Title: Three Tales of Entanglement

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Abstract: Entanglement is one of the most studied features of quantum mechanics and in particular quantum information. Yet its role in quantum information is still not clearly understood. Results such as (R. Josza and N. Linden, Proc. Roy. Soc. Lond. A 459, 2011 (2003)) show that entanglement is necessary, but stabilizer states and the Gottesman-Knill theorem (for example) imply that it is far from sufficient. I will discuss three aspects of entanglement. First, a quantum circuit with a "vanishingly small" amount of entanglement that admits an apparent exponential speed-up over the classical case. Second, I will discuss techniques for lower-bounding the amount of entanglement in bipartite quantum states. Finally, I will discuss the role of entanglement in quantum metrology. Specifically, I will show that entangling ancillas can make no difference to the accuracy of a quantum parameter estimation, regardless of the nature of the coupling Hamiltonian. I will conclude by discussing strategies for improving the scaling of quantum parameter estimation.

Three Tales of Entanglement

Steven T. Flammia
University of New Mexico

Perimeter Institute, September 20, 2006

Two
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Entanglement: What is it good for?

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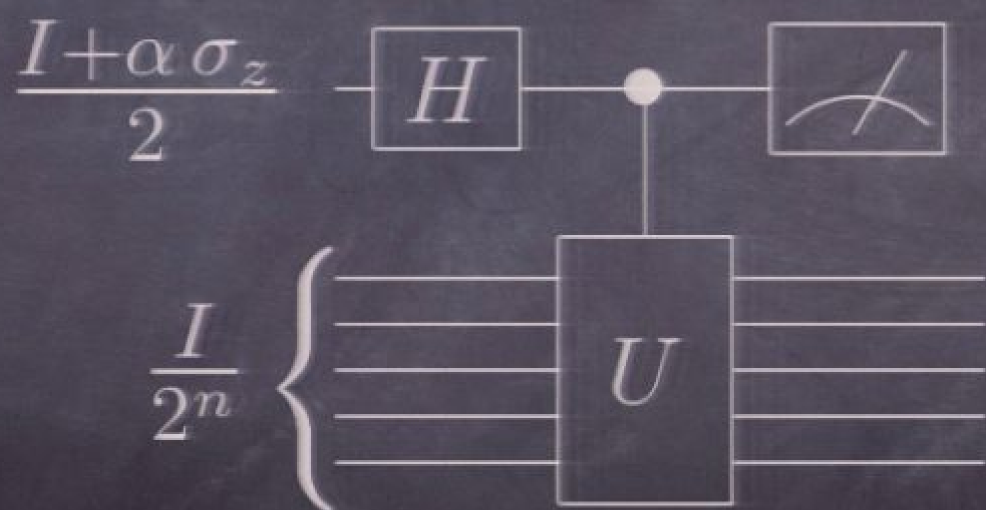
Global entanglement as a necessary resource

- We'd like to understand the role of entanglement in a quantum information.
- Global entanglement is necessary, but not sufficient, to achieve advantages over classical protocols.
- I'll discuss two systems where entanglement matters, but in vastly different ways.
- Power of One Qubit - "Multum ex Parvo"
- Quantum Metrology - "Parvo ex Multum"

R. Jozsa and N. Linden, Proc. R. Soc. Lond. A, **459**, 2003.

What can one qubit do?



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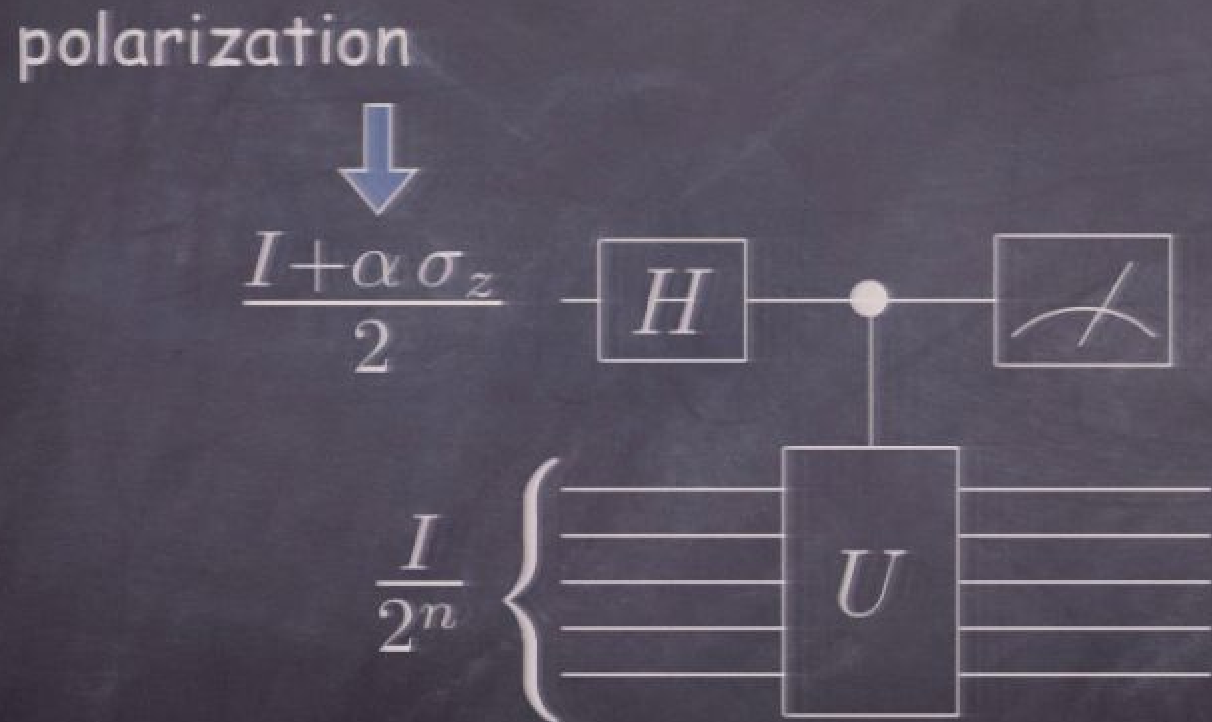

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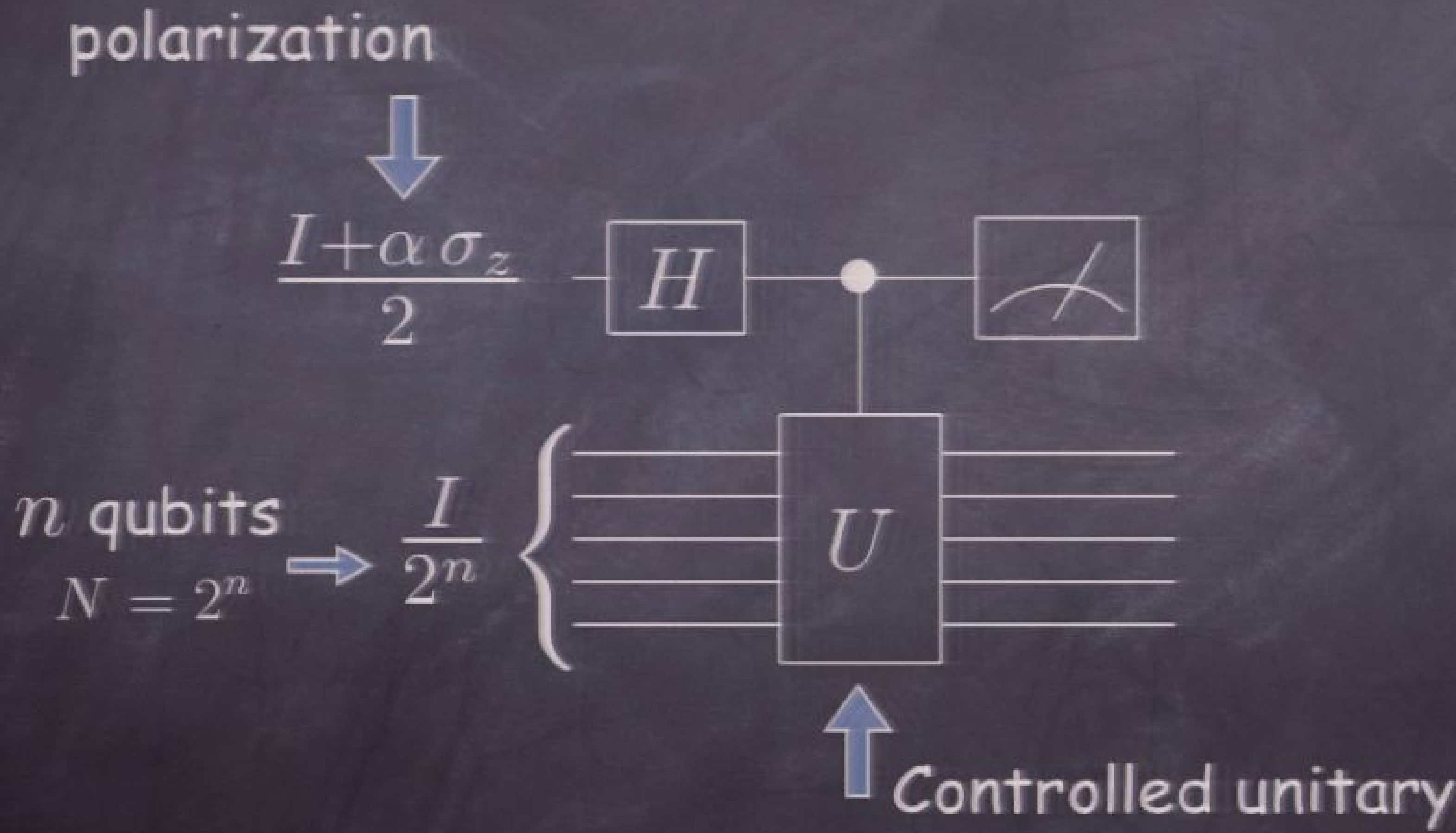

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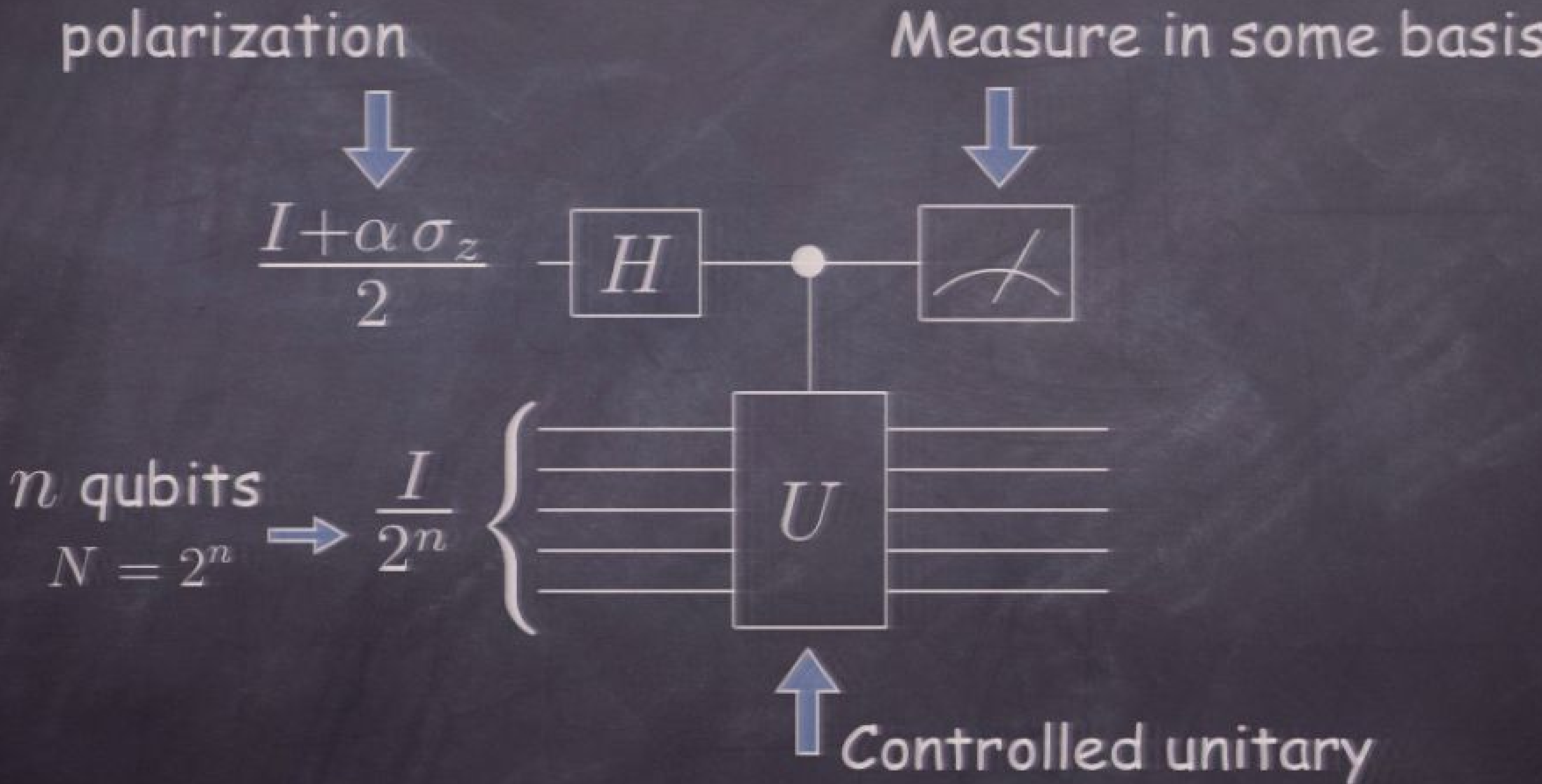

