Title: Indistinguishability or stochastic dependence?   Authors - D. Costantini and U. Garibaldi

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Abstract: Once again the problem of indistinguishability has been recently tackled. The question is why indistinguishability, in quantum mechanics but not in classical one, forces a changes in statistics. Or, what is able to explain the difference between classical and quantum statistics? The answer given regards the structure of their state
spaces: in the quantum case the measure is discrete whilst in the classical case it is continuous. Thus the equilibrium measure on classical phase space is continuous, whilst on Hilbert space it is discrete. Put in other words, this difference goes along the way followed for a long time, it refers to the different nature of elementary particles. Answer of this type completely obscure the probabilistic side of the question. We are able to give in fully probability terms a deduction of the equilibrium probability distribution for the elements of a finite abstract system. Specializing this distribution we reach equilibrium distributions for classical particles, bosons and fermions. Moreover we are able to deduce Gentile's parastatistics too.

# INDISTINGUISHABILITY

or

# STOCHASTIC DEPENDENCE?

D. COSTANTINI and U. GARIBALDI

$Y_1, Y_2, \dots, Y, \dots$ $\{1, \dots, g\}$

$n_j \equiv \# \{Y_i = j ; i = 1, 2, \dots, n\}$

$\underline{n} \equiv (n_1, \dots, n_g), \sum_{j=1}^{g} n_j = n$

$P(\cdot|\cdot)$ exchangeable and invariant

for $i > n$ and all $j$

$$P(Y_i = j | \underline{Y}) = P(j | \underline{n}) = P(j | n_j, n) =$$

$$= \frac{\lambda p_j + n_j}{\lambda + n}$$

$\underline{Y} \equiv Y_1 = j_1, \dots, Y_n = j_n$

$\underline{p_j} \equiv P(Y_i = j$ [illegible]

$\underline{\lambda} =$ [illegible] $, i \neq 0$

$Y_1, Y_2, \ldots, Y_n, \ldots$ $\{1, \ldots, g\}$

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$$P(Y_i = j \mid \underline{Y}) = P(j \mid \underline{n}) = P(j \mid n_j, n) =$$

$$= \frac{\lambda p_j + n_j}{\lambda + n}$$

$\underline{Y} \equiv Y_1 = j_1, \ldots, Y_n = j_n$

$p_j \equiv P(Y_i = j)$

$$\lambda = \frac{P(Y_i = j \mid Y_0 = h)}{p_j - P(Y_i = j \mid Y_0 = h)},\ j \neq h,\ i \neq 0$$

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$S$ $n$ elements and

$g$ cells $1, \dots, g$

individual description (complexion)

$\underline{Y} \equiv X_1 = j_1, \dots, X_n = j_n$, for all $i$, $j_i \in \{1, \dots, g\}$

occupation vector

$\underline{n} \equiv (n_1, \dots, n_g)$, $\sum_{j=1}^{g} n_j = n$

$\mathcal{Y}^{(g,n)}$ the set of all individual descr.

$\mathcal{N}^{(g,n)}$ the set of all occupation vectors

$\underline{C}$ (general condition).

Destruction and creation probabilities are exchangeable and invariant

$\underline{DC}$ (destruction condition).

for the destruction sequence

$D_1, D_2, \dots, D_i$, $i \leq n$, the parameters are $\lambda^D = -n$ and for all $k$, $P_k = \frac{n_k}{n}$.

$\underline{CC}$ (creation condition).

for the creation sequence

$C_1, C_2, \dots, C_i$, $i \leq n$, the parameters are

$$\lambda^C = \lambda + n$$

and for all $j$, $P_j = \frac{\lambda g^{-1} + n_j}{\lambda + n}$

C (general condition).

Destruction and creation probabilities are exchangeable and invariant

DC (destruction condition).

for the destruction sequence $D_1, D_2, \dots, D_i$, $i \leq n$, the parameters are $\lambda^D = -n$ and for all $k$, $p_k = \dfrac{n_k}{n}$.

