Title: Quantum computation as geometry

Date: Aug 02, 2006 02:00 PM

URL: http://pirsa.org/06080007

Abstract: How should we think about quantum computing? The usual answer to this question is based on ideas inspired by computer science, such as qubits, quantum gates, and quantum circuits. In this talk I will explain an alternate geometric approach to quantum computation. In the geometric approach, an optimal quantum computation corresponds to "free falling" along the minimal geodesics of a certain Riemannian manifold. This reformulation opens up the possibility of using tools from geometry to understand the strengths and weaknesses of quantum computation, and perhaps to understand what makes certain physical operations difficult (or easy) to synthesize.

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Finding small quantum circuits.

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Riemannian geometry

Finding shortest paths on a particular manifold.

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I'll overview what we know about the geometry (not much!)

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I'll overview what we know about the geometry (not much!)

I'll speculate on how ideas from geometry may be used to gain insight into quantum computing.

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Virtually all the major problems in computational complexity theory are still open: separate P from NP, or PSPACE.

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Why (loosely): Because a good pseudorandom number generator can be used to generate a pseudorandom function which is (1) easy to compute, and (2) impossible to efficiently distinguish from a truly random function, which is hard to compute.

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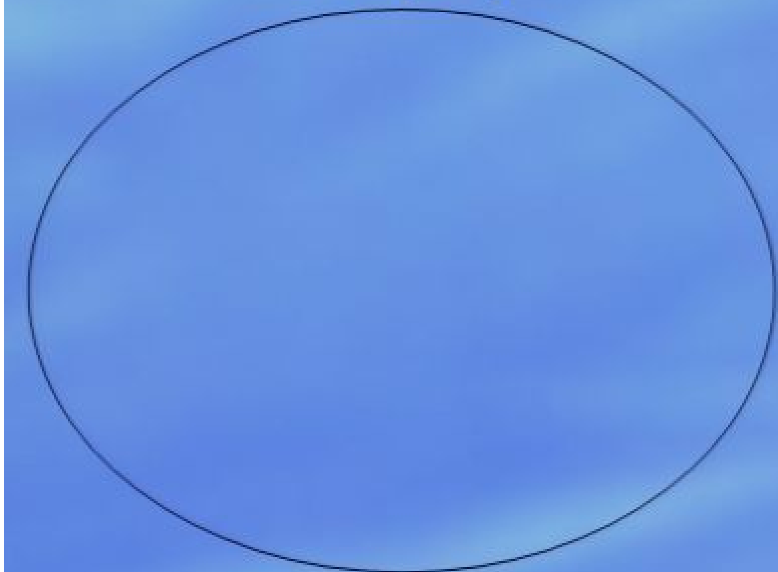
So any proof that (for example) $P \neq NP$ must not rely on being able to efficiently distinguish easy- and hard-to-compute functions.

Quantum computing and Riemannian geometry

$SU(2^n)$



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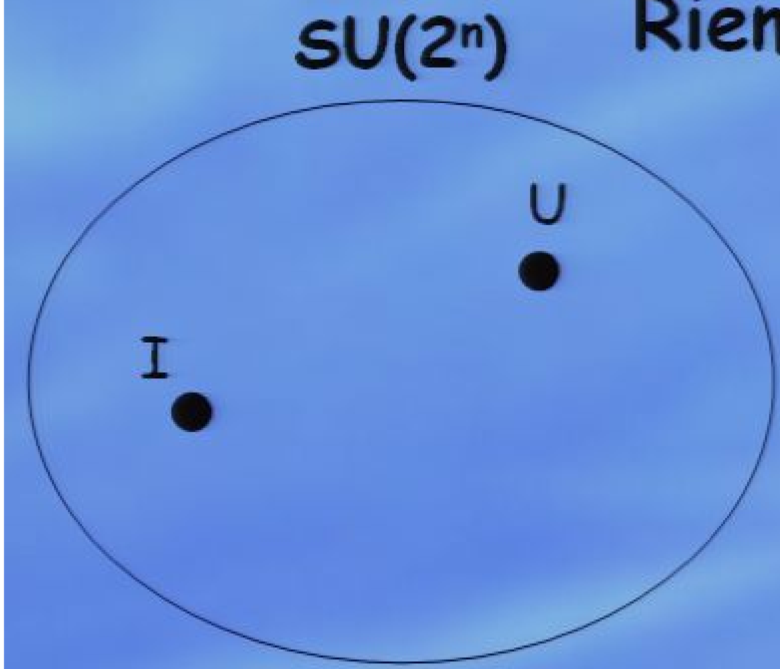

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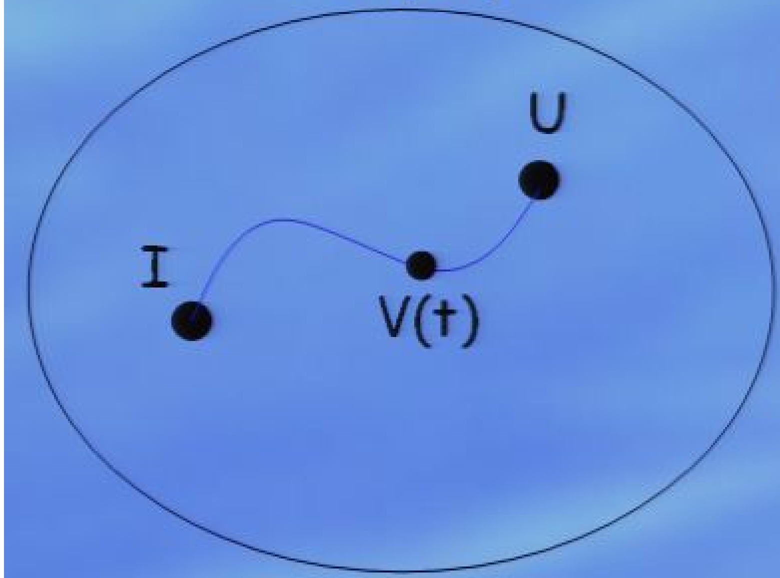


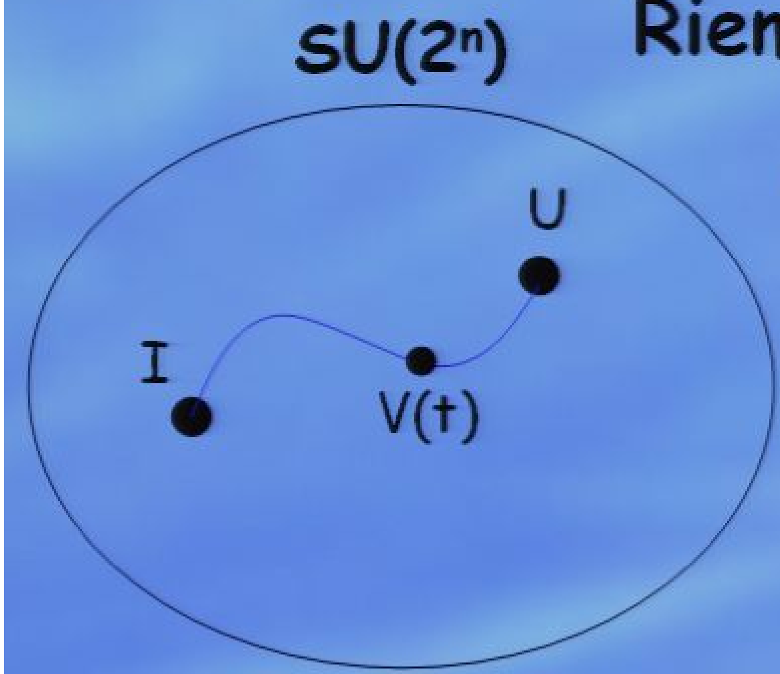
We can think of U as being generated by an n-qubit time-dependent Hamiltonian:

$$H(t) = \sum_\sigma h_\sigma(t)\, \sigma$$

$$V(0) = I; \qquad V(1) = U$$

Notation: σ = tensor product of the Pauli matrices or I.

Quantum computing and Riemannian geometry



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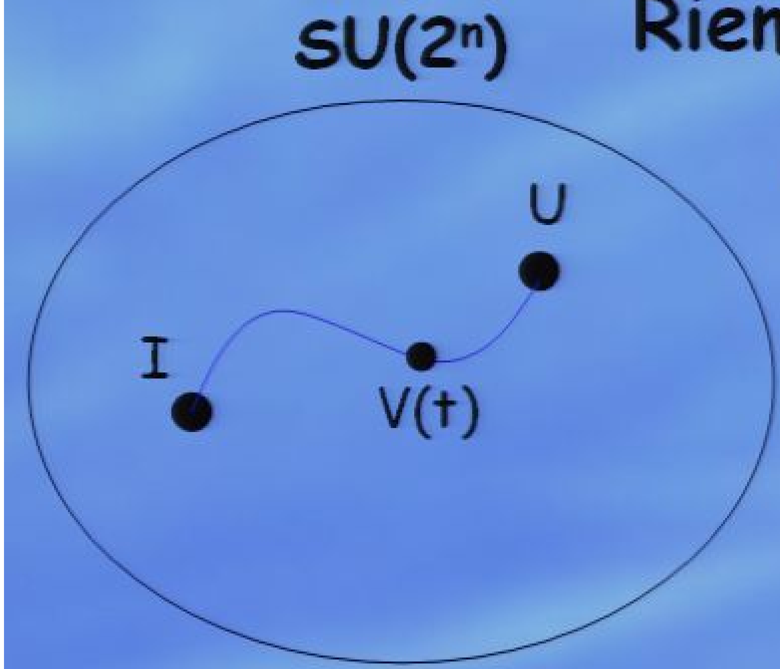

