Title: Geometry of Flat Spacetime

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Abstract: I will show Abner how to construct Minkowski's space-time diagrams directly from Einstein's two postulates and some very elementary plane geometry. This geometric route into special relativity was developed while teaching the subject to nonscientists, but some of its features may be unfamiliar to physicists and philosophers.

Plane geometry in (flat) spacetime

How to construct Minkowski Diagrams (1908)
directly from Einstein's postulates (1905).

Light rectangles

Einstein's Two Postulates (Voraussetzungen) (1905)

1. In electrodynamics, as well as in mechanics, no properties of phenomena correspond to the concept of absolute res⁺

Dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen.

2. Light always propagates in empty space with a definite velocity c, independent of the state of motion of the emitting body.

Sich das Licht im leeren Raume stets mit einer bestimmten, von Bewegungszustande des emittierenden Körpers unabhängigen Geschwindigkeit V fortpflanze.

#2 "only apparently incompatible" with #1
(nur scheinbar unverträgliche)

Einstein's Third Postulate (1905)

3. If a clock at A runs synchronously with clocks at B and C, then the clocks at B and C also run synchronously relative to each other.

Wenn die Uhr in A sowohl mit der Uhr in B als auch mit der Uhr in C synchron läuft, so laufen auch die Uhren in B und C synchron relativ zueinander.

3′. If event A is simultaneous with event B and event C, then events B and C are also simultaneous.

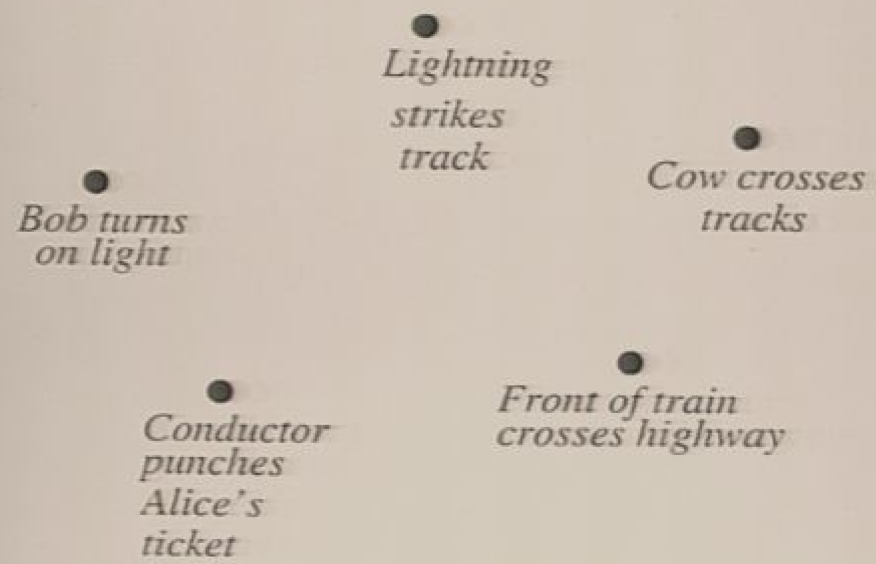
3″. If an event A happens in the same place as event B and event C, then the events B and C also happen in the same place.

An event:

Something happening at definite place and time;
A point in spacetime.

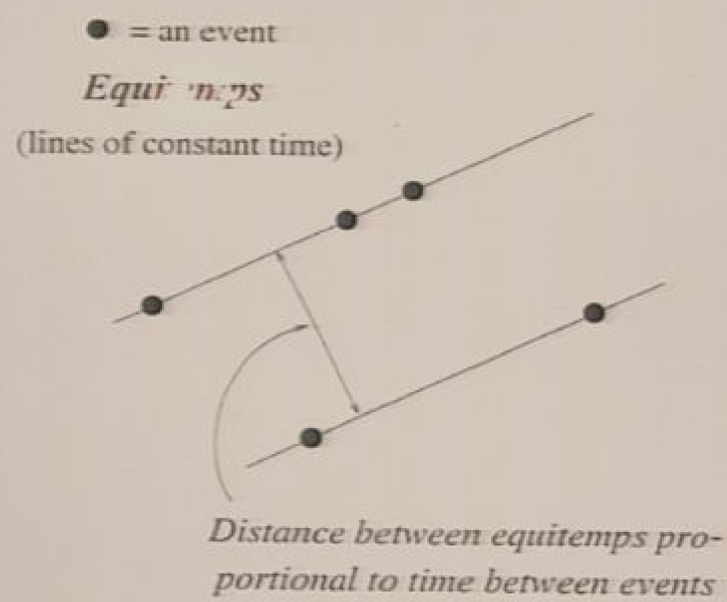
Alice's geometric description of events:

Alice makes a plane diagram depicting events at various times and places in one spatial dimension (e.g. along a long straight railroad track).



Alice organizes events in her diagram by time:

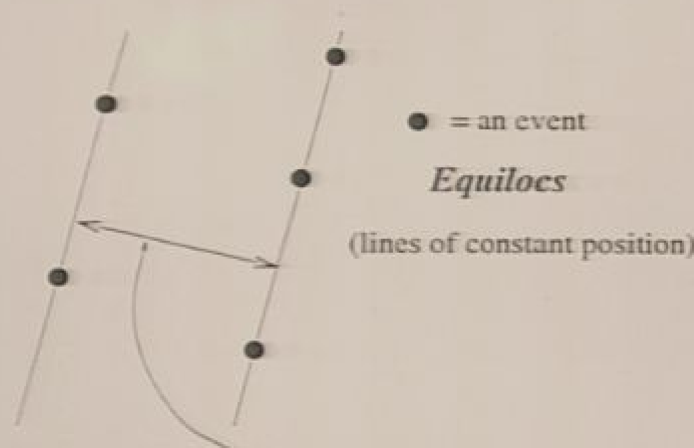
Simultaneous events placed on single straight line



Equitemps must be parallel.

Alice slides events along equitemps
to further organize them by location:

Events in same place lie on same straight line



Distance between equilocs proportional to real space distance between events

Equilocs must be parallel.
Can't be parallel to equitemps, but
otherwise orientation is arbitrary.

Alice redefines the foot:

1 conventional foot* (ft) = 0.3048 m.

1 foot** (f) = [illegible]92458 m.

1 f/ns = 299,792,458 m/s = c, speed of light.
(ns = nanosecond = 10^{-9} sec)

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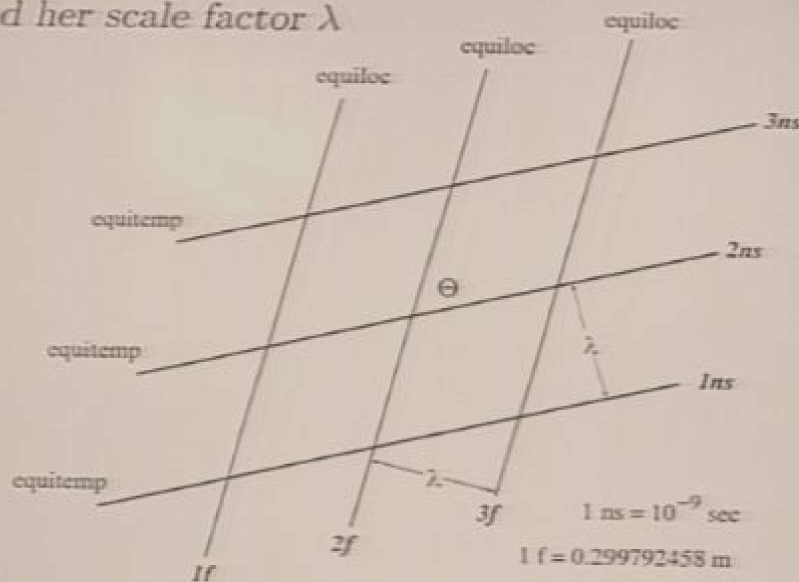
*Archaic unit still used in some backward nations.
**If you prefer, phoot (pronounced "foot").

Alice relates spatial and temporal scales:

Equilocs representing events 1 f apart
are same distance λ apart in diagram as
equitemps representing events 1 ns apart.

Some of Alice's equitemps and equilocs and her scale factor λ

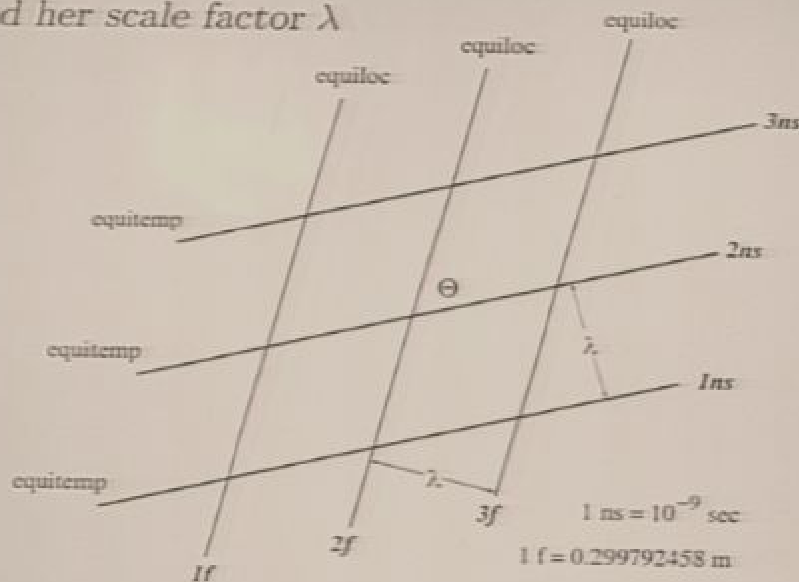


Conventional orientation:

Equilocs more vertical than horizontal;
Equitemps more horizontal than vertical;
Both symmetrically disposed about 45° lines.

