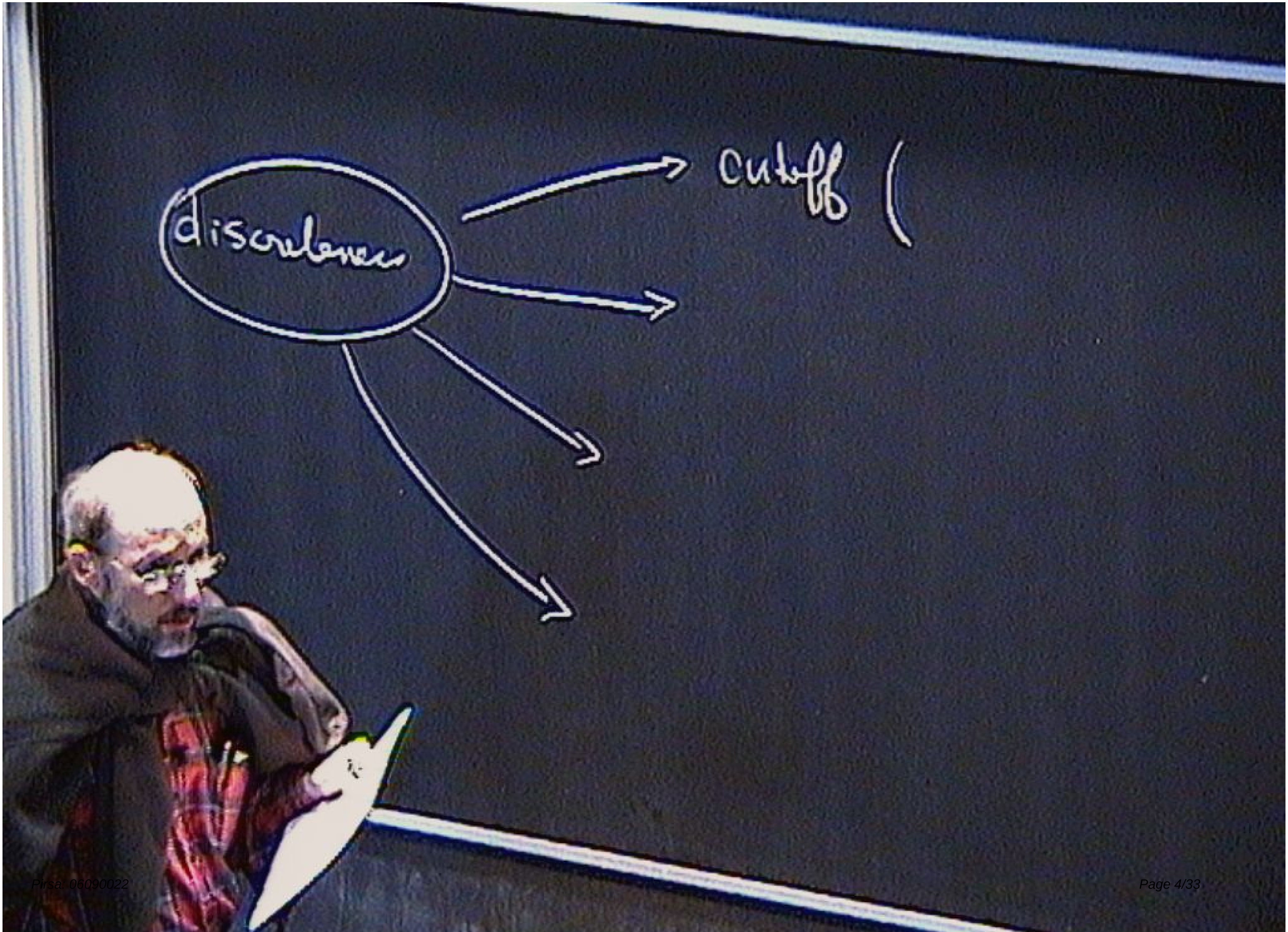
Title: A discrete, Lorentz-invariant wave equation and its continuum limit