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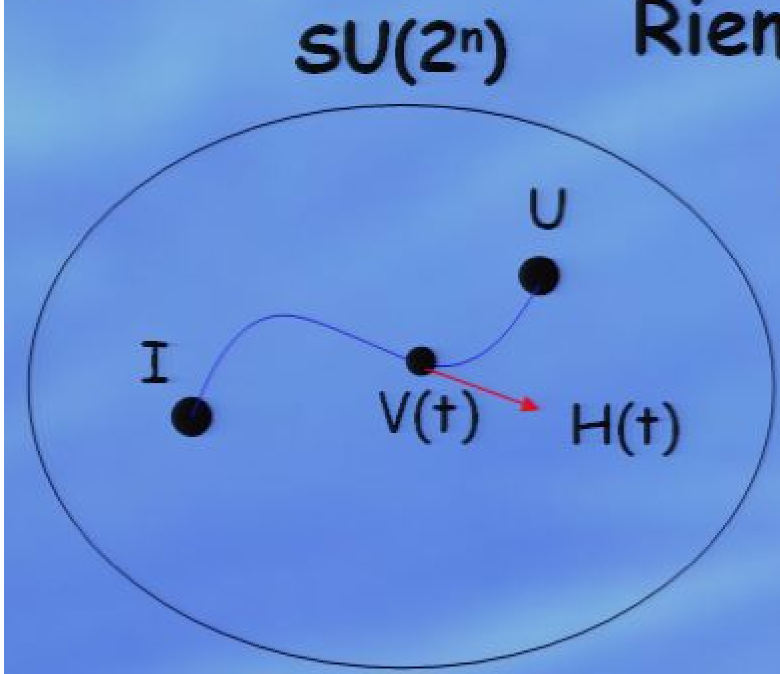

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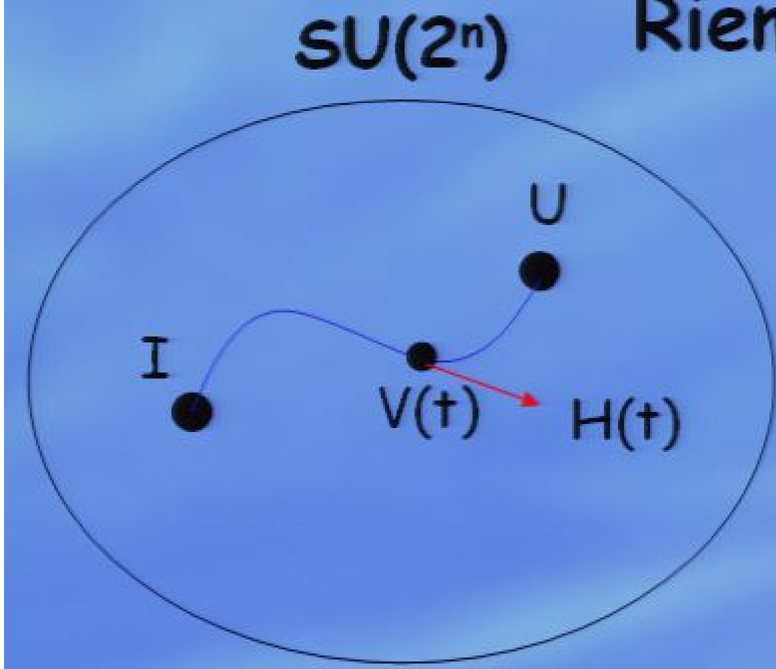

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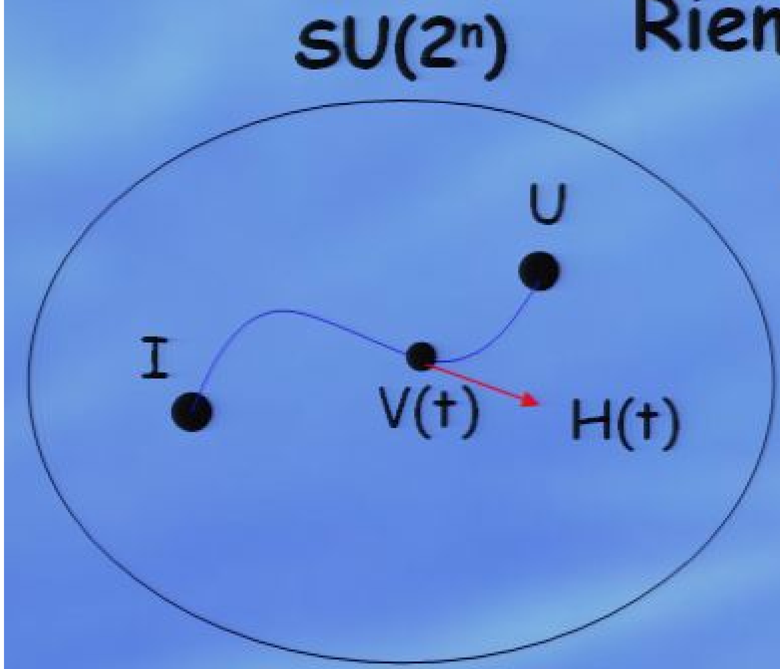

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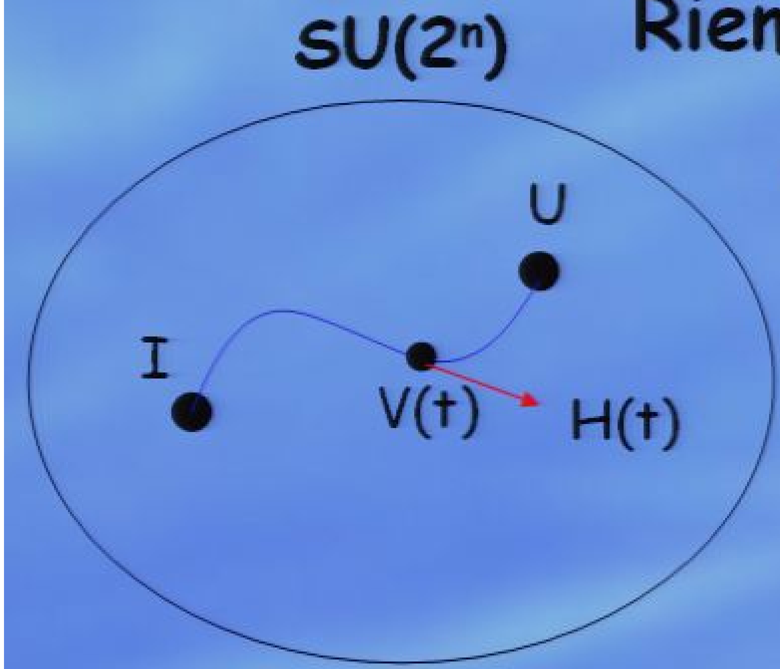

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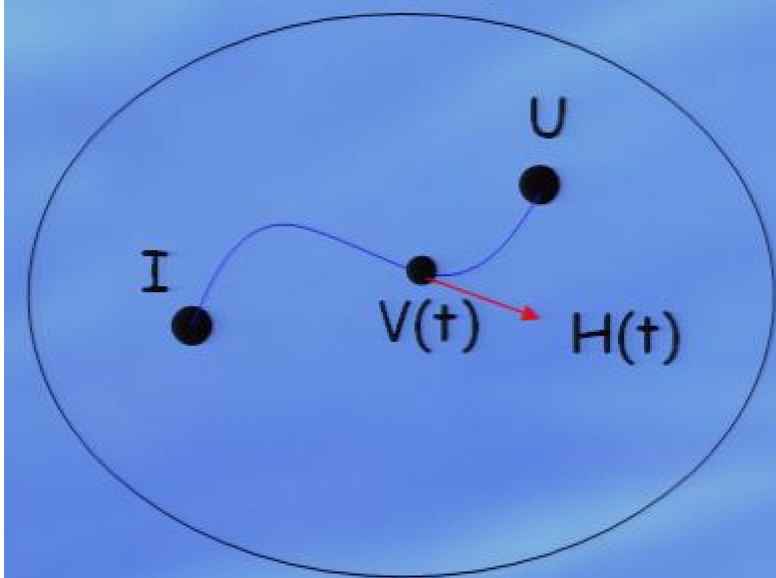

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P(XII+ZZI+XYZ) = XII+ZZI

Quantum computing and Riemannian geometry

Claim: For any $U \in SU(2^n)$:

(a) The minimal number of gates required to exactly synthesize U satisfies:

$$d(I,U) \leq m(U)$$

(b) We can synthesize U to high accuracy using a number of quantum gates polynomial in $d(I,U)$.

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⇒ If we can work on a Riemannian manifold this leads to the geodesic equation.

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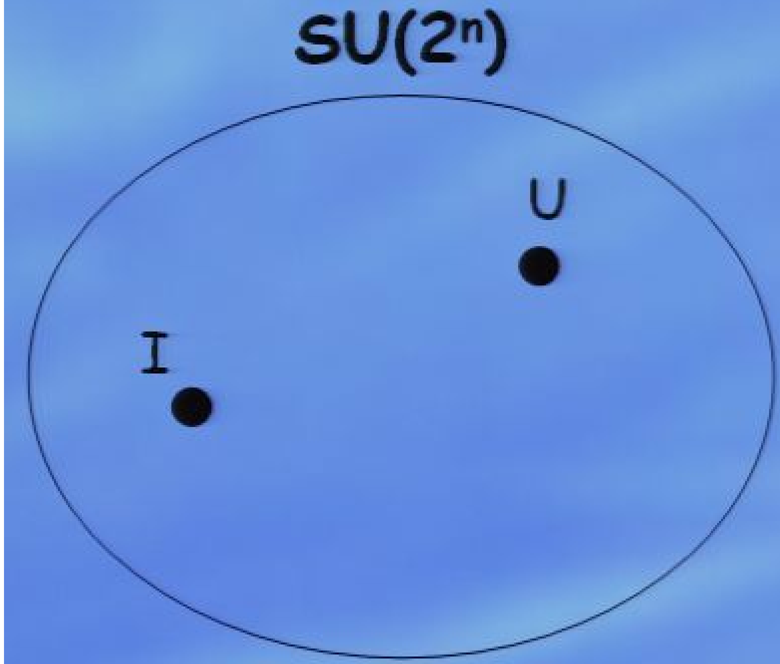
Compare, e.g., with being given part of an optimal circuit.

