Title: Probability theory --classical, quantum and otherwise

Date: Sep 22, 2006 03:00 PM

URL: http://pirsa.org/06090014

Abstract: Quantum mechanics is a non-classical probability calculus -- but hardly the most general one imaginable. In this talk, I'll discuss some familiar non-classical properties of quantum-probabilistic models that turn out to be features of {em all} non-classical models. These include a generic no-cloning theorem obtained in recent work with Howard Barnum, Jon Barrett and Matt Leifer.

1. Intro
2. Framework
3. $\otimes$
4. No Cloning
5. Wild Speculations.

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2. Framework (Mackey '57, etc...)
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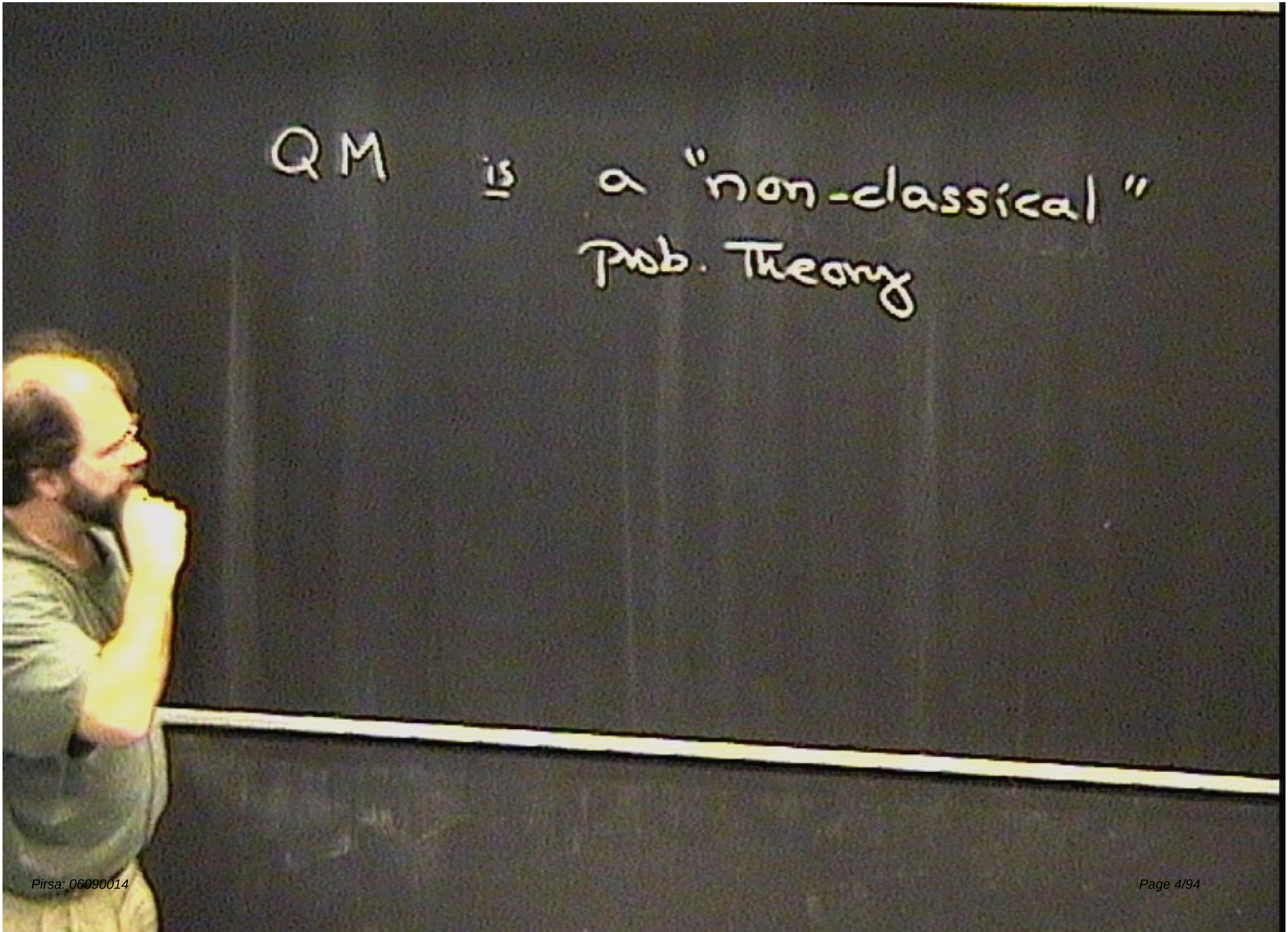
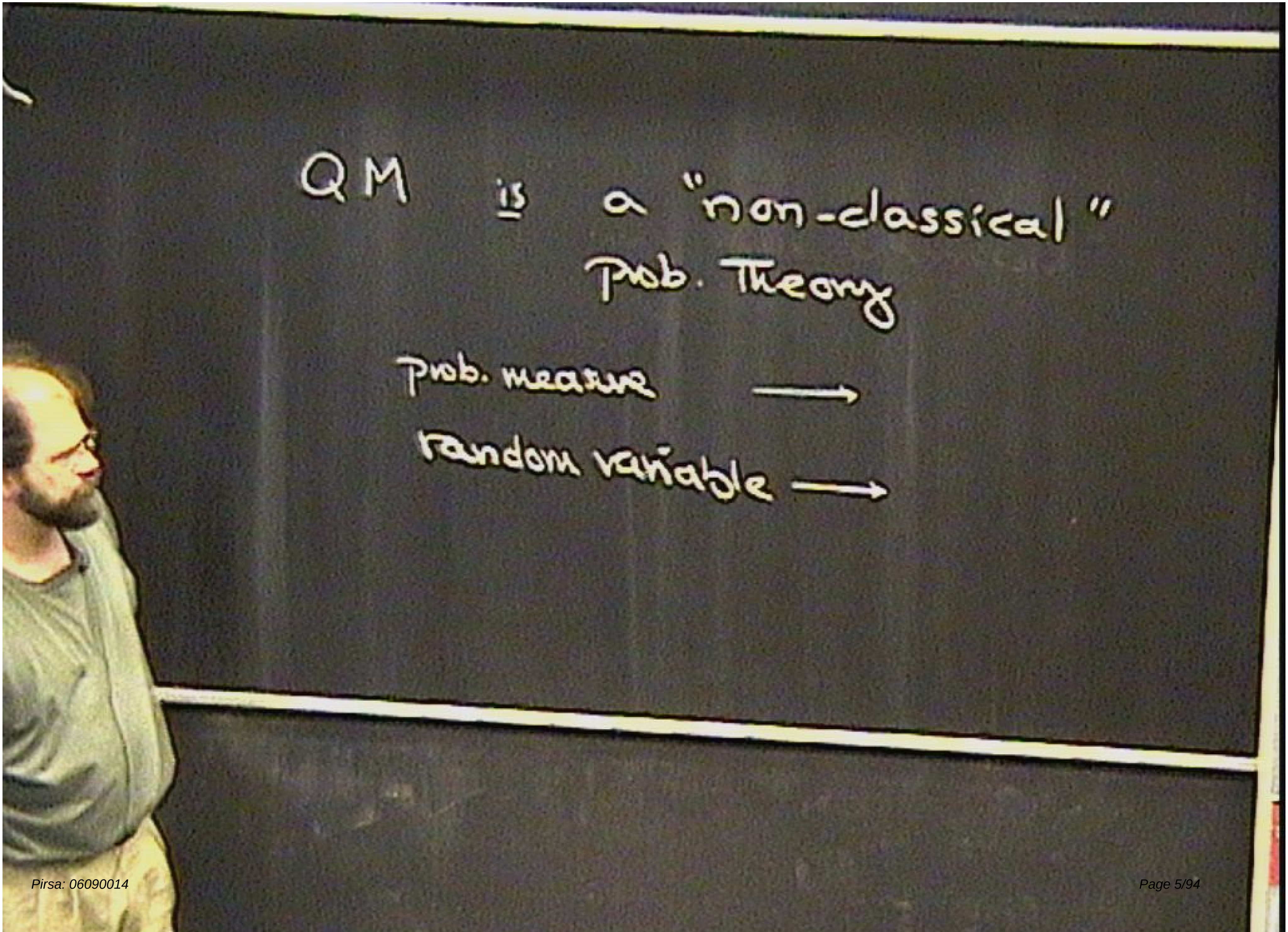

QM is a "non-classical"
Prob. Theory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Prob. Theory
prob. measure ⟶
random variable ⟶

QM is a "non-classical" Prob. Theory

prob. measure $\longrightarrow$ state on a C^* alg

random variable $\longrightarrow$ self-adjoint el't of C^* alg

Rough idea: rvs that don't commute,
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But: If we allow possibility of non-comeasurable "observables" Then we open door to huge range of possible P.T.'s

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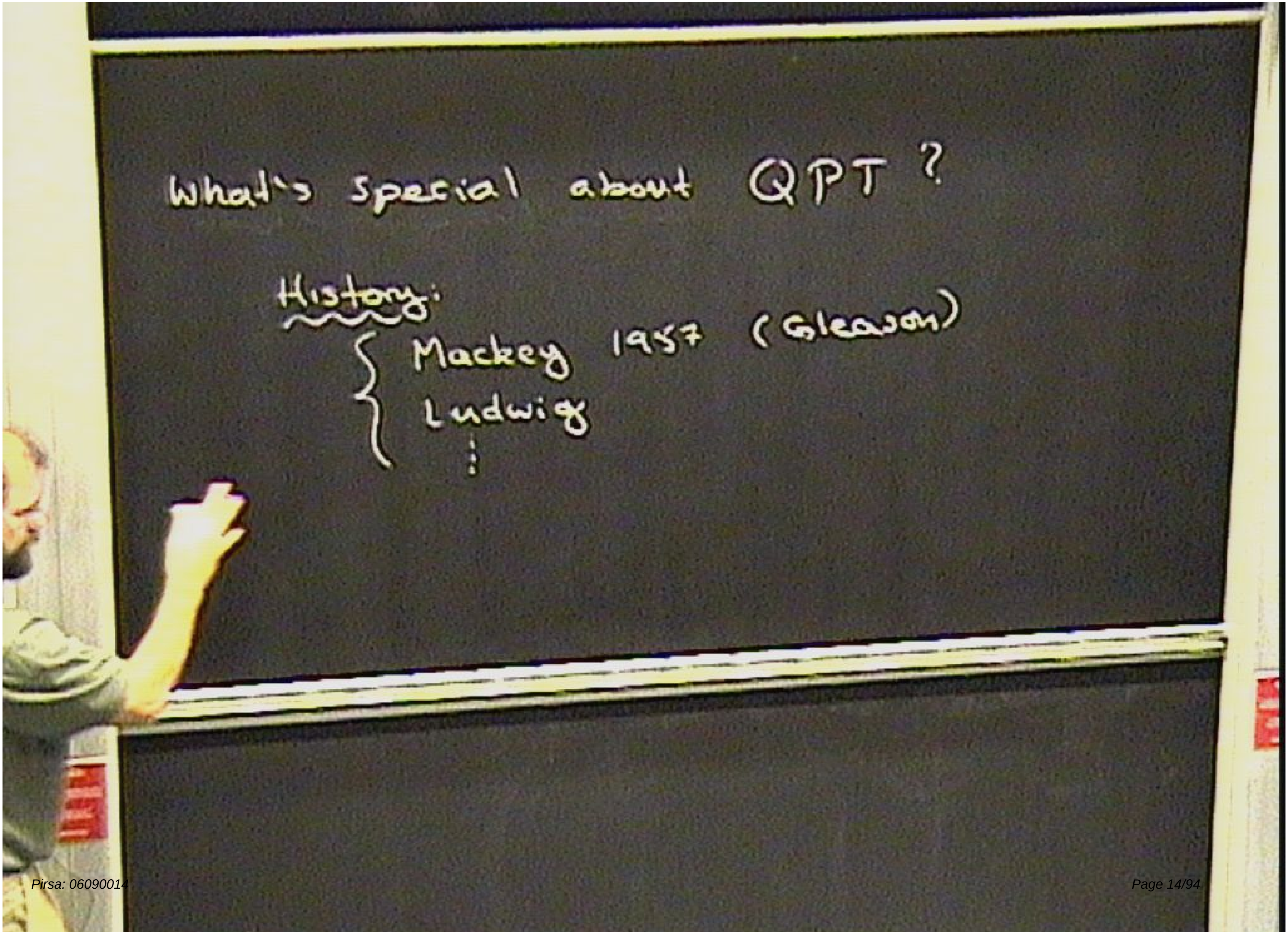
Mackey 1957 (Gleason)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⋮

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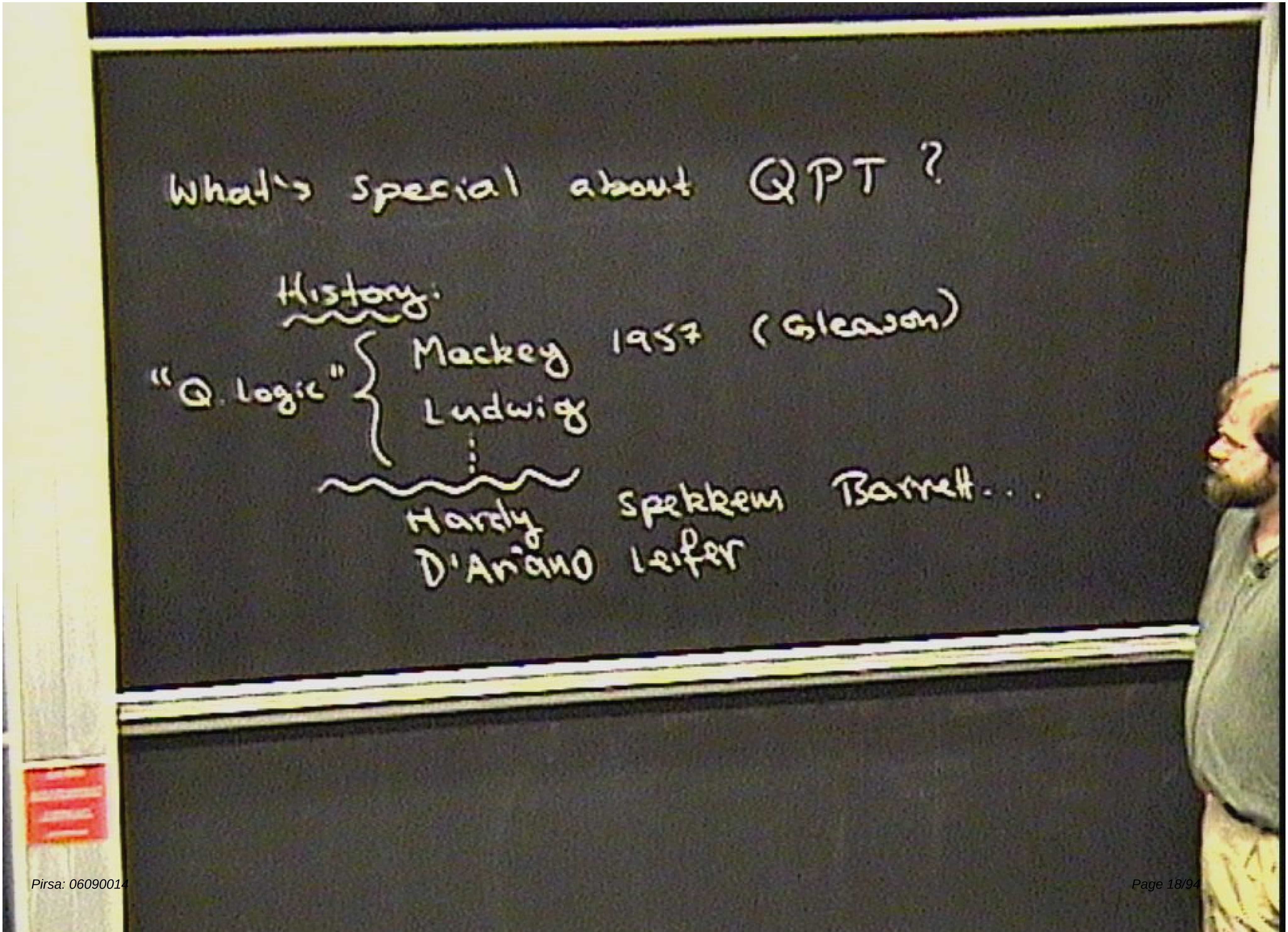
Hardy

D'Ariano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"Q.I." Hardy Spekkens Barrett . . .
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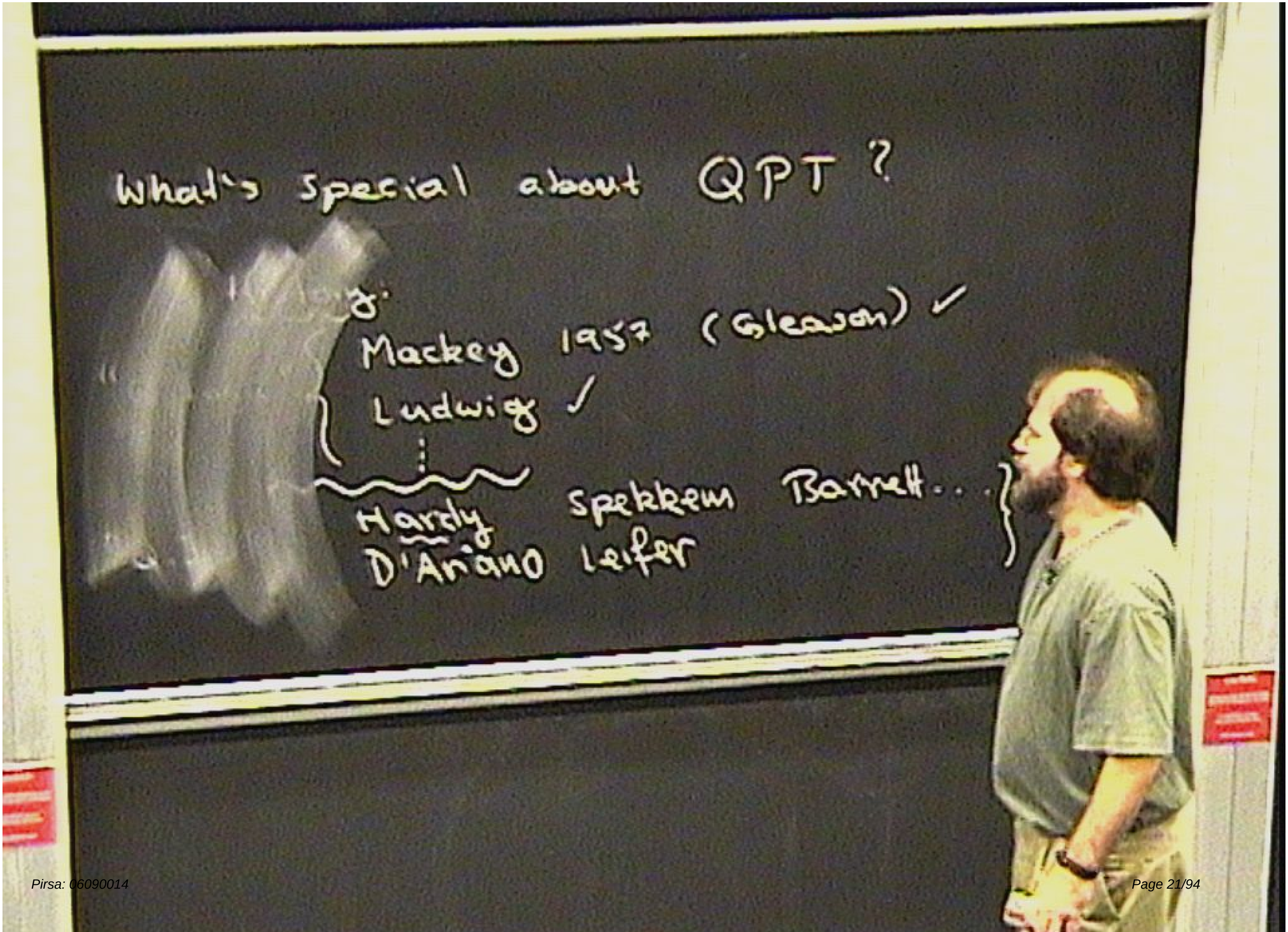
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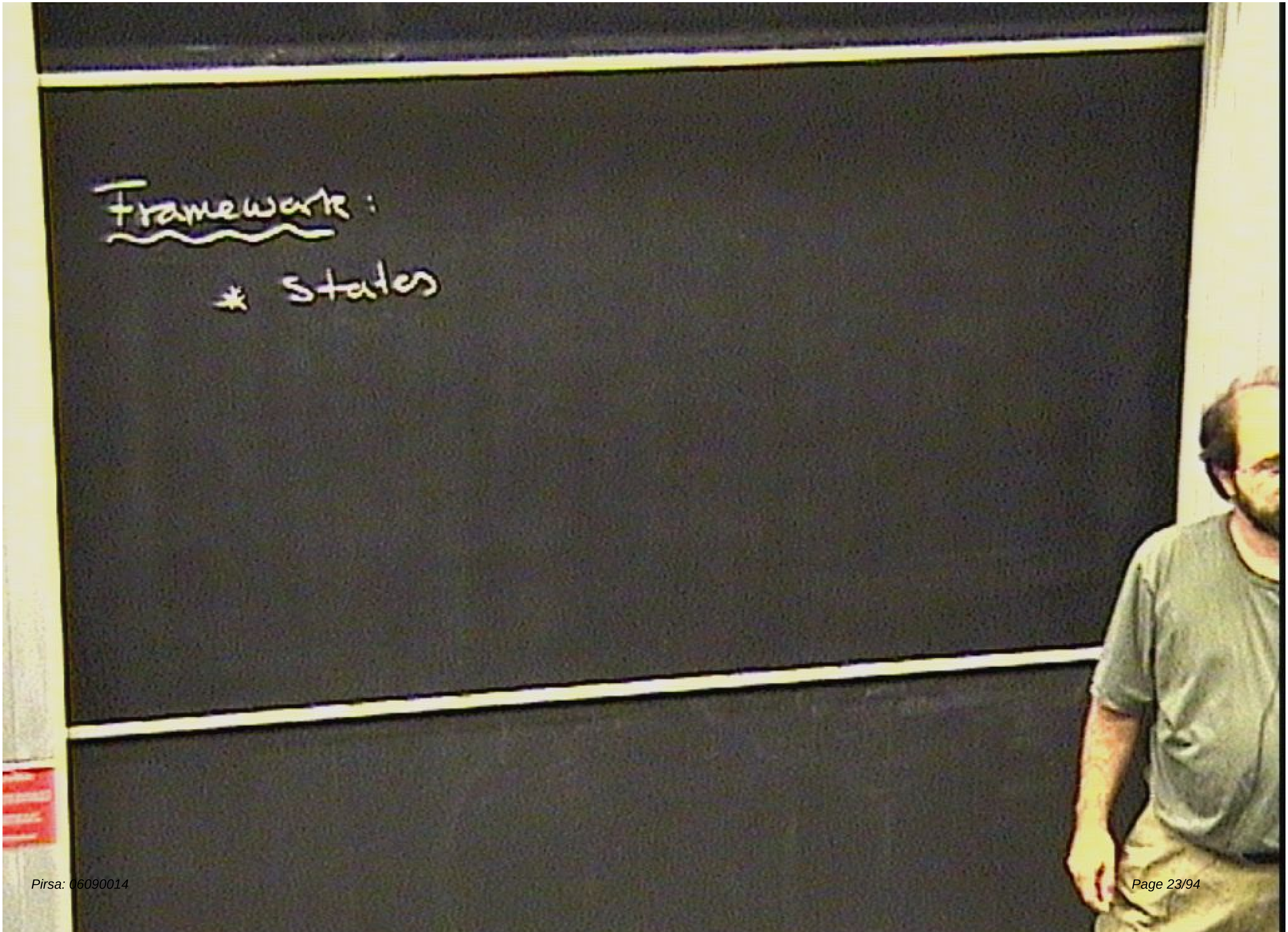
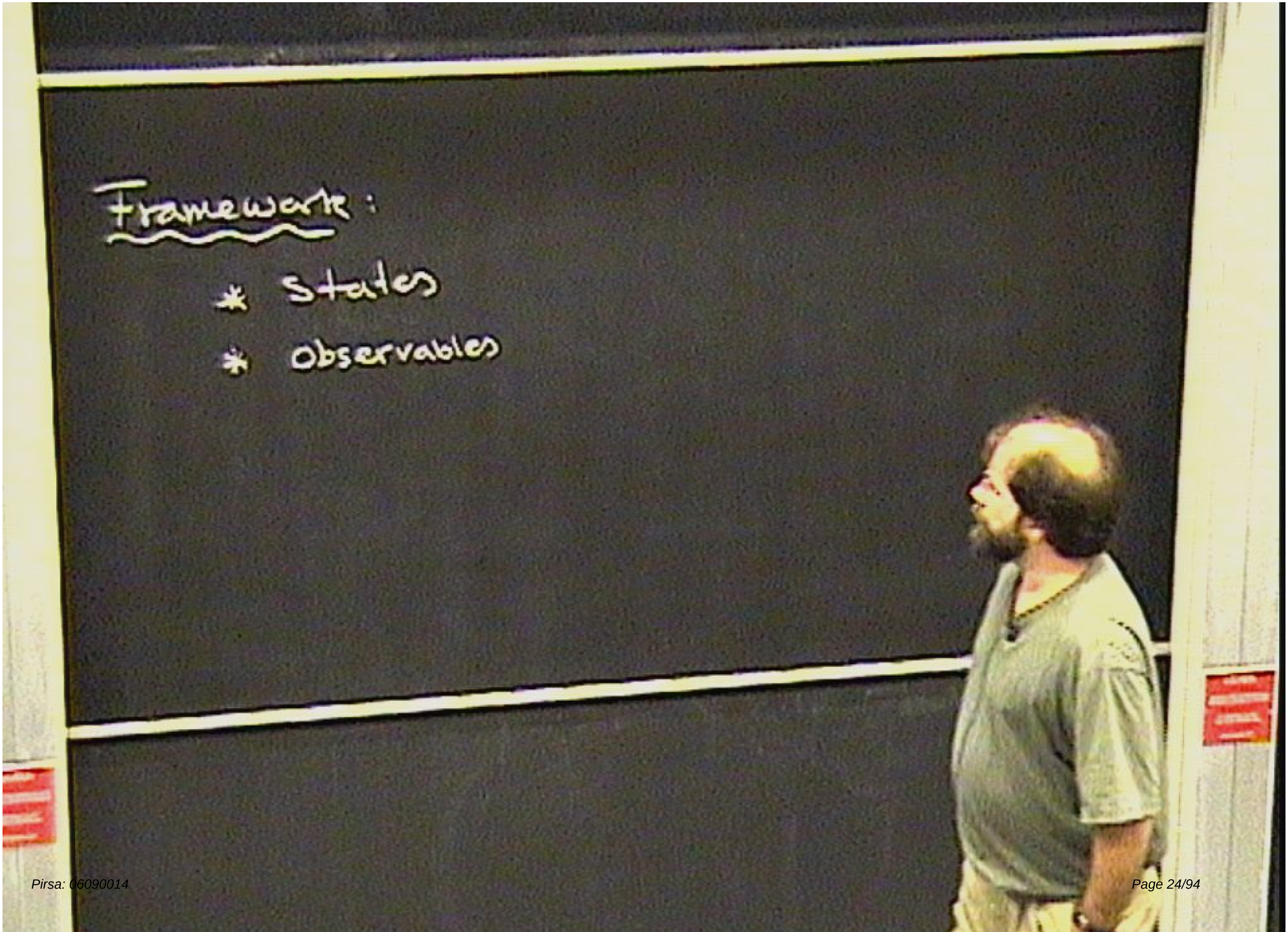
Framework :


