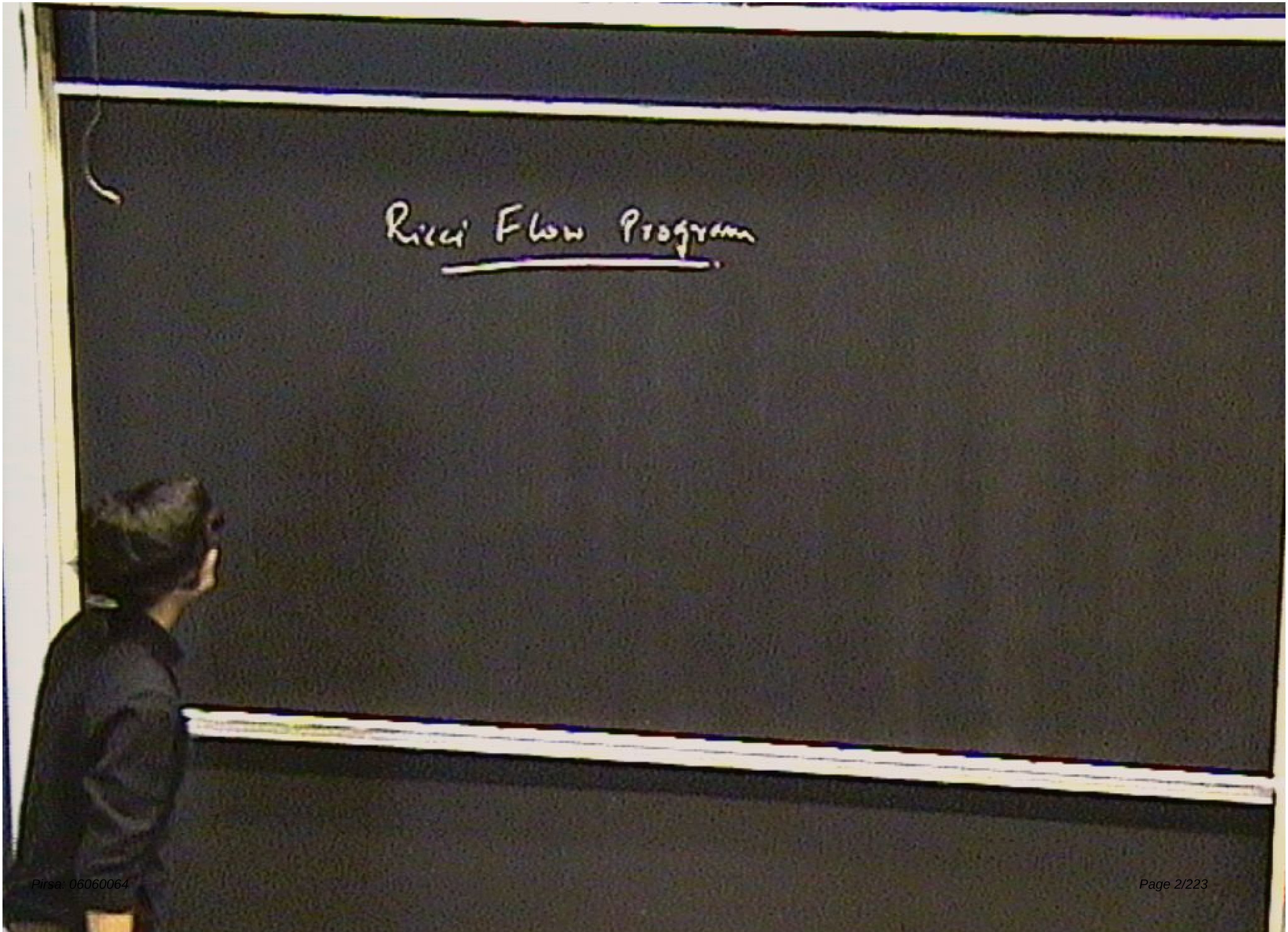
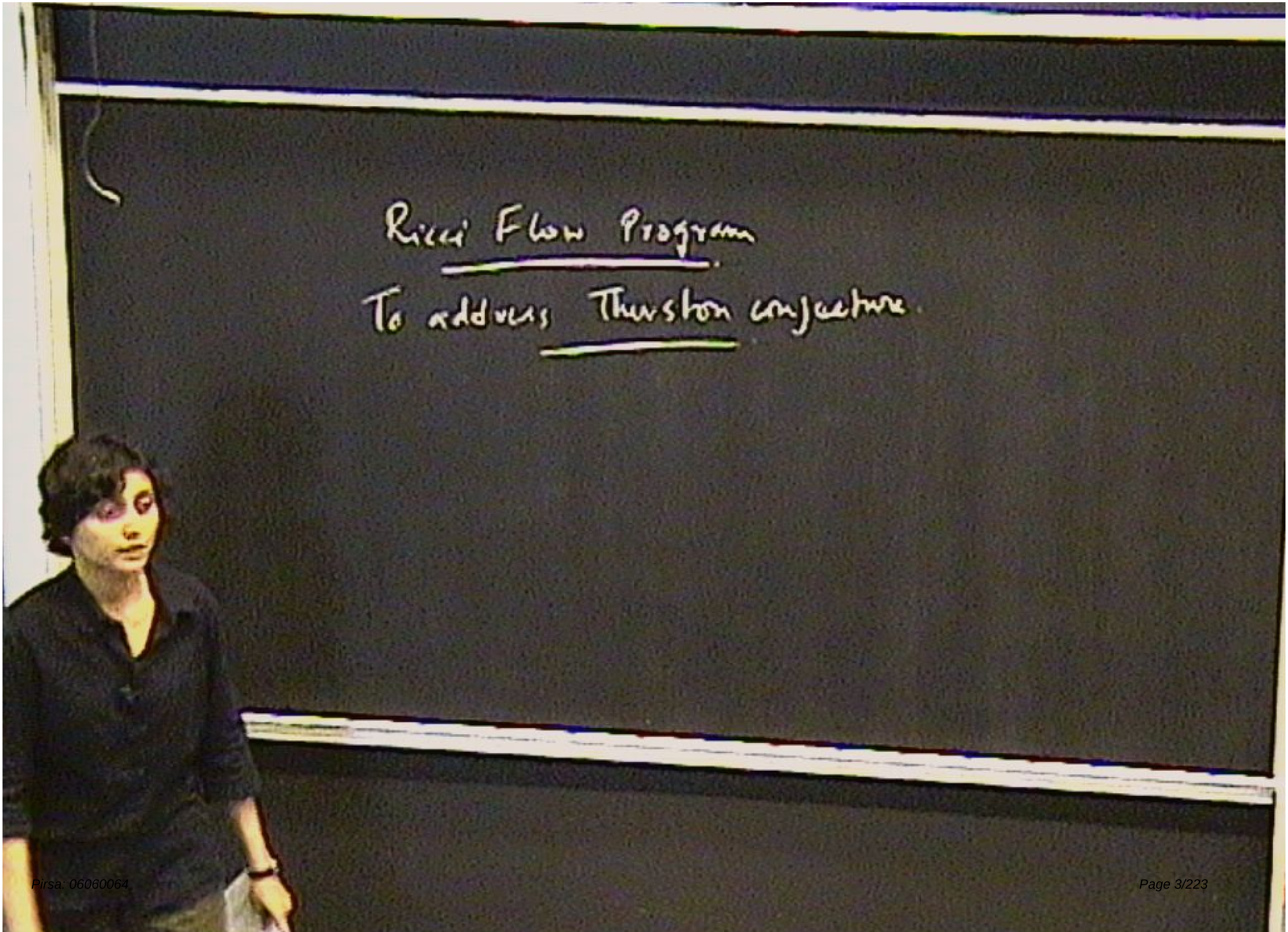
Title: Ricci Flow Program in Math

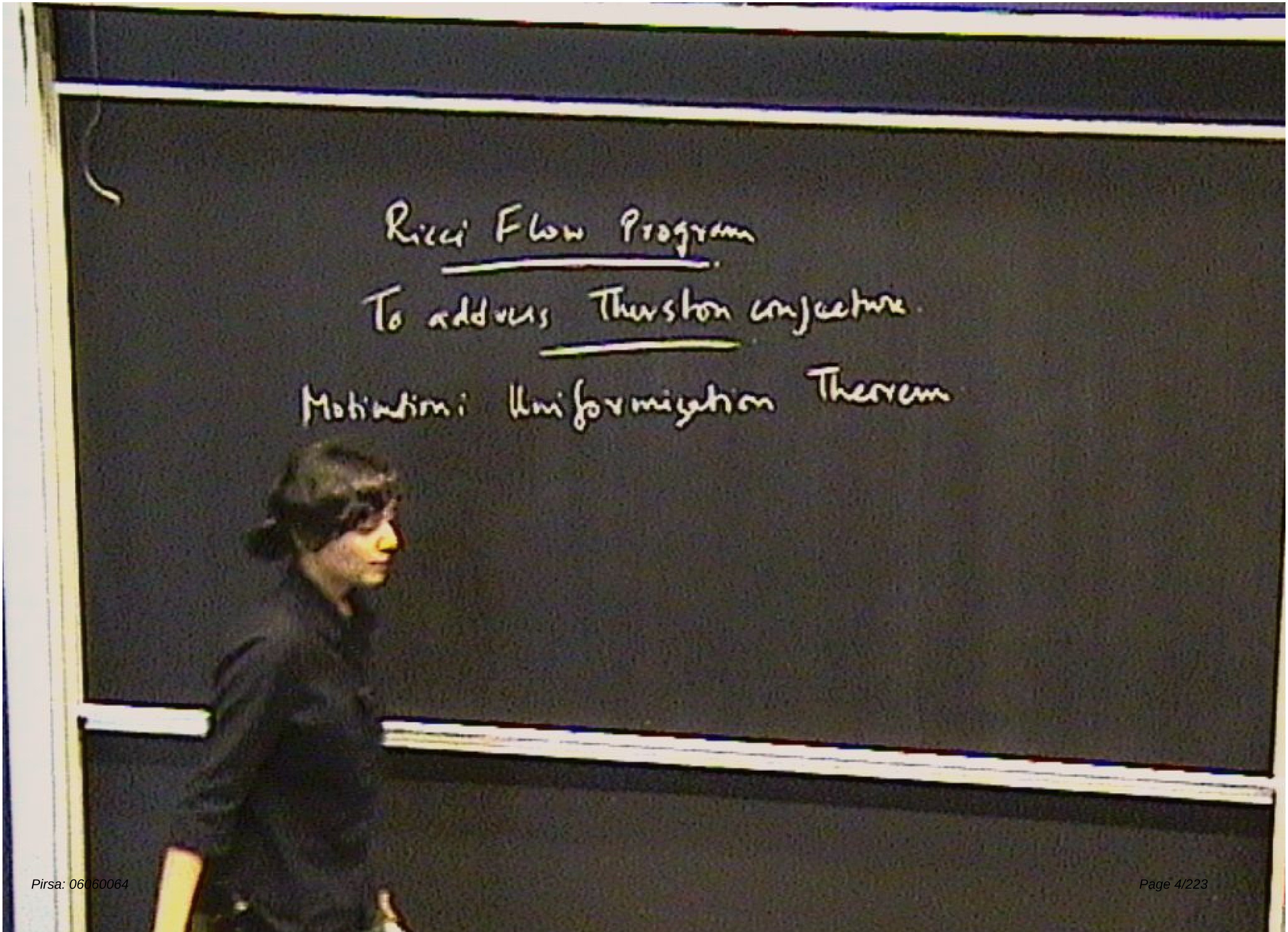
Date: Jun 12, 2006 11:00 AM

URL: http://pirsa.org/06060064

Abstract:


Ricci Flow Program


Ricci Flow Program
To address Thurston conjecture.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Motivation: Uniformization Theorem

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S^2 $\quad$ $\mathbb{R}^2$. $\quad$ $\mathbb{H}^2$

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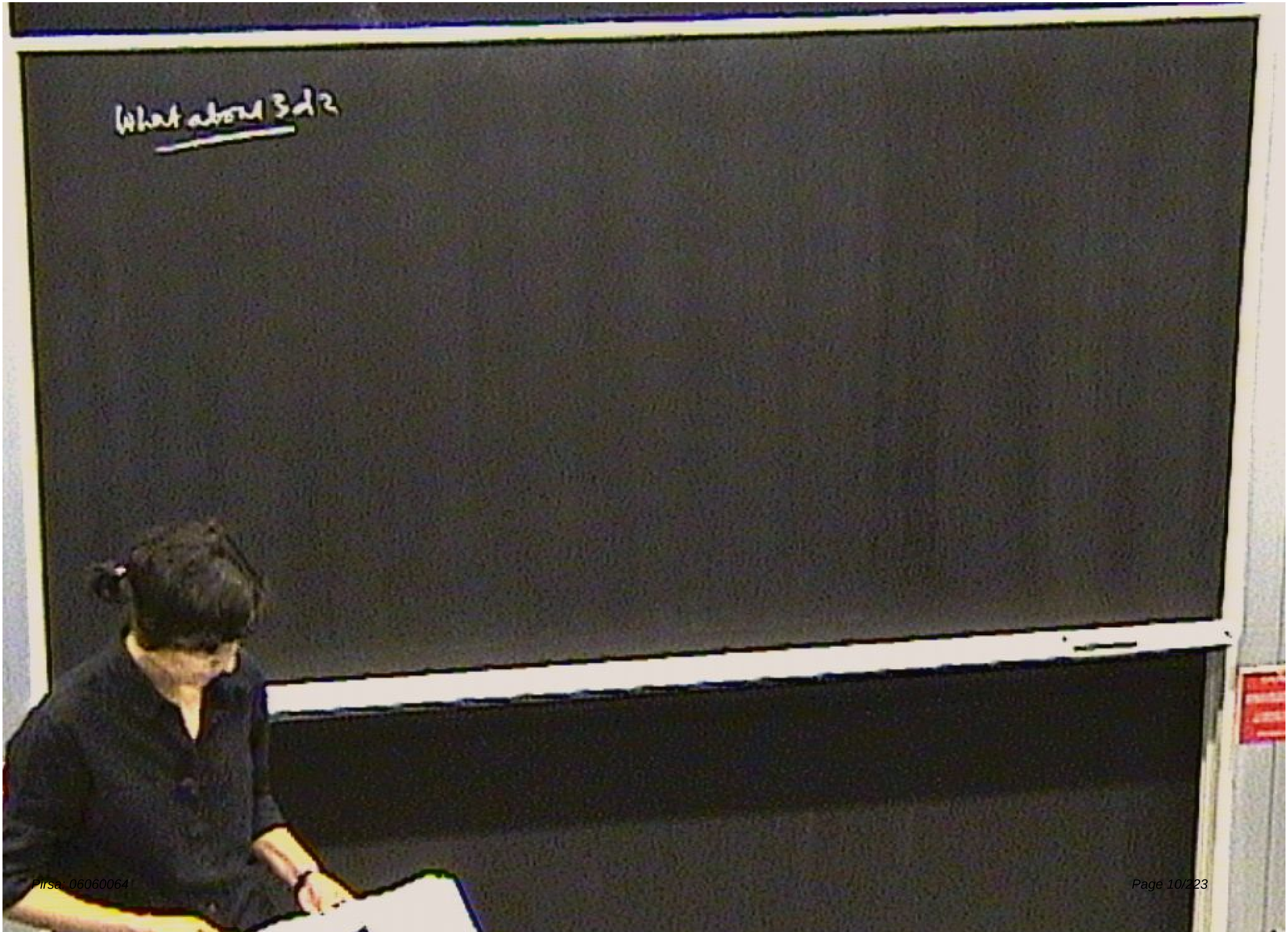
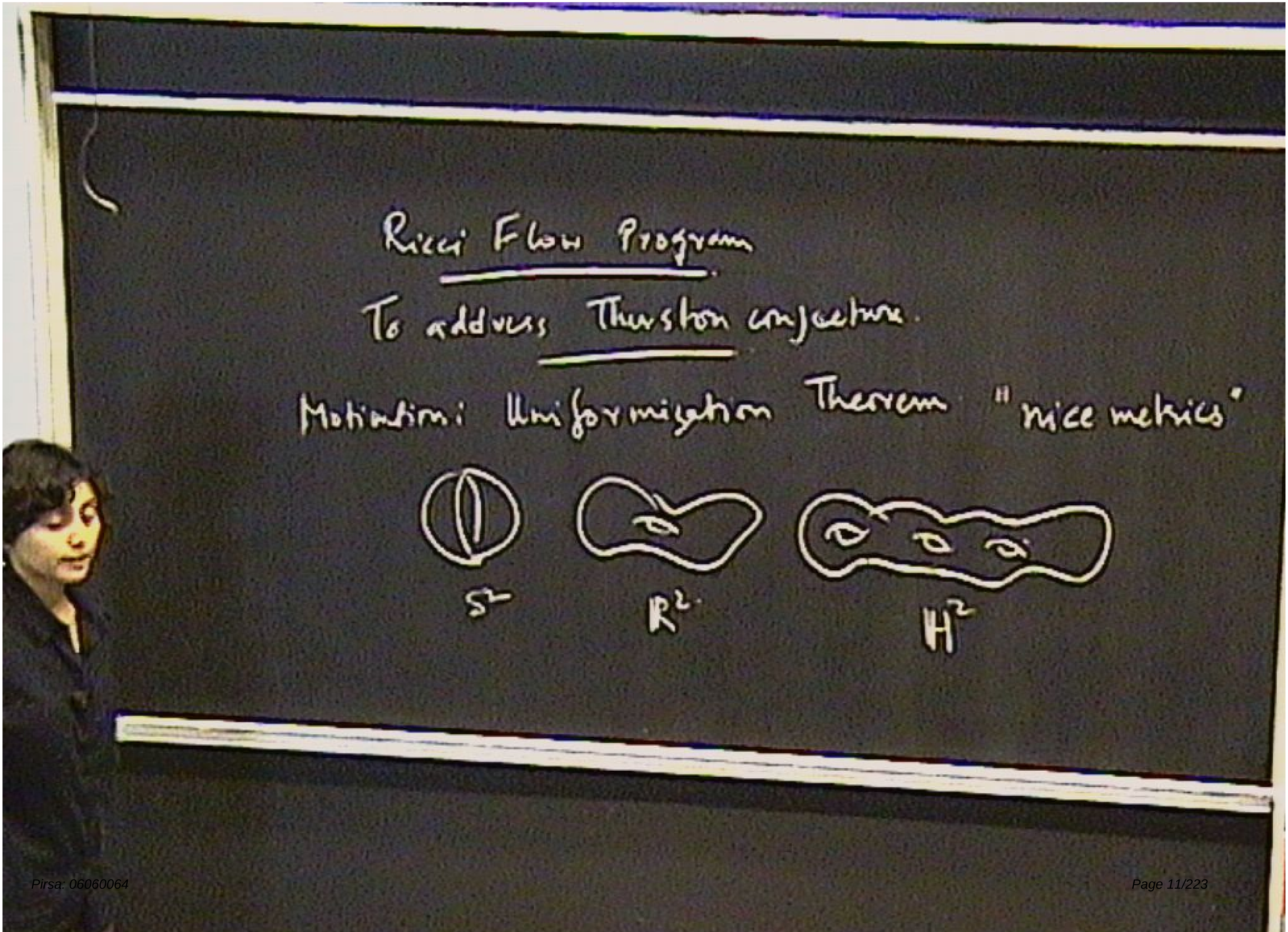

What about 3d?



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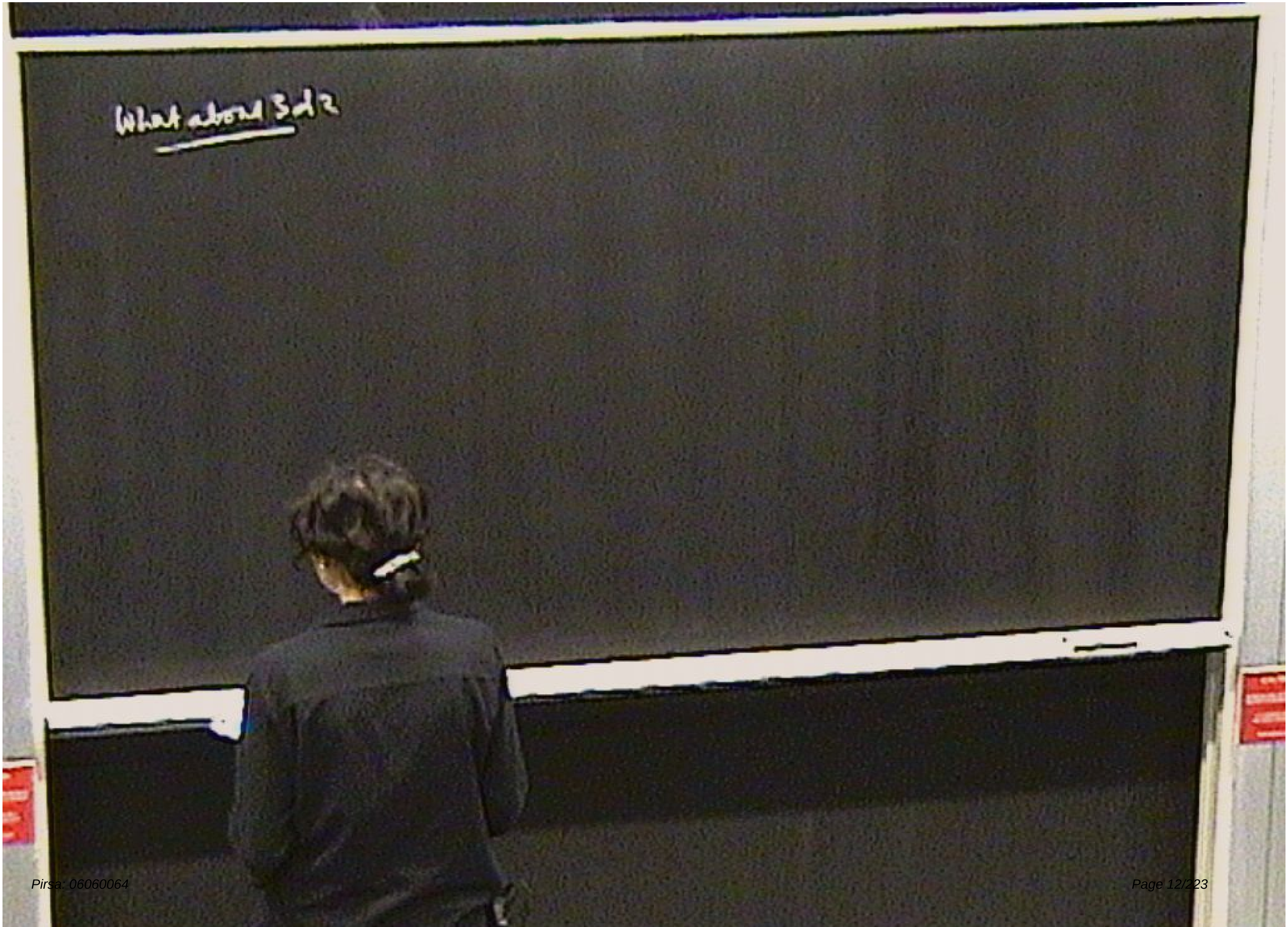
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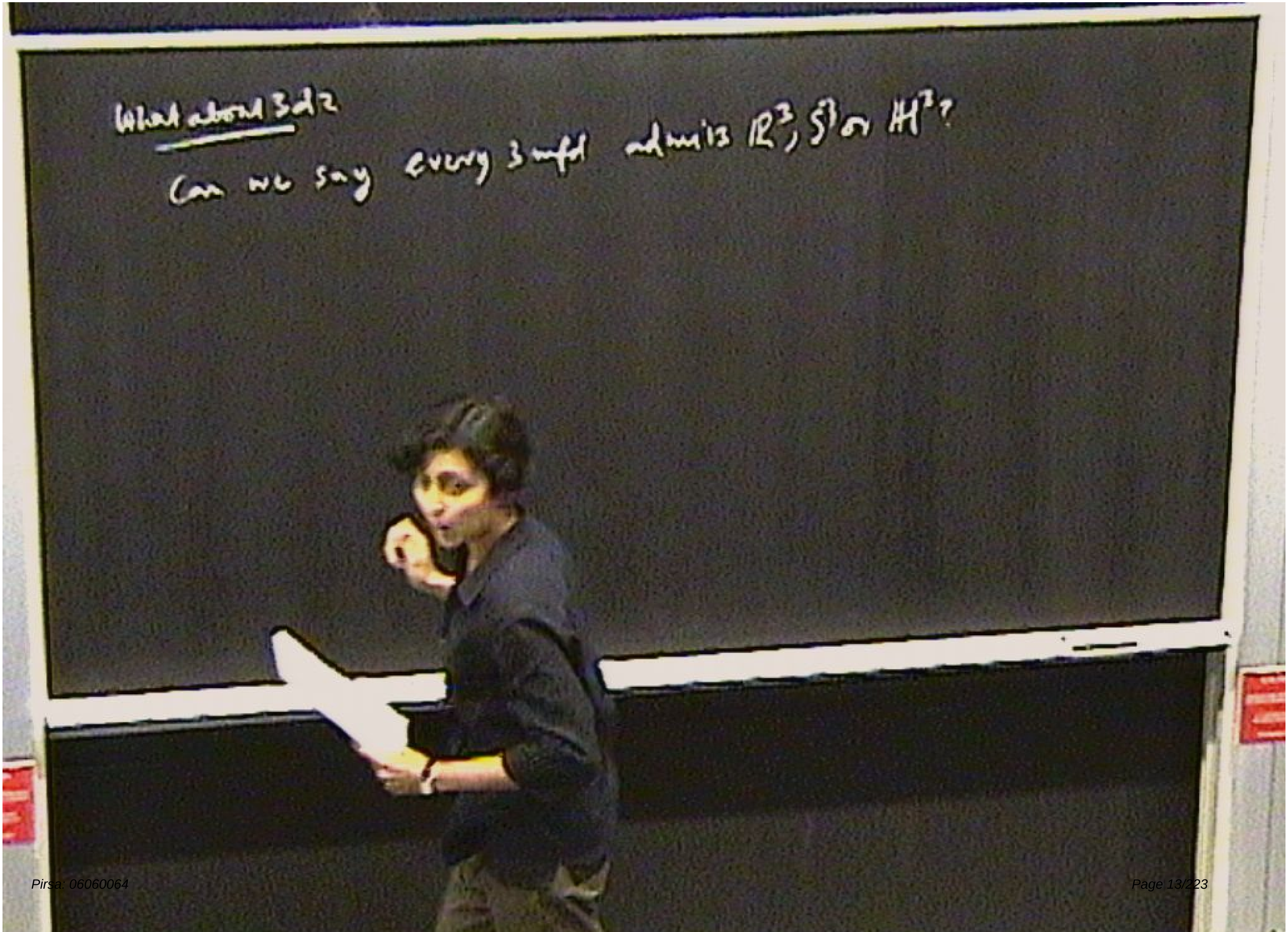
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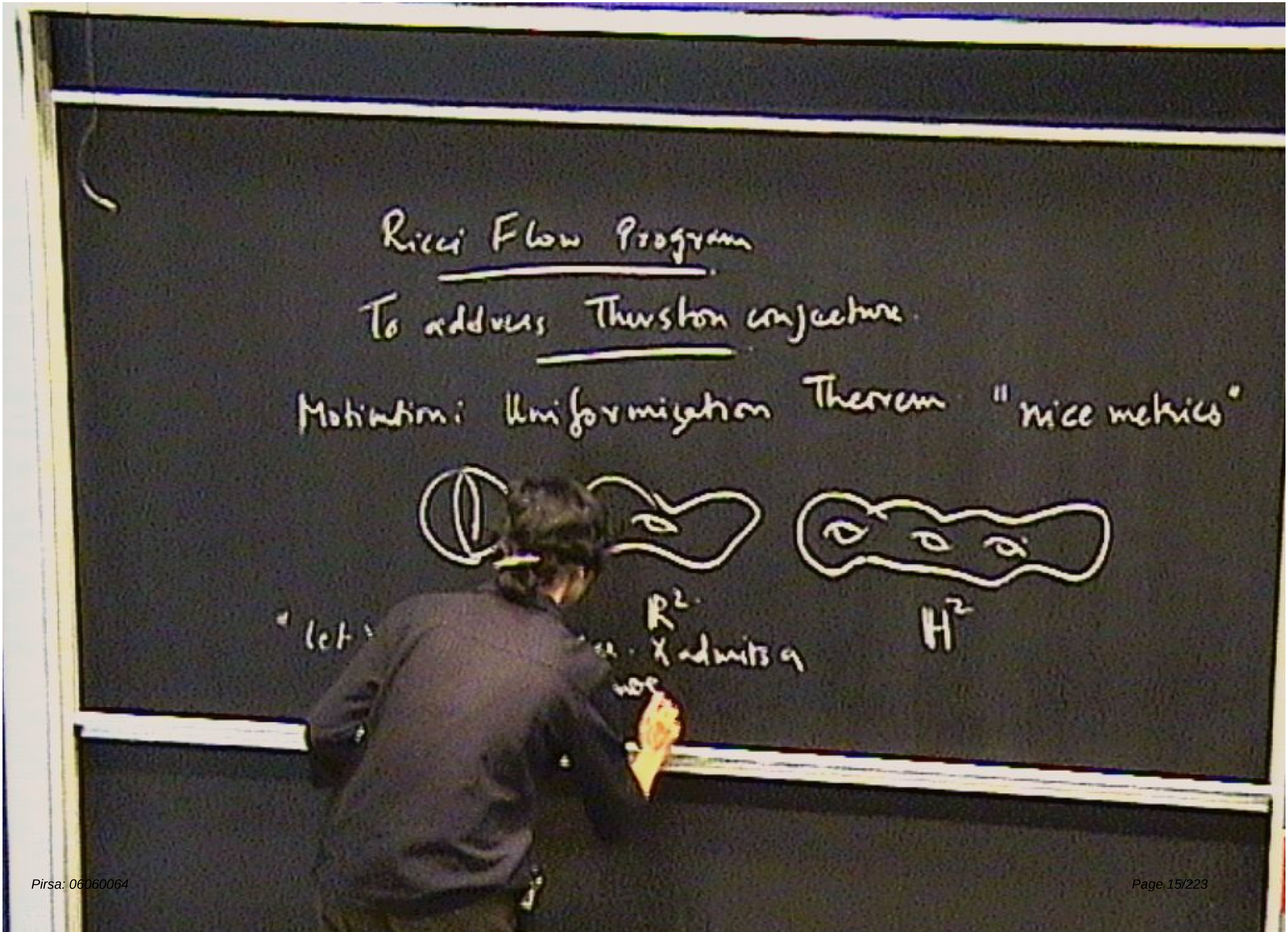
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R²
H²
X admits a