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Let U be a unitary operator on n qubits that can be implemented efficiently in terms of elementary gates. Estimate $\mathrm{Tr}\,(U)/2^n$ with fixed accuracy .

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Applications to testing integrability of chaotic systems and estimating density of states.

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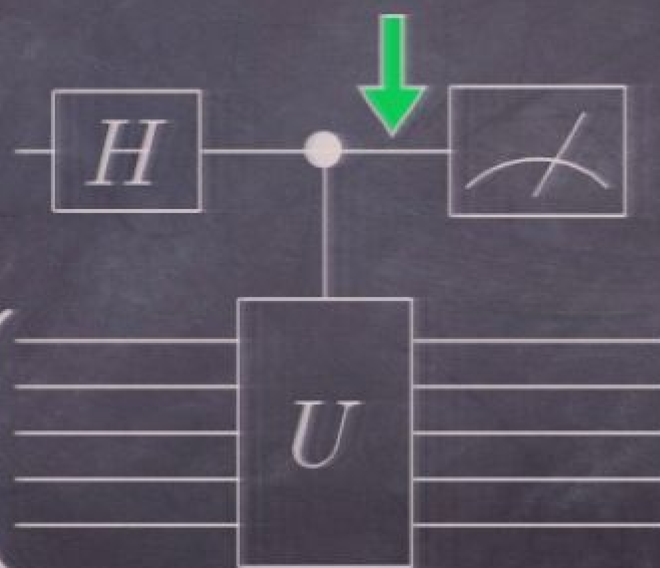
$$\rho_{in} = \frac{I}{2N} + \alpha \frac{\sigma_z^{(1)}}{2N} \rightarrow \rho = \rho_{out} = \frac{1}{2N} \begin{pmatrix} I & \alpha U \\ \alpha U^\dagger & I \end{pmatrix}$$

$$\frac{I + \alpha\, \sigma_z}{2}$$



Measure $\langle \sigma_x \rangle, \langle \sigma_y \rangle$

n qubits $N = 2^n$ $\quad \frac{I}{2^n}$

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$$\langle \sigma_x \rangle = \mathrm{Tr}\,(\sigma_x \rho) = \frac{\alpha}{2N}\mathrm{Tr}\,(U^\dagger + U) = \frac{\alpha\,\mathrm{Re}(\mathrm{Tr}\,(U))}{N}$$

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For a fixed error ϵ, the number of measurements needed scales like $O\left(1/\alpha^2 \epsilon^2\right)$.

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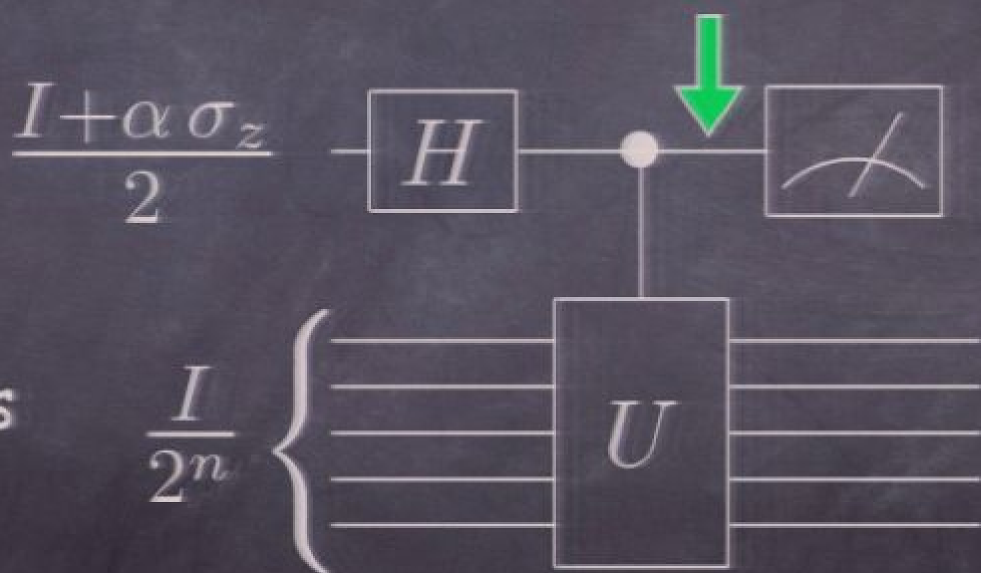

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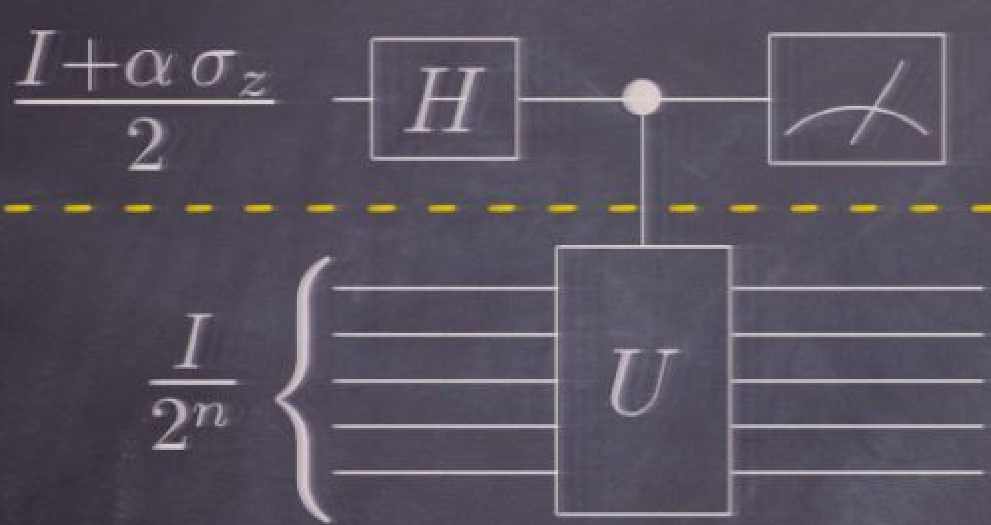

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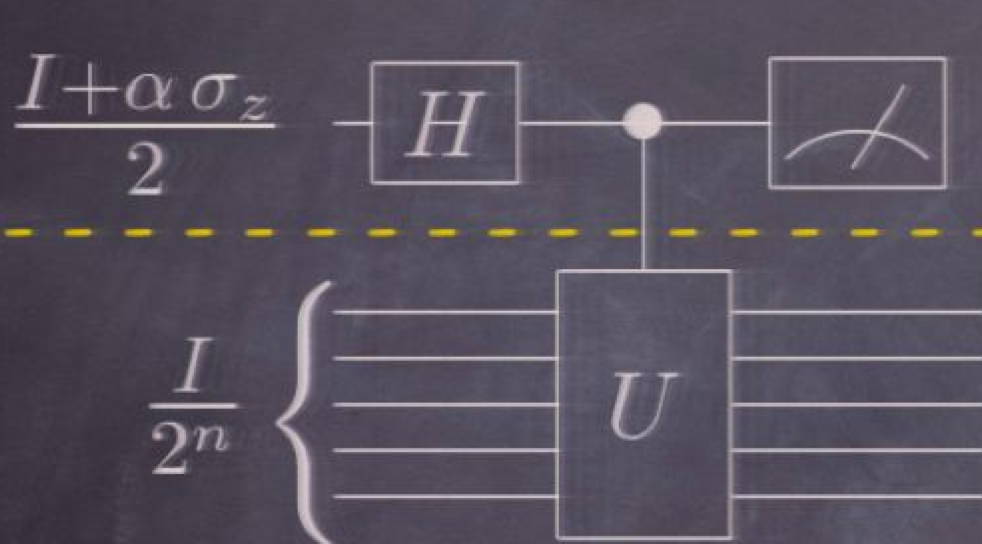

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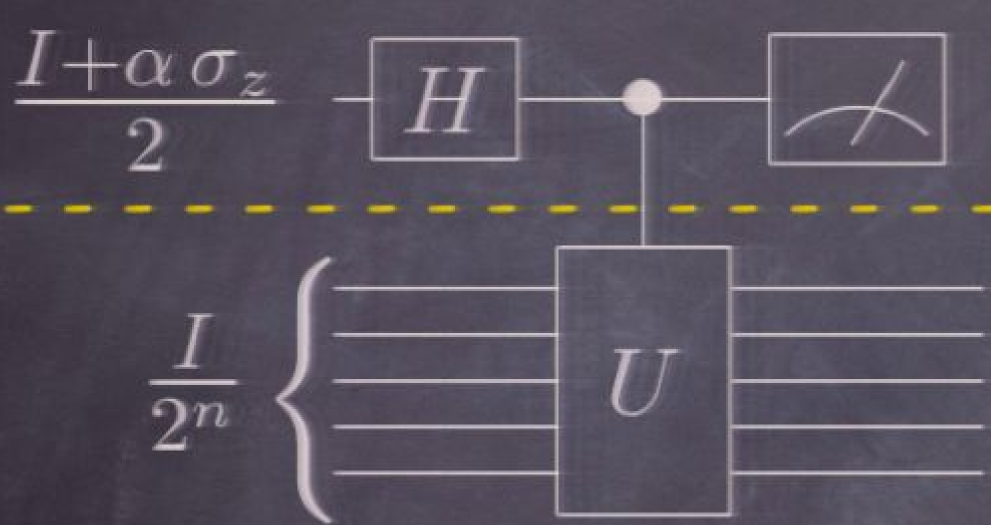


