Title: On the Separability of Physical Systems

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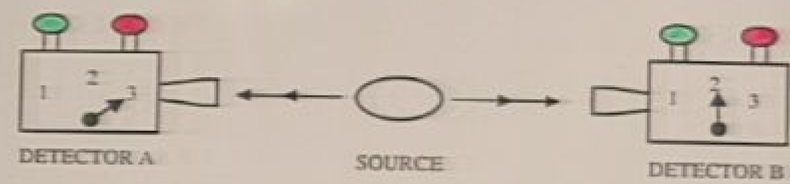
Abstract: In the context of Bell-type experiments, two related notions of "separability" are offered, one of which is logically stronger than the other. It is shown that the weaker of these is logically equivalent to the statistical independence condition widely taken to have been refuted by the results of experiments testing the Bell inequalities. Some consequences of the analysis are discussed.

THE BELL EXPERIMENTAL ARRANGEMENT (*À LA* MERMIN):



COMPOSITE SYSTEM STATES: $\lambda \in \Lambda$

DETECTOR SETTINGS: 1, 2, 3

MEASUREMENT OUTCOMES: GREEN (+ 1), RED (− 1)

JOINT PROBABILITY FUNCTIONS:

$p_\lambda^{AB}(x, y \mid i, j)$

- SWITCH SETTING AT DETECTOR B
- SWITCH SETTING AT DETECTOR A
- OUTCOME AT B
- OUTCOME AT A

THEORETICAL PREDICTIONS:

$$\mathbf{P}^{AB}(x, y \mid i, j) = \int_{\lambda \in \Lambda} \rho(\lambda) p_\lambda^{AB}(x, y \mid i, j)\, d\lambda,$$

where $\rho(\lambda)$ is the distribution of states.

MARGINAL PROBABILITY FUNCTIONS:

(A): $p_\lambda^A(x\,|\,i,j) \equiv p_\lambda^{AB}(x,+1\,|\,i,j) + p_\lambda^{AB}(x,-1\,|\,i,j)$

(B): $p_\lambda^B(y\,|\,i,j) \equiv p_\lambda^{AB}(+1,y\,|\,i,j) + p_\lambda^{AB}(-1,y\,|\,i,j)$

LOCALITY:

(A): $\forall \lambda, x, i, j, j'$,

$$p_\lambda^A(x\,|\,i,j) = p_\lambda^A(x\,|\,i,j')$$

(B): $\forall \lambda, y, i, j, i'$,

$$p_\lambda^B(y\,|\,i,j) = p_\lambda^B(y\,|\,i',j)$$

COMPLETENESS:

$\forall \lambda, x, y, i, j$,

$$p_\lambda^{AB}(x,y\,|\,i,j) = p_\lambda^A(x\,|\,i,j)\,p_\lambda^B(y\,|\,i,j)$$

[or, equivalently,

(A): $p_\lambda^A(x\,|\,i,j,y) = \dfrac{p_\lambda^{AB}(x,y\,|\,i,j)}{p_\lambda^B(y\,|\,i,j)}$, $\left(p_\lambda^B(y\,|\,i,j) \neq 0\right)$

$= p_\lambda^A(x\,|\,i,j)$

(B): $p_\lambda^B(y\,|\,i,j,x) = \dfrac{p_\lambda^{AB}(x,y\,|\,i,j)}{p_\lambda^A(x\,|\,i,j)}$, $\left(p_\lambda^A(x\,|\,i,j) \neq 0\right)$

$= p_\lambda^B(y\,|\,i,j)$]

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COMPLETENESS:

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$$= p_\lambda^A(x \mid i,j)$$

(B): $p_\lambda^B(y \mid i,j,x) = \dfrac{p_\lambda^{AB}(x,y \mid i,j)}{p_\lambda^A(x \mid i,j)}$, $(p_\lambda^A(x \mid i,j) \neq 0)$

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$$= p_\lambda^B(y\,|\,i,j)]$$