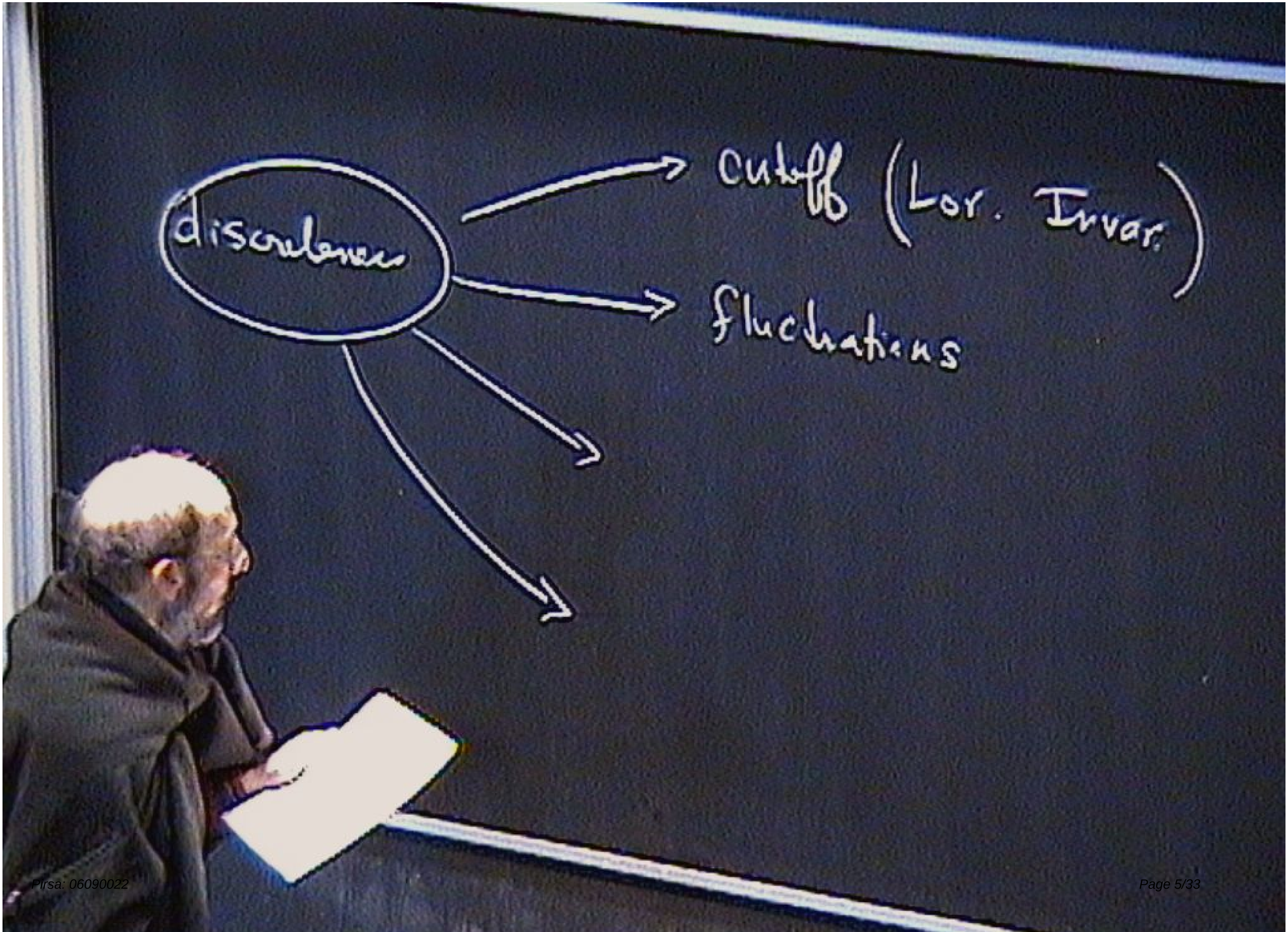
Date: Sep 05, 2006 02:00 PM

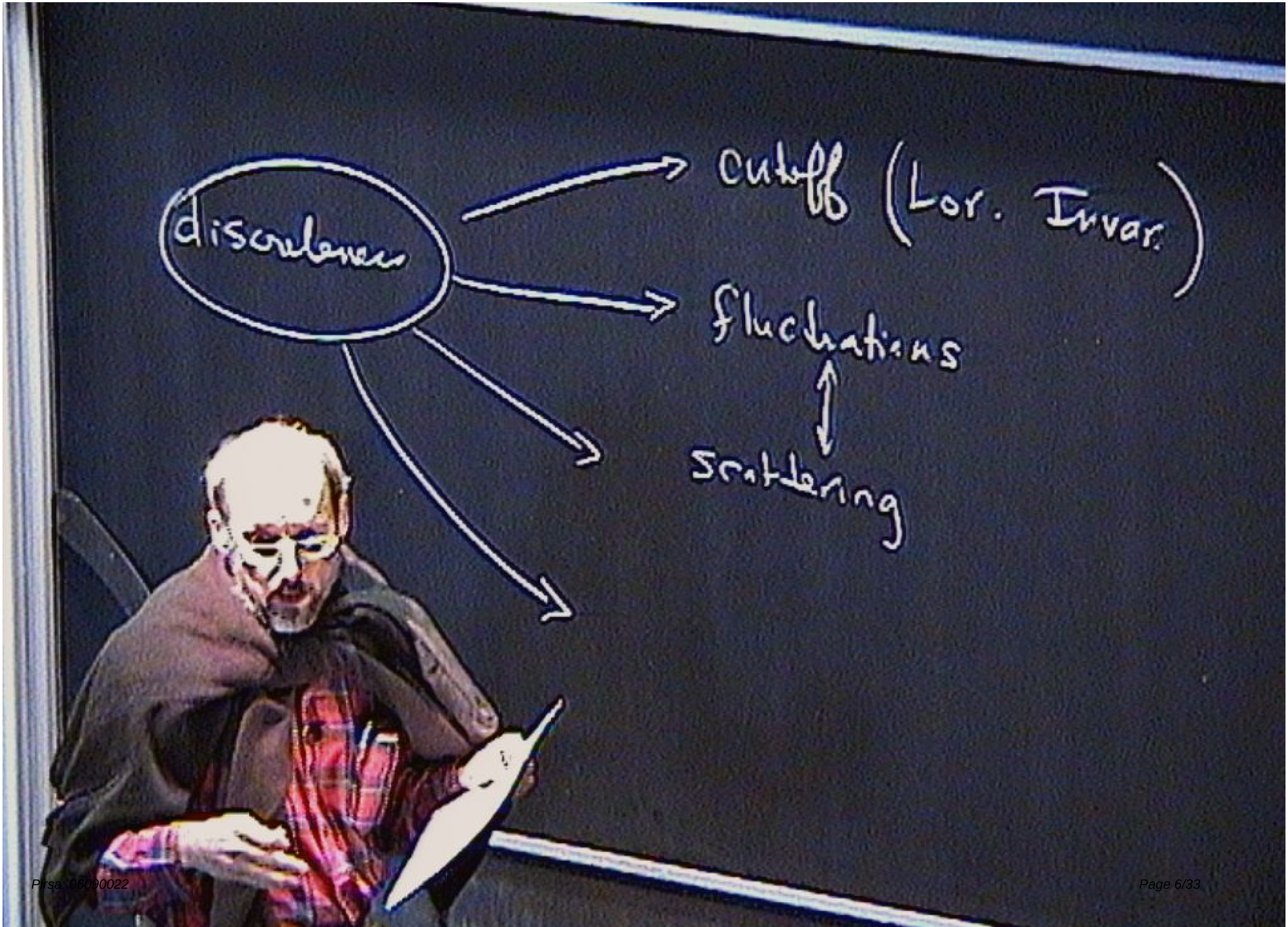
URL: http://pirsa.org/06090022

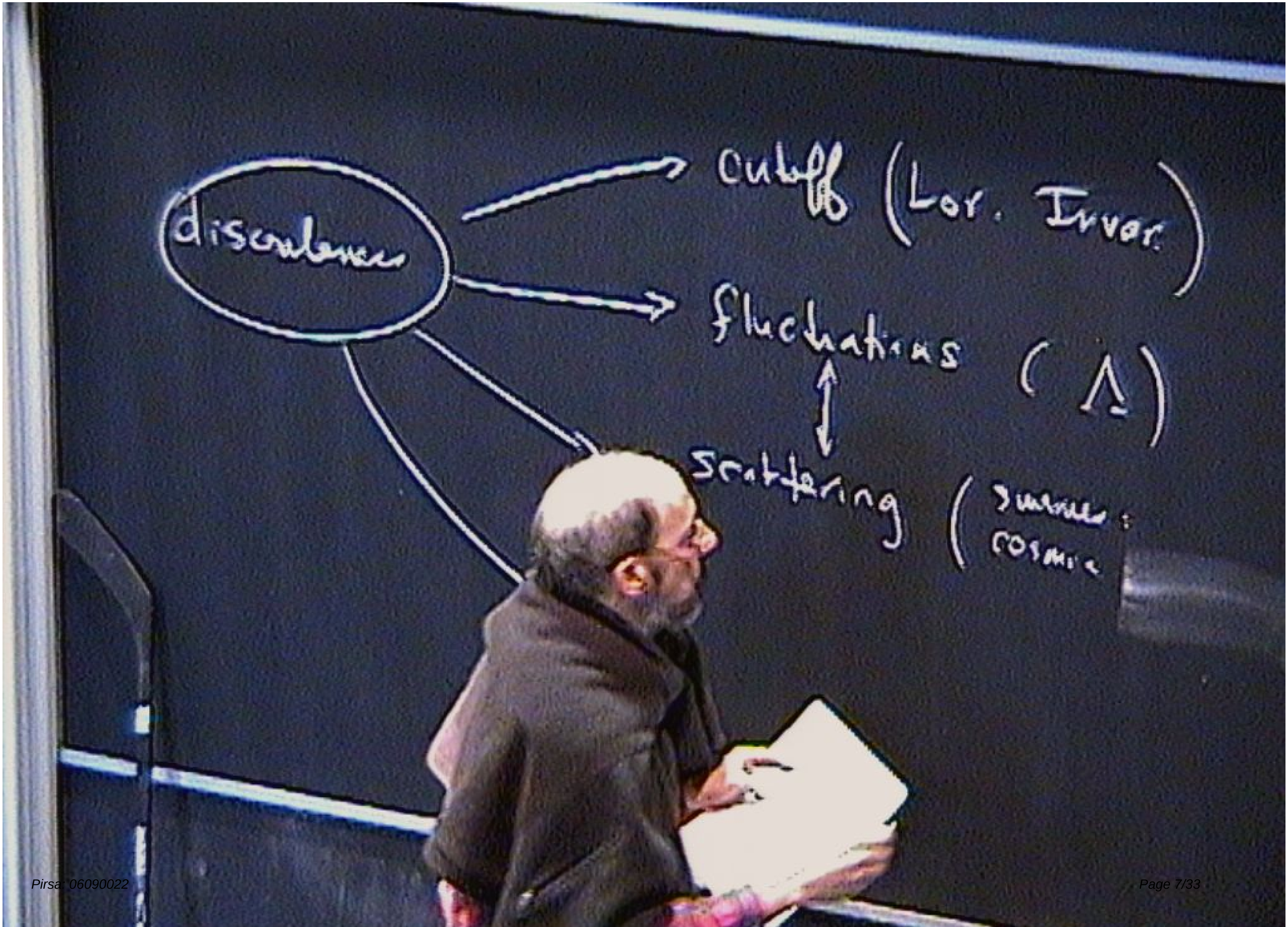
Abstract:

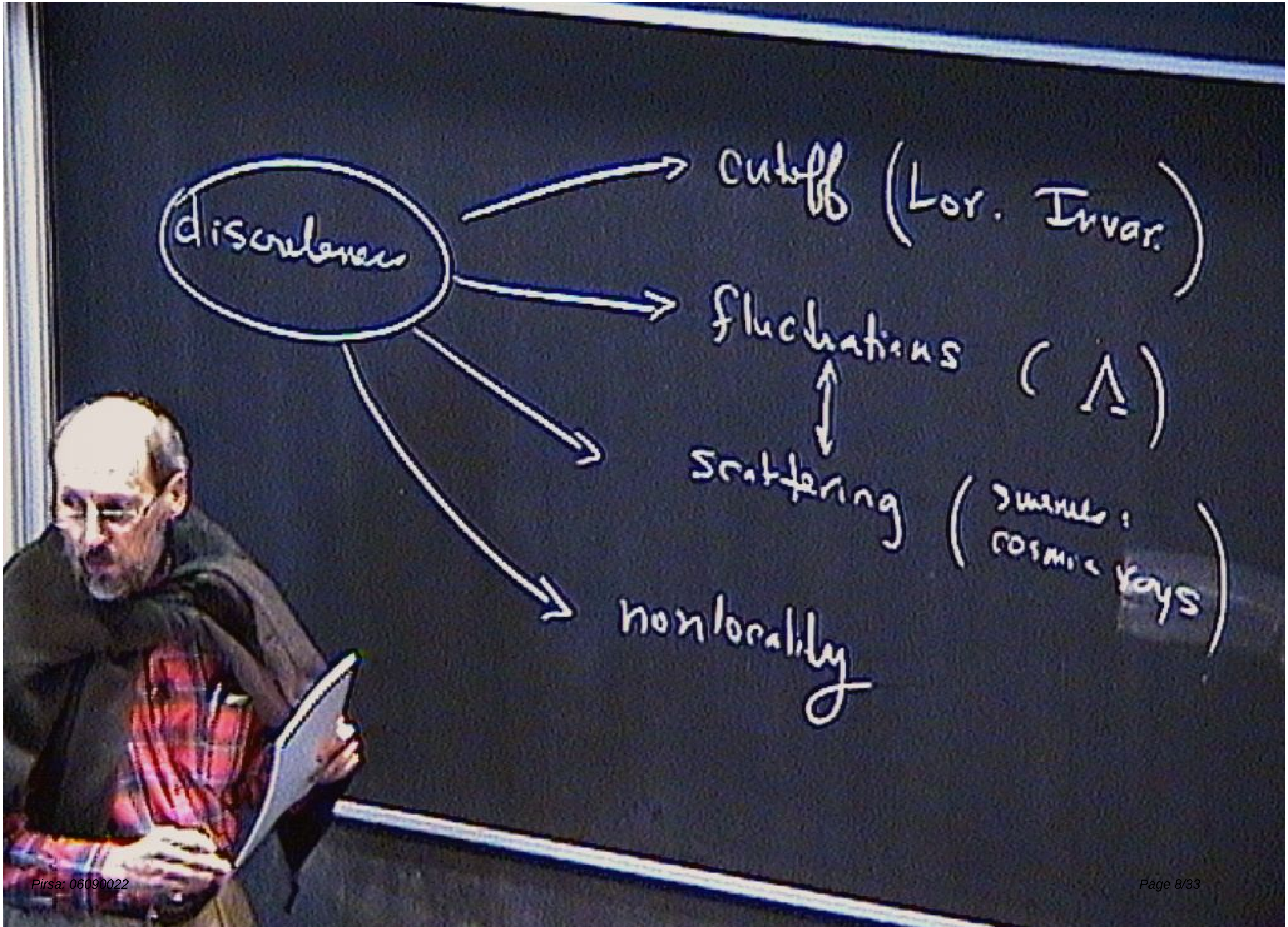
discreteness

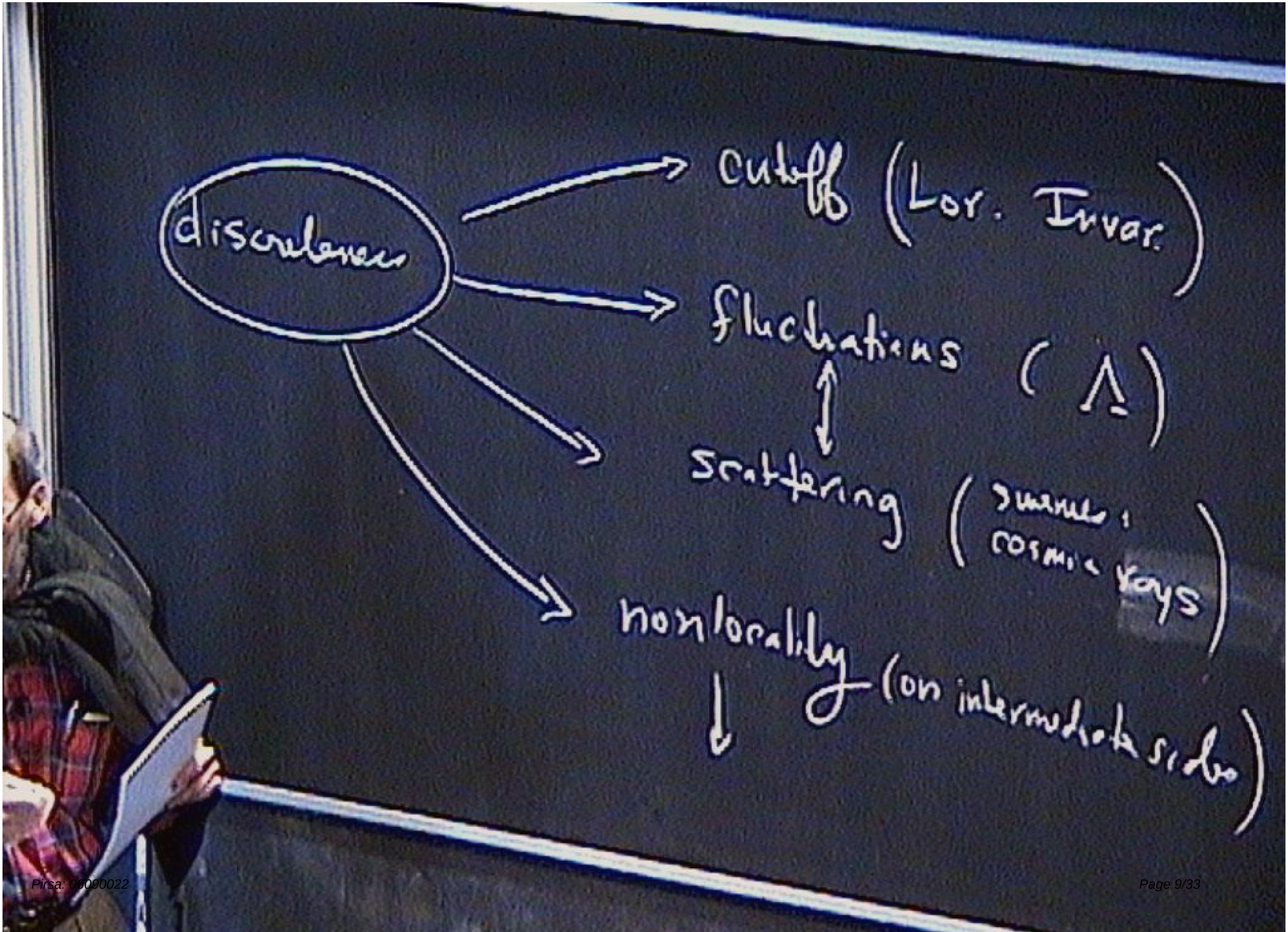

discreteness
cutoff (

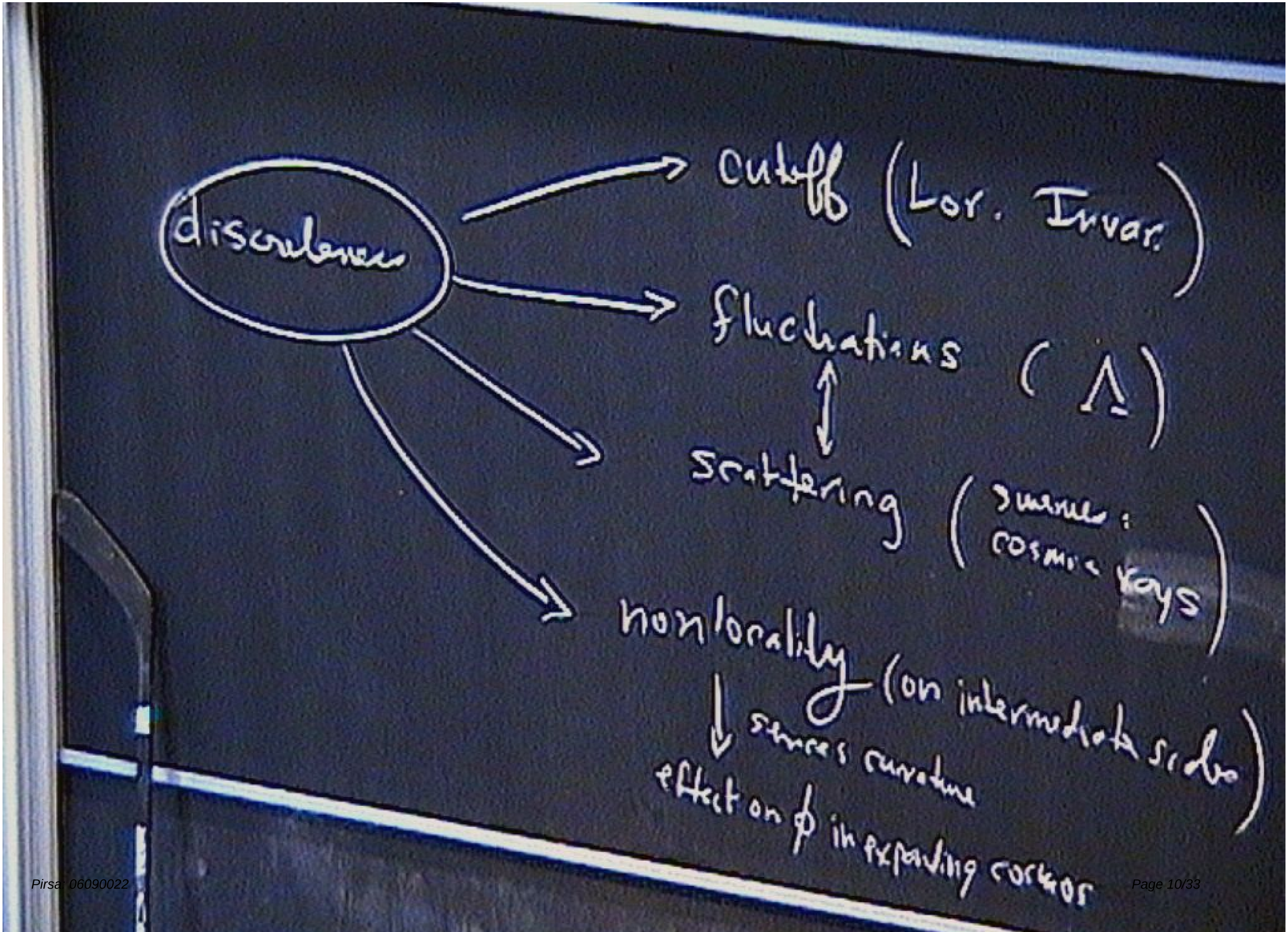

discreteness
cutoff (Lor. Invar.)
fluctuations


discreteness
cutoff (Lor. Invar.)
fluctuations
scattering


discreteness
cutoff (Lor. Invar.)
fluctuations (Λ)
scattering (
cosmic


discreteness
cutoff (Lor. Invar.)
fluctuations (Λ)
scattering
cosmic rays
nonlocality


discreteness
cutoff (Lor. Invar.)
fluctuations (Λ)
scattering
cosmic rays
nonlocality (on intermediate scales)


discreteness
cutoff (Lor. Invar.)
fluctuations (Λ)
scattering (sources: cosmic rays)
nonlocality (on intermediate scales)
sources curvature
effect on φ in expanding cosmos

