Title: New separations in quantum communication complexity

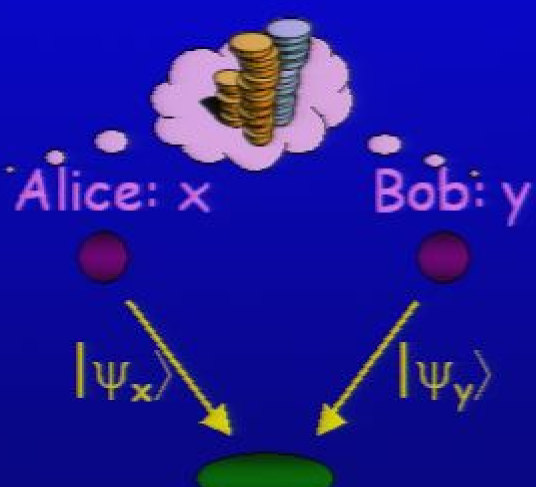
Date: Aug 30, 2006 04:00 PM

URL: http://pirsa.org/06080033

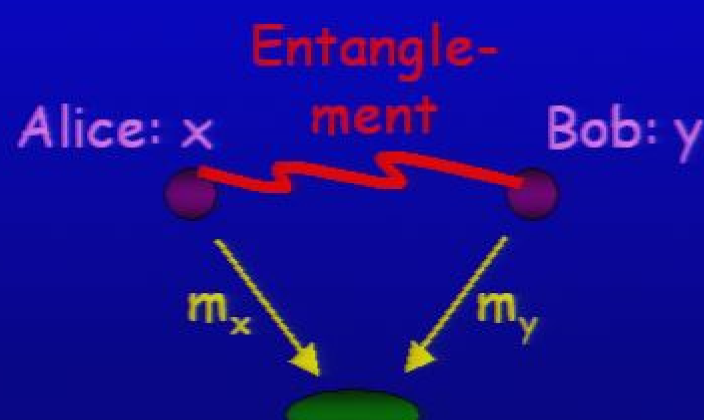
Abstract: In this talk I will present several new results from joint work with Dmitry Gavinsky, Oded Regev and Ronald de Wolf, relating to the model of one-way communication and the simultaneous model of communication. I will describe several separations between various resources (entanglement versus event coin, quantum communication versus classical communication), showing in particular that quantum communication cannot simulate a public coin and that entanglement can be much more powerful than a public coin, even if communication is quantum. I will also present a characterization of the quantum fingerprinting technique.

What is the power of entanglement?

- Determinism vs. Randomness (D vs. R_ε)
- Public Coin vs. Private Coin (R_ε^{pub} vs. R_ε)
- Classical vs. Quantum Communication (R vs. Q)
- **<u>Public Coin vs. Shared Entanglement</u>**
  (R_ε^{pub} vs. R_ε^{ent}, Q_ε^{pub} vs. R_ε^{ent},...)



Referee (quantum): R(x,y)



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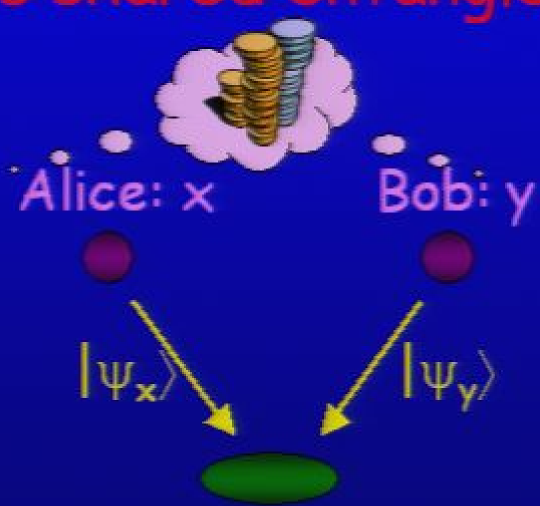
Is shared entanglement stronger than a public coin?



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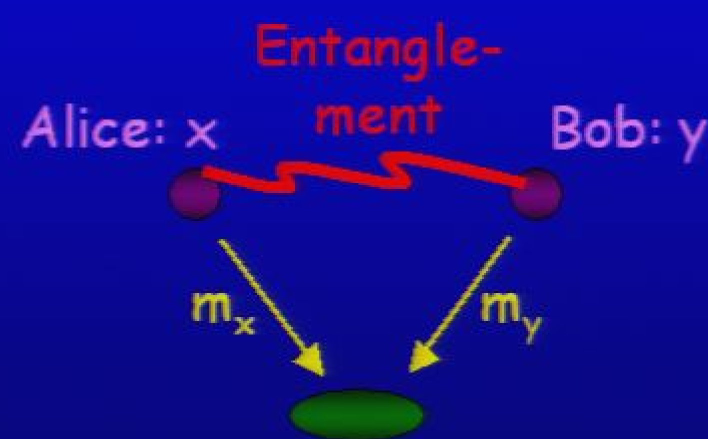

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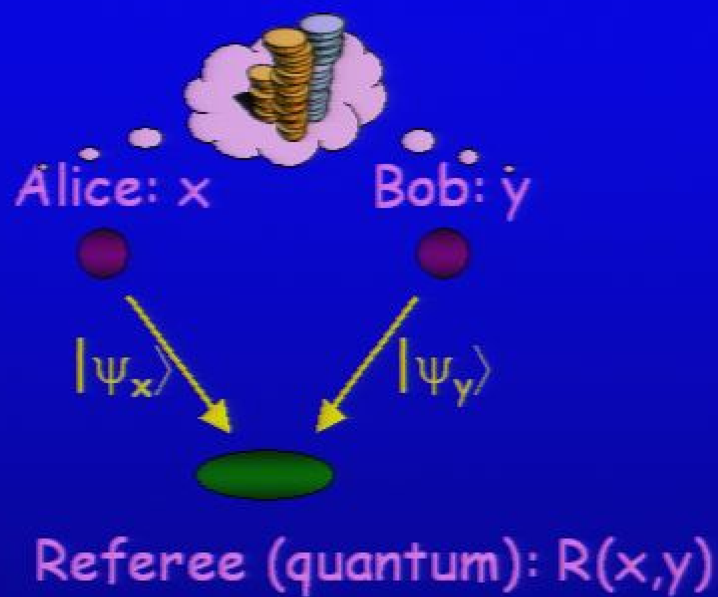
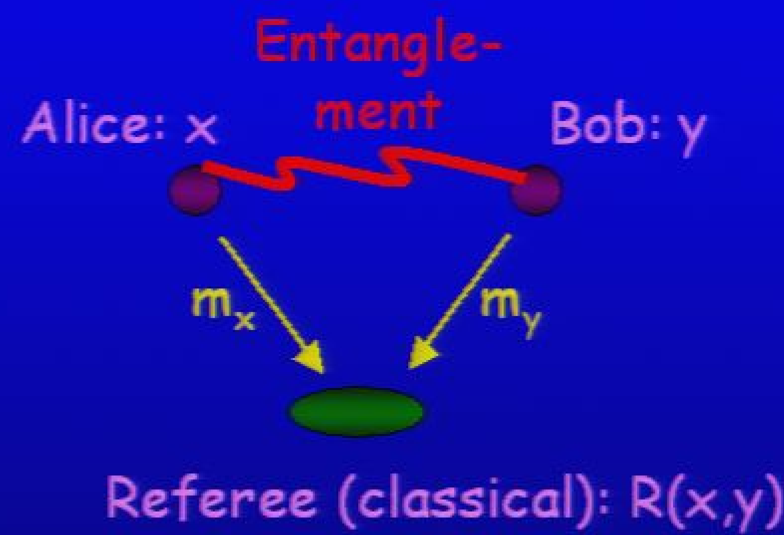
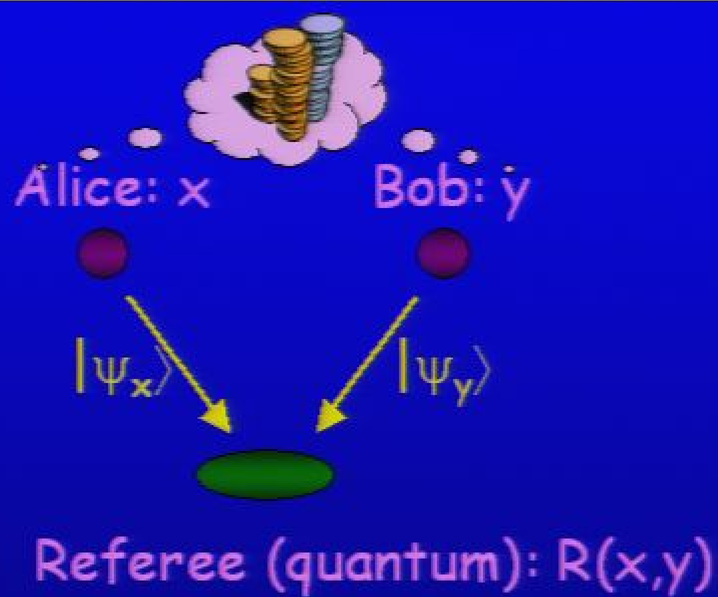