CONDITION C:

$\forall \lambda, i, j$, det C = 0, where

$$C = \begin{bmatrix} p_\lambda^{AB}(+1,+1 \mid i, j) & p_\lambda^{AB}(+1,-1 \mid i, j) \\ p_\lambda^{AB}(-1,+1 \mid i, j) & p_\lambda^{AB}(-1,-1 \mid i, j) \end{bmatrix}$$

CONDITION C $\Leftrightarrow$ COMPLETENESS

STRONG LOCALITY:

$\forall \lambda, x, y, i, j, i', j'$,

$$p_\lambda^{AB}(x, y \mid i, j) = p_\lambda^{A}(x \mid i, j') p_\lambda^{B}(y \mid i', j)$$

STRONG LOCALITY $\Leftrightarrow$ { LOCALITY & COMPLETENESS

$\Updownarrow$

CHSH INEQUALITIES

CONDITION C:

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CONDITION C $\Longleftrightarrow$ COMPLETENESS

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STRONG LOCALITY $\Longleftrightarrow \left\{ \begin{array}{c} \text{LOCALITY} \\ \& \\ \text{COMPLETENESS} \end{array} \right.$

$\Updownarrow$

CHSH INEQUALITIES

EPR - SEPARABILITY:

$\exists f_A, f_B$ such that $\forall \lambda, x, y, i, j$,

1) $\lambda_A = f_A(\lambda)$ and $\lambda_B = f_B(\lambda)$ provides a decomposition of λ into subsystem states.

2) $p_{\lambda_A}(x \mid i, j) = p_\lambda^A(x \mid i, j)$, and
$p_{\lambda_B}(y \mid i, j) = p_\lambda^B(y \mid i, j)$

3) $\overline{H}(\lambda) = \overline{H}(\lambda_A) + \overline{H}(\lambda_B)$,

where $\overline{H}(\lambda) = H(p_\lambda^{AB})$, $\overline{H}(\lambda_A) = H(p_{\lambda_A})$, $\overline{H}(\lambda_B) = H(p_{\lambda_B})$,

and $H(p) = -\sum p \ln p$, where the sum is over the values assumed by the relevant outcome variables, is the Shannon information associated with probability function p.

SEPARABILITY:

$\forall \lambda, i, j$,

$H(p_\lambda^{AB}) = H(p_\lambda^A) + H(p_\lambda^B)$

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$\exists f_A, f_B$ such that $\forall \lambda, x, y, i, j$,

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SEPARABILITY:

$\forall \lambda, i, j$,

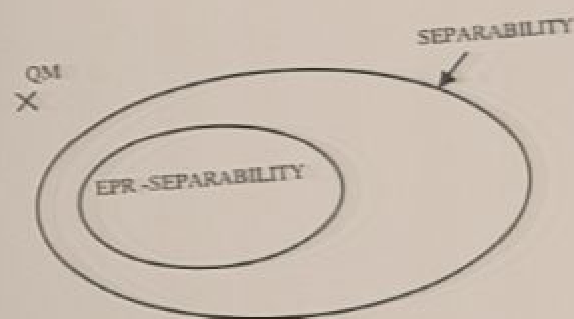
$$H(p_\lambda^{AB}) = H(p_\lambda^A) + H(p_\lambda^B)$$

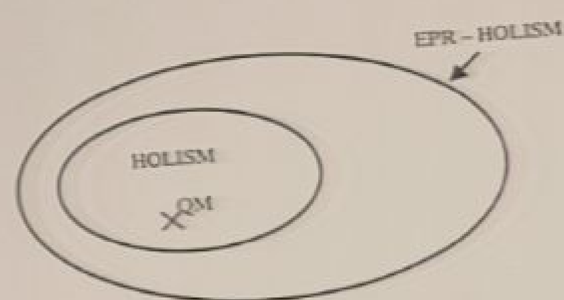
$$\text{EPR – SEPARABILTY} \underset{\not\Leftarrow}{\Longrightarrow} \text{SEPARABILITY}$$