Ricci Flow Program

To address Thurston conjecture

Motivation: Uniformization Theorem "nice metrics"

S^2 $\mathbb{R}^2$ $\mathbb{H}^2$

* let X be compact surface. X admits a locally homogeneous metric.

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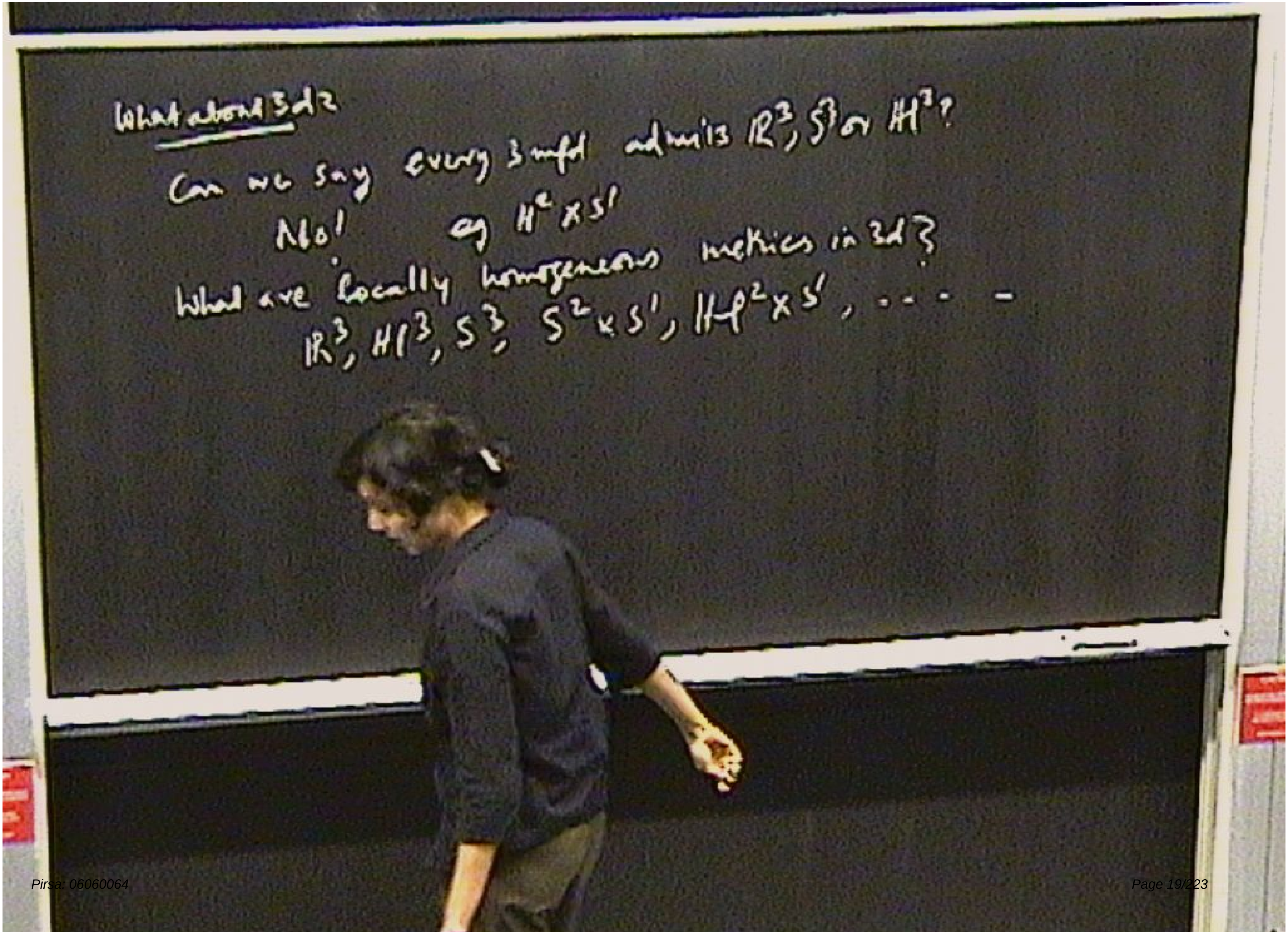
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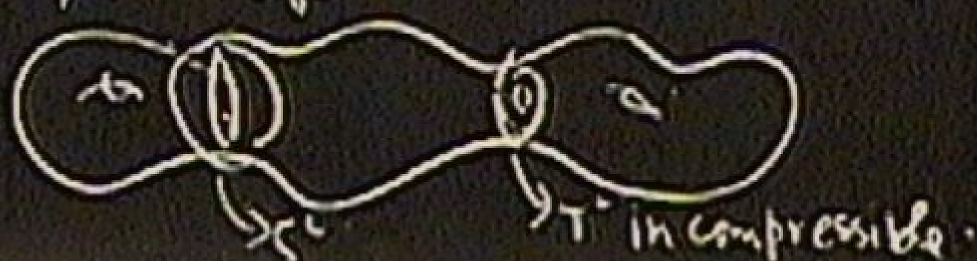
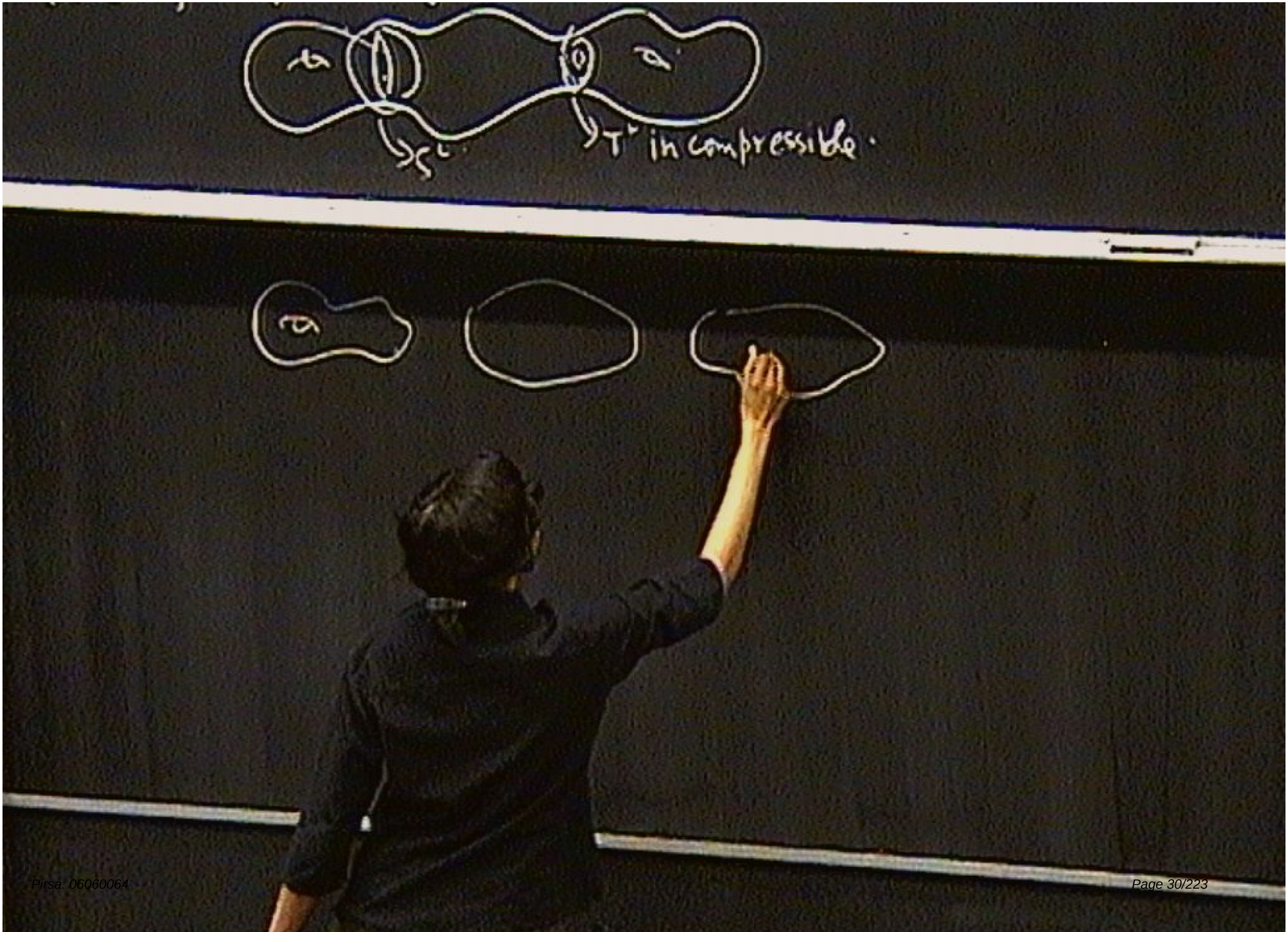
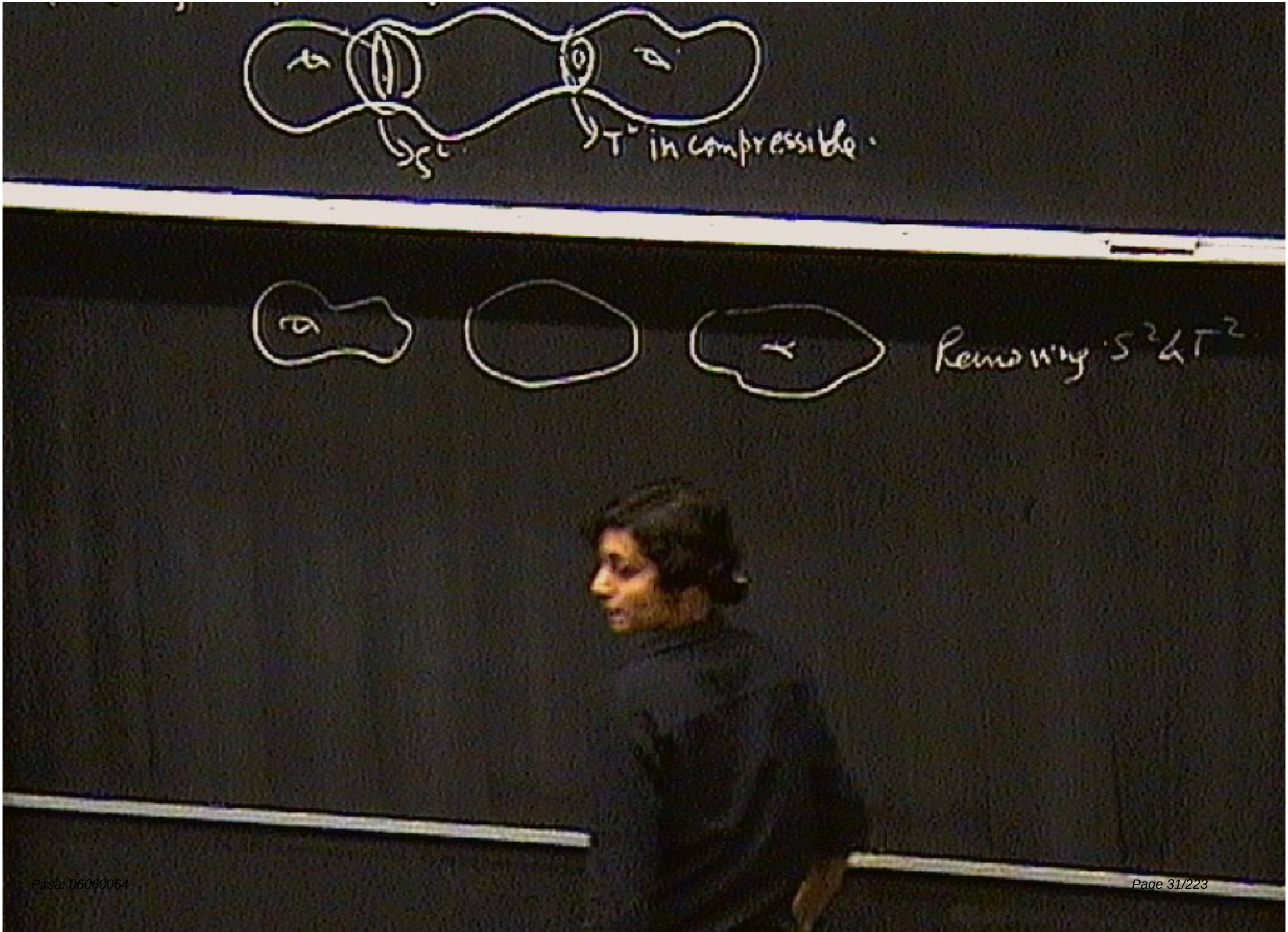
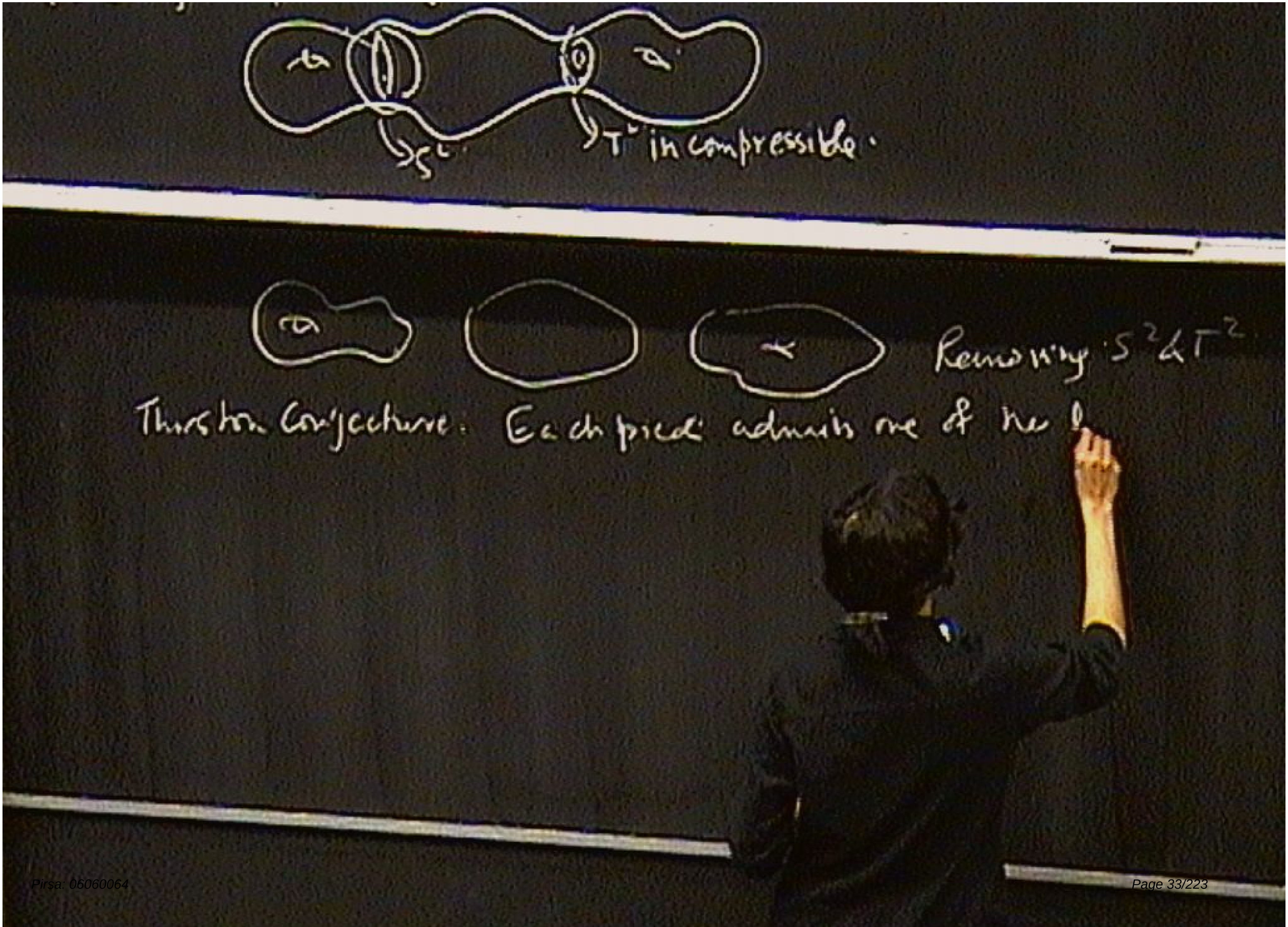



S^2
T^2 in compressible.


