ
- $p^0 = \Omega b$ fixed temporal frequency
- $\Phi(p^0, x)$ (spatial) frequency limited by

$$S_b = \left[-\Omega\sqrt{b^2+1}, \Omega\sqrt{b^2+1}\right] \qquad b^2 \leq 1$$

$$S_b = \left[-\Omega\sqrt{b^2+1}, -\Omega\sqrt{b^2-1}\right] \cup \left[\Omega\sqrt{b^2-1}, \Omega\sqrt{b^2+1}\right] \qquad b^2 \geq 1$$

Sampling in Minkowski spacetime

- D'Alembertian: $-\frac{\partial^2}{\partial t^2}+\Delta$ Eigenfunctions: $e^{i(p^0 t-\mathbf{p}\cdot\mathbf{x})}$

Spectral cutoff: $|(p^0)^2-\mathbf{p}^2|\leq\Omega^2$ Fix $p^0=\Omega b$

$$\Omega\sqrt{1+b^2}\geq|\mathbf{p}|\geq\Omega\sqrt{b^2-1}$$

min. possible sample density $\propto$ bandwidth volume (Landau's thm)

	$b\leq 1$	$b>1$
1D	$2\Omega\sqrt{1+b^2}$	$2\Omega\left(\sqrt{1+b^2}-\sqrt{b^2-1}\right)$
2D	$\pi\Omega^2(1+b^2)$	$2\pi\Omega^2$
3D	$\frac{4\pi}{3}\Omega^3(1+b^2)^{\frac{3}{2}}$	$\frac{4\pi}{3}\Omega^3\left((1+b^2)^{\frac{3}{2}}-(b^2-1)^{\frac{3}{2}}\right)$ $\xrightarrow{b\to\infty}$ ∞

Reconstruction in flat space-time

- Note: in 1+1, 1+2, 1+3 D, each fixed temporal mode obeys a spatial sampling theorem and vice versa
- E.g. $\phi \in B(M,\Omega)$ M:= 1+1 D Minkowski spacetime.
- Φ := temporal Fourier transform of ϕ
- $p^0 = \Omega b$ fixed temporal frequency
- $\Phi(p^0, x)$ (spatial) frequency limited by

$$S_b = \left[-\Omega\sqrt{b^2+1}, \Omega\sqrt{b^2+1}\right] \qquad b^2 \leq 1$$

$$S_b = \left[-\Omega\sqrt{b^2+1}, -\Omega\sqrt{b^2-1}\right] \cup \left[\Omega\sqrt{b^2-1}, \Omega\sqrt{b^2+1}\right] \qquad b^2 \geq 1$$

Reconstruction in flat space-time II

- Maximum bandwidth volume: $2\sqrt{2}\Omega$

  → There could be a set $\Lambda := \{y_n\}_{n=-\infty}^{\infty}$ satisfying Landau's thm and is sampling for every $B(S_b)$ $\forall\, b$ *i.e.* for every fixed temporal mode

- One such a set is $\Lambda := \{y_n\}_{n=-\infty}^{\infty}$ where $y_n := \left\{ \begin{array}{ll} \frac{n\pi}{\sqrt{2}\Omega} & n \text{ even} \\ \frac{(n+\alpha)\pi}{\sqrt{2}\Omega} & n \text{ odd} \end{array} \right\}$ $\alpha \in (0,1)$

- Every temporal mode $\Phi(p^0, x)$ can be reconstructed perfectly from $\{\Phi(p^0, y_n)\}_{y_n \in \Lambda}$

$\Longrightarrow$ every $\phi \in B(M,\Omega)$ can be reconstructed from $\{\phi(t, y_n)\}_{y_n \in \Lambda\ t \in \mathcal{R}}$

- Similarly every $\phi \in B(M,\Omega)$ can be reconstructed from $\{\phi(t_n, x)\}_{t_n \in \Lambda\ x \in \mathcal{R}}$

Reconstruction in flat space-time III

- In 1+3 dimensions, such a spatial sampling theorem is not possible as there is no upper bound on the bandwidth volume for a fixed temporal mode.