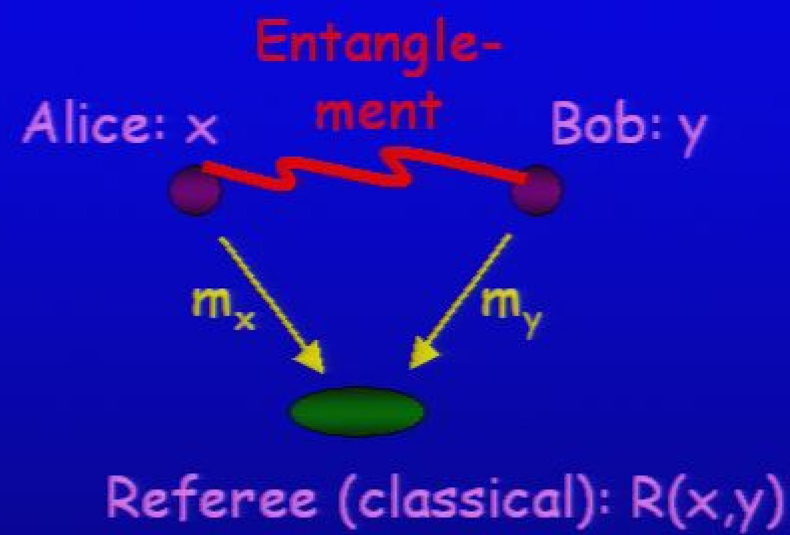


Q_ε^{pub} vs. R_ε^{ent} ???

Result 2 [GKRW'06]: R_ε^{ent} (R2) << Q_ε^{pub} (R2)

There is a relation R2(x,y) s. t. $R_\varepsilon^{ent}(R2) = O(\log n)$ and $Q_\varepsilon^{pub}(R2) = \Omega(\sqrt[3]{n}/\log n)$.





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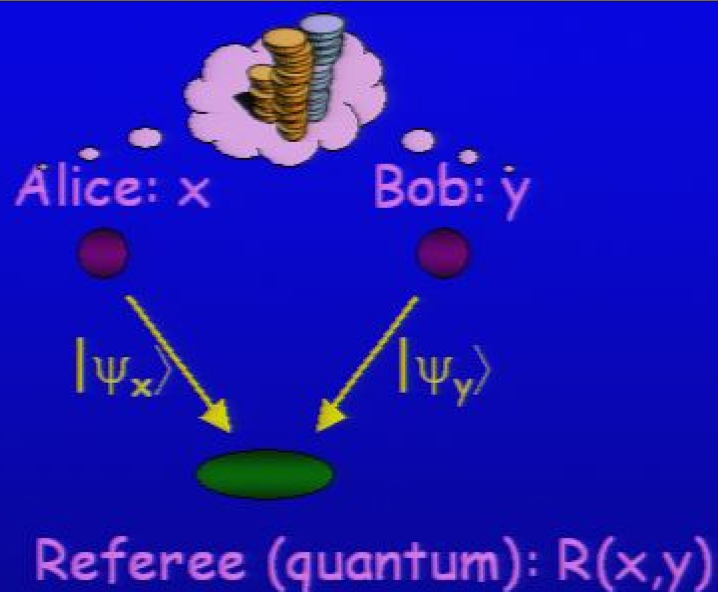
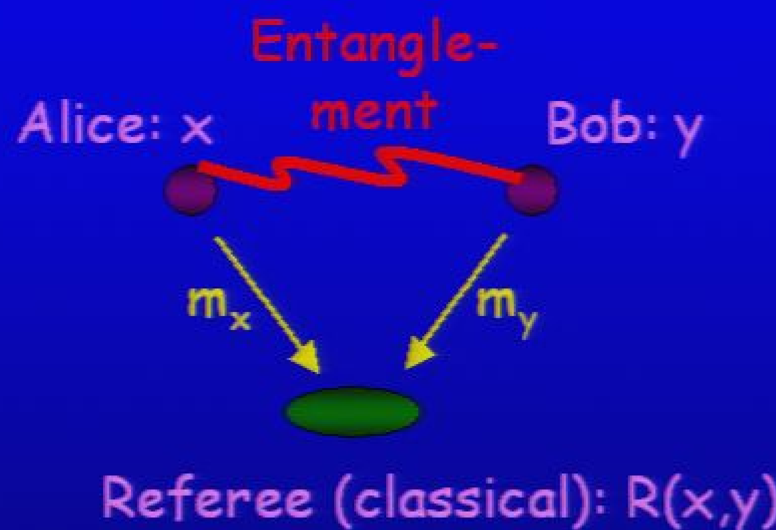




$\Omega(\sqrt[3]{n}/\log n)$ quantum communication

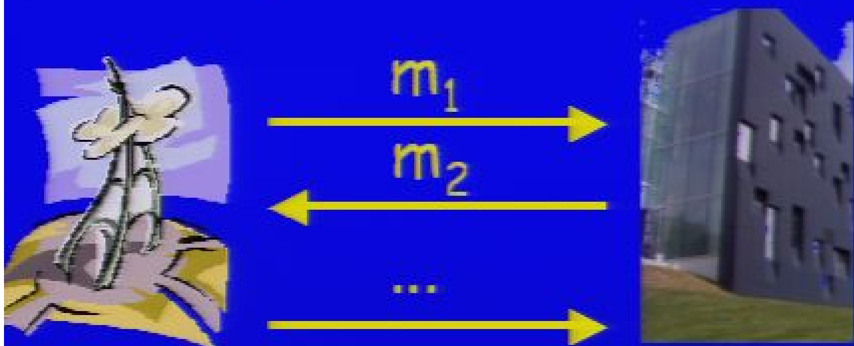
$O(\log n)$ classical communication* (uses log n shared EPR pairs)

Entanglement is much stronger than shared randomness!

* classical protocol due to Harry Buhrman

SMP and other models

Two-way communication model



Bob: y

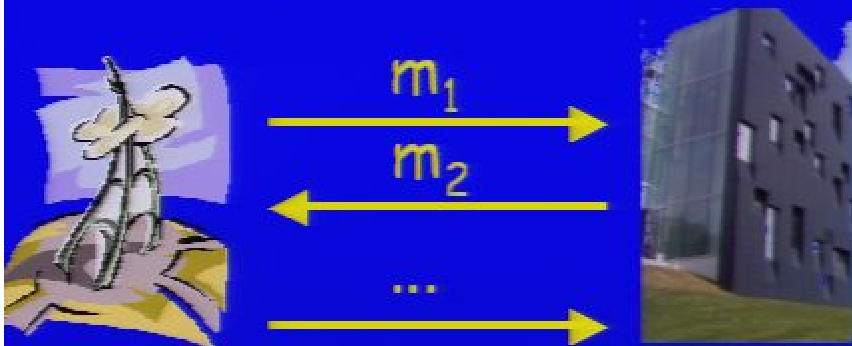
Alice: x

Cost: total communication
needed to compute R(x,y)
in the worst case

Complexities: D^2, R^2_ε, Q^2_ε, $Q^{2,\,ent}_\varepsilon$ ...