S²
T² incompressible.
Removing S² & T²

Removing S^2 & T^2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Removing S^2 & T^2
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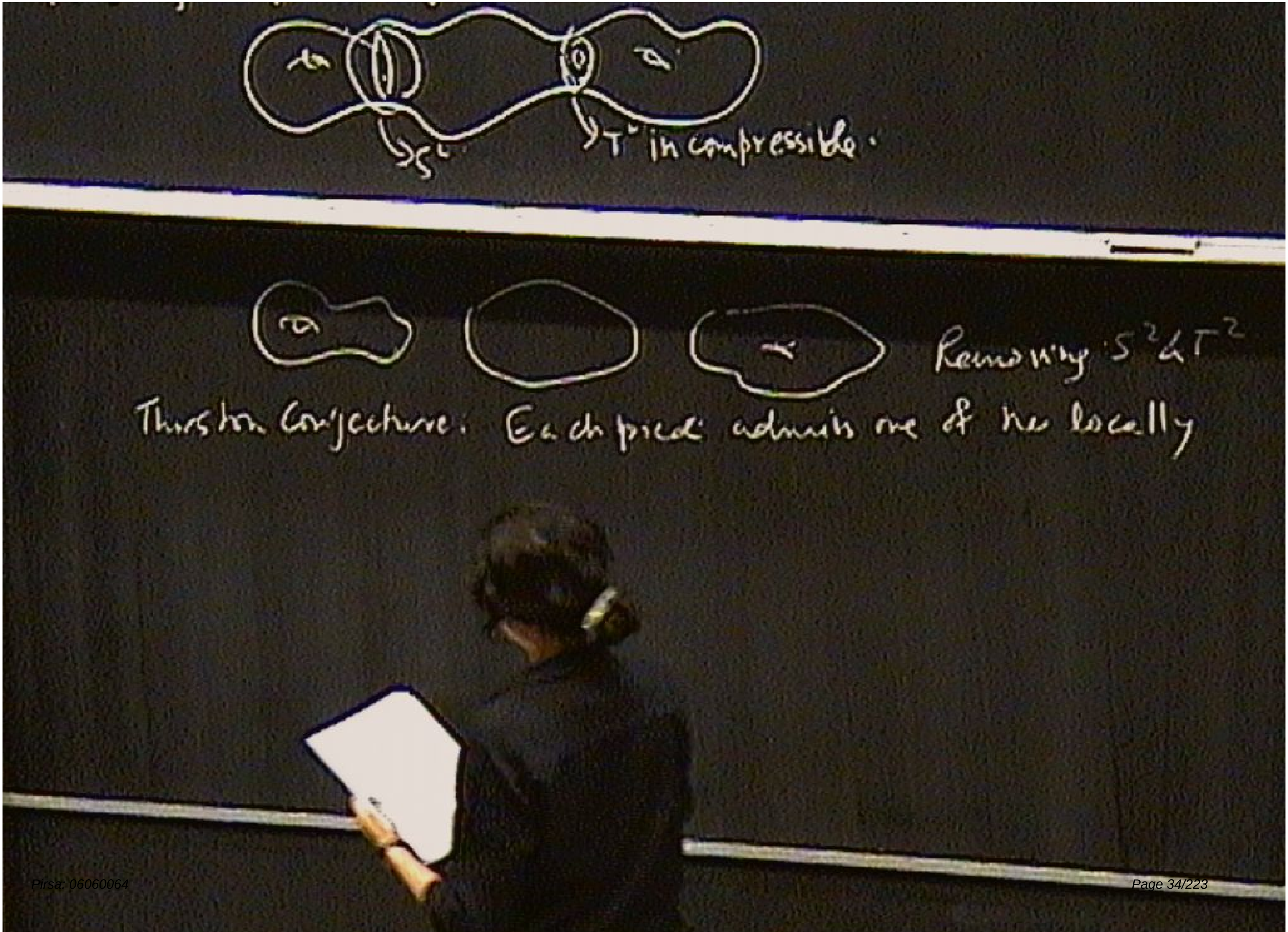
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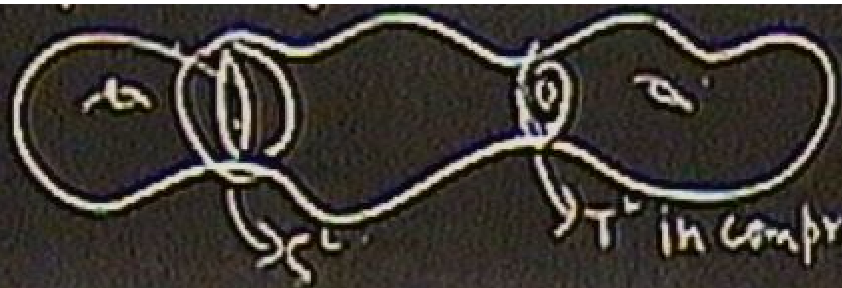

Removing S^2 & T^2

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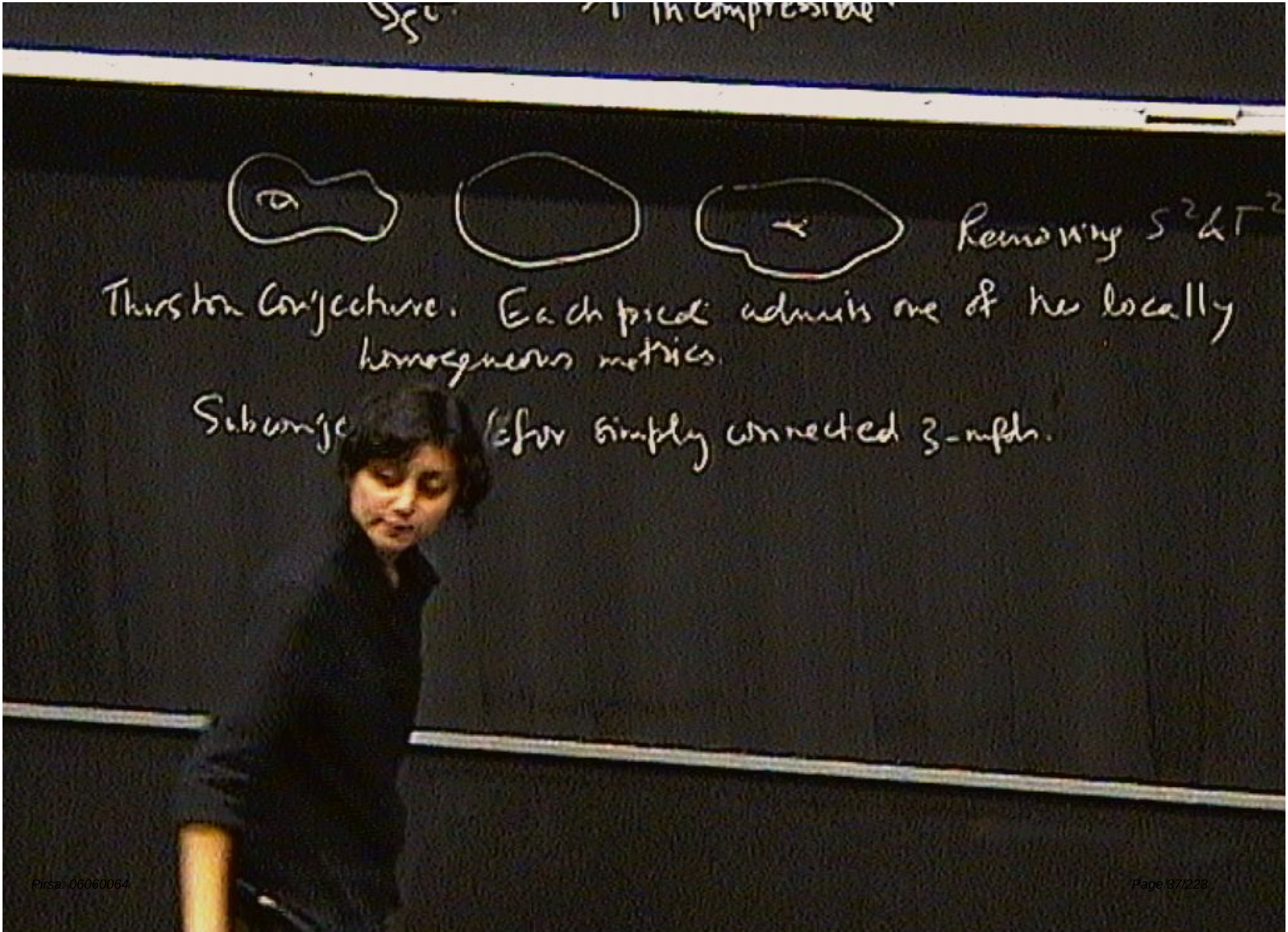
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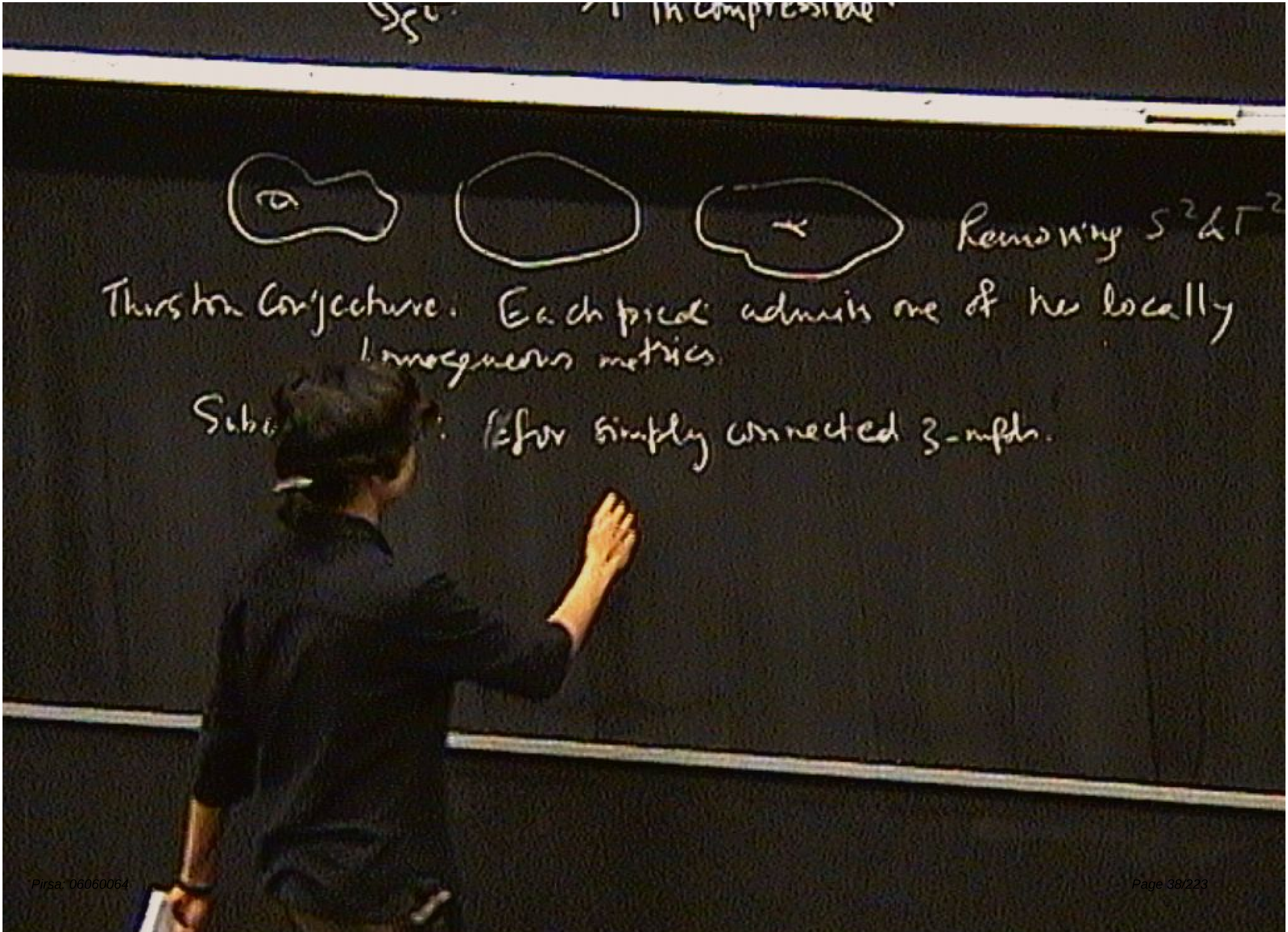
Subconj
for simply connected 3-mfds.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$\hookrightarrow$ admit S^3 metric.

S^2 $\to T$ incompressible

Removing S^2 & T^2

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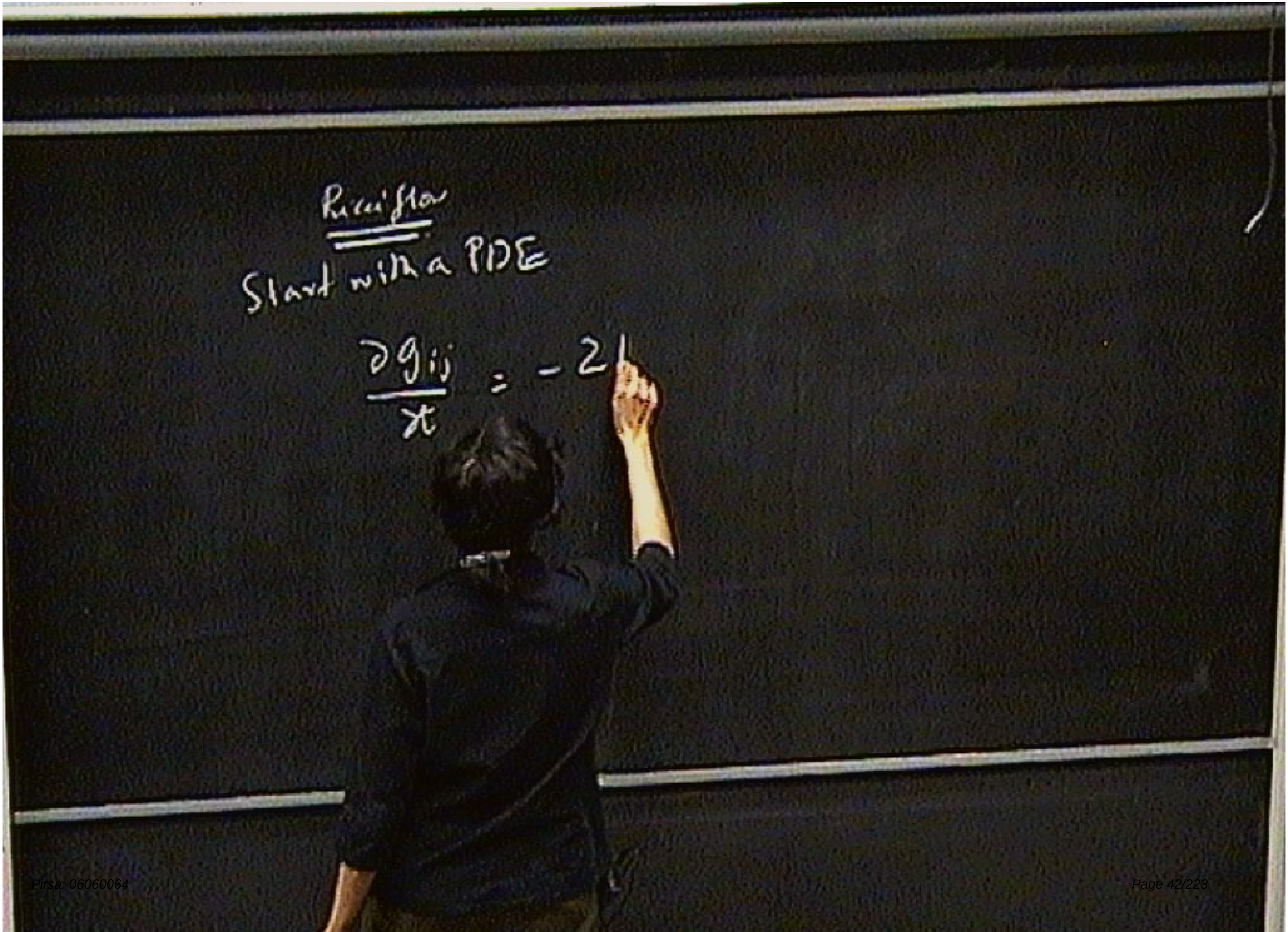
"Poincare Conjecture" $\hookrightarrow$ admit S^3 metric.

Ricci flow


Ricci flow
Start with a PDE
$\frac{\partial g_{ij}}{\partial t} = -2$

Ricci flow

Start with a PDE

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Ricci flow

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Take a 3-mfd M.

Ricci flow

Start with a PDE

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Take a 3-mfd M. $\quad g(t=0) = g_0$

Ricci flow

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Take a 3-mfd M. $g(t=0) = g_0$

Does g "flow" to a nice metric?

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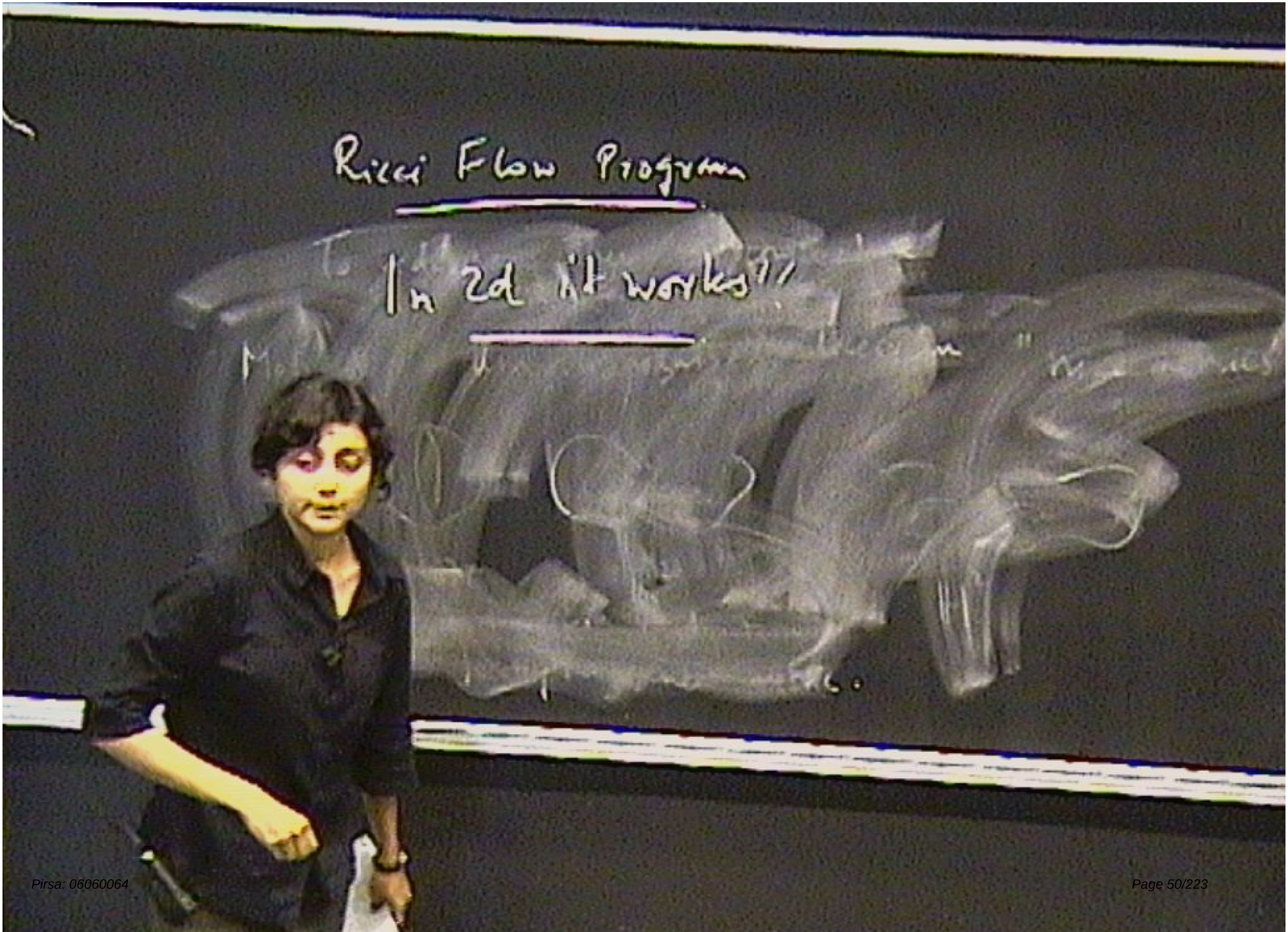
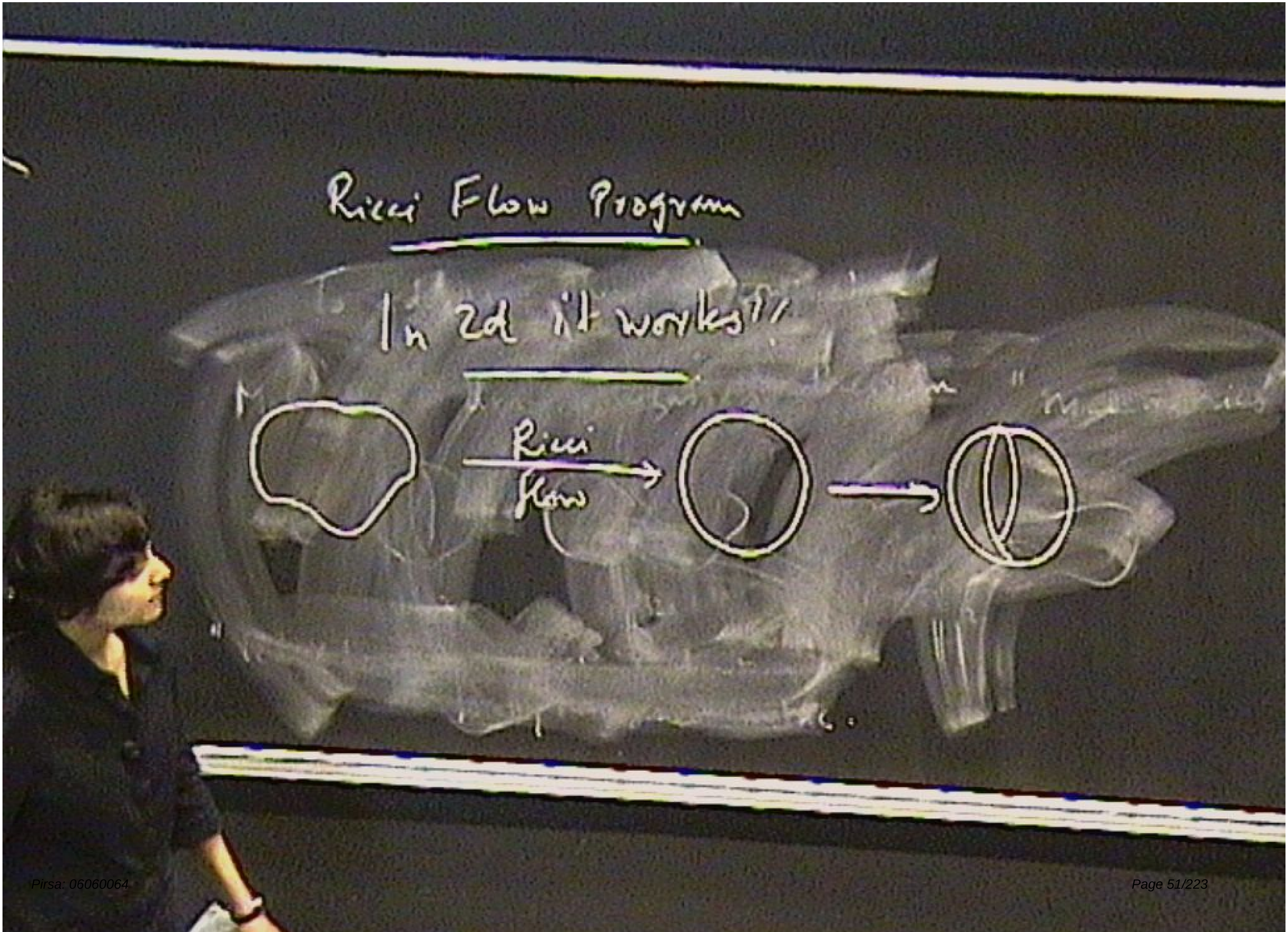