Time increases with height on page

Some of Alice's equitemps and equilocs and her scale factor λ

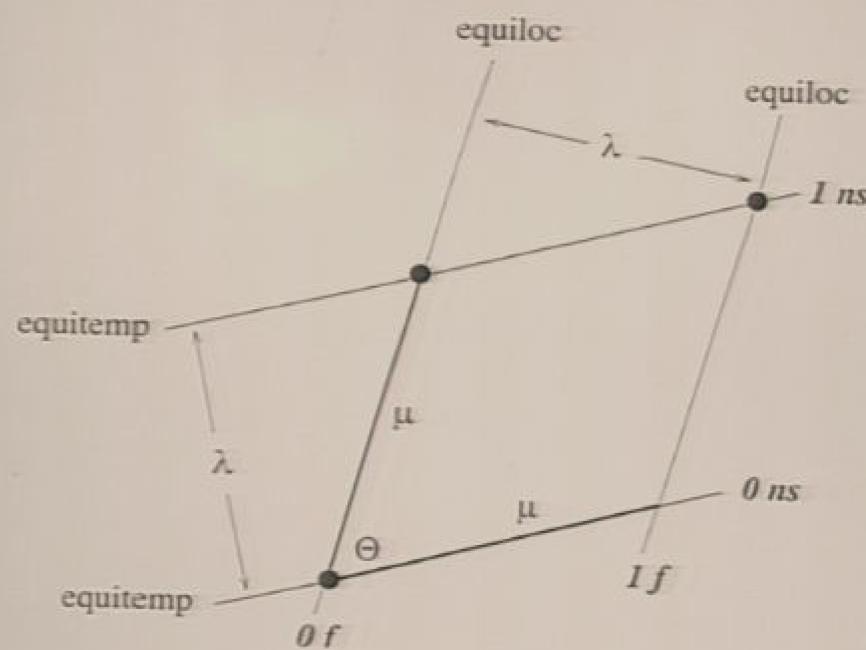


Conventional orientation:

Equilocs more vertical than horizontal;
Equitemps more horizontal than vertical;
Both symmetrically disposed about 45° lines.

Time increases with height on page

Alternative scale factor μ

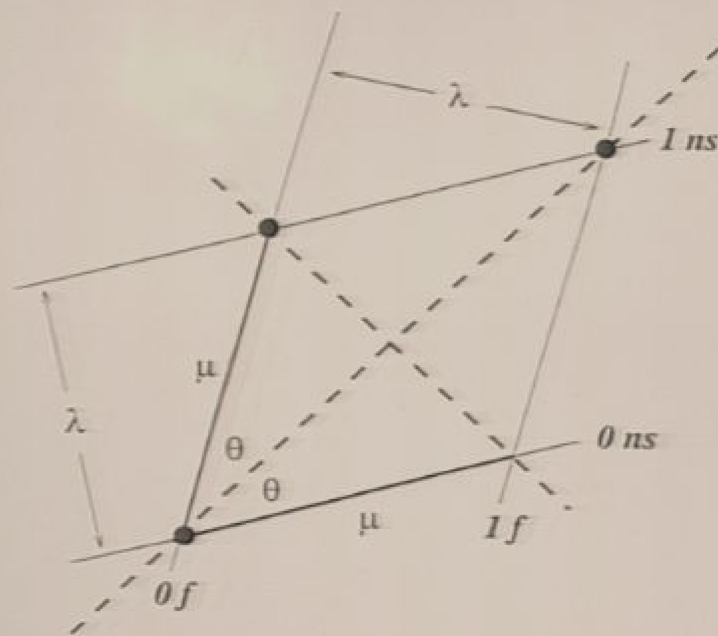


Equilocs and equitemps are characterized by two independent parameters: any two of λ, μ, Θ

Note: Area of *unit rhombus* $= \lambda\mu = \mu^2 \sin\Theta$.

Photon trajectory:

All events in the history of something moving at 1f/ns



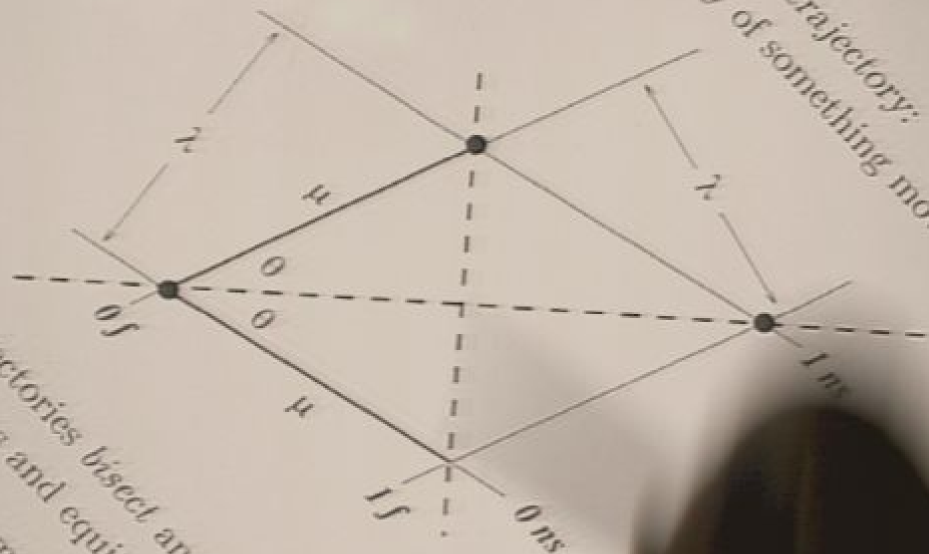
Photon trajectories *bisect* angle $\Theta = 2\theta$
between equilocs and equitemps

(Equilocs and equitemps *symmetrically disposed*
about photon trajectories)

Trajectories of oppositely moving photons
are *perpendicular.*

Photon trajectory:

...nts in the history of something moving at lig...



Photon trajectories bisect angle $\Theta = 2\theta$

between equilocs and equitemps

...uilocs and equitemps symmetrically disposed

...hoton trajectories)

...of oppositely moving photons

...r.

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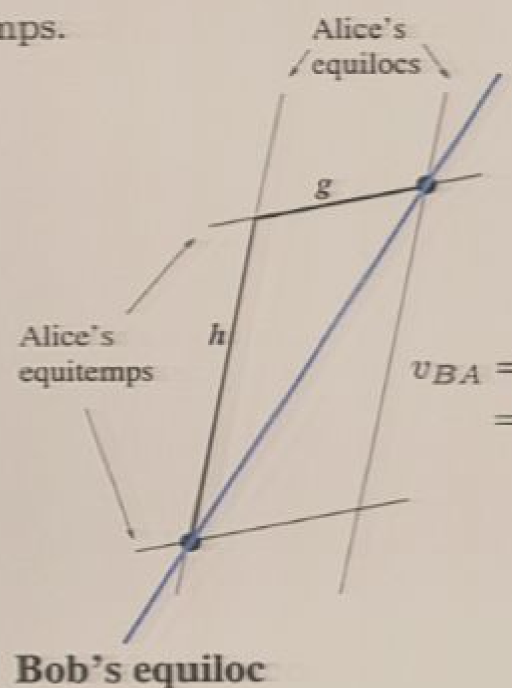
Bob's description of the same events

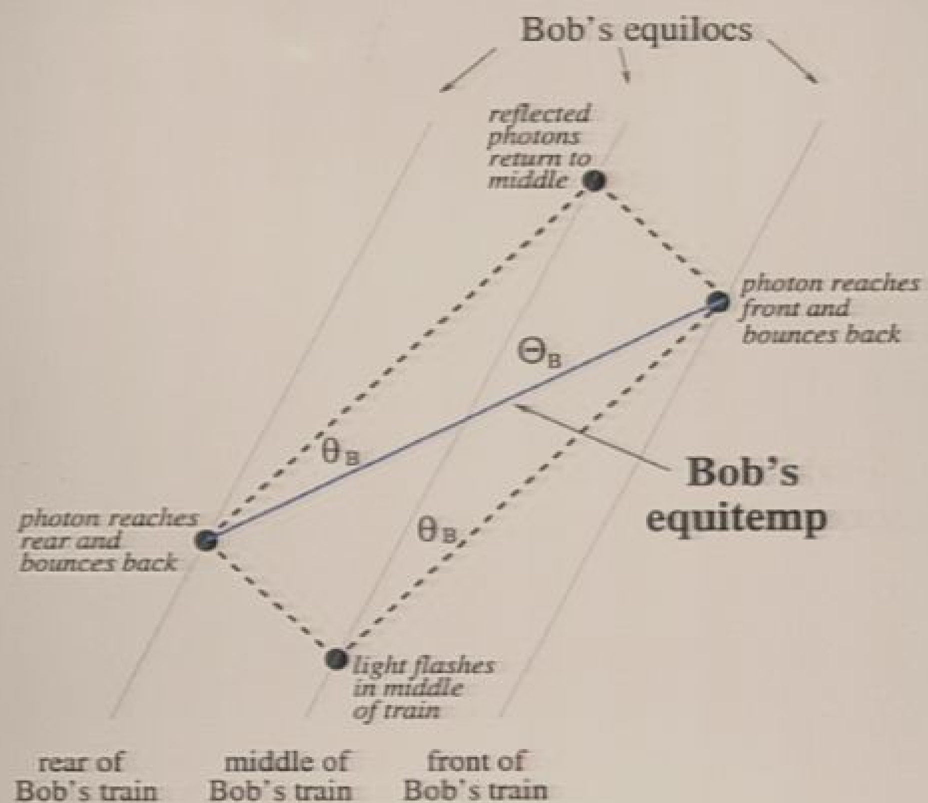
Bob moves uniformly with respect to Alice. He uses Alice's diagram to depict events, but tries to impose on it *his own* equilocs and equitemps.

