

Title: Matching of the Hagedorn temperature in AdS/CFT

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Abstract:

Matching of the Hagedorn Temperature in AdS/CFT

Troels Harmark

Niels Bohr Institute

Perimeter Institute, 26th of September, 2006

Based on:

hep-th/0605234 & hep-th/0608115 with Marta Orselli (Nordita)

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Motivation:

Thermodynamics of $U(N)$ $\mathcal{N} = 4$ Super Yang-Mills (SYM) on $\mathbb{R} \times S^3$

Why interesting?

Consider $U(N)$ $\mathcal{N} = 4$ SYM in the planar limit

't Hooft coupling $\lambda = g_{YM}^{-2} N$ does not run since $\mathcal{N} = 4$ SYM is finite

For $\lambda \ll 1$: There is a critical temperature $T_H \sim 1/R(S^3)$

$$T < T_H : F(T) \sim \mathcal{O}(1) \quad T > T_H : F(T) \sim \mathcal{O}(N^2)$$

Resembles a confinement/deconfinement transition

T_H a Hagedorn temperature

A Hagedorn density of states $\rho(E) \sim E^{-1} \exp(T_H E)$ for $T \lesssim T_H$

AdS/CFT: $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to type IIB string theory on $AdS_5 \times S^5$

AdS/CFT dictionary: **string tension** $T_{\text{str}} = \frac{1}{2}\sqrt{\lambda}$

Conjecture: The Hagedorn temperature of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$
is dual to the Hagedorn temperature of string theory on $AdS_5 \times S^5$

Is it possible to match the Hagedorn temperature in AdS/CFT?

Gauge theory:

We can only compute Hagedorn temperature for $\lambda \ll 1$

Current status: Free part + one-loop part computed

String theory:

No known first quantization of strings on $\text{AdS}_5 \times S^5$

Hagedorn temperature only computable for pp-wave background
(strings on $\text{AdS}_5 \times S^5$ with large R-charge)

Problem:

Matching of spectra in Gauge-theory/pp-wave correspondence
requires $\lambda \gg 1$

→ Seemingly no possibility of match of Hagedorn temperature

Why does matching of spectra in gauge-theory/pp-wave correspondence require $\lambda \gg 1$?

Consider gauge-theory/pp-wave correspondence of BMN

Z, X: two complex scalars

Consider the three single-trace operators:

$$\mathcal{O}_1 = \text{Tr} [\text{sym}(X^2 Z^J)] \quad \leftarrow \quad \text{Chiral primary (BPS)} \Rightarrow \text{Survives the limit}$$

$$\mathcal{O}_2 = \text{Tr}[X^2 Z^J] \quad \leftarrow \quad \text{Conjectured to decouple in the limit}$$

$$\mathcal{O}_3 = \sum_l e^{2\pi i \frac{ln}{J}} \text{Tr}[XZ^l XZ^{J-l}] \quad \leftarrow \quad \text{Near-BPS} \Rightarrow \text{Survives the limit}$$

Gauge-theory/pp-wave correspondence needs $\lambda \gg 1$ since we are expanding around chiral primaries

Conjecture of BMN: The unwanted states for $\lambda \ll 1$ decouple for $\lambda \gg 1$

Weakly coupled gauge theory:

For $\lambda = 0$: All quantum numbers of $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ the same

\Rightarrow They contribute the same in the partition function

One-loop contribution just a perturbation of this result.

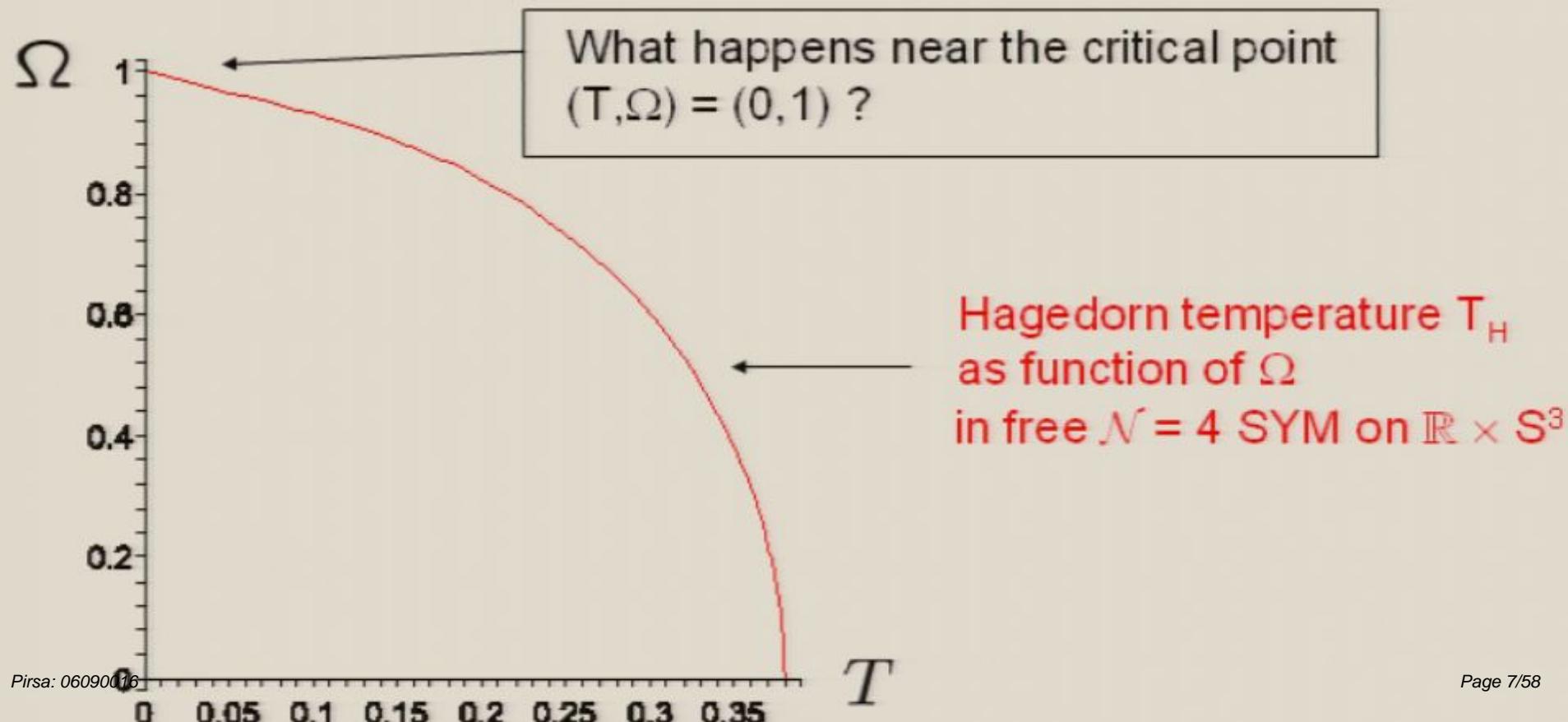
We need a new way to match gauge theory and string theory