- Can show that ∃ sets $\Lambda := \{x_n^1, x_m^2\}$ s.t. the information $\{\phi(t, x_n^1, x_m^2)\}$, $\{\phi(t, x_n^1, x_m^2, x^3)\}$ is sufficient to reconstruct $\phi(t, x^1, x^2)$, $\phi(t, x^1, x^2, x^3)$ everywhere.

- Shows that 2D surfaces have a finite density of degrees of freedom, good for holography

- Expect in more general 1+1 or 1+2 D spacetimes it may be possible to reconstruct $\phi \in B(M, \Omega)$ from its values taken on certain discrete sets of timelike curves.

Sampling theory in FRW Spacetimes

- For simplicity, consider 1+1 D. $ds^2 = -dt^2 + a^2(t)dx^2 = a^2(\eta)\left(-d\eta^2 + dx^2\right)$

  $\eta :=$ conformal time

- D'Alembertian: $\Delta = a^{-2}(\eta)\left(-\partial_\eta^2 + \partial_x^2\right) = -a^{-2}(\eta)\left(\partial_\eta^2 + k^2\right)$

- Expect as in Minkowski case, that for each fixed spatial mode `k' there will be a temporal sampling theorem.

- E.g. De Sitter with a finite ending time: $a(t) = e^t \qquad t \in (-\infty, 0]$

$$\eta = e^{-t} \qquad \eta \in [1, \infty)$$

$$\Delta = -\eta^2\left(\partial_\eta^2 + k^2\right) \qquad L^2\left([1,\infty) \times (-\infty,\infty); \eta^{-2} d\eta dk\right)$$

De Sitter with finite end time

$$a(t) = e^t \quad t \in (-\infty, 0] \qquad \eta = e^{-t} \quad \eta \in [1, \infty)$$

$$\Delta = -\eta^2 \left(\partial_\eta^2 + k^2\right) \quad L^2\left([1,\infty) \times (-\infty,\infty); \eta^{-2} d\eta dk\right)$$

- Spectral cutoff (bandlimit): $|\lambda| \leq \Omega^2 \quad \lambda \in \sigma(\Delta)$
- For each fixed `k' this implies a cutoff on $\Delta_k = -\eta^2 \left(\partial_\eta^2 + k^2\right)$ in $L^2([1,\infty), \eta^{-2} d\eta)$
- **Prelminary results:**
  - The subspace of $L^2([1,\infty), \eta^{-2} d\eta)$ spanned by eigenfunctions $\{f_j^{(k)}\}_{j=1}^{N_k}$ to Δ_k obeying the bandlimit has finite dimension N_k for each fixed k.

$\Rightarrow$ Any set $\{\eta_j\}_{j=1}^{N_k}$ of points s.t. the determinant $det[S] \neq 0$ $\quad S_{jn} := f_j^{(k)}(\eta_n)$
is a set of sampling for the fixed spatial mode k.

$$\phi(\eta, k) = \sum_{n=1}^{N_k} c_n f_n^{(k)}(\eta) \qquad c_n = \sum_{j=1}^{N_k} \phi(\eta_j, k) S_{kn}^{-1}$$

De Sitter with finite end time

- **Further questions:**
  - How does the dimension N_k behave as a function of k, or of the end time of the manifold
  - We conjecture an overall temporal sampling formula that reconstructs $\phi(\eta, x)$ from $\{\phi(\eta_n, x)\}_{n \in Z}$
  - What about the full De Sitter spacetime $t \in (-\infty, \infty)$

- Some techniques used so far:

  -complex analysis

  -operator theory (self adjoint extensions of symmetric operators, spectral theory)

4.

Outlook

Future Work:

- would like to develop a full understanding of reconstruction properties of $B(M,\Omega)$ in physically relevant spacetimes: De Sitter, Power law, Schwarzschild, etc.

- spatial sampling theorems for fixed temporal frequencies, higher dimensions

- given a general pseudo-Riemannian manifold, can we develop general sufficiency conditions on a discrete set of spatial hypersurfaces s.t. bandlimited fields can be reconstructed from their values on those surfaces.