$$d(I, U) \leq m(U)$$



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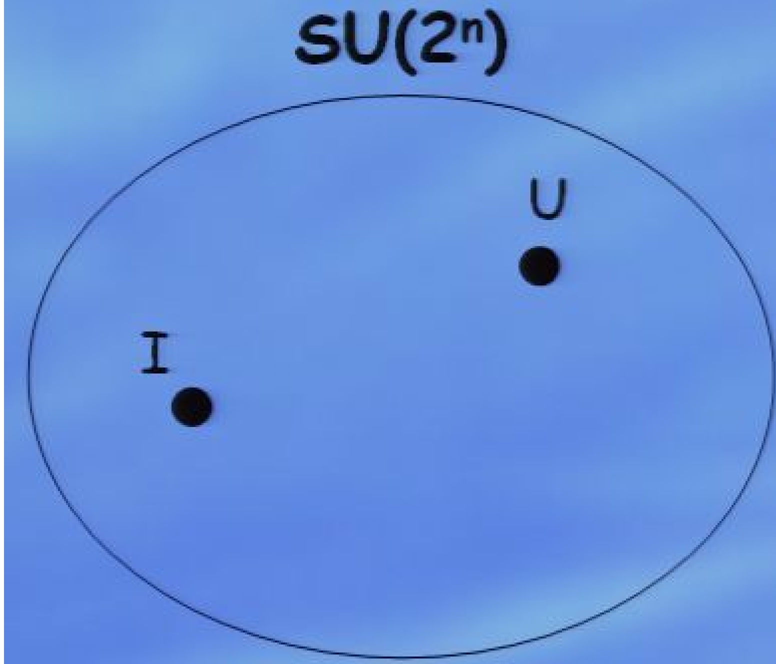

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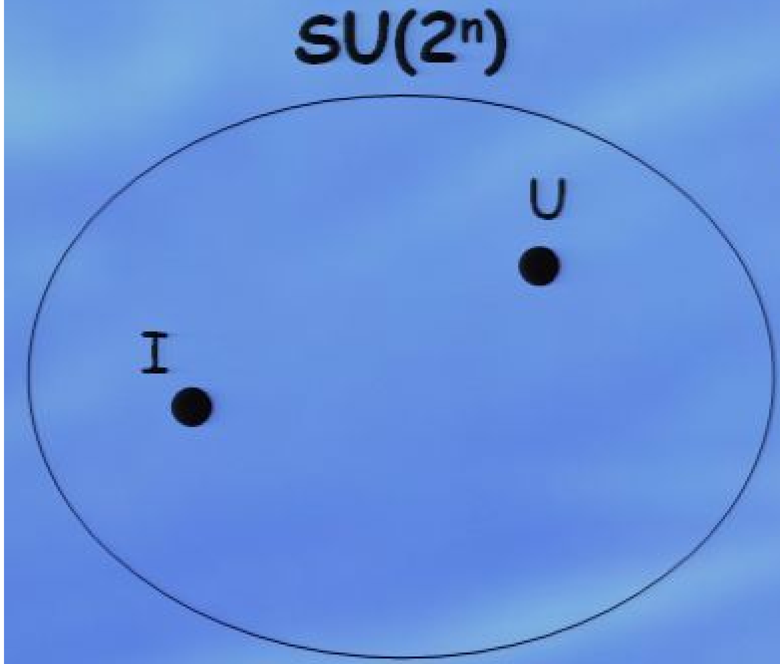

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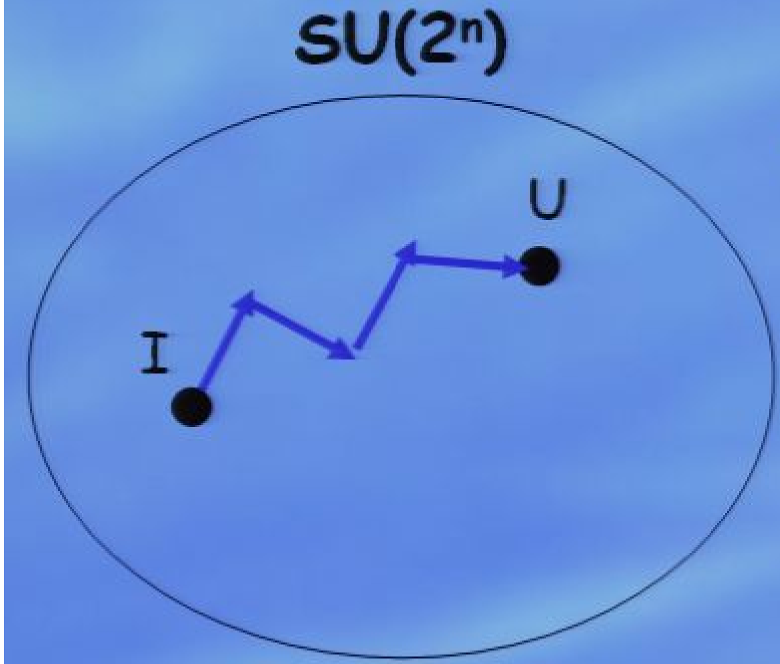

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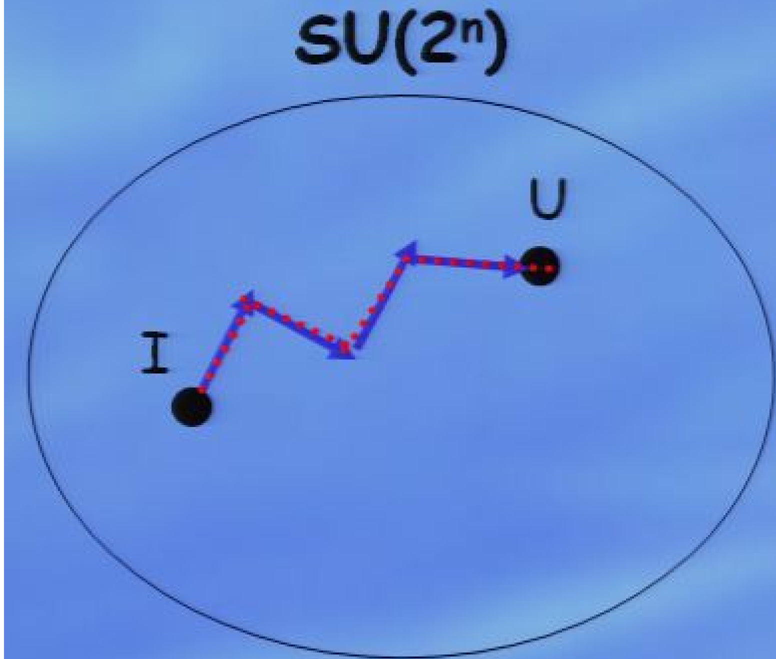

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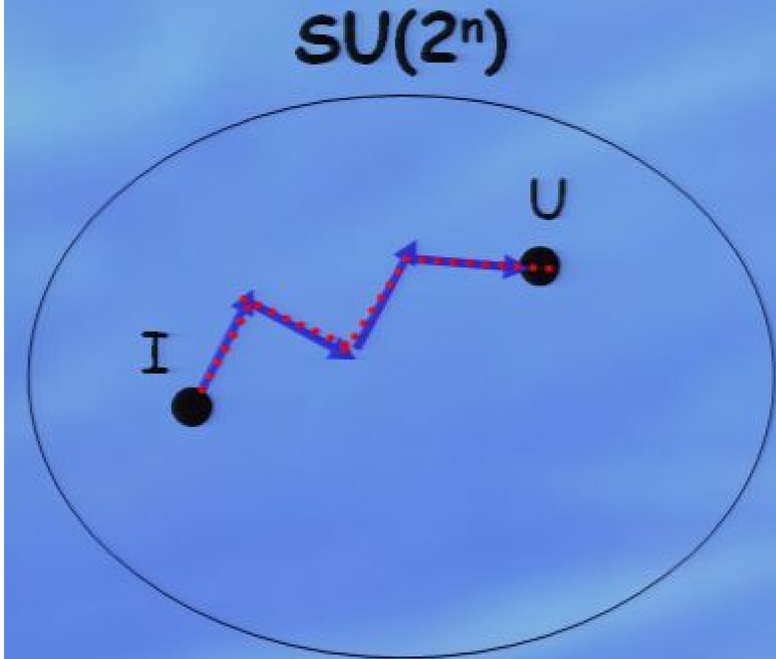

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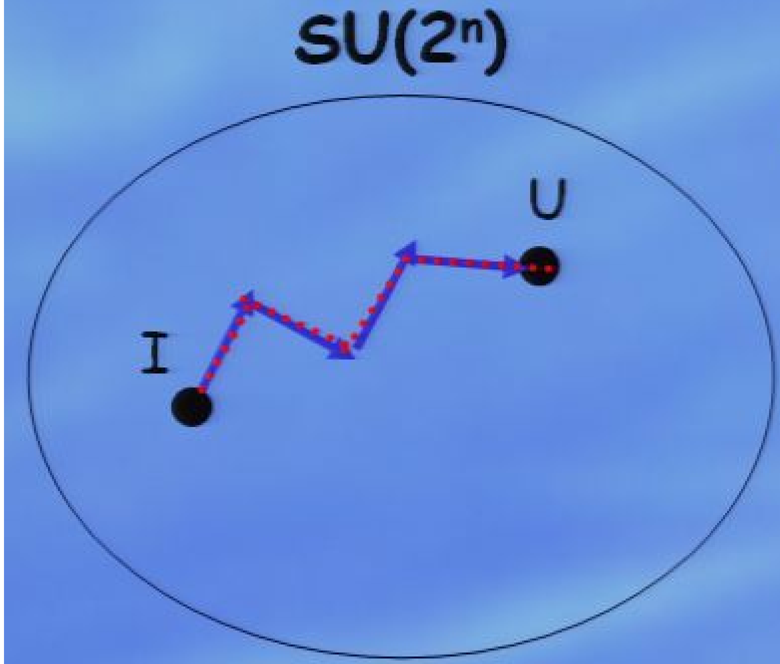