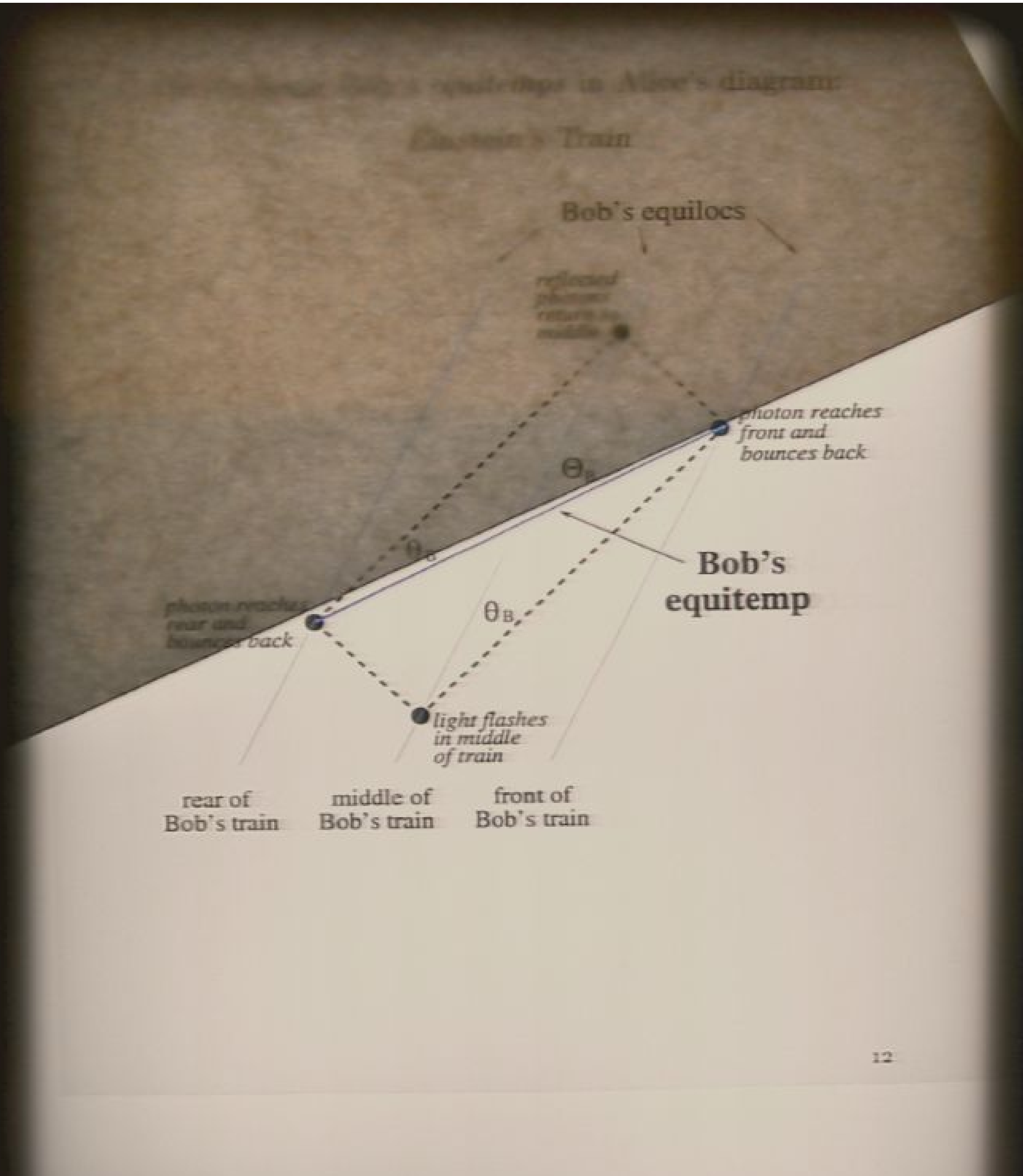


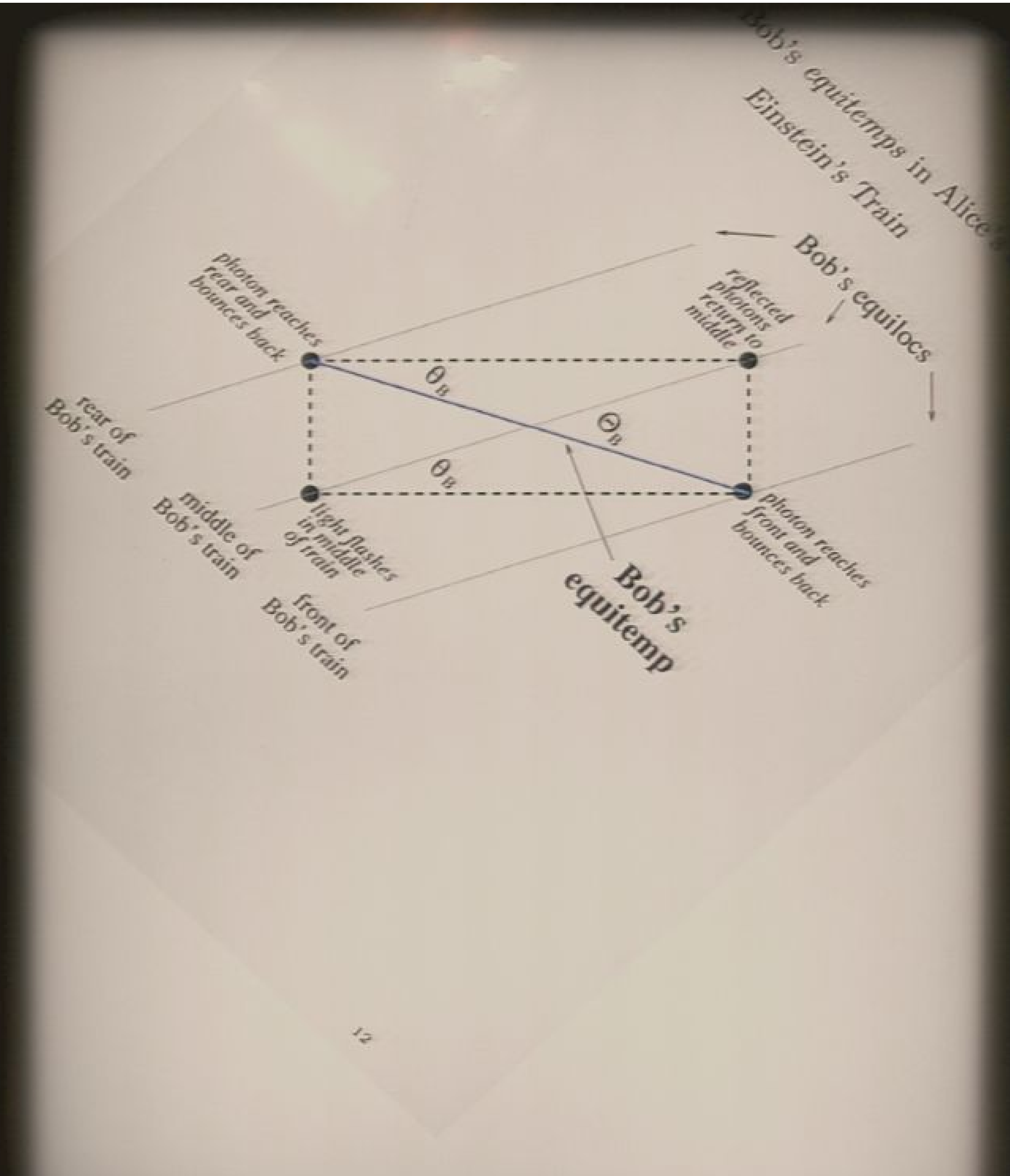
$$v_{BA} = \mu_A g / \mu_A h$$
$$= g/h$$

Determining Bob's *equitemps* in Alice's diagram:

Einstein's Train

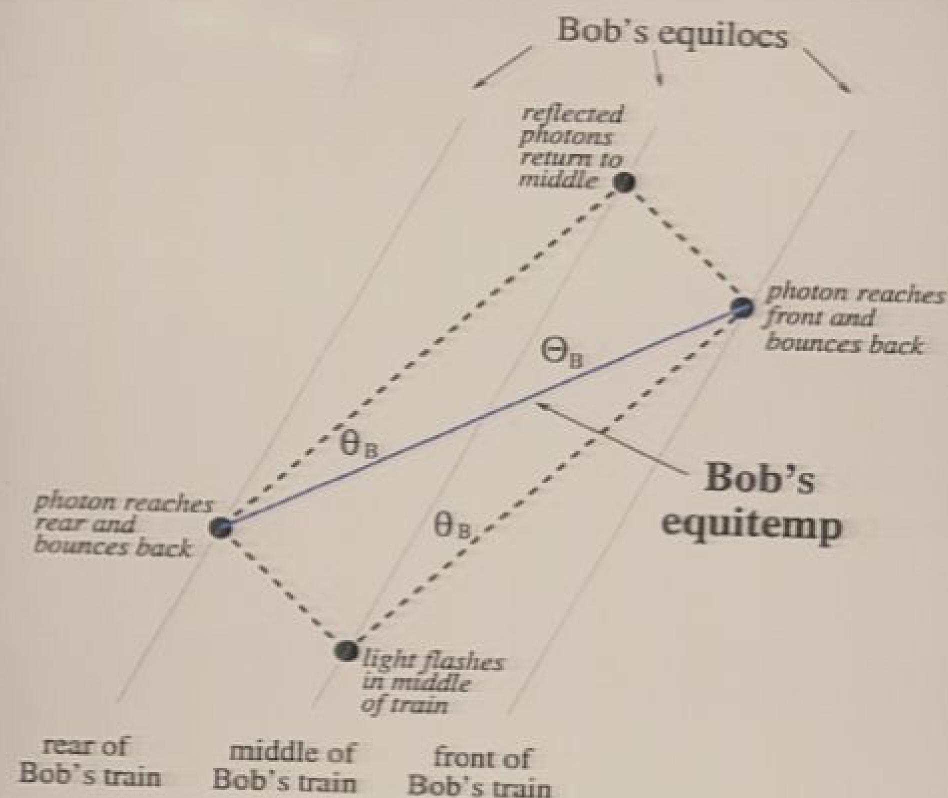



Bob's equilocs
photon reaches front and bounces back
Bob's equitemp
photon reaches rear and bounces back
light flashes in middle of train
rear of Bob's train
middle of Bob's train
front of Bob's train
12