→ Consider $\mathcal{N} = 4$ SYM with non-zero chemical potentials for the R-charges

Partition function $Z(\beta, \Omega_i) = \text{Tr} (e^{-\beta D + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)})$

Special case: $J = J_1 + J_2$ and $\Omega_1 = \Omega_2 = \Omega$, $\Omega_3 = 0$

$$Z(\beta, \Omega) = \text{Tr} (e^{-\beta D + \beta \Omega J})$$



We find that in the limit

$$T \rightarrow 0, \Omega \rightarrow 1, \lambda \rightarrow 0, \tilde{T} \equiv \frac{T}{1-\Omega}, \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega}, N \text{ fixed}$$

the partition function of $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ in the planar limit $N = \infty$ is

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(XXX)}(n\tilde{\beta}) \quad \tilde{\beta} = \frac{1}{\tilde{T}}$$

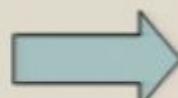
$Z_L^{(XXX)}(\tilde{\beta})$ Partition function for the ferromagnetic Heisenberg $XXX_{1/2}$ spin chain of length L

This is the behavior of planar $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ near the critical point $(T, \Omega) = (0, 1)$

Above limit is a decoupling limit for the Heisenberg chain in $\mathcal{N}=4$ SYM

How does this help?

$\tilde{\lambda} \gg 1$: Only states close to the chiral primaries of the SU(2) sector contributes to the partition function



Opens up possibility of matching spectra of weakly coupled gauge theory and string theory...

Plan for talk:

► Motivation

► Gauge theory side:

Thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

Free thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

Near-critical region in interacting $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

Decoupling limit

Gauge theory spectrum from Heisenberg chain

Hagedorn temperature from Heisenberg chain

► String theory side:

Decoupling limit of string theory on $AdS_5 \times S^5$

Penrose limit, matching of spectra

Computation and matching of the Hagedorn temperature

► Conclusions, Implications for AdS/CFT, Future directions

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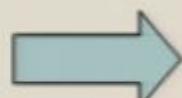
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Thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$:

$$Z(\beta, \Omega_i) = \text{Tr}_M \left(e^{-\beta D + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$$

Partition function
with chemical potentials

D : Dilatation operator

J_i : R-charges for SU(4) R-symmetry of $\mathcal{N} = 4$ SYM

Ω_i : Chemical potentials

State/operator correspondence:

State, CFT on $\mathbb{R} \times S^3$

Energy E

Gauge singlet



Operator, CFT on \mathbb{R}^4

Scaling dimension D

Gauge invariant operator

Gauge singlets:

Because flux lines on S^3 cannot escape

We put $R(S^3) = 1$,
hence $E=D$

M : The set of gauge invariant operators

Given by linear combinations of all possible multi-trace operators

$\text{Tr}(\dots)\text{Tr}(\dots)\dots\text{Tr}(\dots)$

Planar limit $N = \infty$ of $U(N)$ $\mathcal{N} = 4$ SYM

- Large N factorization, traces do not mix
- We can single out the single-trace sector

Single-trace partition function

$$Z_{\text{ST}}(\beta, \Omega_i) = \text{Tr}_S \left(e^{-\beta D + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$$

↑
 S : The set of single-trace operators

Introduce $x = e^{-\beta}$, $y_i = e^{\beta \Omega_i}$

Then we can write $Z_{\text{ST}}(x, y_i) = \text{Tr}_S \left(x^D \prod_{i=1}^3 y_i^{J_i} \right)$

Multi-trace partition function is then

$$\log Z(x, y_i) = \sum_{n=1}^{\infty} \frac{1}{n} Z_{\text{ST}} \left(\omega^{n+1} x^n, y_i^n \right)$$

$\omega = e^{2\pi i}$
Equals -1 when
uplifted to half-integer

Free thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times \mathbf{S}^3$:

$\lambda = 0 : D = D_0 \leftarrow$ The bare scaling dimension

$$\text{Computation of } Z_{\text{ST}}(\beta, \Omega_i) = \text{Tr}_S \left(e^{-\beta D_0 + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$$