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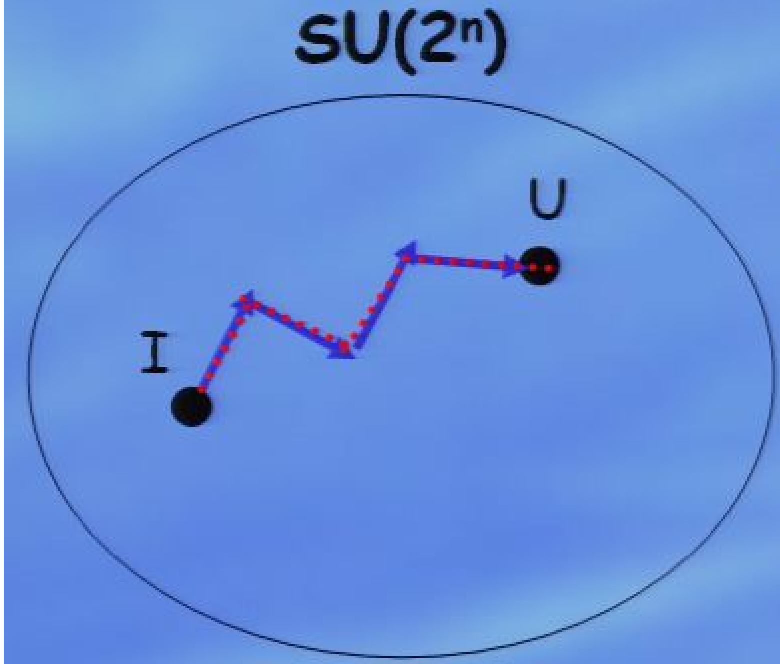

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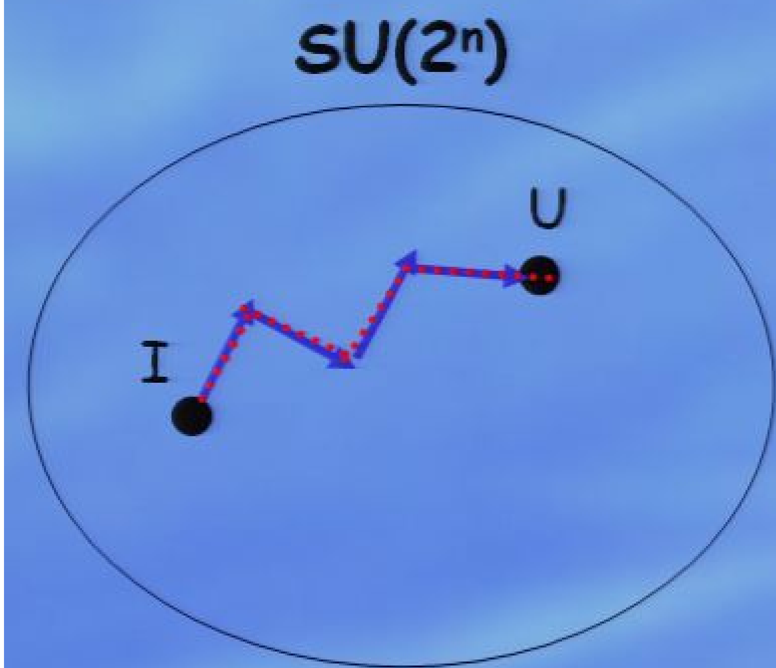

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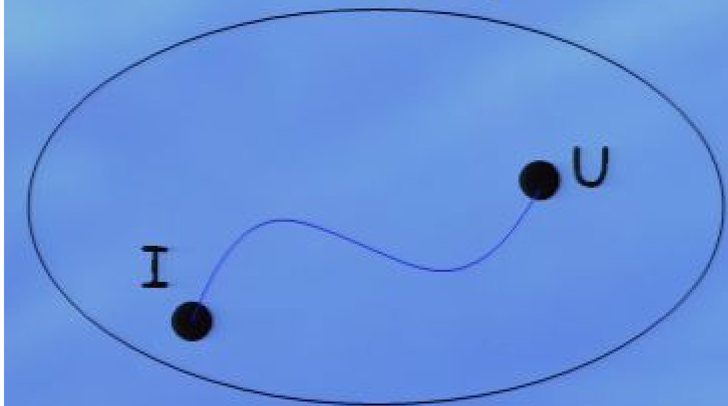
The upper bound: $m(U') \leq \text{poly}(d(I,U))$

$SU(2^n)$



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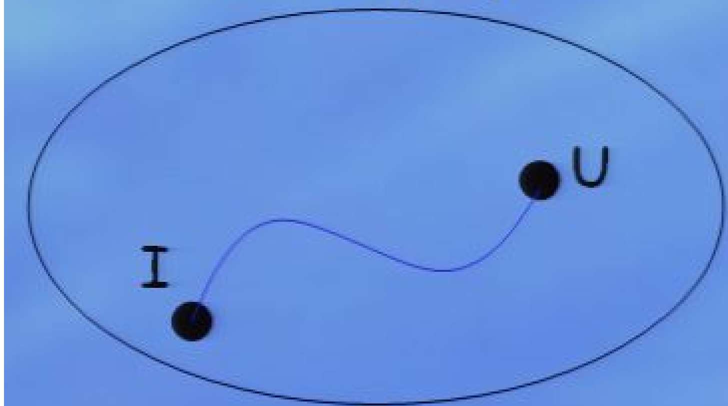

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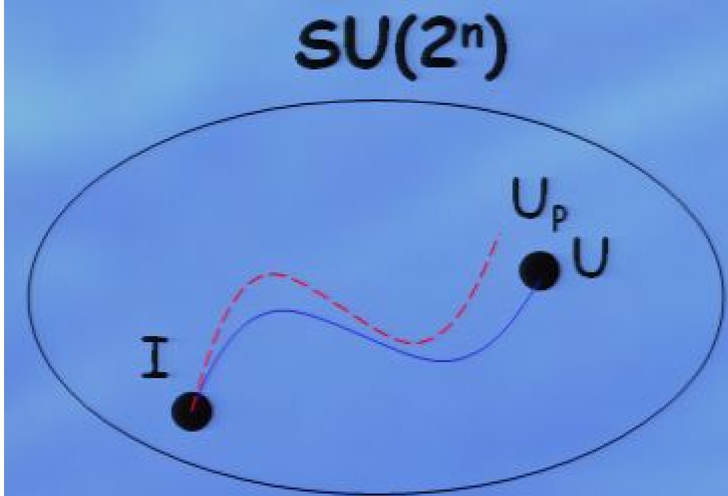

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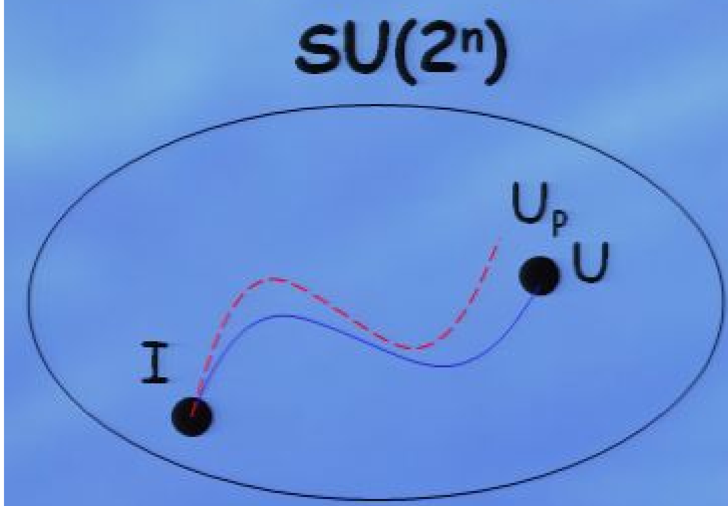

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Idea 2: Break the path to U_p up into small intervals of size Δ. Then approximate evolution by $H_p(t)$ over that interval by evolution due to the "average" Hamiltonian:

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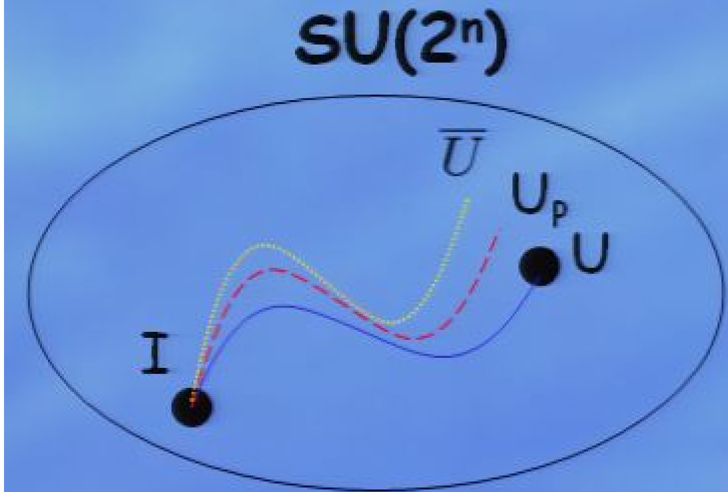

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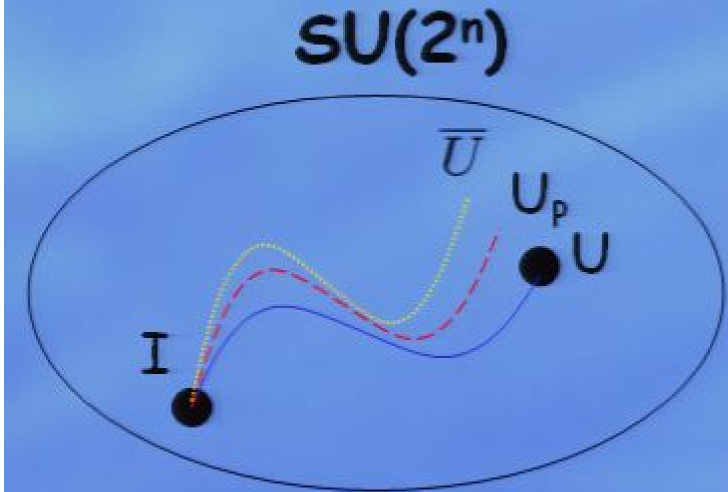

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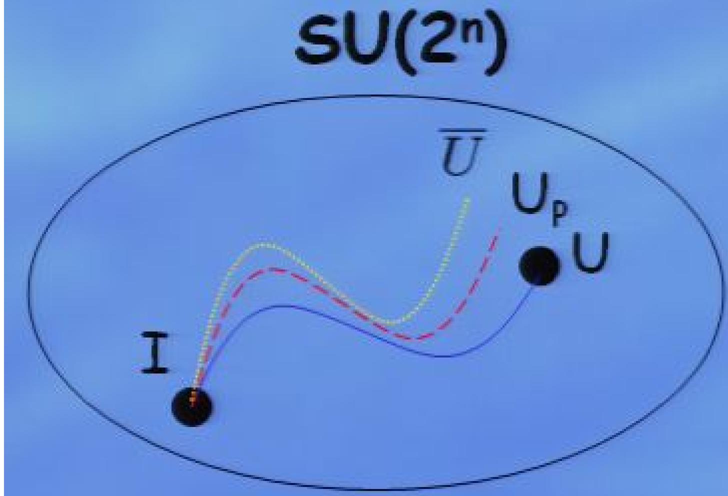

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Easy to do accurately since (a) the average Hamiltonian is two-body, and (b) it has bounded strength.