Ricci Flow Program
In 2d it works!!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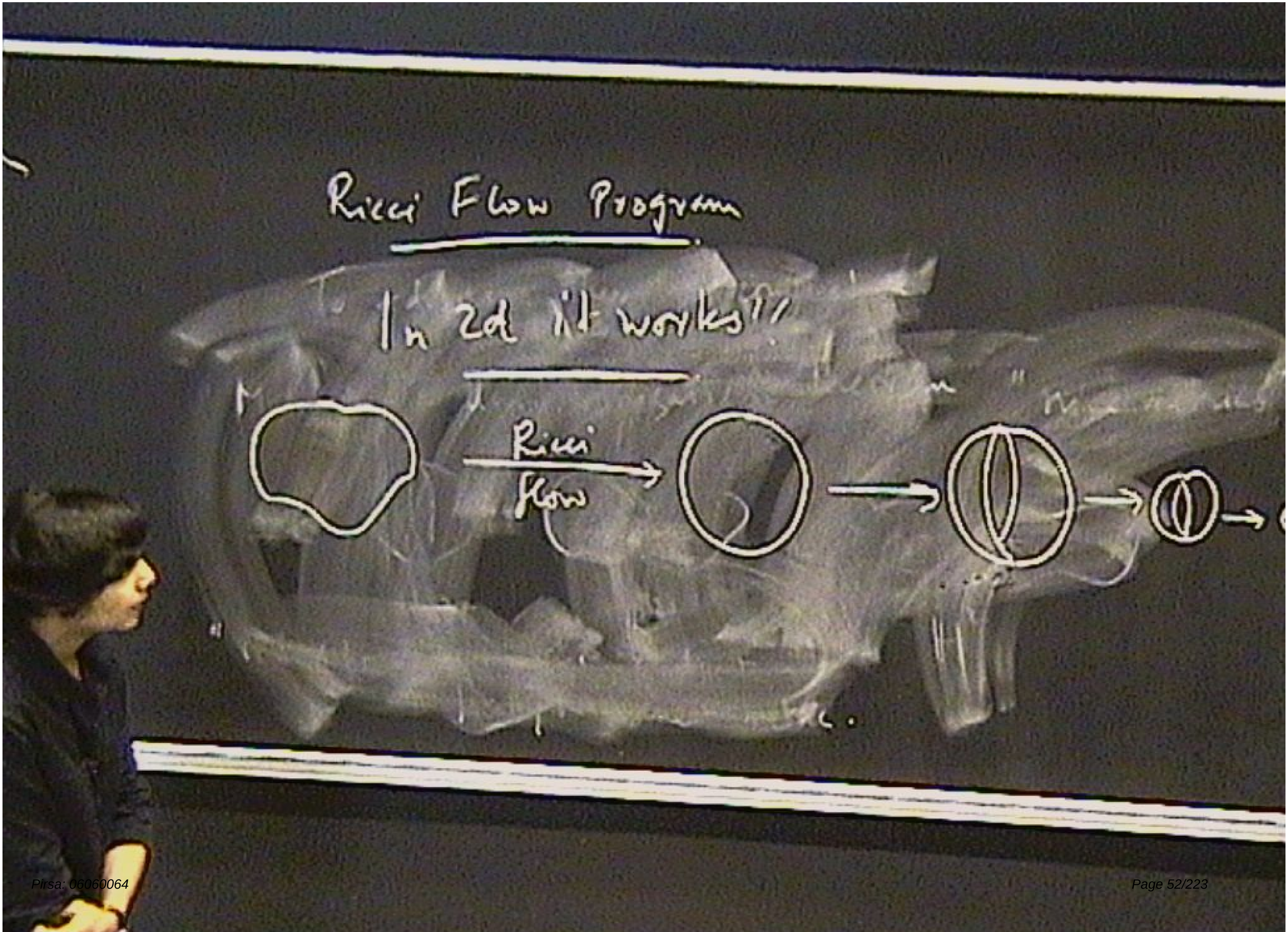
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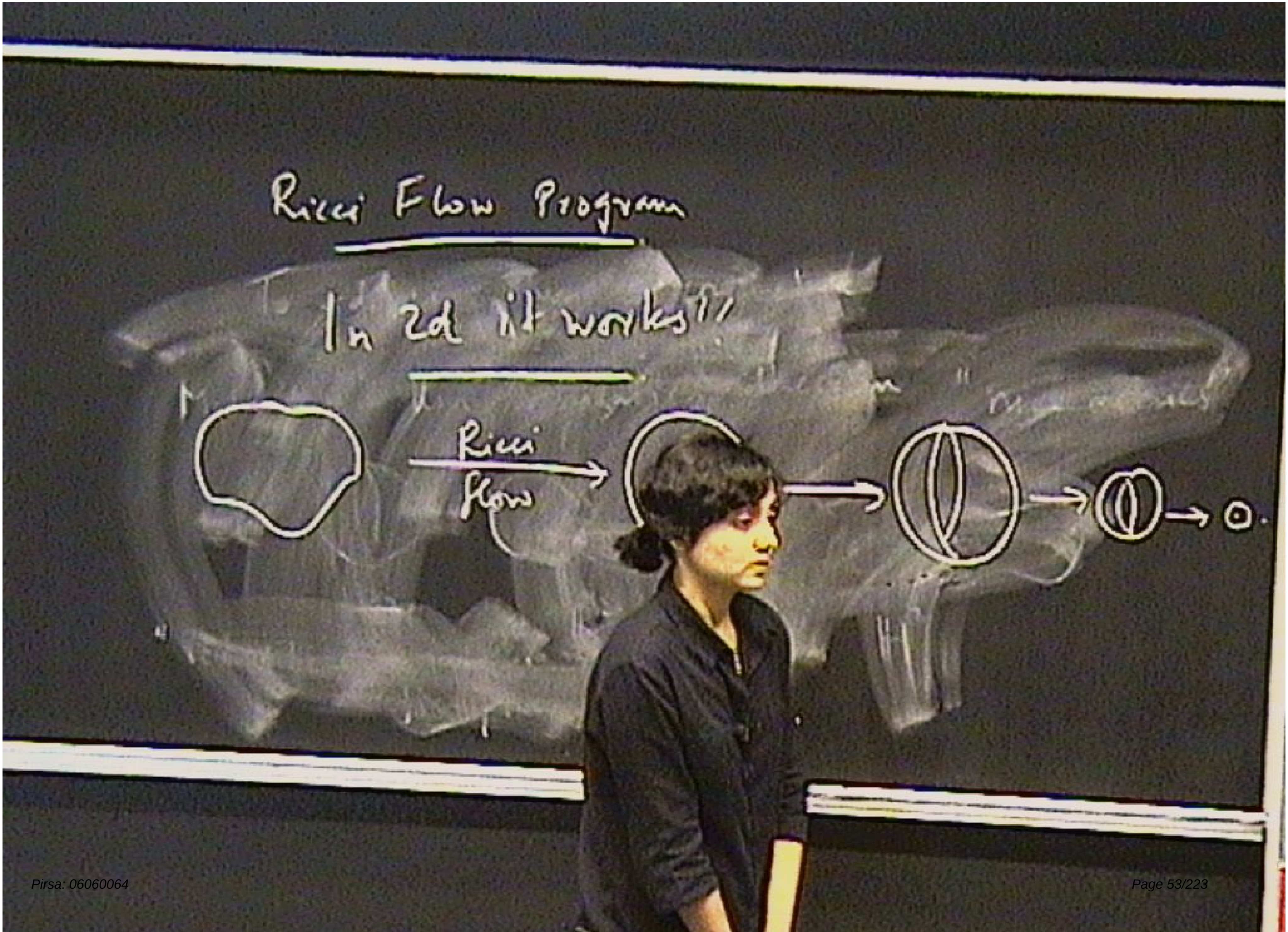
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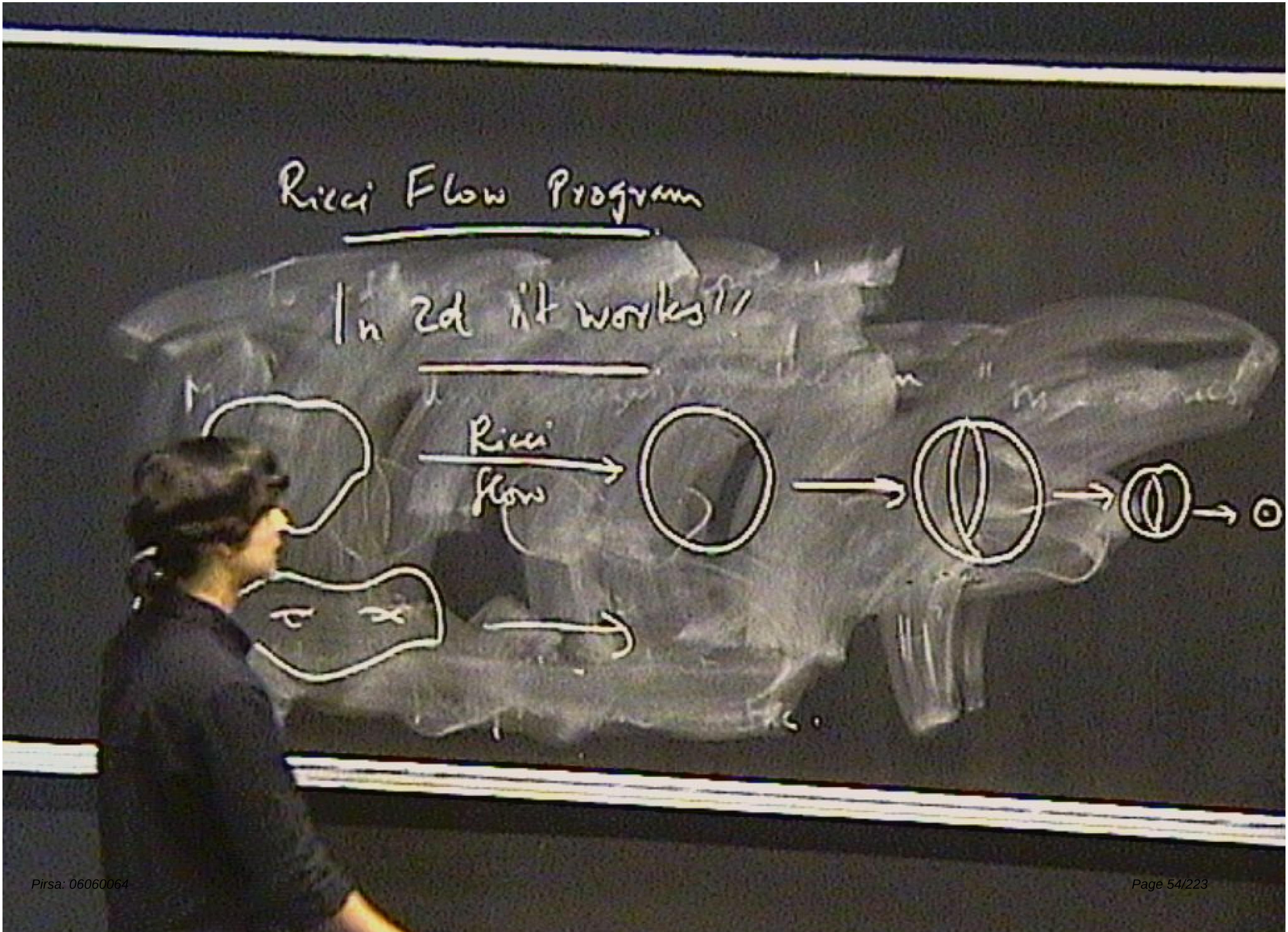
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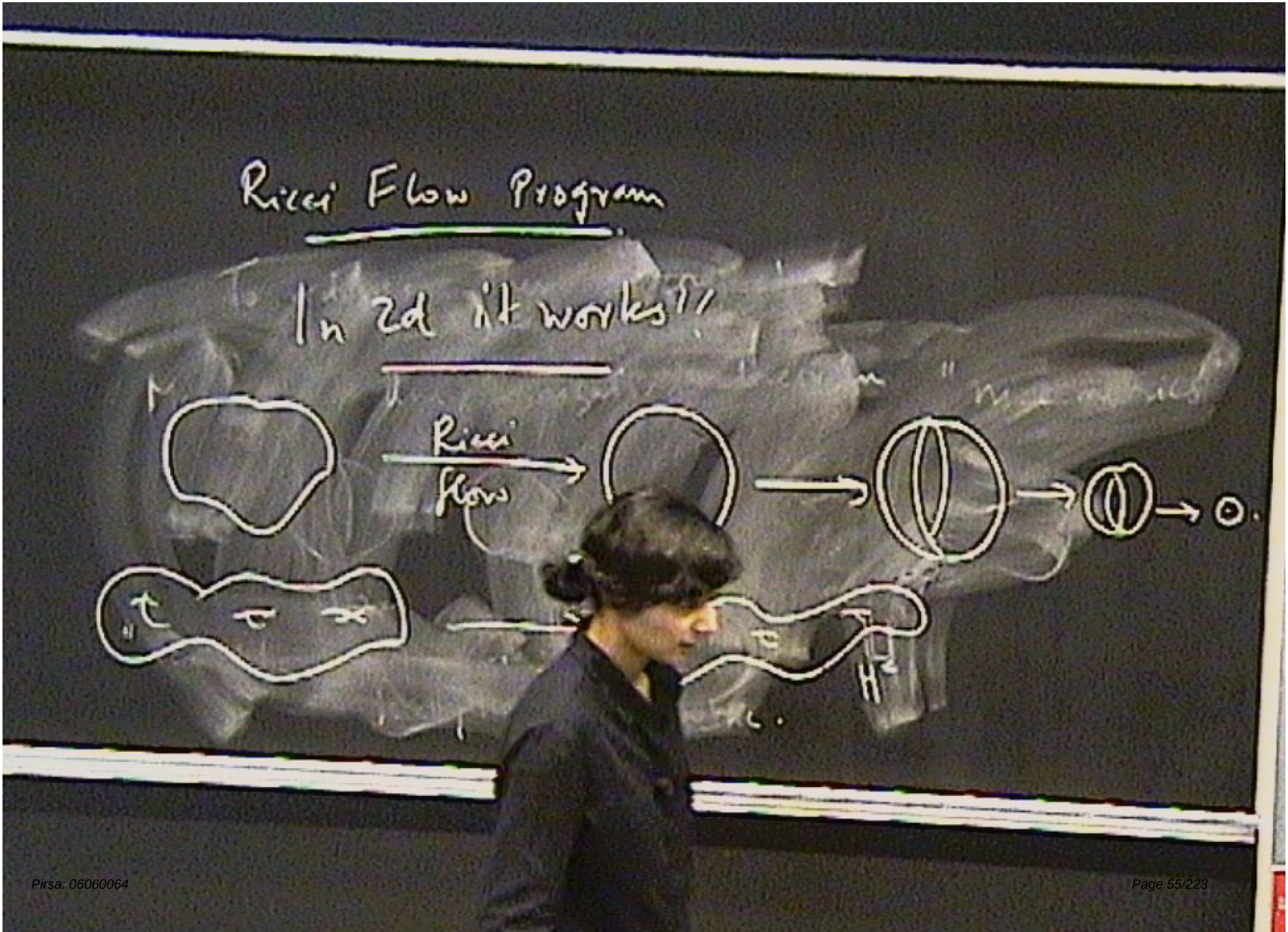
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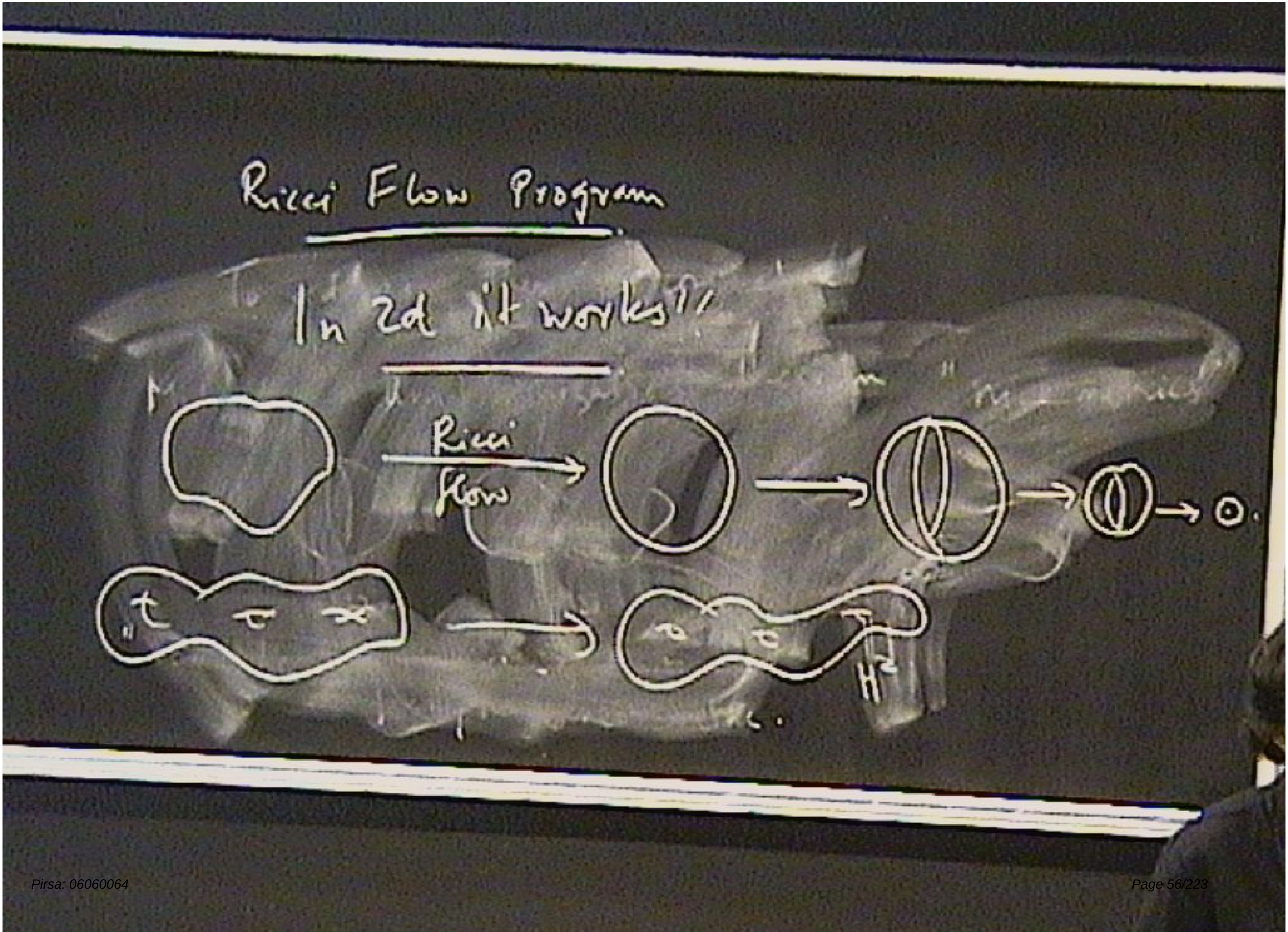
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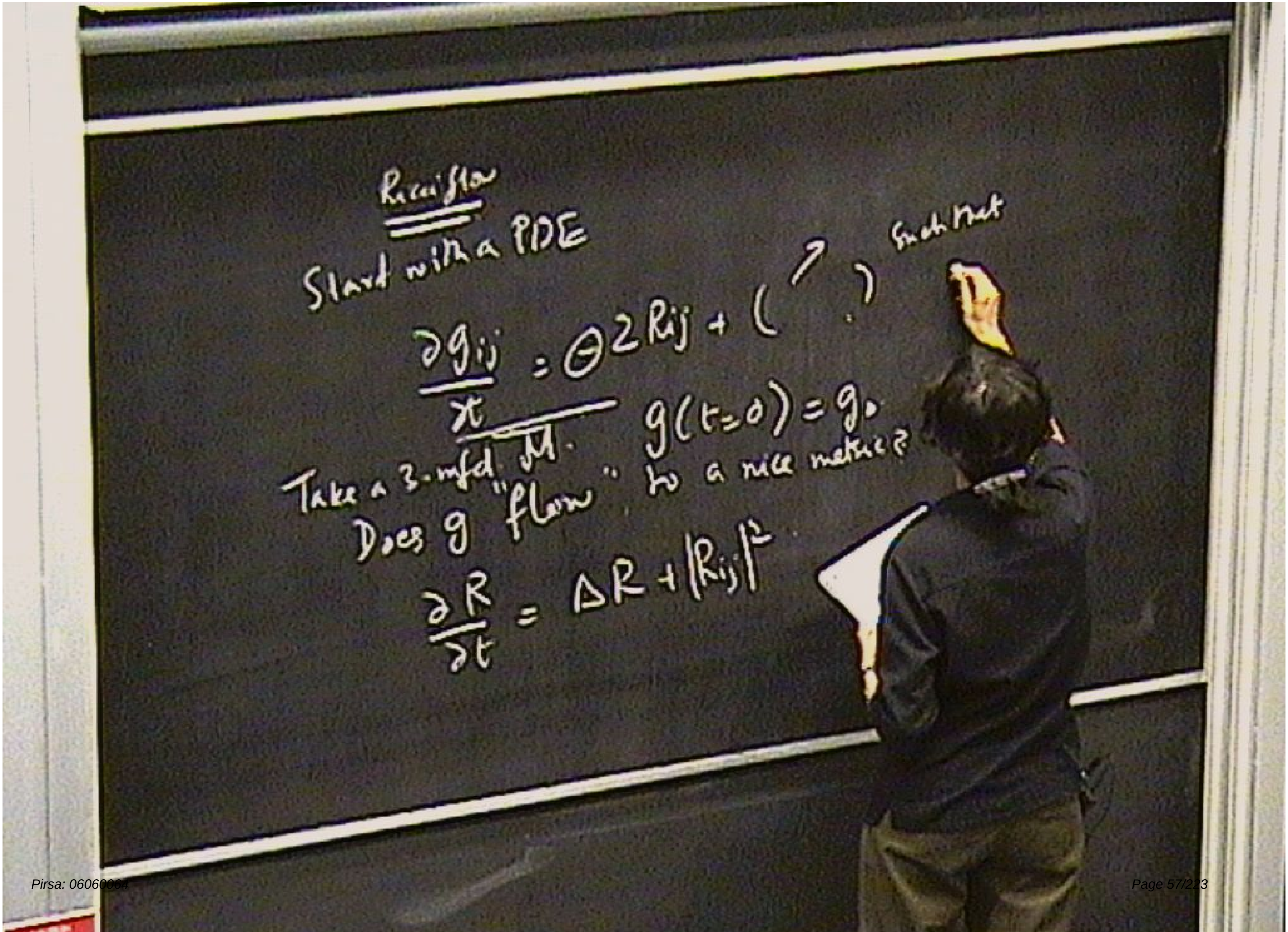
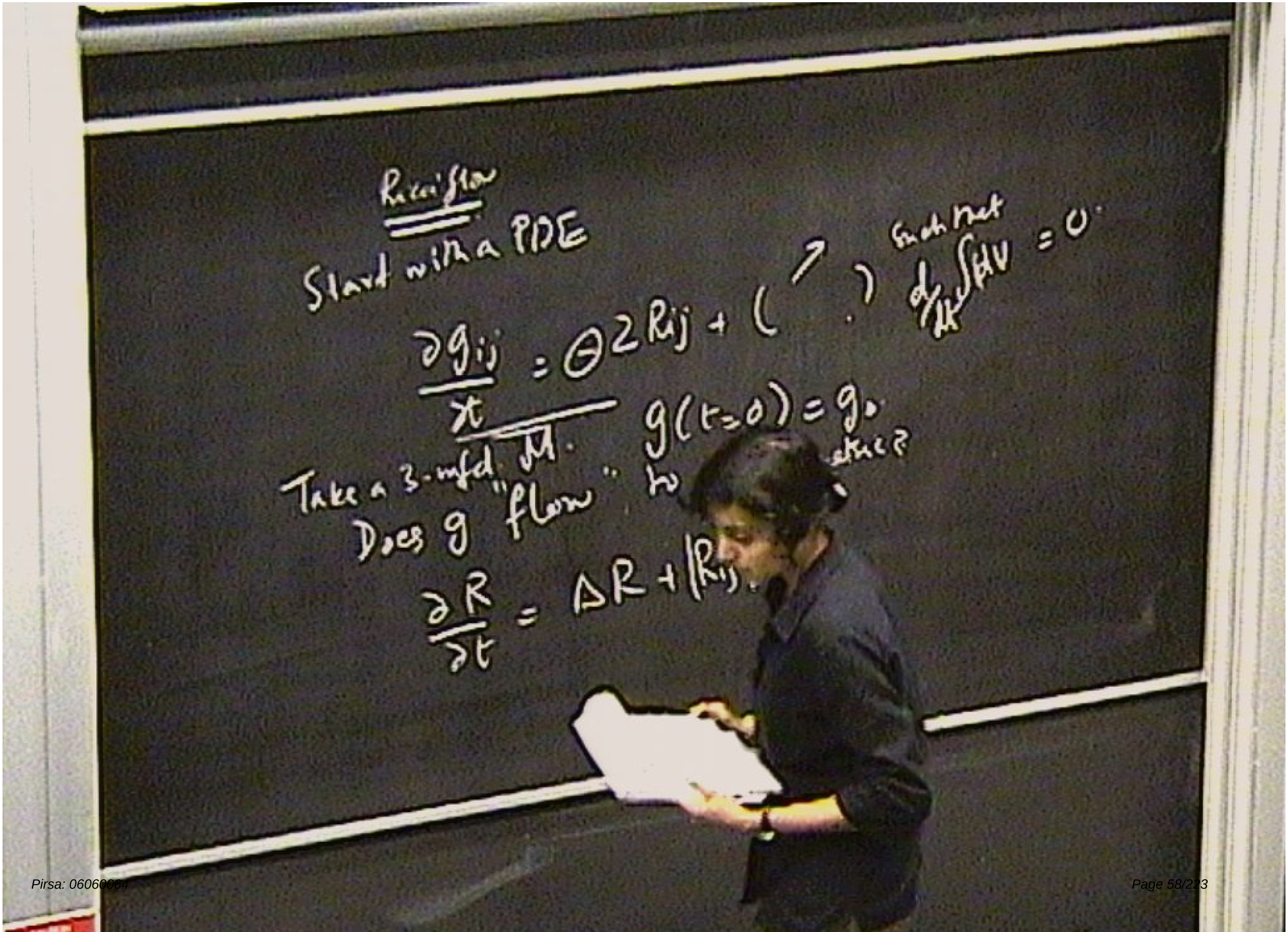

Ricci flow
Start with a PDE
$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + (\quad)$ such that
$g(t=0) = g_0$
Take a 3-mfd M.
Does g "flow" to a nice metric?
$\frac{\partial R}{\partial t} = \Delta R + |R_{ij}|^2$


Ricci flow
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∂gij / ∂t = ⊖ 2 Rij + ( )
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∂R / ∂t = ΔR + |Rij|