Single-trace operators $\text{Tr}(A_1 A_2 \cdots A_L)$, $A_i \in \mathcal{A}$

\mathcal{A} : The set of letters of $\mathcal{N} = 4$ SYM

SU(4) rep

6 real scalars	[0,1,0]
1 gauge boson	[0,0,0]
8 fermions	[1,0,0] \oplus [0,0,1]
plus descendants using the covariant derivative	

$$z(x, y_i) = \text{Tr}_{\mathcal{A}} \left(x^{D_0} \prod_{i=1}^3 y_i^{J_i} \right)$$

$$x = e^{-\beta}, \quad y_i = e^{\beta \Omega_i}$$

$$= \frac{6x^2 - 2x^3}{(1-x)^3} + \frac{x + x^2}{(1-x)^3} \sum_{i=1}^3 (y_i + y_i^{-1}) + \frac{2x^{\frac{3}{2}}}{(1-x)^3} \prod_{i=1}^3 \left(y_i^{\frac{1}{2}} + y_i^{-\frac{1}{2}} \right)$$

From the letter partition function $z(x, y_i)$ we obtain

$$Z_{ST}(x, y_i) = - \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log [1 - z(\omega^{k+1} x^k, y_i^k)]$$

Giving

$$\log Z(x, y_i) = - \sum_{k=1}^{\infty} \log [1 - z(\omega^{k+1} x^k, y_i^k)]$$

Partition function
for free planar
 $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$

Sundborg, Polyakov, Aharony et al.
Yamada & Yaffe, TH & Orselli

Hagedorn temperature:

$Z(x, y_i)$ has a singularity when $z(x, y_i) = 1 \rightarrow$ The Hagedorn singularity

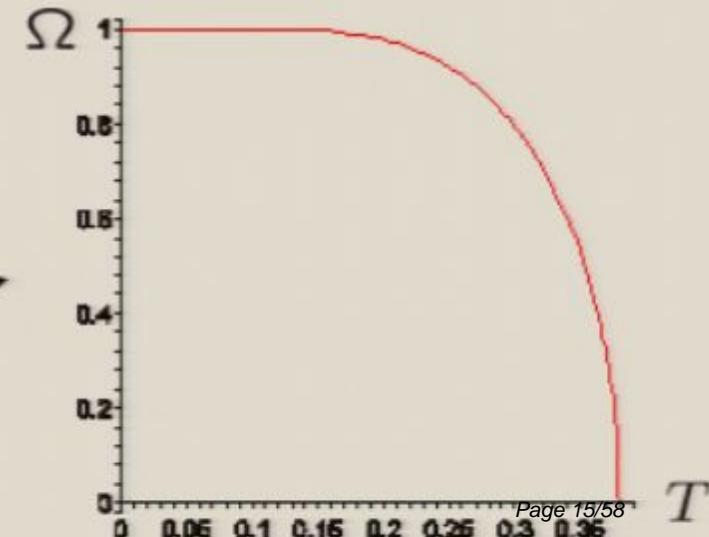
Given the chemical potentials Ω_i :

Defines Hagedorn temperature $T_H(\Omega_1, \Omega_2, \Omega_3)$

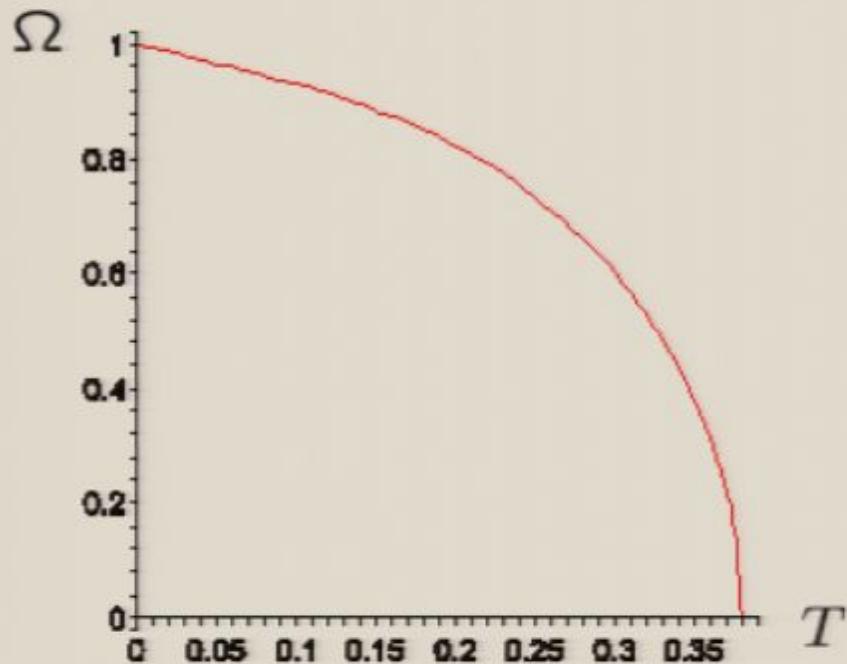
$$\Omega_i = 0 : T_H = \frac{1}{-\log(7 - 4\sqrt{3})}$$

Special cases:

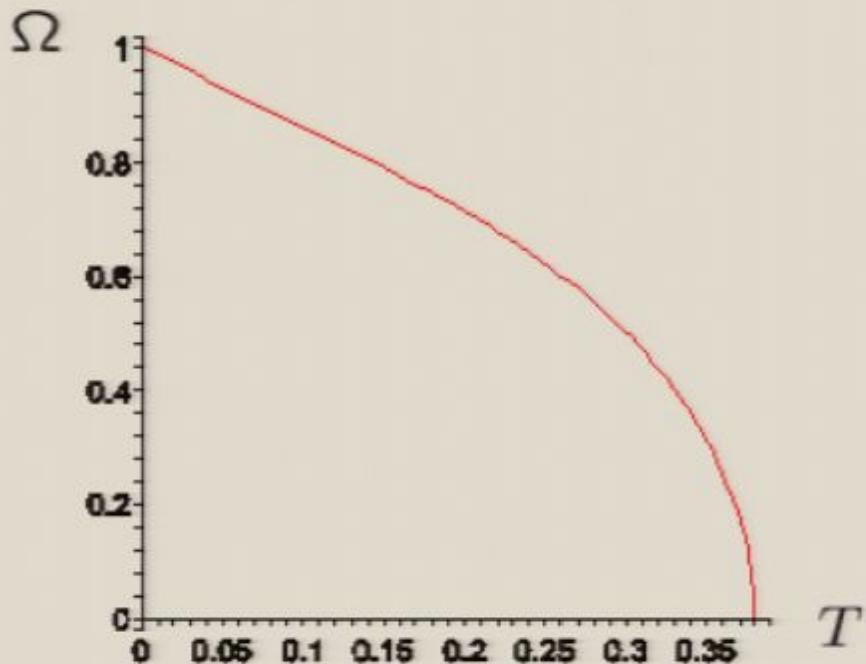
Case 1: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0)$



