Title: Â“No Information Without DisturbanceÂ” Myths and Facts about Quantum Measurements

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Abstract: In this talk I will discuss the question of how to characterize, in an operationally meaningful way, the inevitable Â“disturbanceÂ” of a quantum system in a measurement. I will review some well-known limitations of quantum measurements (facts), and give precise formulations of trade-off relations between information gain and Â“disturbanceÂ”. Famous examples among these limitations are the uncertainty principle, the complementarity principle, and WignerÂ’s theorem on limitations on measurements imposed by conservation laws. The universal validity of each of these has been challenged repeatedly, and no conclusive resolution seems to have been reached.

I will analyze some long-standing conflations and misconceptions (myths) concerning these quantum limitations, such as the reduction of the uncertainty principle to the idea of mechanical disturbance (momentum kicks), the claim that the uncertainty principle has nothing to do with (the impossibility of) simultaneous measurements of noncommuting quantities, and some alleged violations of the uncertainty and complementarity principles. Recent rigorous work has led to apparently contradictory conclusions on these issues. I will show that the contradictions dissolve if due attention is paid to the choice of operationally meaningful notions of measurement accuracy and disturbance.

Dedication...

Abner was here...

- quantum mechanical worldview
- foundational physics as "experimental metaphysics"
- quantum measurement: actualization of potentialities

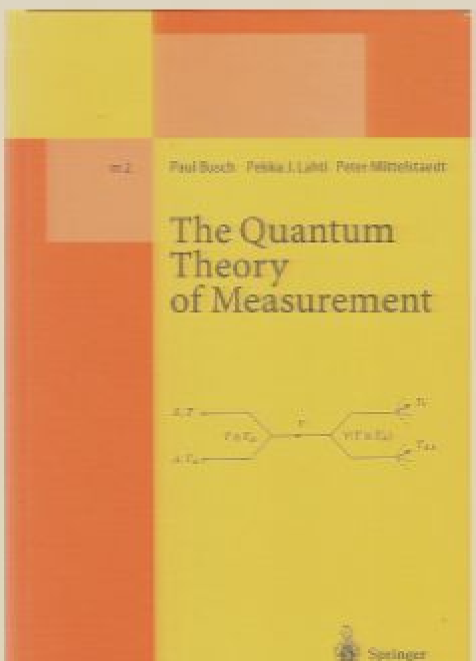
... "what would Abner say?"

Encounters with Abner...

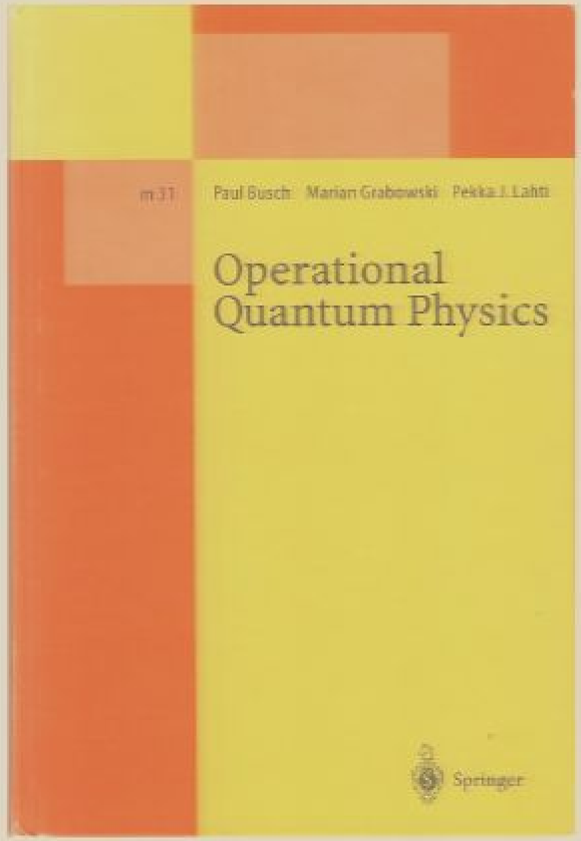
unfinished business

- Limitations on measurements due to conservation laws (WAY Theorem) –

- quantum measurement problem: Insolubility Theorem
  (AS&PB, Insolubility of the quantum measurement problem for unsharp observables, SHPMP, 1996)



(not an advert – out of print)



Quest (Arthurian? Quixotian?):
make sense of unsharp quantum reality (which Hellman box)

"Can quantum mechanical reality be considered sharp?"

Theme and Plan of Talk:

Limitations on Quantum Measurements

- old myths in the light of recent developments
  old and new challenges to complementarity and uncertainty principles:
  Popper, Scully-Englert-Walther, Afshar, Ozawa

- review some disputed no-go claims and trade-off relations

- give operational definitions of (in)accuracy and "disturbance"

- make precise: trade-off between information gain and disturbance

Report some very recent developments, closing important, long-standing gaps

On quantum weirdness......

...... about the minds of quite a few among those concerned with quantum mechanics:

persistence of myths/misconceptions, e.g. about complementarity, uncertainty,...