Bipartite split between the special qubit and the rest

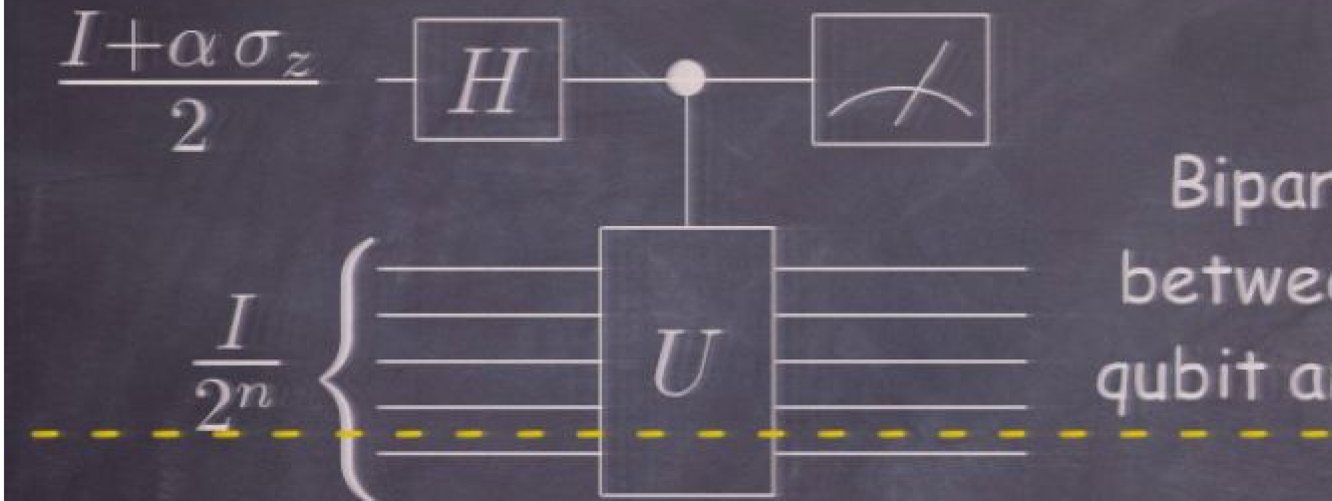
Diagonalize U ➡ First qubit is separable with the other n

Trace out the first qubit ➡ completely mixed state

Where's the entanglement?

$$\frac{I+\alpha\,\sigma_z}{2}$$

$$\frac{I}{2^n}$$



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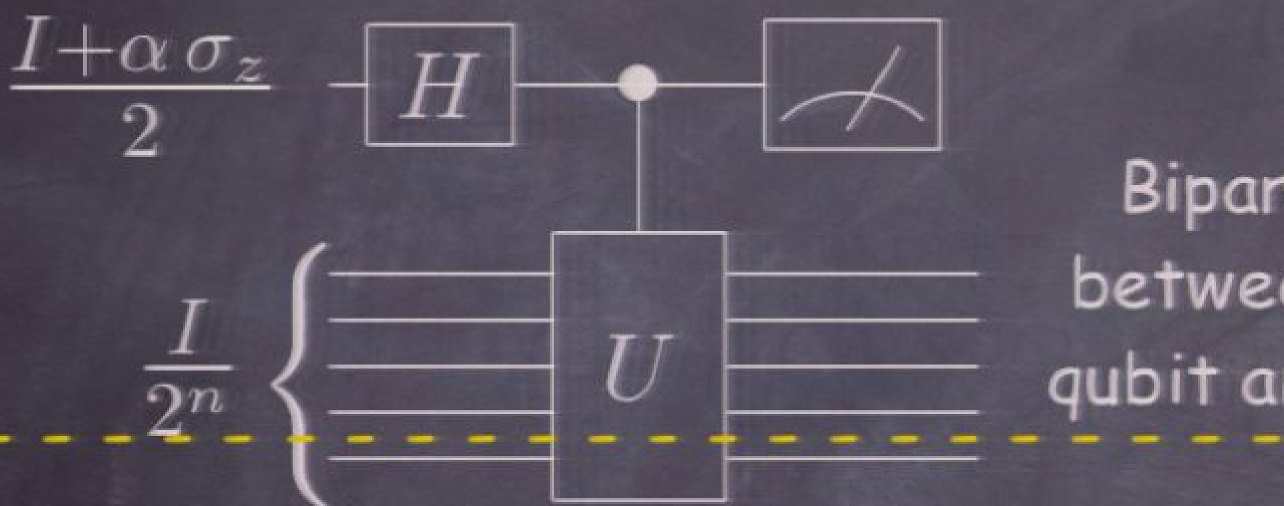
Choose:

$$U = I_{n-2} \otimes \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = 1$$

Where's the entanglement?



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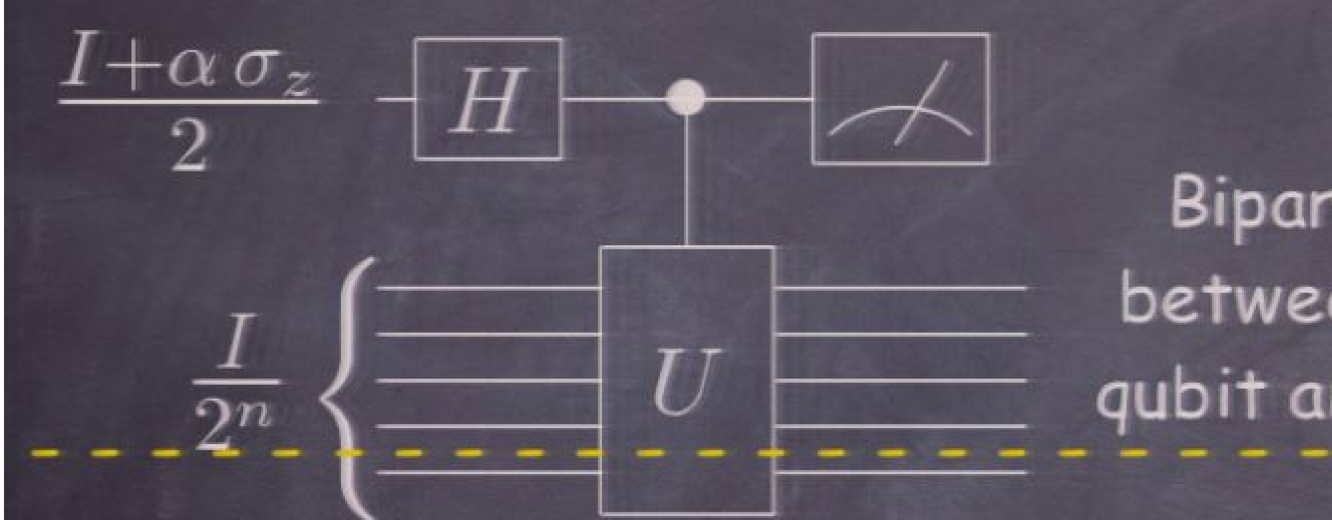

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Definitely entangled!

...But this is the best known U.

Is this typical?

We would like to construct a pseudo-random unitary that is <u>efficiently implementable</u> and calculate the resulting state's negativity.

Probably hard to do analytically.

Quite easy numerically.

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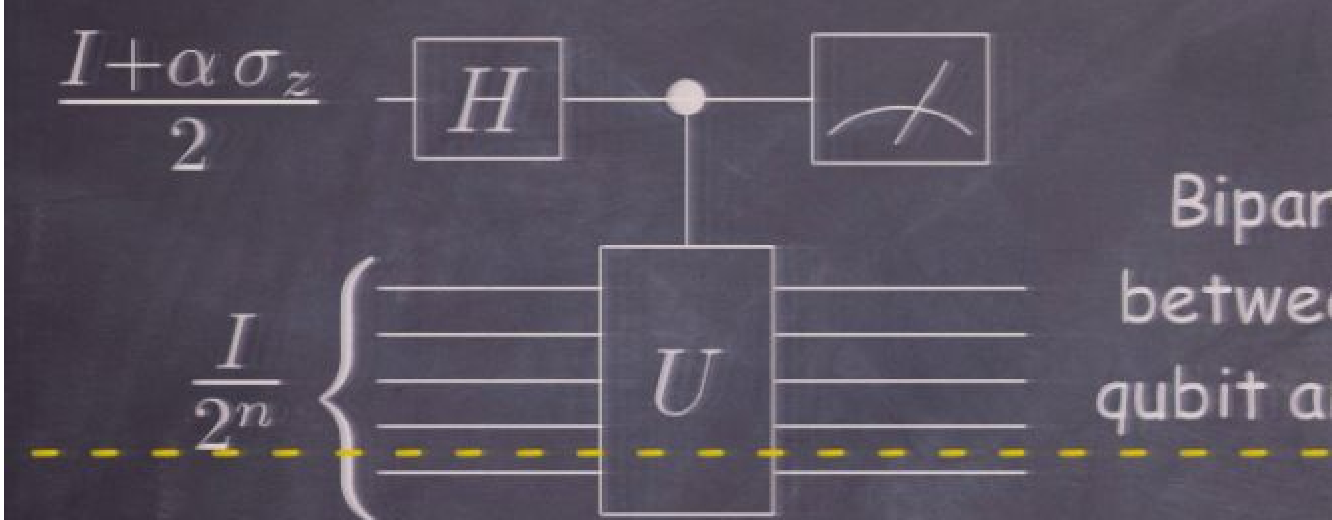

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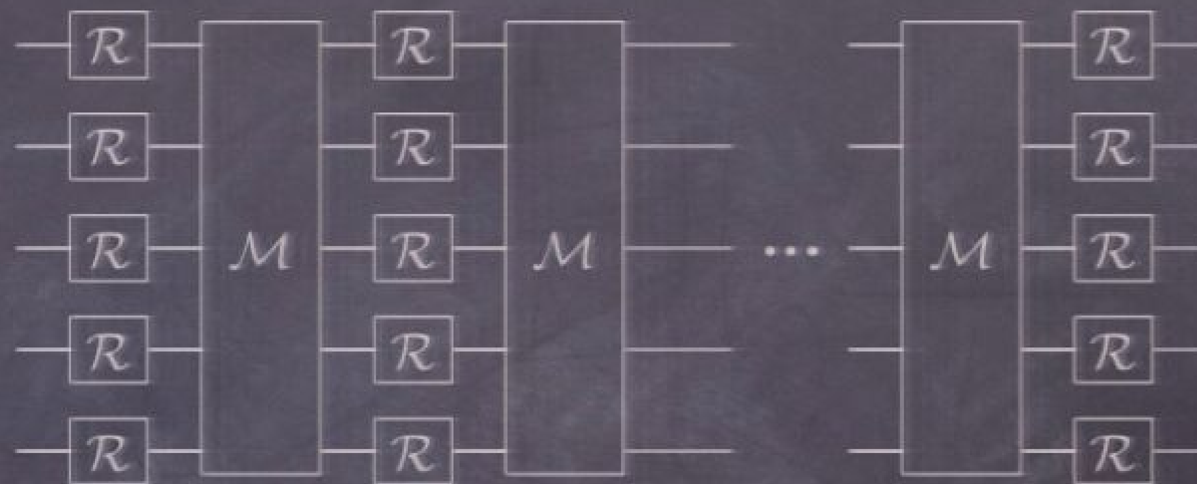

Random SU(2): $\mathcal{R}(\theta,\phi,\chi) = \begin{pmatrix} e^{i\phi}\cos(\theta) & e^{i\chi}\sin(\theta) \\ -e^{-i\chi}\sin(\theta) & e^{-i\phi}\cos(\theta) \end{pmatrix}$

"Mixing matrix" of NN couplings: $\mathcal{M} = \exp\left(i\frac{\pi}{4}\sum_{i=1}^{n-1} \sigma_z^i \otimes \sigma_z^{i+1} \right)$