Case 2: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$



Case 3: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, \Omega)$



$$\Omega \rightarrow 1 : T_H(\Omega) \simeq \frac{1 - \Omega}{\log 2}$$

$$\Omega \rightarrow 1 : T_H(\Omega) \simeq \frac{1 - \Omega}{\log 4}$$

Consider case 2: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$

What happens for $\Omega \rightarrow 1$?

Should also take $T \rightarrow 0$

Try limit: $T \rightarrow 0, \Omega \rightarrow 1, \tilde{T} \equiv \frac{T}{1 - \Omega}$ fixed

Gives finite Hagedorn temperature in the limit:

$$\tilde{T}_H = \frac{1}{\log 2}$$

The limit: $T \rightarrow 0, \Omega \rightarrow 1, \tilde{T} \equiv \frac{T}{1-\Omega}$ fixed

$$(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$$

Corresponds to $x \rightarrow 0, y \rightarrow \infty, \tilde{x} \equiv xy$ fixed

with $x = e^{-\beta}, y = e^{\beta\Omega}$

Take limit of letter partition function

$$z(x, y, y, 1) = \frac{x + x^2}{(1-x)^3} (2 + 2y + 2y^{-1}) + \dots$$

$$\lim_{x \rightarrow 0} z(x, \tilde{x}/x, \tilde{x}/x, 1) = 2\tilde{x}$$

Corresponds to the two complex scalars:

Z : weight (1,0,0)

X : weight (0,1,0)

→ In this limit only the two scalars Z, X survive and the possible operators are:

single-trace operators: $\text{Tr}(A_1 A_2 \cdots A_L), A_i \in \{Z, X\}$
and multi-trace operators by combining these

Therefore: In the above limit we are precisely left with the SU(2) sector of $\mathcal{N} = 4$ SYM

Limit of partition function $\log Z(\tilde{x}) = - \sum_{k=1}^{\infty} \log [1 - 2\tilde{x}^k]$

Hagedorn singularity: $\tilde{x}_H = \frac{1}{2} \Rightarrow T_H = \frac{1}{\log 2}$

Partition function and Hagedorn temperature of the SU(2) sector

The two other cases:

Case 1: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0)$ $\lim_{x \rightarrow 0} z(x, \tilde{x}/x, 1, 1) = \tilde{x}$

Single-trace operators: $\text{Tr}(Z^L)$

half-BPS sector

Case 3: $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, \Omega)$ $\lim_{x \rightarrow 0} z(x, \tilde{x}/x, \tilde{x}/x, \tilde{x}/x) = 3\tilde{x} + 2\tilde{x}^{3/2}$

Single-trace operators: $\text{Tr}(A_1 A_2 \cdots A_L), \quad A_i \in \{Z, X, W, \chi_1, \chi_2\}$

Z,X,W : 3 complex scalars, weights (1,0,0), (0,1,0), (0,0,1)

χ_1, χ_2 : 2 complex fermions, weight (1/2,1/2,1/2)

Near-critical region in interacting $\mathcal{N} = 4$ SYM on $\mathbb{R} \times \mathbf{S}^3$:

We consider weakly coupled $U(N)$ $\mathcal{N} = 4$ SYM on $\mathbb{R} \times \mathbf{S}^3$
near the critical point $(T, \Omega) = (0, 1)$, where $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, \Omega, 0)$

Full partition function:

$$Z(\beta, \Omega) = \text{Tr}_M (e^{-\beta D + \beta \Omega J}) \quad \longleftarrow \quad J = J_1 + J_2$$

Interacting $\mathcal{N} = 4$ SYM:

$$D = D_0 + \lambda D_2 + \lambda^{3/2} D_3 + \lambda^2 D_4 + \dots$$

↑ E.g. Beisert's thesis

Convention here:

$$\lambda = \frac{g_{YM}^2 N}{4\pi^2}$$

Weight factor:

$$e^{-\beta D + \beta \Omega J} = \exp \left(-\beta(D_0 - J) - \beta(1 - \Omega)J - \beta\lambda D_2 - \beta \sum_{n=3}^{\infty} \lambda^{n/2} D_n \right)$$

Near critical region: $T \ll 1, 1 - \Omega \ll 1, \lambda \ll 1$

Since $\beta \gg 1$ and $D_0 - J$ is a non-negative integer

→ Effective truncation to states with $D_0 = J \rightarrow$ The SU(2) sector

Near critical region: $T \ll 1, 1 - \Omega \ll 1, \lambda \ll 1$

Partition function becomes

$$Z(\beta, \Omega) = Z(\tilde{\beta}) = \text{Tr}_{\mathcal{H}} \left(e^{-\tilde{\beta} H} \right)$$

With $\tilde{\beta} \equiv \beta(1 - \Omega), \tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega}$

Hilbert space: $\mathcal{H} = \{\alpha \in M | (D_0 - J)\alpha = 0\}$

Hamiltonian: $H = D_0 + \tilde{\lambda} D_2 + \tilde{\lambda} \lambda \sum_{n=0}^{\infty} \lambda^n D_{4+2n}$