EPR – HOLISM ≡ ~ (EPR – SEPARABILITY)

HOLISM ≡ ~ SEPARABILITY

$$\text{HOLISM} \underset{\not\Leftarrow}{\Longrightarrow} \text{EPR - HOLISM}$$





$$\text{EPR – SEPARABILTY} \underset{\not\Leftarrow}{\Rightarrow} \text{SEPARABILITY}$$

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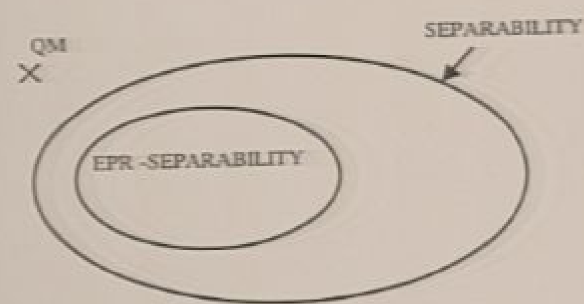

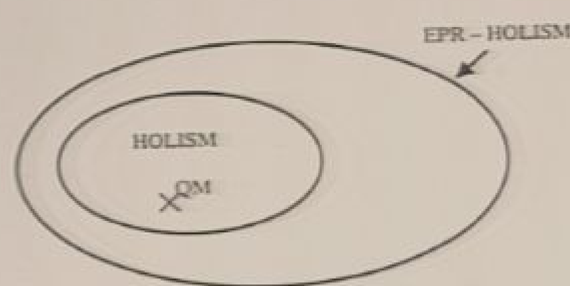

$$\text{EPR – SEPARABILTY} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{SEPARABILITY}$$

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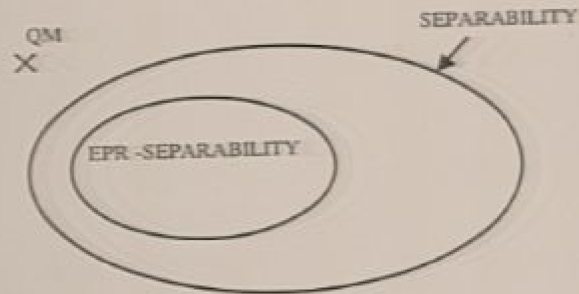

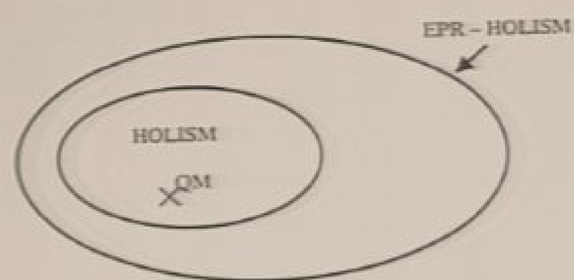

EPR – SEPARABILTY $\underset{\not\Longleftarrow}{\Longrightarrow}$ SEPARABILITY

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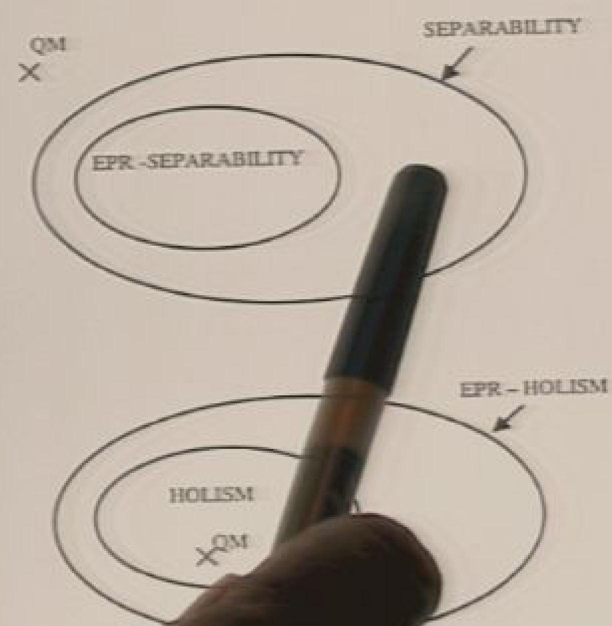
HOLISM $\underset{\not\Longleftarrow}{\Longrightarrow}$ EPR - HOLISM