SMP and other models

Two-way communication model



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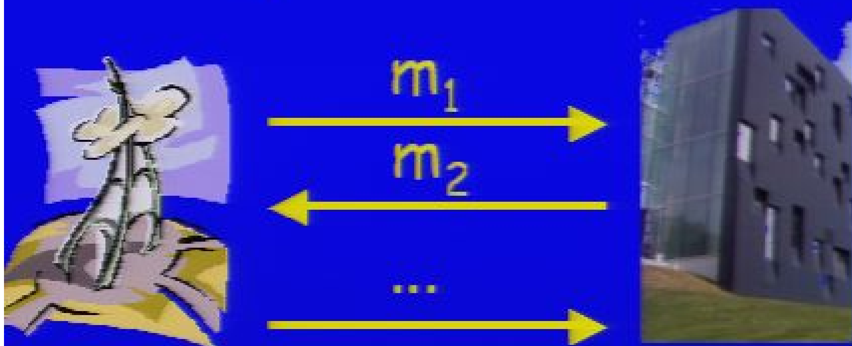

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How well can SMP protocols simulate two-way communication?

SMP and other models

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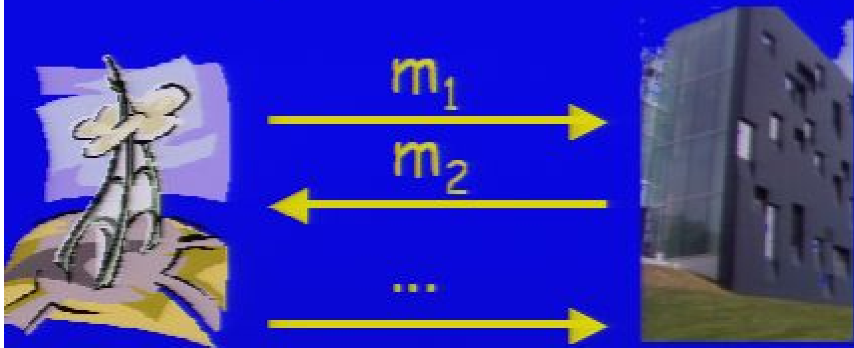

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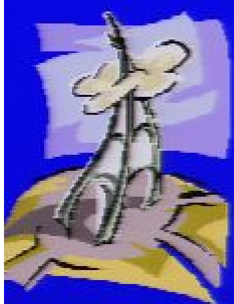
Result 3 [GKW-1'06]: $Q_\varepsilon = O(2^{4Q^{2,\,ent}_\varepsilon} \log n)$

Every multi-round protocol (even with unlimited entanglement) can be simulated by a generalized <u>repeated fingerprint</u> SMP protocol (with exponential overhead).

SMP and other models

Two-way communication model



m_1 →

m_2 ←

... →

Alice: x

Bob: y

Cost: total communication needed to compute $R(x,y)$ in the worst case

Complexities: D^2, R^2_{ε}, Q^2_{ε}, $Q^{2\,ent}_{\varepsilon}$ ...

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Result 3 [GKW-1'06]: $Q_{\varepsilon} = O(2^{4Q^{2\,ent}_{\varepsilon}} \log n)$

Every multi-round protocol (even with unlimited entanglement) can be simulated by a generalized repeated fingerprint SMP protocol (with exponential overhead).

Also Corollary: $R^{pub} = O(2^{4Q^{2\,ent}_{\varepsilon}} \log n)$

The power of fingerprints

All known nontrivial efficient quantum SMP protocols based on (repeated) quantum fingerprints.

Alice: $x \to |a_x\rangle$ Bob: $y \to |b_y\rangle$

SWAP test: $|\langle a_x|b_y\rangle|^2 \leq \delta_0$ if $f(x,y)=0$ $(\delta_0 < \delta_1)$

$|\langle a_x|b_y\rangle|^2 \geq \delta_1$ if $f(x,y)=1$

Repeat $r = \Theta(1/(\delta_1 - \delta_0)^2)$ times to succeed with constant prob.

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Communication Matrix:

$$\begin{array}{c} \leftarrow x \rightarrow \\ M(EQ) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \cdots & & & \cdots & 0 \\ 0 & & & 0 & 1 \end{pmatrix} \updownarrow y \end{array}$$

$$\begin{array}{cc} & |a_{x_1}\rangle, |a_{x_2}\rangle, \ldots, |a_{x_{2^n}}\rangle \\ \begin{array}{c} |b_{y_1}\rangle \\ |b_{y_2}\rangle \\ \cdots \\ |b_{y_{2^n}}\rangle \end{array} & \begin{pmatrix} & \cdots & \\ \cdots & |\langle a_x|b_y\rangle|^2 & \cdots \\ & \cdots & \end{pmatrix} \end{array}$$

$M_{xy}(f)=f(x,y)$

The power of fingerprints

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Learning Theory:

$$L(EQ) = \begin{pmatrix} -1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & & \\ 1 & 1 & -1 & & \\ \cdots & & & \cdots & 1 \\ 1 & & & 1 & -1 \end{pmatrix}$$

$M\ (f)=f(x,y)$

$L\ (f)=(-1)^{f(x,y)}$

The power of fingerprints

Communication Matrix:

$$
\begin{array}{cc}
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$$

margin

$L_{xy}(f)=(-1)^{f(x,y)}$

$\langle a_x|b_y\rangle \geq \gamma$ if $f(x,y)=0$

$\langle a_x|b_y\rangle \leq -\gamma$ if $f(x,y)=1$

The power of fingerprints

Can relate fingerprints to margins [GKW-1'06]

Cost of repeated fingerprints: $\Omega(\log n / \gamma^2)$

Theorem (Foster) : Let the $2^n \times 2^n$ matrix $L_{xy} = (-1)^{f(x,y)}$.

Every realization of f has margin $\gamma \le \|L\|_{op} / 2^n$.

Cost of repeated fingerprints for IP: $\Omega(2^n)$.

Communication Matrix:

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margin

$\langle a_x | b_y \rangle \ge \gamma$ if $f(x,y)=0$

$L_{xy}(f)=(-1)^{f(x,y)}$ $\langle a_x | b_y \rangle \le -\gamma$ if $f(x,y)=1$

The power of fingerprints

Connection:

Quantum Communication ⇄ Learning Theory

New lower bounds on $Q^{2,ent}$ from margin bounds: $Q^{2,ent}(f) = \Omega(\log(1/\gamma(f)))$ (independently obtained by Linial and Shraibman'06, they also show $1/\gamma(f) \approx \mathrm{Disc}(f)$)

The power of fingerprints

Connection:

Quantum Communication ⇄ Learning Theory

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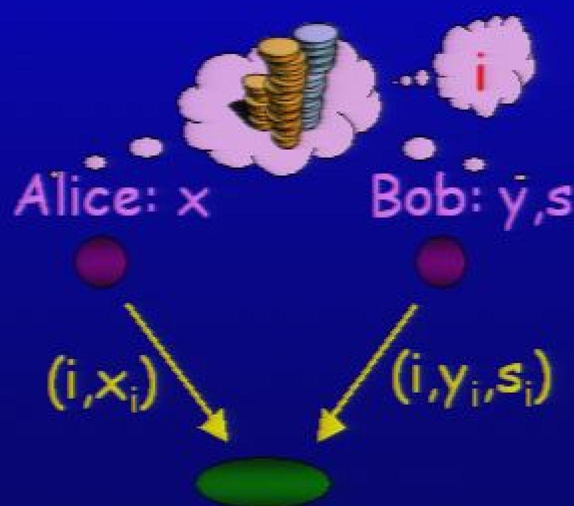
- New upper bounds for SMP and general protocols from margin bounds (embeddings)

- New bounds for margins (embeddings) from SMP upper bounds, possibly coming from simulation of quantum two-way protocols with entanglement (quantum-classical results)

R1: definition and upper bound

Result 1: $R_\varepsilon^{pub}(R1) << Q_\varepsilon(R1)$
There is a relation $R1(x,y)$ s. t. $R_\varepsilon^{pub}(R1) = O(\log n)$ and $Q_\varepsilon(R1) = \Omega(\sqrt[3]{n})$.