→ Hamiltonian for the SU(2) sector

→ N finite (no planar limit necessary)

The above can be used to study $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$
near the critical point $(T, \Omega_1, \Omega_2, \Omega_3) = (0, 1, 1, 0)$

Decoupling limit:

Consider the limit

$$T \rightarrow 0, \Omega \rightarrow 1, \lambda \rightarrow 0, \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed}, \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed}, N \text{ fixed}$$

From the analysis before we see that the full partition function in this limit is

$$Z(\tilde{\beta}) = \text{Tr}_{\mathcal{H}} \left(e^{-\tilde{\beta} H} \right)$$

The SU(2) sector

Hilbert space: $\mathcal{H} = \{\alpha \in M | (D_0 - J)\alpha = 0\}$

Hamiltonian: $H = D_0 + \tilde{\lambda} D_2$

The Hamiltonian truncate → has only the bare + one-loop term

Note also: $\tilde{\lambda}$ can be finite, i.e. it does not have to be small



The exact partition function can in principle be computed for finite $\tilde{\lambda}$ and finite N

Near critical region: $T \ll 1, 1 - \Omega \ll 1, \lambda \ll 1$

Partition function becomes

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Planar limit $\mathcal{N} = \infty$

→ we can focus on the single-trace sector

$$\text{Tr}(A_1 A_2 \cdots A_L), \quad A_i \in \{Z, X\}$$

→ like a spin chain $Z : \uparrow, X : \downarrow$

$$\begin{array}{c} \text{Tr}(XZZX \cdots Z) \\ \downarrow \\ |\downarrow\uparrow\uparrow\downarrow\cdots\uparrow\rangle \end{array}$$

Which spin chain?

$$D_2 = \frac{1}{2} \sum_{i=1}^L (I_{i,i+1} - P_{i,i+1})$$

L: Length of single-trace operator / spin chain

$\tilde{\lambda} D_2$: Hamiltonian of ferromagnetic $XXX_{1/2}$ Heisenberg spin chain

Minahan & Zarembo

Total Hamiltonian: $H = L + \tilde{\lambda} D_2$

In the limit $T \rightarrow 0$, $\tilde{T} \equiv \frac{T}{1 - \Omega}$ fixed, $\tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega}$ fixed, N fixed

planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ has the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(XXX)}(n\tilde{\beta})$$

$$Z_L^{(XXX)}(\tilde{\beta}) = \text{Tr}_L \left(e^{-\tilde{\beta}\tilde{\lambda}D_2} \right)$$

←
Chains of length L

Partition function for the
ferromagnetic $XXX_{1/2}$
Heisenberg spin chain

→ The ferromagnetic Heisenberg model is obtained as a limit of weakly coupled planar $\mathcal{N} = 4$ SYM

Alternative version of limit (not thermal):

We consider $U(N)$ $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ in the limit

$$\epsilon \rightarrow 0, \tilde{H} \equiv \frac{E - J}{\epsilon} \text{ fixed}, \tilde{\lambda} \equiv \frac{\lambda}{\epsilon} \text{ fixed}, J_i \text{ fixed}, N \text{ fixed}$$

In this limit planar $\mathcal{N}=4$ SYM becomes the ferromagnetic $XXX_{1/2}$ Heisenberg model (for the single-trace sector)

\tilde{H} : Hamiltonian for Heisenberg model

Limit very different from pp-wave limits
where $E - J$ is fixed while $J \rightarrow \infty$ and $N \rightarrow \infty$

$\mathcal{N}=4$ SYM is weakly coupled

The decoupled theory is purely bosonic

In the limit $T \rightarrow 0$, $\tilde{T} \equiv \frac{T}{1 - \Omega}$ fixed, $\tilde{\lambda} \equiv \frac{\lambda}{1 - \Omega}$ fixed, N fixed

planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ has the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(XXX)}(n\tilde{\beta})$$

$$Z_L^{(XXX)}(\tilde{\beta}) = \text{Tr}_L \left(e^{-\tilde{\beta}\tilde{\lambda}D_2} \right)$$

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Spectrum of gauge theory from Heisenberg chain:

We can now obtain the low energy part of the spectrum for finite $\tilde{\lambda}$
Hamiltonian: $\tilde{\lambda}D_2$

Spectrum: Vacua ($D_2 = 0$) plus excitations (magnons)

Vacua are given by: $D_2 = 0$

Define the total spin: $S_z = \frac{J_1 - J_2}{2}$

Exists a vacuum for each value of S_z :

$$|S_z\rangle_L \sim \text{Tr}(\text{sym}(Z^{J_1} X^{J_2})) \quad \longleftarrow$$

$$J_1 = \frac{1}{2}L + S_z$$

$$J_2 = \frac{1}{2}L - S_z$$

These $L+1$ states are precisely all the possible states for which $D_2 = 0$, i.e. all the possible vacua

The vacua $|S_z\rangle_L$ are precisely the chiral primaries of $\mathcal{N} = 4$ SYM
obeying $D_0 = J_1 + J_2 (=L)$

→ The low energy excitations are 'close' to BPS

Low energy excitations: Magnons

Assume thermodynamic limit, i.e. Large L

Eigenvalue problem: $\tilde{\lambda}D_2|\Psi\rangle = E|\Psi\rangle$

Ansatz for state with q impurities:

$$|\Psi\rangle = \sum_{l_1, \dots, l_q} \Psi(l_1, \dots, l_q) \prod_{i=1}^q S_{z, l_i} |S_z\rangle_L$$
$$= \sum_{l_1, \dots, l_q} \sum_{s \in Q} \Psi(l_1, \dots, l_q) \prod_{i=1}^q s(l_i) \text{Tr}(A_{s(1)} \cdots A_{s(L)})$$

$$A_{\frac{1}{2}} = Z, \quad A_{-\frac{1}{2}} = X$$

$$Q = \left\{ s = (s(1), s(2), \dots, s(L)) \middle| \sum_{i=1}^L s_i = S_z, \quad s(i) = \pm \frac{1}{2} \right\}$$