– demonstrates need for clarification (new theory needs new words/intuitions)

– Should S Glashow work on it? Maybe not!

– But somebody has to do it (I think).

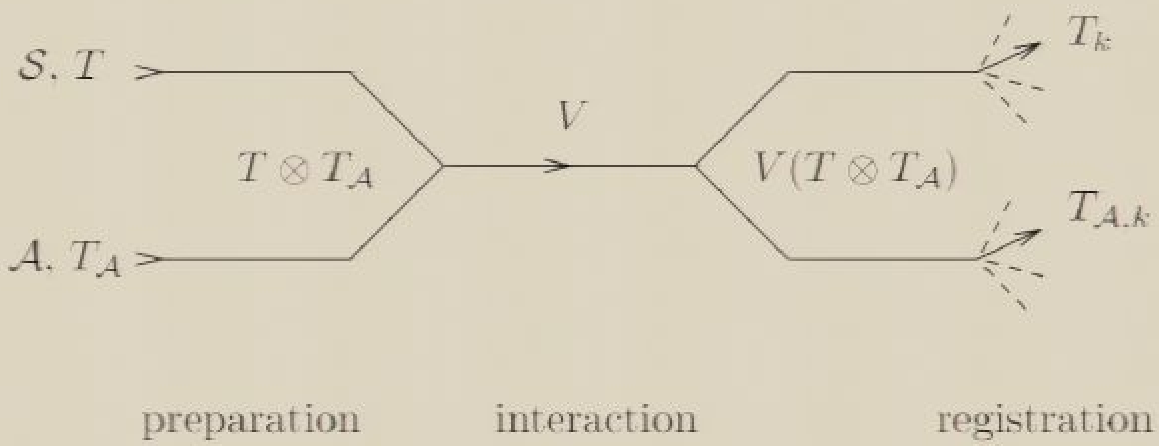
Joint work with:

Pekka Lahti
Teiko Heinonen
Jukka Kiukas
Kari Ylinen
(all at University of Turku)

Preliminaries

(The devil is in the detail.)

Quantum Measurement Theory - Basic Concepts



$\mathcal{M} = \langle \mathcal{H}_{\mathcal{A}}, T_{\mathcal{A}}, U, Z \rangle$ measurement scheme

$\mathcal{H}_{\mathcal{A}}$ = Hilbert space of apparatus (probe)

$T_{\mathcal{A}}$ = initial probe state

$U = U(t_0, t_0 + \Delta t) : \mathcal{H} \otimes \mathcal{H}_{\mathcal{A}} \to \mathcal{H} \otimes \mathcal{H}_{\mathcal{A}}$ measurement evolution (coupling)

$[V(T \otimes T_{\mathcal{A}}) = UT \otimes T_{\mathcal{A}} U^*]$

$Z : X \mapsto Z(X)$ $(X \subseteq \mathbb{R})$ pointer (output) observable measured observable – **POVM** $E : X \mapsto E(X)$:

$\mathrm{tr}\,[UT \otimes T_{\mathcal{A}}U^* \, I \otimes Z(X)] =: \mathrm{tr}\,[T\,E(X)] =: \mathsf{p}_T^E(X)$

state change (“disturbance”)
– **instrument** $X \mapsto \mathcal{I}(X)$, $\mathcal{I}(X) : T \mapsto \mathcal{I}(X)(T) = T_X$

$\mathrm{tr}\,[UT \otimes T_{\mathcal{A}}U^* \, B \otimes Z(X)] =: \mathrm{tr}\,[\mathcal{I}(X)(T)\,B] = \mathrm{tr}\,[T_X B]$

$\mathrm{tr}\,[\mathcal{I}(X)(T)] = \mathrm{tr}\,[T_X] = \mathrm{tr}\,[T\,E(X)]$

THEOREM: (Neumark, Stinespring, Ludwig, Kraus, Davies, Ozawa)
(equivalence class of) $\mathcal{M} \longleftrightarrow$ induced cp $\mathcal{I}$
(equivalence class of (cp)) $\mathcal{I} \longleftrightarrow E$

“No Information Gain Without Disturbance”

THEOREM: No measurement without state change

$\mathcal{I}(\mathbb{R})(T) = T$ (for all T) $\Rightarrow$ measured observable E is trivial

$T = P[\varphi] \mapsto \mathcal{I}(\mathbb{R})(P[\varphi]) = \mathcal{I}(X)(P[\varphi]) + \mathcal{I}(\mathbb{R} \setminus X)(P[\varphi]) = P[\varphi]$
$\Rightarrow \ \mathcal{I}(X)(P[\varphi]) = \lambda(X) P[\varphi]$

linearity of $\mathcal{I}(X) \ \Rightarrow \ \lambda(X)$ independent of φ

measured observable E:

$\mathsf{p}_\varphi^E(X) = \mathrm{tr}\,[\mathcal{I}(X)(P[\varphi])] = \lambda(X)$ – independent of φ

$E(X) = \lambda(X)\, I$ – trivial POVM

no state change $\Rightarrow$ no information gain

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no state change $\Rightarrow$ no information gain