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Geodesic equation: $\dot{L} = -i2^n(1-p^{-2})[L, P(L)]$

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⇒ The system is integrable.

Big picture

What we've learnt

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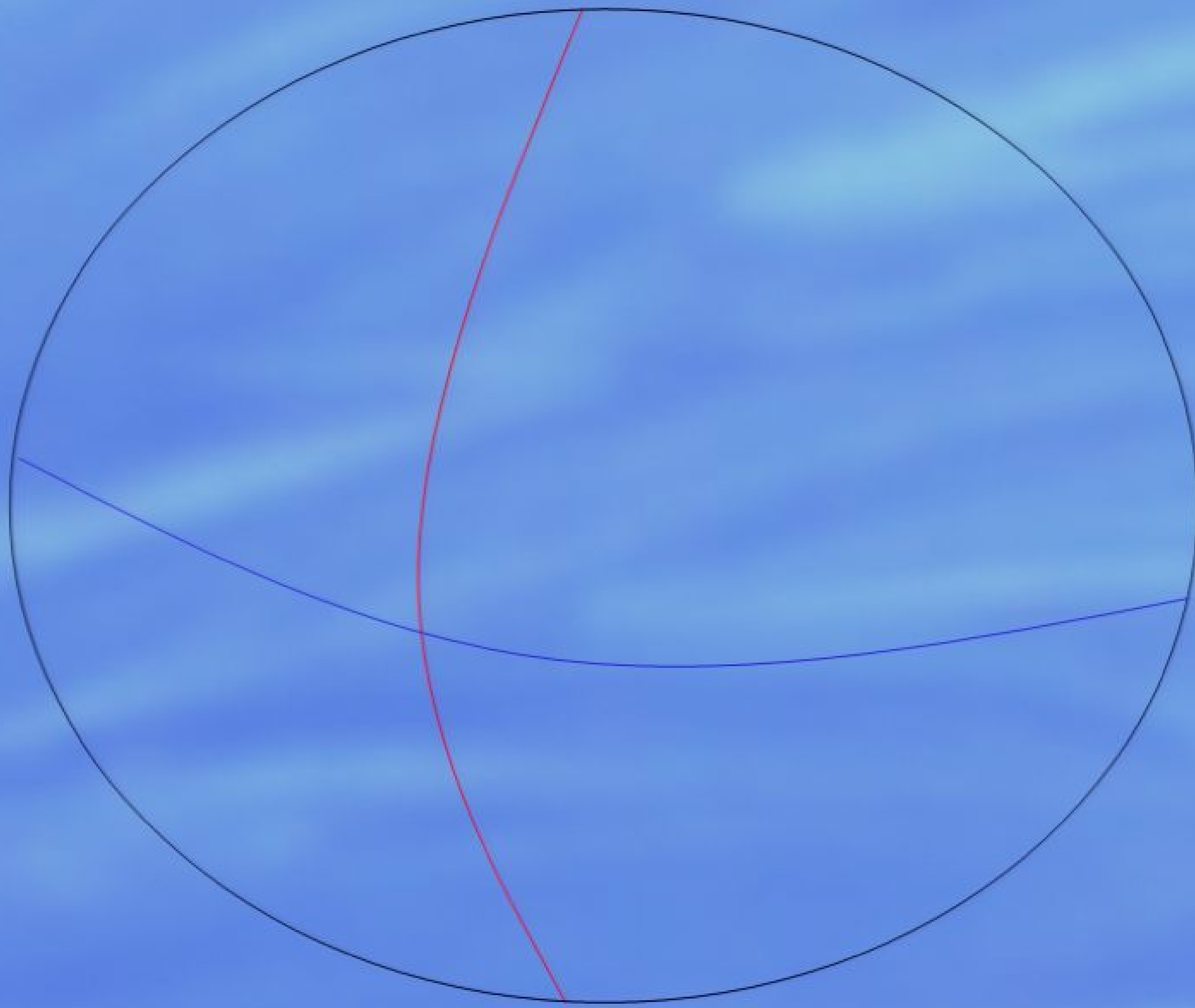

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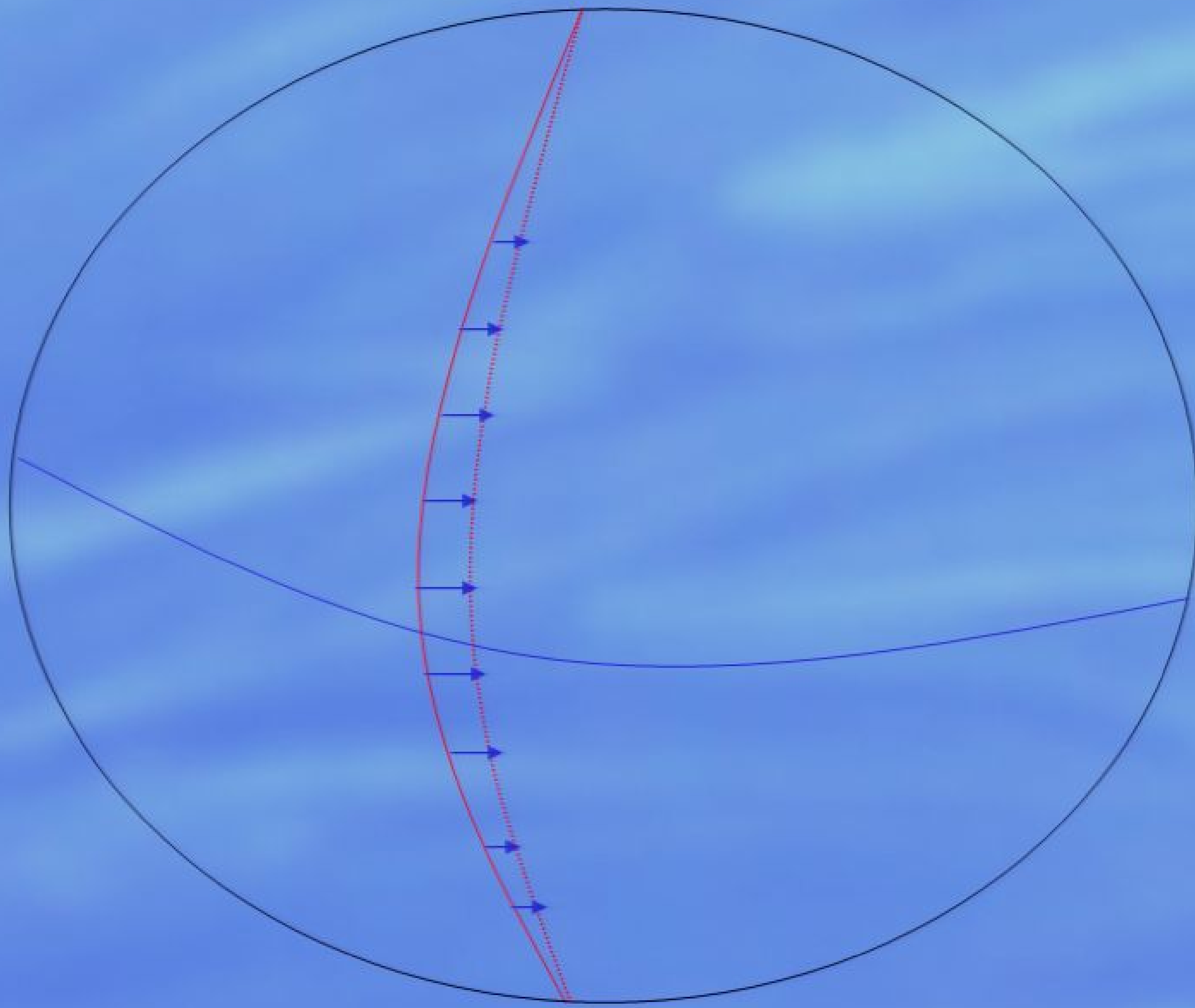