Ricci flow

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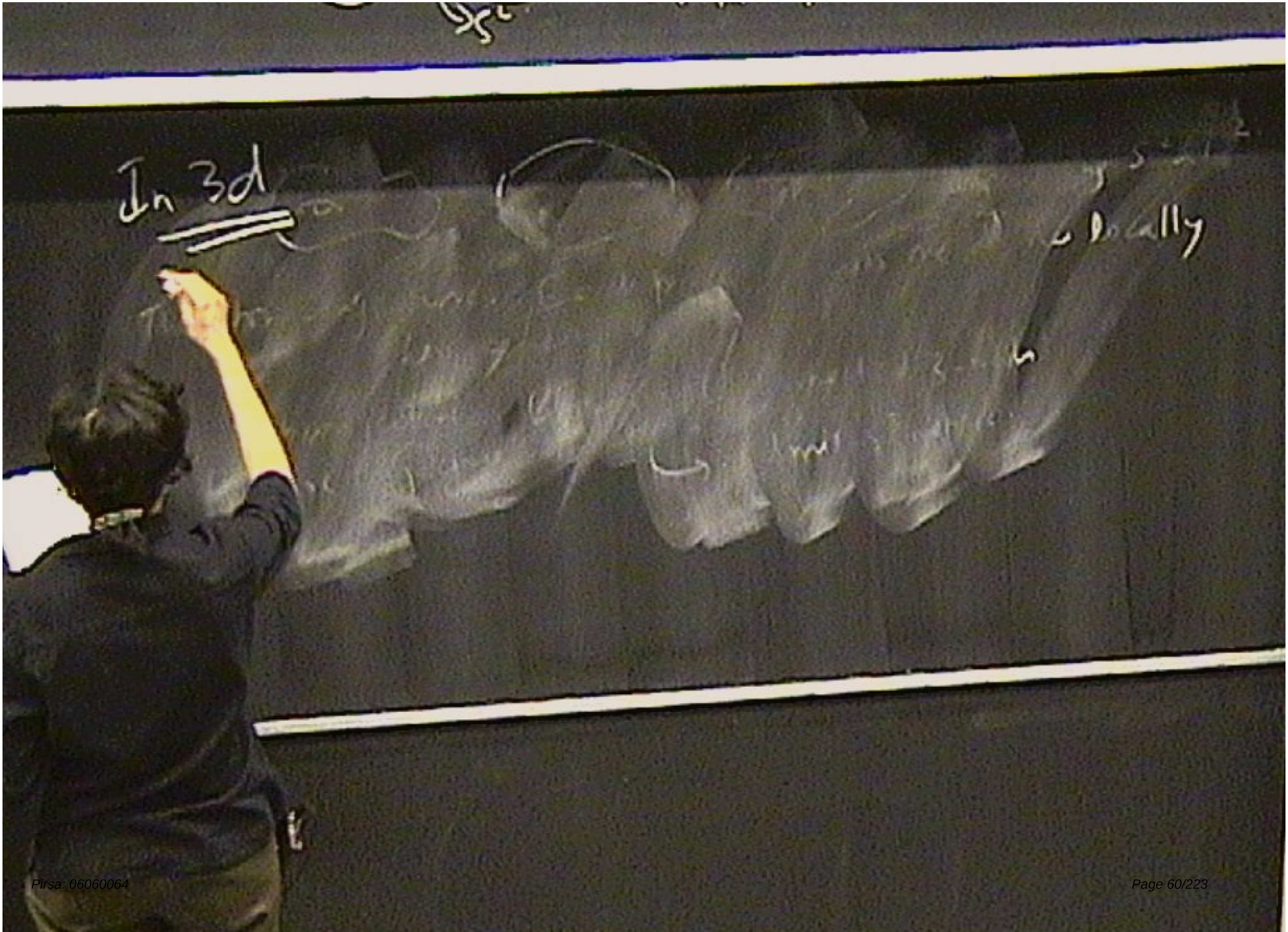
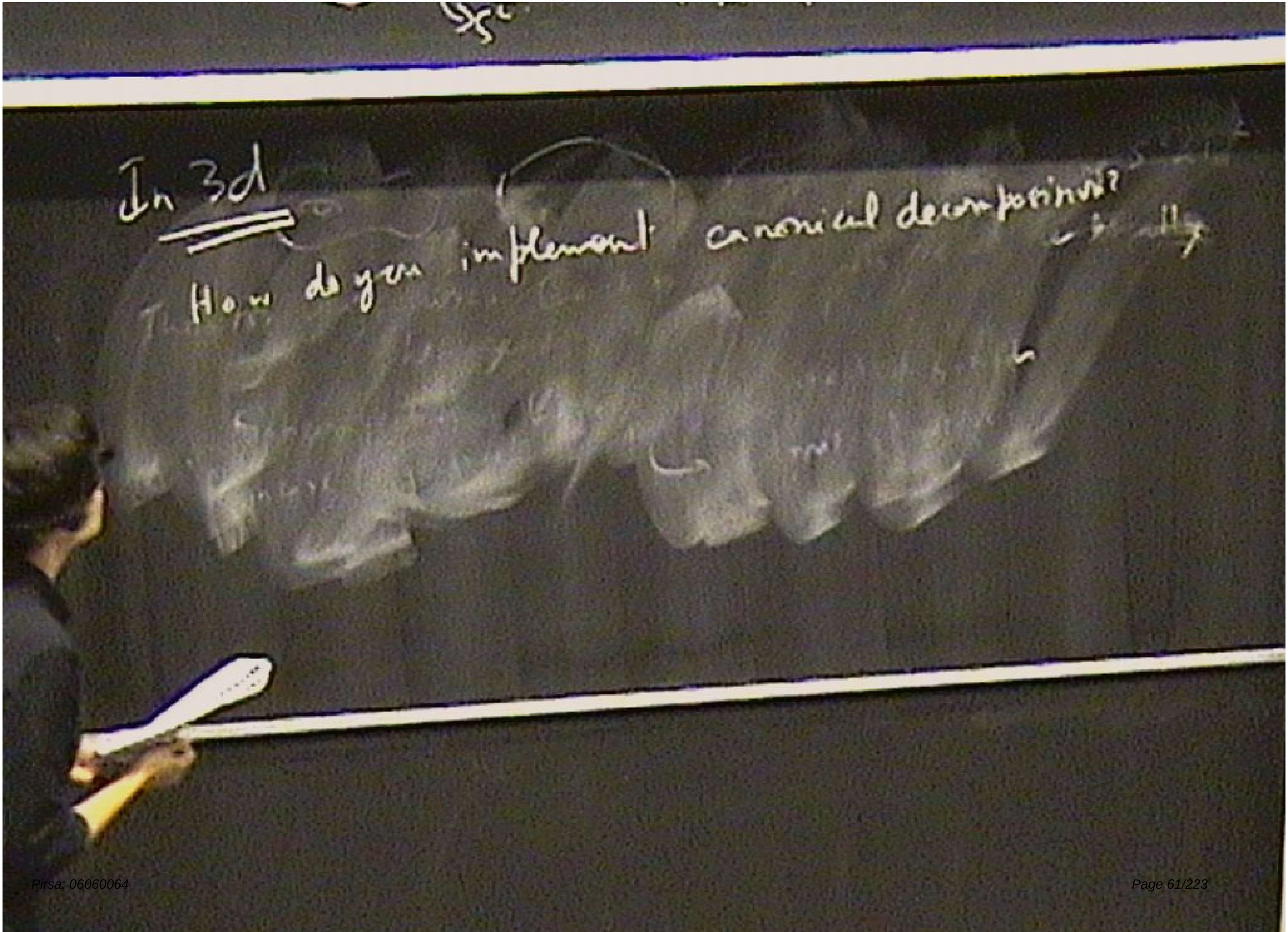

In 3d


In 3d
How do you implement canonical decomposition?

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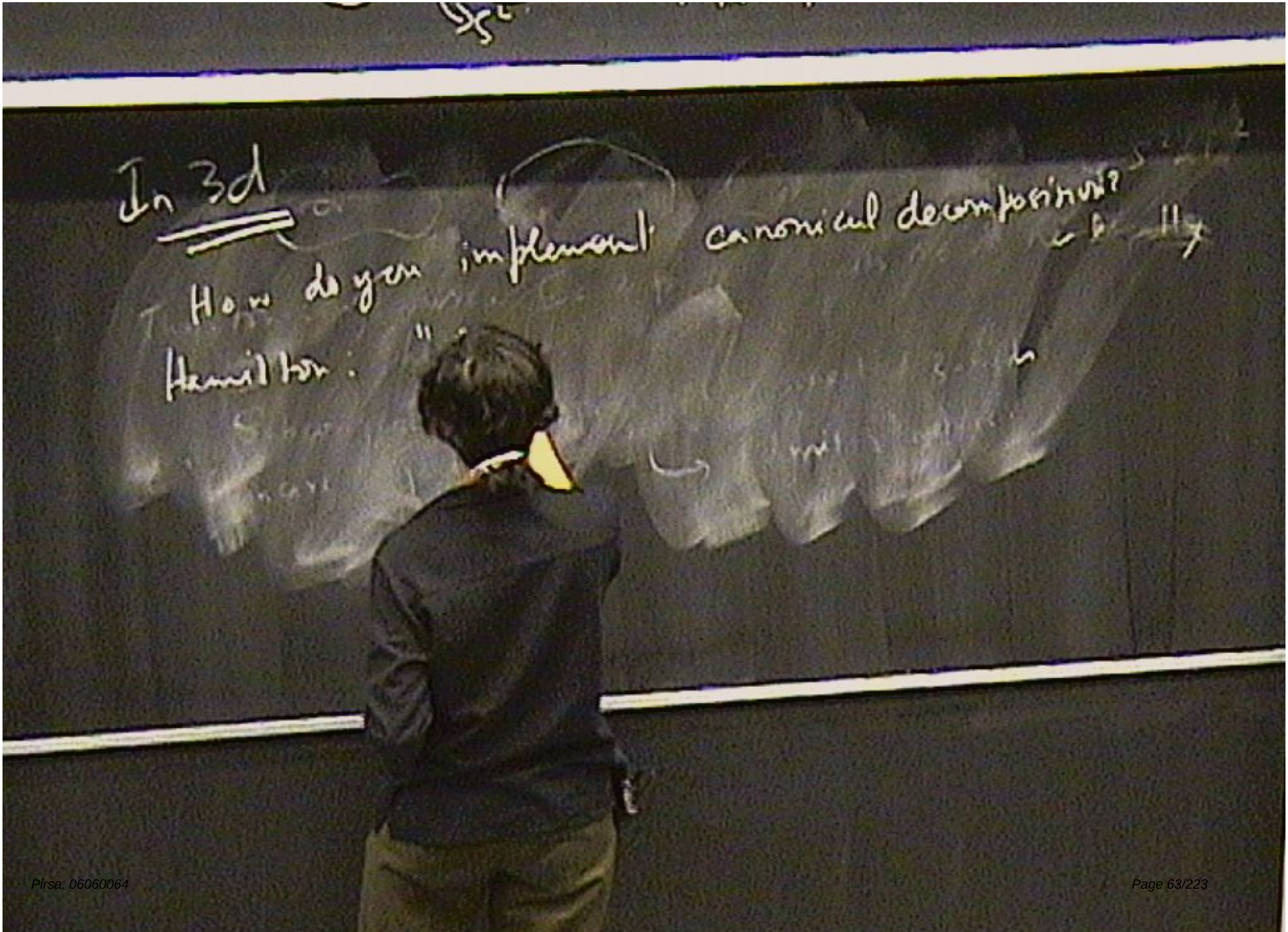
In 3d
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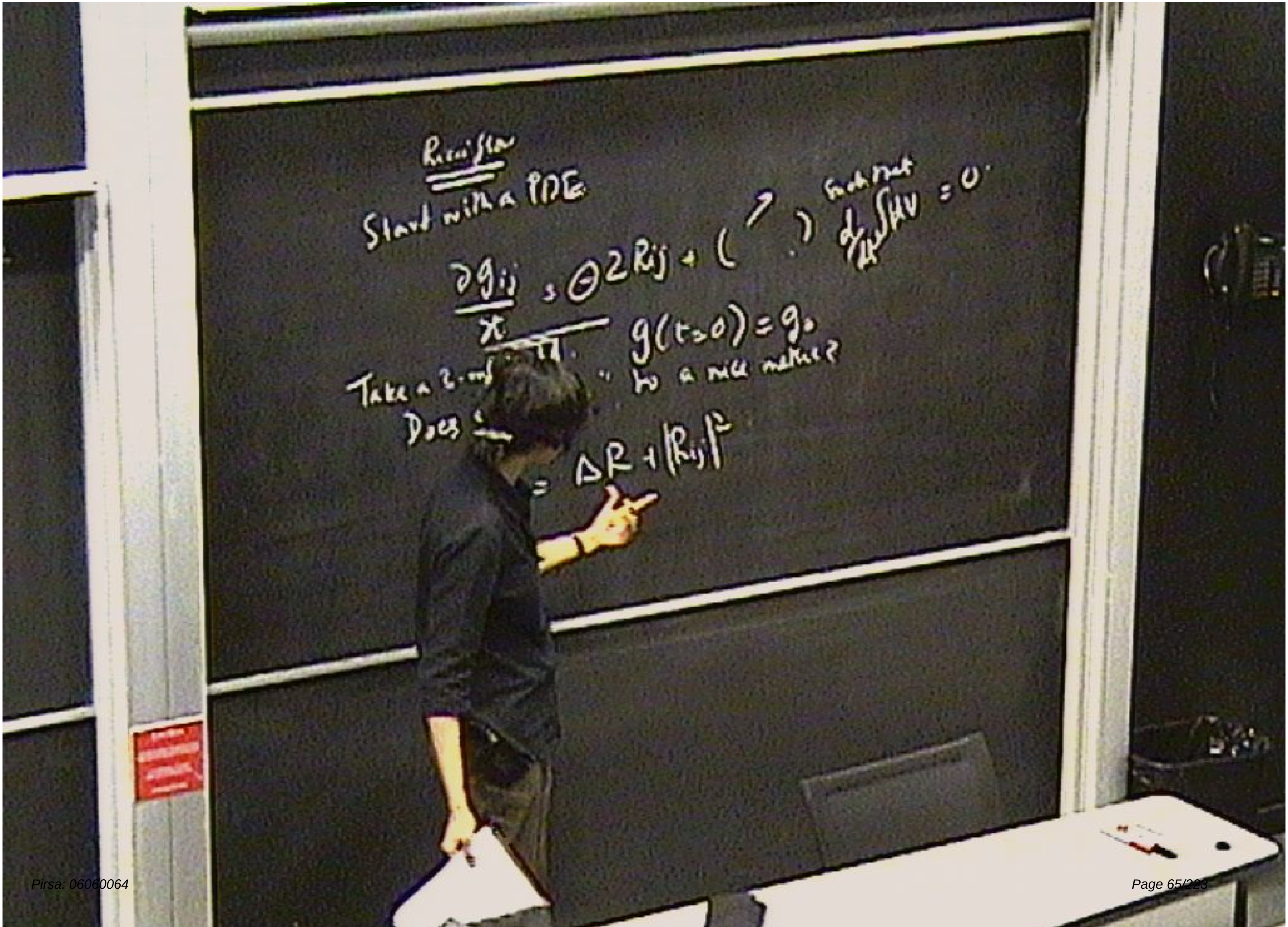
Hamilton:

In 3d

How do you implement canonical decomposition?

Hamilton: "Ricci flow with surgery"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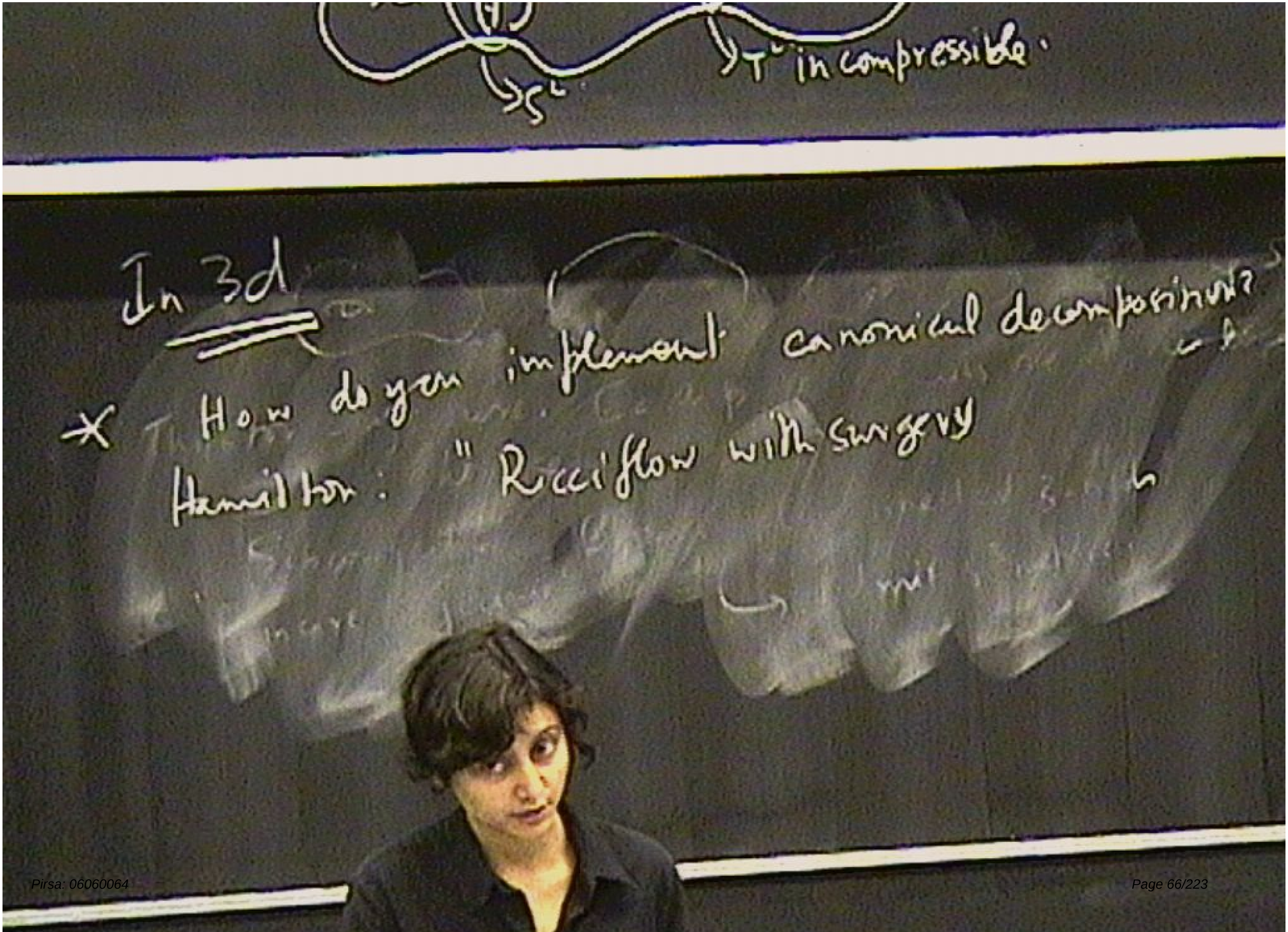
T² incompressible.
S²
In 3d
* How do you implement canonical decompositions?
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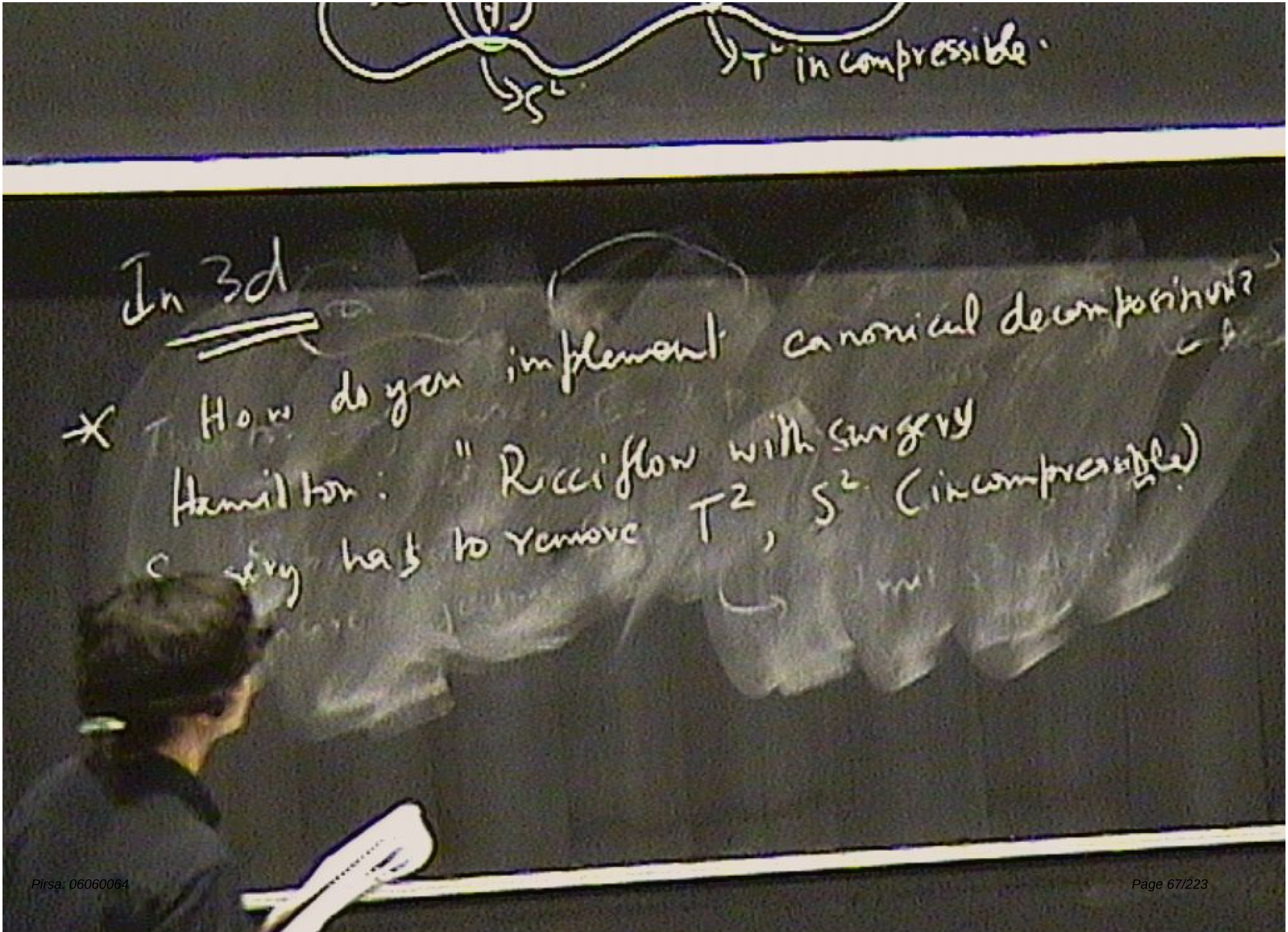
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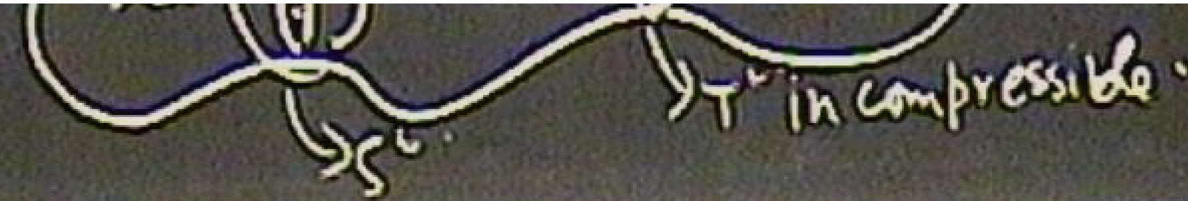
Hamilton: "Ricci flow with surgery
Surgery has to remove T², S² (incompressible)



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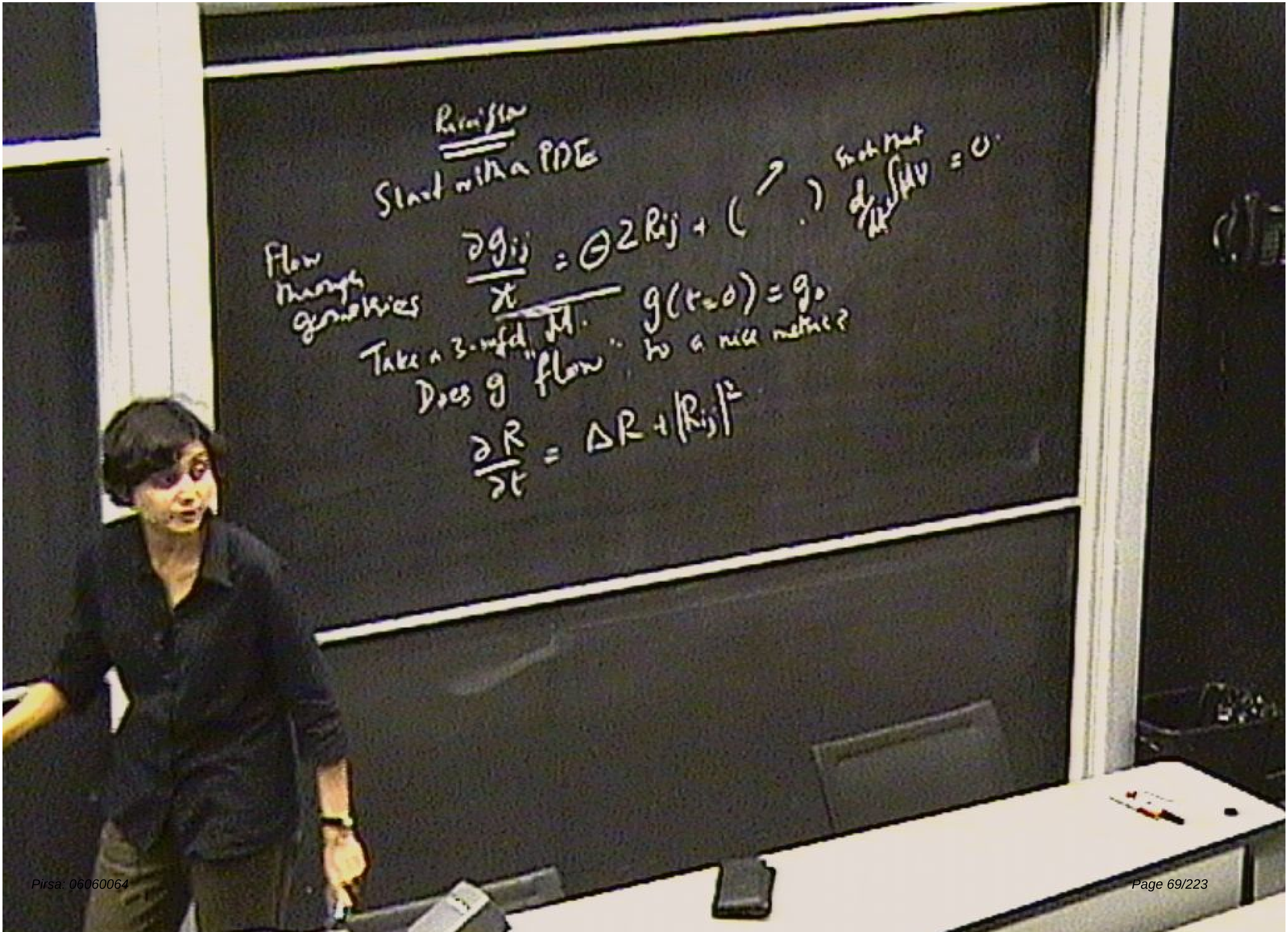
Ricci flow
Start with a PDE
flow through geometries
Take a 3-mfd, M.
Does g "flow" to a nice metric?