Alice: $x \in \{0,1\}^n$
Bob: $y, s \in \{0,1\}^n$ s.t. $|s| = \frac{1}{2}n$ "mask"
Referee: (i, x_i, y_i) s.t. $s_i = 1$



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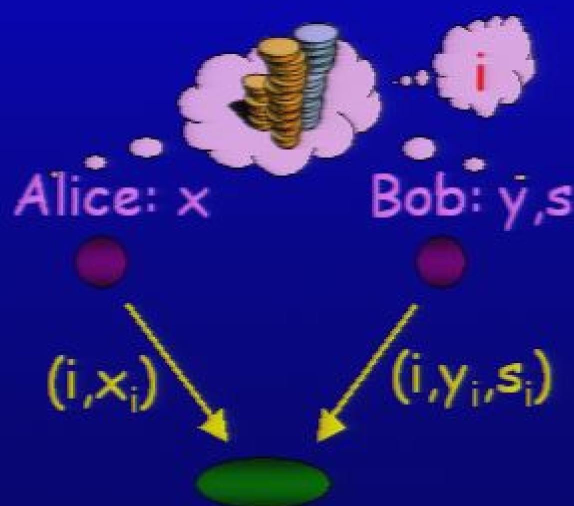
$s = 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1$

$x = 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1$

$y = 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0$



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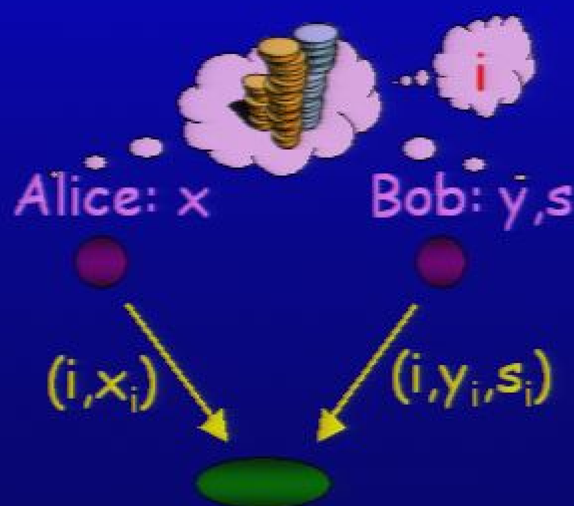


if $s_i=1 \rightarrow$ output (i,x_i,y_i)
(happens with prob. $\frac{1}{2}$)
repeat a few times to boost success prob.

R2: definition and upper bound

Result 2: R_ε^{ent} (R2) << Q_ε^{pub} (R2)

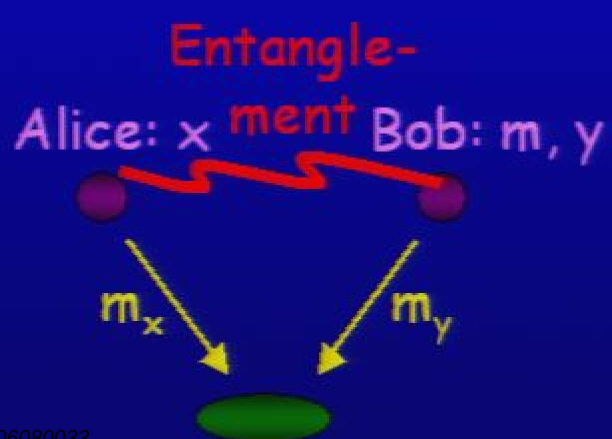
There is a relation R2(x,y) s. t. $R_\varepsilon^{ent}(R2) = O(\log n)$ and $Q_\varepsilon^{pub}(R2) = \Omega(\sqrt[3]{n}/\log n)$.

Alice: $x \in \{0,1\}^n$

Bob: m - matching of n bits

$y \in \{0,1\}^{n/2}$ (a bit y_{ij} for all $(i,j) \in m$)

Referee: $(i, j, x_i \oplus x_j, y_{ij})$ s.t. $(i,j) \in m$



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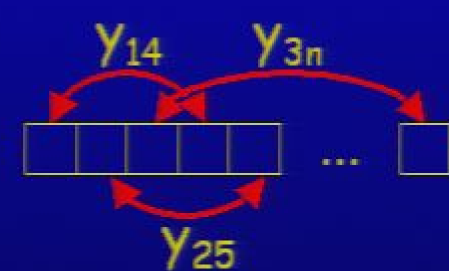

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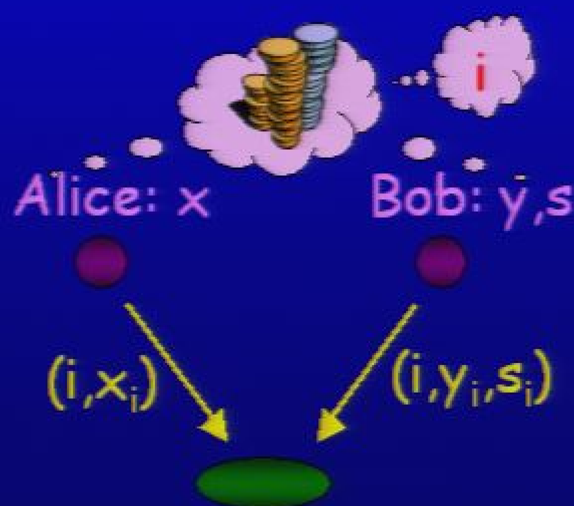

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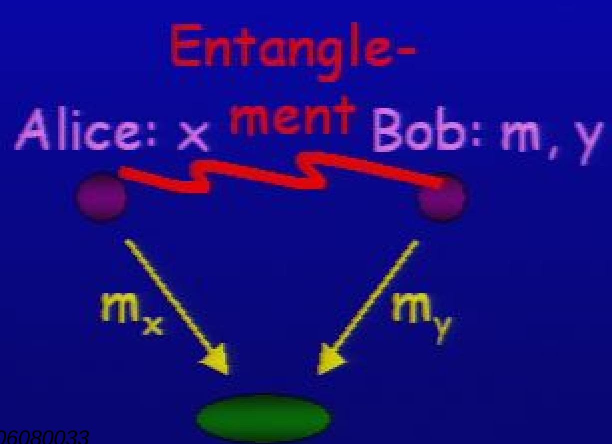

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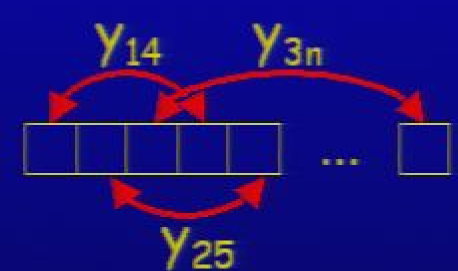
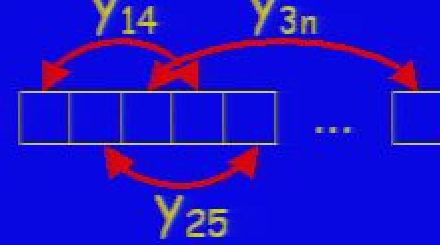


R2: definition and upper bound

$\boxed{O(\log n) \text{ classical bits}}$

Alice: $x = x_1 x_2 \dots x_n$ **Bob:**



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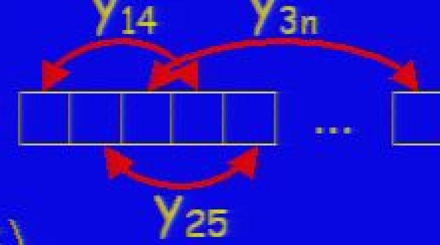

- Share log n EPR pairs

$$|\Psi\rangle_{AB} = \sum_{i=1}^{n} |i\rangle_A |i\rangle_B$$

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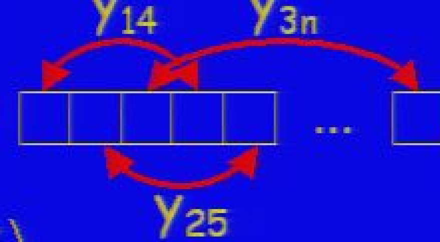

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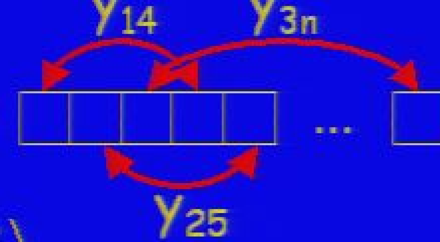

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- Bob: measure with

  $\Pi_{ij} = |i\rangle\langle i| + |j\rangle\langle j|$ for $(i,j) \in m$

  send i, j, y_{ij} (2 log n +1 bits)

$$|\Psi\rangle_{AB} = |i\rangle_A |i\rangle_B + (-1)^{x_i \oplus x_j} |j\rangle_A |j\rangle_B$$

- **Referee:**

output $(i, j, x_i \oplus x_j, y_{ij})$

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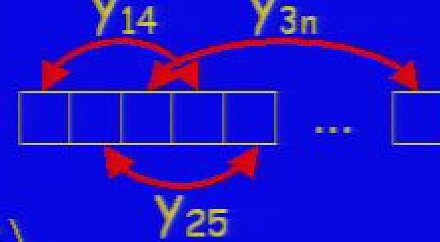