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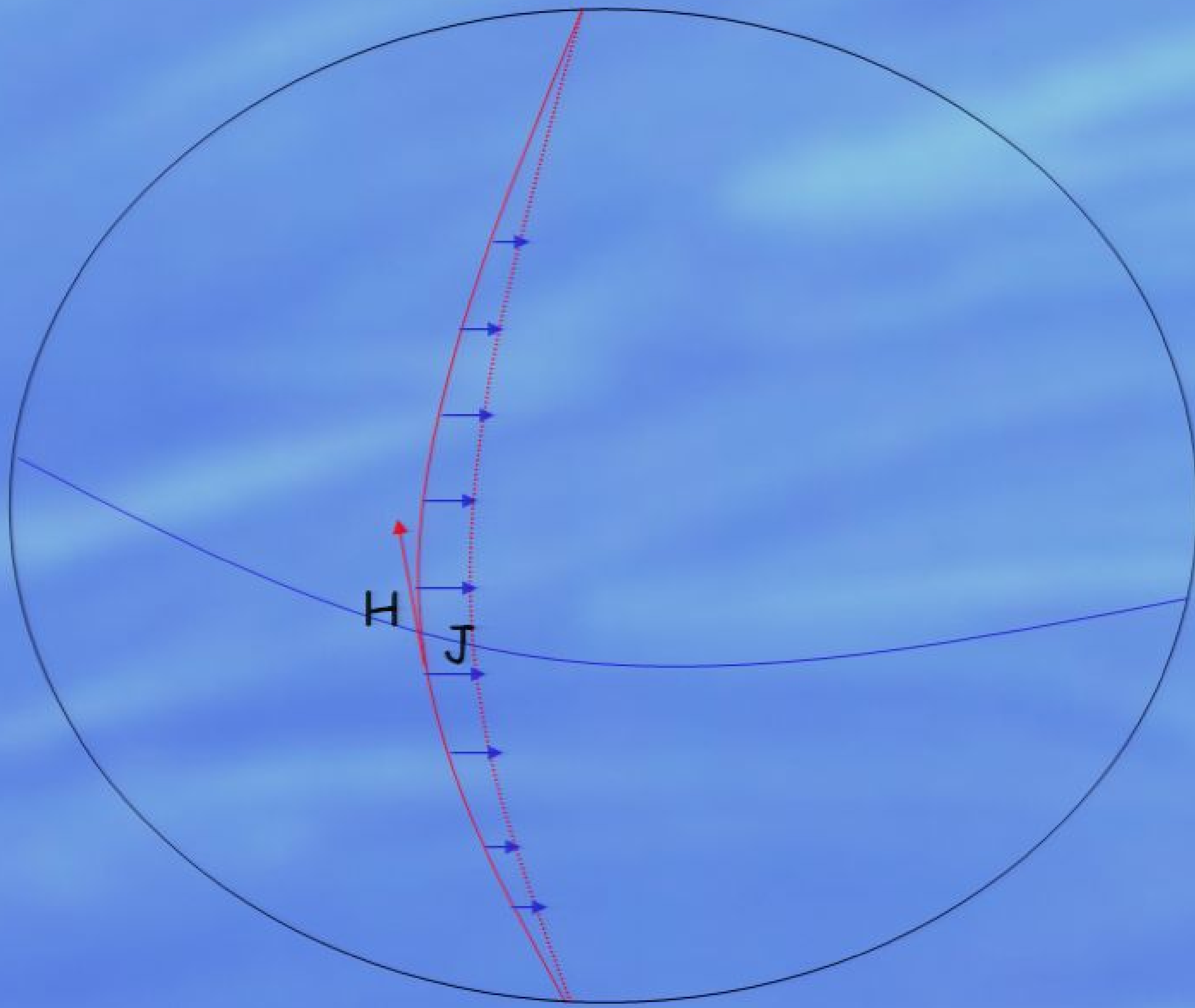

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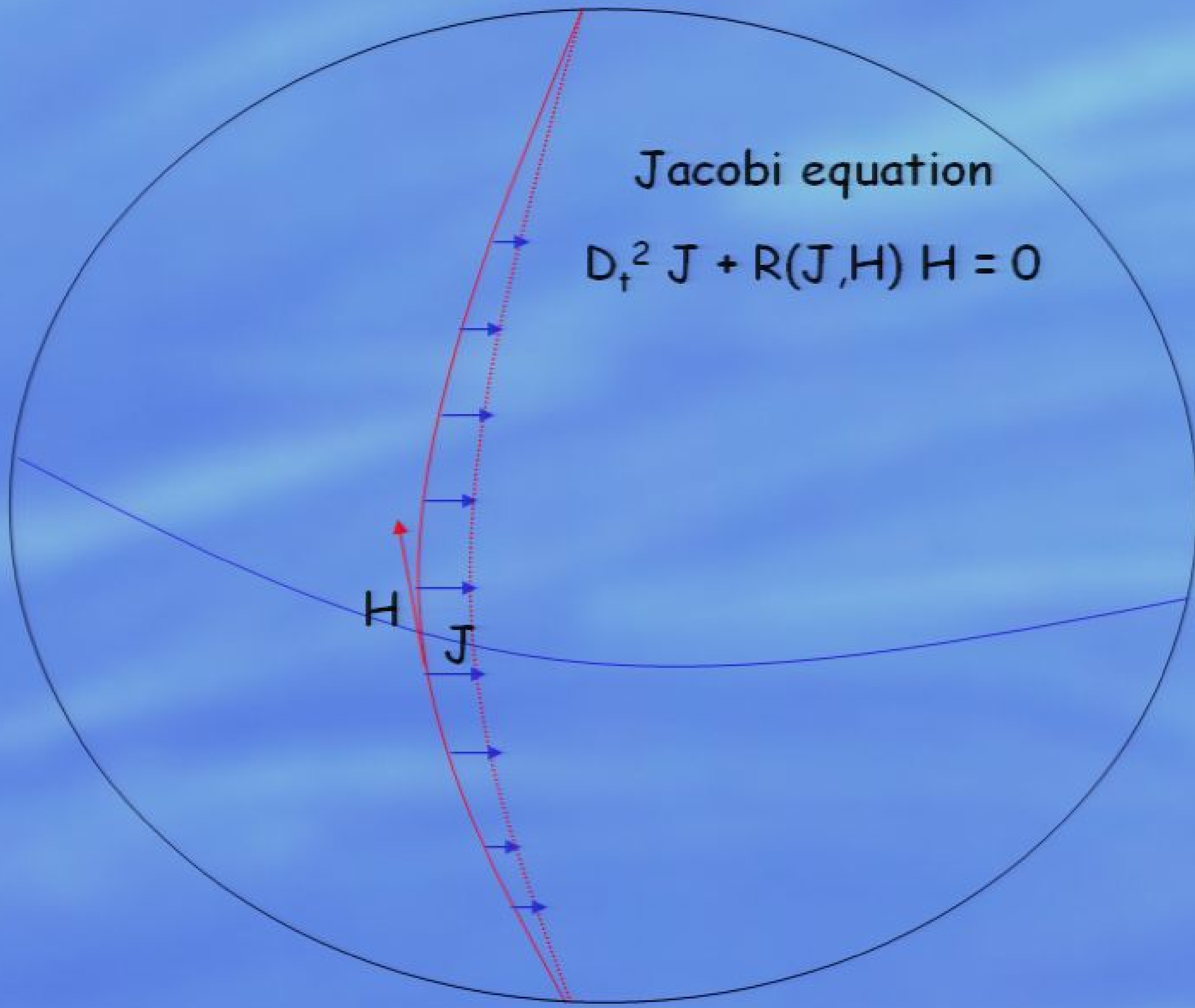

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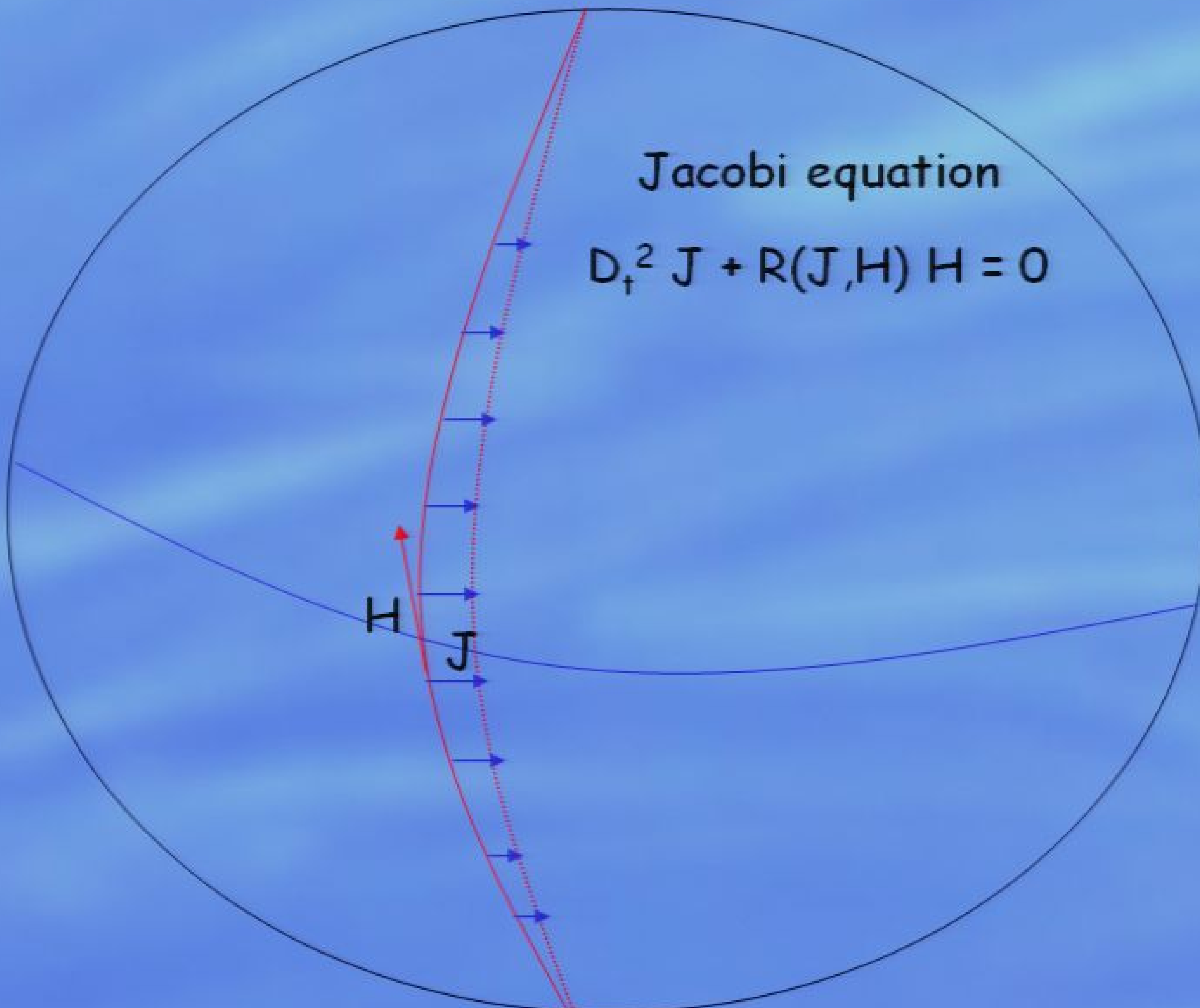