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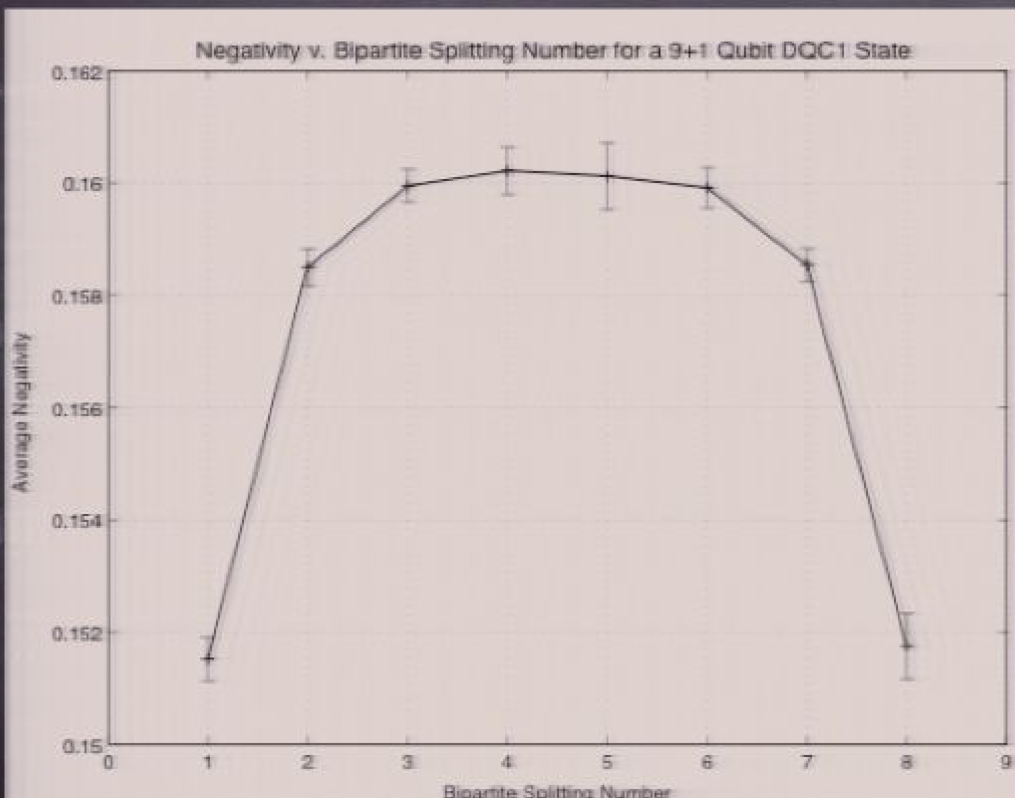
Average negativity



Global entanglement!

Bipartite splitting

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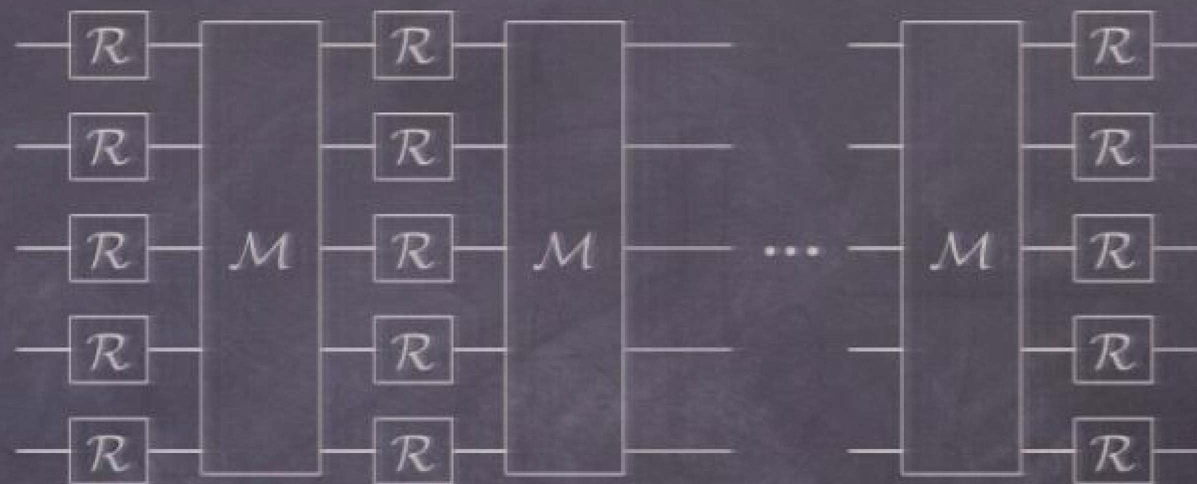

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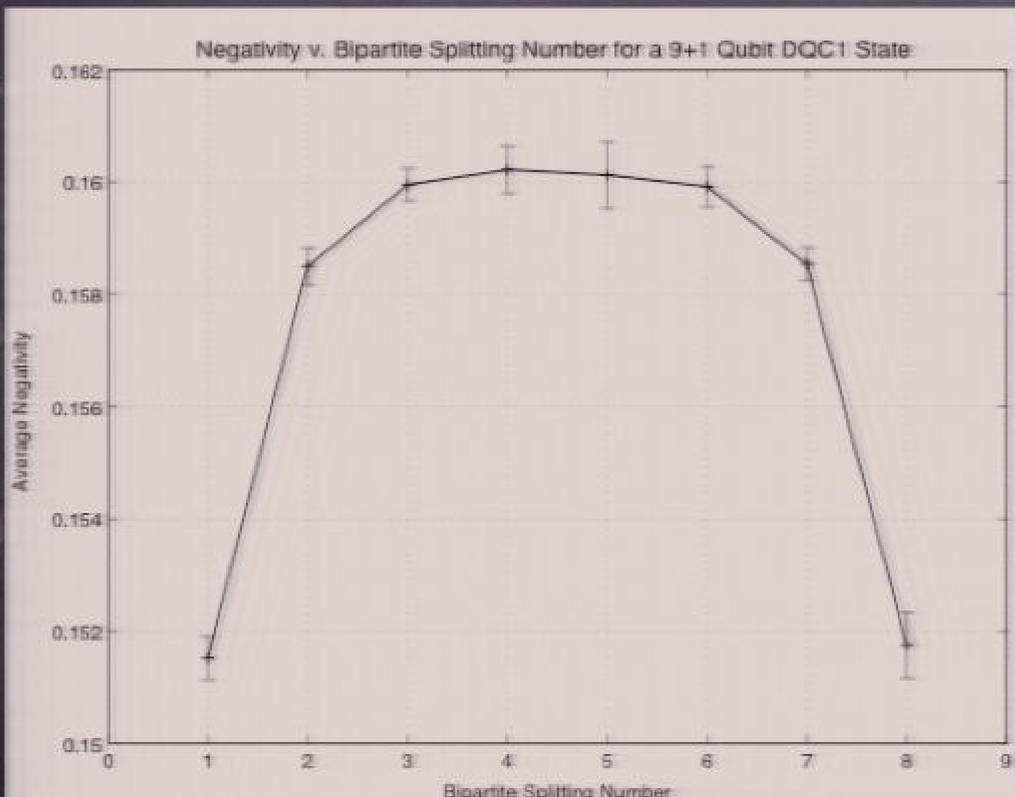
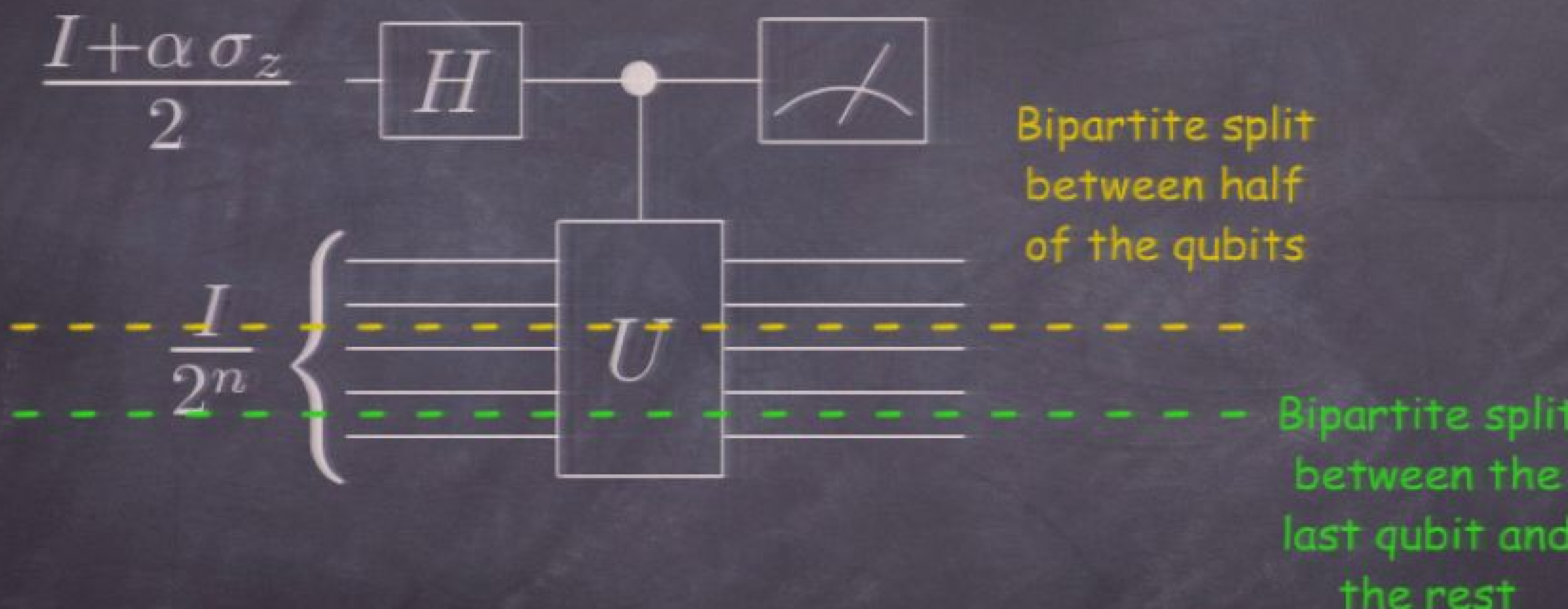


Bipartite splitting

Global entanglement!

The negativity achieves a maximum when the bipartite splitting is done half/half.

Is this typical?