Framework:
* states


Framework:
* States
* Observables

Framework:

* States $\longrightarrow$ convex
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* Observables $\longrightarrow$
  * Let $A(\Omega)$ = affine functionals on Ω
  * $u \in A(\Omega)$: $u(\omega) = 1 \ \forall \omega \in \Omega$
  * $[0, u]$

Framework:

* States $\longrightarrow$ convex set Ω

* discrete Observables $\longrightarrow$

$F : E \to [0,u]$

$x \mapsto F_x$

$$\boxed{\sum_{x \in E} F_x = u}$$

* Let $A(\Omega)$ = affine functionals on Ω

* $u \in A(\Omega)$: $u(\omega) = 1 \;\; \forall \omega \in \Omega$

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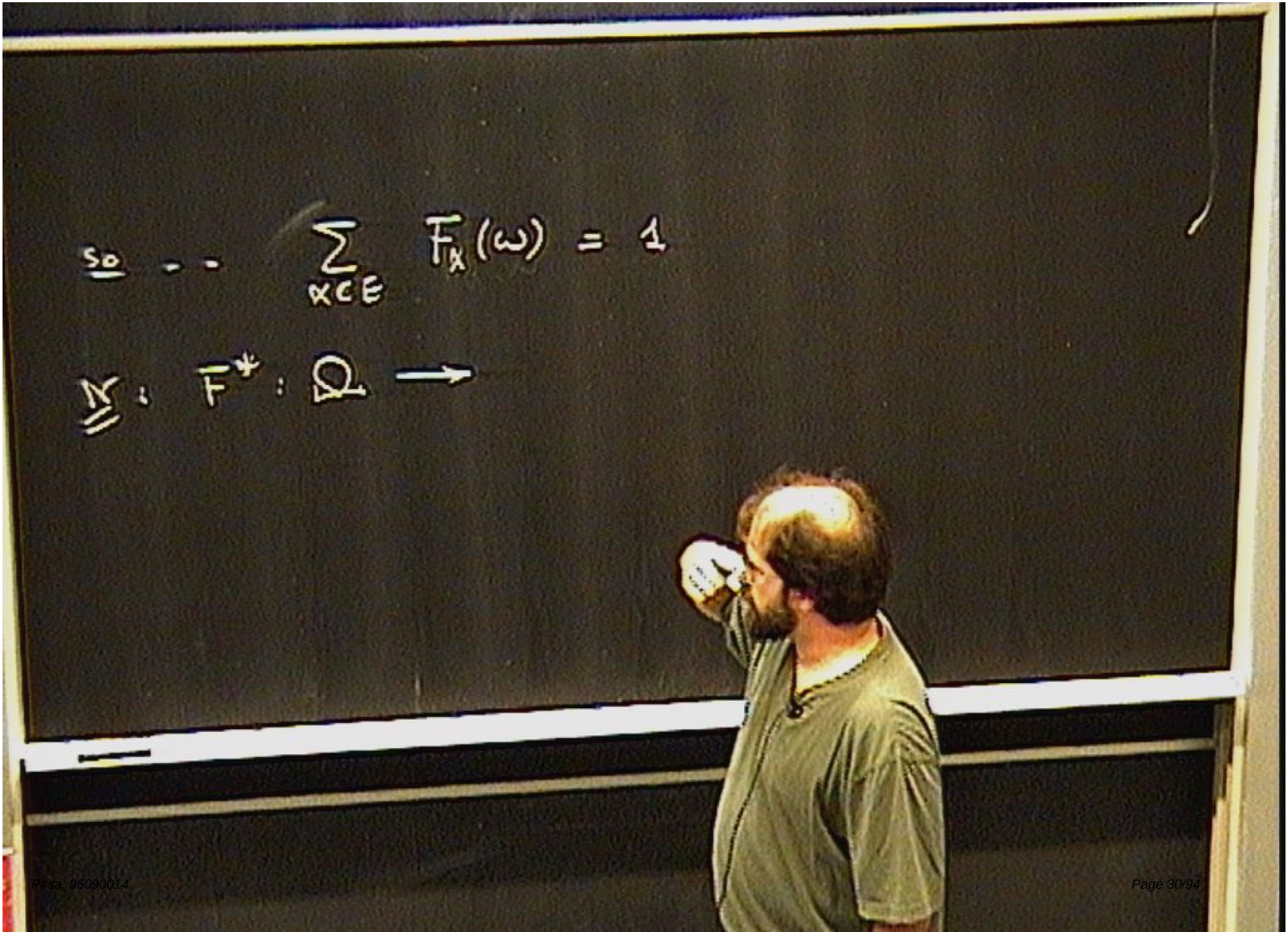
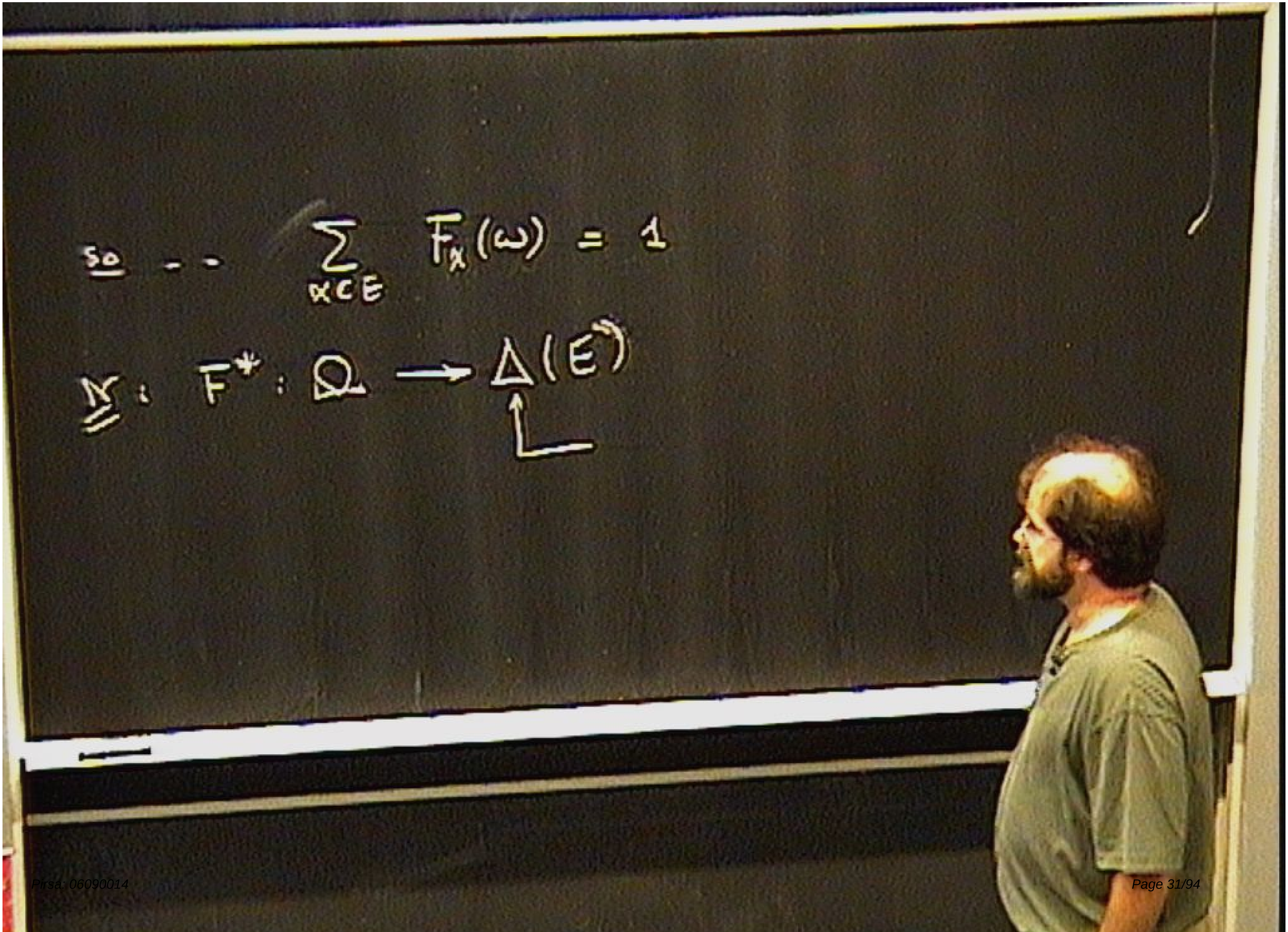

so ... $\sum_{\alpha \subset E} F_\alpha(\omega) = 1$
N : F* : Ω →

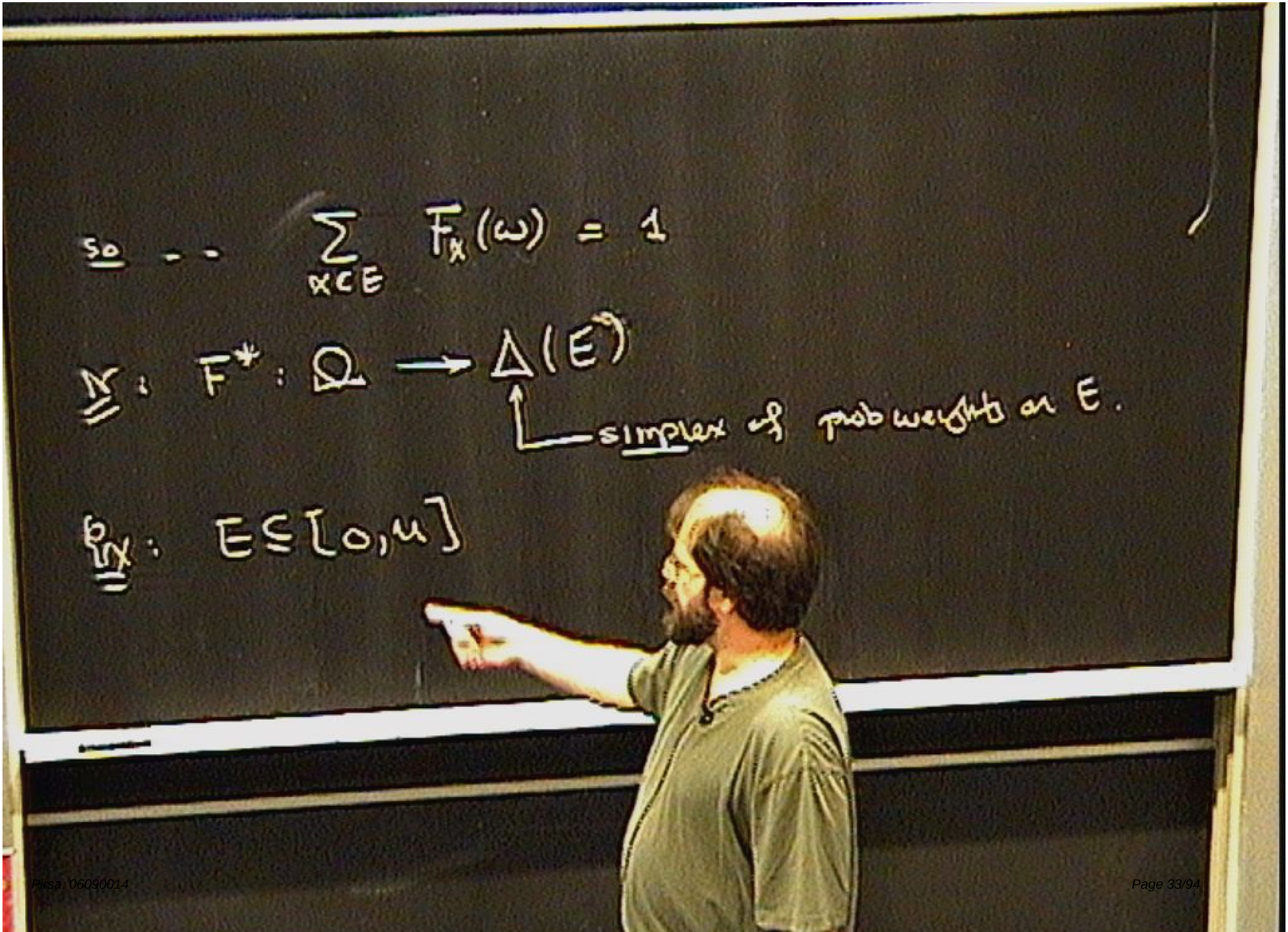

so - - $\sum_{\alpha \subset E} F_\alpha(\omega) = 1$
N: $F^*: \Omega \longrightarrow \Delta(E)$

So -- $\sum_{\alpha \in E} F_\alpha(\omega) = 1$

Nb: $F^* : \Omega \longrightarrow \Delta(E)$

simplex of prob weights on E.

Ex:


so . . . Σ_{x∈E} F_x(ω) = 1
N: F*: Ω → Δ(E)
simplex of prob weights on E.
Ex: E ⊆ [0,u]

so . . . $\sum_{x \in E} F_x(\omega) = 1$

ie: $F^* : \Omega \longrightarrow \Delta(E)$

simplex of prob weights on E.

Ex: $E \subseteq [0, u]$, $\sum_{a \in E} a = u$

Today - all ~~ts~~ finite-dimensional.

<u>So</u> . . . $\sum_{x \in E} F_x(\omega) = 1$

<u>N</u>: $F^* : \Omega \longrightarrow \Delta(E)$

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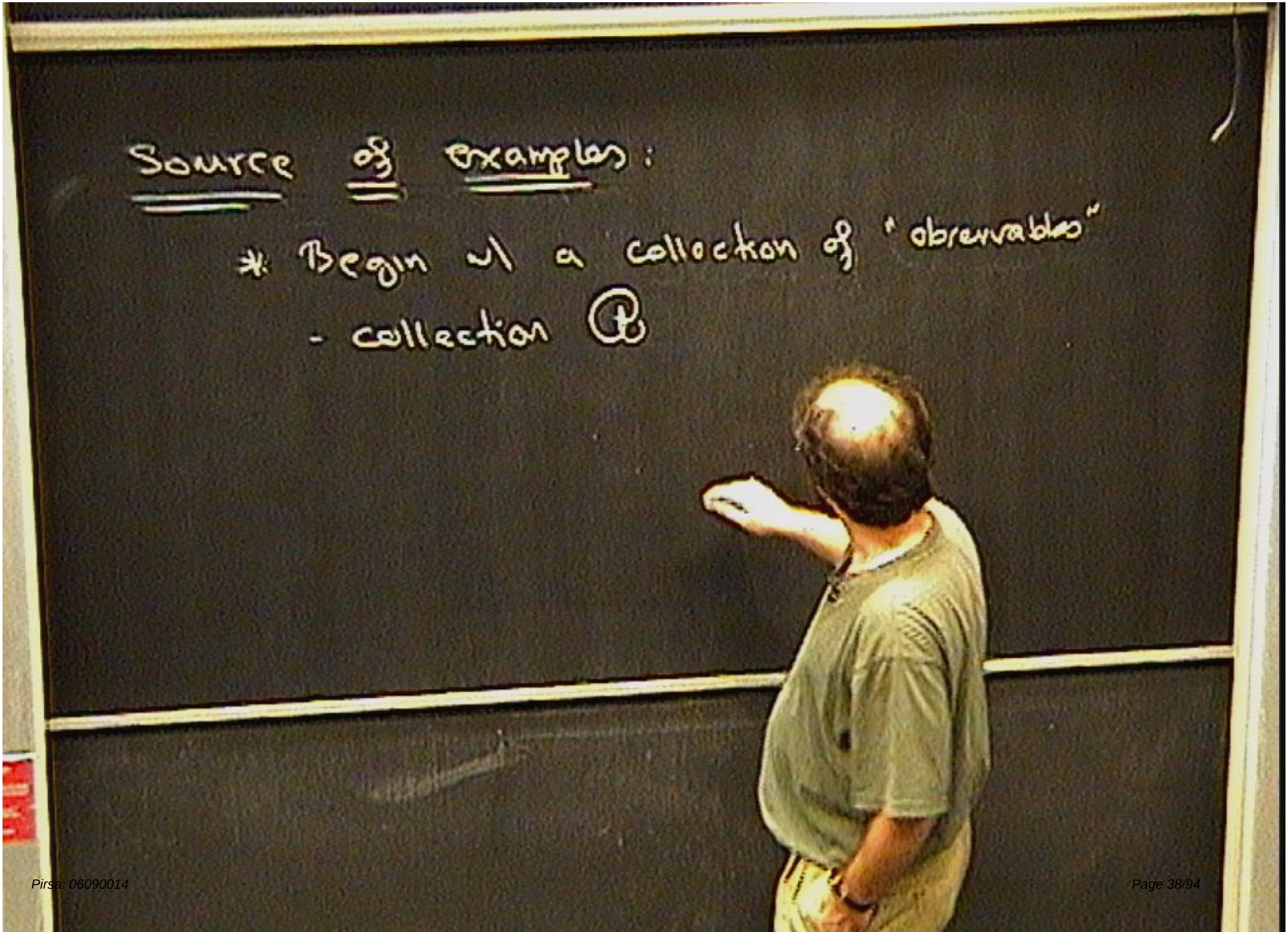
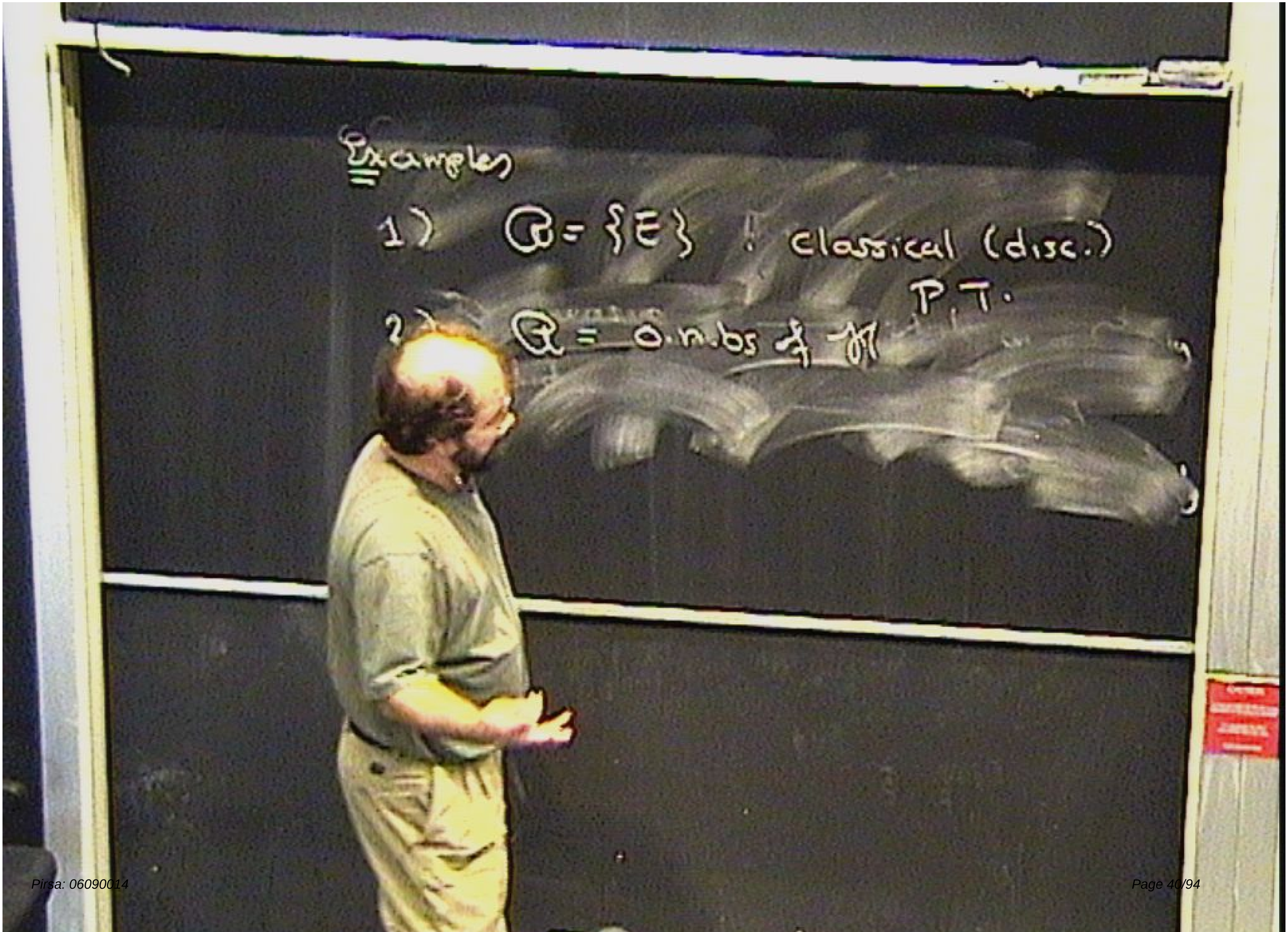
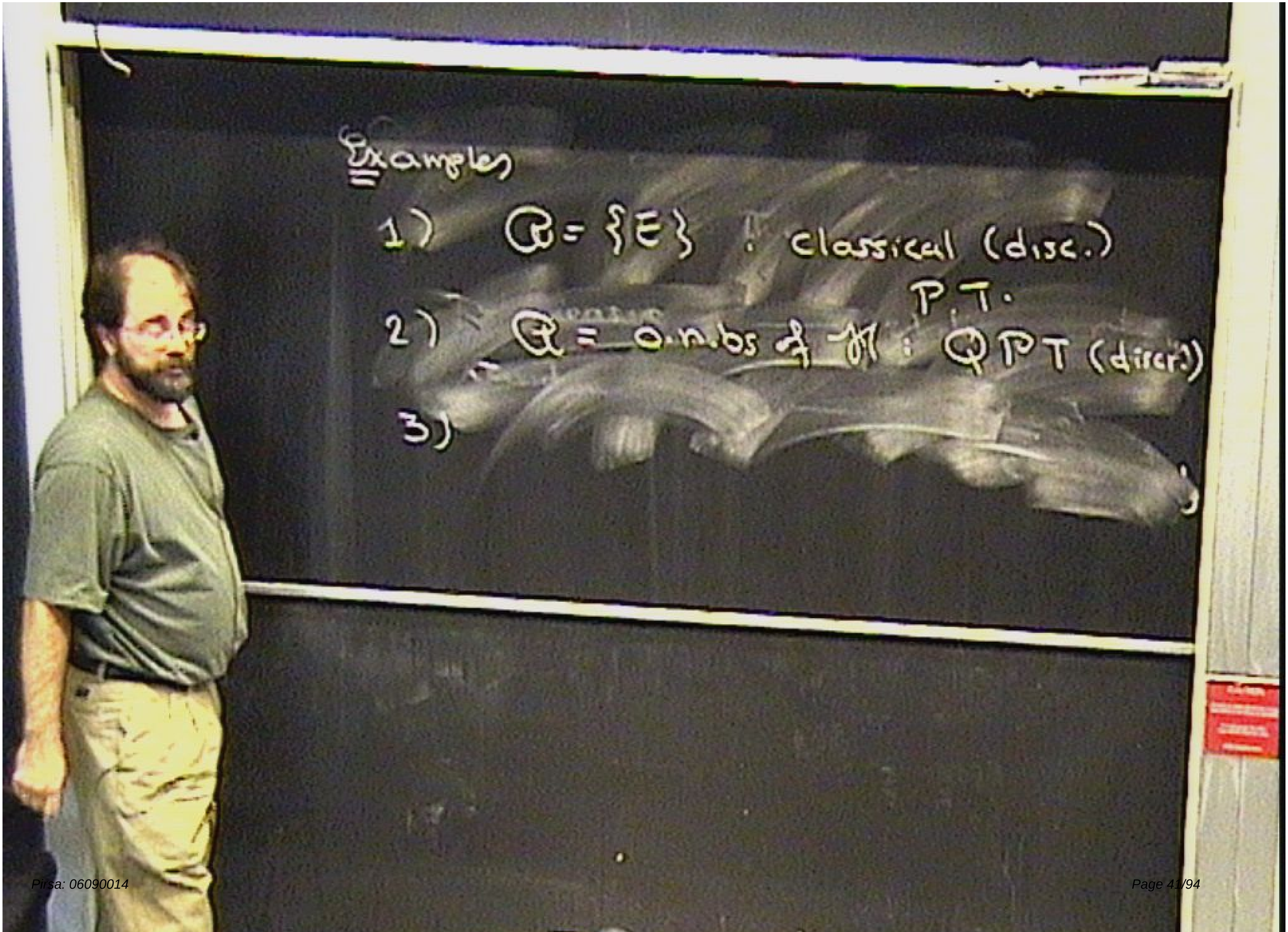