THEOREM:

No measurement without (some transient) entanglement

Let $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ be a non-entangling unitary mapping such that for all vectors $\varphi \in \mathcal{H}_1$, $\phi \in \mathcal{H}_2$, the image of $\mathcal{H}_1 \otimes \mathcal{H}_2$ under U is of the form $U(\varphi \otimes \phi) = \varphi' \otimes \phi'$. Then U is of one of the following types:

(a) $U = V \otimes W$ where $V : \mathcal{H}_1 \to \mathcal{H}_1$ and $W : \mathcal{H}_2 \to \mathcal{H}_2$ are unitary;

(b) $U(\varphi \otimes \phi) = V_{21}\phi \otimes W_{12}\varphi$, where $V_{21} : \mathcal{H}_2 \to \mathcal{H}_1$ and $W_{12} : \mathcal{H}_1 \to \mathcal{H}_2$ are surjective isometries.

The latter case can only occur if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of equal dimensions. (PB, IJTP 2003)

GOAL: quantify/optimize trade-off between information gain and disturbance.

(Quantum information theory: Fuchs, Peres, ...)

Two "classic" quantum limitations of measurements:

Complementarity and Uncertainty

Complementarity Principle:

mutual exclusivity of setups for definition (preparation) and observation (measurement) of canonically conjugate quantities $(\hat{Q}, \hat{P})$

Complementarity: Formalizations

(C-1.) Preparation Complementarity

$Q_T(X) = 1 \Rightarrow 0 < P_T(Y) < 1$ (X, Y bounded intervals)

Popular variant of preparation complementarity:

(C-1a.) Value complementarity

Q sharply defined $\Rightarrow$ P uniformly distributed (and vice versa)

(C-2.) Measurement Complementarity

$Q(X) \wedge P(Y) = O$ (X, Y bounded intervals)
(no joint measurement (probability))

(C-3.) Complementarity for sequential measurements

Sharp measurement of position destroys all prior information about momentum (and vice versa).

THEOREM:

$\mathrm{tr}\left[\mathcal{I}^{\mathsf{Q}}(X)(T)\,\mathsf{P}(Y)\right] = \mathsf{p}_T(X \times Y)$ defines a joint probability distribution

marginals:

$\mathsf{p}_T(X \times \mathbb{R}) = \mathrm{tr}\,[T\mathsf{Q}(X)]$

$\mathsf{p}_T(\mathbb{R} \times Y) = \mathrm{tr}\left[T\widetilde{\mathsf{P}}(Y)\right]$

$\widetilde{\mathsf{P}}(Y)$ are functions of $\hat{Q}$!

Uncertainty Principle:

quantifying limitations of preparations and measurements...

(... and a little more ...)

Uncertainty – the half-... glass

Uncertainty

Complex of ideas:

- measurement "disturbs" the observed system
  – information trade-off
  "classical" momentum kicks; reduction of wave function; entanglement
- limitation of joint preparations and measurements
- cloud chamber tracks: approximate/fuzzy phase space trajectories
- possibility of unsharp joint preparations and measurements of $\widehat{Q}$, $\widehat{P}$

Myths and Facts about Complementarity and Uncertainty

Myth 1: Uncertainty relations are formal expression of complementarity. (Bohr 1928)

Fact: They are more than that. (See above, and Bohr 1928).

Bohr 1928:

"In the language of the relativity theory, the content of the relations (2) [the uncertainty relations] may be summarized in the statement that according to the quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum vectors associated with the individuals. This circumstance may be regarded as a simple symbolical expression for the complementary nature of the space-time description and claims of causality. At the same time, however, the general character of this relation makes it possible to a certain extent to reconcile the conservation laws with the space-time co-ordination of observations, the idea of a coincidence of well-defined events in a space-time point being replaced by that of unsharply defined individuals within finite space-time regions."

Myth2: The position-momentum uncertainty relation is due to “classical” momentum kicks.
(Heisenberg 1927)

Fact: Wrong! (Heisenberg 1927)

Myth 3: The principle of complementarity is much deeper than the uncertainty relation. (Scully, Englert, Walther 1995)

Fact: ... it ain’t that simple. (See above, and PB&C. Shilladay, forthcoming.)

Myth 4: The uncertainty principle has no bearings on possibilities or limitations of joint measurements. (Popper, Margenau, Ballentine)

Fact: That’s wrong!

Myth 5: Quantum mechanics does not allow any joint measurements of $\hat{Q}, \hat{P}$ whatsoever. (Folklore)

Fact: Wigner function 1932; von Neumann 1932: impossibility of sharp joint measurements.
Husimi 1939: positive phase space probability distribution for each quantum state
Arthurs&Kelly 1965: model of joint unsharp measurement.
Since 1970s: notion of phase space observables.
Since 1980s: quantum optical realizations.

Myth 5: Quantum mechanics does allow arbitrarily accurate joint measurements of $\hat{Q}, \hat{P}$. (Popper, Margenau, Ozawa (?))