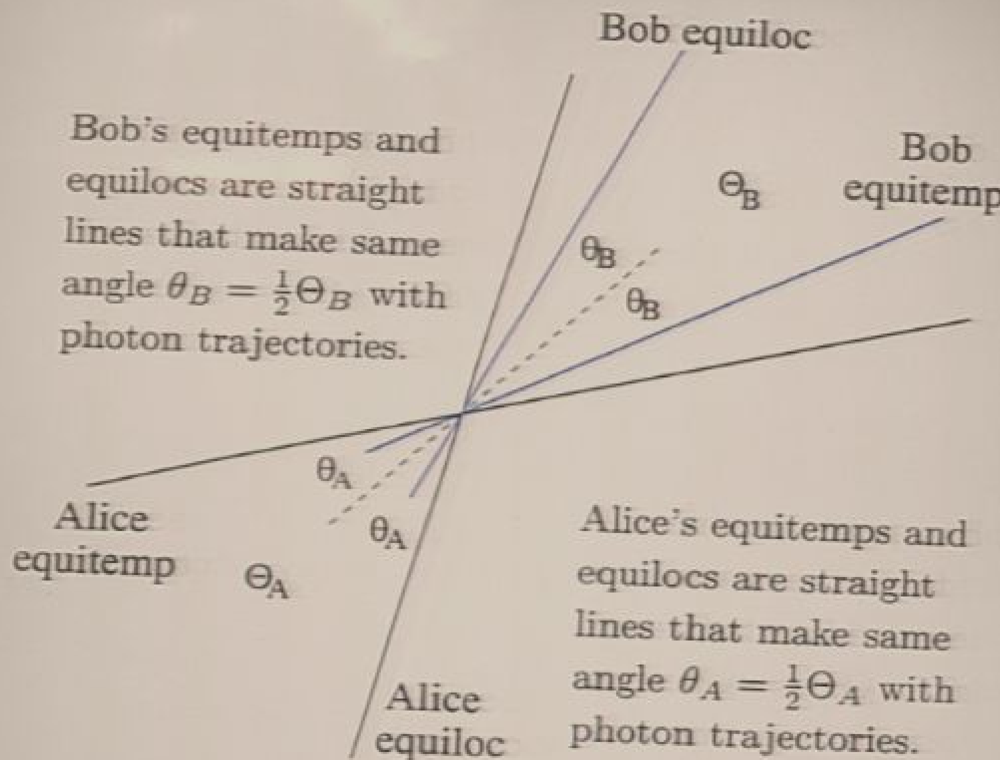

ob's equitemps in Alice's
Einstein's Train
Bob's equilocs
photon reaches rear and bounces back
reflected photons return to middle
θ_B
Θ_B
θ_B
rear of Bob's train
middle of Bob's train
front of Bob's train
light flashes in middle of train
photon reaches front and bounces back
Bob's equitemp
12

Determining Bob's *equitemps* in Alice's diagram:

Einstein's Train



Bob's equitemps and equilocs are straight lines that make same angle $\theta_B = \frac{1}{2}\Theta_B$ with photon trajectories.

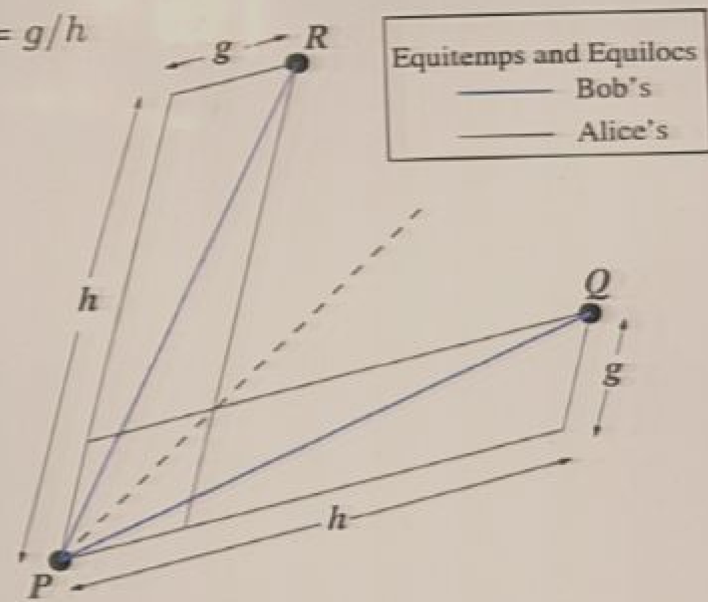


Alice's equitemps and equilocs are straight lines that make same angle $\theta_A = \frac{1}{2}\Theta_A$ with photon trajectories.

Cannot tell who made the diagram first and who later added their own equitemps and equilocs.

"Relativity of Simultaneity"

$v_{BA} = g/h$

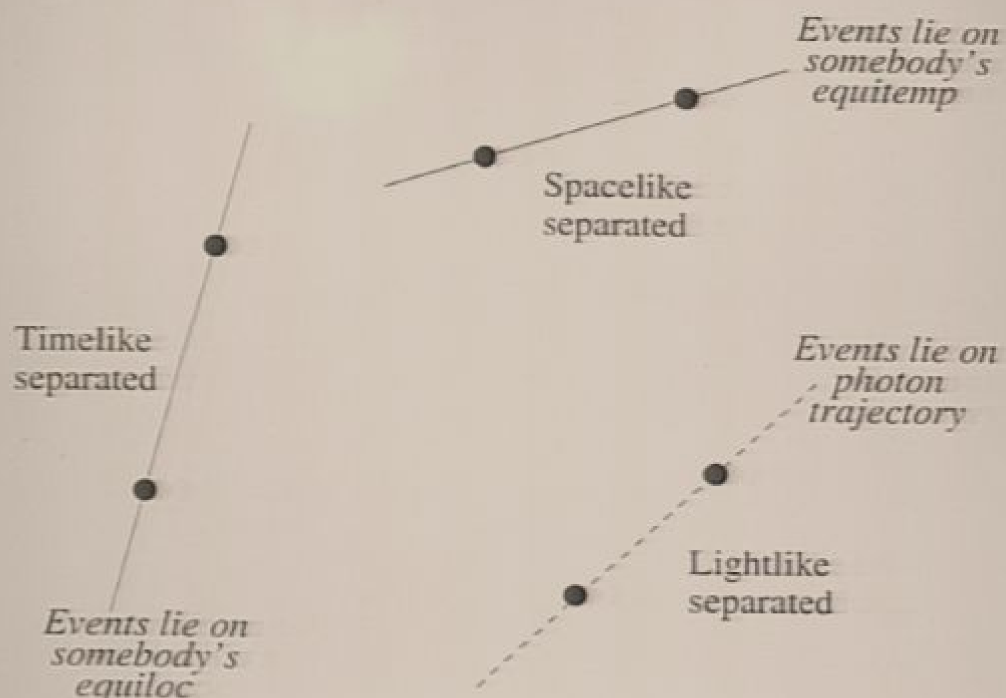


Bob: **P**, **R** at same place. **P**, **Q** at same time.

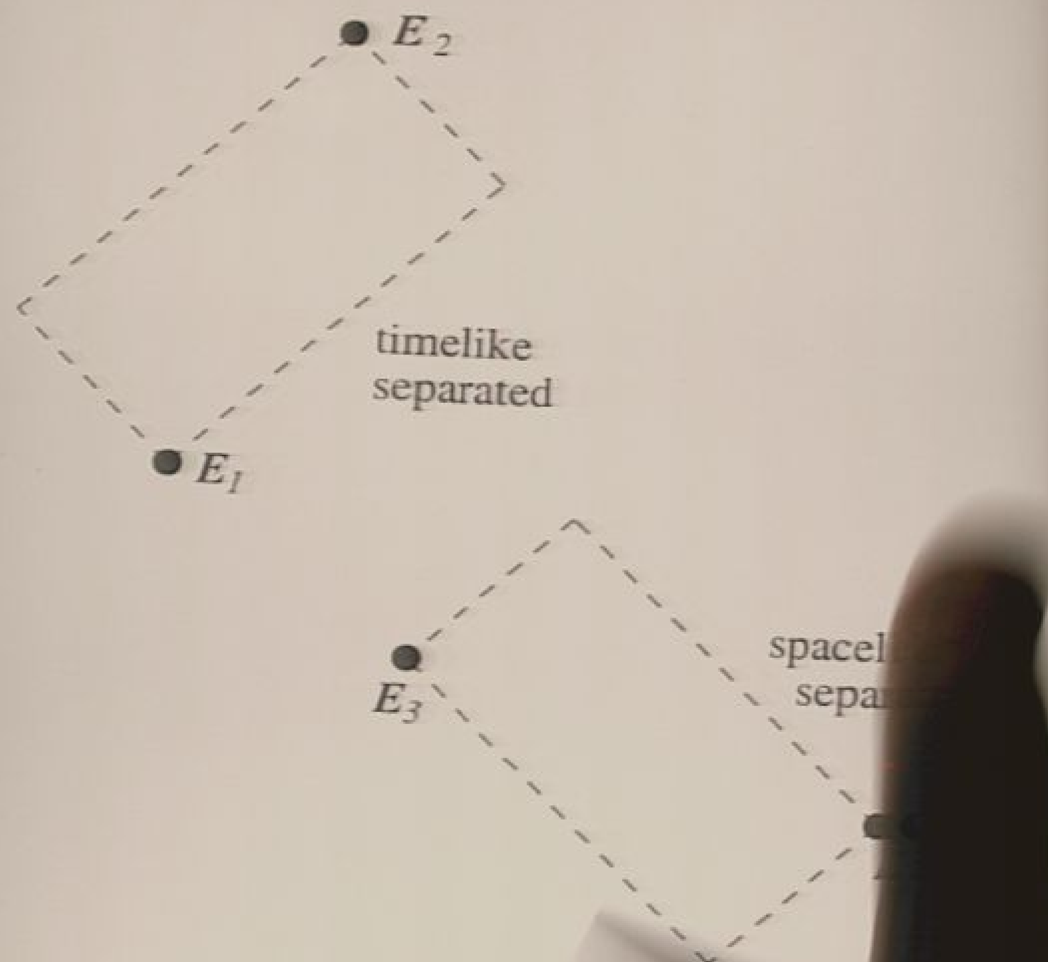
Alice: $D_{PR} = v_{BA} T_{PR}$. $T_{PQ} = v_{BA} D_{PQ}$.