1. Discreteness *can respect* Lorentz-transformations

   (Kinematic randomness plays a role – Poisson processes)

2. But locality must be abandoned

   Implies radical nonlocality at fundamental level (*micro*-scale l)

3. One can recover locality approximately at large scales (*macro*-scale)

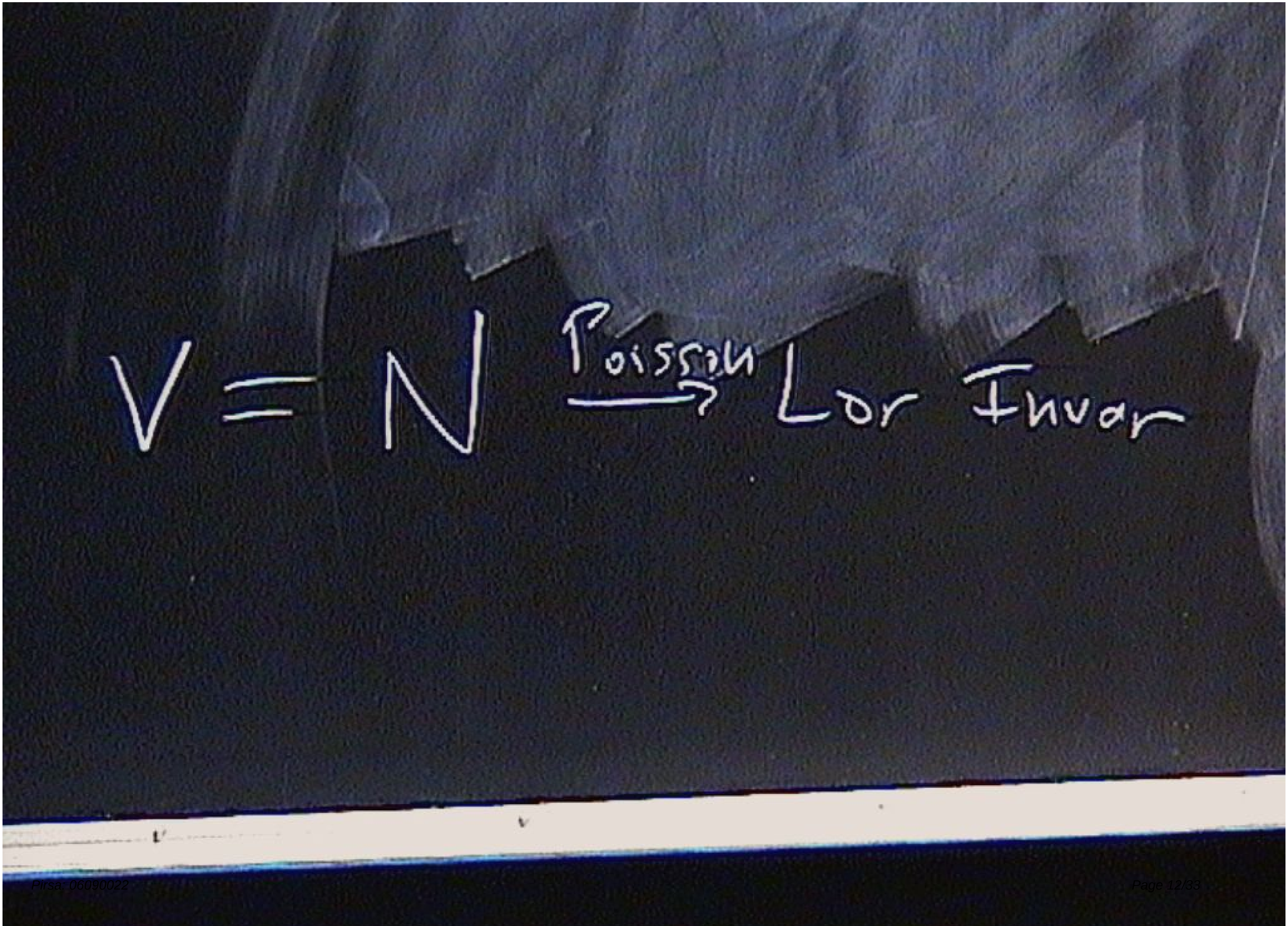
4. But residual nonlocality survives at *intermediate* length-scales (*meso*-scale, below λ_0)

5. An effective meso-theory would be *continuous* but *nonlocal*

Illustrate these claims with scalar field ϕ on a *fixed* causet C:

Recovery of $\Box\phi$.

($\delta\Lambda$ is also a nonlocal effect of discreteness; I'll not discuss it)