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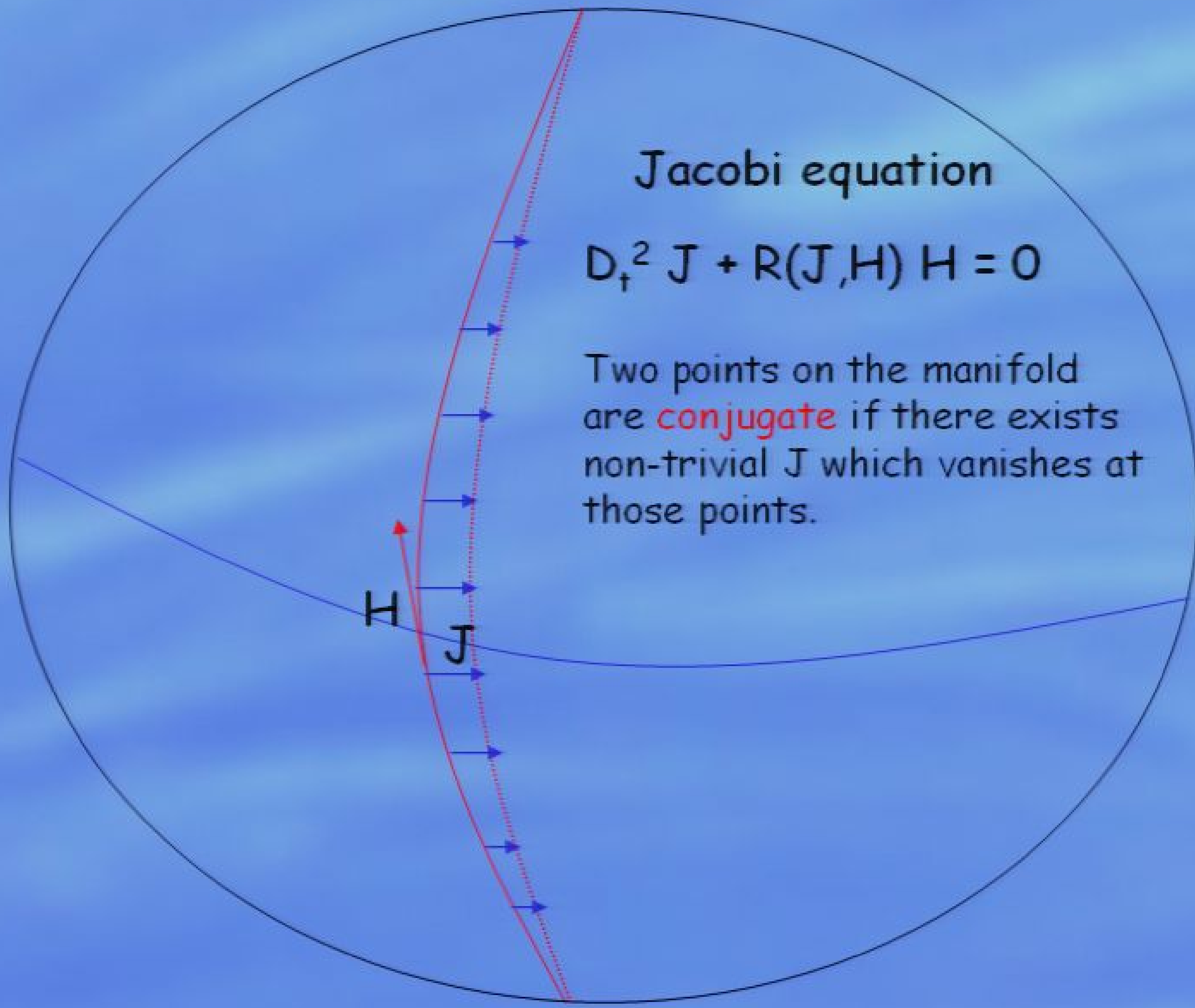

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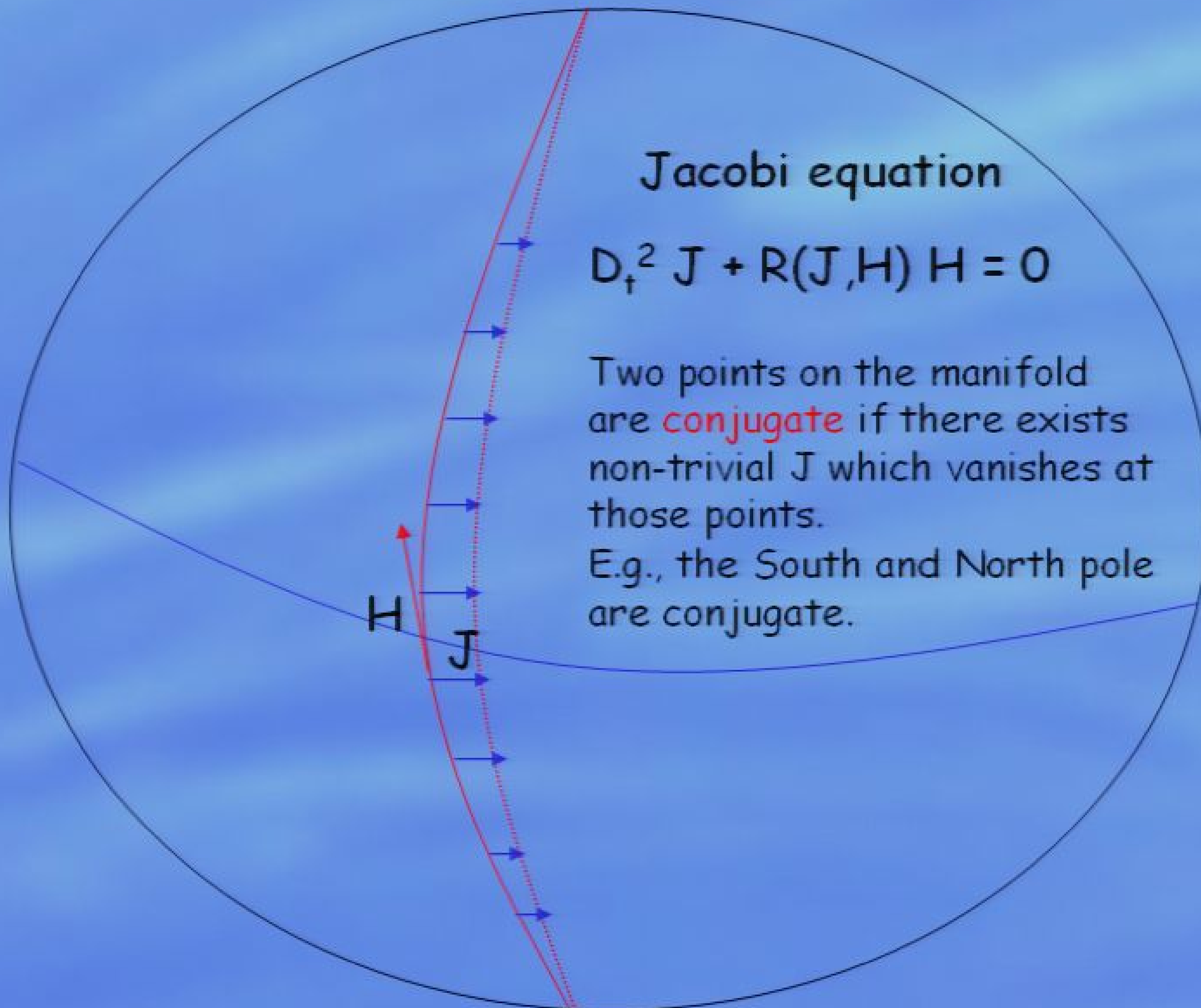

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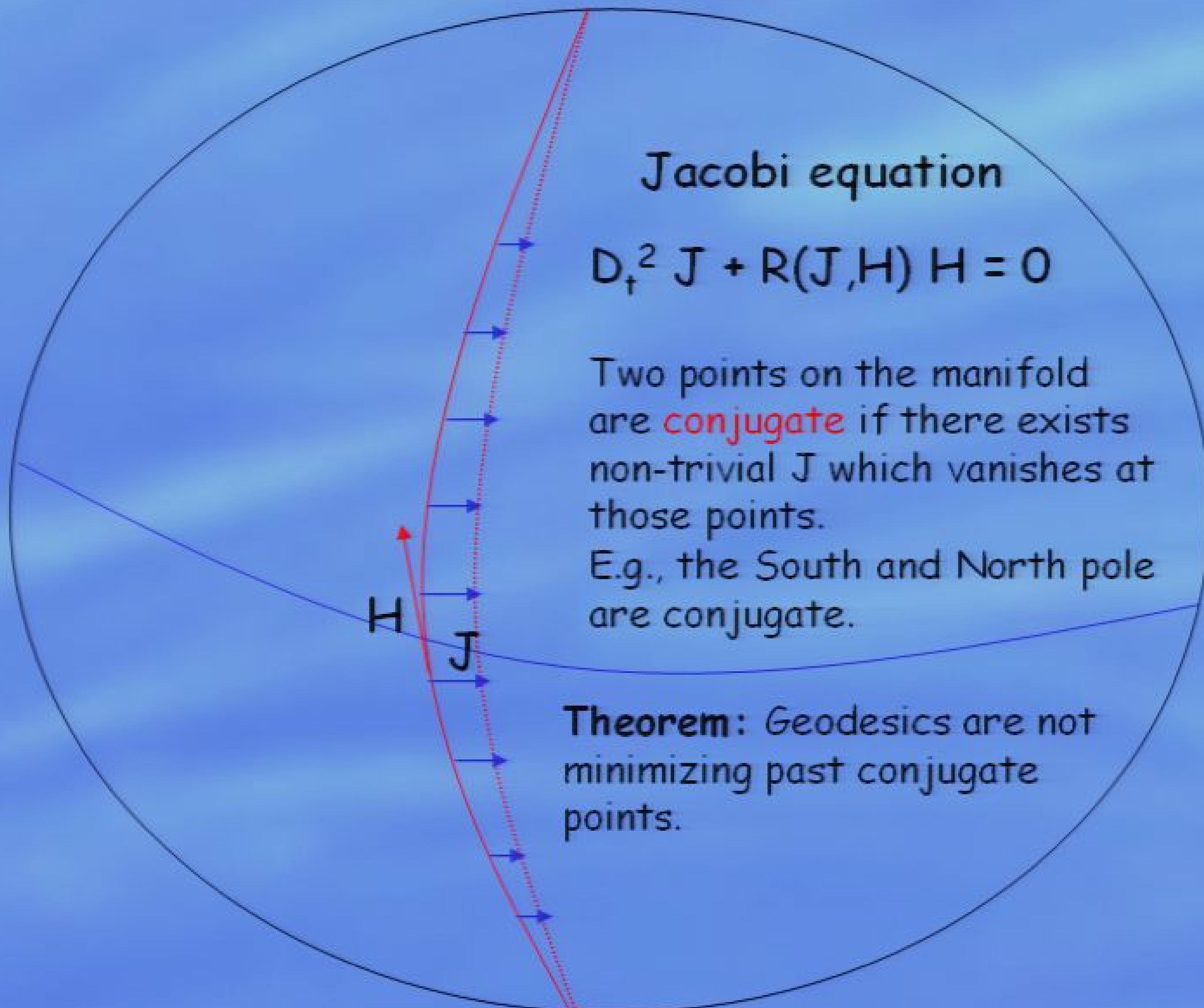