Source of examples:
* Begin w/ a collection of "observables"
- collection

Source of examples:

* Begin w/ a collection of "observables"
  - collection $\mathcal{Q} = \{E, F, \ldots\}$ of classical outcome-sets.
* Define a state to be a map $\omega : \bigcup \mathcal{Q} \to [0,1] \subseteq \mathbb{R}$

$$\forall E \in \mathcal{Q} \quad \sum_{x \in E} \omega(x) = 1$$


Examples
1) $\mathcal{Q} = \{E\}$! Classical (disc.)
P.T.
2) $\mathcal{Q}$ = o.n.bs of $\mathcal{H}$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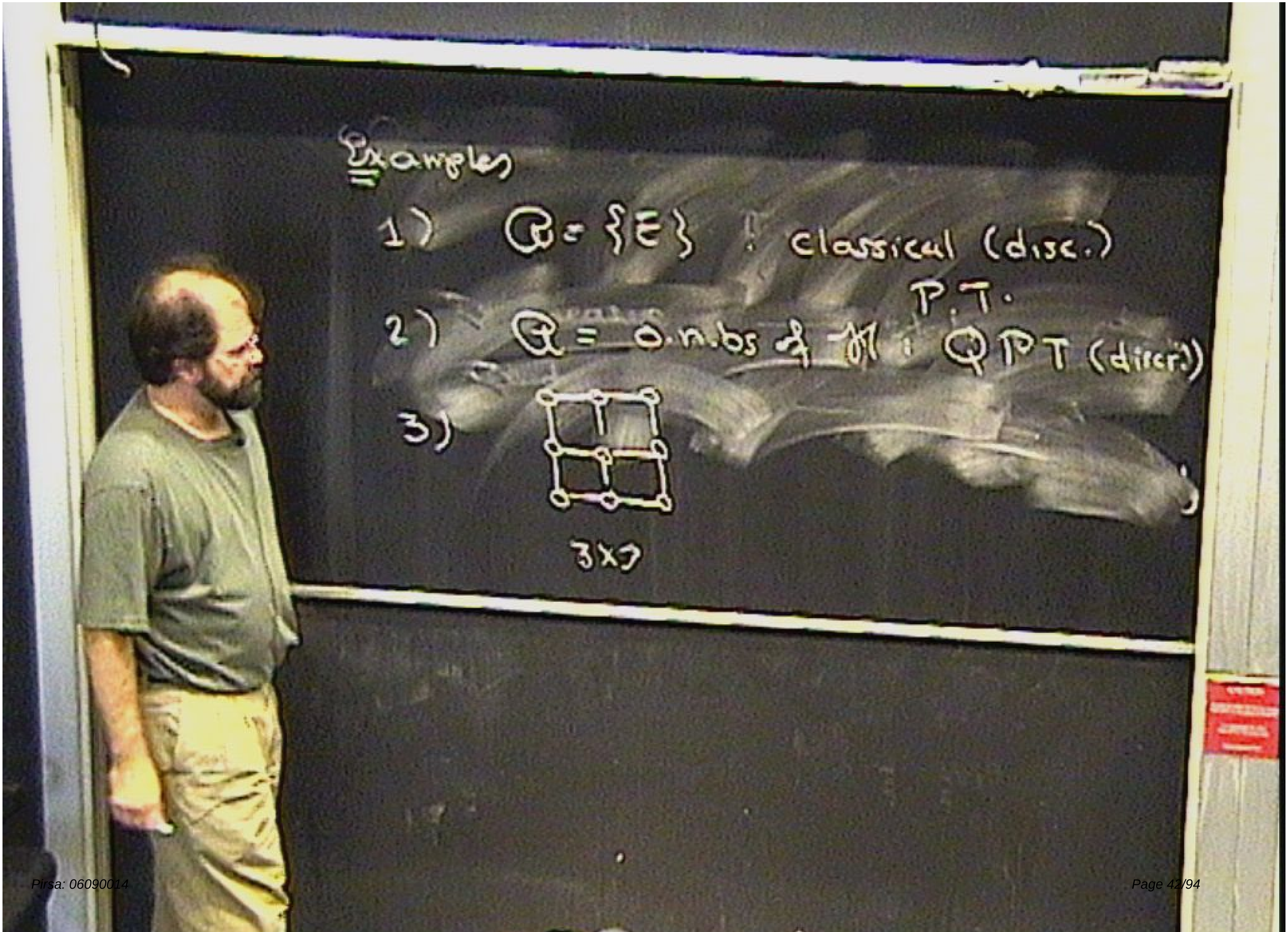
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1) Q = {E} : Classical (disc.) P.T.
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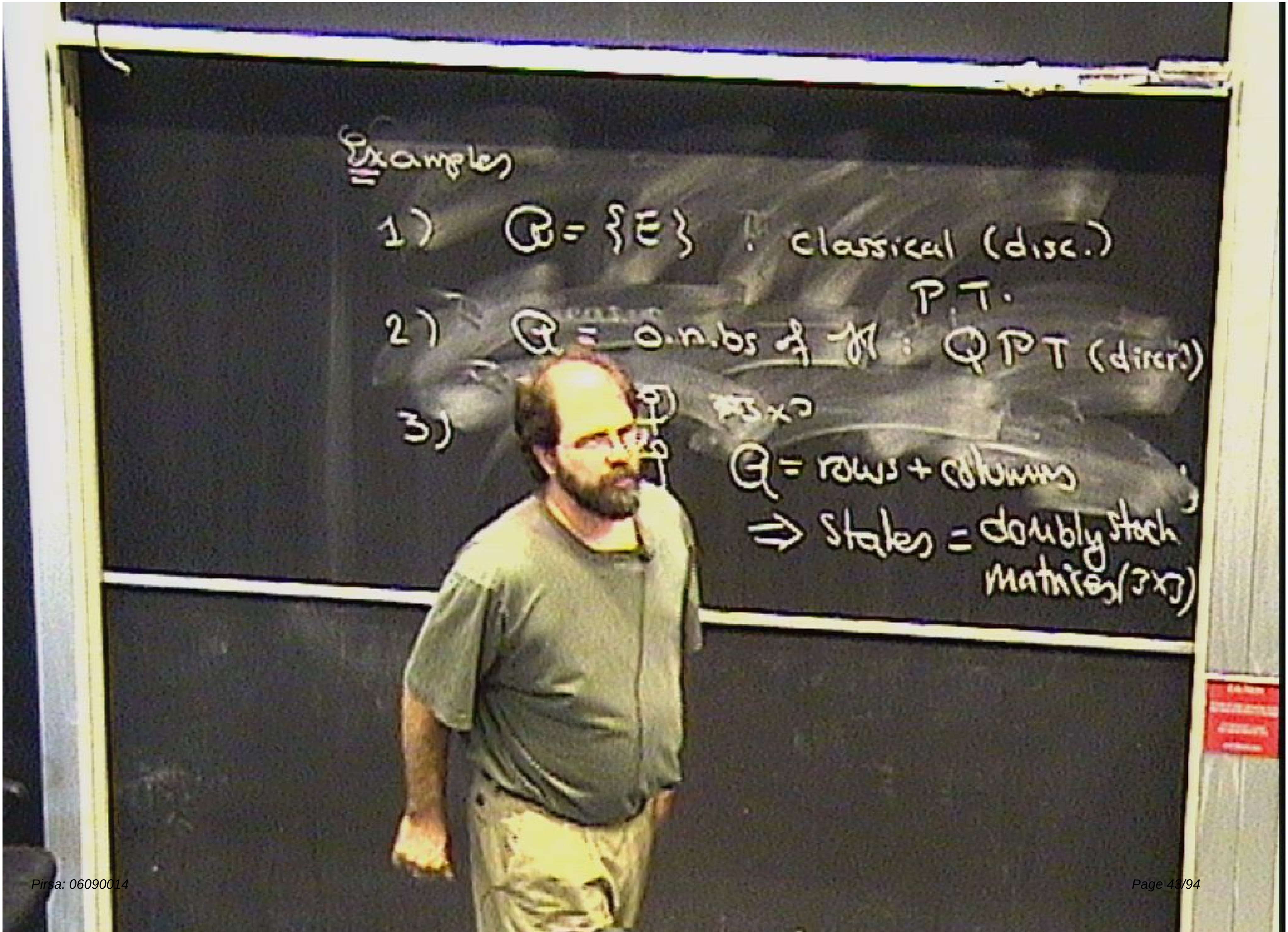
3x3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Q = rows + columns
⇒ States = doubly stoch
matrices (3x3)

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$\mathcal{Q}$ = rows + columns,

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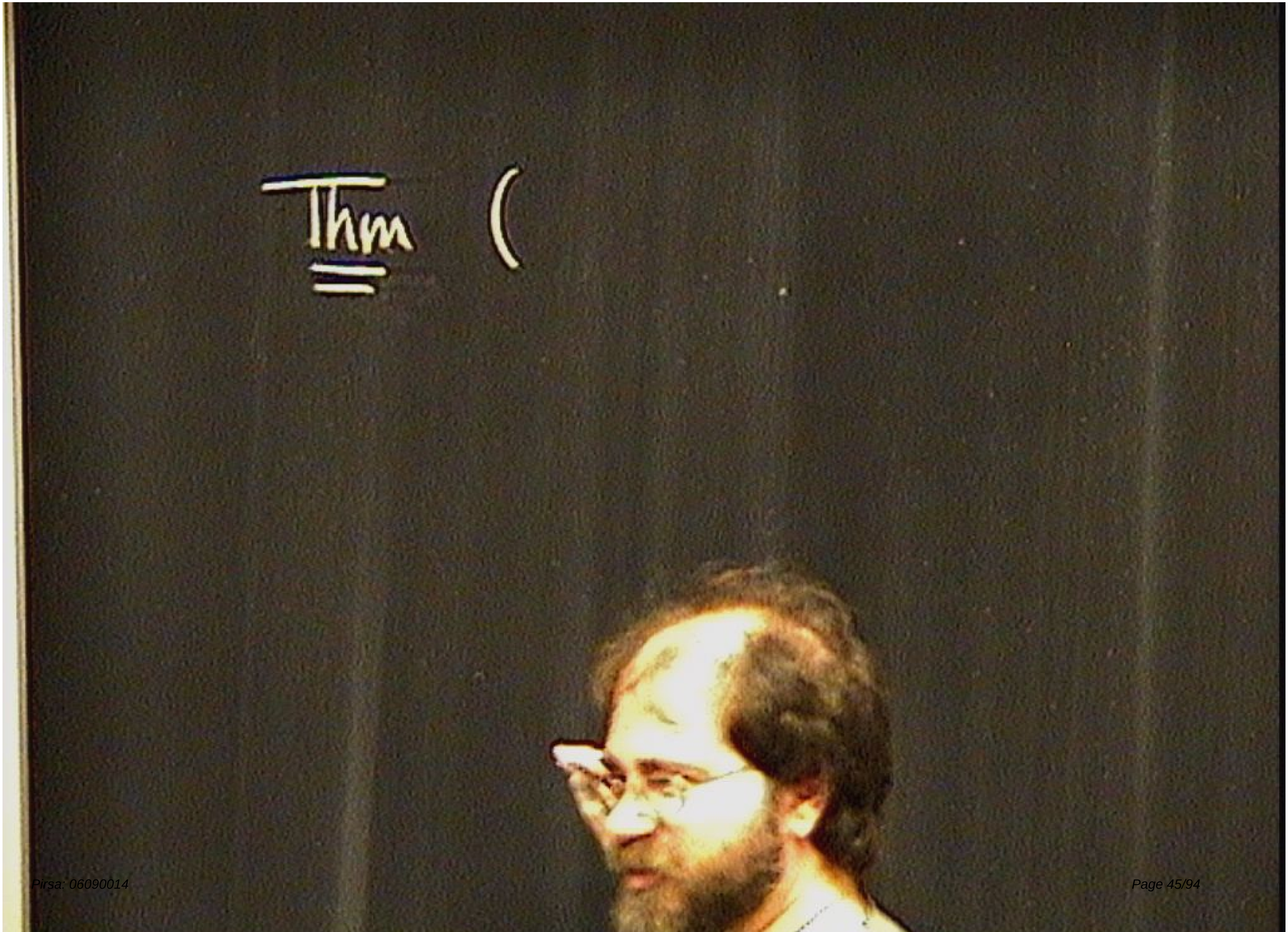
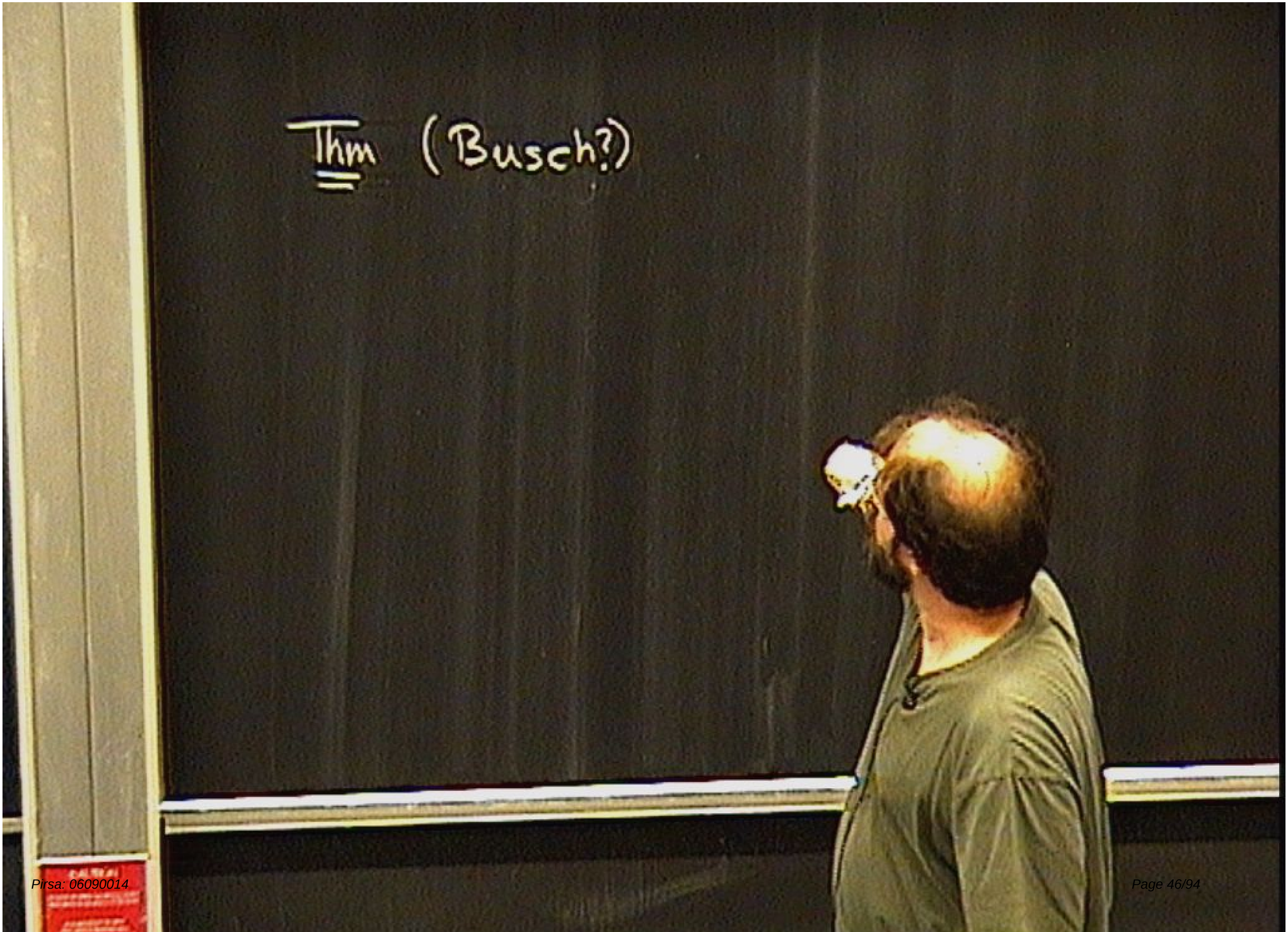
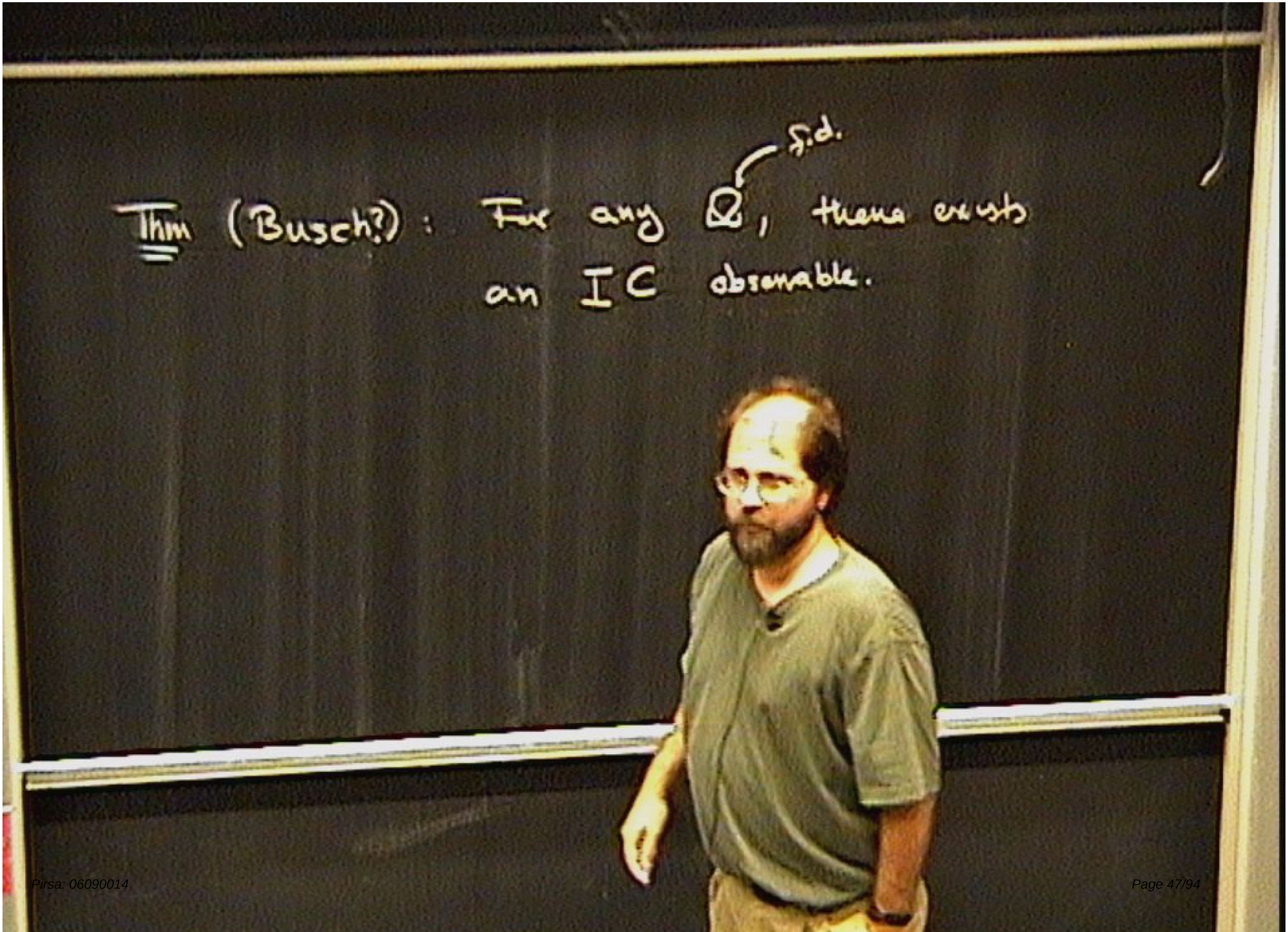