V = N —Poisson→ Lor Invar

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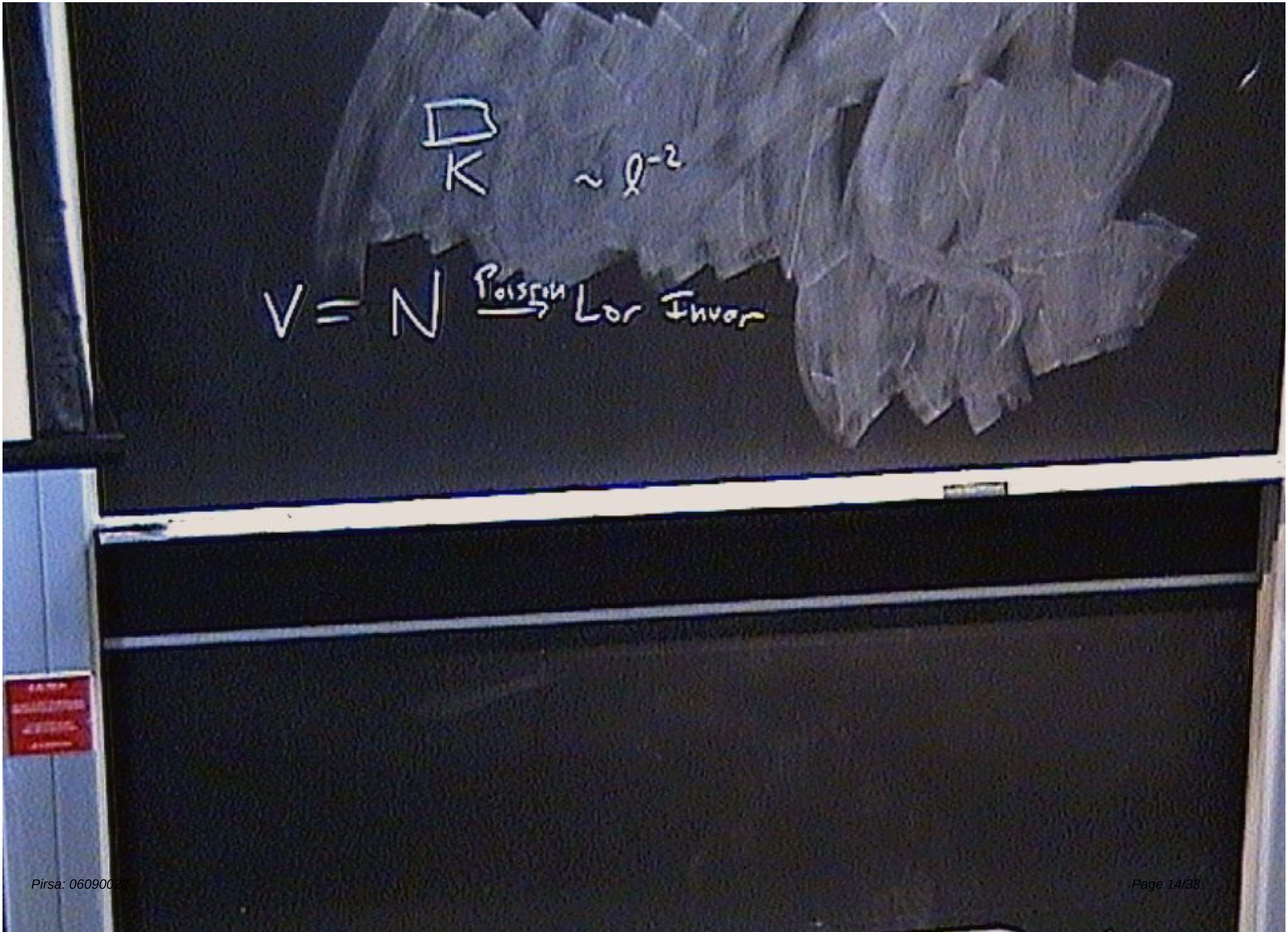
K ~ ℓ^-2
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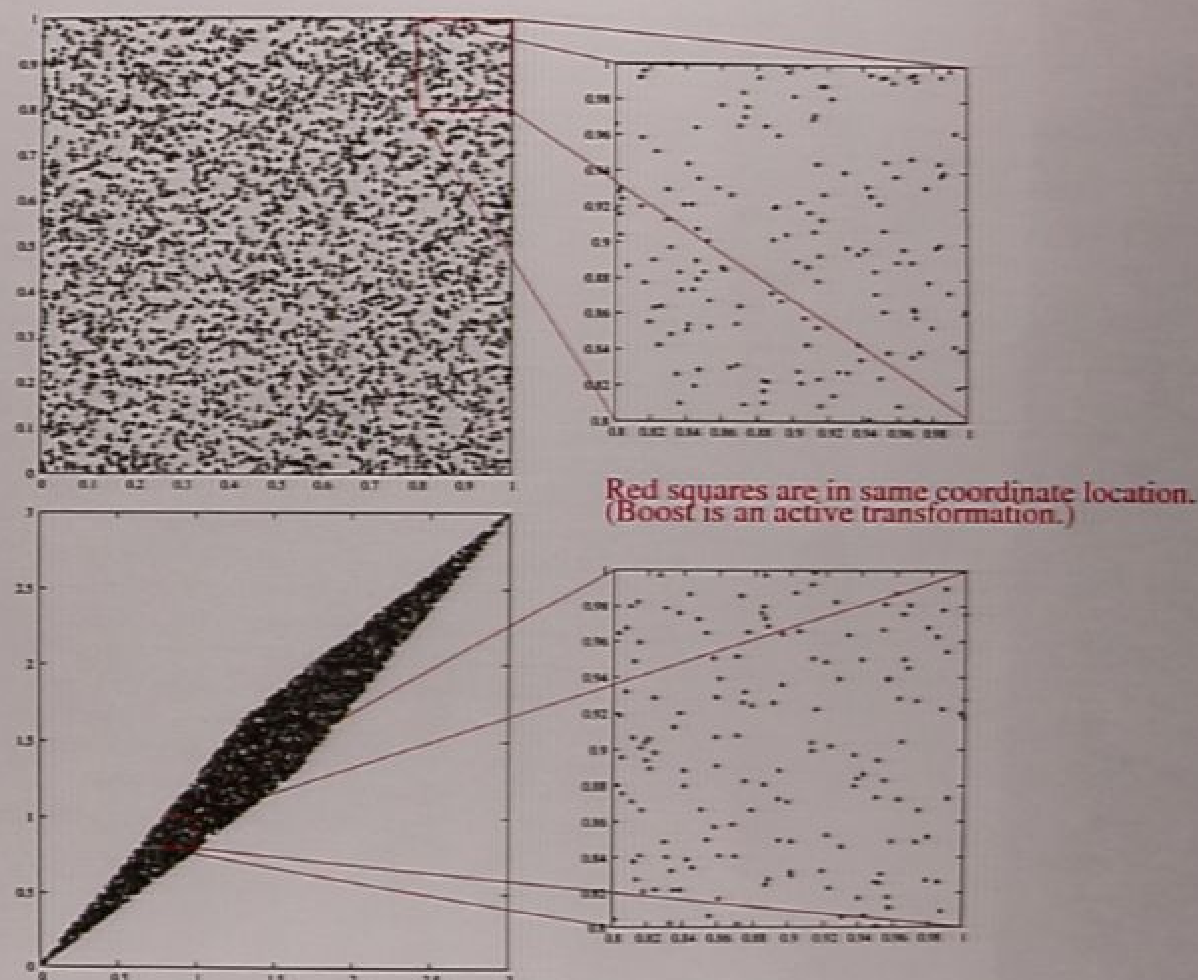
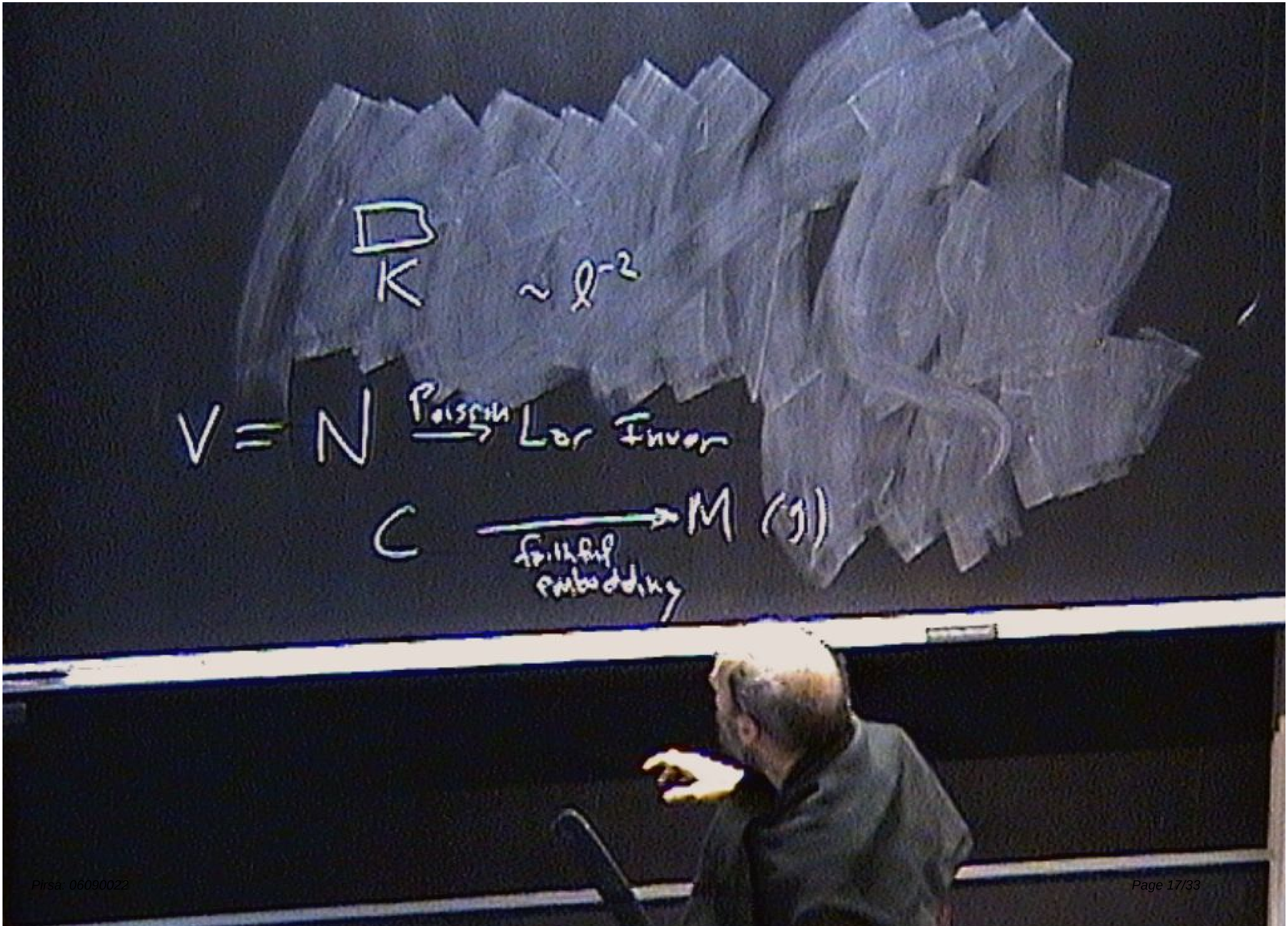
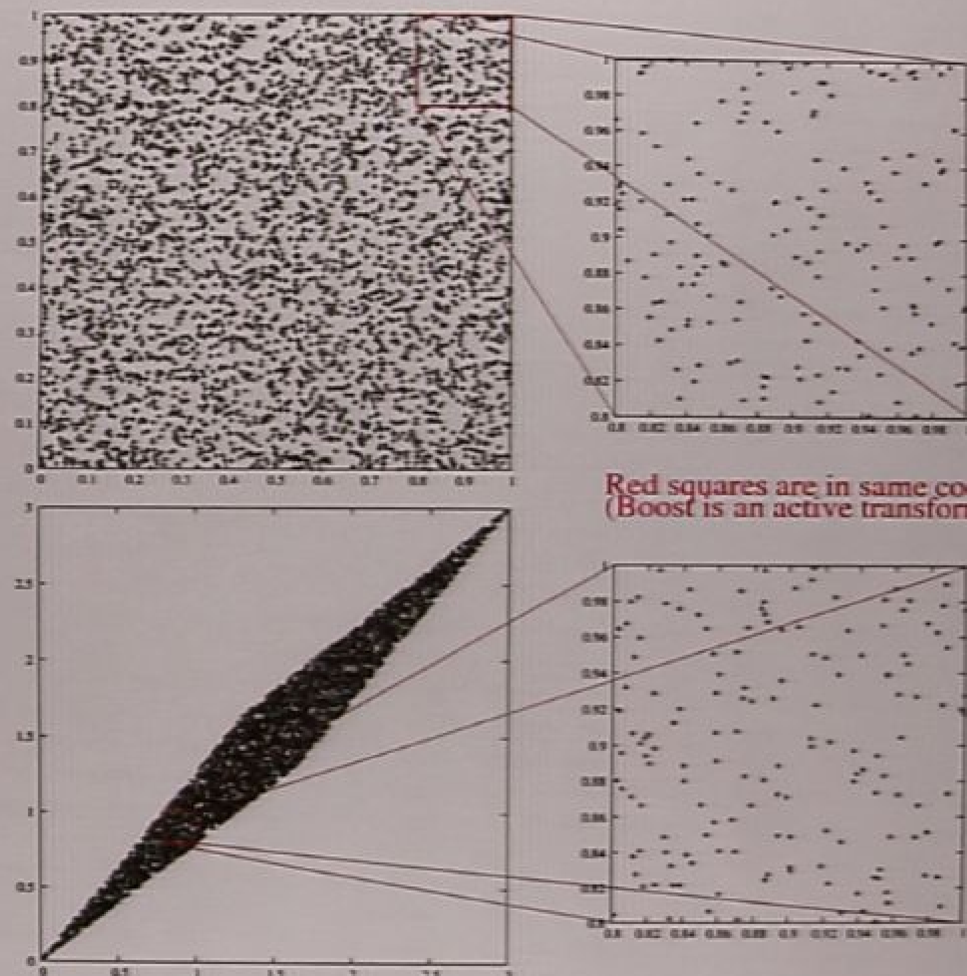

Red squares are in same coordinate location.
(Boost is an active transformation.)