$(\mu_A g)$ $(\mu_A h)$ $(\mu_A g)$ $(\mu_A h)$

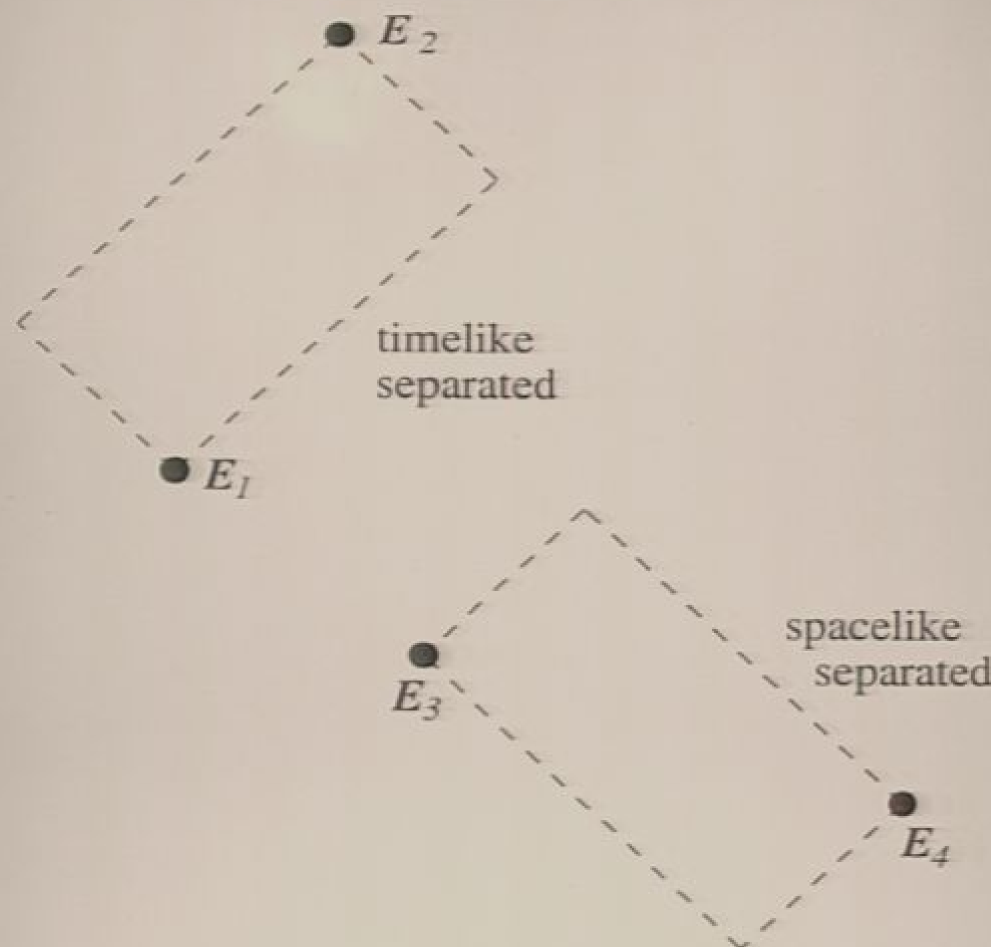
Relations between events



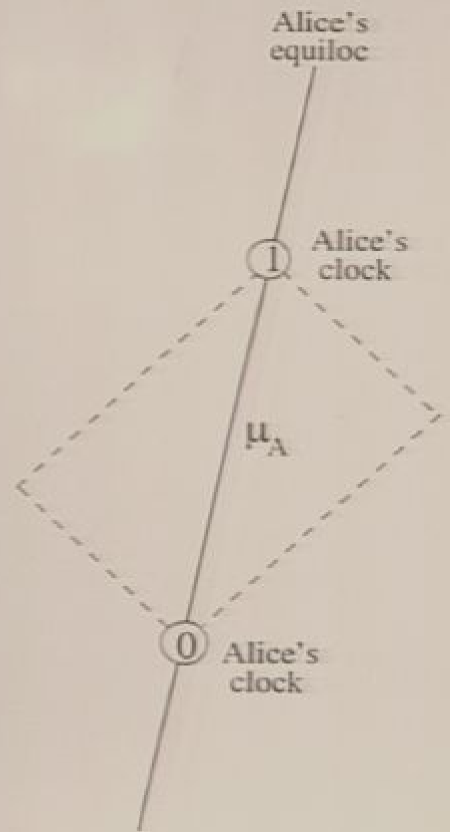
Two events determine a *light rectangle.*



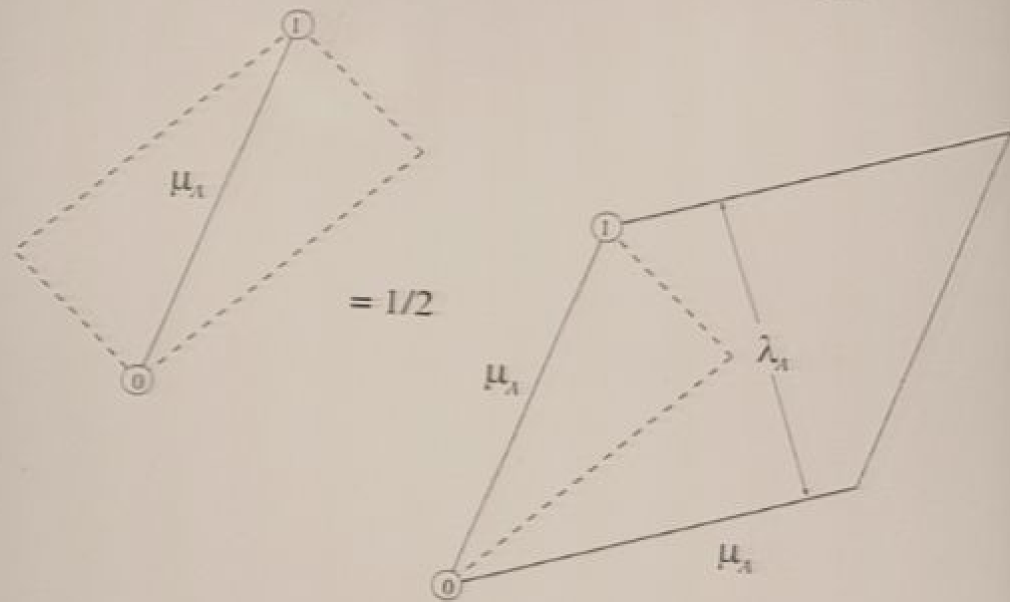
Two events determine a *light rectangle*.



Alice's unit light rectangle



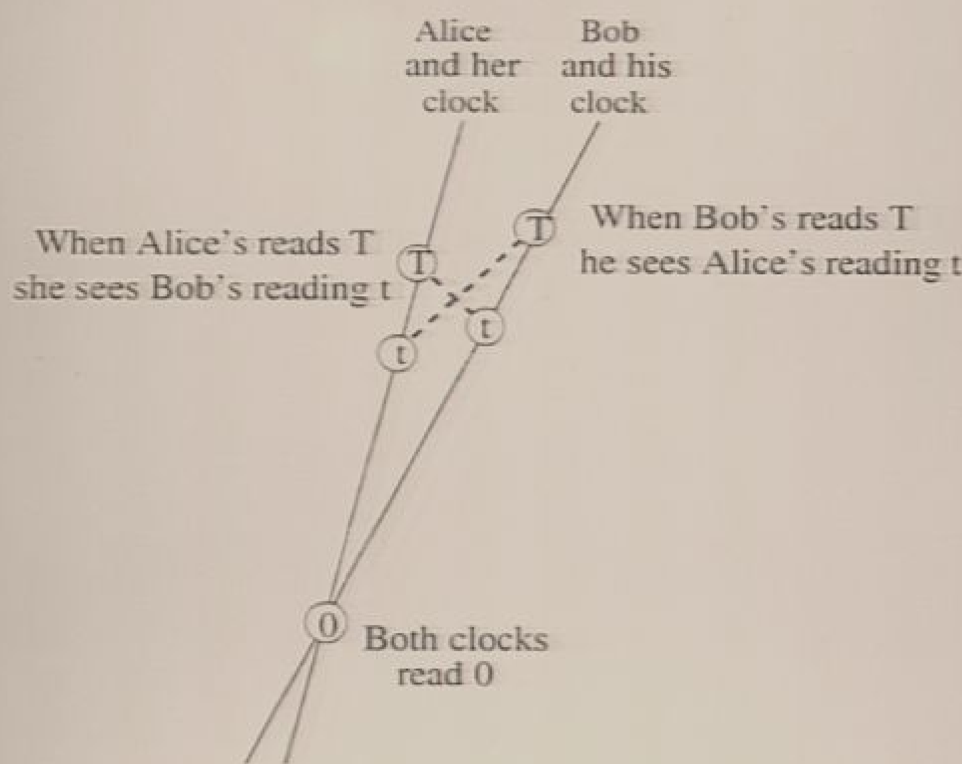
Area Ω_0 of Alice's unit light rectangle