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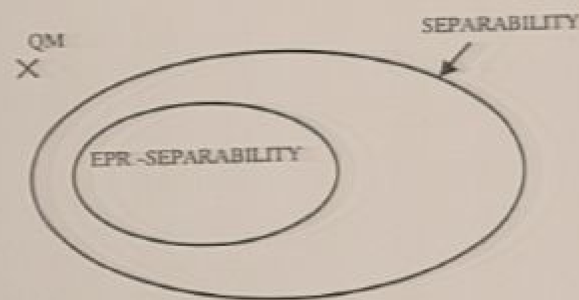

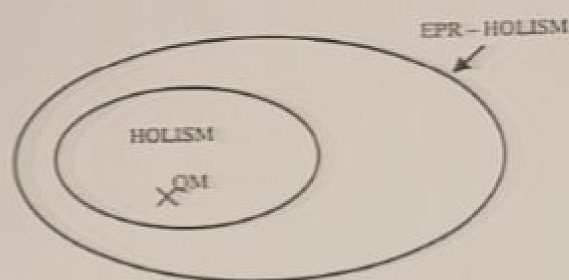

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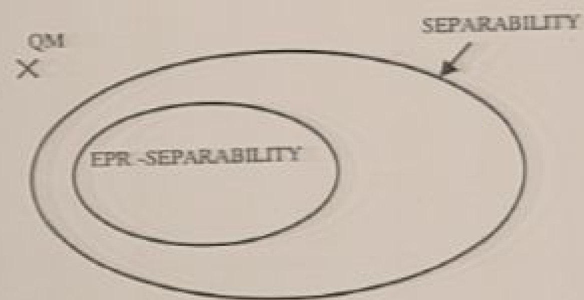

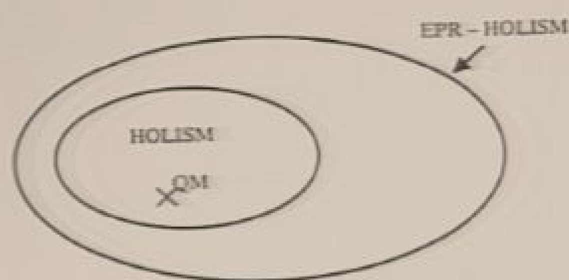

SEPARABILITY

$\Updownarrow$

(*) $\forall \lambda, i, j$,

$$c_{11}^{c_{11}} c_{12}^{c_{12}} c_{21}^{c_{21}} c_{22}^{c_{22}} = (c_{11} - \Delta)^{c_{11}} (c_{12} + \Delta)^{c_{12}} (c_{21} + \Delta)^{c_{21}} (c_{22} - \Delta)^{c_{22}},$$

where $\Delta = \det C$

$\Updownarrow$

$\forall \lambda, i, j$, $\Delta = 0$

$\Updownarrow$

CONDITION C

$\Updownarrow$

COMPLETENESS

SEPARABILITY

$\Updownarrow$

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CONDITION C

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$\forall \lambda, i, j$, $\Delta = 0$

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CONDITION C

$\Updownarrow$

COMPLETENESS

WINSBERG – FINE SEPARABILITY:

$\exists f$ such that $\forall \lambda, x, y, i, j$,

$$p_\lambda^{AB} = f\left(p_\lambda^A, p_\lambda^B\right)$$

SEPARABILITY (construed in the information-theoretic sense) holds just in case

$$f(\alpha, \beta) = \alpha\beta$$

MUTUAL INFORMATION:

$$I[X:Y]_{\lambda,i,j} = \sum_{\substack{x=\pm1 \\ y=\pm1}} p_\lambda^{AB}(x,y \mid i,j) \ln\left[\frac{p_\lambda^{AB}(x,y \mid i,j)}{p_\lambda^A(x \mid i,j)\, p_\lambda^B(y \mid i,j)}\right] = I[Y:X]_{\lambda,i,j}$$

$$= \sum_{\substack{x=\pm1 \\ y=\pm1}} p_\lambda^{AB}(x,y \mid i,j) \ln\left[\frac{p_\lambda^A(x \mid i,j,y)}{p_\lambda^A(x \mid i,j)}\right]$$

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(for $p_\lambda^A(x \mid i,j)\, p_\lambda^B(y \mid i,j) \neq 0$)

Clearly,

COMPLETENESS $\Longleftrightarrow \forall \lambda, i, j,\ I[X:Y]_{\lambda,i,j} = I[Y:X]_{\lambda,i,j} = 0$

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Clearly,

$$\text{COMPLETENESS} \Longleftrightarrow \forall \lambda, i, j,\ I[X:Y]_{\lambda,i,j} = I[Y:X]_{\lambda,i,j} = 0$$

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$$I[X;Y]_{\lambda,i,j} = \sum_{\substack{x=\pm1 \\ y=\pm1}} p_\lambda^{AB}(x,y\,|\,i,j)\ln\left[\frac{p_\lambda^{AB}(x,y\,|\,i,j)}{p_\lambda^A(x\,|\,i,j)\,p_\lambda^B(y\,|\,i,j)}\right] = I[Y;X]_{\lambda,i,j}$$

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Clearly,

COMPLETENESS $\Longleftrightarrow$ $\forall \lambda, i, j$, $I[X;Y]_{\lambda,i,j} = I[$[illegible]$ = 0$

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$$f(\alpha, \beta) = \alpha\beta$$

MUTUAL INFORMATION:

$$I[X:Y]_{\lambda,i,j} = \sum_{\substack{x=\pm 1 \\ y=\pm 1}} p_\lambda^{AB}(x, y \mid i, j) \ln\left[\frac{p_\lambda^{AB}(x, y \mid i, j)}{p_\lambda^A(x \mid i, j)\, p_\lambda^B(y \mid i, j)}\right] = I[Y:X]_{\lambda,i,j}$$

$$= \sum_{\substack{x=\pm 1 \\ y=\pm 1}} p_\lambda^{AB}(x, y \mid i, j) \ln\left[\frac{p_\lambda^A(x \mid i, j, y)}{p_\lambda^A(x \mid i, j)}\right]$$

$$= \sum_{\substack{x=\pm 1 \\ y=\pm 1}} p_\lambda^{AB}(x, y \mid i, j) \ln\left[\frac{p_\lambda^B(y \mid i, j, x)}{p_\lambda^B(y \mid i, j)}\right].$$

(for $p_\lambda^A(x \mid i, j)\, p_\lambda^B(y \mid i, j) \neq 0$)

Clearly,

COMPLETENESS $\Longleftrightarrow \forall \lambda, i, j,\ I[X:Y]_{\lambda,i,j} = I[Y:X]_{\lambda,i,j} = 0$

GENERALIZATION

Let s be the number of subsystems and n be the number of possible outcomes at each detector site.