Fact: Werner 2004, Cassinelli et al 2004: uncertainty relations are necessary for joint measurability. (Or you have a weird notion of joint measurement.)

Myth 6: The accuracy-disturbance uncertainty relation is not universally valid. (Ozawa)

Fact: That's wrong.

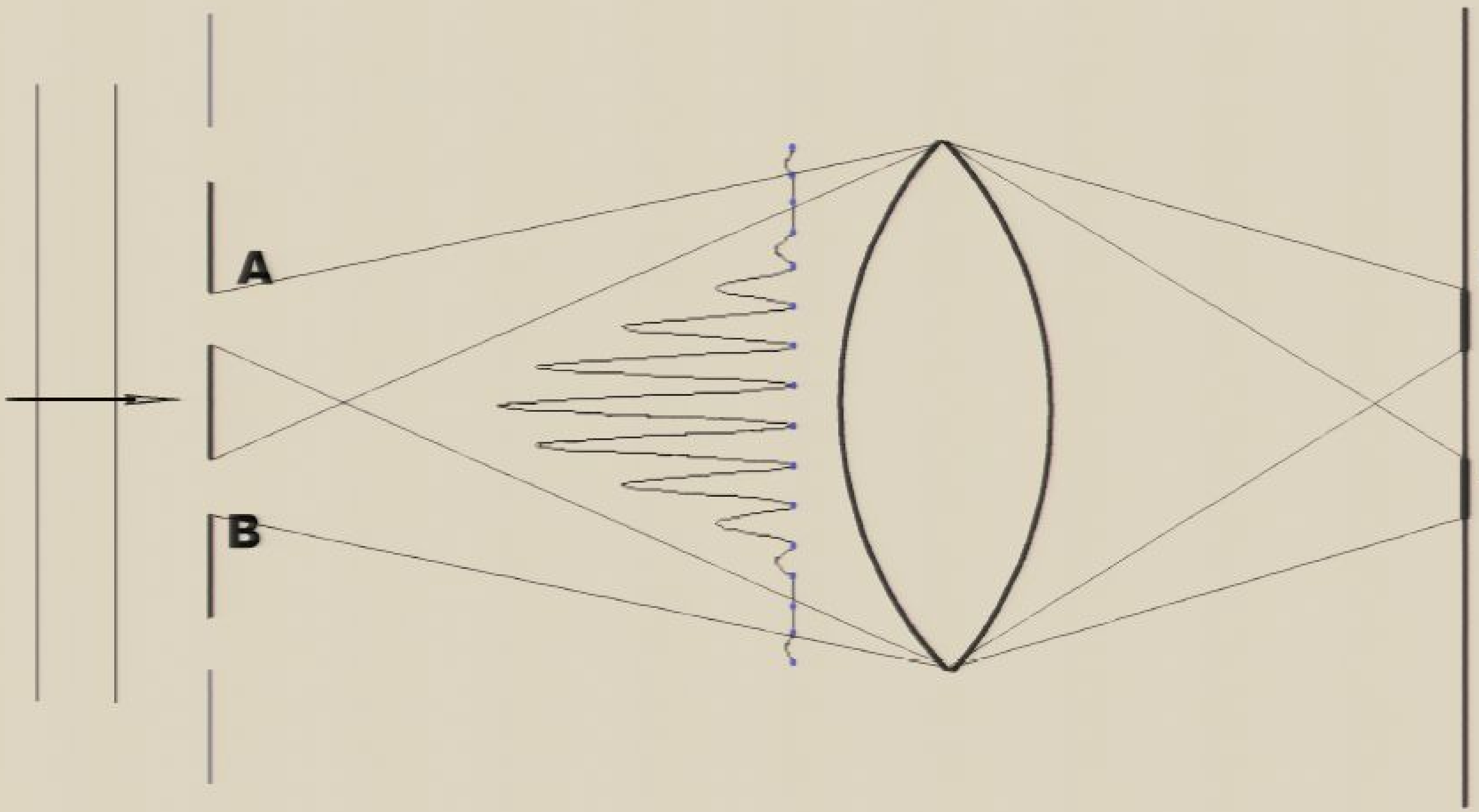
Myth 7: Bohr's complementarity principle is wrong. (Popper, Afshar)

Fact: That's wrong (as far as I can tell).

On a failed attack on the Complementarity Principle

(nice experiment, not-so-good interpretation)

Afshar's experiment



THEOREM: (PB & P Lahti, 1986)