K ~ l^-2
V = N Poisson Lor Invar
C → M (g)
faithful embedding



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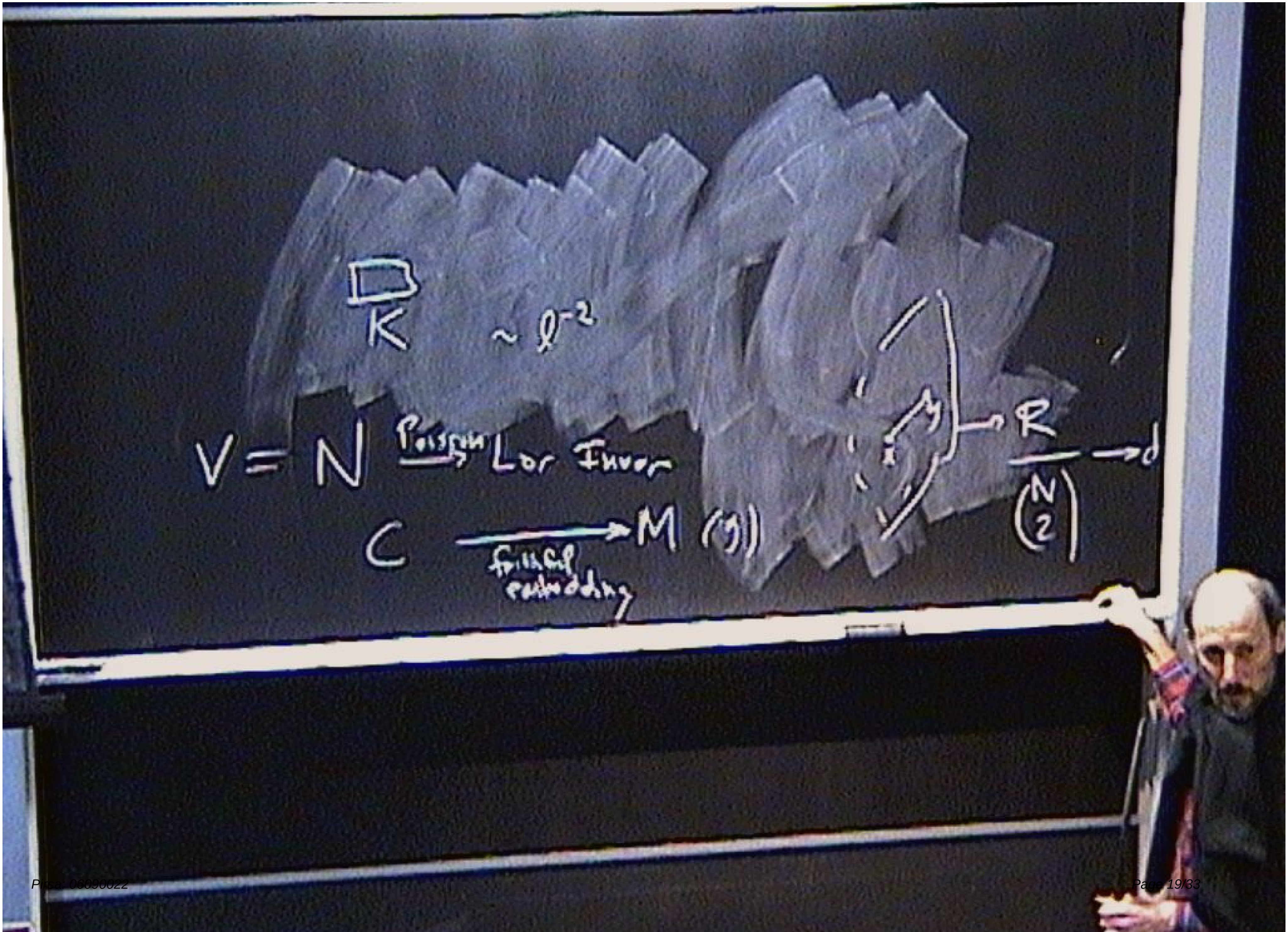
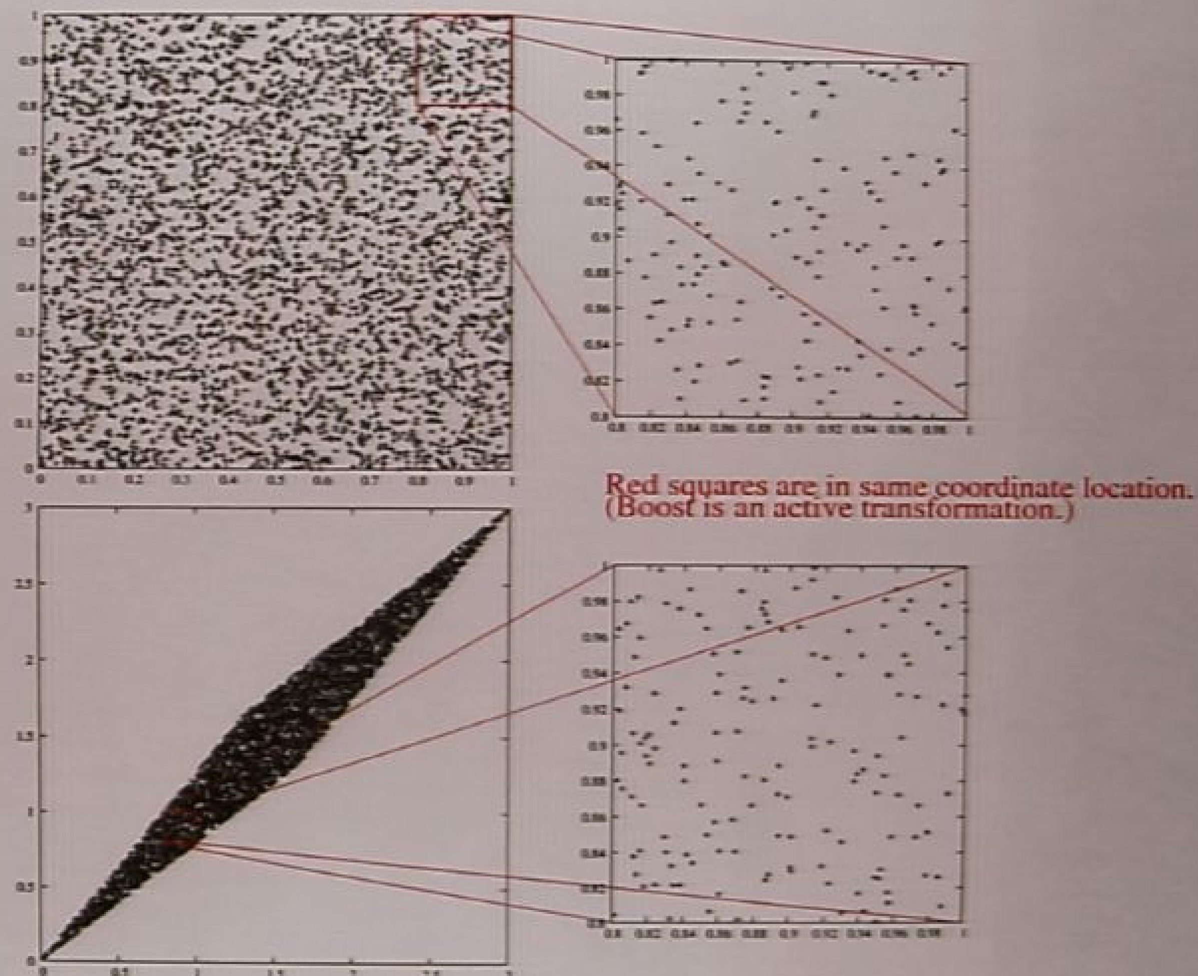

K
~ ℓ^-2
V = N
Poisson
C
M
faithful embedding
R
d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A theorem on Poisson processes

$\Omega =$ space of all sprinklings of $\mathbb{M}^d$ (sample space)

Poisson process induces a measure μ on Ω

Let f be a rule for deducing a direction from a sprinkling
$f : \Omega \to H =$ unit vectors in $\mathbb{M}^d$