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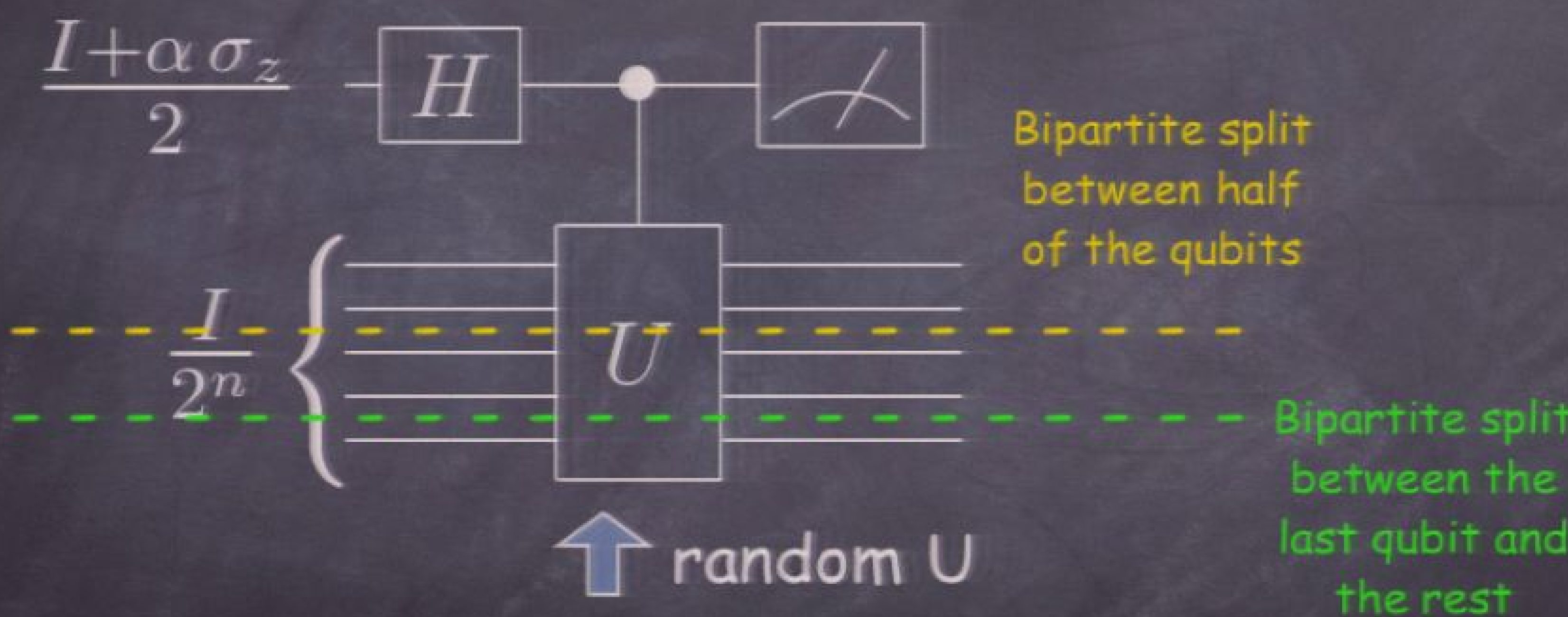
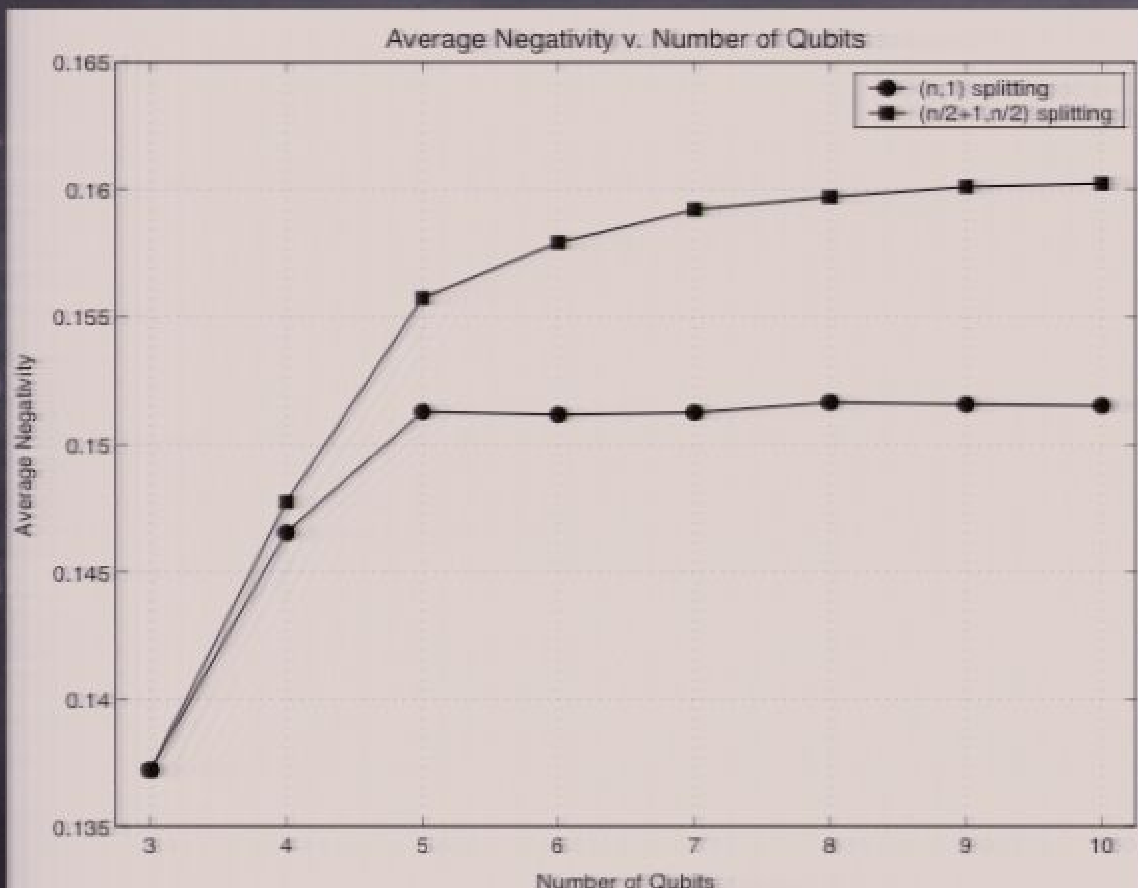


Generate lots of statistics and see what the negativity is....

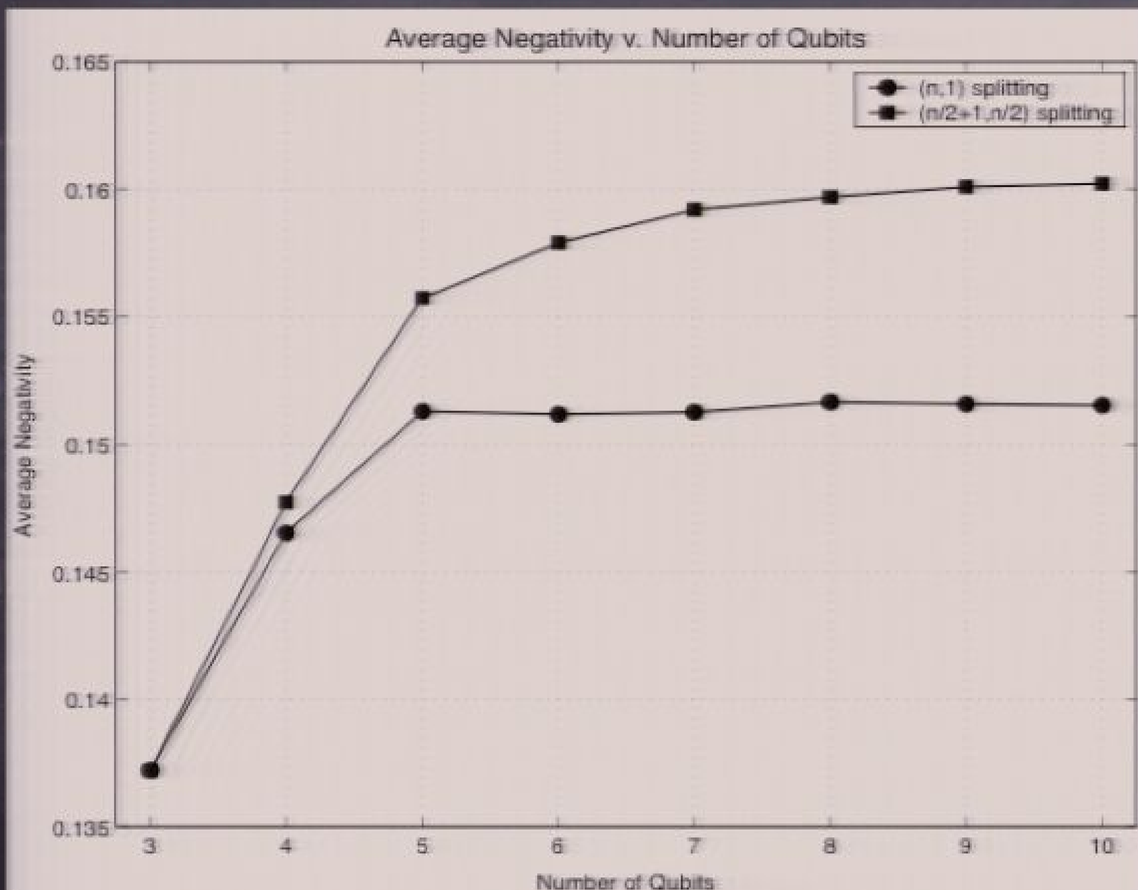
Is this typical?

Average negativity



Number of qubits

Is this typical?



Average negativity

Number of qubits

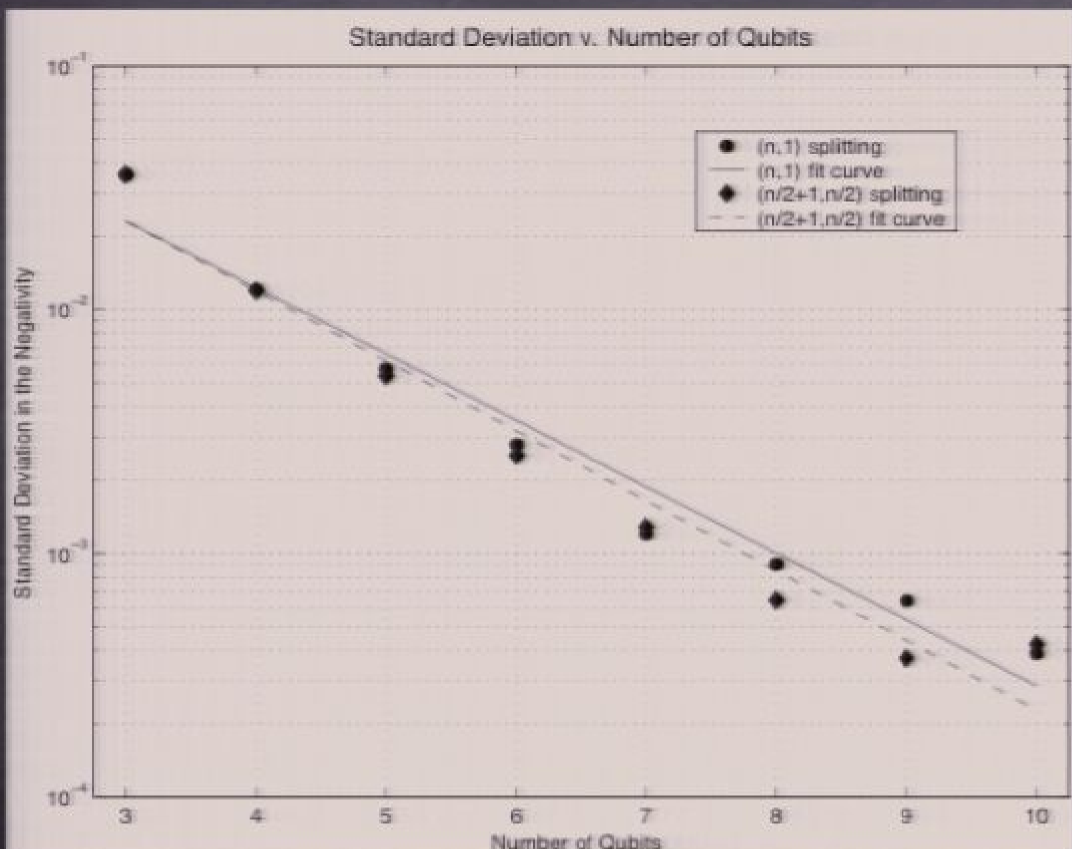
The (n,1) splitting quickly levels off, but...

the half/half splitting rises slowly.

Can it achieve negativity > .25 ?

Is this typical?