$\mathsf{Q}(X)\mathsf{P}(Y) = \mathsf{P}(Y)\mathsf{Q}(X) \Leftrightarrow$

X, Y are periodic sets with minimal periods $\alpha, \beta > 0$ satisfying

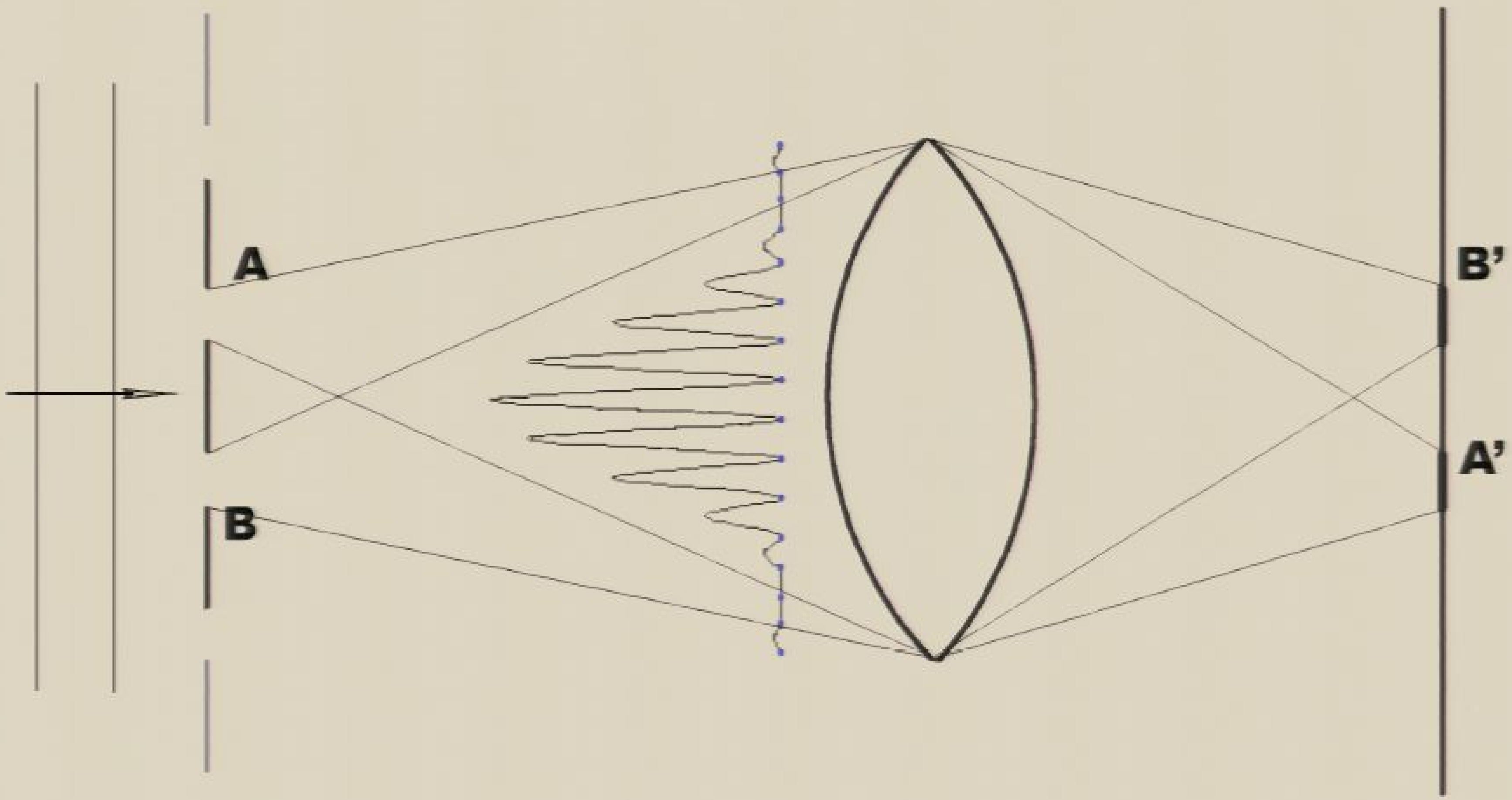
$$\frac{2\pi\hbar}{\alpha\beta} = n \in \mathbb{N}$$

$\varphi(q)$, $\tilde{\varphi}(p)$ in two-slit experiment satisfy this periodicity with $n = 1$

φ is approximate simultaneous eigenstate of the above $\mathsf{Q}(X_\alpha)$ and $\mathsf{P}(Y_\beta)$

Afshar's experiment is an approximate simultaneous measurement of $\mathsf{Q}(X_\alpha)$ and $\mathsf{P}(Y_\beta)$

Afshar's experiment



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Uncertainty principle: 3 faces

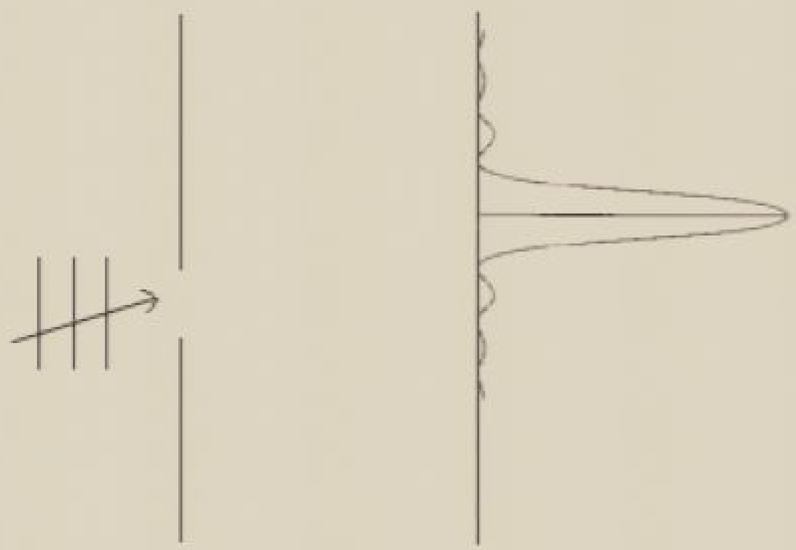
(U-1.) $\Delta_T \hat{Q} \cdot \Delta_T \hat{P} \geq \frac{\hbar}{2}$ preparation UR