In 3d

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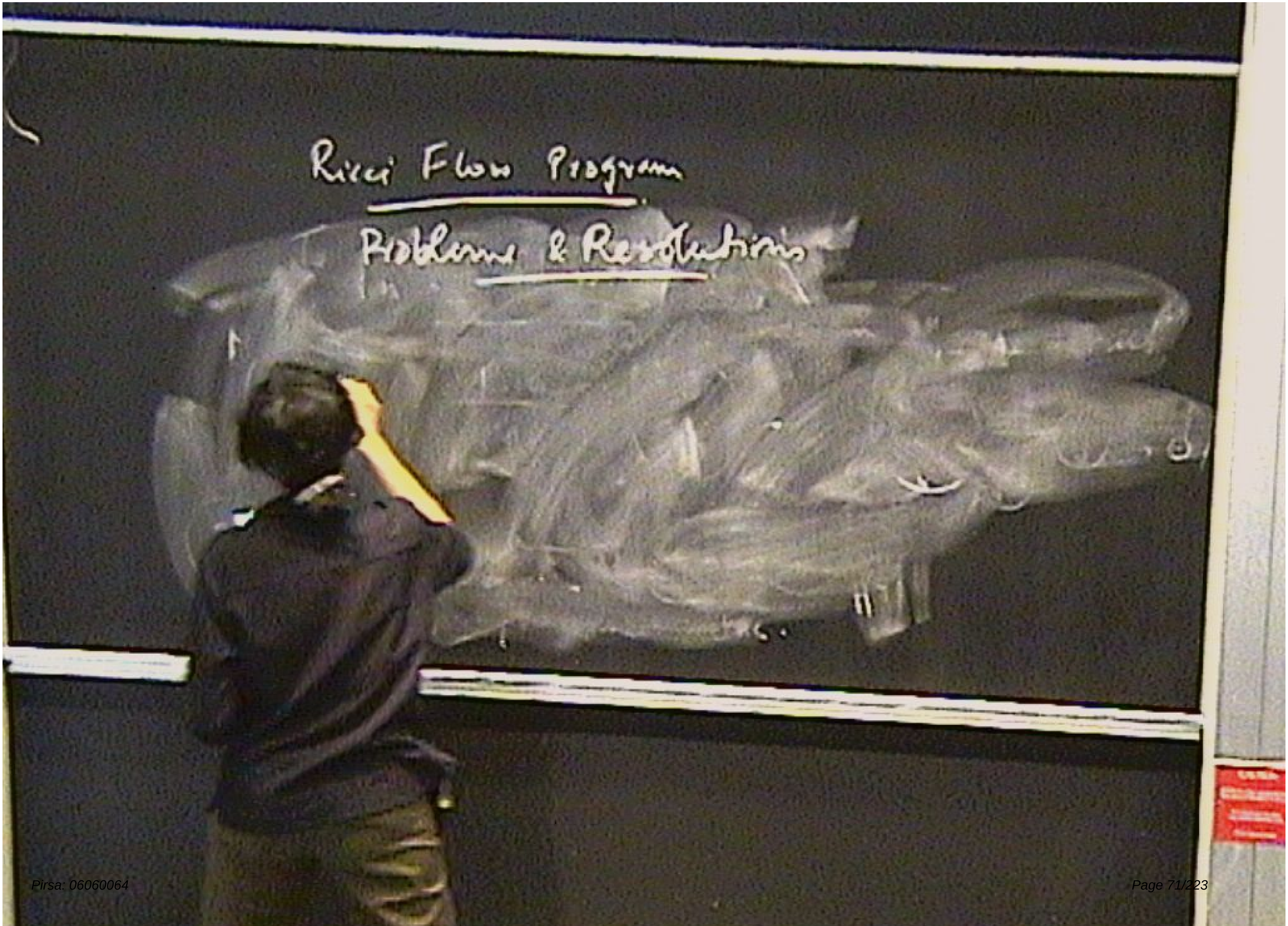
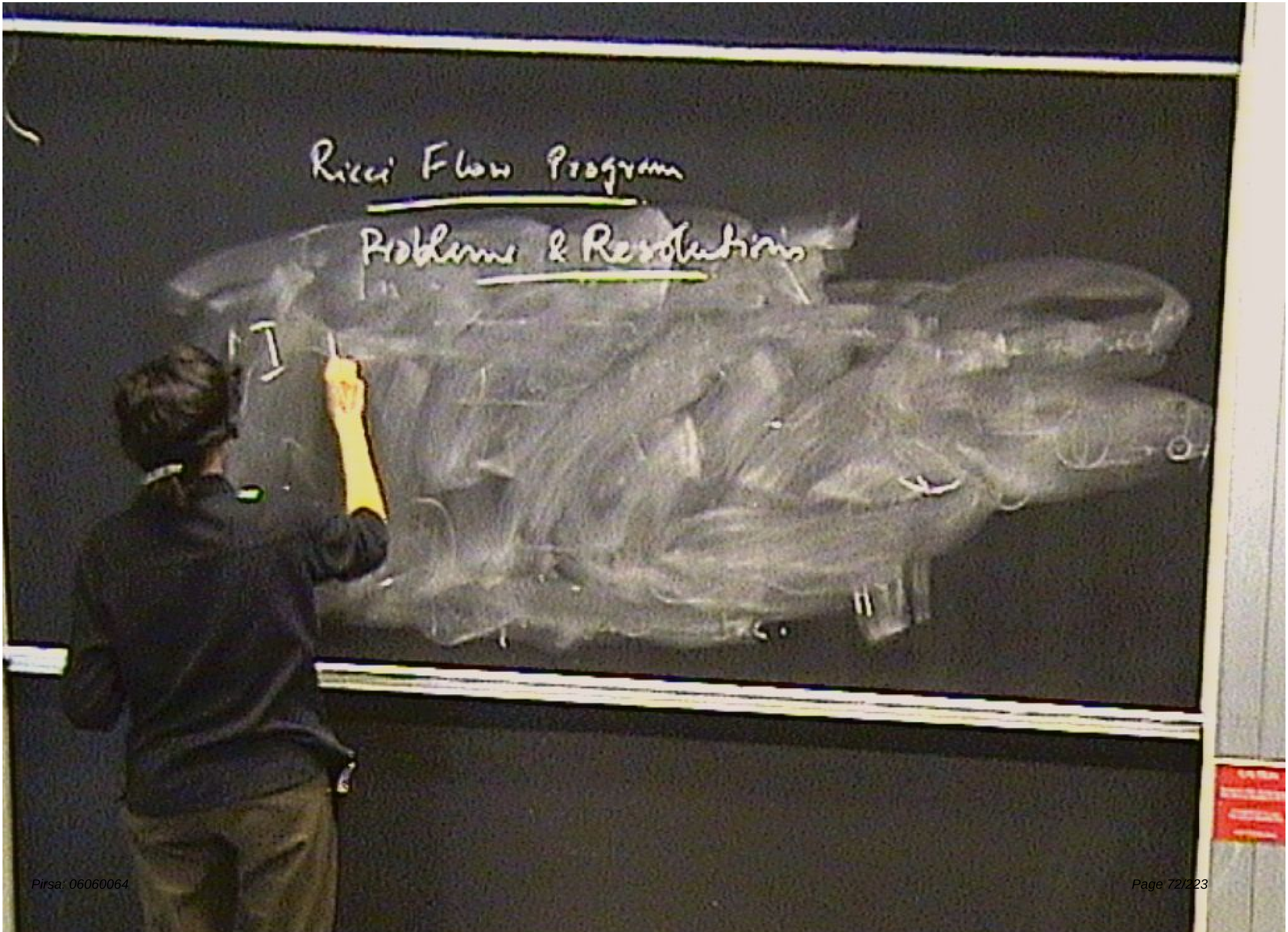
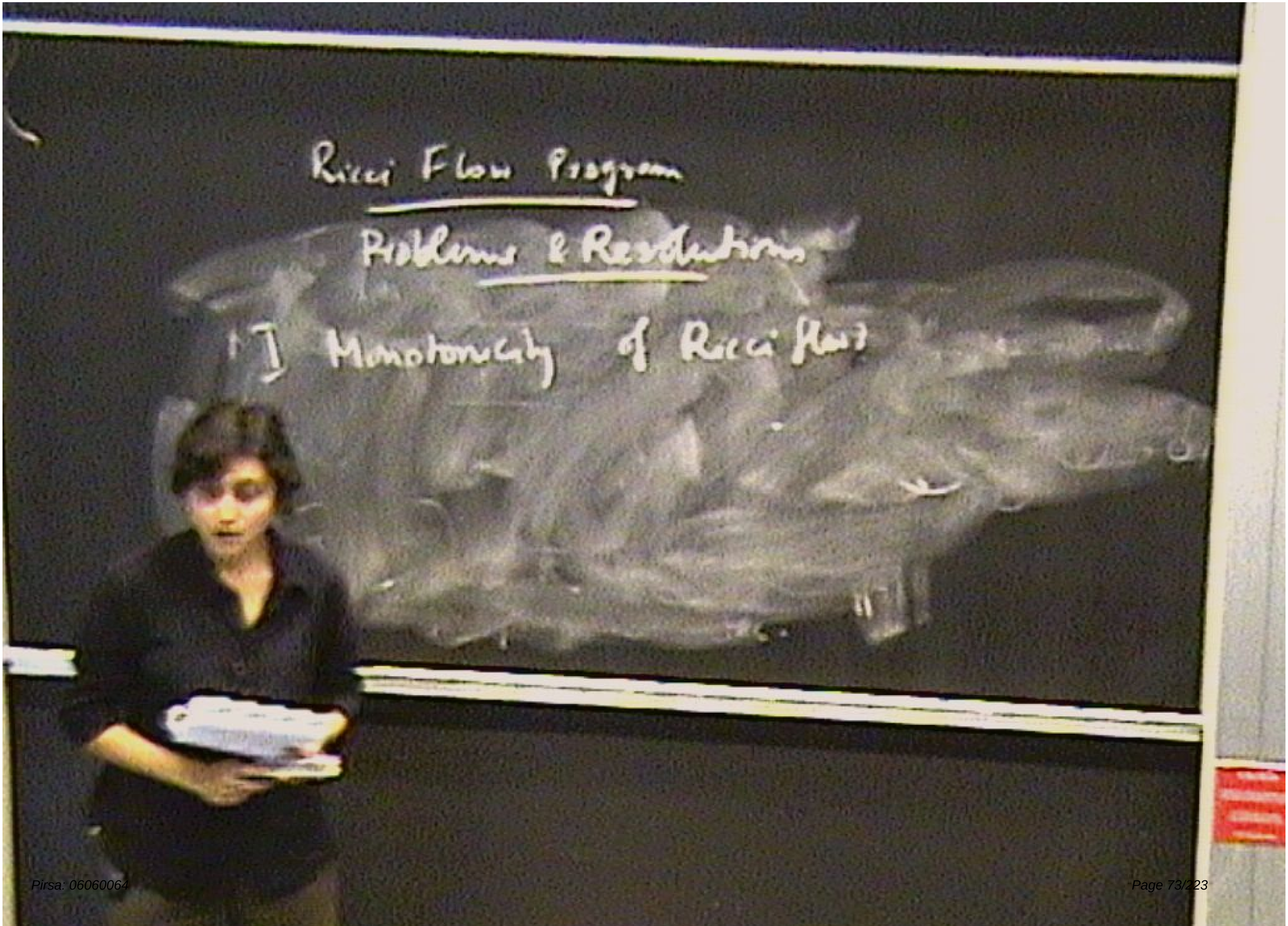
Hamilton: "Ricci flow with surgery"

Surgery has to remove T^2, S^2 (incompressible)


Ricci Flow Program
Problems & Resolutions

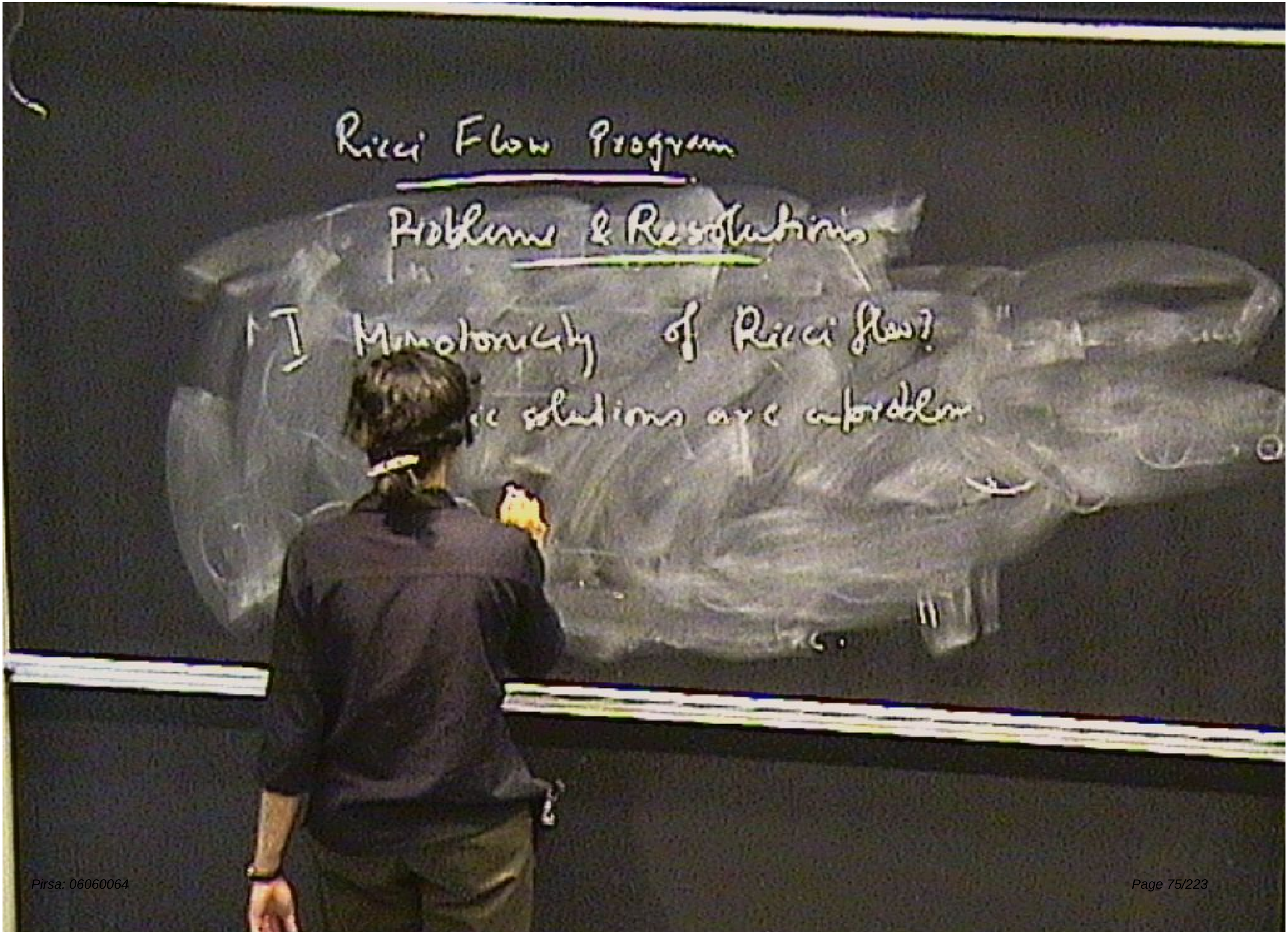

Ricci Flow Program
Problems & Resolutions
I


Ricci Flow Program
Problems & Resolutions
I] Monotonicity of Ricci flow

Ricci Flow Program

Problems & Resolutions

[I] Monotonicity of Ricci flow?


Ricci Flow Program
Problems & Resolutions
I Monotonicity of Ricci flow?
solutions are a problem.

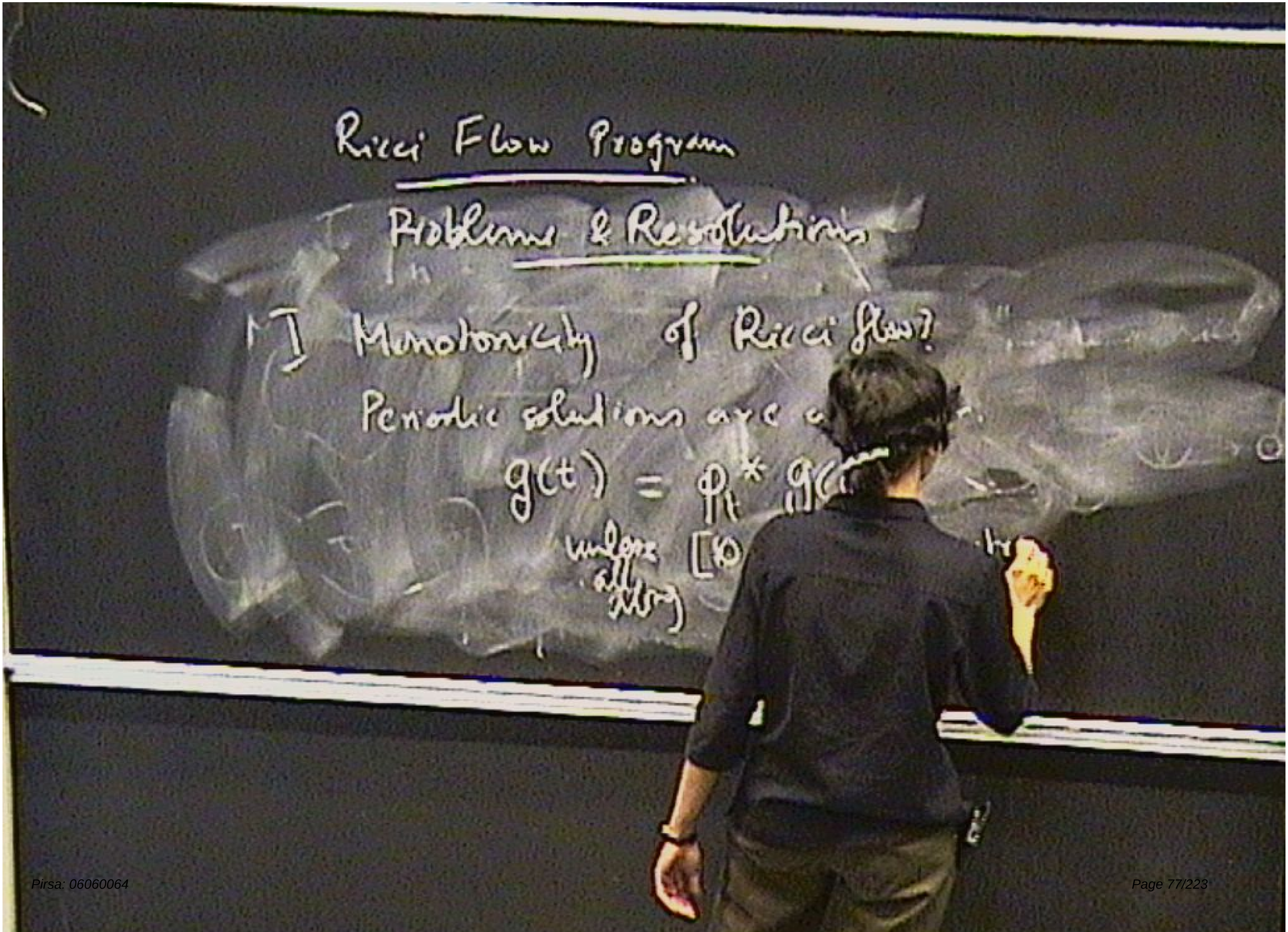
Ricci Flow Program

Problems & Resolutions

I. Monotonicity of Ricci flow?

Periodic solutions are a problem.

$$g(t) = \varphi_t^* g(0)$$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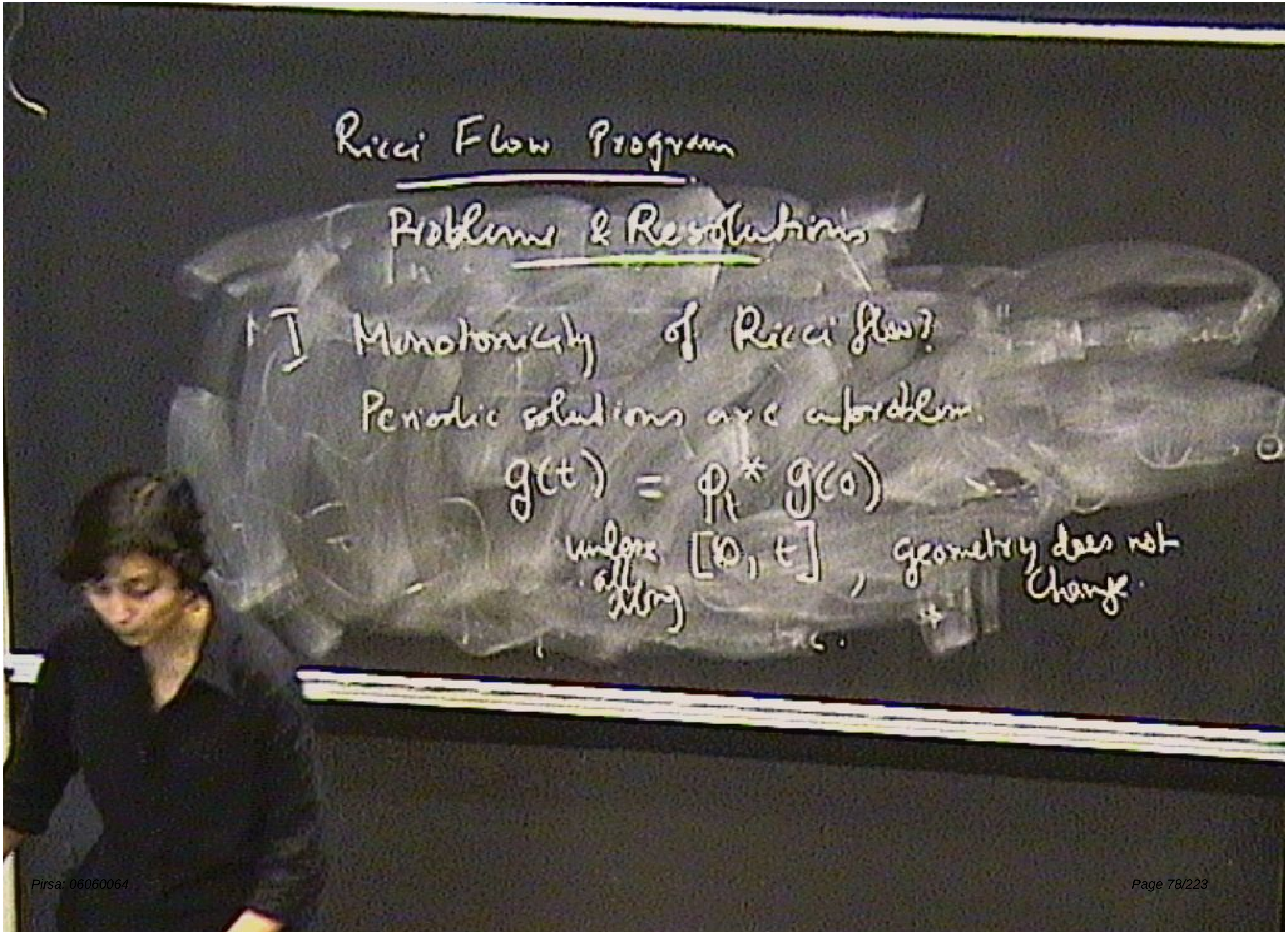
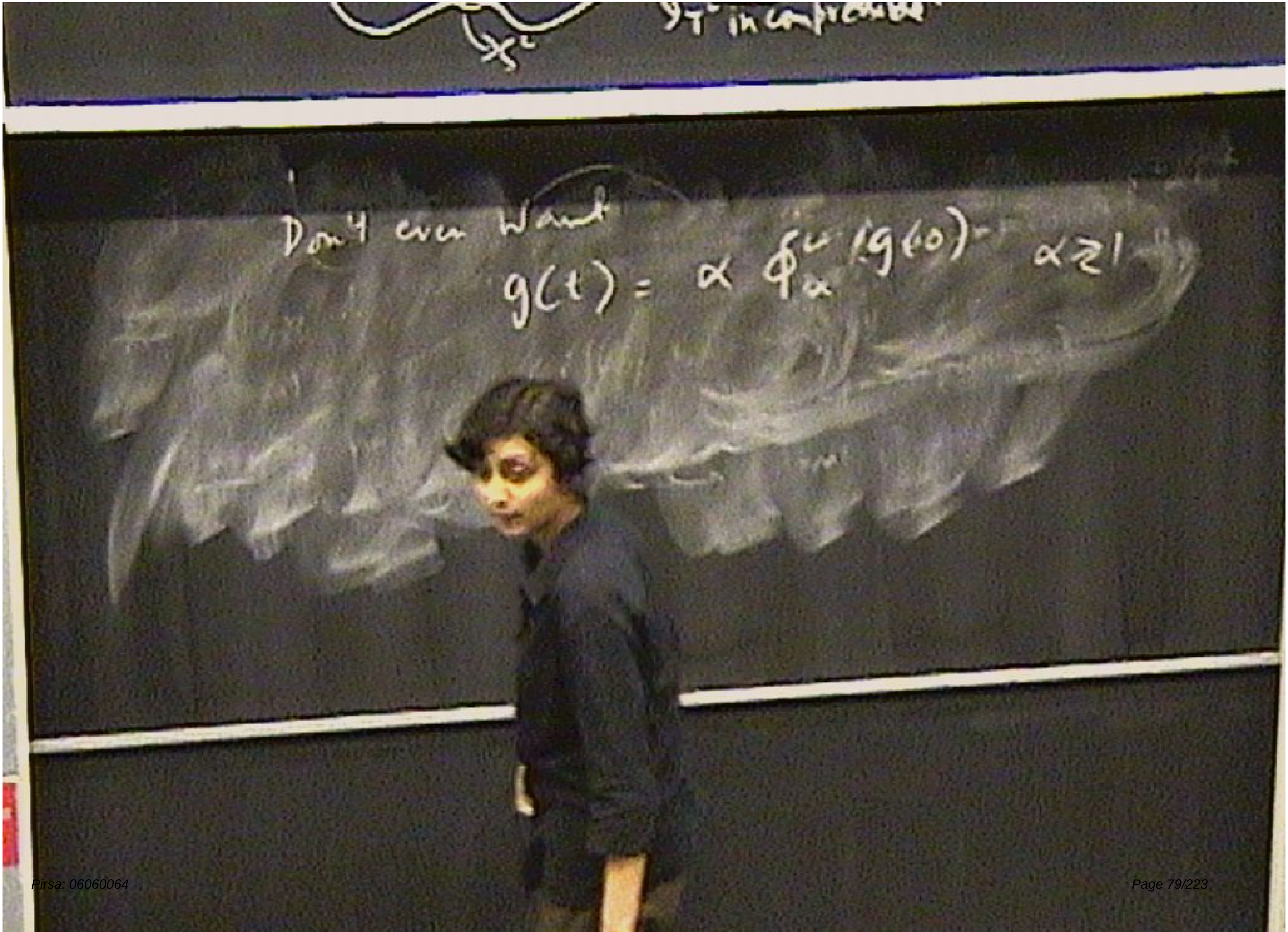
Periodic solutions are
g(t) = φ_t* g(


Ricci Flow Program
Problems & Resolutions
1] Monotonicity of Ricci flow?
Periodic solutions are a problem.
g(t) = φt* g(0)
where [0, t], geometry does not change.
all along


Don't even want
g(t) = α φ_α^ω (g(0))
α ≥ 1

incompressible

Don't even want

$$g(t) = \alpha\, \phi_{\alpha}^{t}(g(0)) \qquad \alpha \geq 1$$

Don't even want

$$g(t) = \alpha\, \phi_\alpha^{*}(g(t_0)) \qquad \alpha \geq 1$$

Rule out periodic solutions.