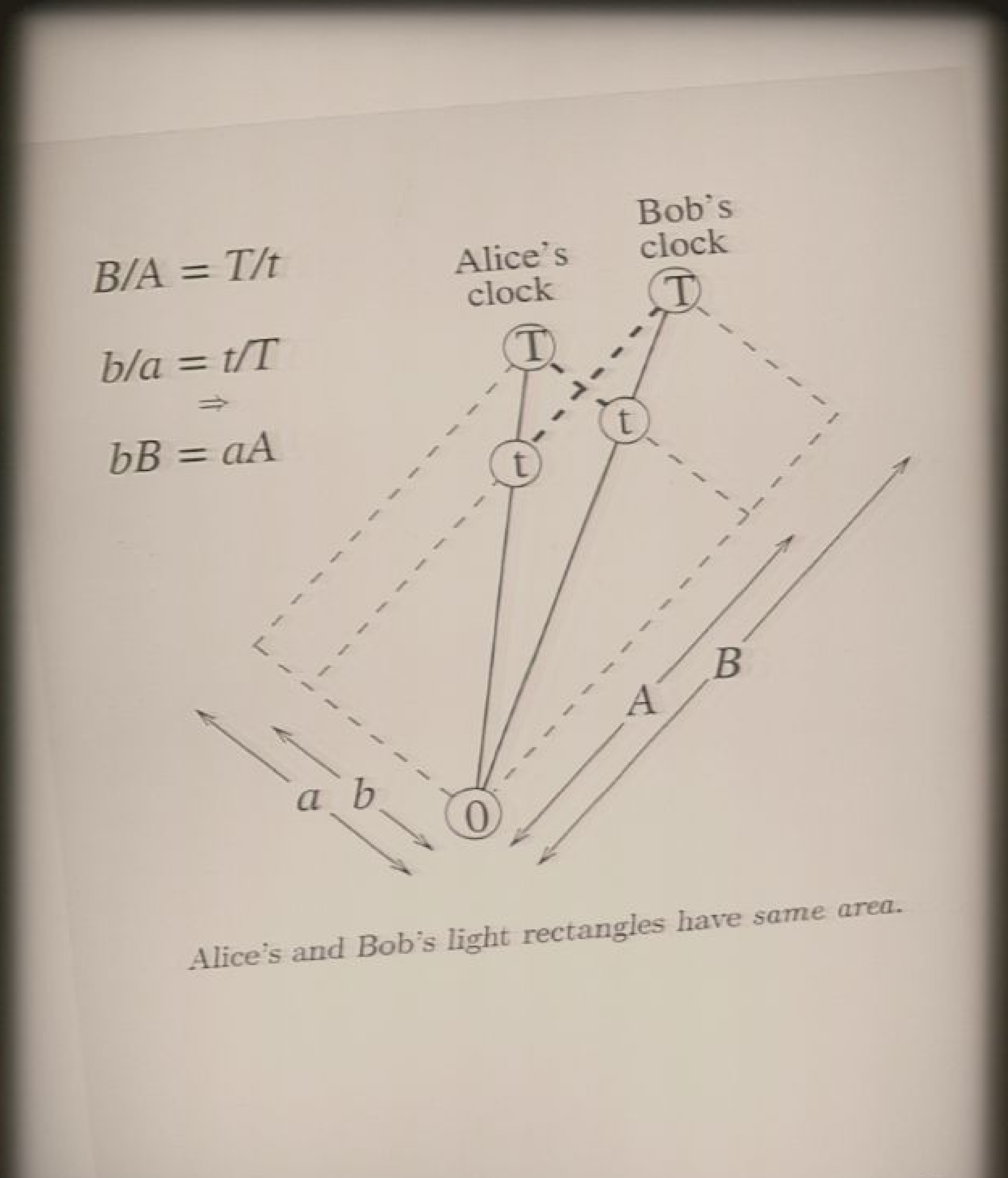


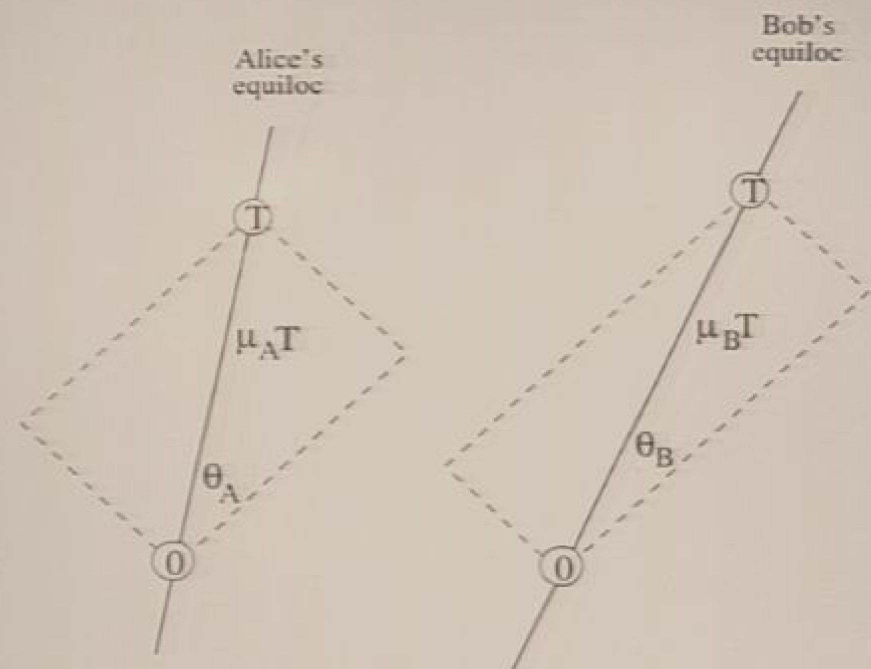
$$\Omega_0 = \frac{1}{2}\lambda_A \mu_A$$

Relation between Alice's and Bob's *scale factors*:

Reciprocity of Doppler Effect




B/A = T/t
b/a = t/T
⇒
bB = aA
Alice's clock
Bob's clock
T
t
T
t
0
a
b
A
B
Alice's and Bob's light rectangles have same area.



Light rectangles have same area.

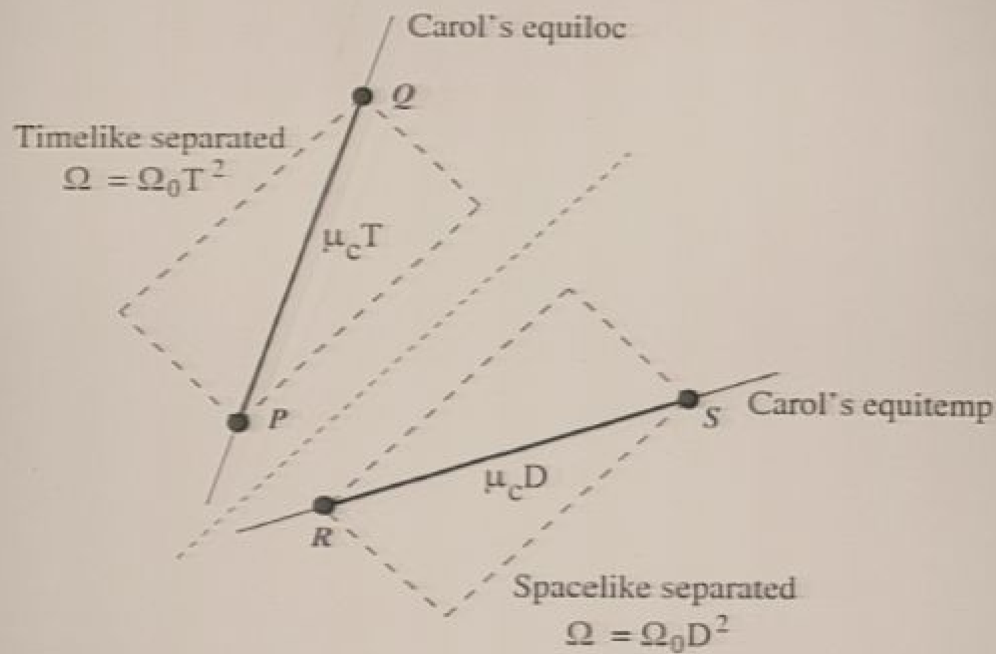
$T = 1 \Longrightarrow$ *unit light rectangles have same area*

$$\Omega_0 = \tfrac{1}{2}\mu\lambda$$

Product $\mu\lambda$ of scale factors is the same for everyone:

$$\mu_A\lambda_A = \mu_B\lambda_B = \mu_C\lambda_C = \cdots$$

Meaning of area Ω of light rectangle for *any* pair of events:



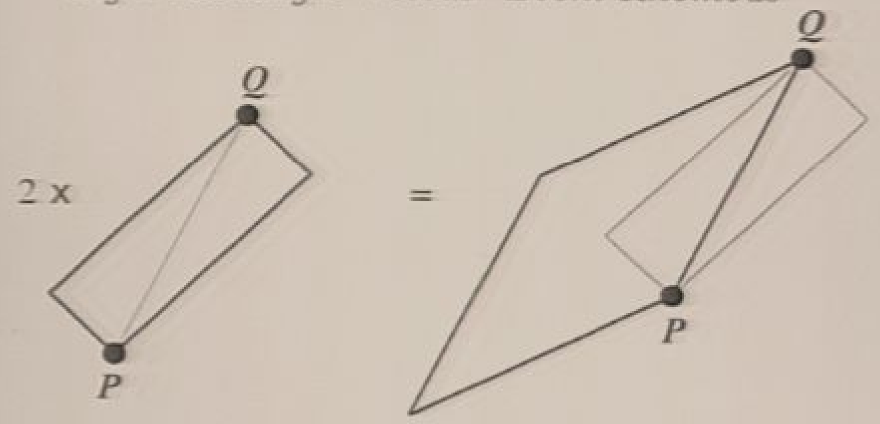
Timelike separated: Ω/Ω_0 is square of time between events in frame in which events at same place.

Spacelike separated: Ω/Ω_0 is square of distance between events in frame in which events at same time.

Ω/Ω_0 *is squared interval* I^2

What about $I^2 = |T^2 - D^2|$?