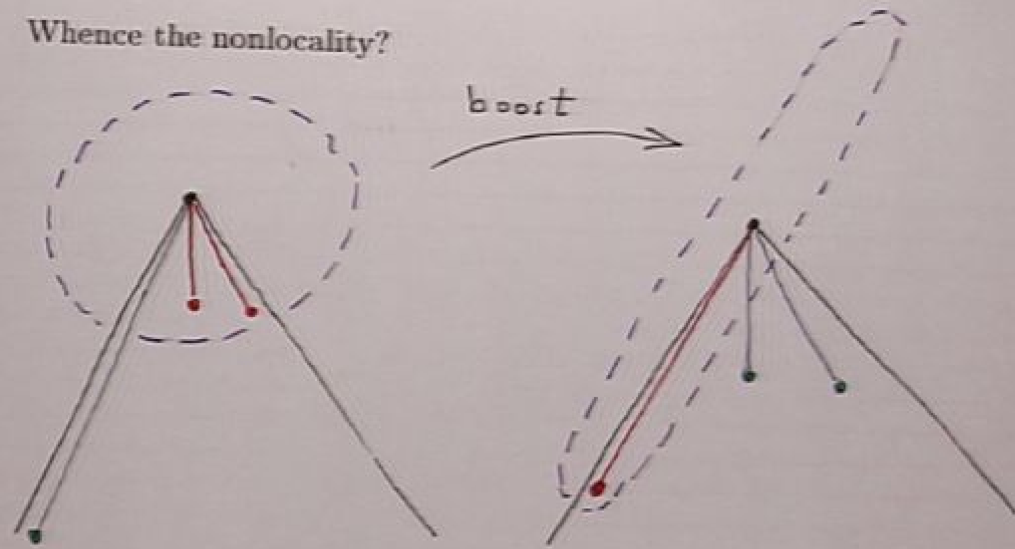
Require f *equivariant* ($f\Lambda = \Lambda f$, $\Lambda \in$ Lorentz)

Assume that f is measurable (hardly an assumption)

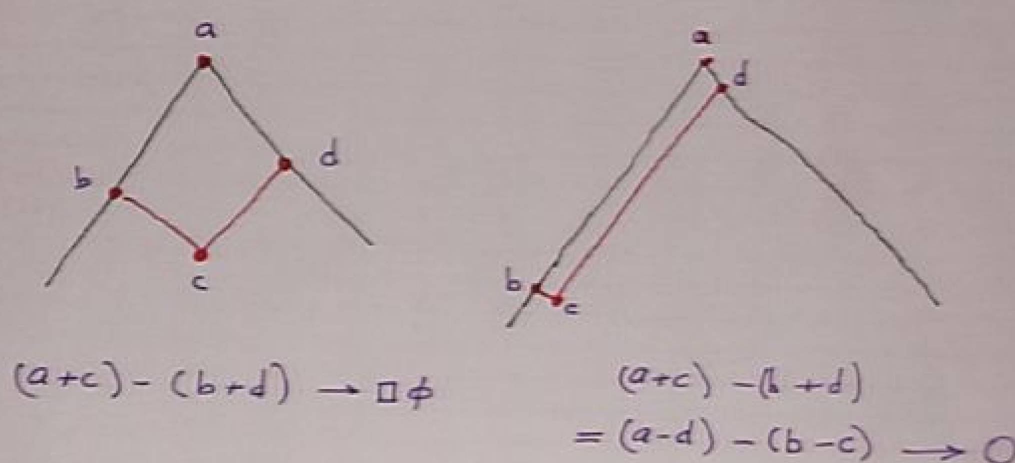
Theorem *No such f exists*
(not even on a partial domain of positive measure)

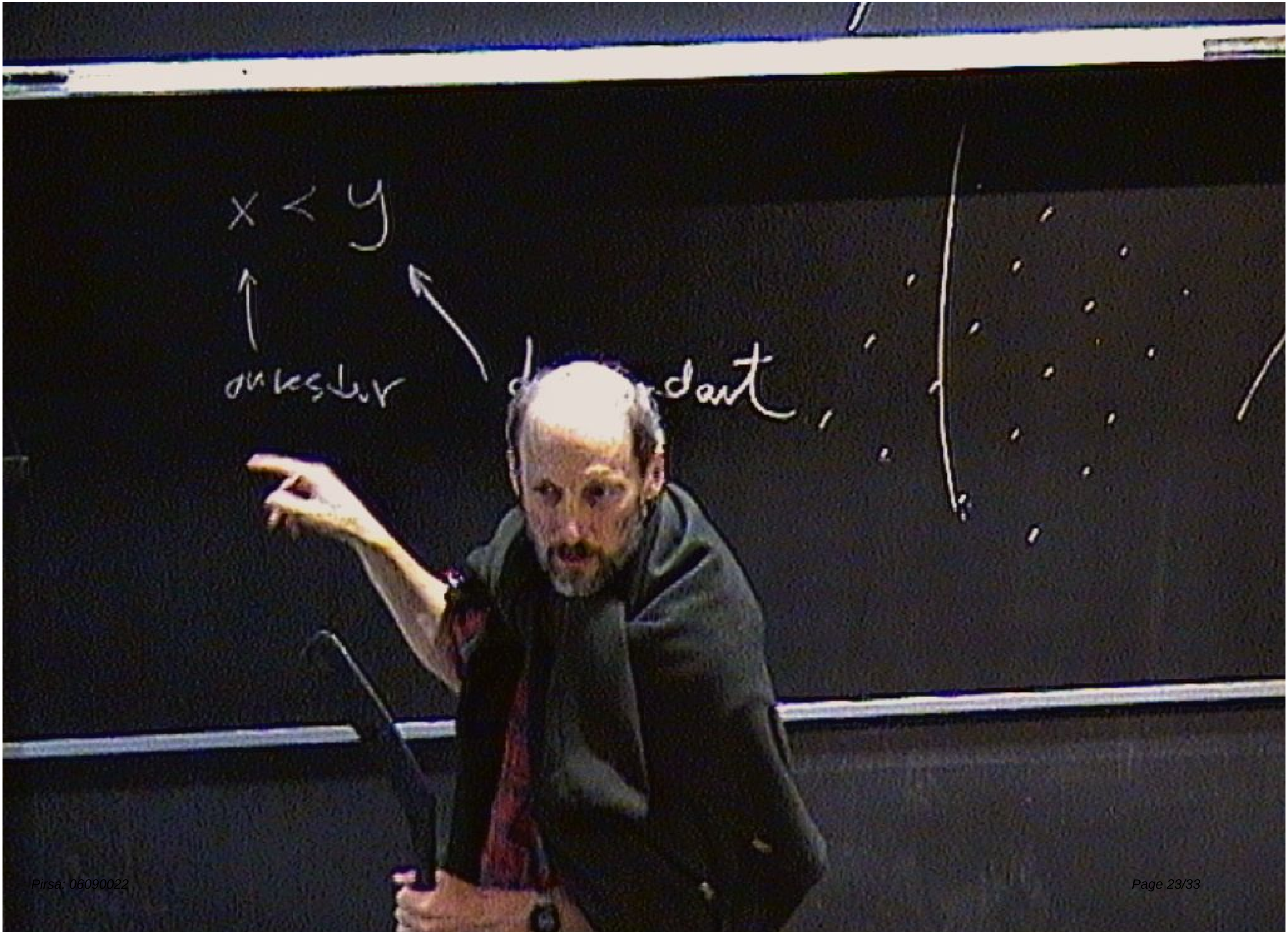
(So with probability 1, a sprinkling will not determine a frame.)