OUTCOME VARIABLES: $x_1, x_2, \ldots, x_s$

POSSIBLE VALUES FOR VARIABLE x_k : $x_{k1}, x_{k2}, \ldots, x_{kn}$

DETECTOR SETTING VARIABLE FOR DETECTOR AT A_k : ℓ_k

JOINT PROBABILITY FUNCTIONS:

$$p_\psi^{12\ldots s}(x_1, x_2, \ldots, x_s \mid \ell_1, \ell_2, \ldots, \ell_s)$$

MARGINALS:

$$p_\psi^{k}(x_k \mid \ell_1, \ell_2, \ldots, \ell_s) = \sum_{\substack{X_j \\ j \neq k}} p_\psi^{12\ldots s}(x_1, x_2, \ldots, x_s \mid \ell_1, \ell_2, \ldots, \ell_s),$$

where $X_j \equiv (x_{j1}, x_{j2}, \ldots, x_{jn})$

and $\forall j, m, m',\ m < m' \Rightarrow x_{jm} \leq x_{jm'}$

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$$p_{\lambda}^{A_1 A_2 \ldots A_s}(x_1, x_2, \ldots, x_s \mid i_1, i_2, \ldots, i_s)$$

MARGINALS:

$$p_{\lambda}^{A_k}(x_k \mid i_1, i_2, \ldots, i_s) = \sum_{\substack{x_j \in X_j \\ j \neq k}} p_{\lambda}^{A_1 A_2 \ldots A_s}(x_1, x_2, \ldots, x_s \mid i_1, i_2, \ldots, i_s),$$

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where $X_j \equiv \{x_{j1}, x_{j2}, \ldots, x_{jn}\}$

and $\forall j, m, m',\ m < m' \Rightarrow x_{jm} \leq x_{jm'}$

COMPLETENESS:

$\forall \lambda, x_j, i_j,$

$$p_\lambda^{k_1 k_2 \ldots k_s}(x_1, x_2, \ldots, x_s \mid i_1, i_2, \ldots, i_s) = \prod_{k=1}^{s} p_\lambda^{k_k}(x_k \mid i_1, i_2, \ldots, i_s)$$

SEPARABILITY:

$\forall \lambda, i_j,$

$$H\left(p_\lambda^{k_1 k_2 \ldots k_s}\right) = \sum_{k=1}^{s} H(p_\lambda^{k_k})$$

CONDITION C:

(hyper)det $C = 0$,

where $C(\lambda; i_1, i_2, \ldots, i_s)$ is the s-dimensional (hypercubic) matrix with n elements along each side and whose general element is:

$$c_{k_1 k_2 \ldots k_s}(\lambda; i_1, i_2, \ldots, i_s) = p_\lambda^{k_1 k_2 \ldots k_s}\left(x_{1k_1}, x_{2k_2}, \ldots, x_{sk_s} \mid i_1, i_2, \ldots, i_s\right)$$

CONDITION Q:

$\forall \lambda, i_j,$

$Q = 0,$

where $Q(\lambda; i_1, i_2, \ldots, i_s)$ is the s-dimensional (hypercubic) matrix with n elements along each side and whose general element is given by:

$$q_{k_1 k_2 \ldots k_s}(\lambda; i_1, i_2, \ldots, i_s) = \sum_{\substack{k'_1 \neq k_1 \\ k'_2 \neq k_2 \\ \vdots \\ k'_s \neq k_s}} (-1)^{2\delta} (\text{hyper}) \det C[k_1, k'_1; k_2, k'_2; \ldots; k_s, k'_s],$$

where $\delta = \left| \sum_{j=1}^{s} (k_j - k'_j) \right|$ and $C[k_1, k'_1; k_2, k'_2; \ldots; k_s, k'_s]$ is the s-dimensional (hypercubic) matrix with two elements on each side produced by striking from C all but the k_j and k'_j "paths" in dimension j, for each of $j = 1, 2, \ldots, s$.