Using Bethe ansatz techniques + integrability of the Heisenberg chain
we get the spectrum for $L \gg 1$:

$$E = \frac{2\pi^2 \tilde{\lambda}}{L^2} \sum_{n \neq 0} n^2 M_n, \quad \sum_{n \neq 0} n M_n = 0$$

Hagedorn temperature from Heisenberg chain:

Consider the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} \text{Tr}_L \left(e^{-n\tilde{\beta}\tilde{\lambda}D_2} \right)$$

Define

$$V(t) \equiv \lim_{L \rightarrow \infty} \frac{1}{L} \log \text{Tr}_L \left(e^{-t^{-1}D_2} \right)$$

Notice: $f(t) = -tV(t)$ ← f(t) is the thermodynamic limit of the free energy per site for the Heisenberg chain

We see then that

$$e^{-nL\tilde{\beta}} \text{Tr}_L \left(e^{-n\tilde{\beta}\tilde{\lambda}D_2} \right) \simeq \exp \left(-nL\tilde{\beta} + LV((n\tilde{\beta}\tilde{\lambda})^{-1}) \right) \text{ for } L \rightarrow \infty$$

Therefore we have the Hagedorn singularity for temperature $\tilde{T} = \tilde{T}_H$ given by

$$\tilde{T}_H = \frac{1}{V(\tilde{\lambda}^{-1}\tilde{T}_H)}$$

A general relation between thermodynamics of Heisenberg chain and the Hagedorn temperature

n=1 gives the first singularity

$$\tilde{T}_H = \frac{1}{V(\tilde{\lambda}^{-1}\tilde{T}_H)}$$



Defines \tilde{T}_H as function of $\tilde{\lambda}$

Large $\tilde{\lambda}$	\longleftrightarrow	Low temperatures $t \ll 1$
Small $\tilde{\lambda}$	\longleftrightarrow	High temperatures $t \gg 1$

Small $\tilde{\lambda}$ /high temperatures:

$$V(t) = \log 2 - \frac{1}{4t} + \frac{3}{32t^2} - \frac{1}{64t^3} - \frac{5}{1024t^4} + \frac{3}{1024t^5} + \mathcal{O}(t^{-6})$$

Obtained from the integral equation:

$$u(x) = 2 + \oint_C \frac{dy}{2\pi i} \left\{ \frac{1}{x-y-2i} \exp \left[-\frac{2t^{-1}}{y(y+2i)} \right] + \frac{1}{x-y+2i} \exp \left[-\frac{2t^{-1}}{y(y-2i)} \right] \right\} \frac{1}{u(y)}$$

$$V(t) = \log [u(0)]$$

Shiroishi & Takahashi

Using the general formula we get

$$\begin{aligned} \tilde{T}_H = & \frac{1}{\log 2} + \frac{1}{4 \log 2} \tilde{\lambda} - \frac{3}{32} \tilde{\lambda}^2 + \left(\frac{3}{128} + \frac{\log 2}{64} \right) \tilde{\lambda}^3 + \left(-\frac{3}{512} - \frac{17 \log 2}{1024} + \frac{5(\log 2)^2}{1024} \right) \tilde{\lambda}^4 \\ & + \left(\frac{3}{2048} + \frac{39 \log 2}{4096} + \frac{3(\log 2)^2}{4096} - \frac{3(\log 2)^3}{1024} \right) \tilde{\lambda}^5 + \mathcal{O}(\tilde{\lambda}^6) \end{aligned}$$

large $\tilde{\lambda}$ / low temperatures:

Using the low-energy spectrum $E = \frac{2\pi^2 \tilde{\lambda}}{L^2} \sum_{n \neq 0} n^2 M_n$, $\sum_{n \neq 0} n M_n = 0$

we find for $t \ll 1$: $V(t) = \zeta\left(\frac{3}{2}\right) \sqrt{\frac{t}{2\pi}}$

This gives $\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-2/3} \tilde{\lambda}^{1/3}$ for $\tilde{\lambda} \gg 1$

Sensible that $\tilde{T}_H \rightarrow \infty$ for $\tilde{\lambda} \rightarrow \infty$ since from the Hamiltonian $\tilde{\lambda} D_2$ we see that the vacua gives the dominant contribution

→ Partition function becomes the trace over chiral primaries

Correction computed in the Heisenberg chain: $V(t) = \zeta\left(\frac{3}{2}\right) \sqrt{\frac{t}{2\pi}} - t$

Takahashi

Gives correction: $\tilde{T}_H = \frac{(2\pi)^{1/3}}{\zeta(\frac{3}{2})^{2/3}} \tilde{\lambda}^{1/3} + \frac{4\pi}{3\zeta(\frac{3}{2})^2} + \mathcal{O}(\tilde{\lambda}^{-1/3})$

Decoupling limit of string theory:

$\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ dual to type IIB string theory on $AdS_5 \times S^5$

$$T_{\text{str}} = \frac{1}{2}\sqrt{\lambda}$$

$$g_s = \frac{\lambda}{N}$$

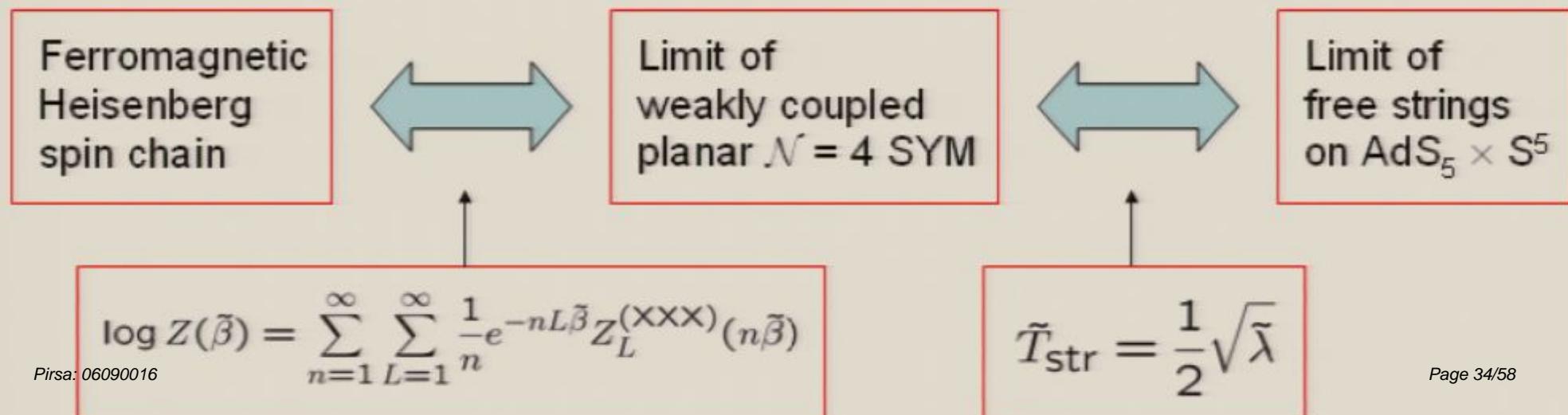
with $T_{\text{str}} = \frac{R^2}{4\pi l_s^2}$

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$$\epsilon \rightarrow 0, \tilde{H} \equiv \frac{E - J}{\epsilon} \text{ fixed}, \tilde{T}_{\text{str}} \equiv \frac{T_{\text{str}}}{\sqrt{\epsilon}} \text{ fixed}, \tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed}, J_i \text{ fixed}$$

A zero string-tension, zero string-coupling limit

Consider planar limit $N = \infty$ / free strings $g_s = 0$:



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Light-cone string spectrum:

$$\begin{aligned} l_s^2 p^+ H_{\text{lc}} &= 2fN_0 + \sum_{n \neq 0} [(\omega_n + f)N_n + (\omega_n - f)M_n] + \sum_{n \in Z} \sum_{I=3}^8 \omega_n N_n^{(I)} \\ &\quad + \sum_{n \in Z} \left[\sum_{b=1}^4 \left(\omega_n - \frac{1}{2}f \right) F_n^{(b)} + \sum_{b=5}^8 \left(\omega_n + \frac{1}{2}f \right) F_n^{(b)} \right] \end{aligned}$$

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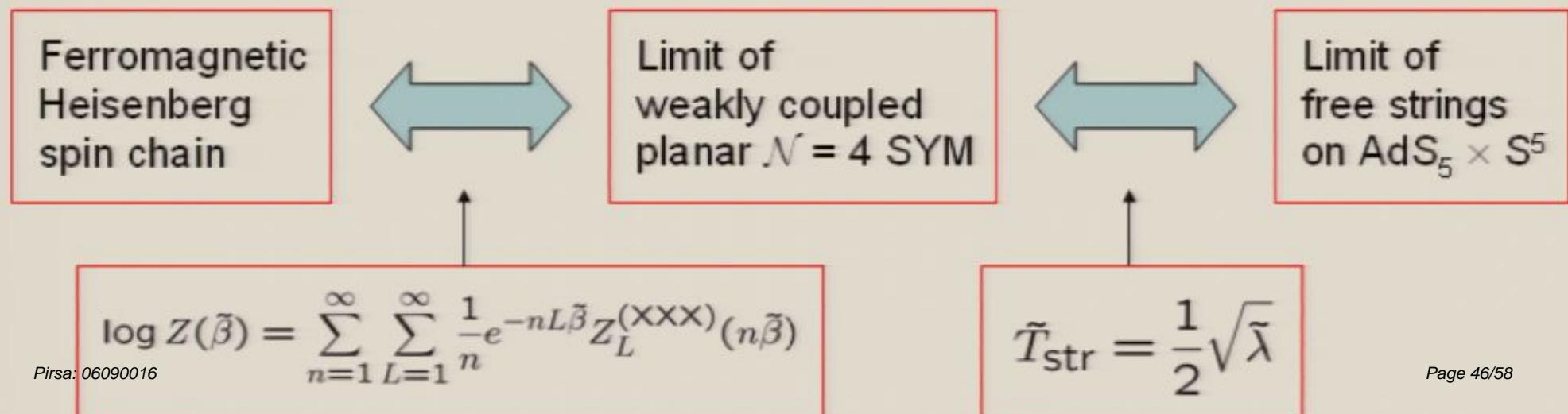
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Computation of Hagedorn temperature, I:

Computation using spectrum after decoupling limit

Multi-string partition function:

$$\log Z(\tilde{a}, \tilde{b}) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(e^{-\tilde{a}n\tilde{H}_{\text{lc}} - \tilde{b}np^+})$$

Trace over single-string states

$Z(a,b)$ has singularity for $\tilde{b}\sqrt{\tilde{a}} = l_s^2 \zeta(3/2) \sqrt{2\pi}$

From the Penrose limit one finds $\tilde{a} = \tilde{\beta}$, $\tilde{b} = \mu T_{\text{str}} l_s^2 \tilde{\beta}$