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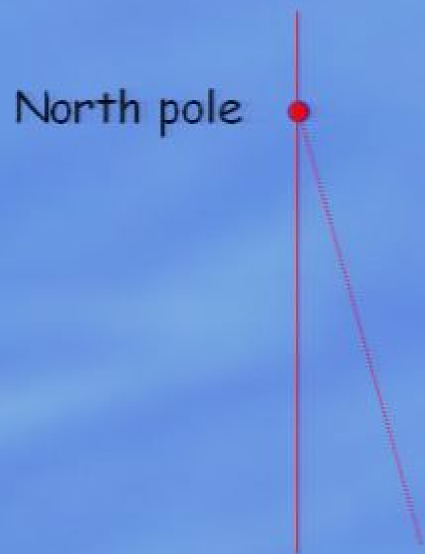

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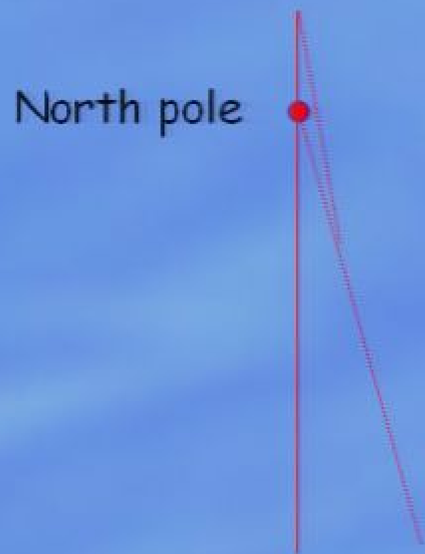

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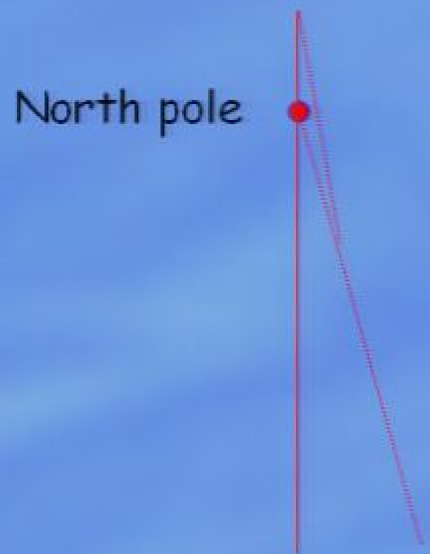

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Therefore, the global minimum must be elsewhere.

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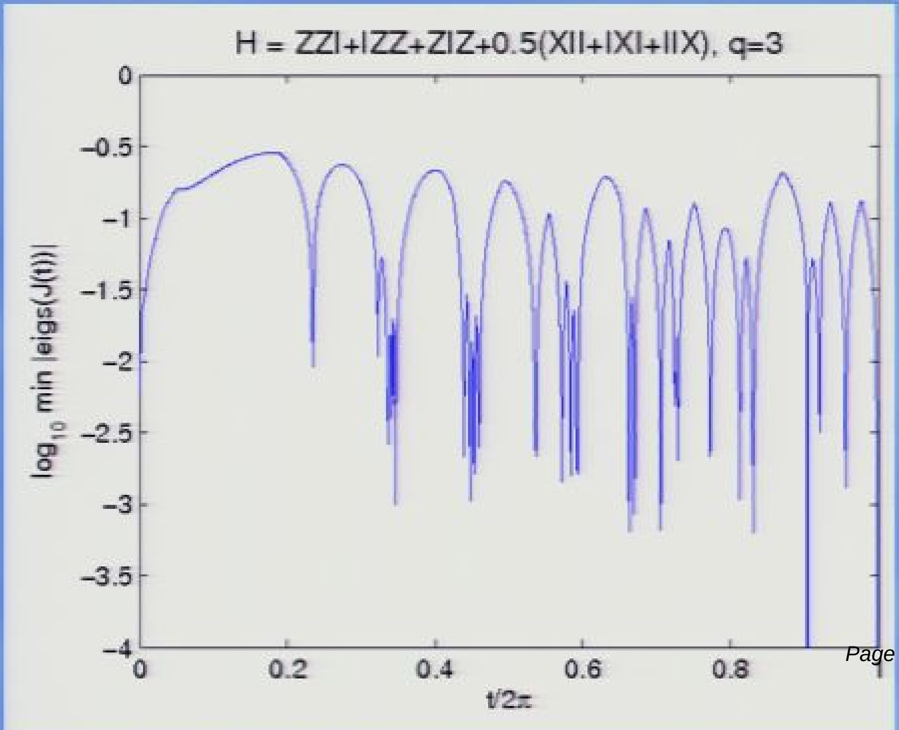

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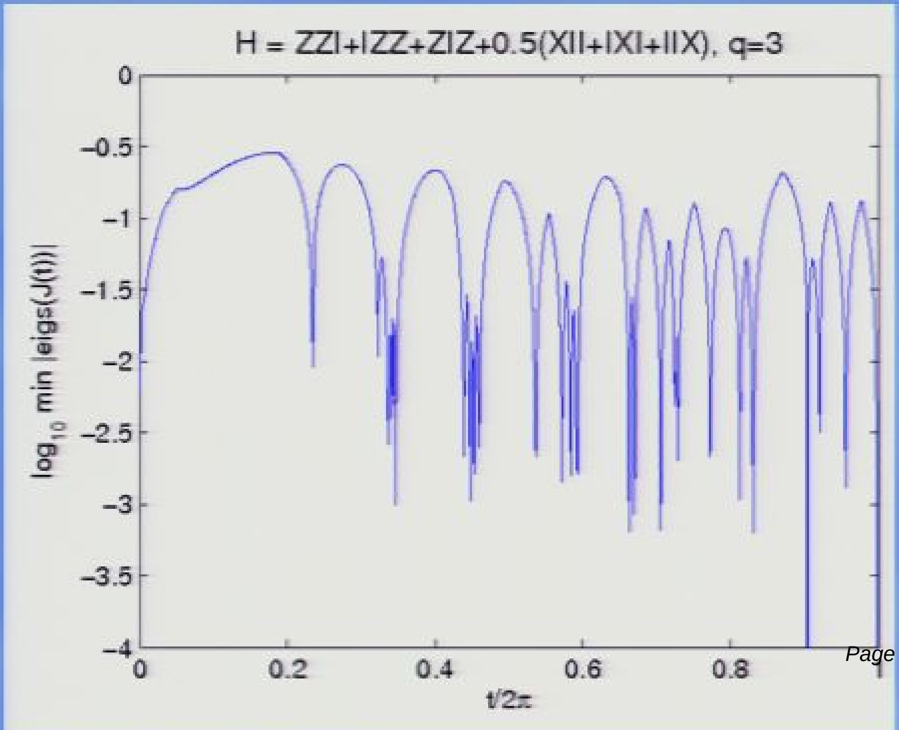

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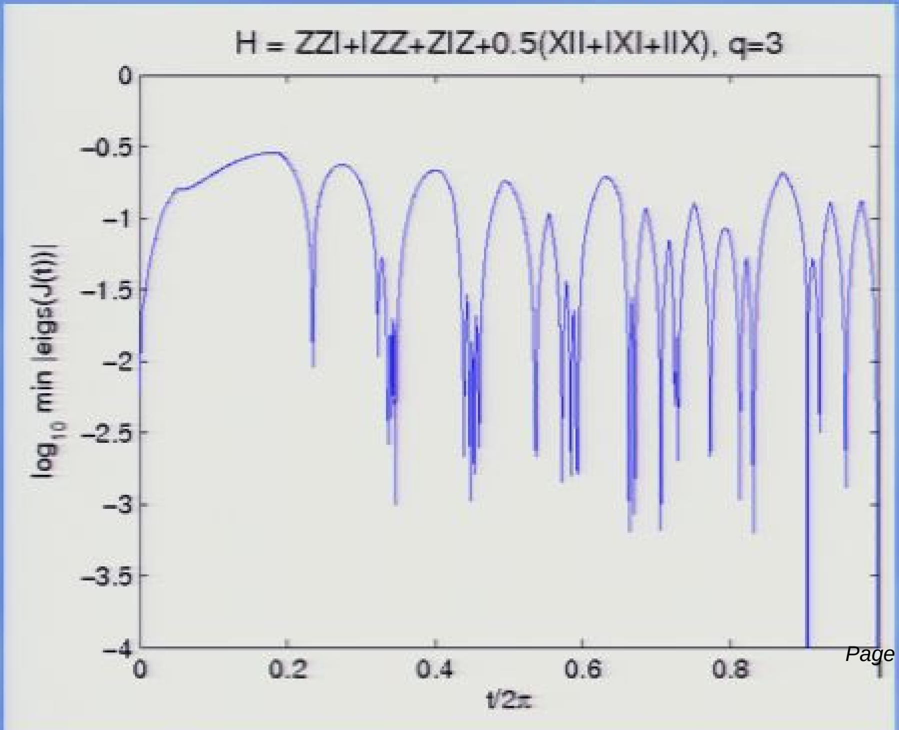

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