Standard deviation



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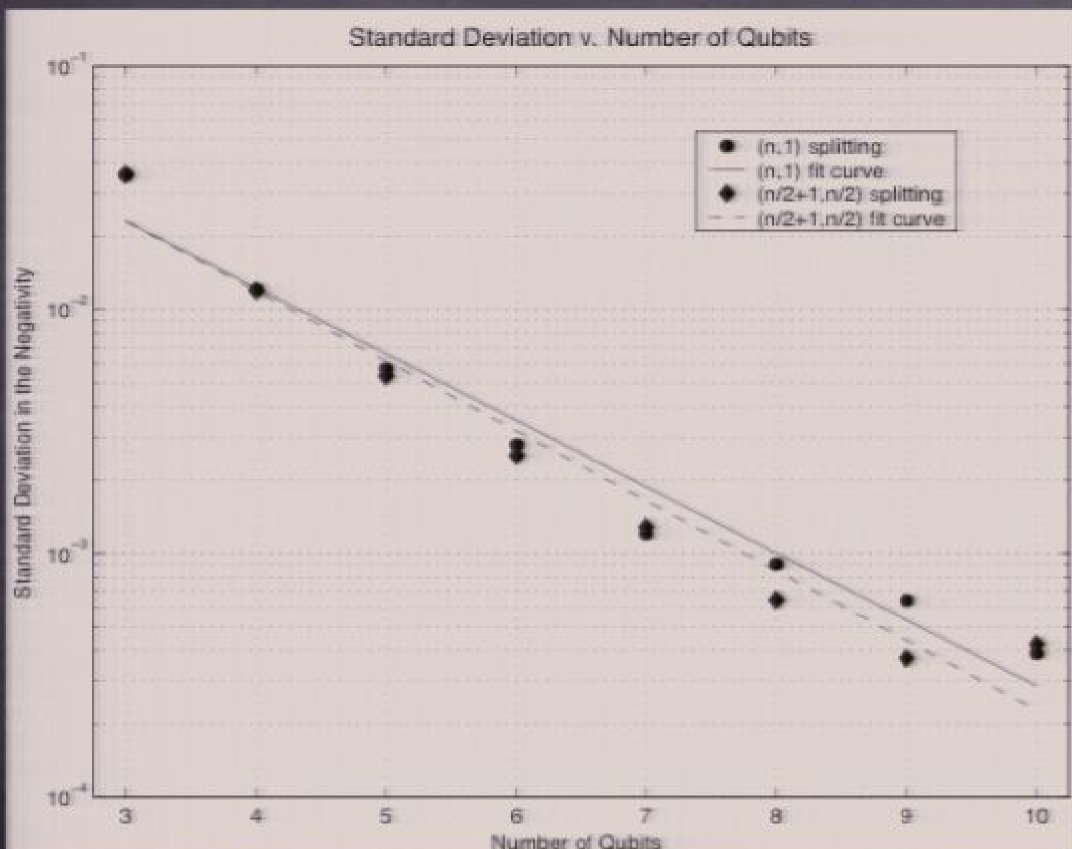

Number of qubits

Standard deviation decreases exponentially.

Almost all unitaries have the same negativity

Bounding the negativity

Recall the form of rho: $\rho = \frac{1}{2N}\begin{pmatrix} I & \alpha U \\ \alpha U^\dagger & I \end{pmatrix}$

Strategy: find trace invariants of $\breve{\rho}$ to constrain the eigenvalues.

Trace invariants:

$\mathrm{Tr}\left(\breve{\rho}^s\right)$ s=1,2,3,...

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Then maximize the negativity subject to these constraints.

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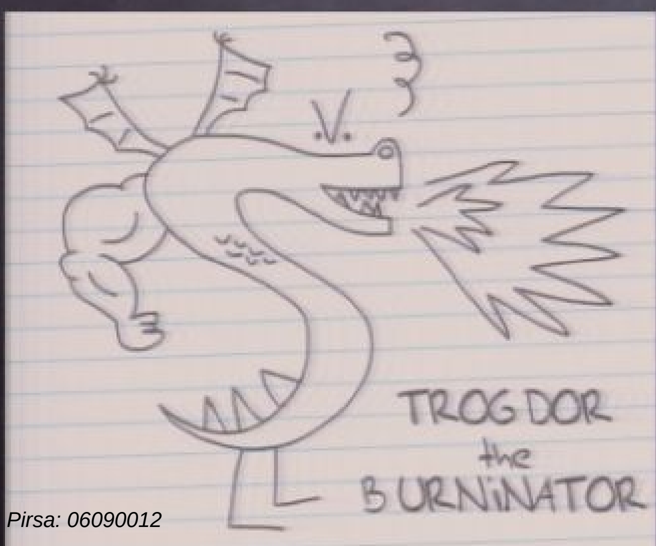

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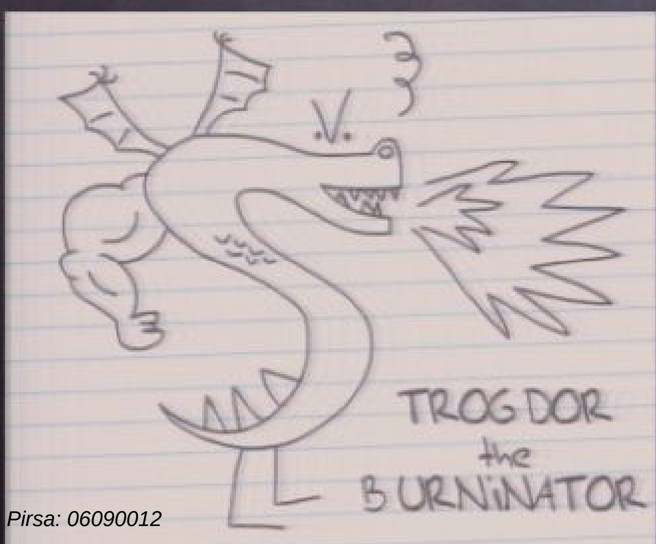

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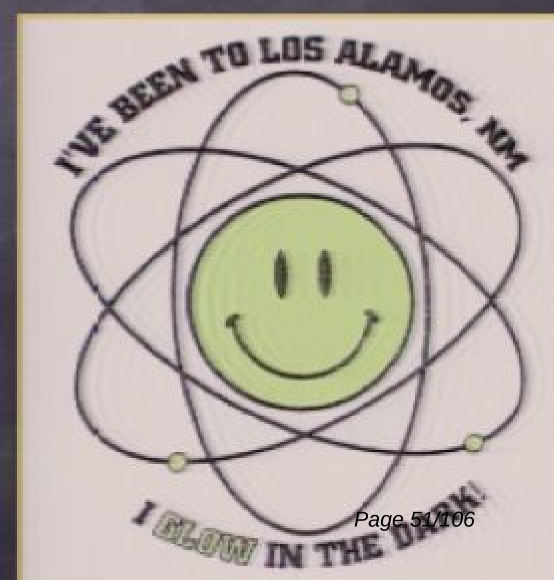

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Lemma 1: $\mathrm{Tr}\,(\breve{A}\breve{B}) = \mathrm{Tr}\,(AB)$.

Proof is simple; just pick A and B and write it out.

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Bounding the negativity

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when k≥4, the lemma can't help us!

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Use Lagrange multipliers to do the maximization. Symmetry yields:

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A,B,C eigenvalues, u,v,w degeneracies, $\alpha = 1$.

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Hard to solve.
Can be done
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Number of Qubits

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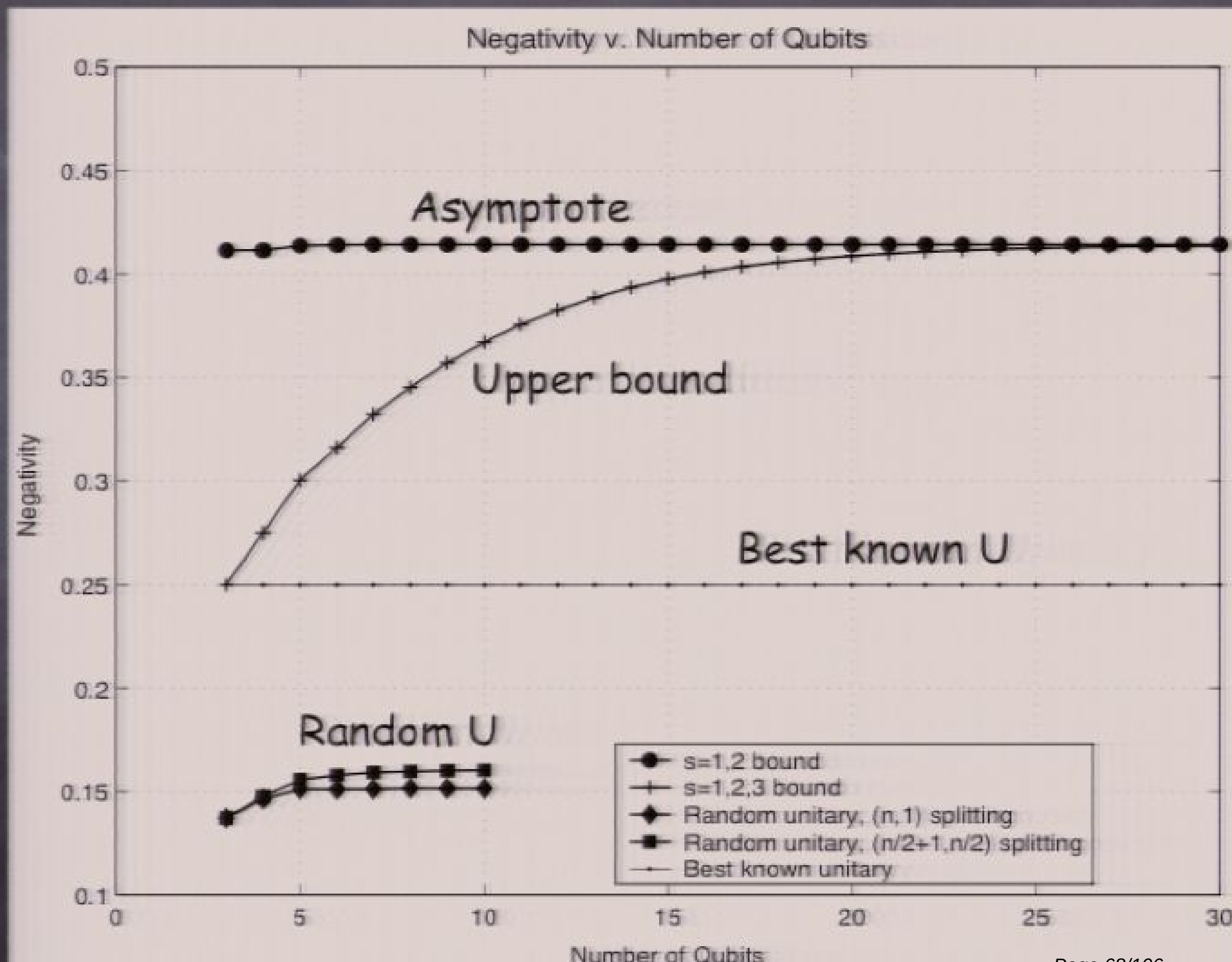

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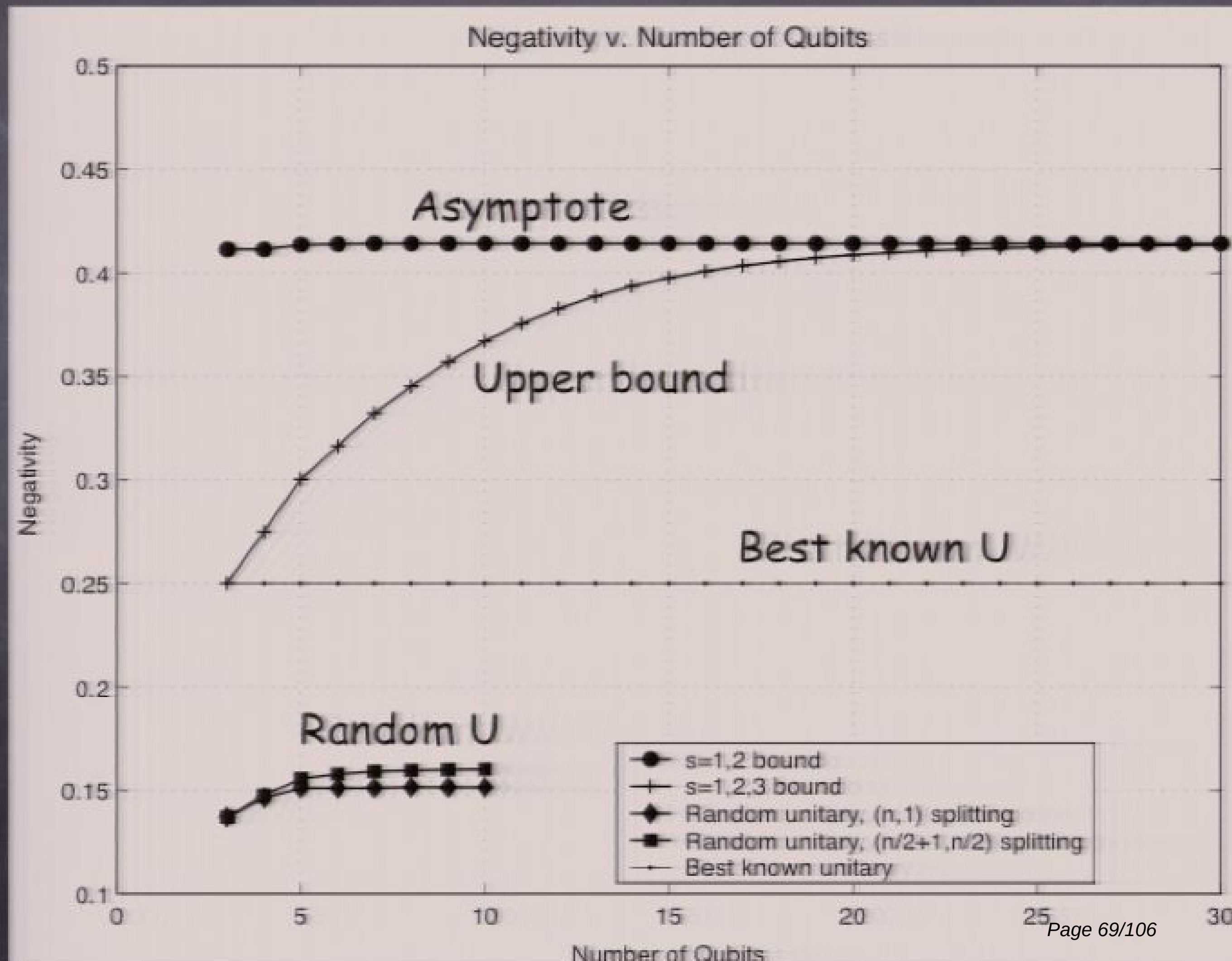
Best known U achieves the upper bound for n=3. Is this optimal?