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$$\lambda^C = \lambda + n$$

and for all $j$, $p_j = \dfrac{\lambda g^{-1} + n_j}{\lambda + n}$

$$n_{kl} \equiv (n_1, \ldots, n_k - 1, \ldots, n_l - 1, \ldots, n_g)$$

$$n^{mo}_{kl} \equiv (n_1, \ldots, n_k - 1, \ldots, n_l - 1, \ldots, n_m + 1, \ldots, n_o + 1, \ldots, n_g)$$

$$P(D_1 = k, D_2 = l; \underline{n}, \lambda^D) =$$

$$= \frac{n_k}{n} \cdot \frac{n_l}{n-1}$$

$$P(C_1 = m, C_2 = o; \underline{n}^{mo}_{kl}, \lambda^C) =$$

$$= \frac{\lambda g^{-1} + n_m}{\lambda + n - 2} \cdot \frac{\lambda g^{-1} + n_o}{\lambda + n - 1}$$

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$$\underline{n}_{ke} \equiv (n_1, \ldots, n_k - 1, \ldots, n_e - 1, \ldots, n_g)$$

$$\underline{n}_{ke}^{mo} \equiv (n_1, \ldots, n_k - 1, \ldots, n_e - 1, \ldots, n_m + 1, \ldots, n_o + 1, \ldots, n_g)$$

$$P(D_1 = k, D_2 = e; \underline{n}, \lambda^D) =$$

$$= \frac{n_k}{n} \cdot \frac{n_e}{n-1}$$

$$P(C_1 = m, C_2 = o; \underline{n}_{ke}^{mo}, \lambda^C) =$$

$$= \frac{\lambda g^{-1} + n_m}{\lambda + n - 2} \cdot \frac{\lambda g^{-1} + n_o}{\lambda + n - 1}$$

(7)

$\varepsilon(j)$ is the energy of the cell $j$

for the four cells involved in an elastic collision

$$\varepsilon(m) + \varepsilon(o) = \varepsilon(k) + \varepsilon(e)$$

for all $j$, $p_j = g^{-1}$

$$c \equiv \frac{g}{\lambda}$$

$$P(\underline{n}^{mo}_{ke} \mid \underline{n}) = P(\underline{n}_{ke} \mid \underline{n})\, P(\underline{n}^{mo}_{ke} \mid \underline{n}_{ke}) =$$

$$= A^{mo}_{ke}(\underline{n})\, n_k n_e (1 + c n_m)(1 + c n_o)$$

$$P(\underline{n} \mid \underline{n}^{mo}_{ke}) = P(\underline{n}_{ke} \mid \underline{n}^{mo}_{ke})\, P(\underline{n} \mid \underline{n}_{ke}) =$$

$$= A^{ke}_{mo}(\underline{n})\,(n_m + 1)(n_o + 1)(1 + c(n_k - 1))(1 + c(n_e - 1))$$

$$A^{mo}_{ke}(\underline{n}) = A^{ke}_{mo}(\underline{n})$$

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$$= A^{mo}_{ke}(\underline{n})\, n_k\, n_e\, (1 + c\, n_m)(1 + c\, n_o)$$

$$P(\underline{n} \mid \underline{n}^{mo}_{ke}) = P(\underline{n}_{ke} \mid \underline{n}^{mo}_{ke})\, P(\underline{n} \mid \underline{n}_{ke}) =$$

$$= A^{ke}_{mo}(\underline{n})(n_m + 1)(n_o + 1)(1 + c(n_k - 1))(1 + c(n_e - 1))$$

$A^{m\ldots}_{k\ldots}($[illegible]$\,A^{ke}_{mo}(\underline{n})$

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$$P(\underline{n} \mid \underline{n}^{mo}_{ke}) = P(\underline{n}_{ke} \mid \underline{n}^{mo}_{ke})\, P(\underline{n} \mid \underline{n}_{ke}) =$$

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$$A^{mo}_{ke}(\underline{n}) = A^{ke}_{mo}(\underline{n})$$

$\varepsilon(j)$ is the energy of the cell $j$

for the four cells involved in an elastic collision

$$\varepsilon(m) + \varepsilon(o) = \varepsilon(k) + \varepsilon(\ell)$$

for all $j$, $p_j = g^{-1}$

$$c \equiv \frac{g}{\lambda}$$

$$P(\underline{n}^{mo}_{k\ell} \mid \underline{n}) = P(\underline{n}_{k\ell} \mid \underline{n})\, P(\underline{n}^{mo}_{k\ell} \mid \underline{n}_{k\ell}) =$$

$$= A^{mo}_{k\ell}(\underline{n})\, n_k\, n_\ell\, (1 + c\, n_m)(1 + c\, n_o)$$

$$P(\underline{n} \mid \underline{n}^{mo}_{k\ell}) = P(\underline{n}_{k\ell} \mid \underline{n}^{mo}_{k\ell})\, P(\underline{n} \mid \underline{n}_{k\ell}) =$$

$$= A^{k\ell}_{mo}(\underline{n})\, (n_m + 1)(n_o + 1)(1 + c(n_k - 1))(1 + c(n_\ell - 1))$$

$$A^{mo}_{k\ell}(\underline{n}) = A^{k\ell}_{mo}(\underline{n})$$

the stochastic dynamic of $S$

$$\underline{X}(0), \underline{X}(\ ), \ldots, \underline{X}(t), \underline{X}(t+1), \ldots$$

a probability distribution on $\eta^{(s,n)}$ is an equilibrium probability distribution, when

$$\lim_{t \to \infty} P\left(\underline{X}(t) = \underline{n}(t) \mid \underline{X}(0) = \underline{n}(0)\right)$$

exists whatever may be the initial distribution $\underline{n}(0)$.