(U-2.) $\delta q \cdot \delta p \geq C\hbar$ joint measurement inaccuracy trade-off

(U-3.) $\delta q \cdot D\mathsf{P} \geq C\hbar$ accuracy-disturbance trade-off

(Formalizations to follow shortly....)

EXAMPLE: single slit diffraction (Heisenberg)



$$W(|\varphi|^2) \cdot W(|\widetilde{\varphi}|^2) \geq C\hbar$$

Confirmation: neutron interferometry, atom interferometry

Case studies:

inaccuracy and disturbance in measurements

A. Some "classic" examples

1. Heisenberg: slit diffraction

2. Heisenberg: γ-microscope
   inaccuracy $\leftrightarrow$ wavelength; disturbance $\leftrightarrow$ Compton scattering

3. Heisenberg: momentum measurement via Doppler effect
   inaccuracy $\leftrightarrow$ 1/frequency, disturbance $\leftrightarrow$ ..... collapse

Note emphasis on (approximate) repeatability!

B. Sharp observables – repeatable measurements

repeatability: $\mathrm{tr}\,[\mathcal{I}(X)\,(\mathcal{I}(X)(T))] = \mathrm{tr}\,[\mathcal{I}(X)(T)]$

THEOREM: (Davies, Ozawa, Luczak)
$\mathcal{M}$ repeatable $\Rightarrow$ measured observable E is discrete.

EXAMPLE: von Neumann measurement
$\hat{A} = \sum_k a_k P_k$, $\mathcal{I}_k^N(T) = \sum_\ell P[\varphi_{k\ell}] T P[\varphi_{k\ell}]$ $(\sum_\ell P[\varphi_{k\ell}] = P_k)$

EXAMPLE: Lüders measurement
$\mathcal{I}_k^L(T) = P_k T P_k$
Consequence of ideality: $\forall k, T : \mathrm{tr}\,[T P_k] = 1 \Rightarrow \mathcal{I}_k(T) = T$
(QND property)

LÜDERS THEOREM:

$\hat{A} = \sum_k a_k P_k$, $\hat{B} = \sum_\ell b_\ell Q_\ell$:

$$\forall T, \ell \; \mathrm{tr}\left[\sum_k \mathcal{I}_k^{L,A}(T) Q_\ell\right] = \mathrm{tr}\left[T Q_\ell\right] \;\Leftrightarrow\; P_k Q_\ell = Q_\ell P_k$$

INTERPRETATION:

A Lüders measurement of A disturbs all observables B that do not commute with A.

OBSERVATION:

This extends to some pairs of POVMs (PB, J Singh 1998),
but not all! (Arias, Gheondea, Gudder, JMP 2002)

$$E_1, \ldots, E_n \geq O, \; \sum_k E_k = I: \; \mathcal{I}_k^{L,E}(T) = \mathrm{tr}\left[E_k^{1/2} T E_k^{1/2}\right]$$

Myth: All measurements are repeatable.
(Underlying the Projection Postulate)

Myth: Repeatable measurements are "bad". (E.g. unrealistic)

Fact: Repeatable measurements are good for some purposes,
bad for others.

Fact: Repeatable measurements can be pretty well approximated.

(E.g. QND, quantum Zeno)

C. Von Neumann model of unsharp position measurement

("Standard Model of Quantum Measurement Theory" ......)

$\mathcal{H} = L^2(\mathbb{R}) \qquad \mathcal{H}_{\mathcal{A}} = L^2(\mathbb{R}) \qquad T_{\mathcal{A}} = P[\phi]$

$U = \exp(-\frac{i}{\hbar}\lambda\hat{Q}\otimes\hat{P}_{\mathcal{A}}) \qquad Z = \mathsf{P}_{\mathcal{A}}$

measured observable:

$E = \mathsf{Q}_e : X \mapsto \mathsf{Q}_e(X) = \chi_X * e(\hat{Q}) = \int_{\mathbb{R}} \chi_X * e(q)\, \mathsf{Q}(dq)$

$e(q) = \lambda|\phi(-\lambda q)|^2$

instrument:

$\mathcal{I}^{\mathsf{Q}_e}(X)(T) = \int_X K_q T K_q^*\, dq \qquad (K_q\varphi)(x) = \sqrt{\lambda}\phi\,(\lambda(q-x))\,\varphi(x)$

von Neumann (1932) did not discover the POVM Q_e but noted that this is an approximately repeatable measurement of position (correlation with pointer values).