NOTE:

The generalized form of (*) is as follows:

$$\prod_{k_1,k_2,\dots,k_s} c_{k_1k_2\dots k_s}^{c_{k_1k_2\dots k_s}} = \prod_{k_1,k_2,\dots,k_s} \left(c_{k_1k_2\dots k_s} - q_{k_1k_2\dots k_s}\right)^{c_{k_1k_2\dots k_s}}$$

For $s = n = 2$,

$$q_{11} = q_{22} = \det C = -q_{12} = -q_{21}$$

and we have the following chain of equivalences:

SEPARABILITY

$\Updownarrow$

CONDITION Q

$\Updownarrow$

CONDITION C

$\Updownarrow$

COMPLETENESS,

all for $s=n=2$.

$$\text{COMPLETENESS} \equiv \begin{cases} \forall \lambda, x, y, i, j, \\ p_\lambda^{AB}(x,y|i,j) = p_\lambda^{A}(x|i,j)\, p_\lambda^{B}(y|i,j) \end{cases}$$

$$\Updownarrow$$

$\forall \lambda, i, j,$

$$\sum_{\substack{x=\pm1 \\ y=\pm1}} p_\lambda^{AB}(x,y|i,j) \ln p_\lambda^{AB}(x,y|i,j) =$$

$$\sum_{x=\pm1} p_\lambda^{A}(x|i,j) \ln p_\lambda^{A}(x|i,j) + \sum_{y=\pm1} p_\lambda^{B}(y|i,j) \ln p_\lambda^{B}(y|i,j)$$

[illegible]

$$\Updownarrow$$

$\forall \lambda, i, j$, the following pair of equations has a non-trivial solution

$$z_1 p_\lambda^{AB}(+1,+1|i,j) + z_2 p_\lambda^{AB}(+1,-1|i,j) = 0$$
$$z_1 p_\lambda^{AB}(-1,+1|i,j) + z_2 p_\lambda^{AB}(-1,-1|i,j) = 0$$

Equivalently, $\forall \lambda, i, j$, the following pair of equations has a non-trivial solution

[illegible]

Alternatively,

$$\text{CONDITION C} \Longleftrightarrow \begin{cases} \forall \lambda, i, j, \text{ there exists a zero-eigenvalue} \\ \text{solution to the equation} \\ Cz = \gamma z \end{cases}$$

$$\text{COMPLETENESS} \equiv \begin{cases} \forall \lambda, x, y, i, j, \\ p_\lambda^{AB}(x,y|i,j) = p_\lambda^{A}(x|i,j)\, p_\lambda^{B}(y|i,j) \end{cases}$$

$$\Updownarrow$$

CONDITION S

$\forall \lambda, i, j,$

$$\sum_{\substack{x=\pm1 \\ y=\pm1}} p_\lambda^{AB}(x,y|i,j) \ln p_\lambda^{AB}(x,y|i,j) =$$

$$\sum_{x=\pm1} p_\lambda^{A}(x|i,j) \ln p_\lambda^{A}(x|i,j) + \sum_{y=\pm1} p_\lambda^{B}(y|i,j) \ln p_\lambda^{B}(y|i,j)$$

CONDITION C

$$\Updownarrow$$

$\forall \lambda, i, j,$ the following pair of equations has a non-trivial solution:

[illegible]

[illegible]

Equivalently, $\forall \lambda, i, j,$ the following pair of equations has a non-trivial solution:

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CONJECTURE:

$\forall n \geq 2, \forall s \geq 2$, the following are logically equivalent:

SEPARABILITY
CONDITION Q
CONDITION C
COMPLETENESS

SEPARABILITY	$\overset{(3)}{\Longrightarrow}$	CONDITION Q
(2) $\Downarrow$ $\Uparrow$ (1)	$\underset{(4)}{\Longleftarrow}$	$\Downarrow$ (5)
COMPLETENESS		CONDITION C

(1) is trivial; it follows from properties of logarithms

(2) is non-trivial; it follows from the limiting case of the generalized Shannon-Gibbs inequality

? (3) is highly non-trivial; it can be established via a network of series expansions

(4) is relatively straightforward, just a bit messy

? (5) is highly non-trivial; it follows from properties of hyperdeterminants and a generalization of the Dodgson Condensation

? (6) CONDITION C $\underset{?}{\Rightarrow}$ CONDITION Q