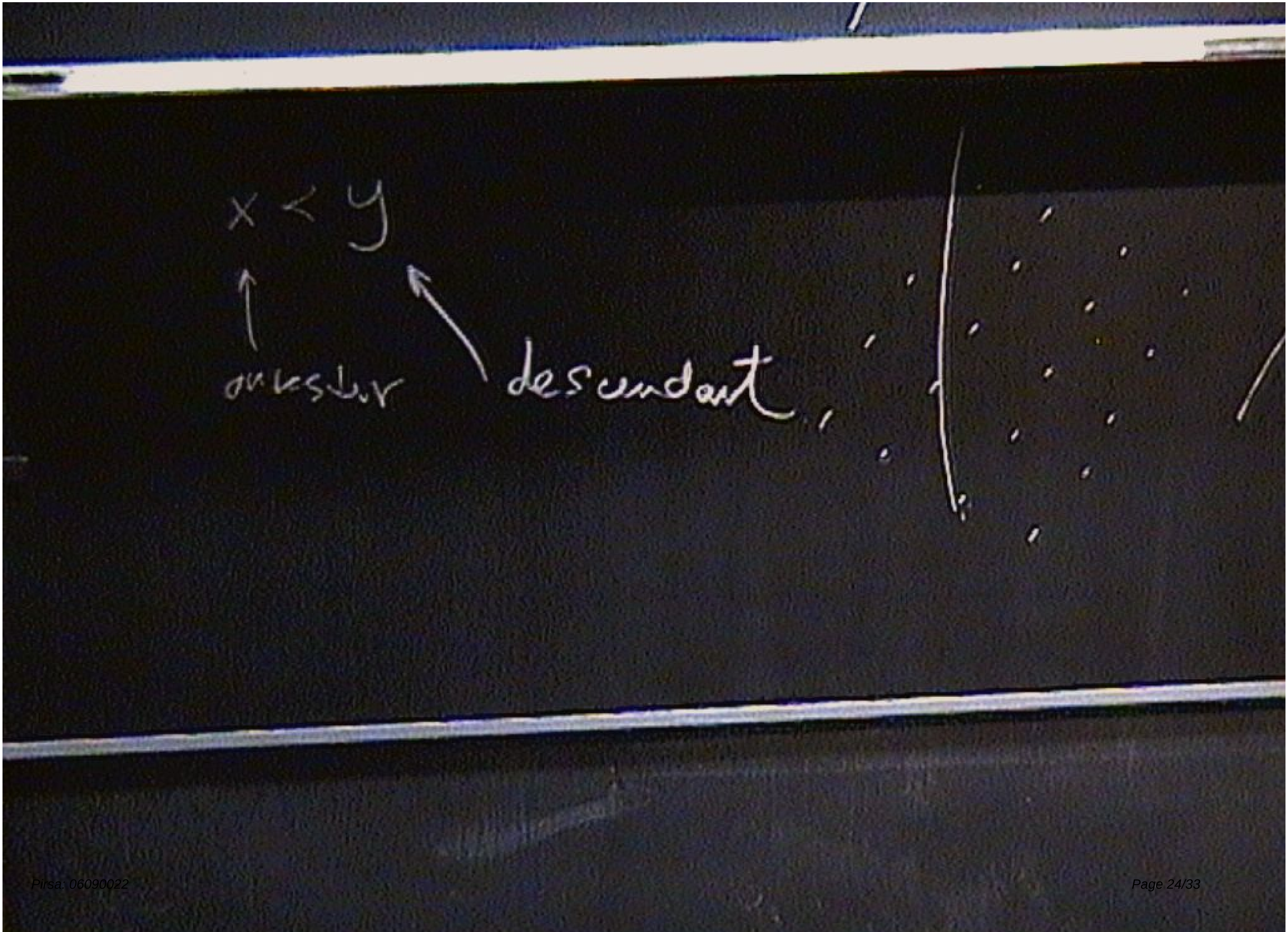
Whence the nonlocality?



Needs a miracle. (consider eg $\phi = t^2 - x^2$, invariance $\Rightarrow \infty$?)




x < y
ancestor


x < y
descendant

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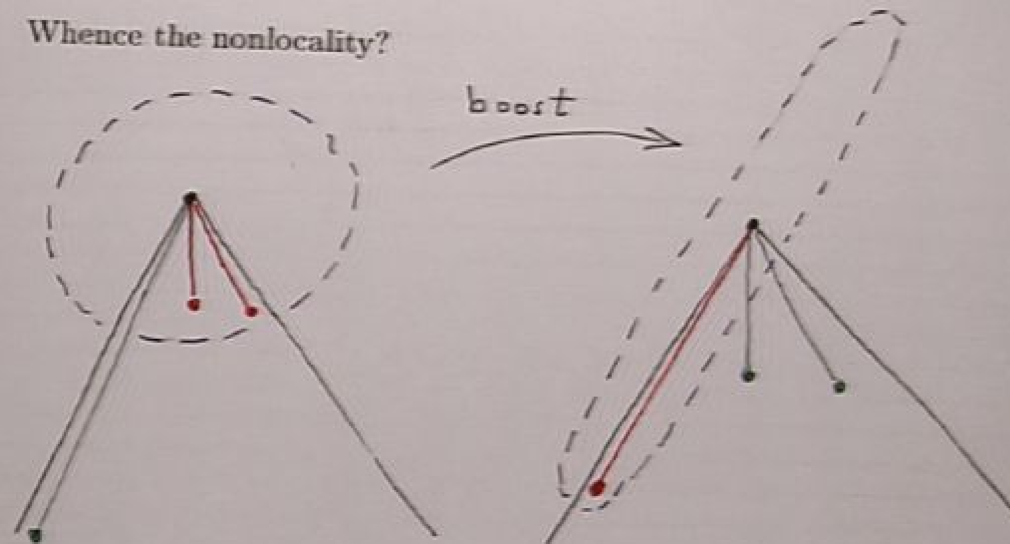

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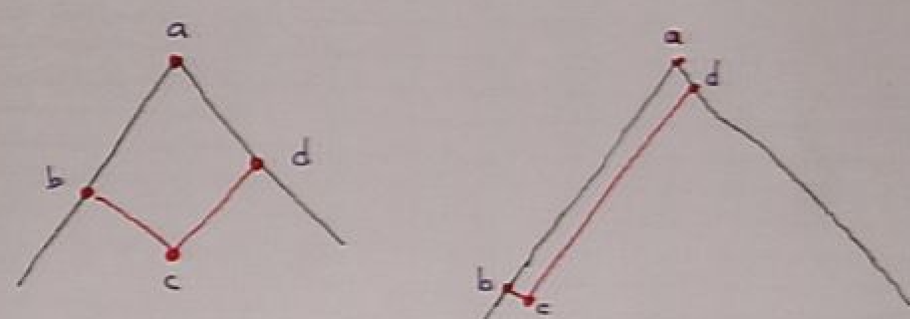


$(a+c) - (b+d) \rightarrow \Box\phi$

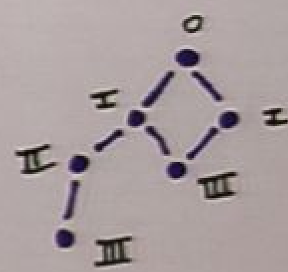
$$(a+c) - (b+d) = (a-d) - (b-c) \rightarrow 0$$

These ideas lead to expressions like

$$\frac{4}{l^2}\left(-\frac{1}{2}\phi(0)+\sum_{x\in I}\phi(x)-2\sum_{x\in II}\phi(x)+\sum_{x\in III}\phi(x)\right)$$

i.e.

$$\Box\phi(i) \leftrightarrow \sum_k B(i,k)\phi(k)$$



where

$$\frac{l^2}{4}B(i,k)=\begin{cases} -\frac{1}{2} \text{ if } i=k \\ 1 \text{ if } i\prec k \text{ is a link (NN) } |<i,k>|=0 \\ -2 \text{ if } i\prec k \text{ and (NNN) } |<i,k>|=1 \\ 1 \text{ if } i\prec k \text{ and (NNNN) } |<i,k>|=2 \end{cases}$$

Can prove that, as $l\to 0$

$$S\equiv \mathbf{E}\sum_k B_{ik}\phi_k \qquad\qquad \to \Box\phi(x_i)$$

using e.g.

$$\mathbf{E}\sum_{x\in I}\phi(x)=\int\frac{dudv}{l^2}\exp\{-uv/l^2\}\,\phi(u,v)$$

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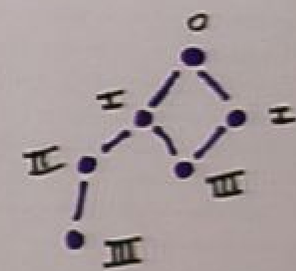

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IDEA: Our averaged sum is a *continuum* expression,

$$\int B(x-x')\,\phi(x')\,d^2x' \,,$$

where

$$B(x) = \theta(x)\left(-2K\delta(x) + 4K^2\,p(\xi)\,e^{-\xi}\right) \,,$$

with $p(\xi) = 1 - 2\xi + \frac{1}{2}\xi^2$, $\xi = Kuv$, and $K = 1/l^2$.

But can *decouple* K from l^2. We get a nonlocal continuum analog of the D'alembertian! Call it $\underset{K}{\Box}$.

Umkehren: approximate $\int$ by $\sum$ over sprinkled points!
This produces the causet expression,

$$\frac{4\varepsilon}{l^2}\left(-\frac{1}{2}\phi(y) + \varepsilon \sum_{x \prec y} p(\xi)\,e^{-\xi}\,\phi(x)\right) \,,$$

where $\xi = \varepsilon\,|\langle x, y\rangle|$ and $\varepsilon = l^2 K$.

The "trick" drives down the fluctuations, but nonlocality survives at the intermediate scale $\lambda_0 = 1/\sqrt{K}$.

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Remarks and applications

- Analogous expressions exist in other dimensions. In $d = 4$

$$p(\xi) = 1 - 3\xi + (3/2)\xi^2 - (1/6)\xi^3$$

$$\leftrightarrow \sum_{I} - 3\sum_{II} + 3\sum_{III} - \sum_{IV}$$

- Can now study propagation on sprinkled causet (Rideout) cf. swerves

- The continuum theory's free field is *stable*: ($\ker \Box_K = \ker \Box$) But response to sources differs

- Quantum Field Theory version? New approach to renormalization? Our nonlocality does *not* remove ∞'s, but perhaps it will allow an invariant (Lorentzian) cutoff.

- How big is λ_0? Must balance fluctuations vs. nonlocality. $L = \text{Hubble}^{-1}$, $l =$ Planck length.

$$\lambda_0 \gtrsim (l^2 L)^{1/3}$$

if want $\Box_K$ pointwise accurate. $\Rightarrow$ nuclear size!!