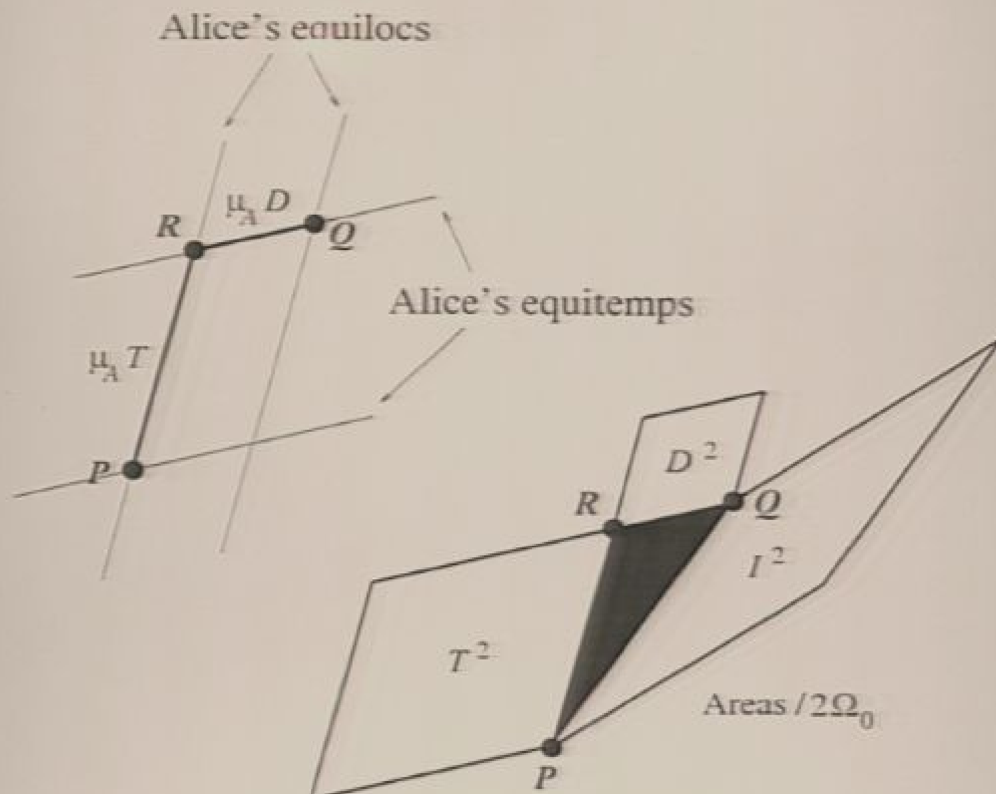
Light Rectangle versus *Event Rhombus*



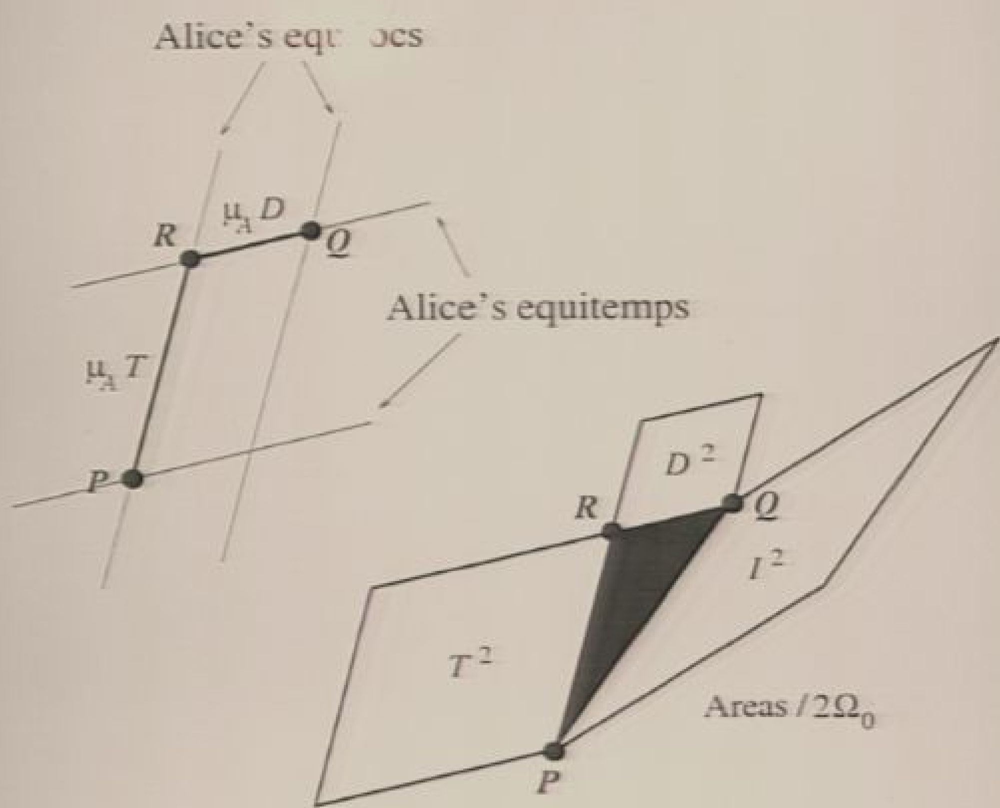
$I_{PQ}^2 = \frac{\text{Area}}{\Omega_0}$

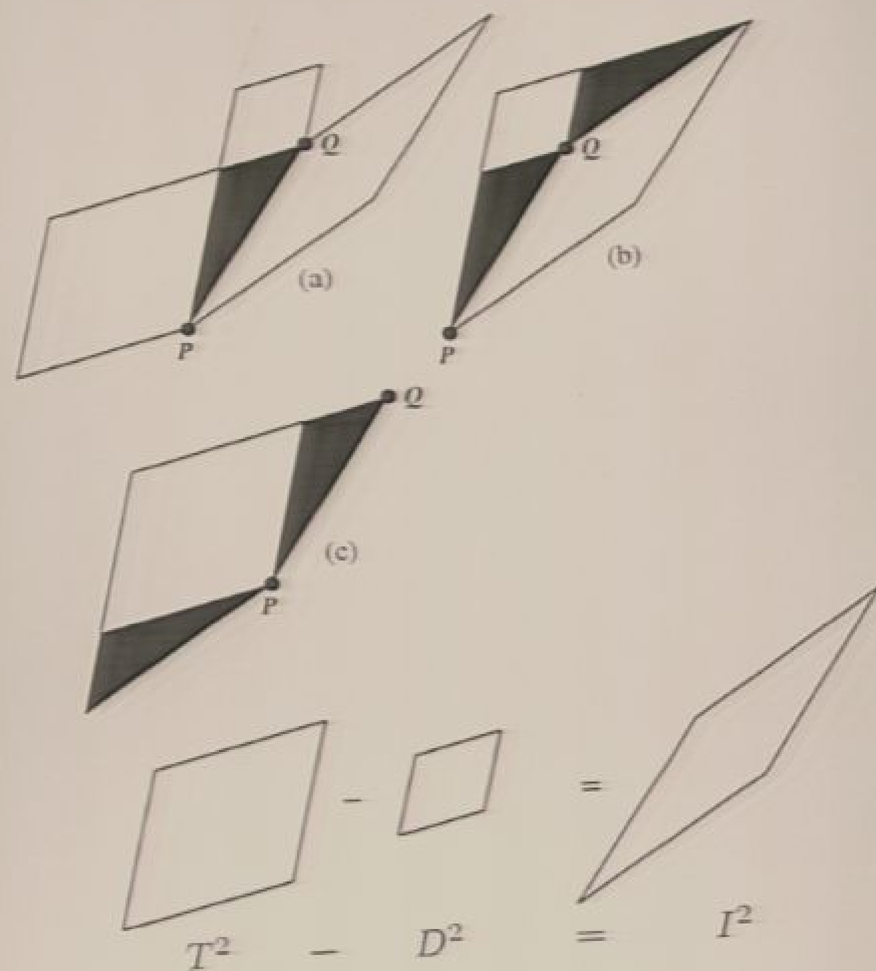
$I_{PQ}^2 = \frac{\text{Area}}{2\Omega_0}$

Interval I between events P and Q in terms of Alice's time T and distance D between them:



Interval I between events P and Q in terms of Alice's time T and distance D between them:




Q
P
(a)
Q
P
(b)
Q
P
(c)
$T^2 - D^2 = I^2$

Application (in 3+1 dimensions)

How to measure the interval between P and Q
using only light signals and a single clock:

Alice moves uniformly with her clock;
Alice and her clock are both present at P.
Bob is present at Q.

When P happens Alice's clock reads T_0.

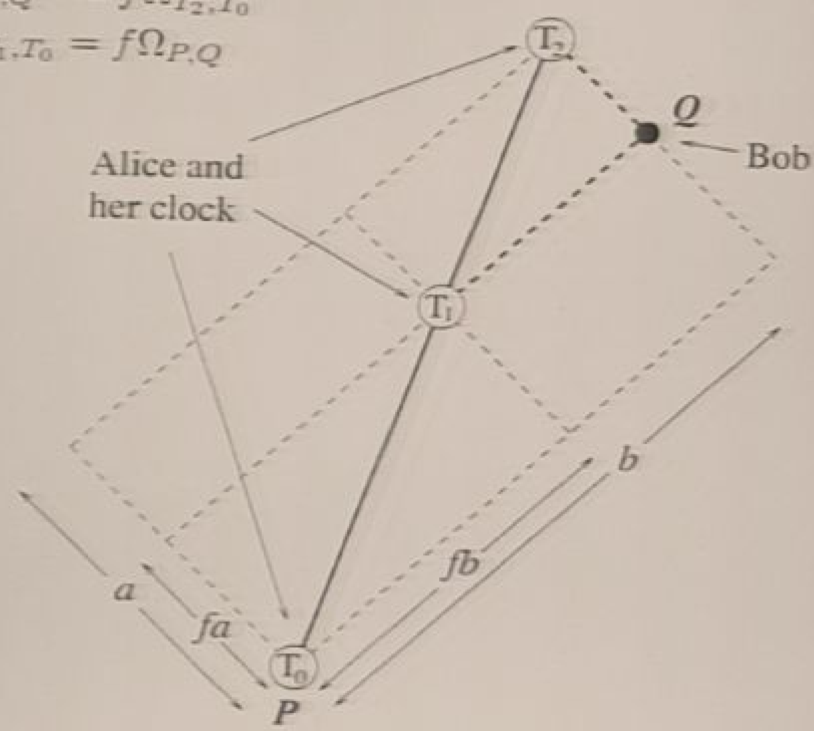
When Q happens, Bob *sees* Alice's clock reading T_1.