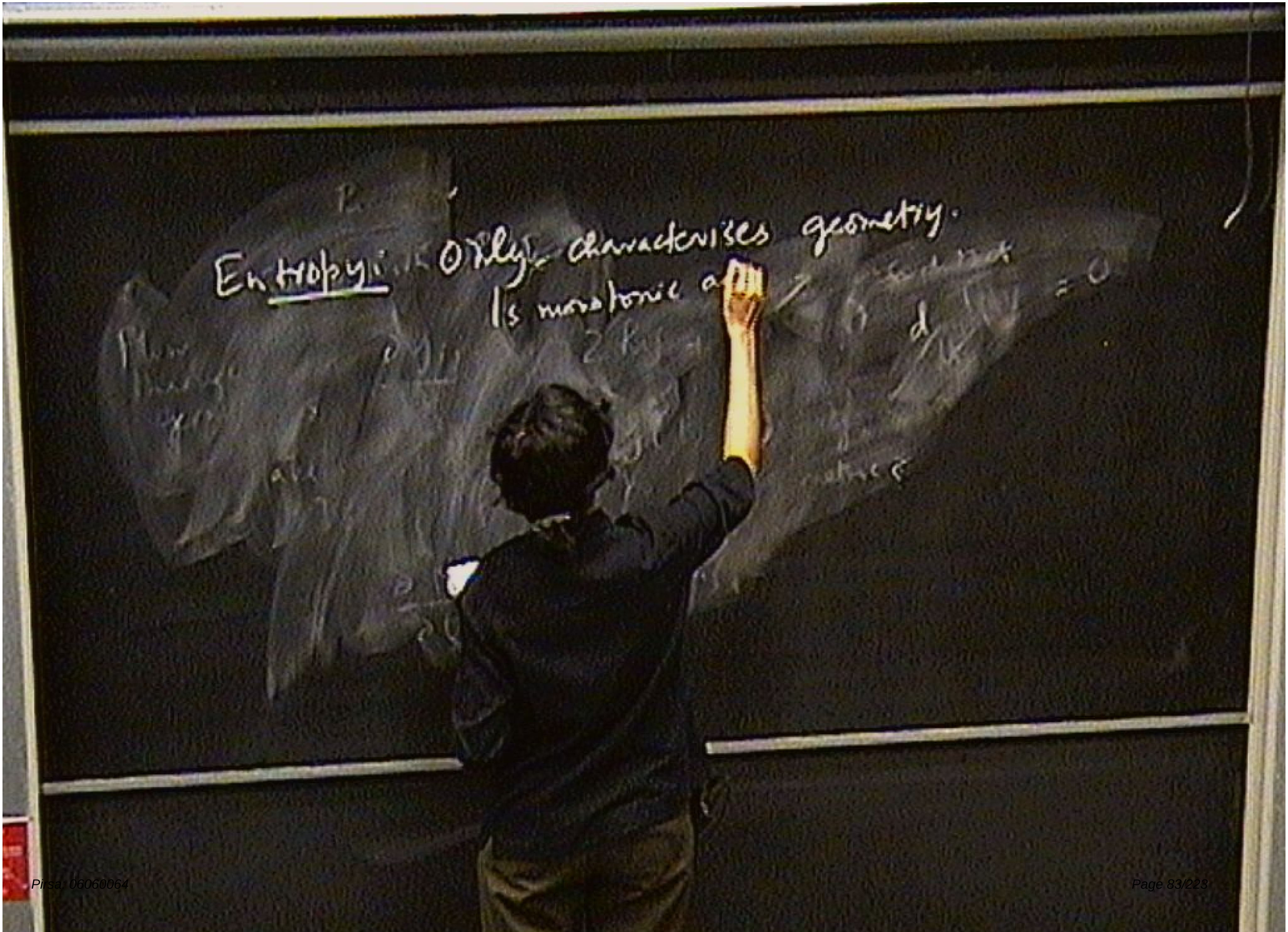
Strategy: Find an "entropy"

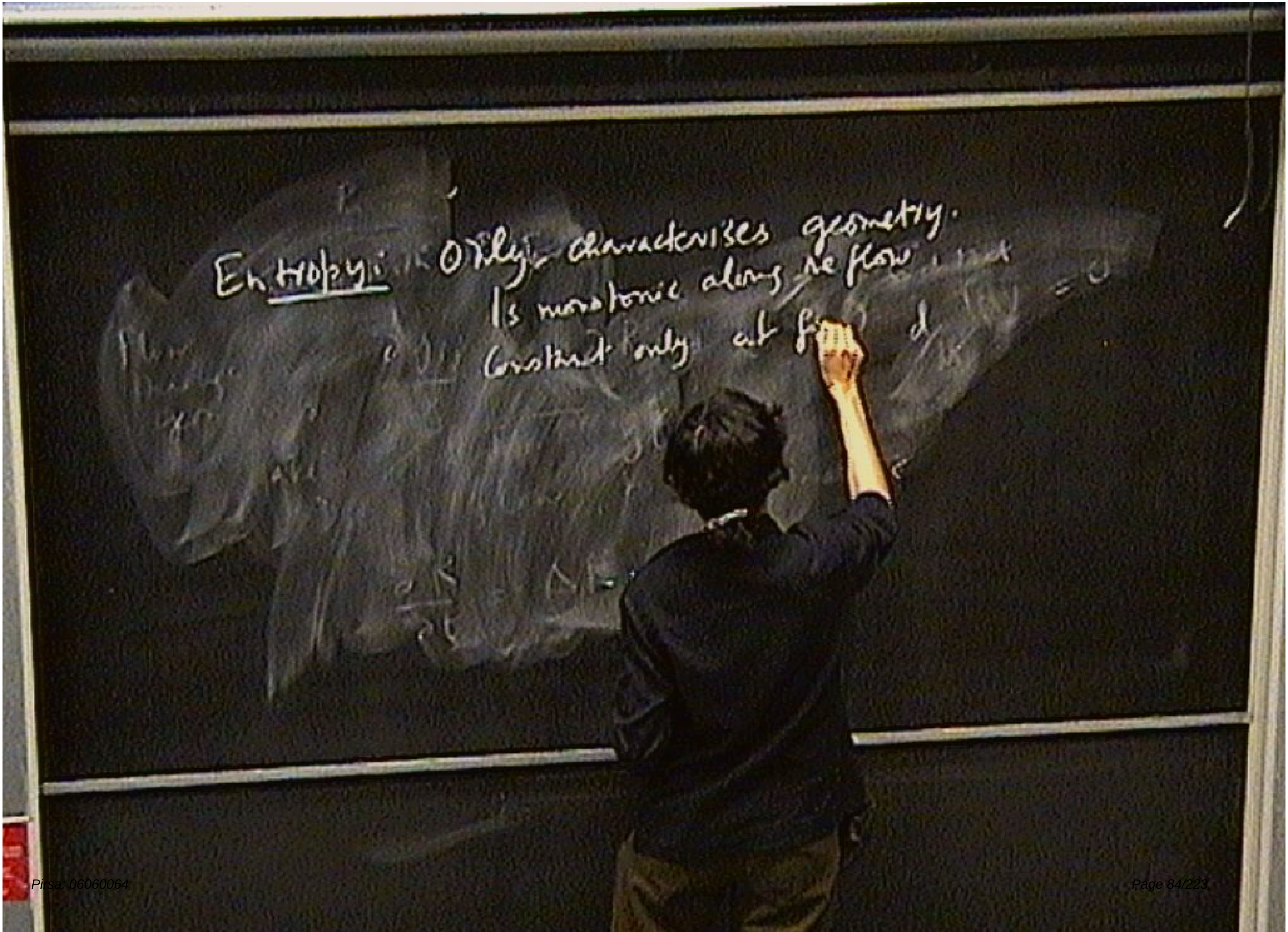
Don't even want

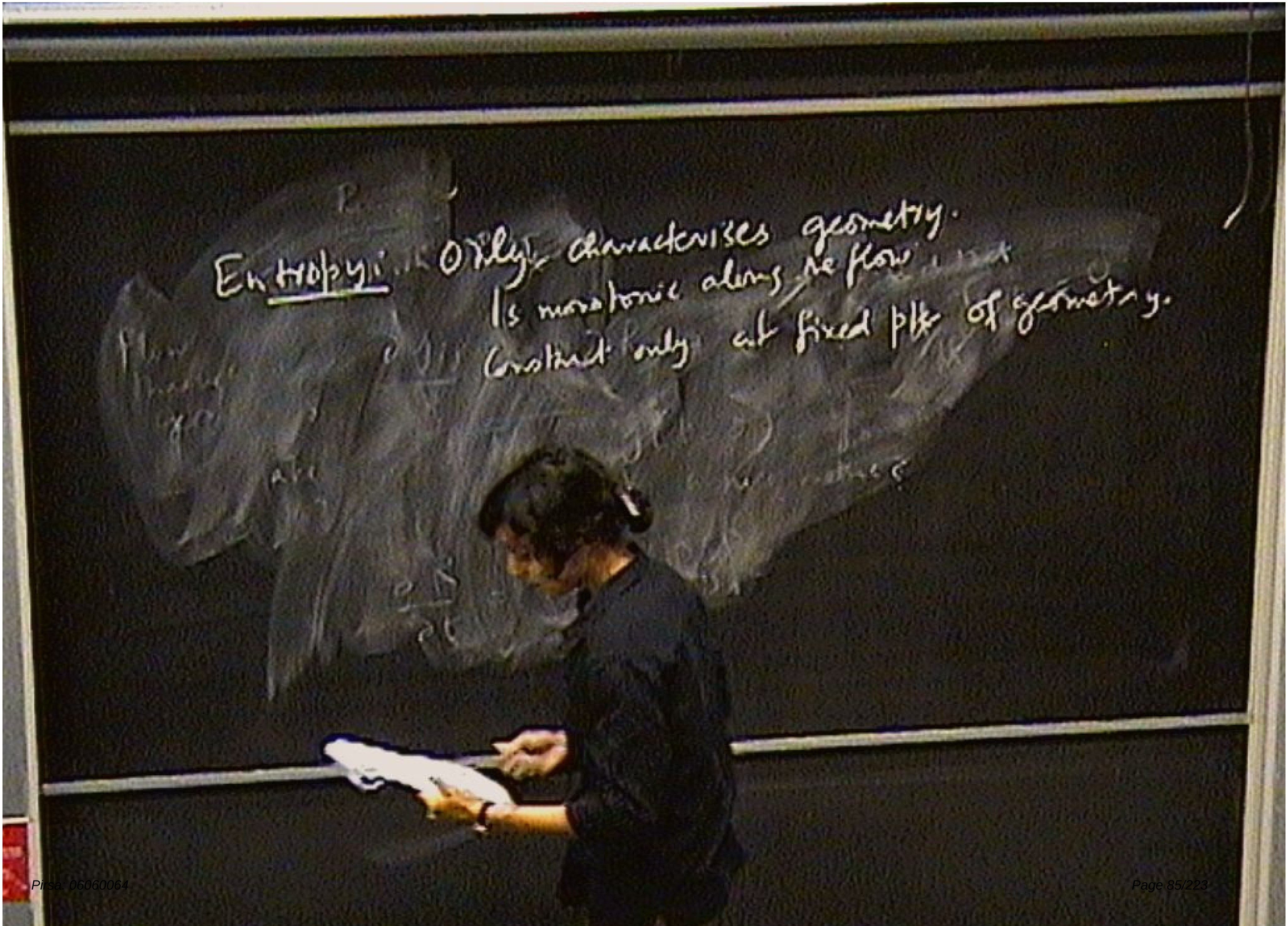
$$g(t) = \alpha\, \phi_{\alpha}^{*}(g(t_0)) \qquad \alpha \geq 1$$

Rule out periodic solutions $\rightarrow$ $\Big\{$ Perelman (2002)

Strategy: Find an "entropy"


characterises geometry.
Is monotonic


Entropy:
characterises geometry.
Is monotonic along the flow
Constant only at


Entropy: Only characterises geometry.
Is monotonic along the flow
Constant only at fixed pts of geometry.

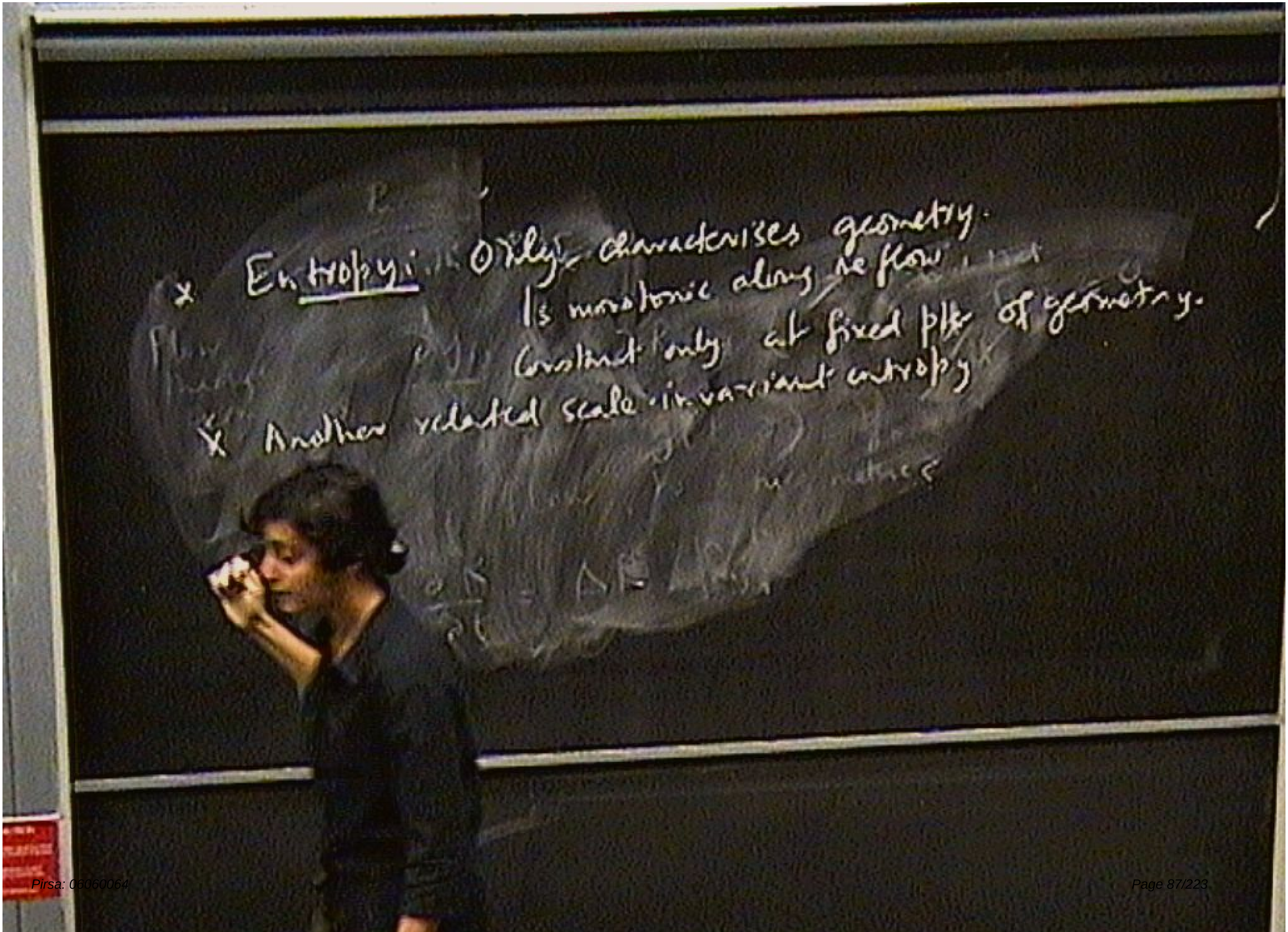
* Entropy: Only characterises geometry.

  Is monotonic along the flow

  Constant only at fixed pts of geometry.

  (invariant entropy)

* Another related scale-invariant


* Entropy: Only characterises geometry.
Is monotonic along the flow
Constant only at fixed pts of geometry.
* Another related scale invariant entropy

* Entropy: Only characterises geometry.
Is monotonic along the flow
Constant only at fixed pts of geometry.

* Another related scale-invariant entropy

II. Singular behaviour under Ricci flow.

* Entropy: Only characterises geometry.
Is monotonic along the flow
Constant only at fixed pts of geometry.

* Another related scale-invariant entropy

II Singular behaviour under Ricci flow

* Entropy: Only characterises geometry.
  Is monotonic along the flow
  Constant only at fixed pts of geometry.
* Another related scale-invariant entropy

II : Singular behaviour under Ricci flow:

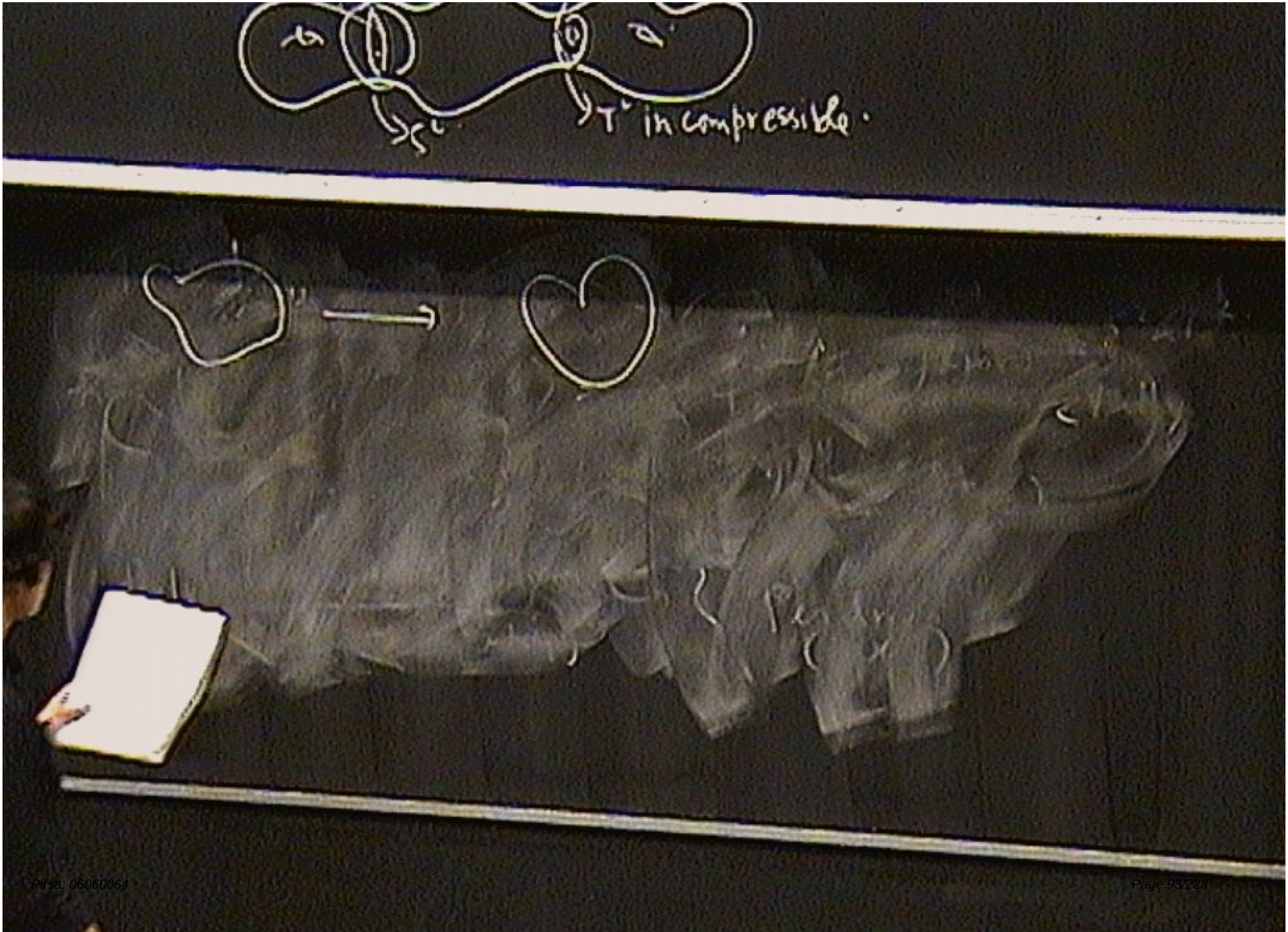
* Entropy: Only characterises geometry.
  Is monotonic along the flow.
  Constant only at fixed pts of geometry.
* Another related scale-invariant entropy