Thm (


Thm (Busch?)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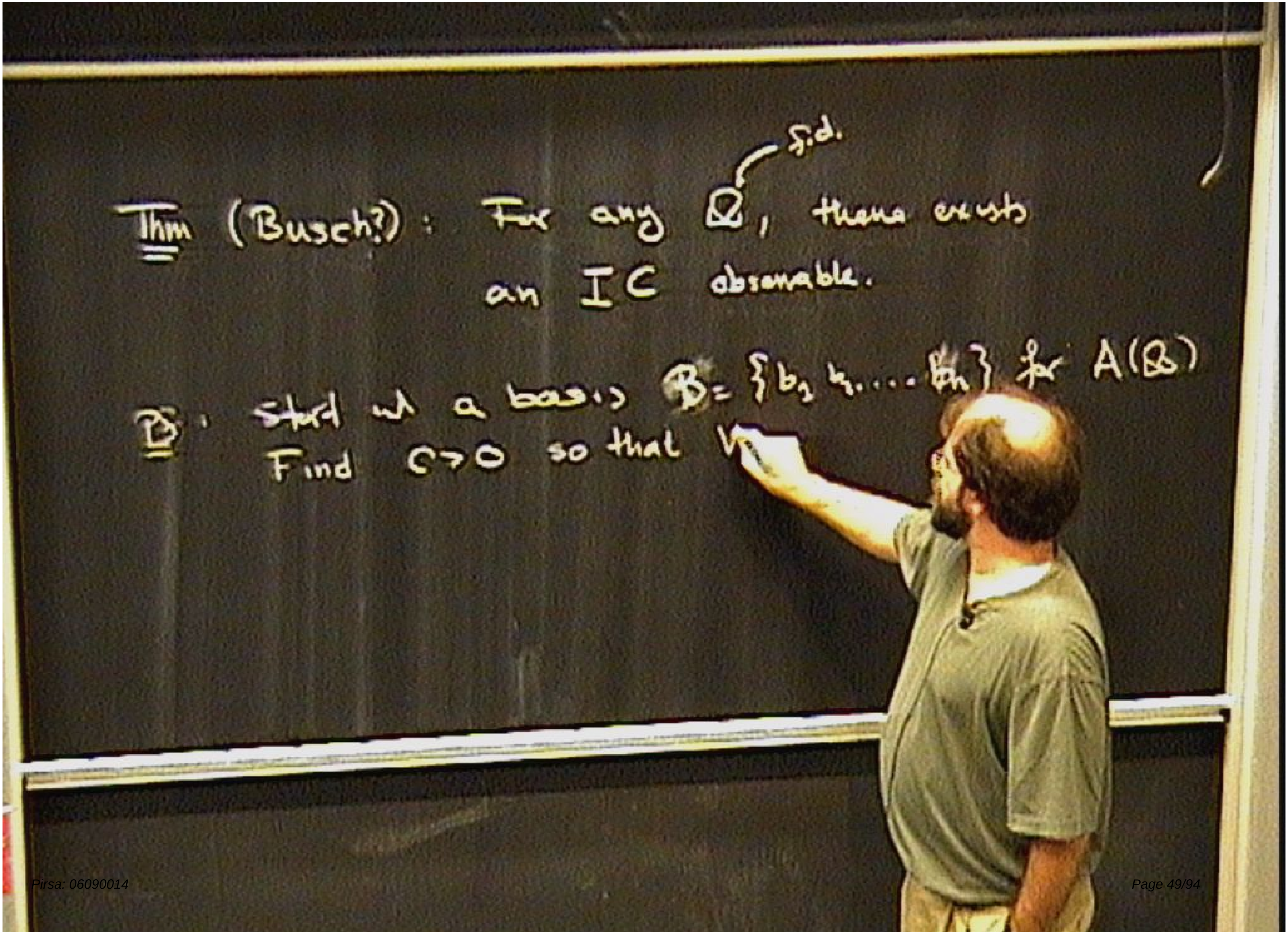
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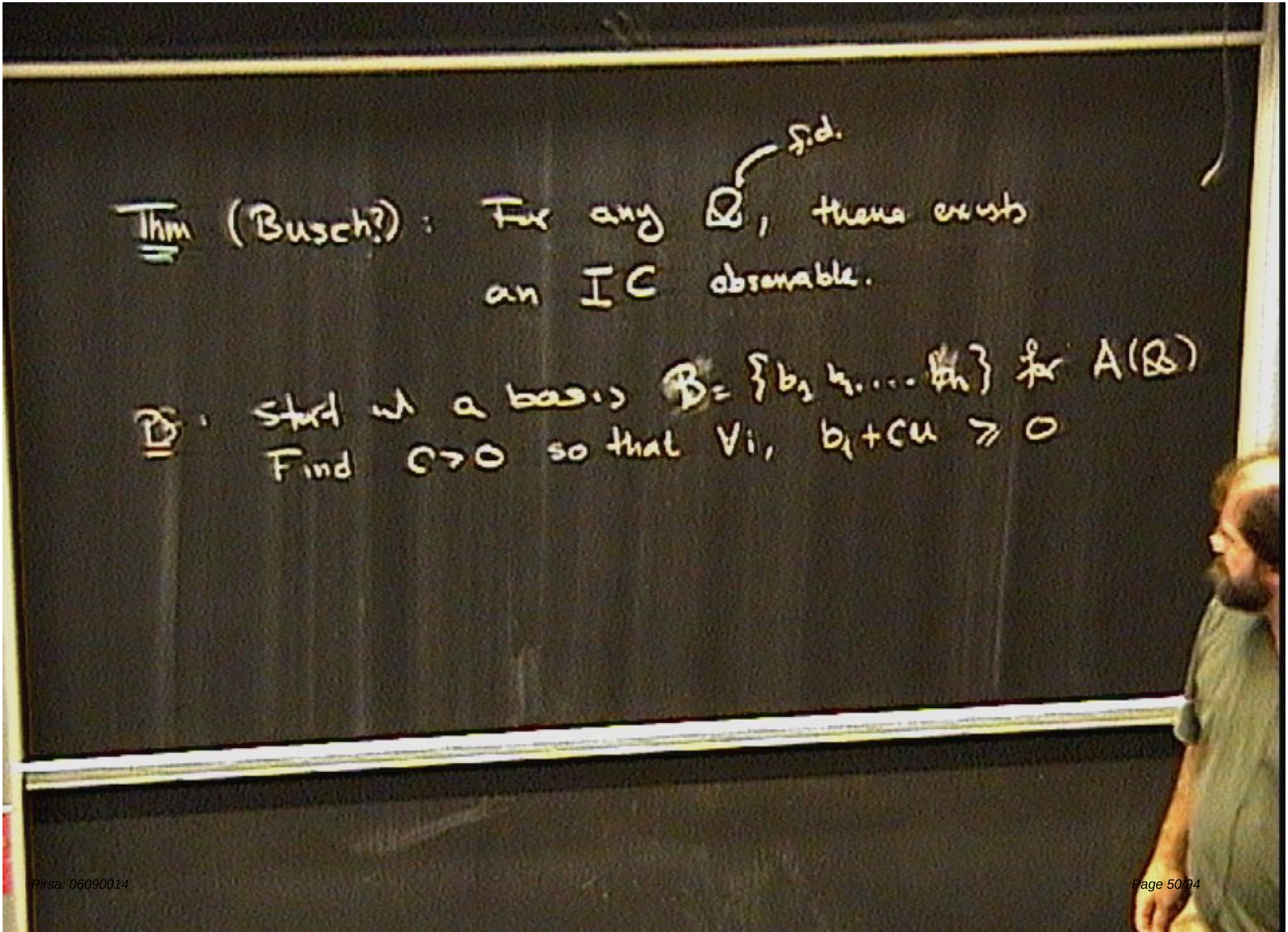
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Thm (Busch?): For any Ω, there exists
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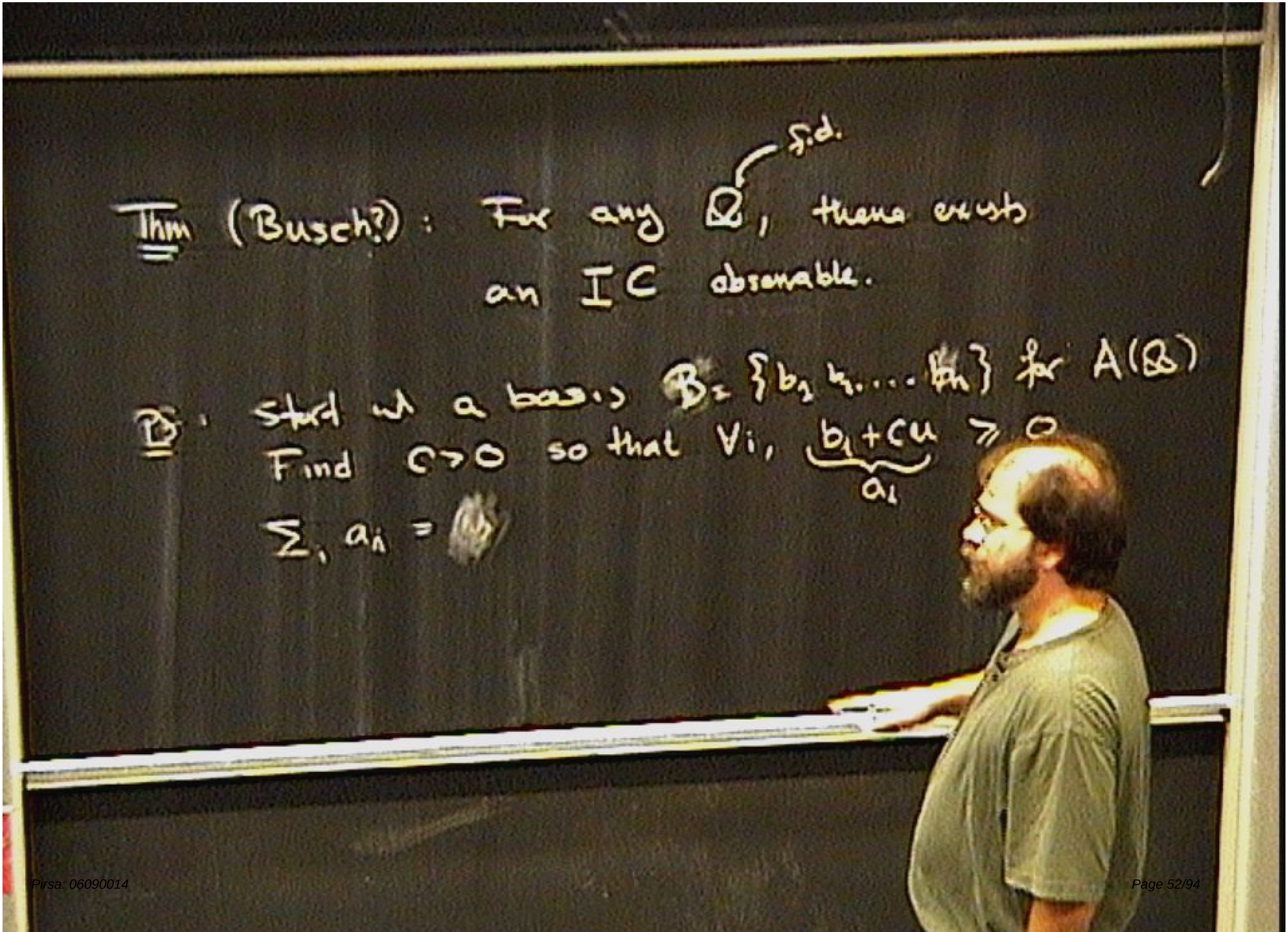
Start w/ a basis B = {b₁, b₂, ... bₙ} for A(Ω)
Pf: Find c > 0 so that ∀i, bᵢ + cu ≥ 0

Thm (Busch?): For any $\mathcal{Q}$ (f.d.), there exists an IC observable.

Pf: Start w/ a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ for $A(\mathcal{Q})$.
Find $c > 0$ so that $\forall i$, $b_i + cu \geq 0$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$\sum_i a_i =$

wlog $\sum b_i = u$

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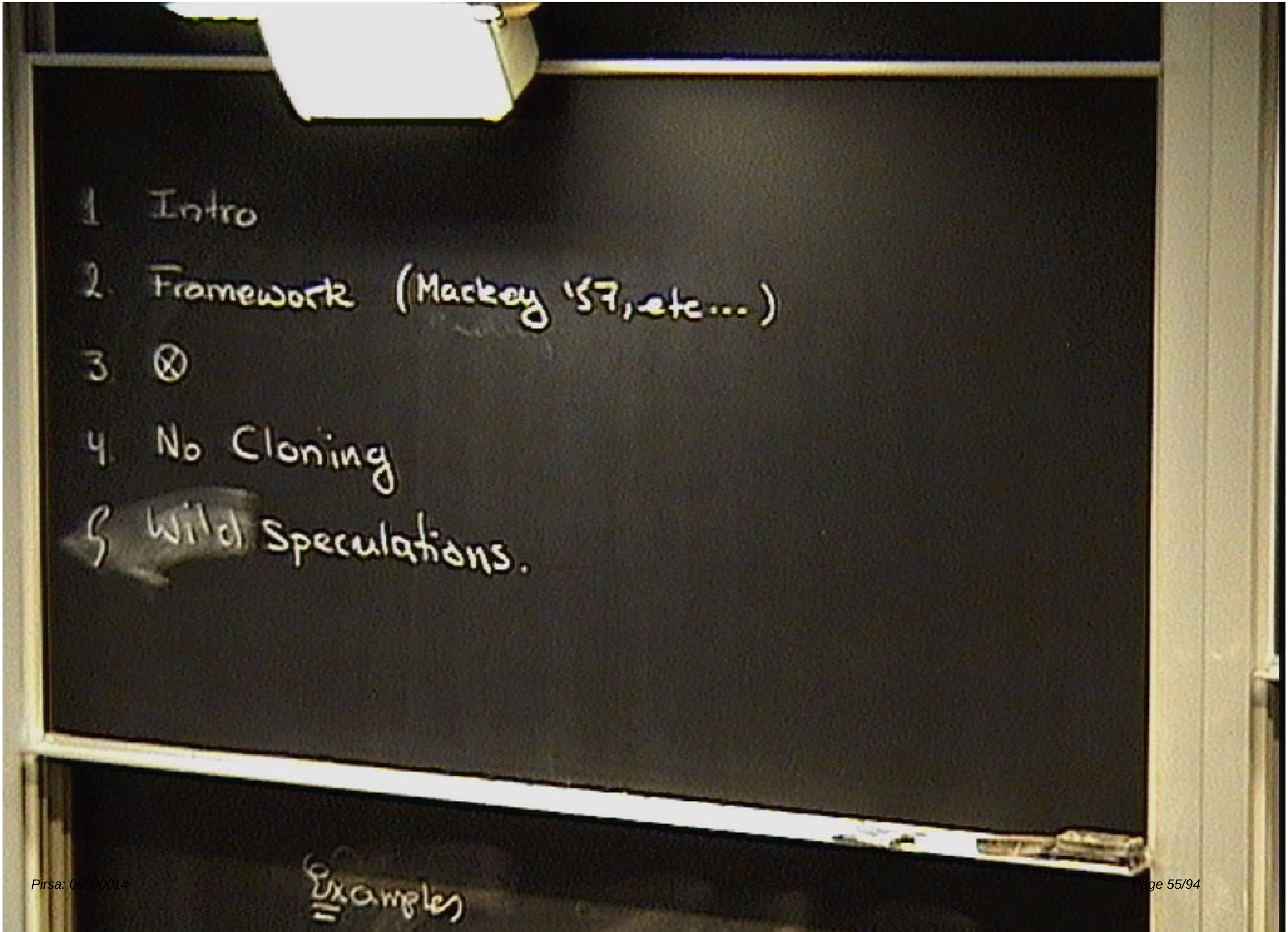
wolog
$\sum b_i = u$

$\sum_i a_i = u + ncu = (1+nc)u$

Let $\hat{a}_i = \frac{a_i}{1+nc}$ ✓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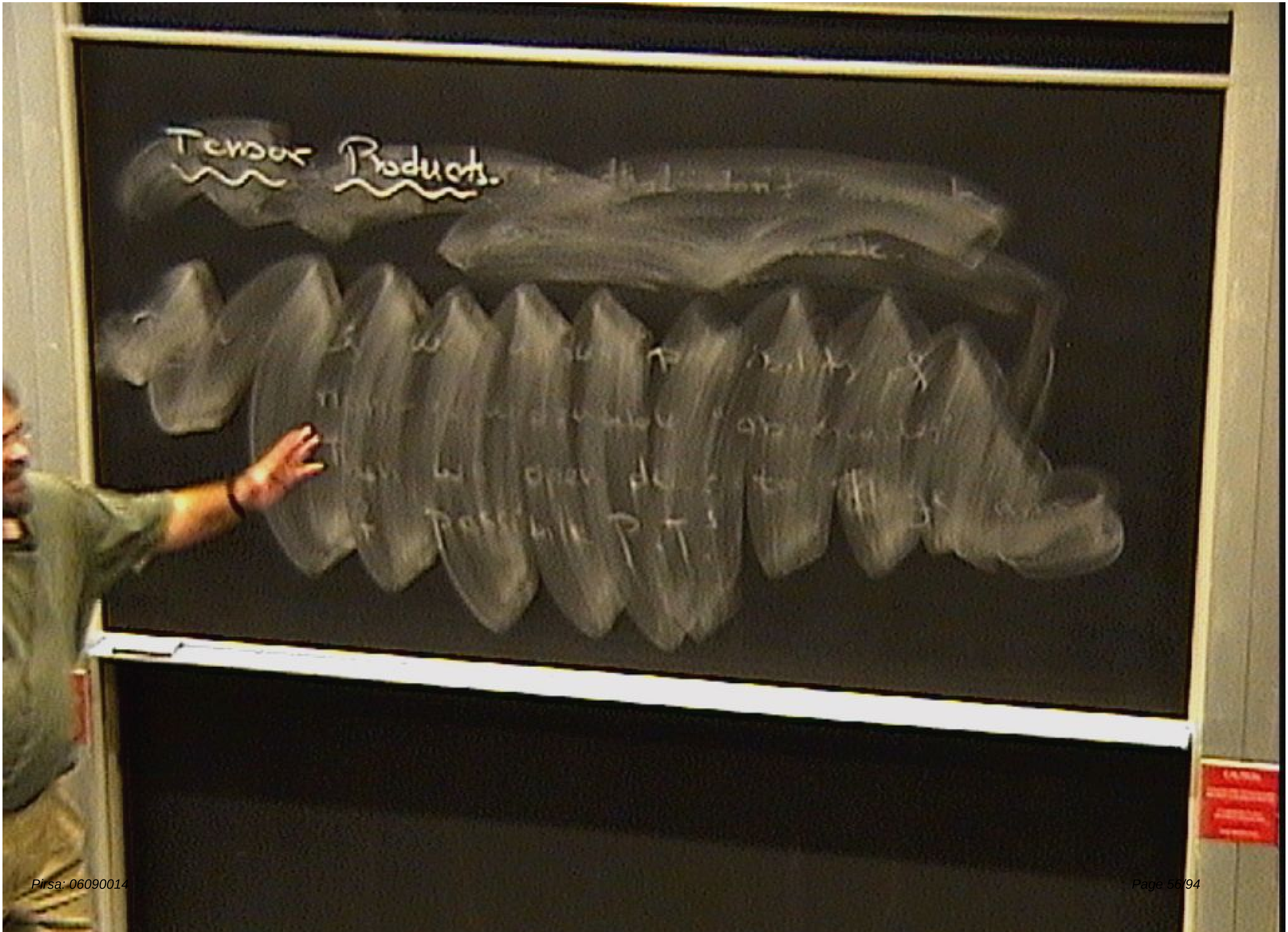
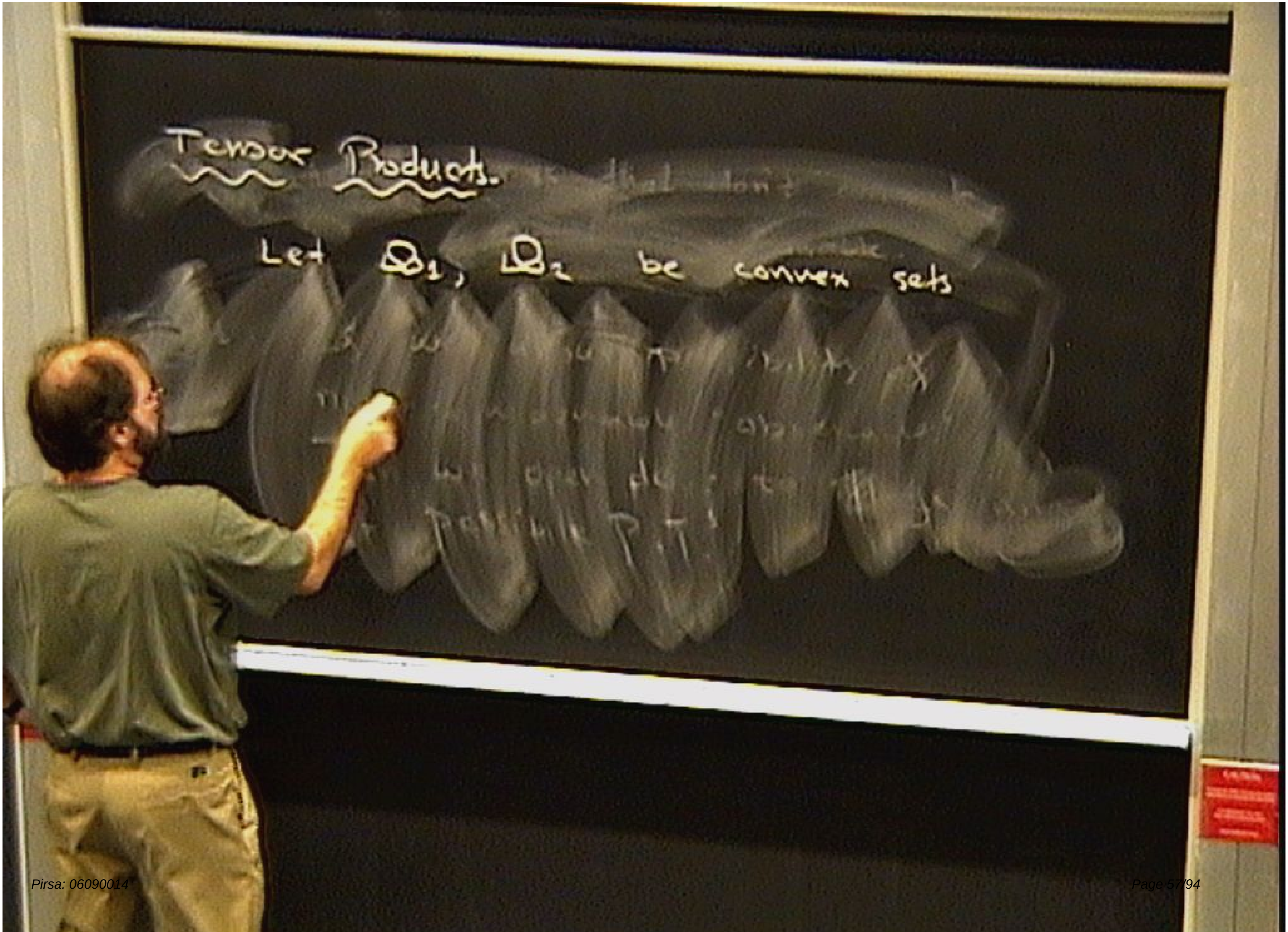
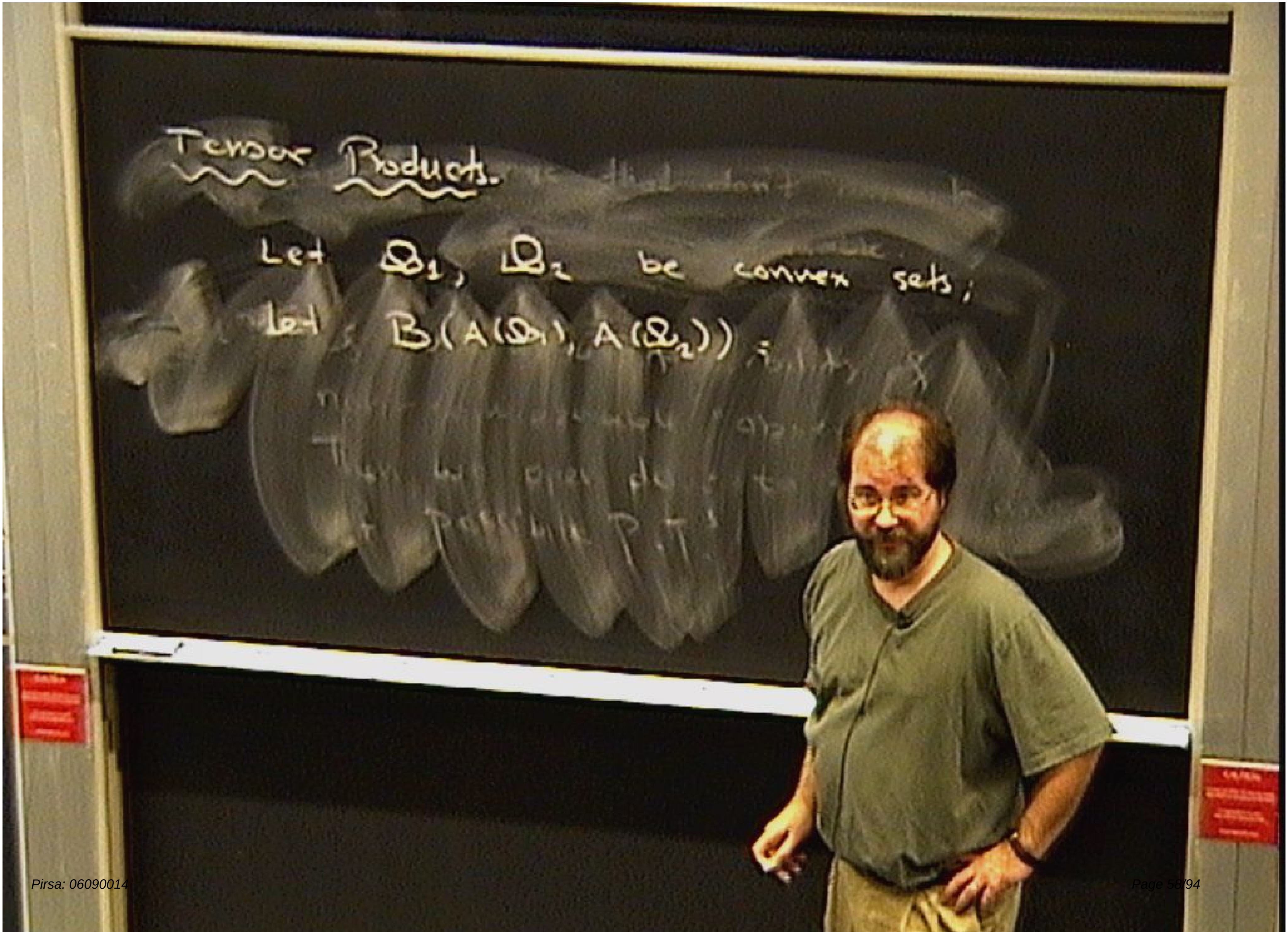

Tensor Products.