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D. Ozawa's model of a sharp position measurement

(E.g., Phys. Lett. A 299 (2002) 1-7)

$$
\begin{aligned}
U &= \exp\left[-\frac{i\pi}{3\sqrt{3}\hbar}(2\hat{Q}\otimes\hat{P}_{\mathcal{A}} - 2\hat{P}\otimes\hat{Q}_{\mathcal{A}} + \hat{Q}\hat{P} - \hat{Q}_{\mathcal{A}}\hat{P}_{\mathcal{A}})\right] \\
&= \exp\left(-\frac{i}{\hbar}\hat{Q}\otimes\hat{P}_{\mathcal{A}}\right)\exp\left(\frac{i}{\hbar}\hat{P}\otimes\hat{Q}_{\mathcal{A}}\right)
\end{aligned}
$$

$T_{\mathcal{A}} = P[\phi]$, pointer: $\mathsf{Q}_{\mathcal{A}}$

measured observable: Q (sharp position!)

instrument:

$$
\mathcal{I}(X)(P[\varphi]) = \int_X \langle\varphi|\mathsf{Q}(dq)|\varphi\rangle\, e^{-\frac{i}{\hbar}q\hat{P}}|\phi\rangle\langle\phi|e^{\frac{i}{\hbar}q\hat{P}}
$$

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$$= \exp\left(-\frac{i}{\hbar}\widehat{Q}\otimes\widehat{P}_{\mathcal{A}}\right)\exp\left(\frac{i}{\hbar}\widehat{P}\otimes\widehat{Q}_{\mathcal{A}}\right)$$

$T_{\mathcal{A}} = P[\phi]$, pointer: $\mathsf{Q}_{\mathcal{A}}$

measured observable: Q (sharp position!)

instrument:

$$\mathcal{I}(X)(P[\varphi]) = \int_X \langle\varphi|\mathsf{Q}(dq)|\varphi\rangle\, e^{-\frac{i}{\hbar}q\widehat{P}}|\phi\rangle\langle\phi|e^{\frac{i}{\hbar}q\widehat{P}}$$

Limitations due to conservation laws

THEOREM (W.A.Y.)

If a sharp observable admits a repeatable measurement then it commutes with any bounded additive conserved observable of the object-apparatus system.

Shimony and Stein:
– dropped repeatability – raised question about continuous/unbounded (conserved) quantities

EXAMPLE:

measurement of $\hat{Q}$, conserved quantity: $\hat{P} + \hat{P}_{\mathcal{A}} = \hat{P} \otimes I + I \otimes \hat{P}_{\mathcal{A}}$:

use modified von Neumann coupling $U = \exp\left(-i\frac{\lambda}{2}\left[(Q - Q_{\mathcal{A}})P_{\mathcal{A}} + P_{\mathcal{A}}(Q - Q_{\mathcal{A}})\right]\right)$

– satisfies momentum conservation

– get measured observable $\mathsf{Q}_e = e * \mathsf{Q}$, $e(q) = (e^{\lambda} - 1)\left|\phi(-(e^{\lambda} - 1)q)\right|^2$

– unsharp pointer $\mathsf{Q}_{\mathcal{A},h}$ gives measured observable Q_{e*h}

Abner: good enough?

Accuracy and Disturbance – Operational Measures

I. General Idea

accuracy

$$\left.\begin{array}{llll} T & \to & <\mathcal{M}(\mathsf{Q})>: & \mathsf{Q}_T \\ T & \to & <\mathcal{M}(\mathsf{Q}')>: & \mathsf{Q}'_T \end{array}\right\} \quad \mathsf{Q}' \leftrightarrow \mathsf{Q}$$

disturbance

$$\left.\begin{array}{ll} T & \to <\mathcal{M}(\mathsf{P})>: \quad \mathsf{P}_T \\ T & \to <\mathcal{M}(\mathsf{Q})>: T' \to <\mathcal{M}(\mathsf{P})>: \ \mathsf{Q}_T \ , \ \mathsf{P}_{T'} = \tilde{\mathsf{P}}_T \\ T & \to <\mathcal{M}(\mathsf{Q}')>: T'' \to <\mathcal{M}(\mathsf{P})>: \ \mathsf{Q}_T \ , \ \mathsf{P}_{T''} = \mathsf{P}'_T \end{array}\right\} \quad \begin{array}{l} \tilde{\mathsf{P}} \leftrightarrow \mathsf{P} \\ \mathsf{P}' \leftrightarrow \mathsf{P} \end{array}$$

$$T' = \mathcal{I}^{\mathsf{Q}}(\mathbb{R})(T), \quad T'' = \mathcal{I}^{\mathsf{Q}'}(\mathbb{R})(T)$$

Accuracy and Disturbance – Operational Measures

II. Intrinsic Noise Operator

$\Rightarrow$ unsharp joint measurement! (Davies 1976) $D(\mathsf{P}) = \delta p$

$$\delta q \cdot \delta p \geq \frac{\hbar}{2}$$

Can also do: Q_{e_0} followed by P_{f_0}

Gives joint observable for Q_e, P_f

Inaccuracy relation is automatically satisfied:

$$\Delta(e) \cdot \Delta(f) \geq \frac{\hbar}{2}$$

Equivalent to Arthurs-Kelly model.