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  send i, j, y_{ij} (2 log n +1 bits)

$$|\Psi\rangle_{AB} = |i\rangle_A |i\rangle_B + (-1)^{x_i \oplus x_j} |j\rangle_A |j\rangle_B$$

- Alice and Bob: apply $H^{\otimes \log n}$

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- **Referee:** output $(i, j, x_i \oplus x_j, y_{ij})$ ✓ ✓

R2: definition and upper bound

$\boxed{O(\log n) \text{ classical bits}}$ **Alice:** $x = x_1 x_2 \ldots x_n$ **Bob:** y_{14} y_{3n} y_{25}

- Share log n EPR pairs $\qquad |\Psi\rangle_{AB} = \sum_{i=1}^{n} |i\rangle_A |i\rangle_B$
- Alice: $|i\rangle_A \rightarrow (-1)^{x_i} |i\rangle_A \qquad |\Psi\rangle_{AB} = \sum_{i=1}^{n} (-1)^{x_i} |i\rangle_A |i\rangle_B$
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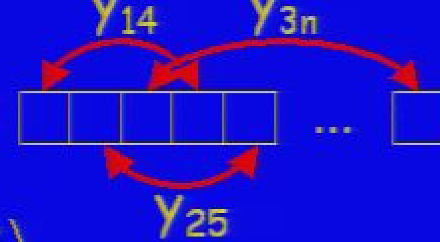

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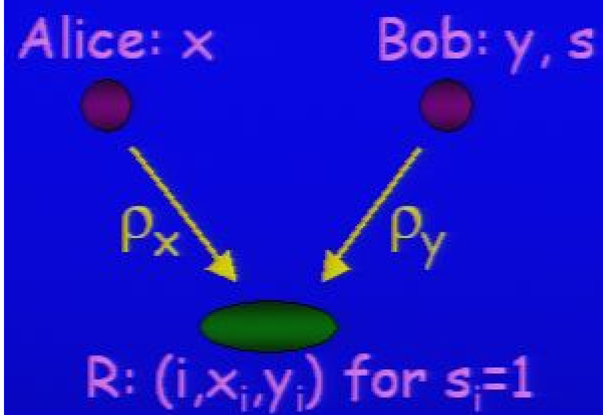
$$\boxed{Q_\varepsilon(R1) = \Omega(\sqrt[3]{n})}$$



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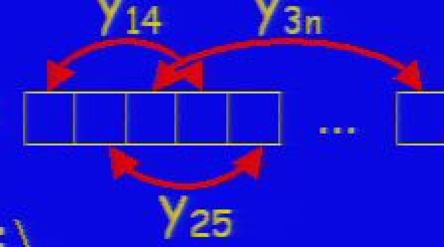

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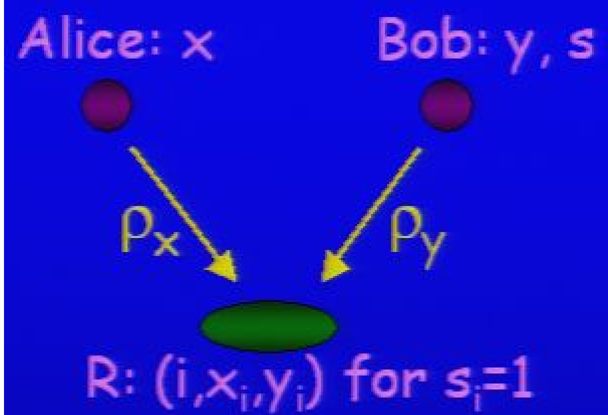

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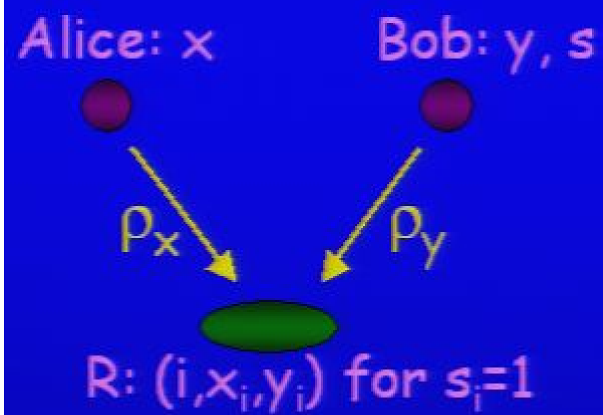

Information Theory:

extract subproblem for fixed index j

$$\rho_0^j = \frac{1}{2^{n-1}} \sum_{x:x_j=0} \rho_x \qquad \rho_1^j = \frac{1}{2^{n-1}} \sum_{x:x_j=1} \rho_x$$

Given $\rho_{x_j}^j \otimes \rho_{y_j}^j$ output

(x_i, y_i) if $i=j$ (correctly with prob. $1-\varepsilon$)

"don't know" if $i \neq j$

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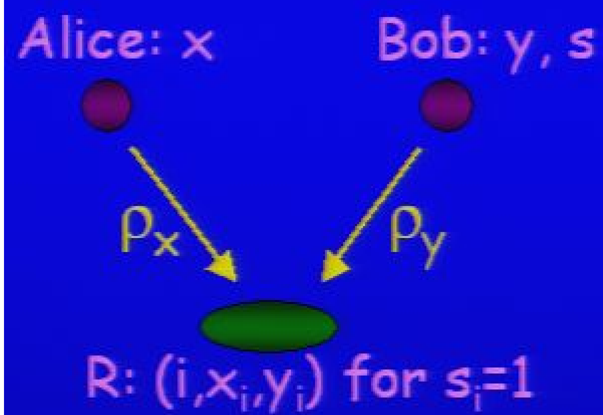

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Relate to known bounds on Random Access Codes [Nayak'99]

given ρ_x output x_j and given ρ_y output y_j

Bounded Error State Identification

State Identification

Given ρ_0 or ρ_1, identify which one.

Optimal success probability given by $\frac{1}{2} + \frac{1}{2}\,|\rho_0 - \rho_1|_{tr}$

Trace distance is too small $\rightarrow$ error prob. too close to $\frac{1}{2}$.

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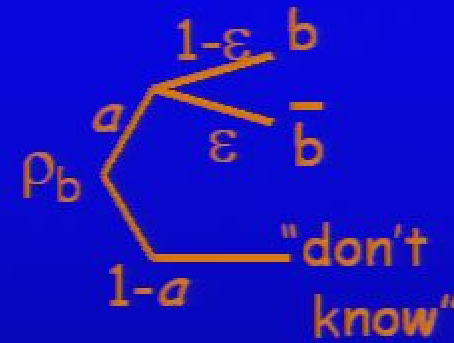

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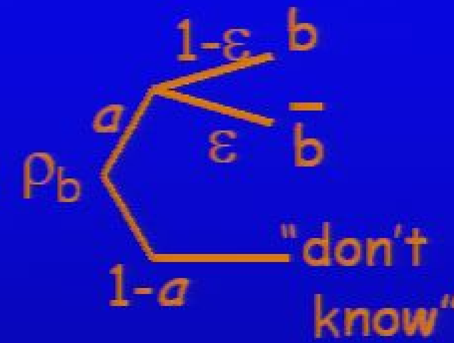

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a_ε is not determined by $\rho_0 - \rho_1$ but can be written as a semidefinite program (SDP).

State Identification

Example:

$|\phi_0\rangle = \sqrt{a}|0\rangle + \sqrt{1-a}\,|2\rangle$
$|\phi_1\rangle = \sqrt{a}\,|1\rangle + \sqrt{1-a}\,|2\rangle$

$|\phi_0 - \phi_1|_{tr} \approx \sqrt{a}$ (small) error probability $\frac{1}{2}(1-\sqrt{a})$

Measure in computational basis:

observe $|0\rangle \rightarrow$ output "0"
observe $|1\rangle \rightarrow$ output "1"
observe $|2\rangle \rightarrow$ output "don't know"

Gain: error probability reduced to 0

Cost: we get an answer only with probability a

Tensor Lemma

Suppose we are given two independent problems:

ρ_0, ρ_1 with a_ε = max. prob. of guess

σ_0, σ_1 with b_ε = max. prob. of guess

Tensored problem: given $\rho_0 \otimes \sigma_0$, $\rho_0 \otimes \sigma_1$, $\rho_1 \otimes \sigma_0$ or $\rho_1 \otimes \sigma_1$ identify which one in the bounded error setting. Let $p_{\varepsilon'}$ be the maximum probability of making a guess.