Tensor Products.
Let Ω_1, Ω_2 be convex sets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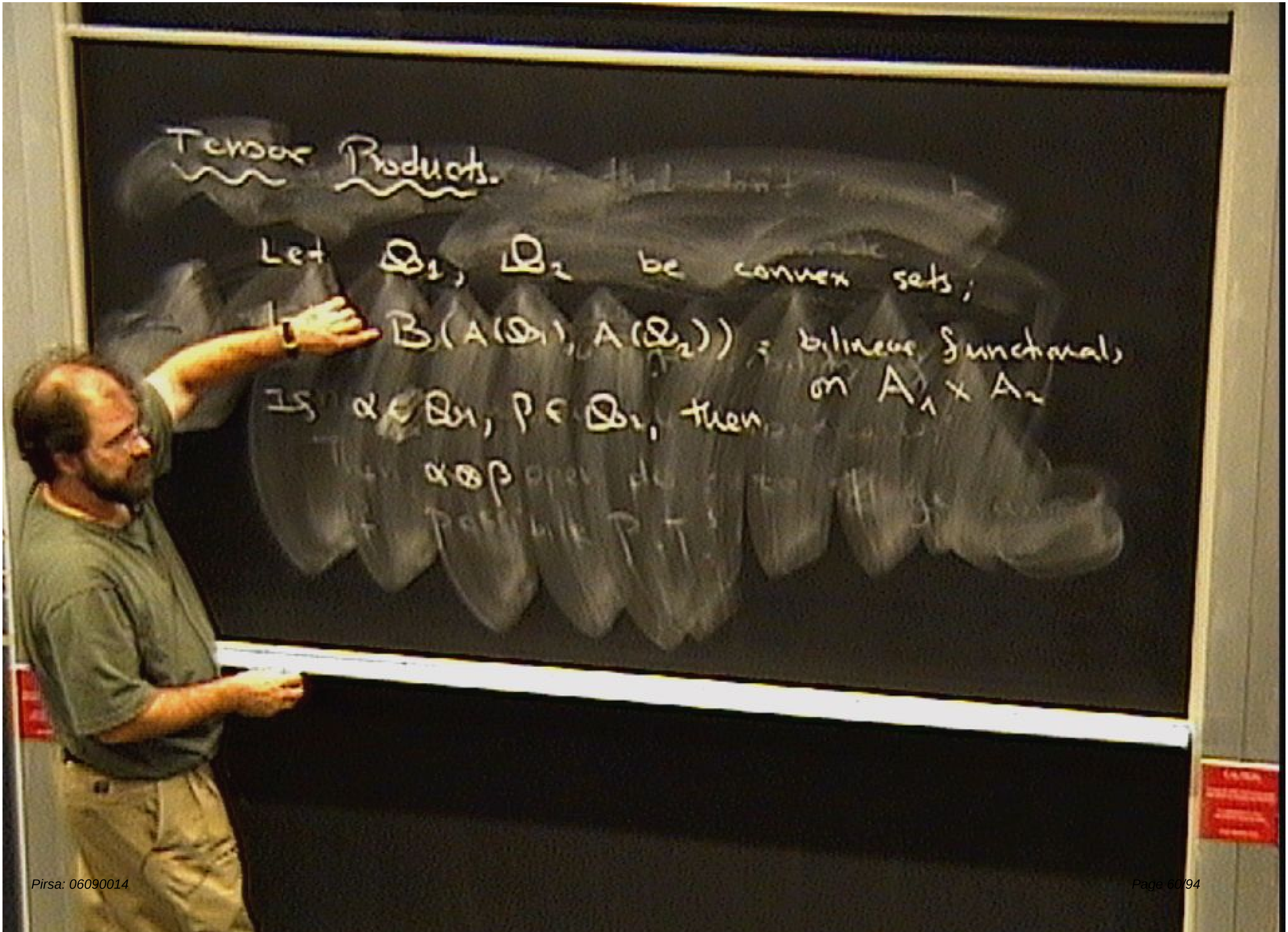
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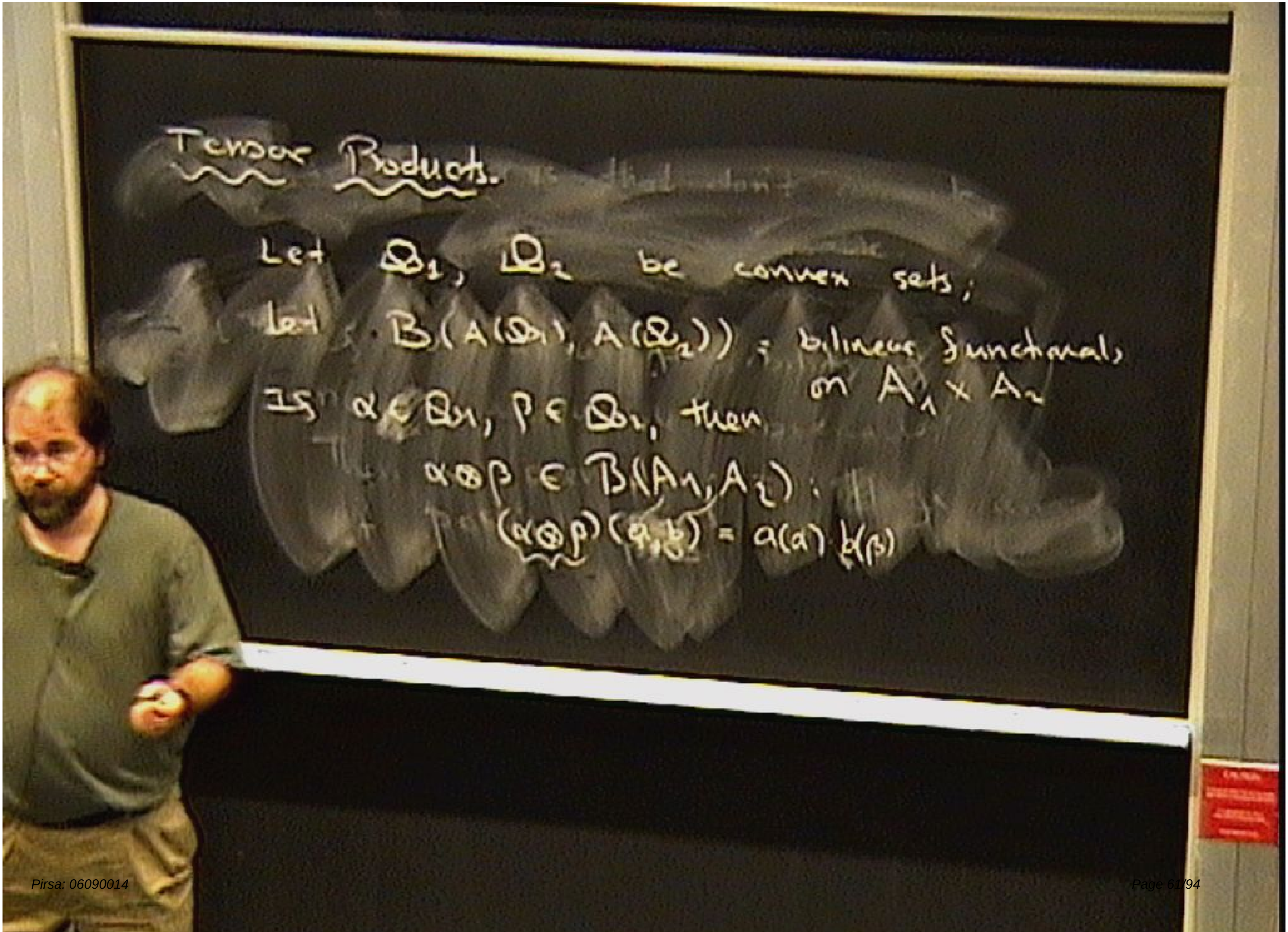
If $\alpha \in \Omega_1$, $\beta \in \Omega_2$, then
$\alpha \otimes \beta$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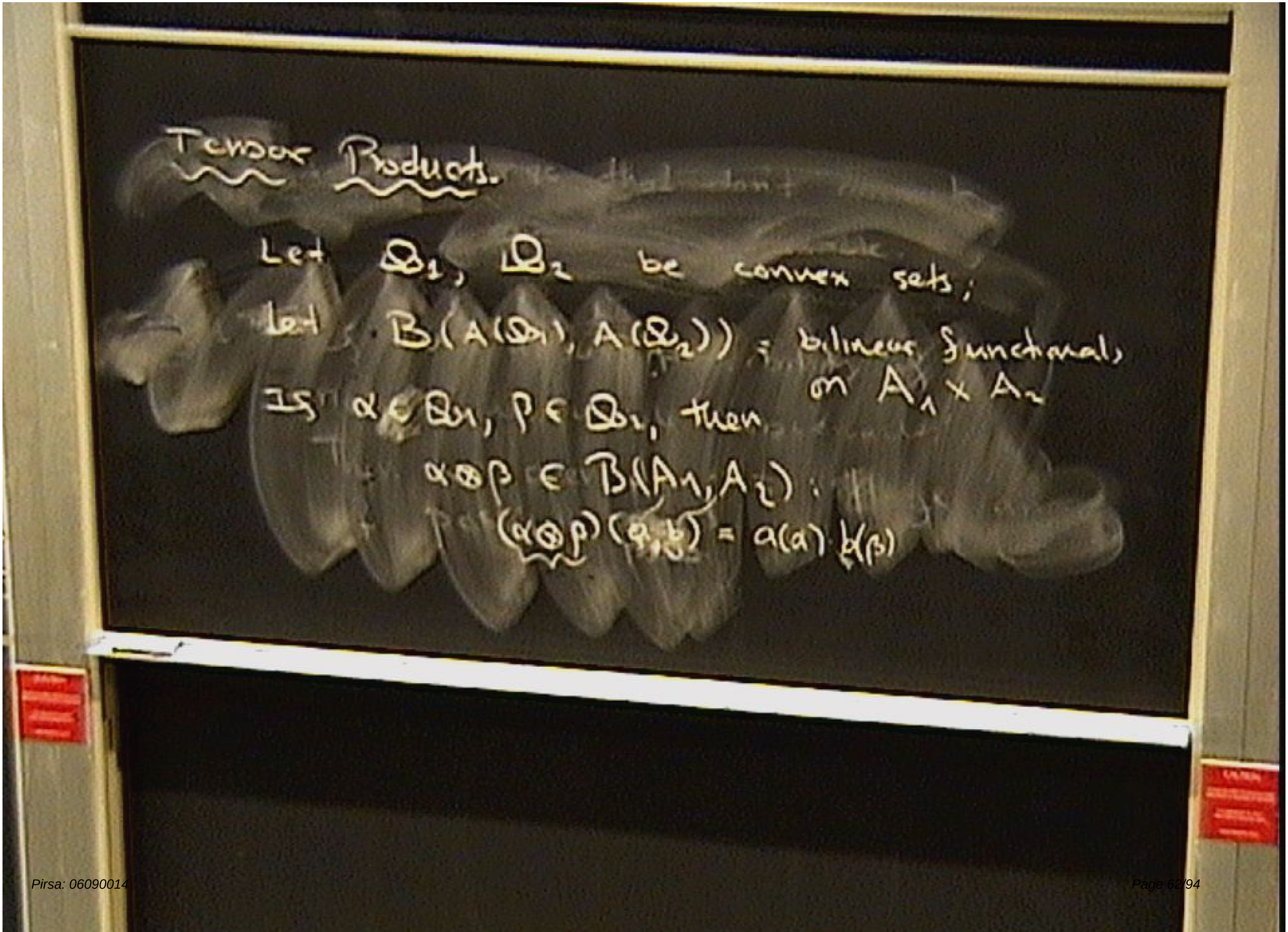
If $\alpha \in \Omega_1$, $\beta \in \Omega_2$, then
$\alpha \otimes \beta \in B(A_1, A_2)$:
$(\alpha\otimes\beta)(a,b) = a(\alpha)\, b(\beta)$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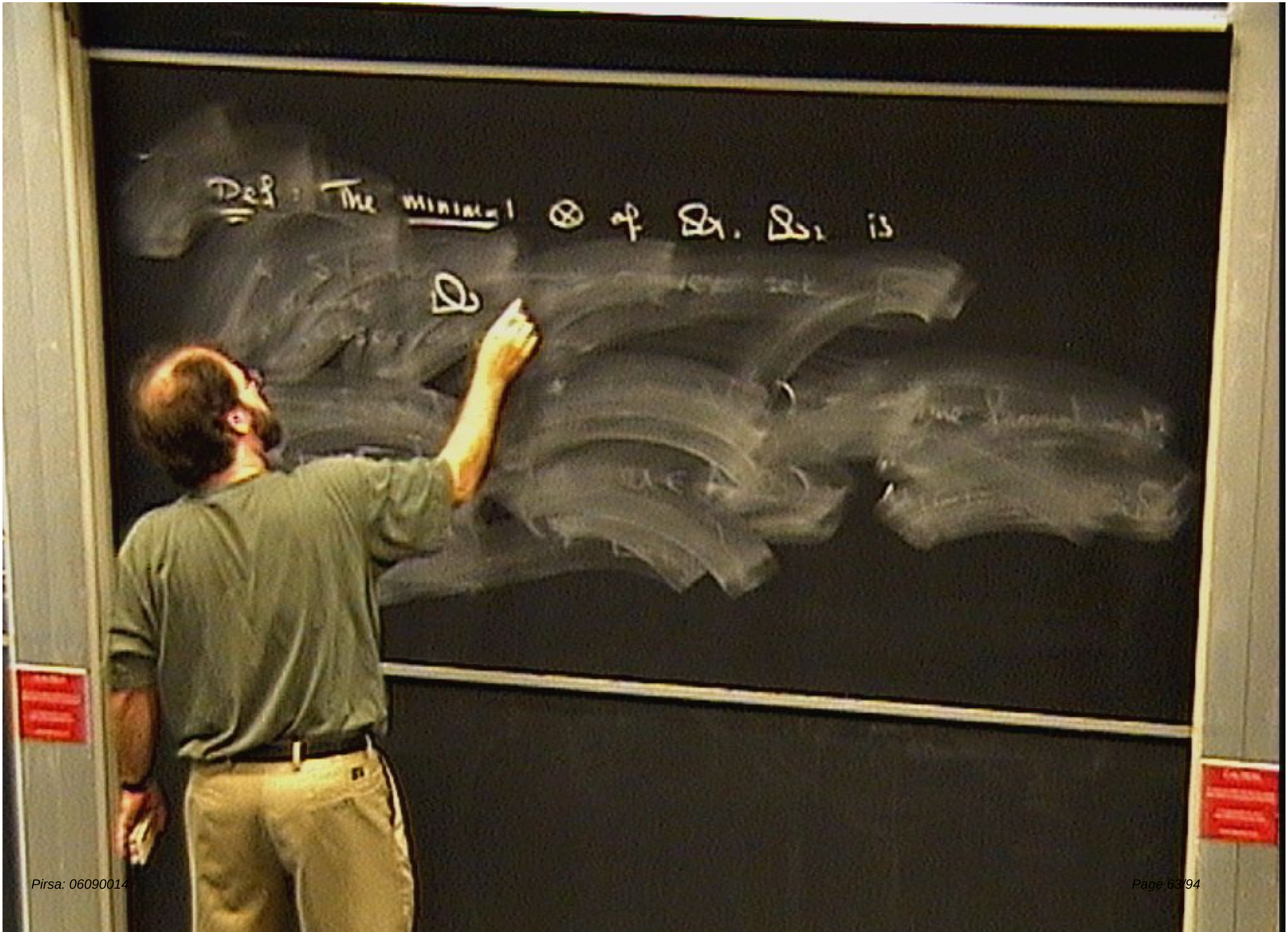
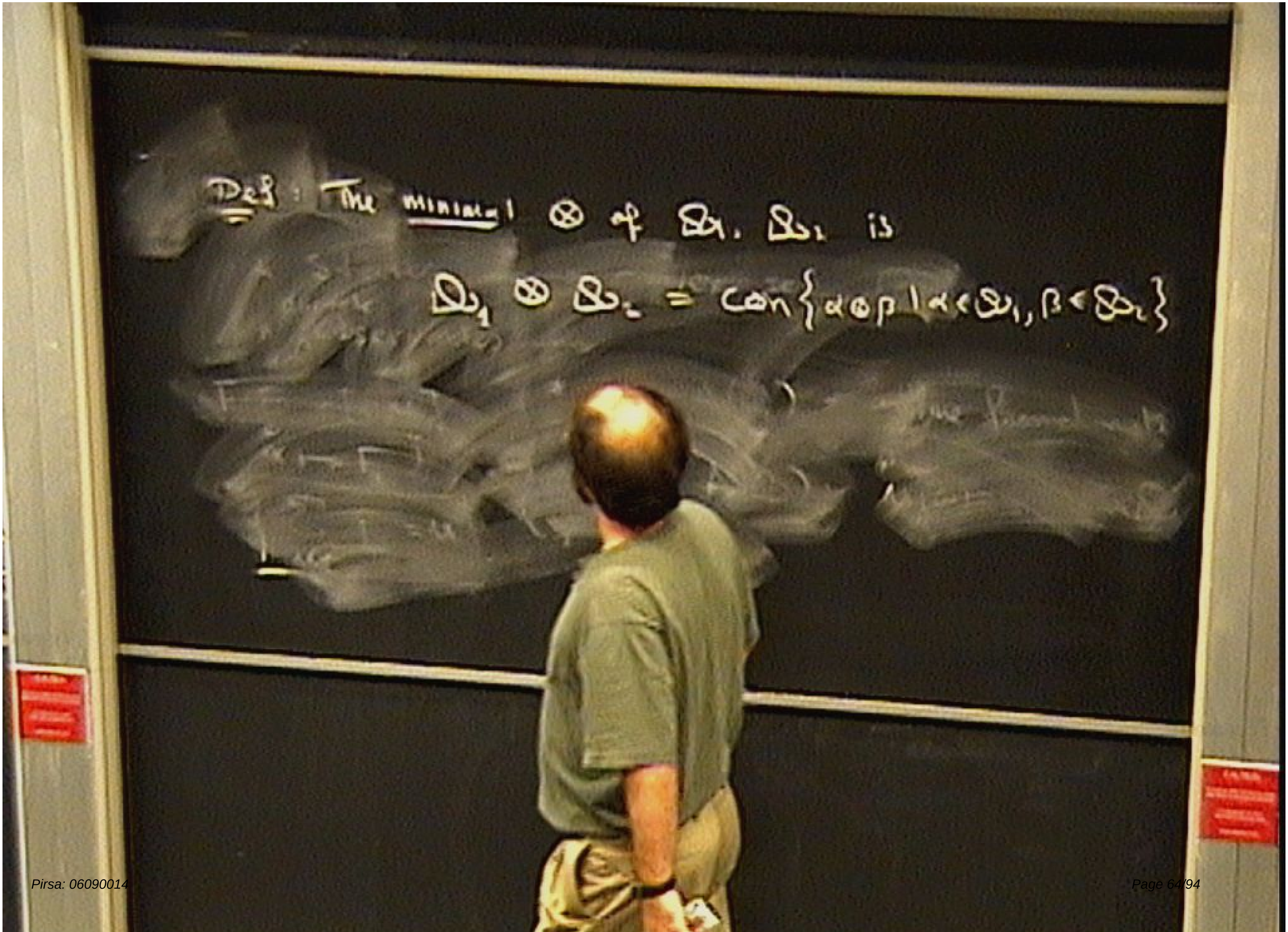

Def: The minimal $\otimes$ of $\mathfrak{A}_1, \mathfrak{A}_2$ is
$\mathfrak{A}$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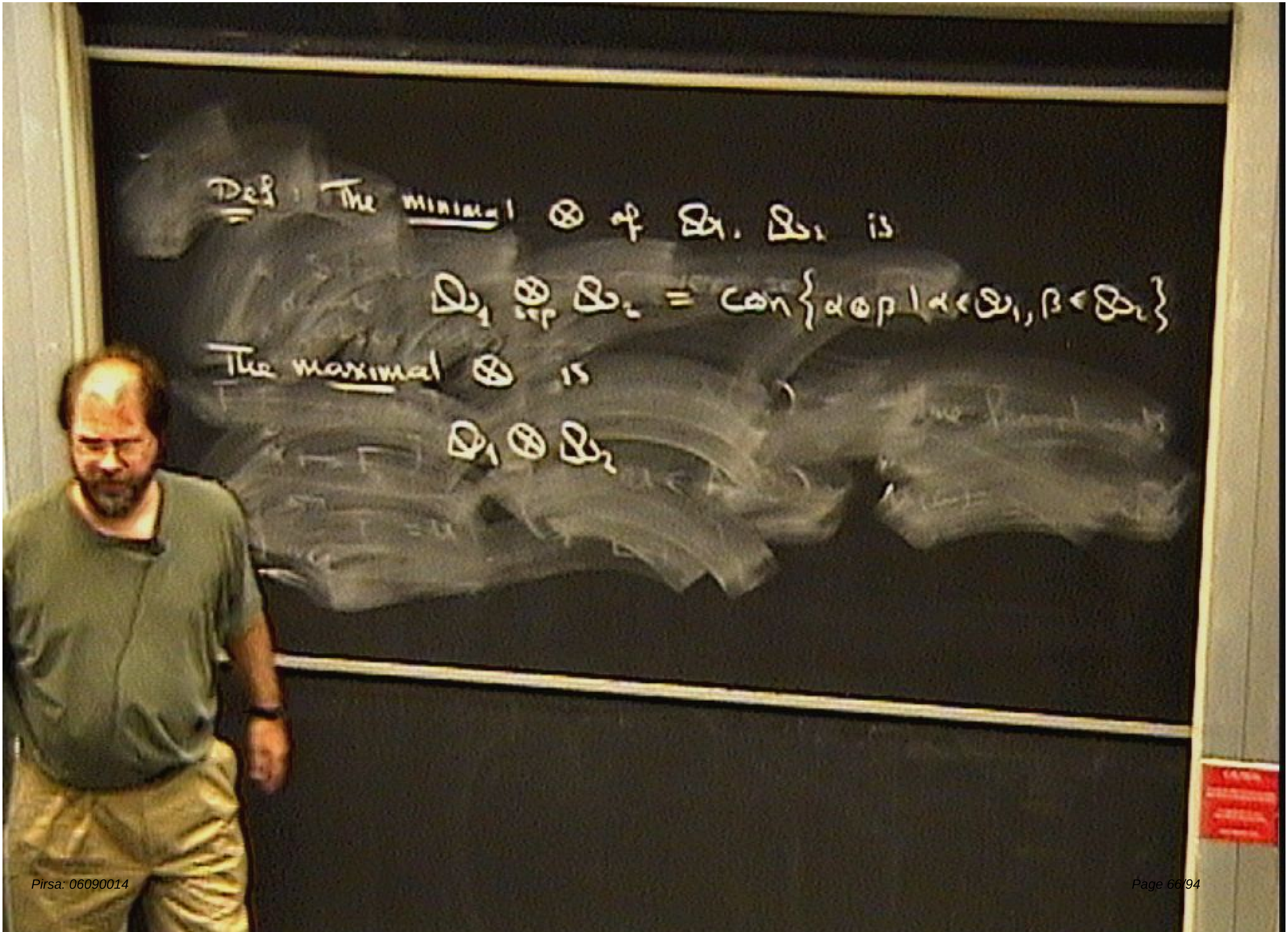
$\mathfrak{B}_1 \otimes \mathfrak{B}_2 = \mathrm{Con}\{\alpha\otimes\beta \mid \alpha\in\mathfrak{B}_1, \beta\in\mathfrak{B}_2\}$