Using $\mu = \frac{1}{\sqrt{1-\Omega}}$, $\tilde{T}_{\text{str}} = \mu T_{\text{str}}$, $\tilde{T}_{\text{str}} = \frac{1}{2} \sqrt{\tilde{\lambda}}$

we get $\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-2/3} \tilde{\lambda}^{1/3}$

Matches the Hagedorn temperature computed
in gauge theory/Heisenberg chain

Computation of Hagedorn temperature, II:

We can also consider the Hagedorn temperature as computed using the full pp-wave spectrum. Consider the partition function

$$\log Z(a, b) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}((-1)^{(n+1)F} e^{-anH_{\text{lc}} - bnp^+})$$

This has a Hagedorn singularity for

$$b = 4l_s^2 \mu \sum_{p=1}^{\infty} \frac{1}{p} \left[3 + \cosh(\mu ap) - 4(-1)^p \cosh\left(\frac{1}{2}\mu ap\right) \right] K_1(\mu ap)$$

Sugawara

Using now $a = \mu \tilde{\beta}$, $b = \mu T_{\text{str}} l_s^2 \tilde{\beta}$

we can take the large μ limit, obtaining again

$$\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-2/3} \tilde{\lambda}^{1/3}$$

Check on the validity of the decoupling limit

→ verifies commutativity of limits

We have matched spectrum and Hagedorn temperature of weakly coupled gauge theory and free string theory, in a sector of AdS/CFT

Why it worked?

- ▶ Because on the gauge theory side we could consider $\tilde{\lambda} \gg 1$
Corresponds to looking at states near chiral primaries
 - We can ignore most states in the SU(2) sector,
only the magnon states important for low energies
- ▶ Because we have a pp-wave with the same vacuum structure
as for the gauge theory side

A non-trivial match between weakly coupled
gauge theory and weakly coupled string theory

Can either be understood as matching of spectra (non-thermal)
or matching of thermal partition function (thermodynamics)

Conclusions:

We found the decoupling limit

$$T \rightarrow 0, \Omega \rightarrow 1, \lambda \rightarrow 0, \tilde{T} \equiv \frac{T}{1-\Omega}, \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega}, N \text{ fixed}$$

of $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ in which the partition function becomes

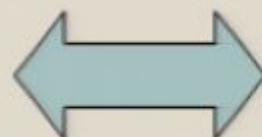
$$Z(\tilde{\beta}) = \text{Tr}_{\mathcal{H}} \left(e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2)} \right)$$

where \mathcal{H} corresponds to SU(2) sector of $\mathcal{N}=4$ SYM.

A well-defined theory for all $\tilde{\lambda}, N$

In the planar limit $N = \infty$:

Physics of Heisenberg model



Physics of decoupled planar $\mathcal{N}=4$ SYM

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(\times \times \times)}(n\tilde{\beta})$$



$$\tilde{T}_H = \frac{1}{V(\tilde{\lambda}^{-1}\tilde{T}_H)}$$

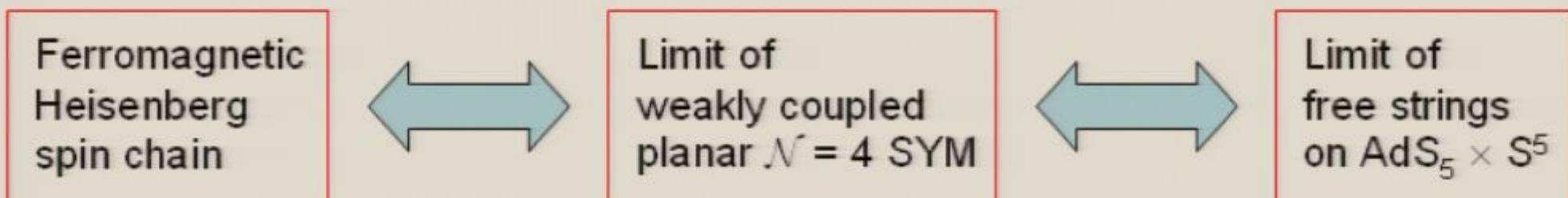
- A manifestly integrable decoupled sector of planar $\mathcal{N} = 4$ SYM
Integrability in planar $\mathcal{N} = 4$ SYM without conjectures!
- Describes $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near
the critical point $(T, \Omega_1, \Omega_2, \Omega_3) = (0, 1, 1, 0)$

Other interesting case: The $SU(2|3)$ sector

- $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near the critical point $(T, \Omega_1, \Omega_2, \Omega_3) = (0, 1, 1, 1)$

Implications for AdS/CFT:

- Dual limit a zero string tension, zero string coupling limit of type IIB string theory on $\text{AdS}_5 \times \text{S}^5$
- Planar limit/zero string coupling: A solvable sector of AdS/CFT



A spin chain/gauge theory/string theory triality

- Explicit matching for spectrum and Hagedorn temperature using pp-wave

$$E = \frac{2\pi^2 \tilde{\lambda}}{L^2} \sum_{n \neq 0} n^2 M_n , \quad \sum_{n \neq 0} n M_n = 0$$

$$\tilde{T}_H = (2\pi)^{1/3} \left[\zeta \left(\frac{3}{2} \right) \right]^{-2/3} \tilde{\lambda}^{1/3}$$

Future directions:

- ▶ Study Hagedorn transition on gauge theory side
- ▶ λ corrections to spectrum/thermodynamics
- ▶ $\tilde{\lambda}^{-1/3}$ corrections on the string side
- ▶ SU(2|3) sector: Which pp-wave?

We have matched spectrum of weakly coupled gauge theory and limit of string theory on pp-wave

- Could put new light on three-loop discrepancy
(no possibility of unknown interpolating functions)
Could explain why the match at two loops – but not at three loops
- 1/N corrections and pp-wave string interactions

$$\gamma' = \frac{3}{J^2}$$