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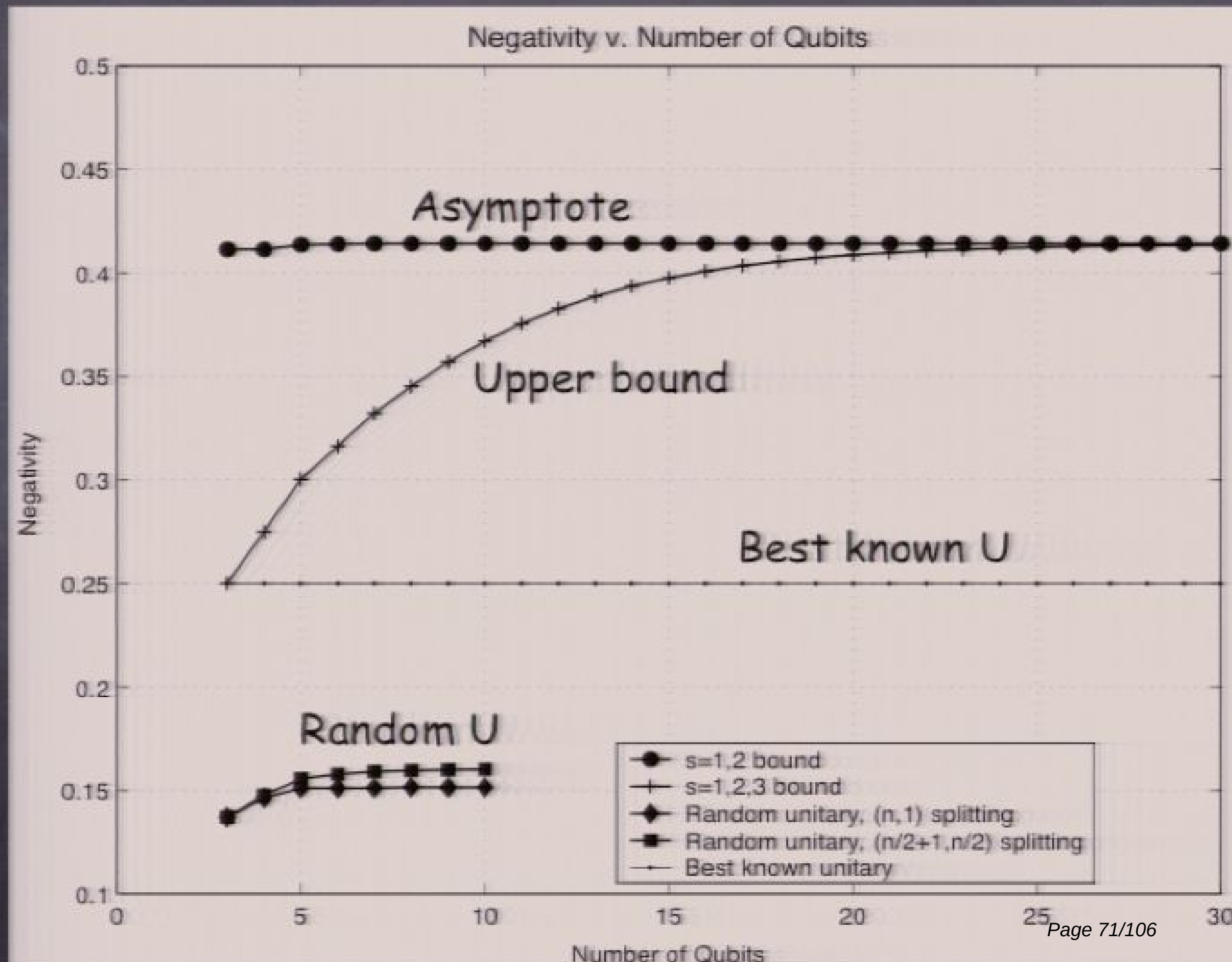

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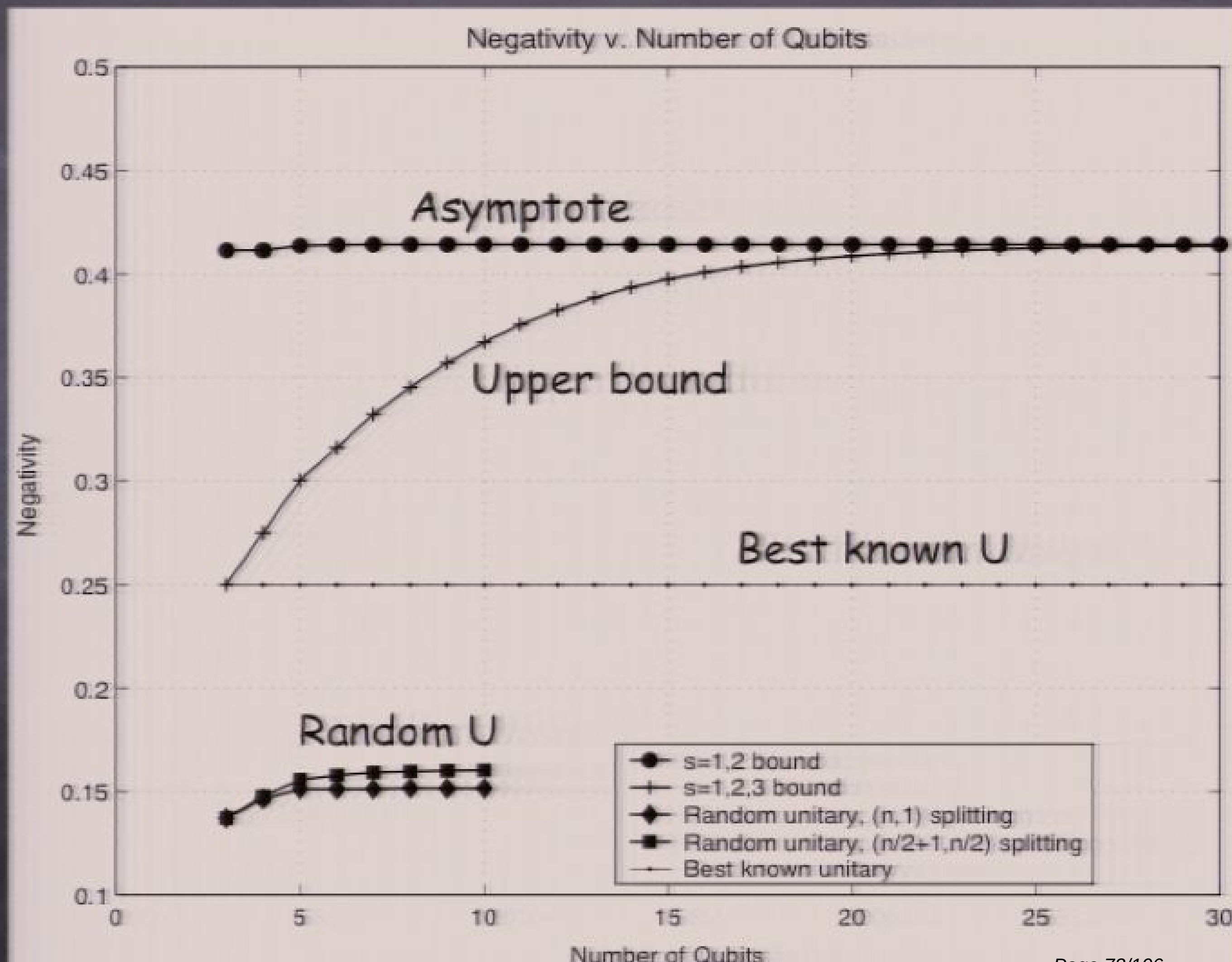

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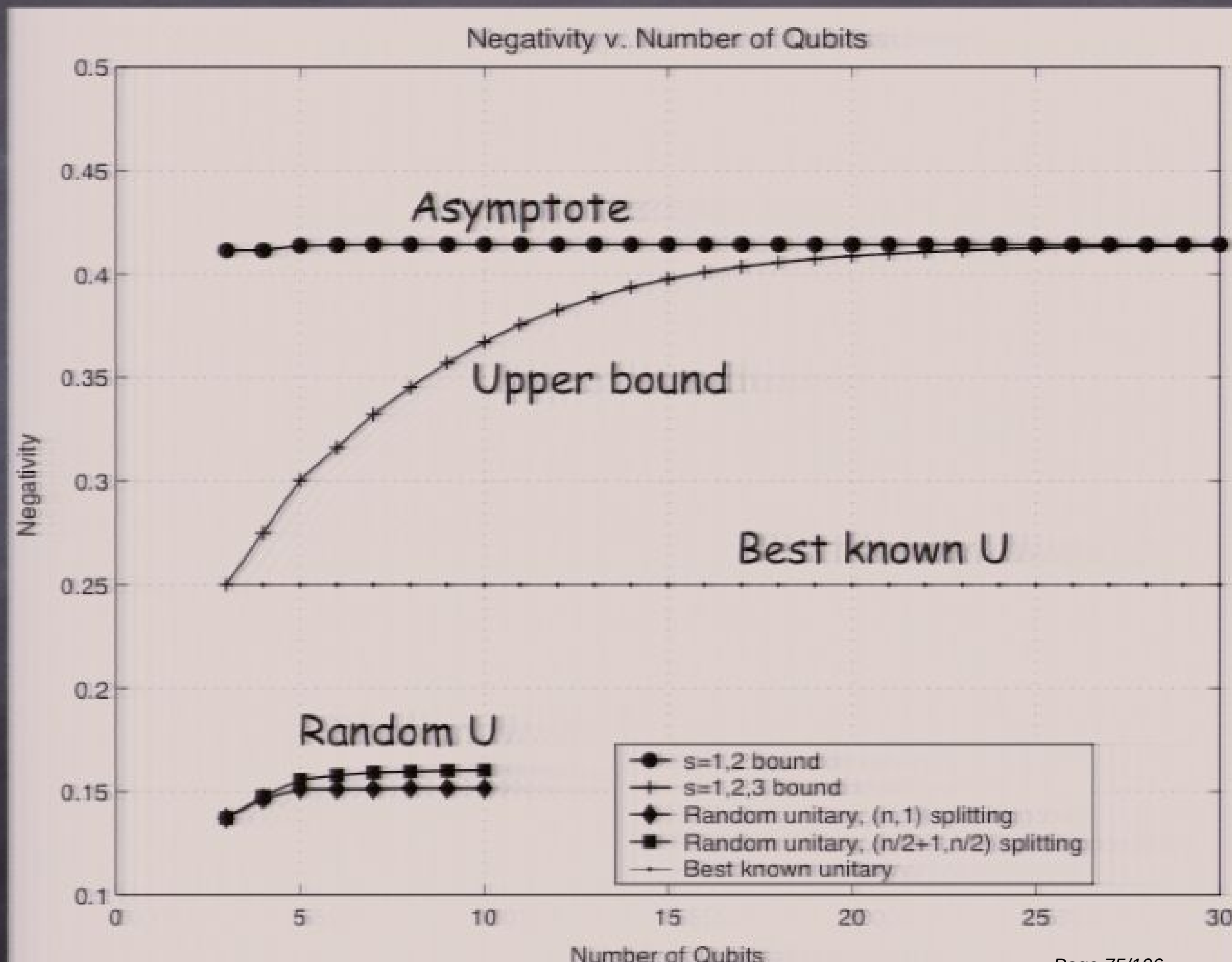