Def: The <u>minimal</u> $\otimes$ of Ω_1, Ω_2 is

$$\Omega_1 \otimes_{sep} \Omega_2 = \mathrm{Con}\{\alpha \otimes \beta \mid \alpha \in \Omega_1, \beta \in \Omega_2\}$$



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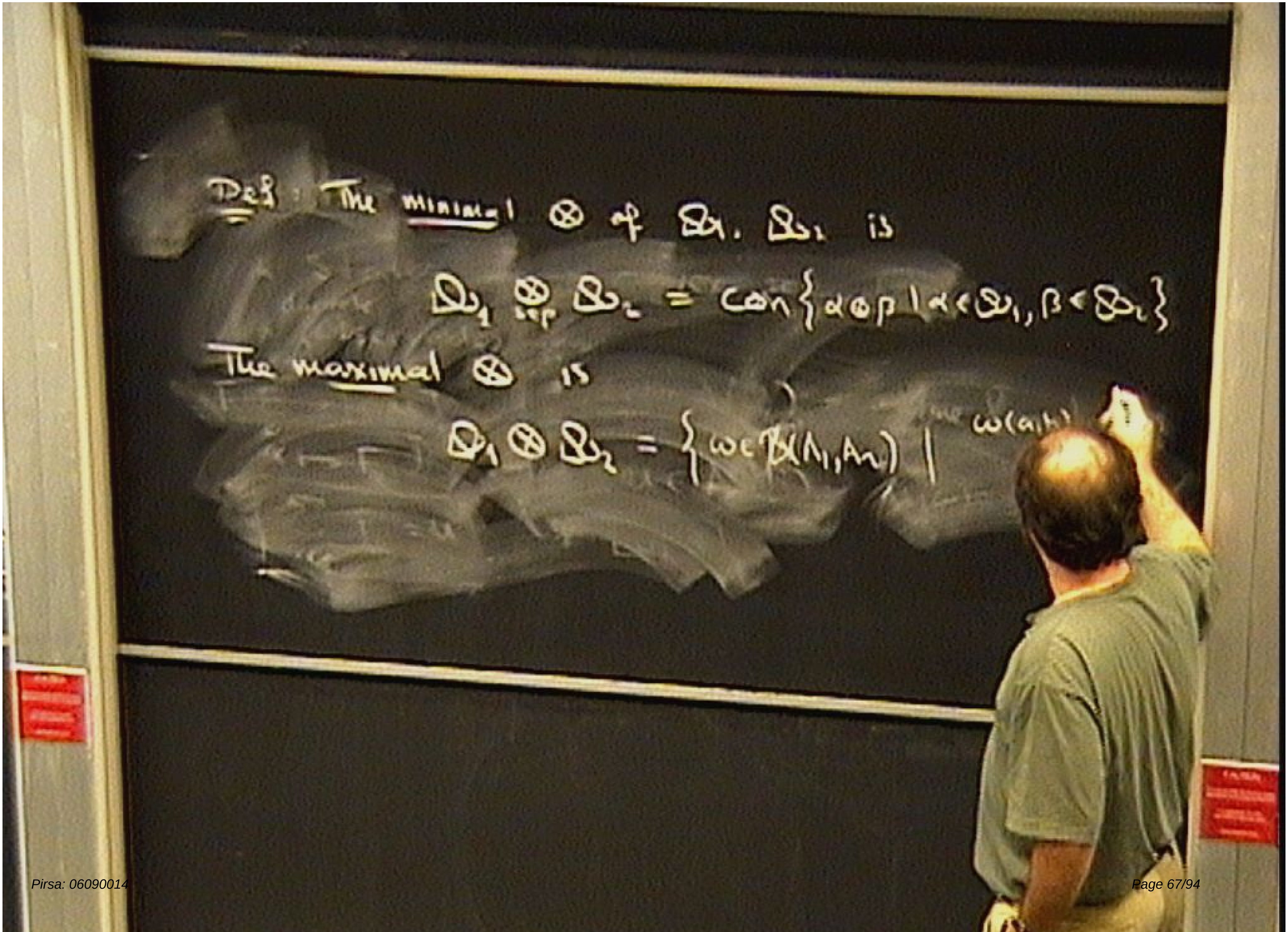
Def: The minimal $\otimes$ of Ω_1, Ω_2 is
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$$\Omega_1 \otimes \Omega_2 = \left\{ \omega \in \mathcal{B}(\Lambda_1, \Lambda_2) \;\middle|\; \begin{array}{l} \forall a,b \in [0,u] \\ \omega(a,b) \geq 0 \\ \omega(u_1, u_2) \end{array} \right.$$

Def: The **minimal** $\otimes$ of $\mathfrak{B}_1, \mathfrak{B}_2$ is

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$$\Omega_1 \otimes_{sep} \Omega_2 \subseteq \Omega_1 \otimes \Omega_2$$

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Foulis/Randall ca. 1980 $\longrightarrow$

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A(B

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$A(\mathcal{S}_1 \otimes \mathcal{S}_2)$

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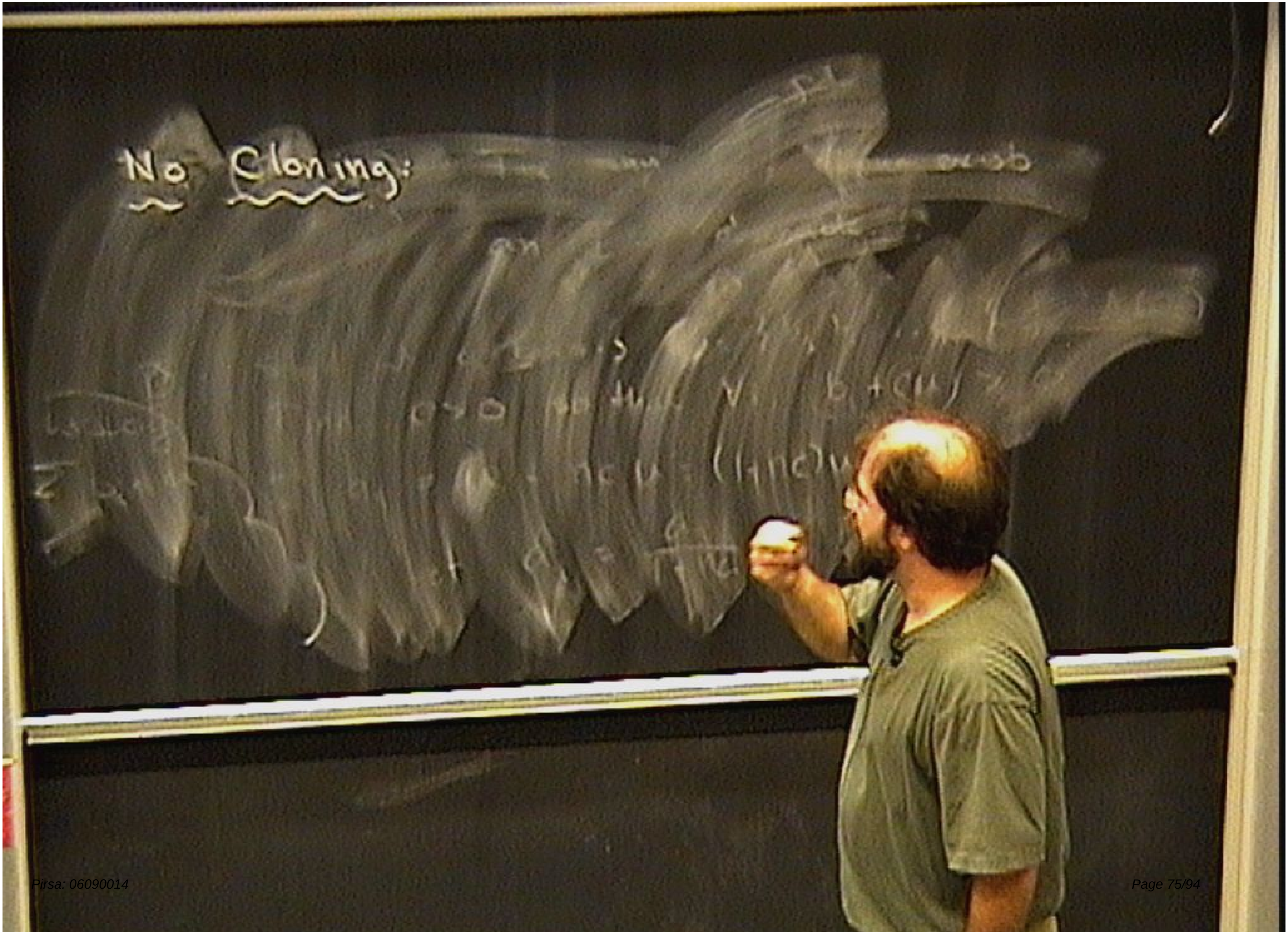
The maximal $\otimes$ is

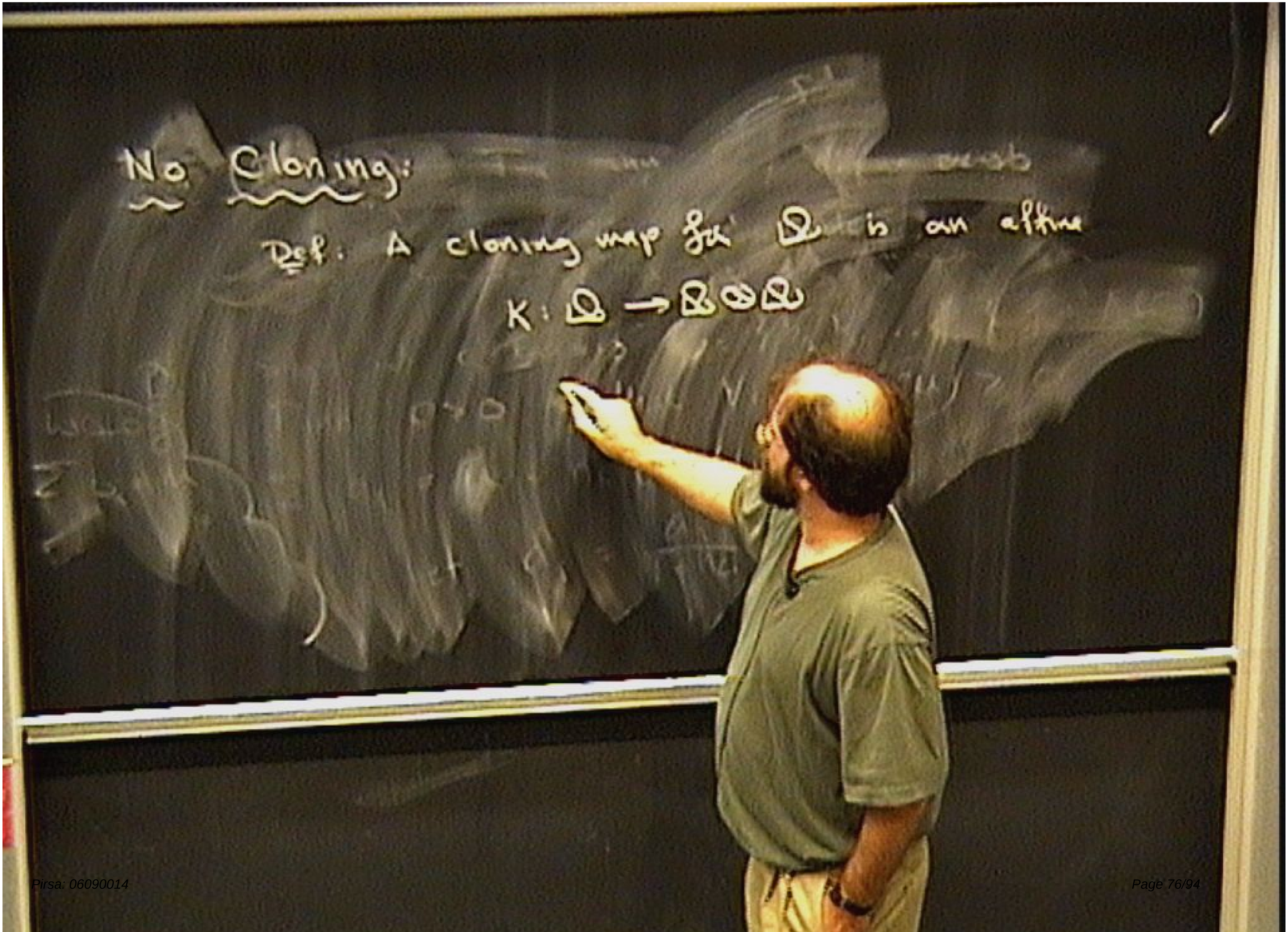
Foulis/Randall ca. 1980 w/ Kläy

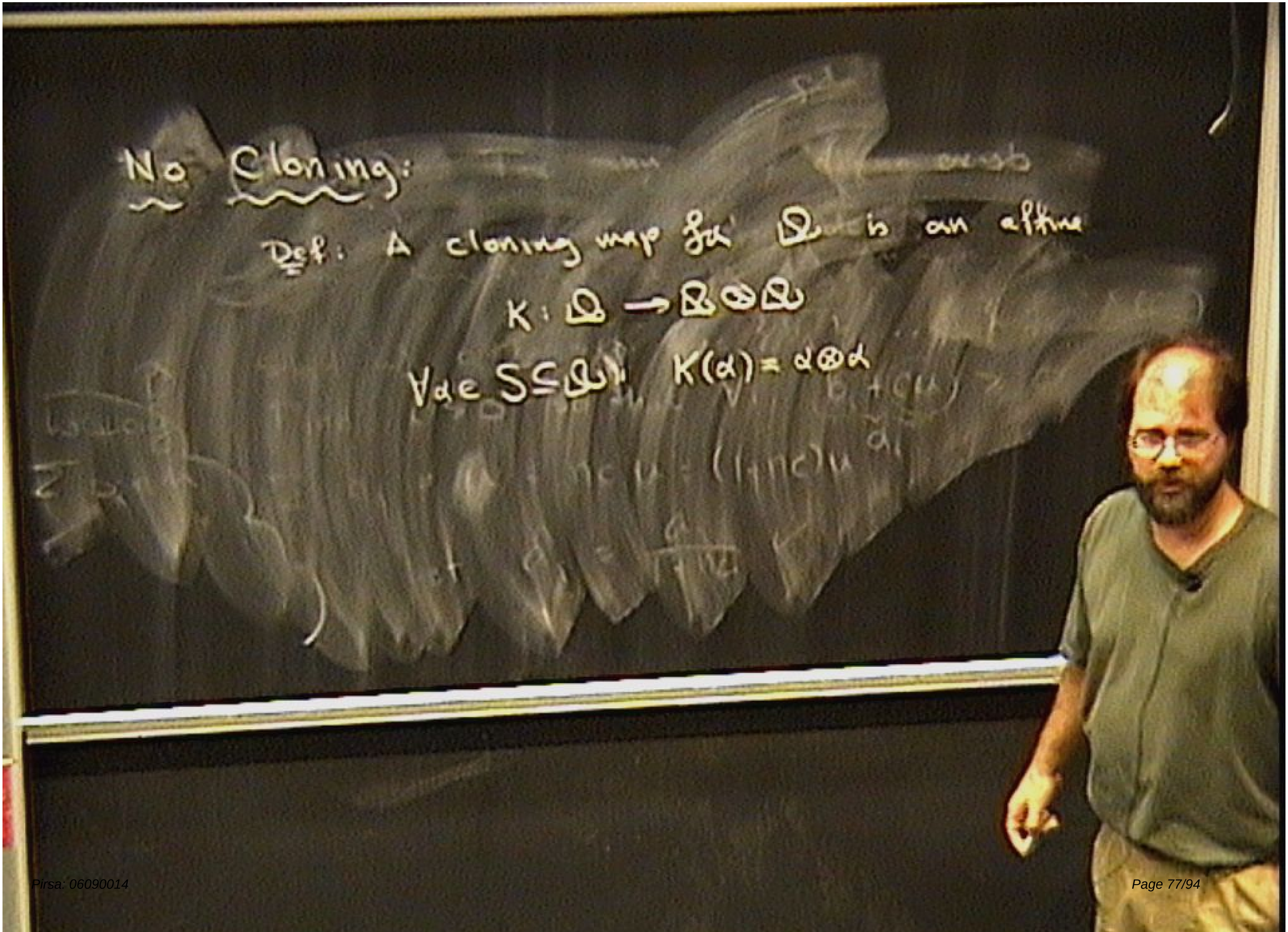
$$\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \left\{ \omega \in \mathcal{B}(A_1, A_2) \;\middle|\; \begin{array}{l} \forall a,b \in [0,u] \\ \omega(a,b) \geq 0 \\ \omega(u_1,u_2) = 1 \end{array} \right.$$

$A(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \cong A(\mathfrak{A}_1) \otimes A(\mathfrak{A}_2)$

$$\mathfrak{A}_1 \otimes_{sep} \mathfrak{A}_2 \subsetneq \mathfrak{A}_1 \otimes \mathfrak{A}_2$$


No Cloning:


No Cloning:
Def: A cloning map for Ω is an affine
K: Ω → Ω⊗Ω


No Cloning:
Def: A cloning map for Ω is an affine
K: Ω → Ω ⊗ Ω
∀α ∈ S ⊆ Ω, K(α) = α ⊗ α

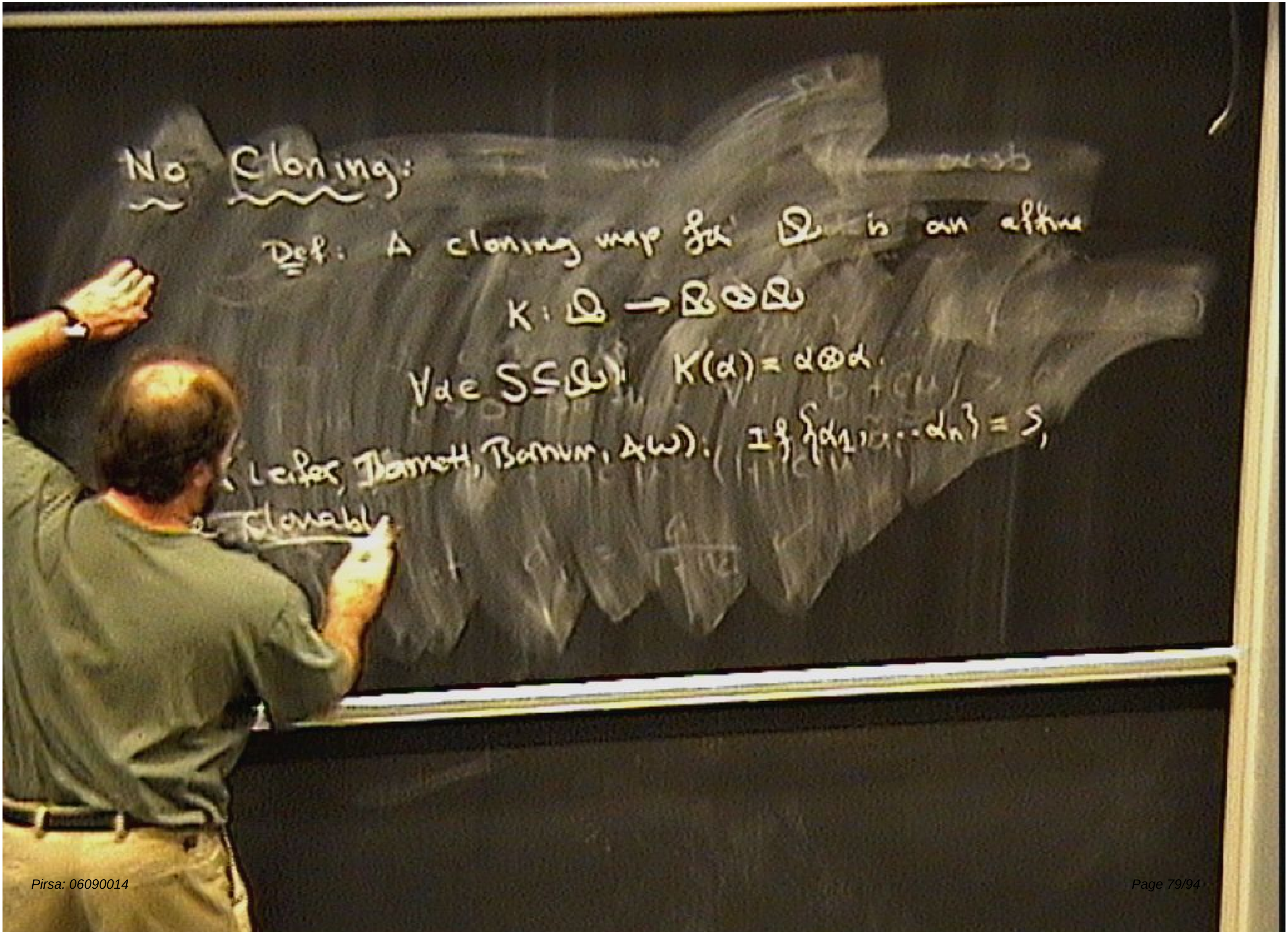
No Cloning:

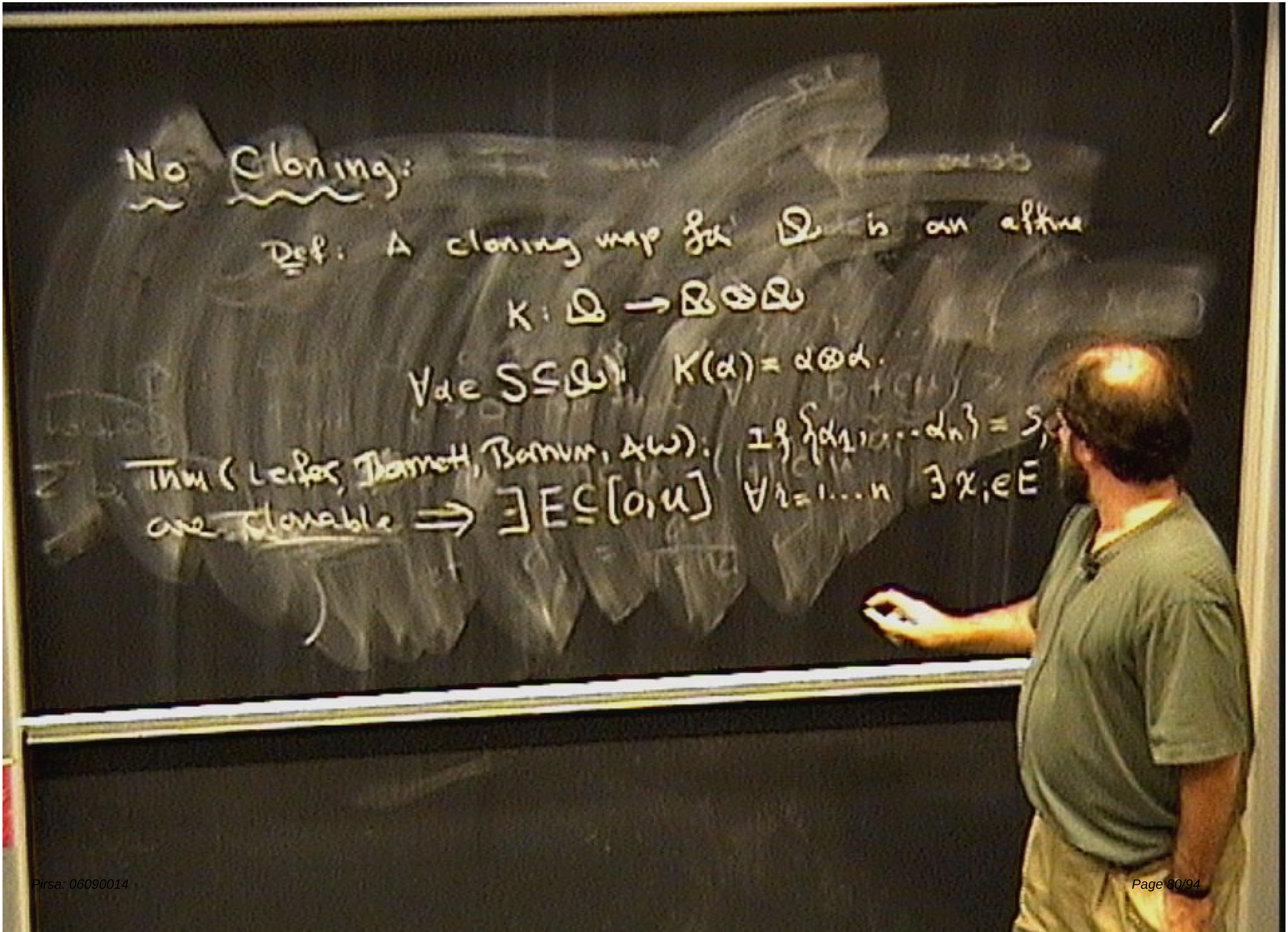
Def: A cloning map for Ω is an affine

$$K : \Omega \to \Omega \otimes \Omega$$

$$\forall \alpha \in S \subseteq \Omega, \quad K(\alpha) = \alpha \otimes \alpha.$$