Expect: $p_{\varepsilon'} = O(a_\varepsilon \cdot b_\varepsilon)$ ("Direct Product Theorem")

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Subtle:

- True classically, but optimal 2-register measurement is NOT a tensor measurement
- Not true if $\varepsilon' > \frac{1}{2}\varepsilon$
- Not true if we want to identify only $\rho_0 \otimes \sigma_0$, $\rho_1 \otimes \sigma_1$ vs. $\rho_0 \otimes \sigma_1$, $\rho_1 \otimes \sigma_0$ (can be $p_{\varepsilon'} = O(\sqrt{a_\varepsilon \cdot b_\varepsilon})$)

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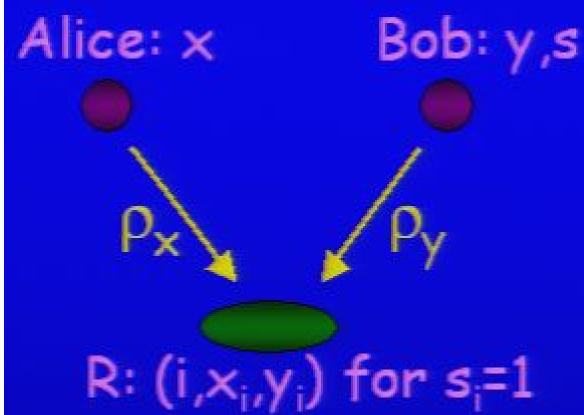
We show (using semidefinite programming duality):

1. If ρ_0, ρ_1 pure and σ_0, σ_1 mixed, then $p_{\varepsilon/2} = O(a_\varepsilon \cdot b_\varepsilon)$.
2. In general $p_{\varepsilon/2} = O(|\rho_0 - \rho_1|_{tr} \cdot b_\varepsilon)$ (purify and use 1.)

R1, R2: lower bound

$$\boxed{Q_\varepsilon(R1) = \Omega(\sqrt[3]{n})}$$



$\frac{1}{2}$ n "state identification problems"
Referee must solve at least one

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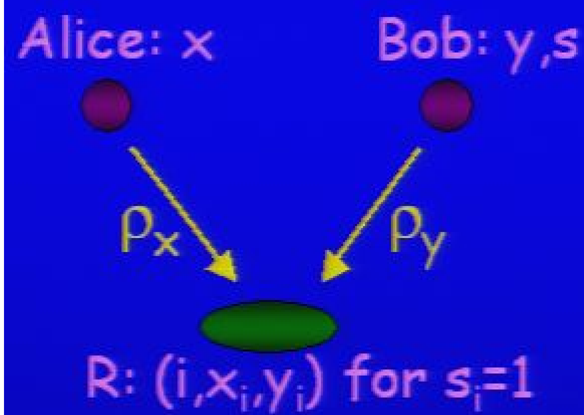

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A separation for a Boolean <u>function</u>

So far (nearly) all exponential separations for communication are for a relation (multi-valued):

- *One*-round: classical vs. quantum comm. [BJK'04]
- SMP: quantum comm. vs. classical w. public coin
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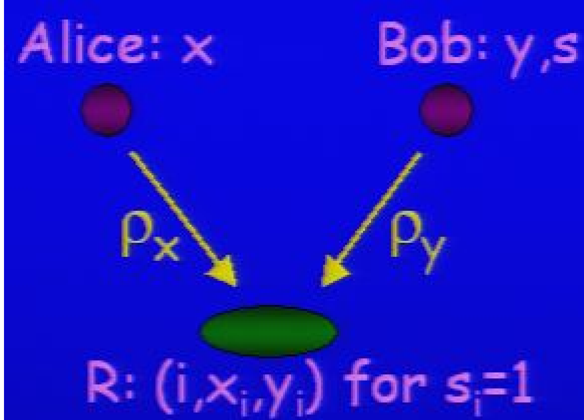

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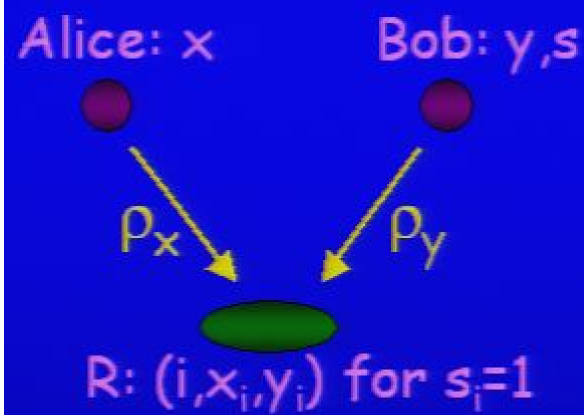

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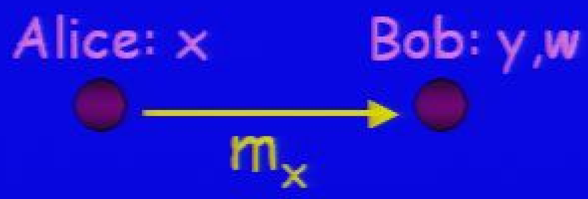
Result 4 [GKW'06]: One-round: $Q^1_\varepsilon(f) << R^{1,pub}_\varepsilon(f)$
There is a partial function $f(x,y)$ s. t. $Q^1_\varepsilon(f) = O(\log n^{3/2})$ and $R^{1,pub}_\varepsilon(f) = \Omega(\sqrt{n}\log n^{\frac{1}{4}})$.

Was independently proved for a slightly modified problem by Kerenidis and Raz in quant-ph/0607173.

The function

Variant of the hidden matching problem:



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Variant of the hidden matching problem:

Alice: x $\xrightarrow{m_x}$ Bob: y, w

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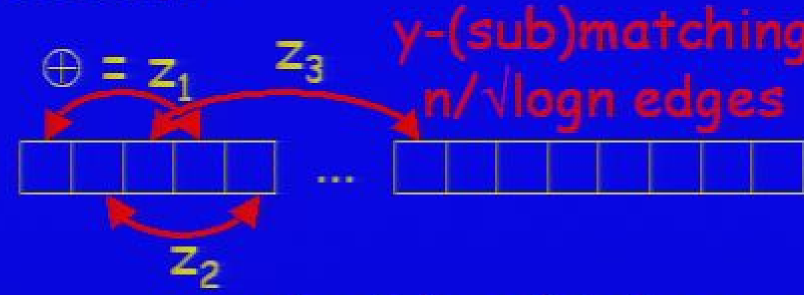

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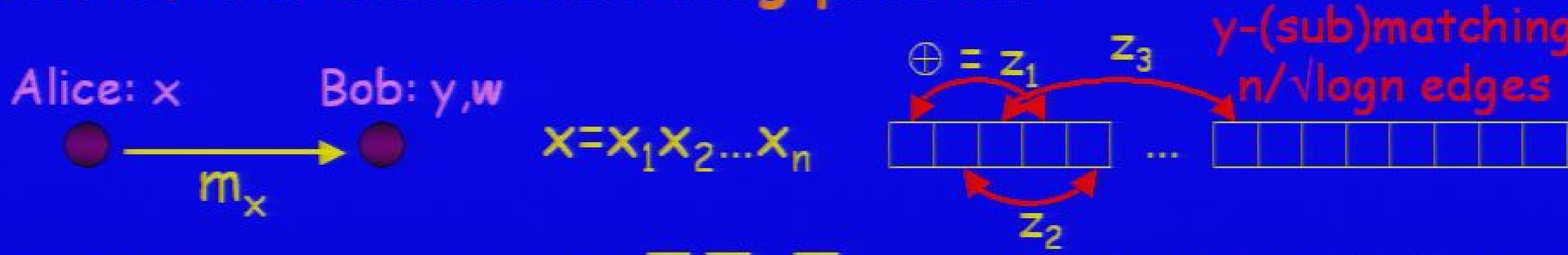