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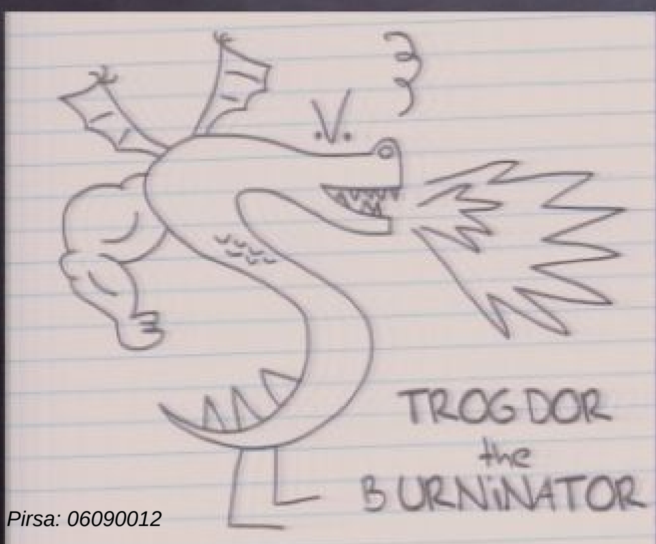
Digression: Why switch notation?

$\left(\rho^{T_B}\right)^s$ Uggh. Unwieldly.

$\dot{\rho}, \tilde{\rho}, \bar{\rho}$ etc. already taken.



$\Longleftrightarrow \check{\rho}$

$\breve{\rho}$

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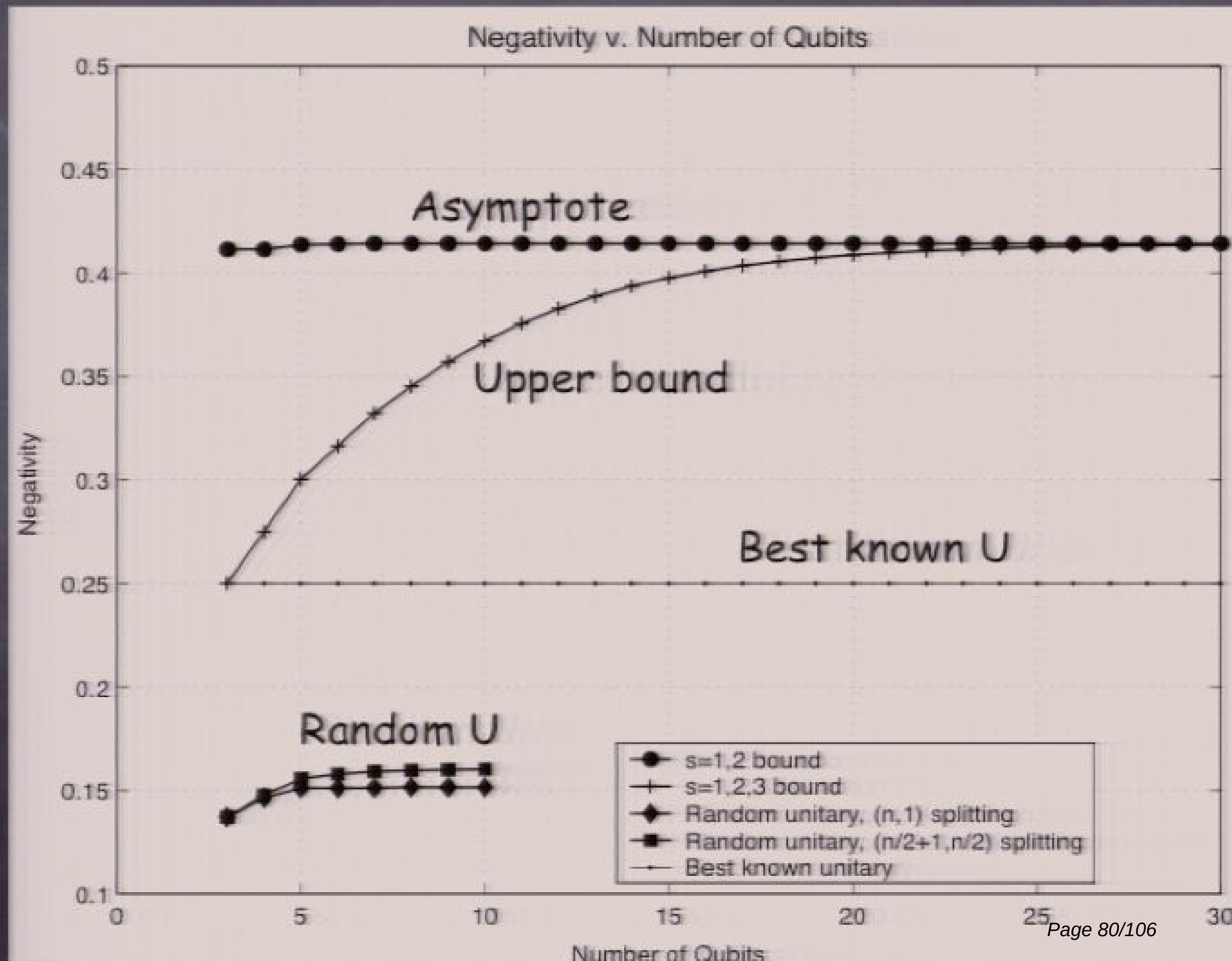

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Quantum Metrology

Goal: Take a one-parameter Hamiltonian $H_\gamma = \gamma h_0$ and estimate the coupling constant.

Usually the Hamiltonian has the form $h_0 = \sum_{j=1}^{N} h_j$

For separable input states, the shot-noise limit says the optimal scaling of the standard deviation is

$$\delta\gamma \sim \frac{1}{t\sqrt{N}(\lambda_M - \lambda_m)}$$

Use Entanglement

If we are allowed to input a “cat” state,

$$\frac{1}{\sqrt{2}}\Big(|\lambda_M,\ldots,\lambda_M\rangle + |\lambda_m,\ldots,\lambda_m\rangle\Big)$$

Quadratic Improvement!

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This was shown to be optimal for
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If we violate these assumptions, can we do better?

Can we do better?

Now consider the Hamiltonian

$$H_\gamma(t) = \gamma h_0 + \tilde{H}(t)\ , \quad h_0 = \sum_{\{j_1,\ldots,j_k\}} h^{(k)}_{j_1,\ldots,j_k}$$

The superscript k denotes k-body coupling terms.

Arbitrary coupling to ancillas and within the probe systems are allowed by the auxiliary Hamiltonian.

First derive a bound for arbitrary h_0, then plug in the special form to see the scaling with N.

Generalized Metrological Precision Bounds

Begin with an intial state ρ_0.

Time evolve under the Hamiltonian to

$$\rho_\gamma(t) = U_\gamma(t)\rho_0 U_\gamma^\dagger(t)$$

The equation of motion for the unitary is

$$i\frac{\partial U_\gamma(t)}{\partial t} = H_\gamma(t)U_\gamma(t)$$

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Quantum Cramér-Rao bound: $\delta\gamma^2 \geq \frac{1}{\mathcal{I}_\gamma(t)}$

Quantum Fisher Information $\mathcal{I}_\gamma(t) = \mathrm{tr}\big(\rho_\gamma(t)\mathfrak{L}_\gamma^2(t)\big) = \langle \mathfrak{L}_\gamma^2(t)\rangle$

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A. S. Holevo, *Probabilistic and statistical aspects of quantum theory.*

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The Fisher Information now relates to the variance of K and its operator semi-norm

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(inequalities hold for mixed states; for pure states, the first two are tight)

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Generalized Metrological Precision Bounds

Introduce a new operator F

$$F_\gamma(t) = U_\gamma^\dagger(t) K_\gamma(t) U_\gamma(t) = i U_\gamma^\dagger(t) \frac{\partial U_\gamma(t)}{\partial \gamma}$$

F satisfies the equation

$$\frac{\partial F_\gamma(t)}{\partial t} = U_\gamma^\dagger(t) h_0 U_\gamma(t)$$

Integrating and putting back in terms of K, we get K as a function of h_0

$$K_\gamma(t) = \int_0^t ds\, U_\gamma(t) U_\gamma^\dagger(s) h_0 U_\gamma(s) U_\gamma^\dagger(t)$$

Generalized Metrological Precision Bounds

Now the triangle inequality and unitary invariance yield

$$\|K_\gamma(t)\| \leq \int_0^t ds\, \|U_\gamma(t)U_\gamma^\dagger(s)h_0U_\gamma(s)U_\gamma^\dagger(t)\| = t\|h_0\|$$

Recall that for the case of no auxiliary Hamiltonian,

$$K_\gamma(t) = th_0$$

so this bound is achievable.

To summarize, what has been shown is that for arbitrary dynamics with arbitrary ancillas, the precision is limited by $\delta\gamma \geq 1/t\|h_0\|$.

Is there an improvement?

Fully general bound: $\delta\gamma \geq 1/t\|h_0\|$

Now we can plug in specific forms for h_0 and see what optimal precisions we can obtain.

Assume the Hamiltonian is k-body, symmetric and separable. $h^{(k)}_{j_1,\ldots,j_k} = h_{j_1}\cdots h_{j_k}$

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New Metrological Limit! $\delta\gamma \geq \dfrac{1}{t\binom{N}{k}(\lambda_M^k - \lambda_m^k)}$

But is it physical?

For the case of 2-body couplings, we get another quadratic speed-up to obtain a new metrological limit:

$$\delta\gamma \geq \frac{2}{tN(N-1)(\lambda_M^2 - \lambda_m^2)} \sim \frac{1}{tN^2(\lambda_M^2 - \lambda_m^2)}$$

2-body couplings are certainly physical, but I've assumed a spatially non-local Hamiltonian!

\begin{speculation}

Cornish et. al., PRL 85 1795 (2000)

\end{speculation}

Summary

- Vanishingly small amounts of entanglement can still lead to an exponential speed-up.
- There appears to be a discontinuity between what is possible with **zero** entanglement, and what is possible with **non-zero** entanglement.

But...

- Even highly entangled ancillas won't buy you <u>anything</u> in quantum metrology, though a cat state is still necessary to beat the standard quantum limit.
- Non-(nearest neighbor) Hamiltonians lead to quadratic (or better) improvements in metrological precision. It remains to be seen how physical this is, but current theory is not pessimistic and some (namely me and JM) might say optimistic.

References

- Datta, Flammia, Caves; "Entanglement and the Power of One Qubit", Phys Rev A, 72, 042316 (2005).
- Boixo, Flammia, Geremia, Caves; "Generalized Limits for Single-Parameter Quantum Estimation", to appear soon (1-2 weeks) on quant-ph.

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