Thm (Leifer, Barrett, Barnum, AW):


No Cloning:
Def: A cloning map for Ω is an affine
K: Ω → Ω⊗Ω
∀α ∈ S ⊆ Ω, K(α) = α⊗α.
(Leifer, Barnett, Barnum, A.W.): If {α_1, ... α_n} = S,
clonable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Thm (Leifer, Barnett, Barnum, AW): If {α1, ... αn} = S,
are clonable ⟹ ∃ E ⊆ [0,u] ∀i = 1...n ∃ xi ∈ E

No Cloning:

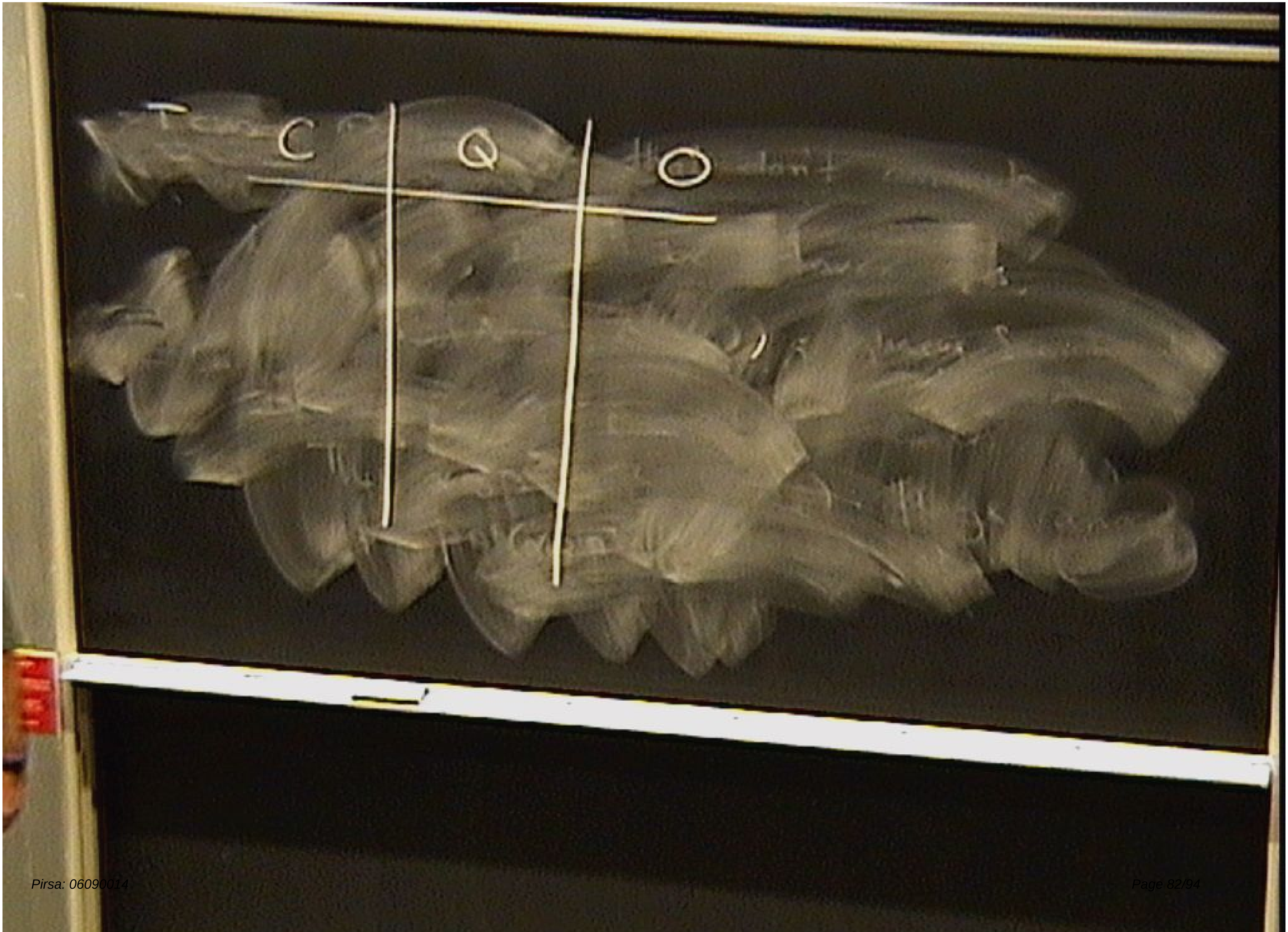
Def: A cloning map for Ω is an affine

$$K : \Omega \to \Omega \otimes \Omega$$

$$\forall \alpha \in S \subseteq \Omega, \quad K(\alpha) = \alpha \otimes \alpha.$$

Thm (Leifer, Barnett, Barnum, AW). If $\{\alpha_1, \ldots, \alpha_n\} = S$, are clonable $\Longrightarrow \exists E \subseteq [0,u] \ \forall i = 1 \ldots n \ \exists x_i \in E$

$$x_i(\alpha_i) = 1$$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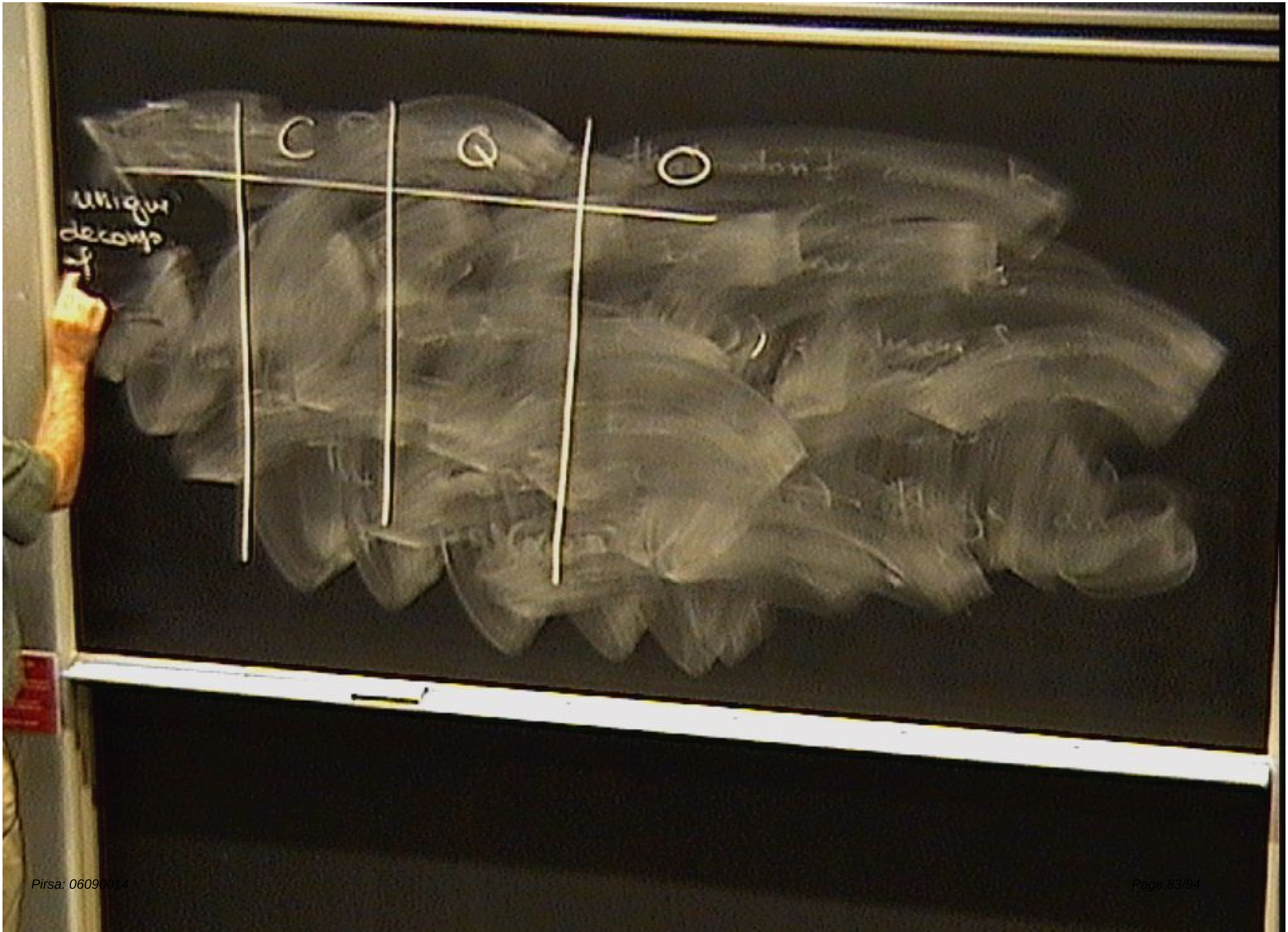
C
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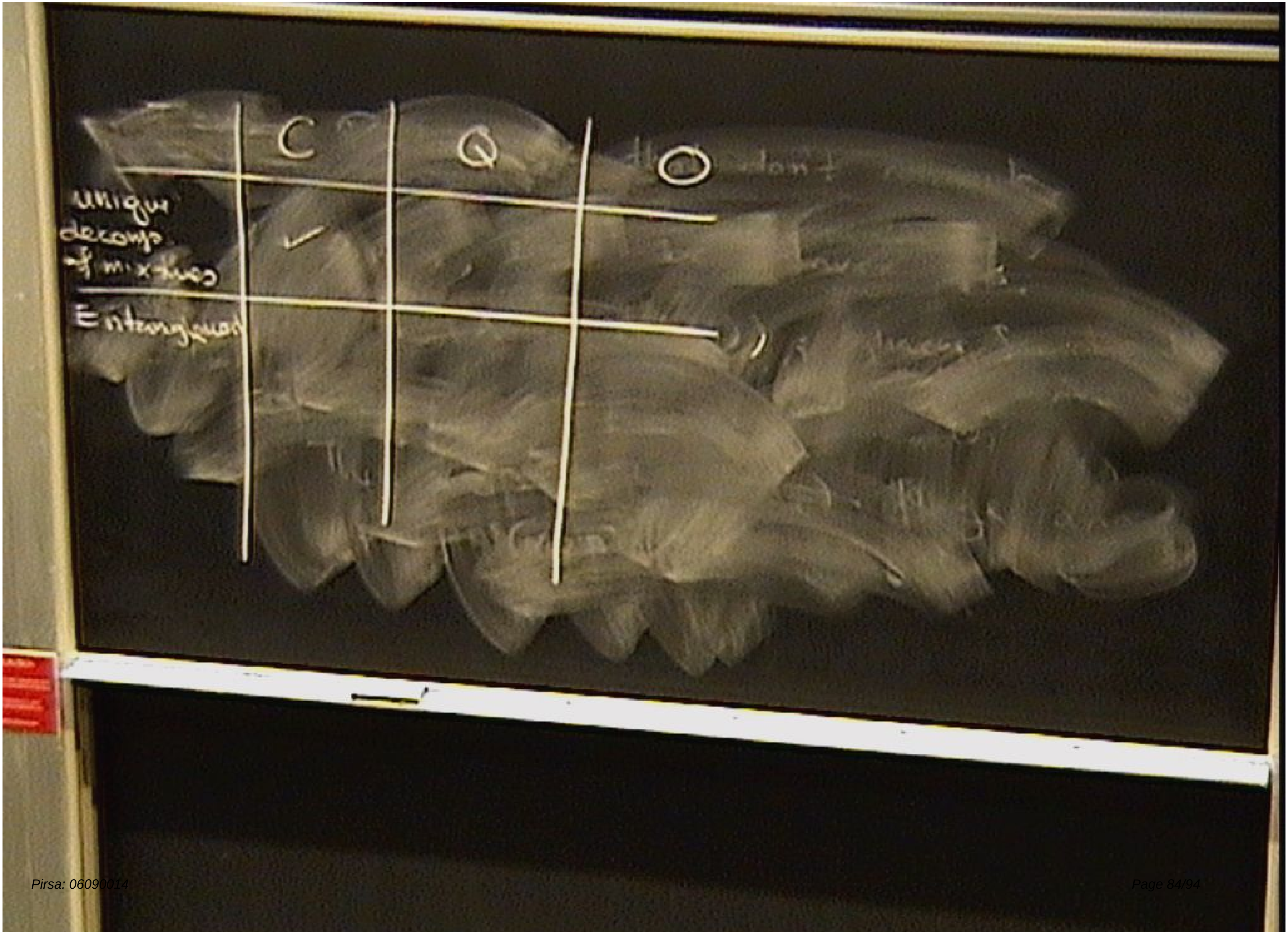
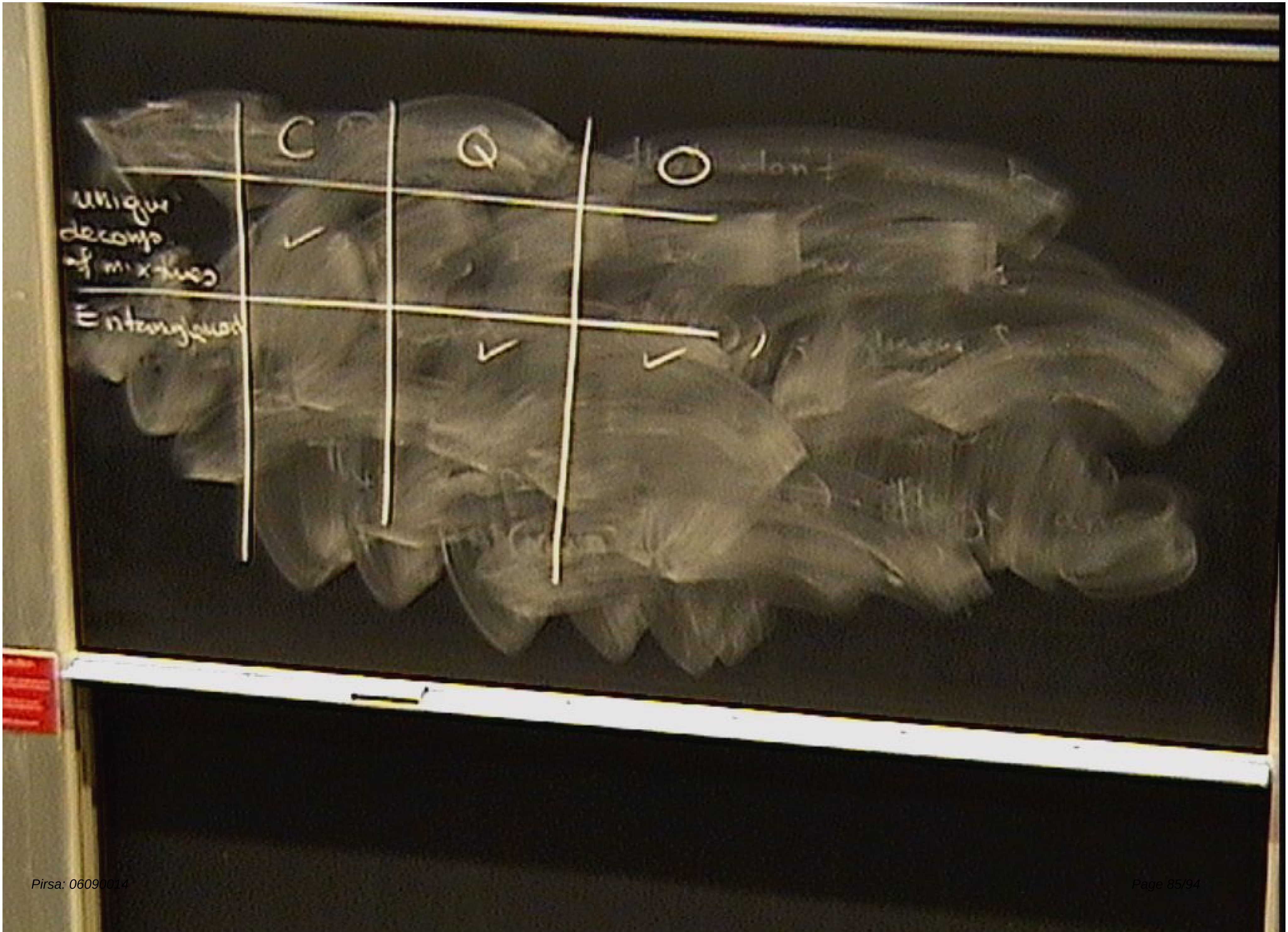