When Alice *sees* Q happen, her clock reads T_2.

$$I_{PQ}^2 = |(T_1 - T_0)(T_2 - T_0)|$$

P and Q timelike separated

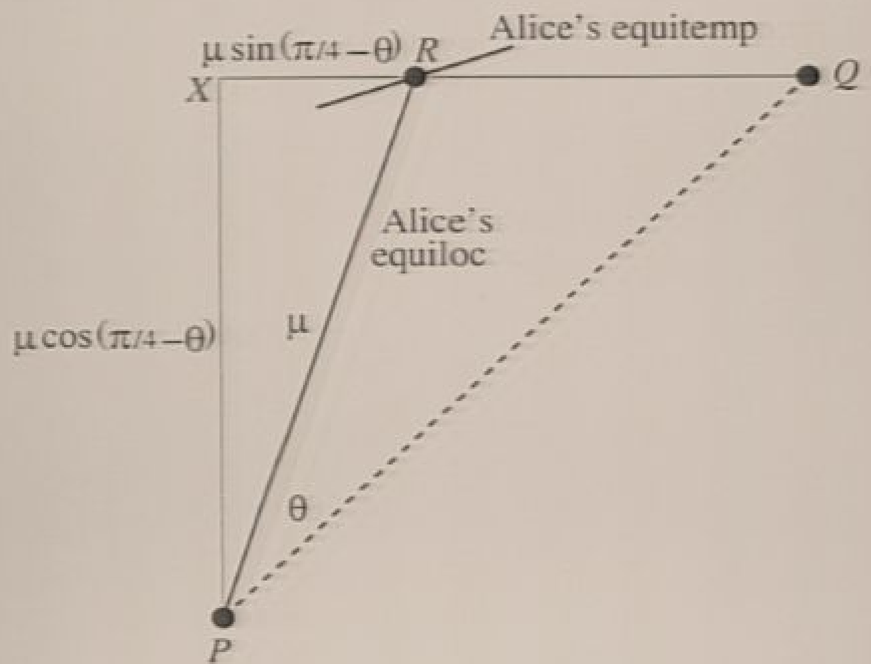
$$\Omega_{P,Q} = f\Omega_{T_2,T_0}$$
$$\Omega_{T_1,T_0} = f\Omega_{P,Q}$$



$$\Omega^2_{P,Q} = \Omega_{T_2,T_0} \; \Omega_{T_1,T_0} \implies$$
$$I^2_{P,Q} = (T_2 - T_0)(T_1 - T_0)$$

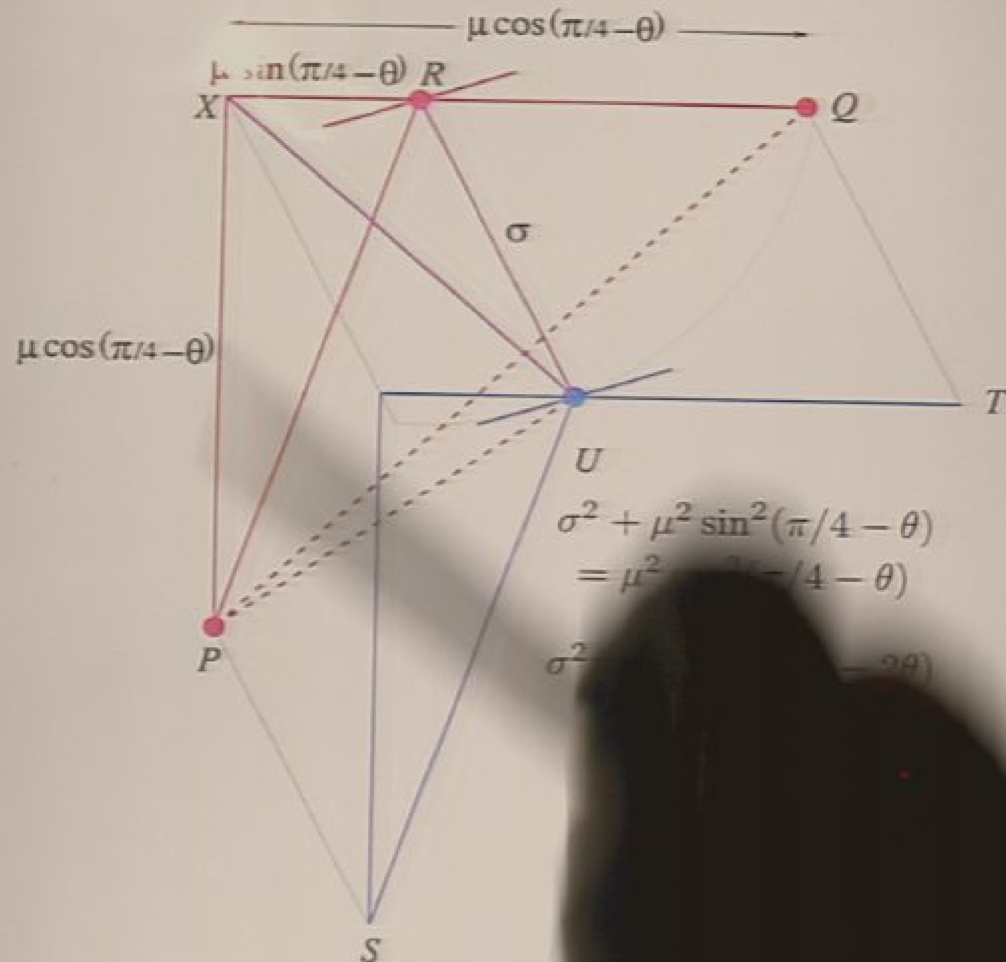
Stacking plane diagrams in orthogonal direction.

Isotropy: When Alice adds *second spatial dimension* perpendicular to plane, photon trajectories through a point would expand to right circular cone.



Sets scale factor σ for perpendicular dimension.

Determination of perpendicular scale factor σ



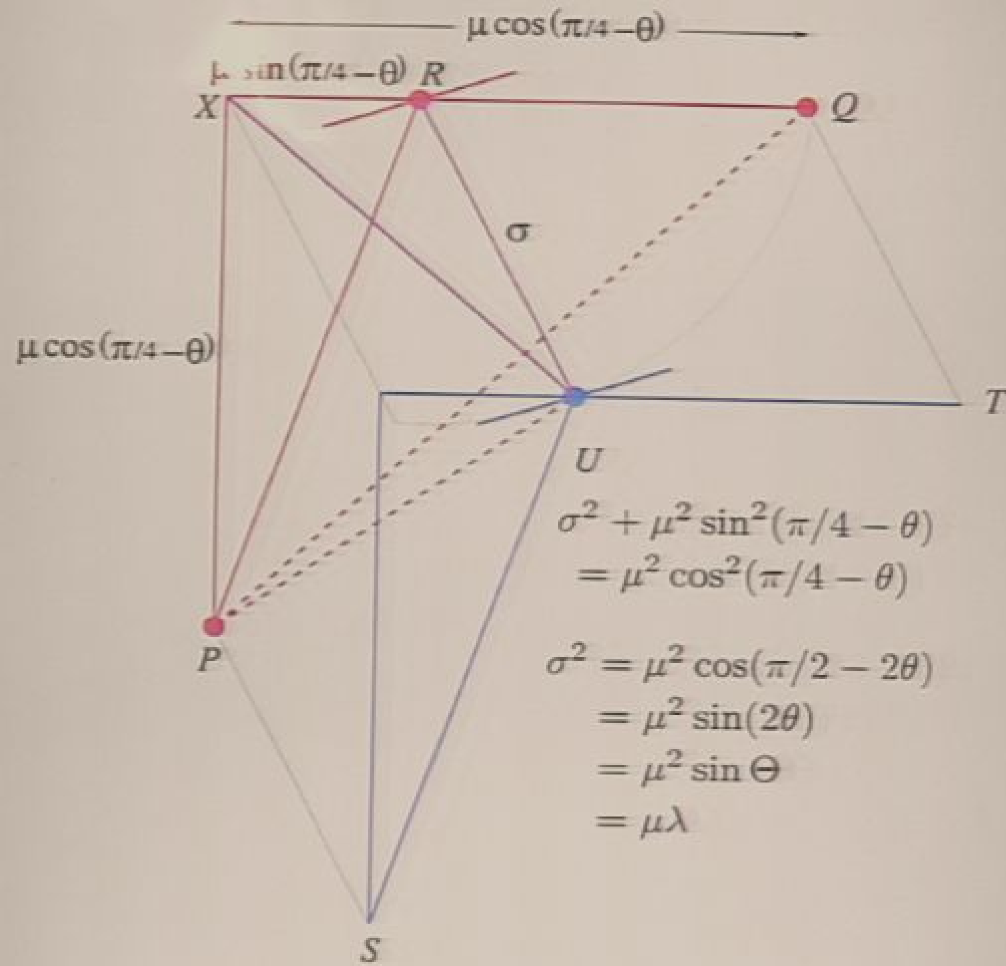
$\sigma^2 + \mu^2 \sin^2(\pi/4 - \theta)$

$= \mu^2$ [illegible] $/4 - \theta)$

σ^2 [illegible] $2\theta)$

σ *is (invariant!) geometric me*[illegible]

Determination of perpendicular scale factor σ



$$\sigma^2 + \mu^2 \sin^2(\pi/4 - \theta) = \mu^2 \cos^2(\pi/4 - \theta)$$

$$\sigma^2 = \mu^2 \cos(\pi/2 - 2\theta) = \mu^2 \sin(2\theta) = \mu^2 \sin \Theta = \mu\lambda$$

σ is (invariant!) geometric mean of μ and λ.

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