Chapman - Kolmogorov eq.

$$P(\underline{X}(t+1) = \underline{n}),$$

$$= \sum_{\underline{n}'} P(\underline{X}(t+1) = \underline{n} \mid \underline{X}(t) = \underline{n}')\, P(\underline{X}(t) = \underline{n}'),\ t = 0, 1, \ldots$$

$$P(\underline{X}(t+1) = \underline{n}) - P(\underline{X}(t) = \underline{n}) =$$

$$= \sum_{\underline{n}'} P(\underline{X}(t+1) = \underline{n} \mid \underline{X}(t) = \underline{n}')\, P(\underline{X}(t) = \underline{n}') -$$

$$\sum_{n'} P(X(t+1) = \underline{n}' \mid \underline{X}(t) = \underline{n})\, P(\underline{X}(t) = \underline{n})$$

when for any pair $\underline{n}' \neq \underline{n}$, the equality

$$P(\underline{X}(t+1) = \underline{n} \mid \underline{X}(t) = \underline{n}')\, P(\underline{X}(t) = \underline{n}') =$$

$$= P(\underline{X}(t+1) = \underline{n}' \mid \underline{X}(t) = \underline{n})\, P(\underline{X}(t) = \underline{n})$$

holds, then

$$P(\underline{X}(t+1) = \underline{n}) = P(X(t) = \underline{n}) = \pi(\underline{n})$$

dynamic of S

$\ldots, \underline{X}(t), \underline{X}(t+1), \ldots$

istributi

m probability

u

$\underline{n}(0))$

e the initial

## detailed balance

ensure that the equilibrium distribution is reached when

$$\frac{\pi_c(\underline{n}_{kl}^{mo})}{\pi_c(\underline{n})} =$$

$$= \frac{n_k n_l (1 + c\, n_m)(1 + c\, n_o)}{(n_m + 1)(n_o + 1)(1 + c(n_k - 1))(1 + c(n_l - 1))} \quad (+)$$

$$\pi_c(\underline{n}) = \begin{cases} \text{const} & \text{for } \underline{n} \in E(0) \quad \text{if } c = +1 \\ \text{const} & \text{for } \underline{n} \in E(0),\ n_j = 0,1 \quad \text{if } c = -1 \\ \text{const} \times \left(\prod_{j=1}^{q} n_j!\right)^{-1} & \text{for } \underline{n} \in E(0) \quad \text{if } c = 0 \end{cases}$$

belonging to $E(0)$ means belonging to the set for which $\sum_{j=1}^{q} n_j\, \varepsilon(j) = E(0)$

$$\binom{M+d-1}{M} = \frac{(M+d-1)!}{M!\,(d-1)!}$$

for $c = +1$, $(+) = 1$

this means that the equilibrium probability distribution is uniform on $\mathcal{N}^{(q,n)}$.
This is the Bose-Einstein statistics

for $c = -1$, $(+) = 1$ if $n_k = n_\ell = 1$, $n_m = n_0 = 0$

this means that the equilibrium probability distribution is uniform on $\mathcal{N}^{(q,n)}$ if its occupation vectors satisfy Pauli's principle.
This is the Fermi-Dirac statistics

for $c = 0$, $(+) = \dfrac{n_k\, n_\ell}{(n_m+1)(n_0+1)}$

this means that the equilibrium probability distribution allots the same probability to all individual descriptions (complexions), that is it is uniform on $\mathcal{Y}^{(q,n)}$

This is the Maxwell-Boltzmann statistics

$Y_1, Y_2, \dots, Y_n, \dots$ $\quad \{1, \dots, g\}$

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$\underline{p_j} \equiv P(Y_i = j)$

$$\underline{\lambda} = \frac{P(Y_i = j \,|\, Y_0 = h)}{p_j - P(Y_i = j \,|\, Y_0 = h)}, \quad j \neq h, \ i \neq 0$$