Necessity of this uncertainty relation for joint measurability:
Werner 2004; Cassinelli et al 2004.

Ozawa model of sharp position measurement

accuracy: $N(\mathsf{Q}) = O, \qquad \delta q = 0$

recall instrument:

$$\mathcal{I}(X)(P[\varphi]) = \int_X \langle\varphi|\mathsf{Q}(dq)|\varphi\rangle \, e^{-\frac{i}{\hbar}q\hat{P}}|\phi\rangle\langle\phi|e^{\frac{i}{\hbar}q\hat{P}}$$

find "distorted" momentum:

$\mathrm{tr}\,[\mathcal{I}(\mathbb{R})\mathsf{P}(Y)] = \mathrm{tr}\left[T\widetilde{\mathsf{P}}(Y)\right] \;\Rightarrow$

$\widetilde{\mathsf{P}}(Y) = \langle\phi|\mathsf{P}(Y)|\phi\rangle\, I$ – trivial POVM

still find noise operator $N(\widetilde{\mathsf{P}}) = (\Delta_\phi \hat{P})^2\, I$

disturbance: $D(\mathsf{P}) = N(\widetilde{\mathsf{P}})$? NO! Need judicious choice.

Accuracy and Disturbance – Operational Measures

III. Ozawa's (non-operational!) measures

Ozawa's noise operator: $N^{no}(\hat{Q}) = U^* I \otimes \hat{Z} U - \hat{Q} \otimes I$

Ozawa noise: $\varepsilon^{no}(\hat{Q}) = \langle N(\hat{Q})^2 \rangle^{1/2} = \| U^* I \otimes \hat{Z} U \varphi \otimes \phi - \hat{Q} \otimes I \varphi \otimes \phi \|$

Ozawa's disturbance operator: $D^{no}(\hat{P}) = U^* \hat{P} \otimes I U - \hat{P} \otimes I$

Ozawa's disturbance: $\eta^{no}(\hat{P}) = \langle D(\hat{P})^2 \rangle^{1/2} = \| U^* \hat{P} \otimes I U \varphi \otimes \phi - \hat{P} \otimes I \varphi \otimes \phi \|$

$$\varepsilon^{no}(\hat{Q}) \cdot \eta^{no}(\hat{P}) + \varepsilon^{no}(\hat{Q}) \cdot \Delta_\varphi \hat{P} + \Delta \hat{Q} \cdot \eta^{no}(\hat{P}) \geq \frac{\hbar}{2}$$

von Neumann model: $\varepsilon^{no}(\hat{Q}) = \Delta(e), \ \eta^{no}(\hat{P}) = \Delta(f), \ \varepsilon^{no}(\hat{Q}) \cdot \eta^{no}(\hat{P}) \geq \hbar/2$

Ozawa model: $\varepsilon^{no}(\hat{Q}) = 0, \ \eta^{no}(\hat{P}) = ??? < \infty$

"no" = not operational

Accuracy and Disturbance – Operational Measures

IV. Werner's measures

distance between two POVMs E and F (both on $\mathbb{R}$):

(some definitions beforehand:)

$L(g,E) := \int_{\mathbb{R}} g(x) E(dx)$

$\Lambda = \{g : \mathbb{R} \to \mathbb{R} \,:\, g \text{ bounded}, \ |g(x) - g(y) \leq |x - y|\}$

$$d(E,F) := \sup \left\{ \|L(g,E) - L(g,F)\| \,:\, g \in \Lambda \right\}$$

sequential "phase space" measurement with marginals E, F

position measurement accuracy $\delta q = d(E, \mathsf{Q})$

momentum accuracy/disturbance $D(\mathsf{P}) = d(F, \mathsf{P})$

$$\delta q \cdot D(\mathsf{P}) \geq C\hbar$$

$C\hbar = E_0^2/4ab$ is the smallest eigenvalue (> 0!) of $a|\hat{Q}| + b|\hat{P}|$, $a, b > 0$

optimal value: $C \approx 0.3$

– applies to von Neumann model AND Ozawa model

Conclusion & Summary

Things to remember

- Why it makes sense to use T or W to denote state operators
- THE THREE FACES of the complementarity and uncertainty principles
- Caution with non-operational measures of accuracy and disturbance, and with classical intuition
- Useful operational measures of accuracy and disturbance

$$\Delta\sigma_x^2 \, \Delta\sigma_z^2 \geq \frac{1}{2}|\langle[\sigma_x,\sigma_z]\rangle|^2 + \frac{1}{2}\langle\{\sigma_x,\sigma_z\}\rangle^2$$

$$\Longleftrightarrow \quad \Delta\sigma_x^2 + \Delta\sigma_y^2 + \Delta\sigma_z^2 \geq 1$$