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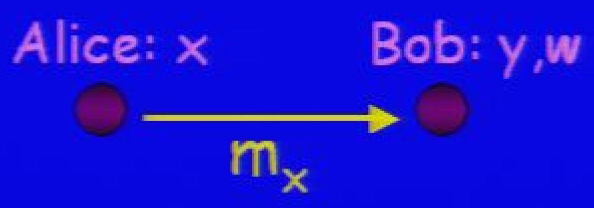

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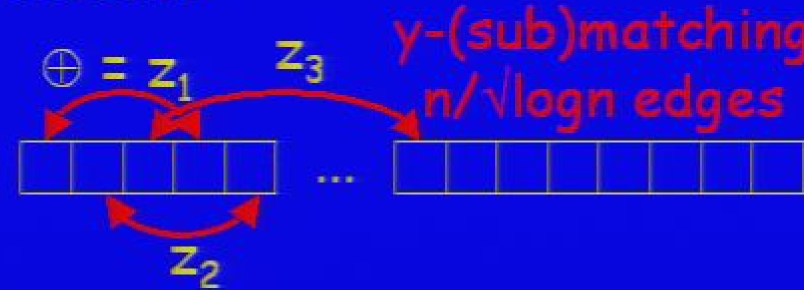

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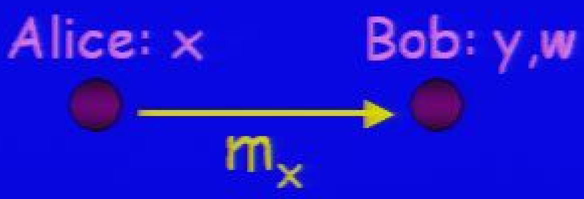

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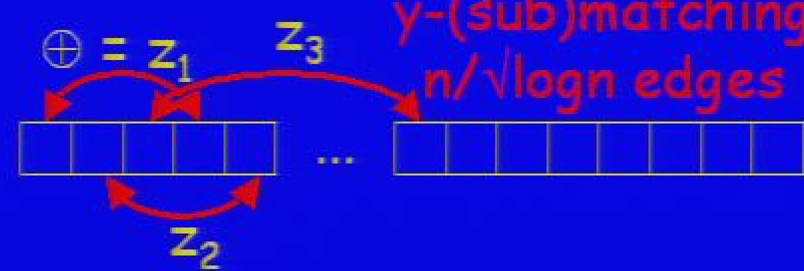

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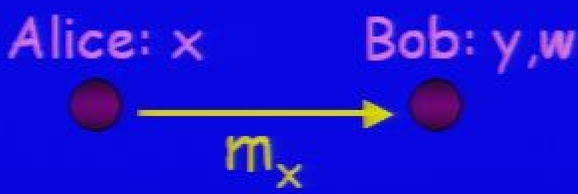

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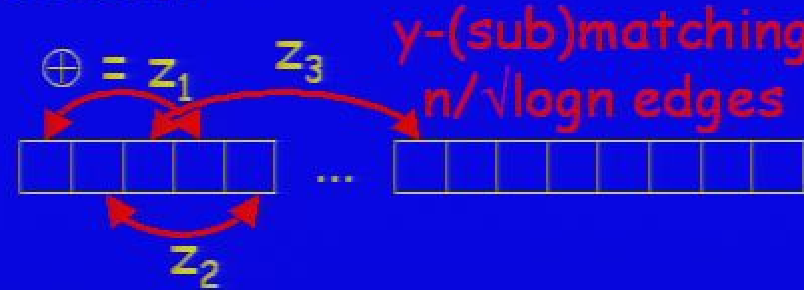

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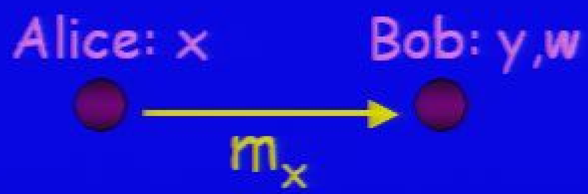

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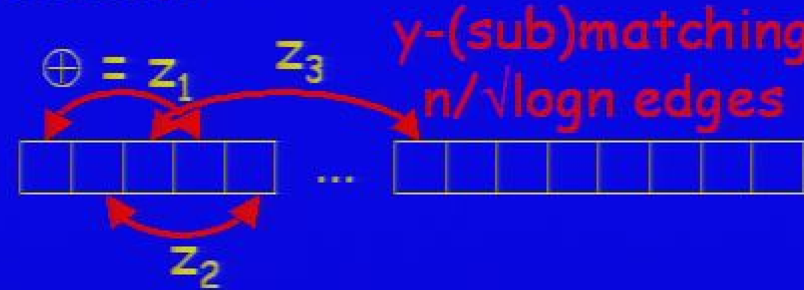

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Let $z_\ell = x_{i\ell} \oplus x_{j\ell}$ string of XORs of the edges and $b \in \{0,1\}$.
Then $w = z \oplus b^{n/\sqrt{\log n}}$.

Output:
$f(x,y,w) = b$

Quantum protocol: Alice sends $|m_x\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{x_i} |i\rangle$

Bob: measures in basis spanned by the edges: $\Pi_\ell = |i_\ell\rangle\langle i_\ell| + |j_\ell\rangle\langle j_\ell|$

if outcome ℓ: $\frac{1}{\sqrt{2}}\left(|i_\ell\rangle + (-1)^{x_{i_\ell} \oplus x_{j_\ell}} |j_\ell\rangle\right)$

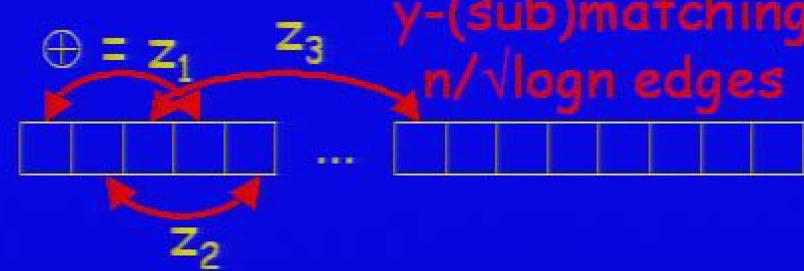
measure in $|\pm\rangle$ basis to determine $z_\ell = x_{i\ell} \oplus x_{j\ell}$, compare with w_ℓ

The function

Variant of the hidden matching problem:

Alice: x $\xrightarrow{m_x}$ Bob: y,w

$x = x_1 x_2 \ldots x_n$



$w = z_1 z_2 \ldots z_{n/\sqrt{\log n}}$ or $w = \overline{z_1}\,\overline{z_2} \ldots \overline{z}_{n/\sqrt{\log n}}$ decide which one

Input:	Promise:	Output:
x – n bit string y – $n/\sqrt{\log n}$ edges (i_ℓ, j_ℓ) w – $n/\sqrt{\log n}$ bit string	Let $z_\ell = x_{i\ell} \oplus x_{j\ell}$ string of XORs of the edges and $b \in \{0,1\}$. Then $w = z \oplus b^{n/\sqrt{\log n}}$.	$f(x,y,w) = b$

zero-error

Quantum protocol: Alice sends $|m_x\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{x_i} |i\rangle$ logn qubits

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measure in $|\pm\rangle$ basis to determine $z_\ell = x_{i\ell} \oplus x_{j\ell}$, compare with w_ℓ

Classical lower bound for f

$Q^1_\varepsilon(f) = O(\log n^{3/2})$ $R^{1,pub}_\varepsilon(f)=\Omega(\sqrt{n}\ \log n^{\frac{1}{4}})$ (tight)

Lower bound – ideas:

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Lower bound - ideas:

1. By Yao's principle assume deterministic protocol and uniform distribution on x,y and b (this fixes w)

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1. By Yao's principle assume deterministic protocol and uniform distribution on x,y and b (this fixes w)
2. Assume c bits of communication, error 1/10. The 2^c messages partition the x's into 2^c sets (pre-images). At least half the x's appear in sets of size at least 2^{n-c-1}. Hence there is a message m with > 2^{n-c-1} x's in its pre-image and error at most 1/5. Fix this message m.