C
Q
O
unique decomp of mixtures
Entanglement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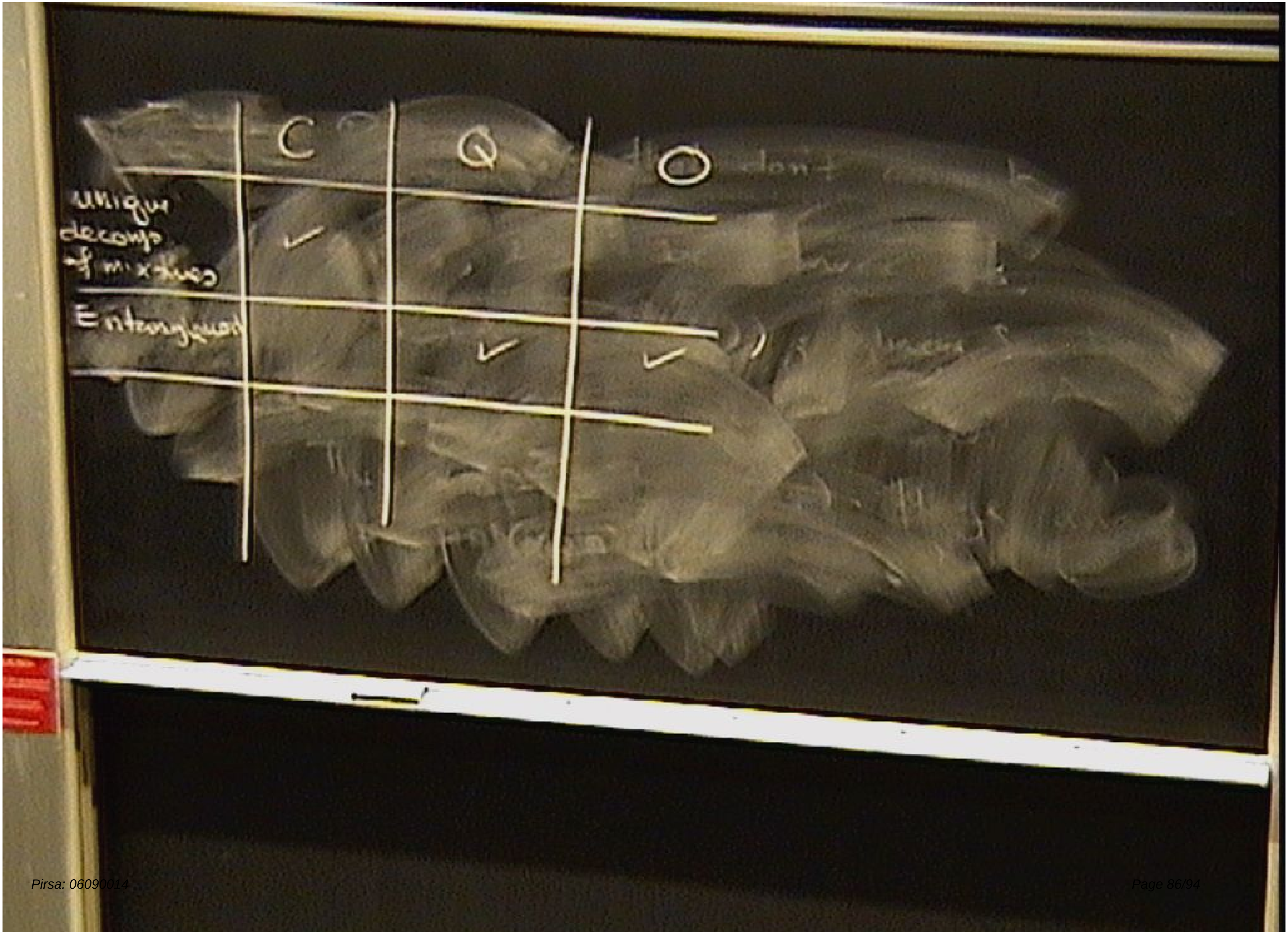
Entanglement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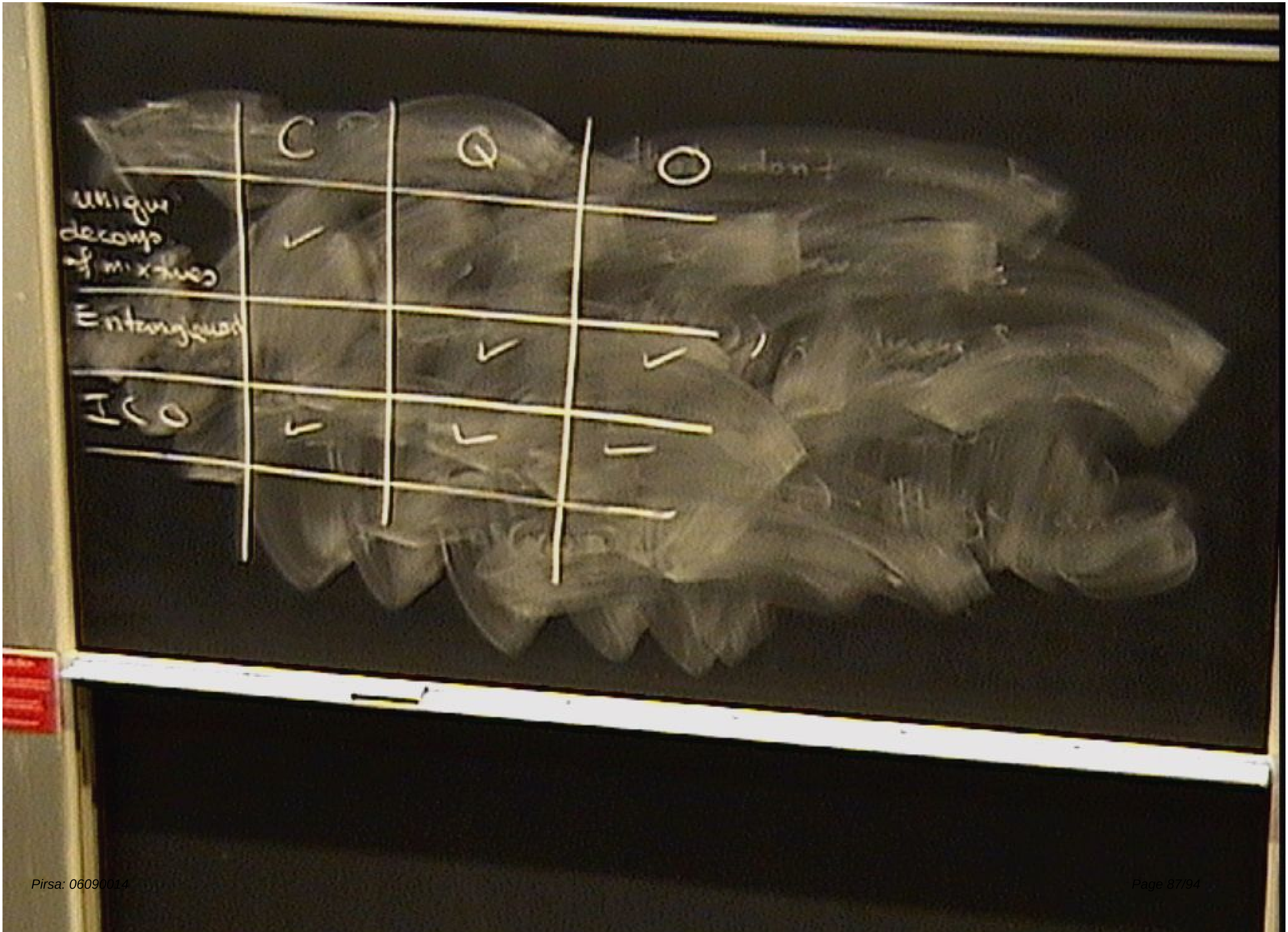
Entanglement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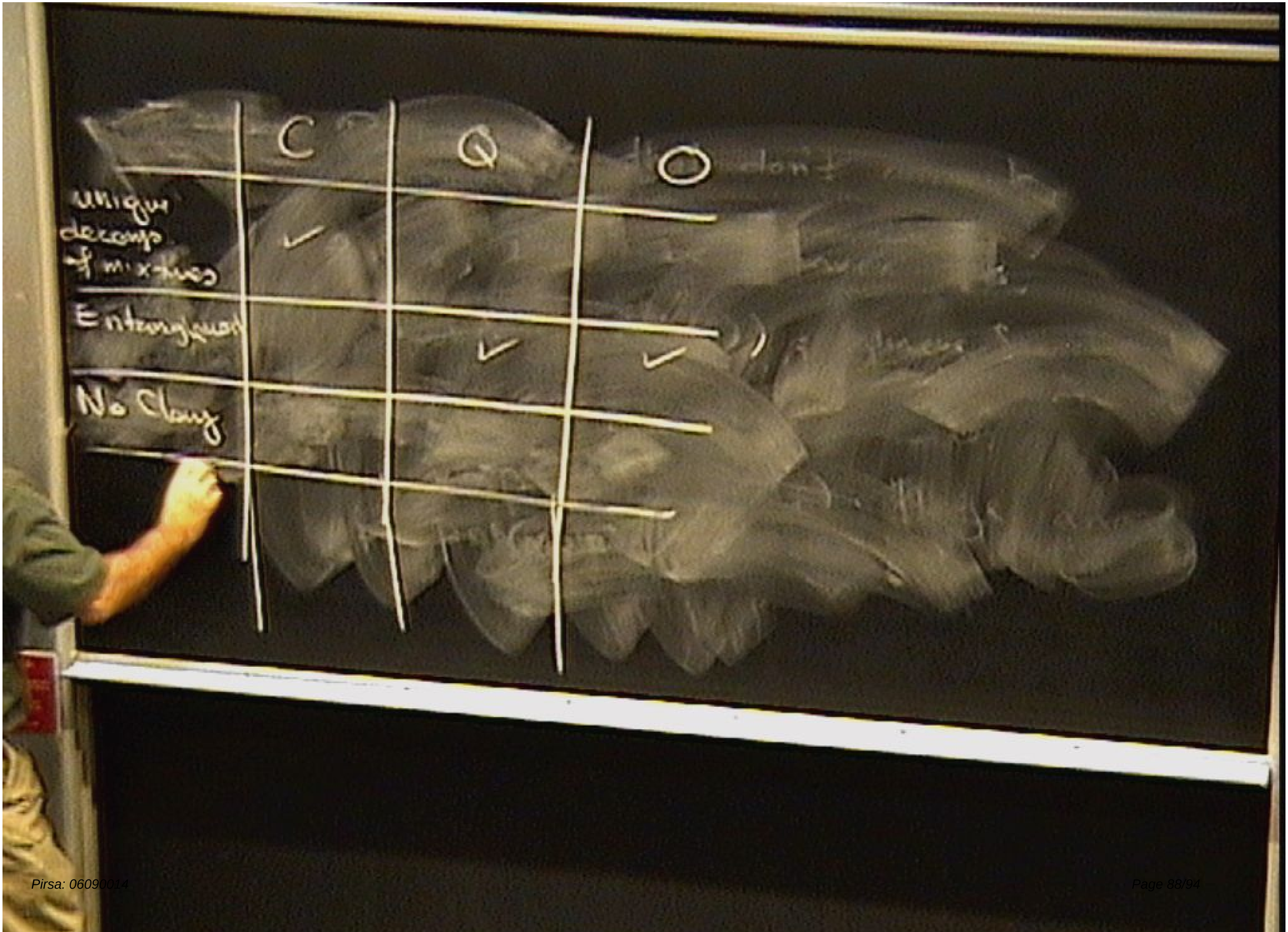
C
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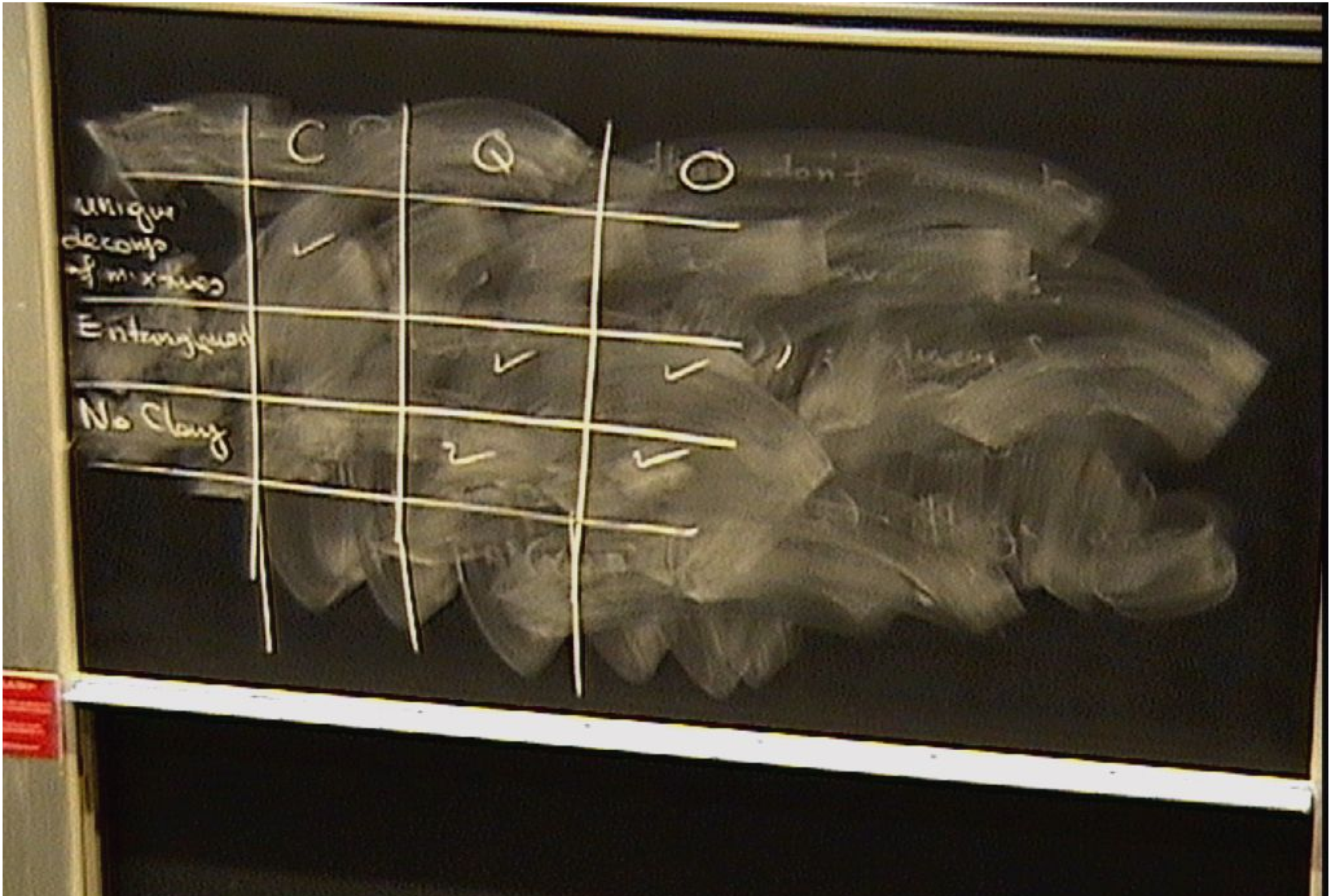
unique decomp of mixtures
Entanglement
No Cloning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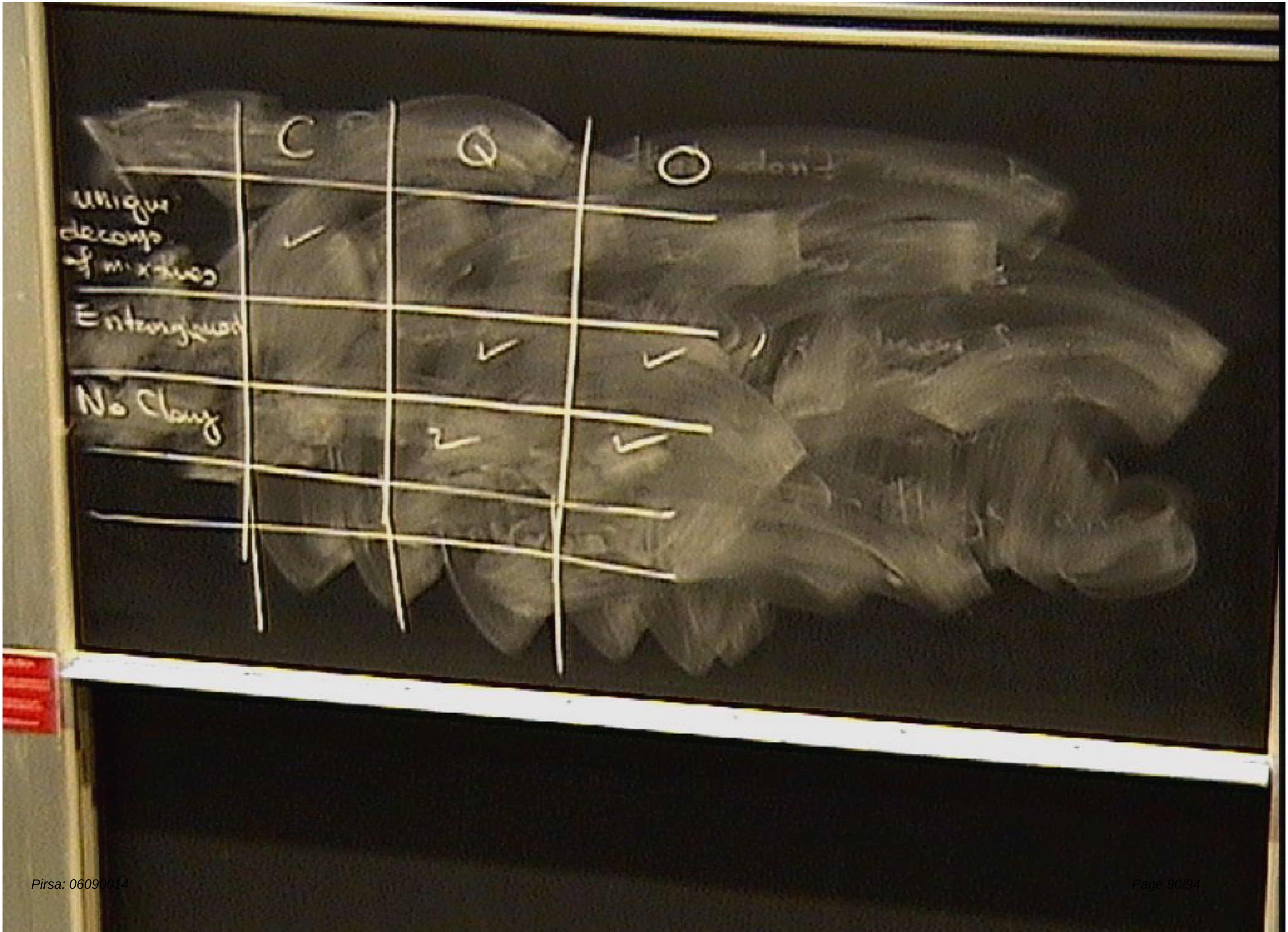
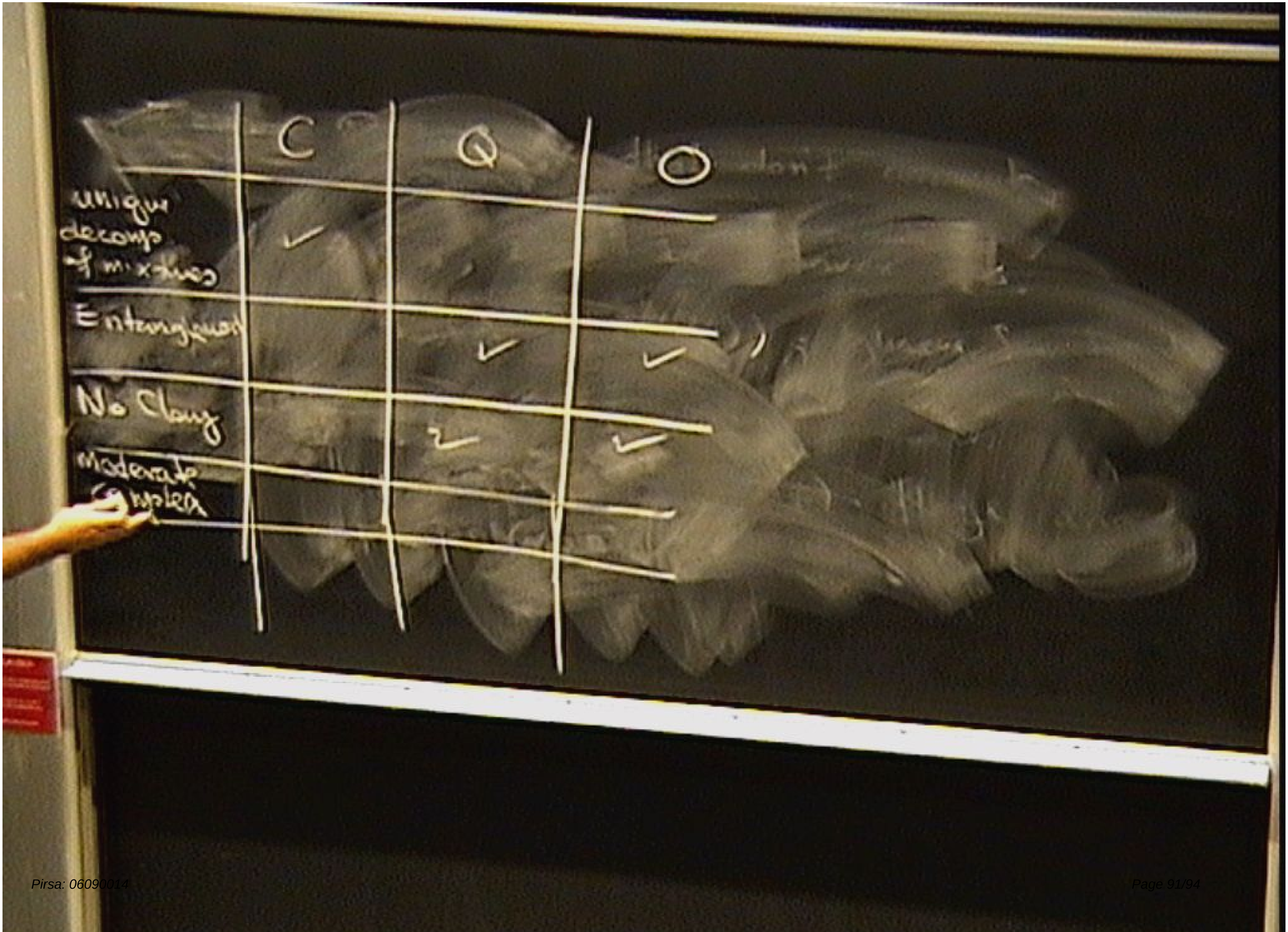
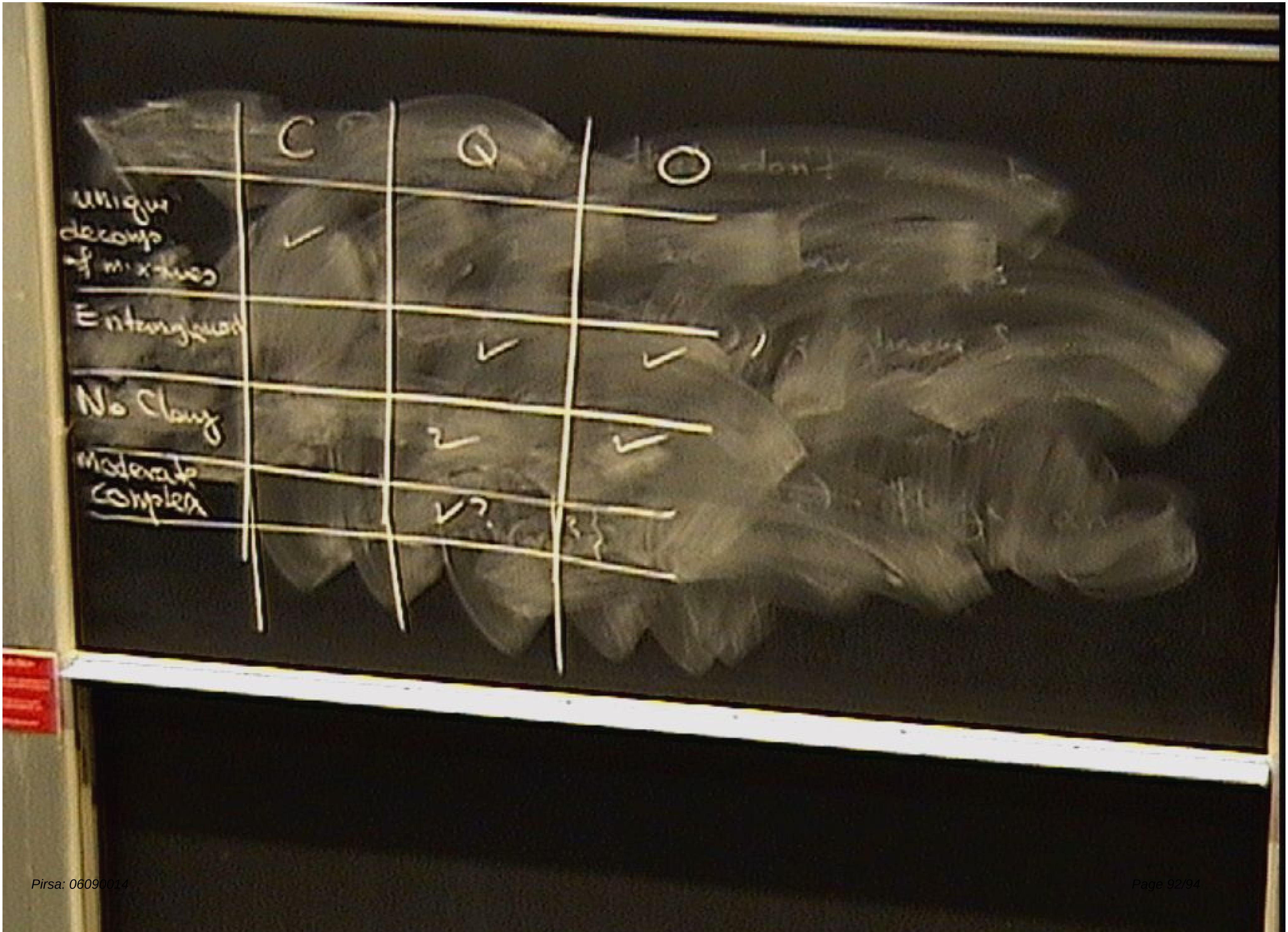
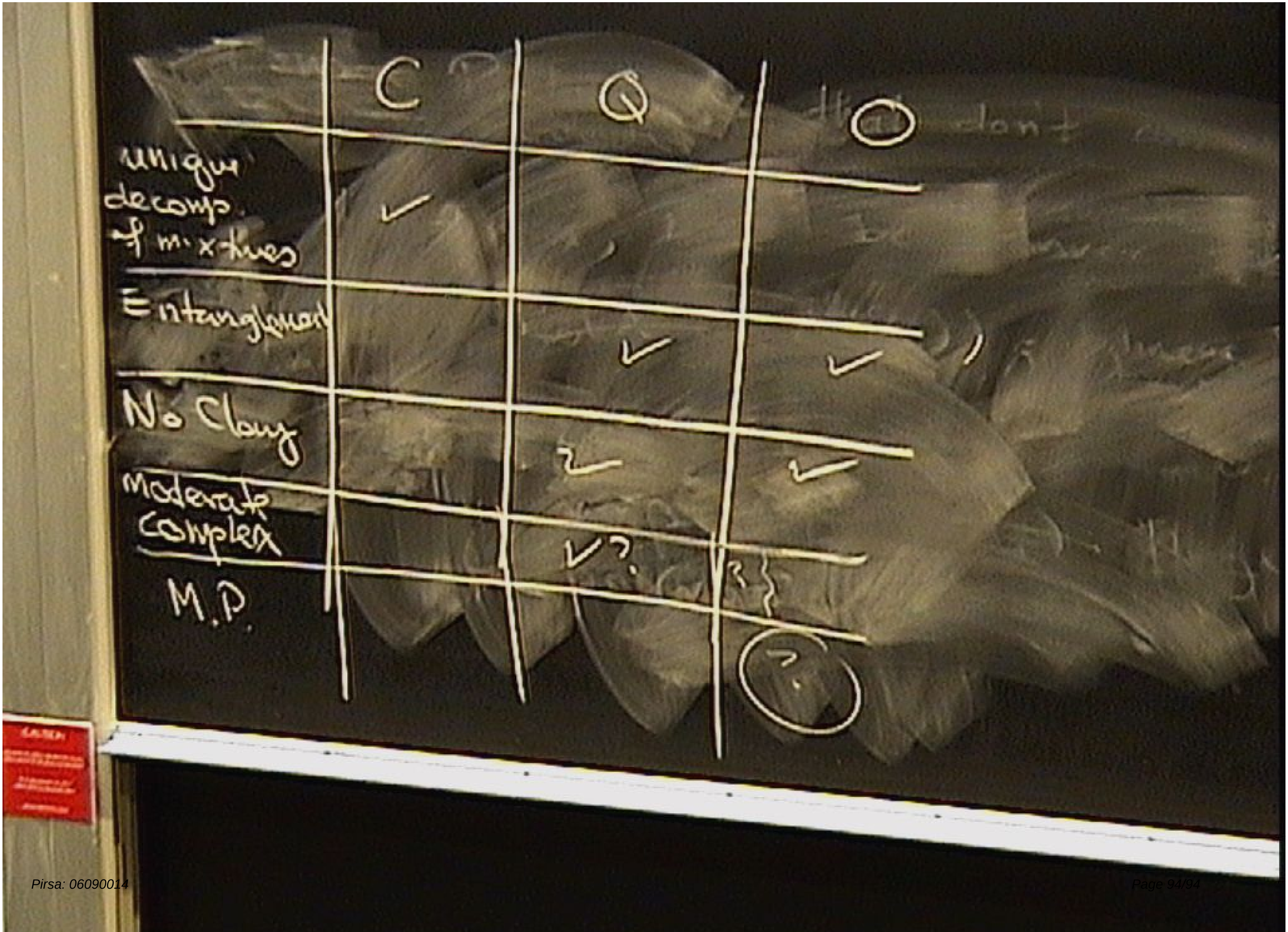

C
Q
O
unique decomp of mixtures
Entanglement
No Cloning


C
Q
O
unique decomp of mixtures
Entanglement
No Cloning
moderate complexity


C
Q
O
unique decomps of mixtures
Entanglement
No Cloning
moderate complex

	C	Q	O
unique decomp of mixtures	✓		
Entanglement		✓	✓
No Cloning		? ✓	✓
moderate complex		✓?	


C
Q
?
unique decomp. of mixtures
Entanglement
No Cloning
Moderate Complex
M.P.
?