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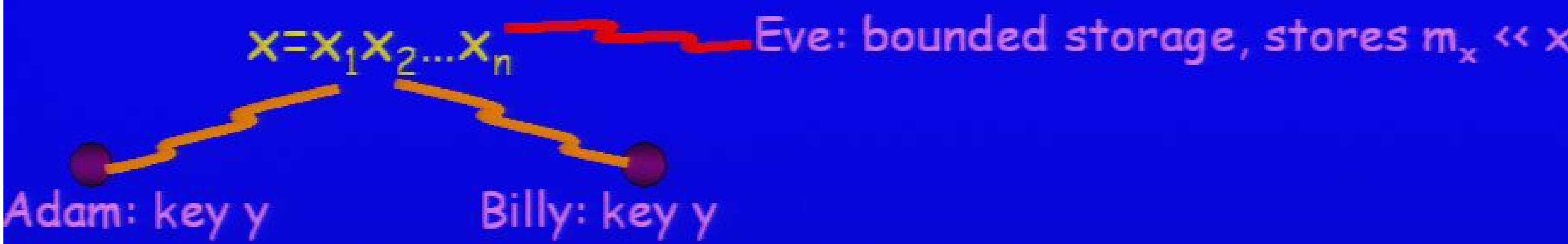
Bounded storage model

Bounded storage model - secure secret key generation:



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Bounded storage model - secure secret key generation:

$x = x_1 x_2 \ldots x_n$ — Eve: bounded storage, stores $m_x \ll x$

Adam: key y $\downarrow$ z

Billy: key y $\downarrow$ z

Adam and Billy use y to extract secret shared key z from x

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"Everlasting security": z is secure

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i.e. distribution of z given m_x and y is close to uniform for uniform y and x (in pre-image of m_x)

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Question: Does z remain secure if storage is quantum?

Corollary:

NO (at least in certain settings there is a counterexample).

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One-way communication:

Alice: x $\xrightarrow{m_x}$ Bob: y

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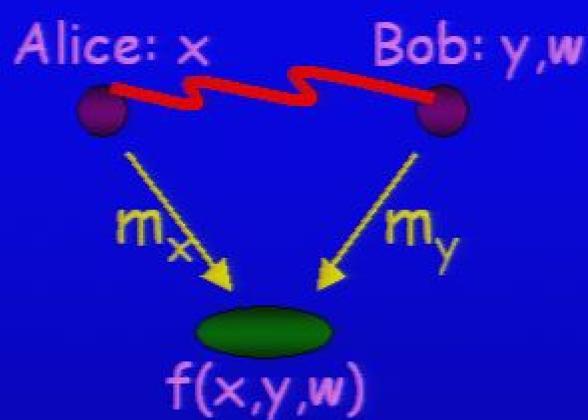
Our function gives counterexample:

if classical storage $= \sqrt{n} \log n^{\frac{1}{4}}$ then z is close to uniform

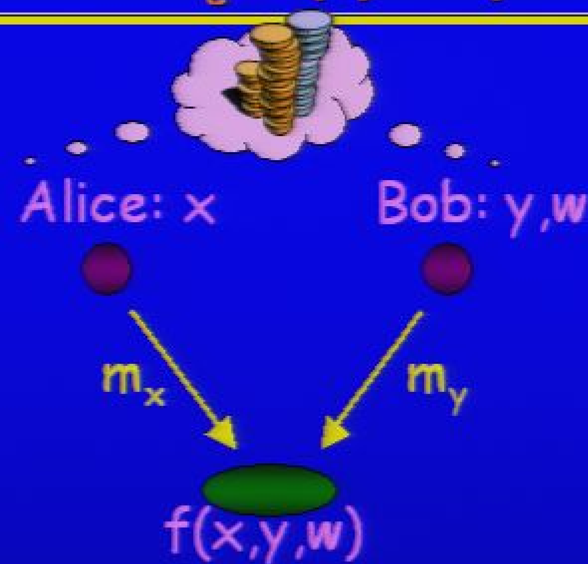
if quantum storage $= \sqrt{n} \log n^{\frac{1}{4}}$ z is far from uniform

SMP separation for f

One way: $Q^1_\varepsilon(f) = O(\log n^{3/2})$ and $R^{1,pub}_\varepsilon(f) = \Omega(\sqrt{n} \log n^{\frac{1}{4}})$



vs.



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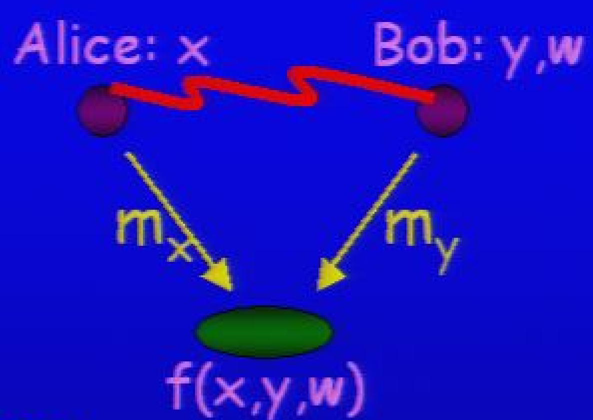

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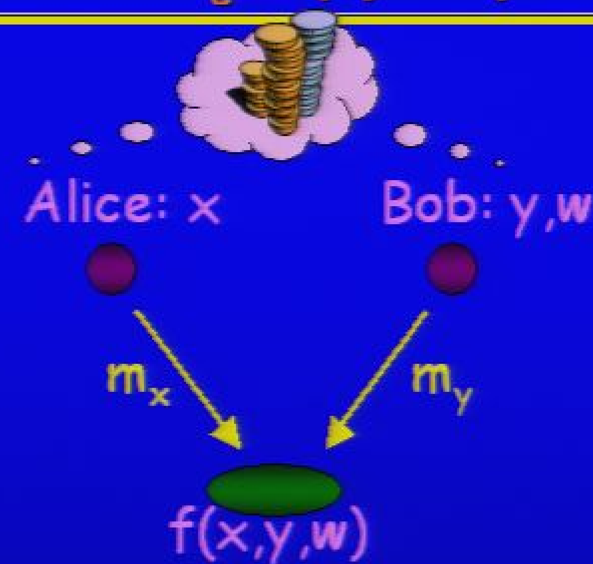

Corollary:

Quantum protocol also works in SMP model with shared entanglement (similar to the Buhrman protocol)

Classical lower bound also holds in the SMP model for classical communication with public coin.

$R_\varepsilon^{ent}(f) = O(\log n^{3/2})$ $R_\varepsilon^{pub}(f) = \Omega(\sqrt{n} \log n^{\frac{1}{4}})$

Summary

- One round separation for a Boolean function ($Q^1_\varepsilon(f) \ll R^1_\varepsilon{}^{pub}(f)$ and $R_\varepsilon^{ent}(f) \ll R_\varepsilon^{pub}(f)$); example where quantum bounded storage becomes insecure
- Bounded error quantum state identification pb.
- Tensor lemma (direct product theorem)
- Classical communication + shared randomness beats qubit communication ($R_\varepsilon^{pub}(R1) \ll Q_\varepsilon(R1)$)
- Classical communication + shared entanglement beats qubit communication ($R_\varepsilon^{ent}(R2) \ll Q_\varepsilon^{pub}(R2)$)
- Fingerprints in the SMP model can simulate multi-round protocols with unlimited entanglement (with exponential overhead)

Open Questions

- An exponential separation for a *total* Boolean function (instead of a partial one)?
- Prove or disprove the general tensor lemma (would imply $Q_\varepsilon(R1)=\Omega(\sqrt{n})$, which is tight)
- Is there a relation (or function) R such that $R_\varepsilon^{ent}(R) >> Q_\varepsilon^{pub}(R)$?
- Other applications of the tensor lemma?

Thank you!

joint work with

Dmitry Gavinsky

Oded Regev

Ronald de Wolf

[GKWR'06] **D. Gavinsky, J. Kempe, O. Regev, R. de Wolf**: "*Bounded-Error Quantum State Identification and Exponential Separations in Communication Complexity*", **STOC'06**, p. 594-603 (2006), quant-ph/0511013

[GKW-1'06] **D. Gavinsky, J. Kempe, R. de Wolf**: "*Strengths and Weaknesses of Quantum Fingerprinting*", **Complexity'06**, p. 288-195 (2006), quant-ph/0603173

[GKW-2'06] **D. Gavinsky, J. Kempe, R. de Wolf**: "*Exponential Separation of Quantum and Classical One-Way Communication Complexity for a Boolean Function*", quant-ph/0607174

[KR'06] **I. Kerenidis, Ran Raz**: "*The one-way communication complexity of the Boolean Hidden Matching Problem*", quant-ph/0607173