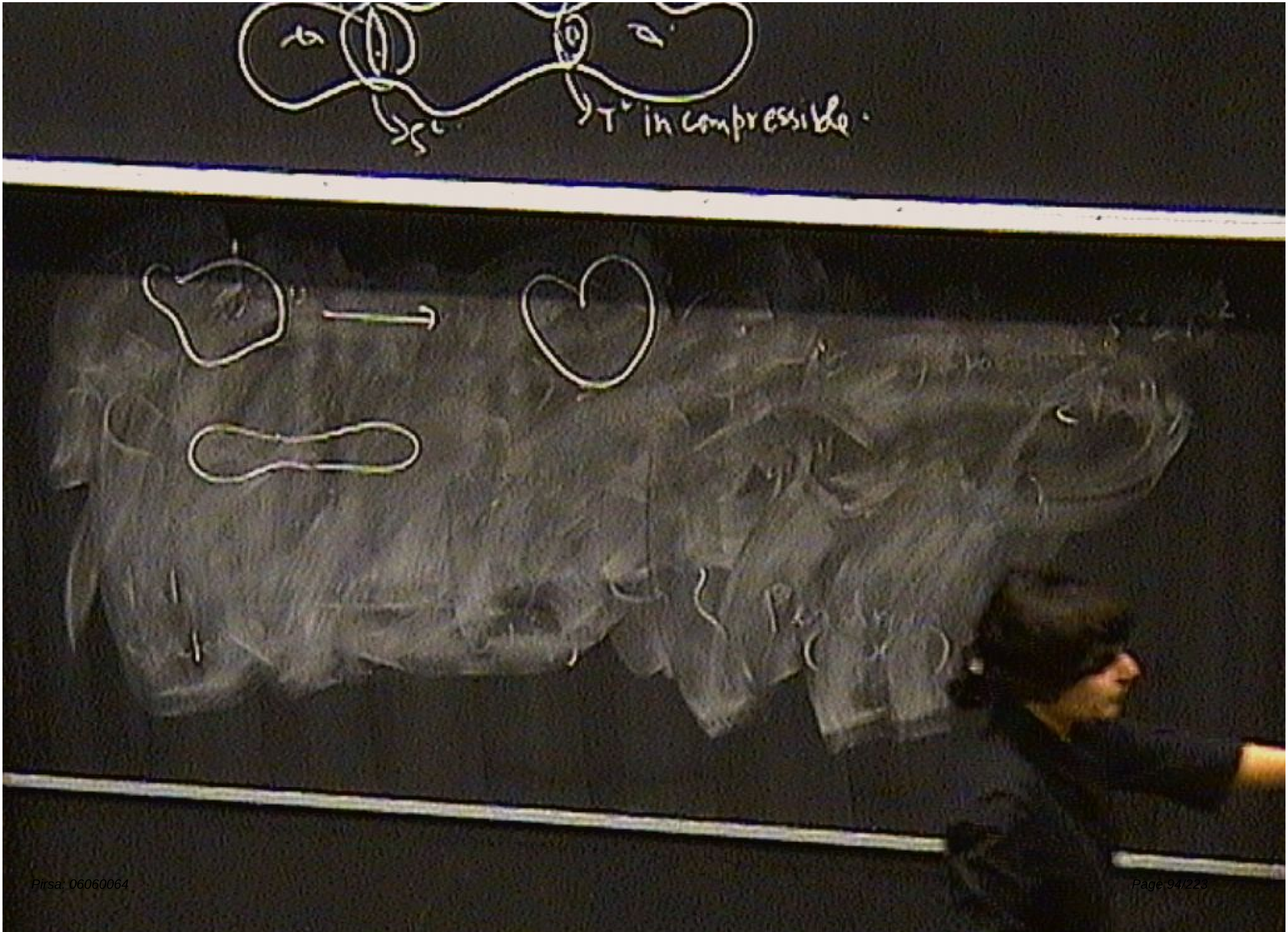
II. Singular behaviour under Ricci flow:

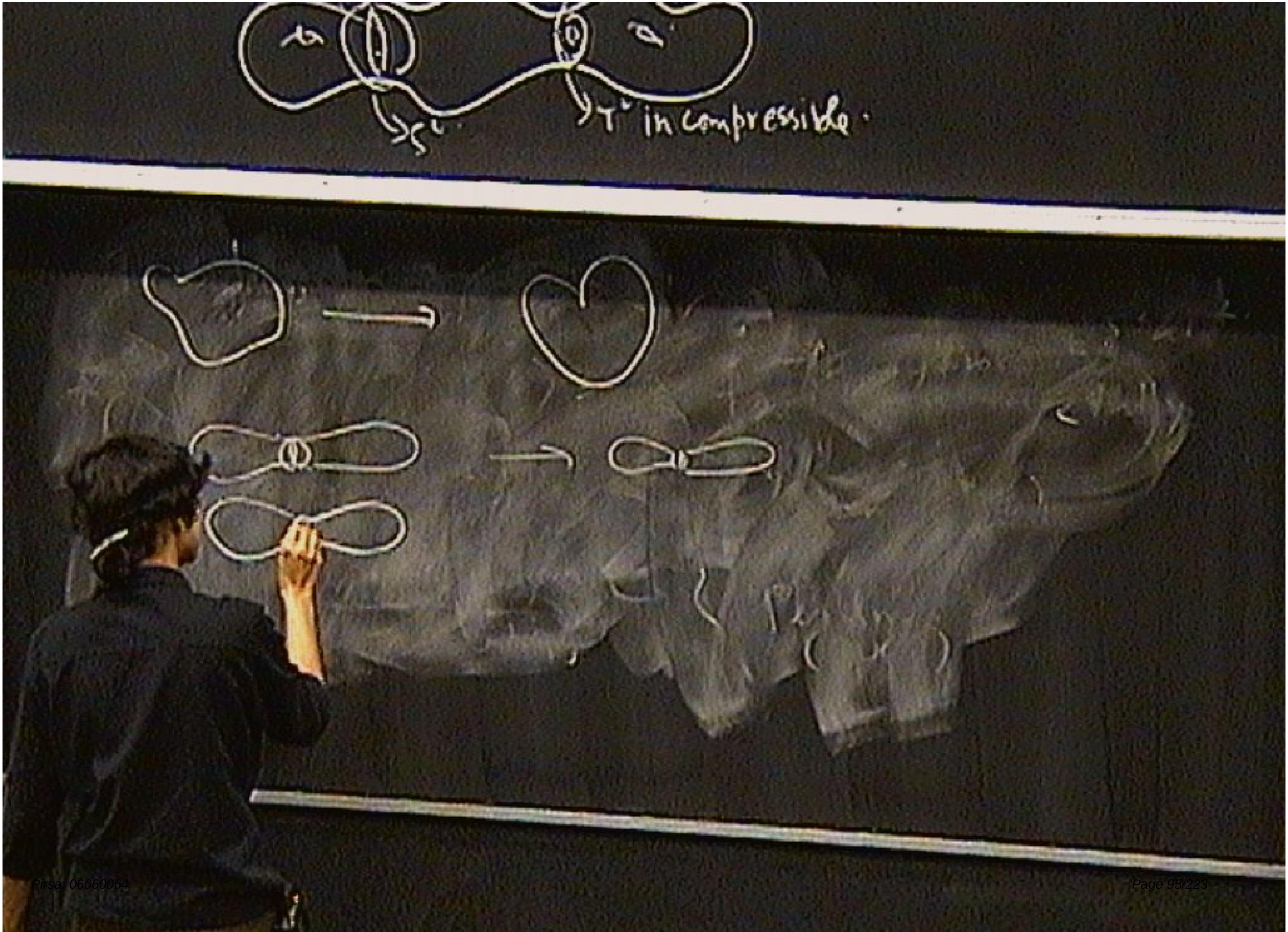
* Entropy: Only characterises geometry.
Is monotonic along the flow
Constant only at fixed pts of geometry.

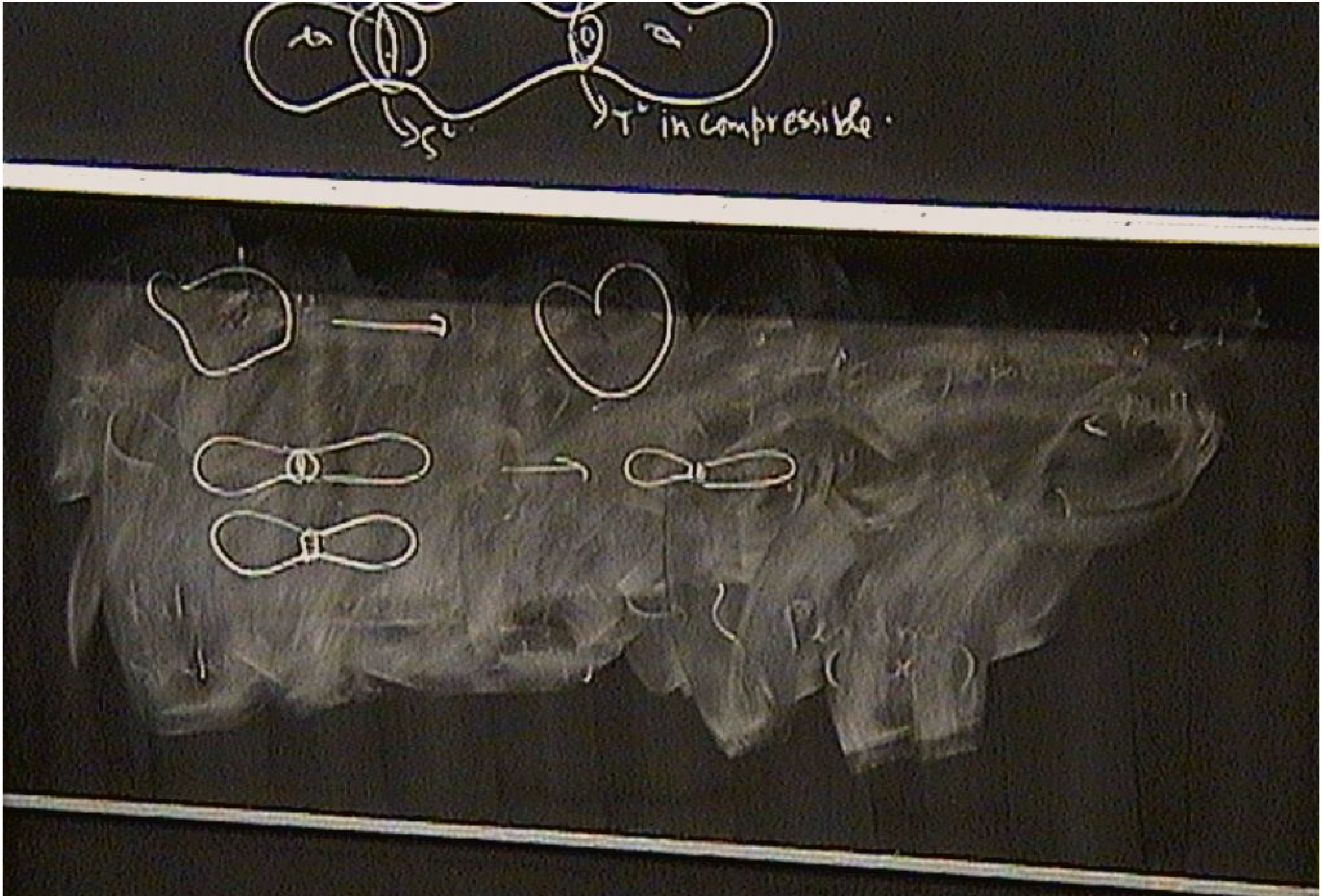
* Another related scale-invariant entropy

II: Singular behaviour under Ricci flow

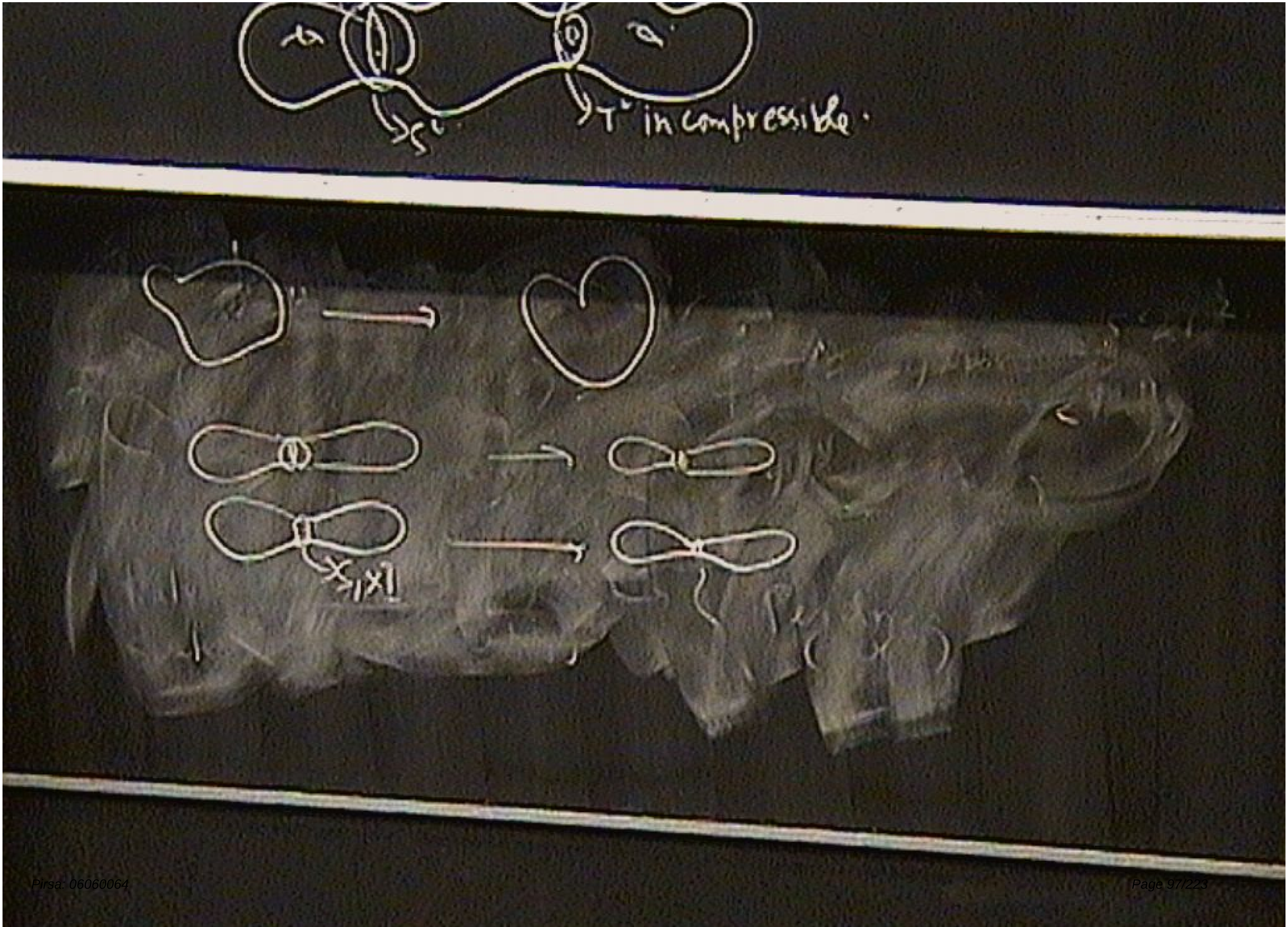

S^1
T^2 incompressible.

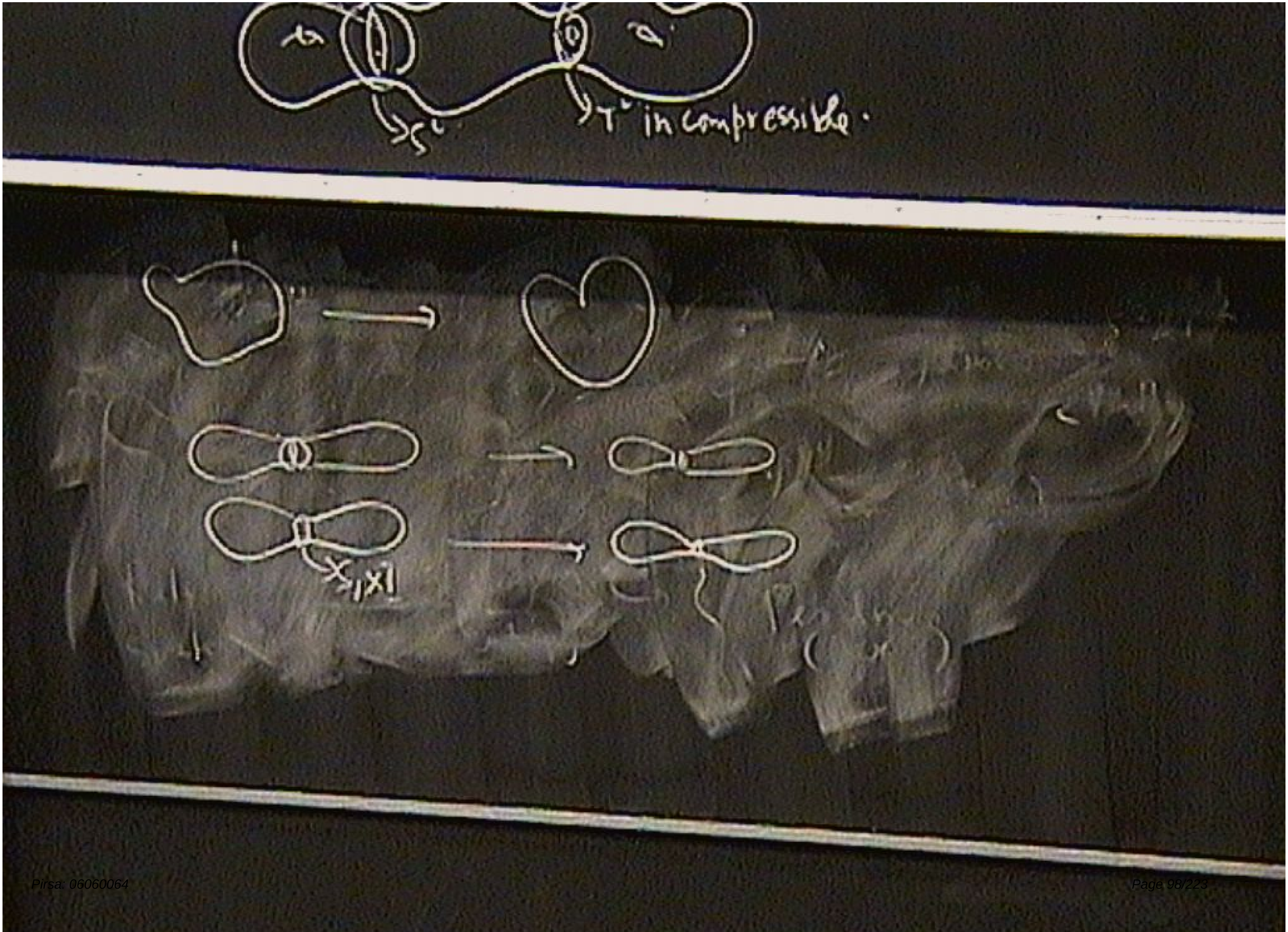

incompressible.

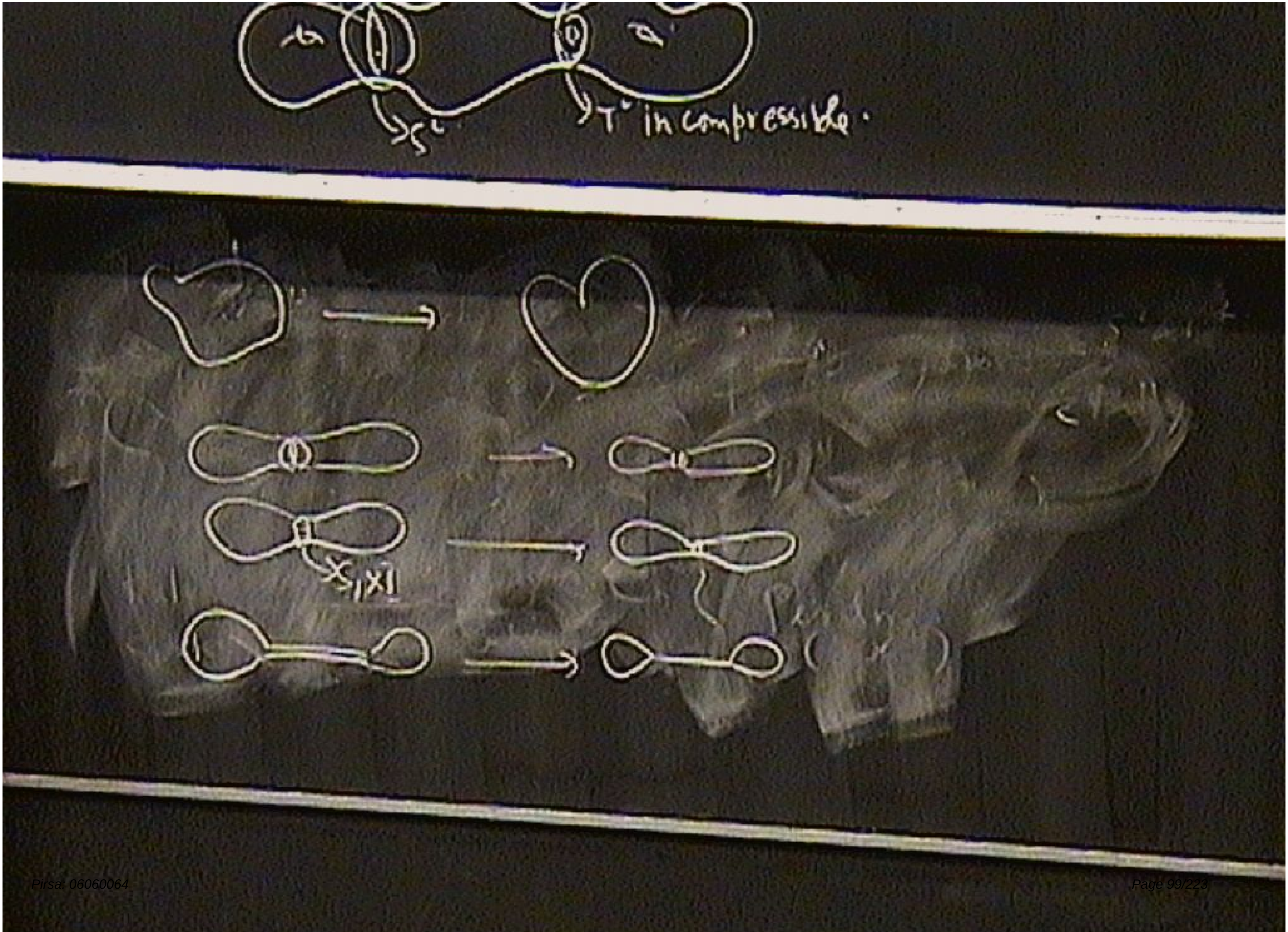

S^1
T^2 incompressible.

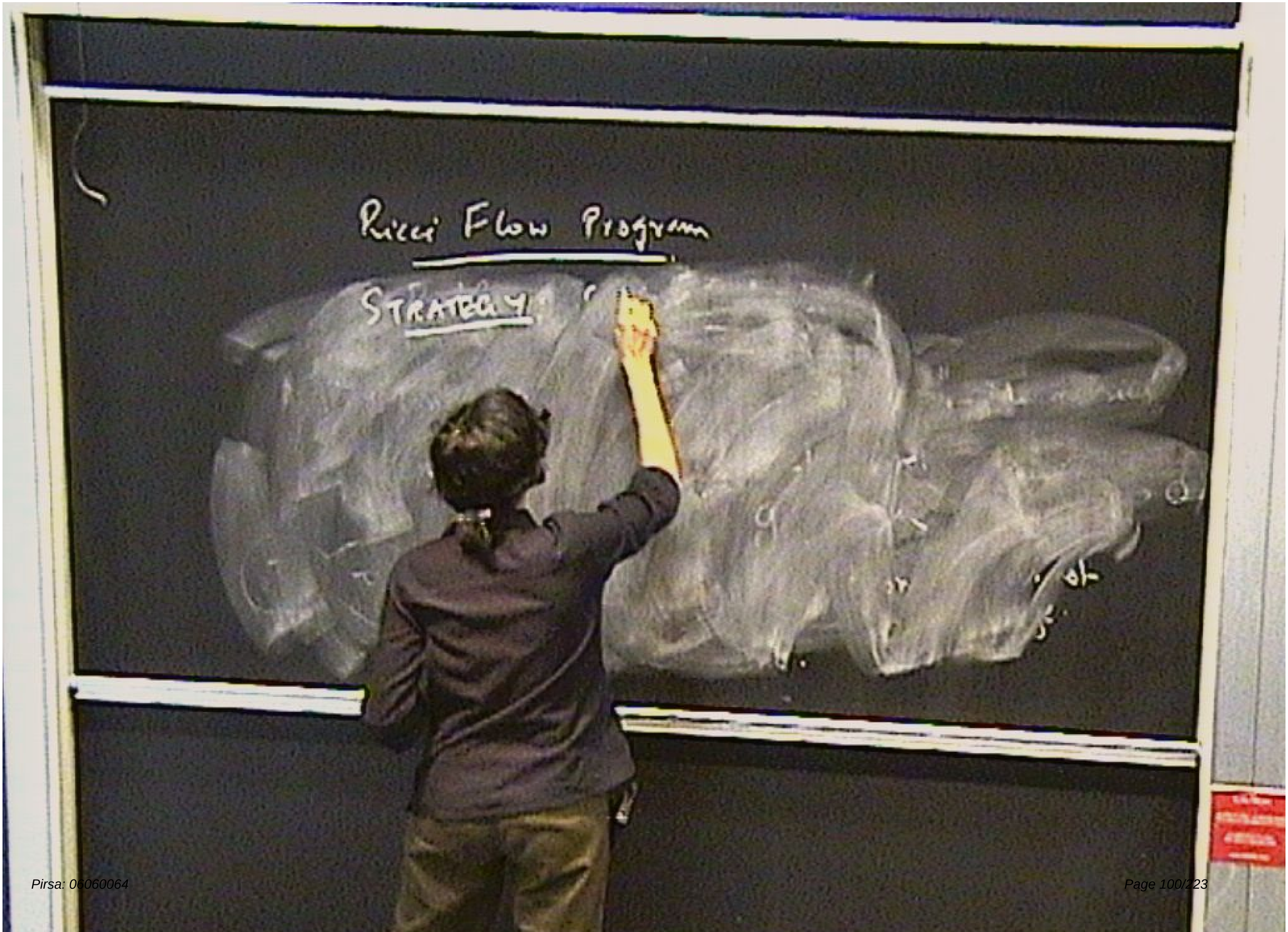


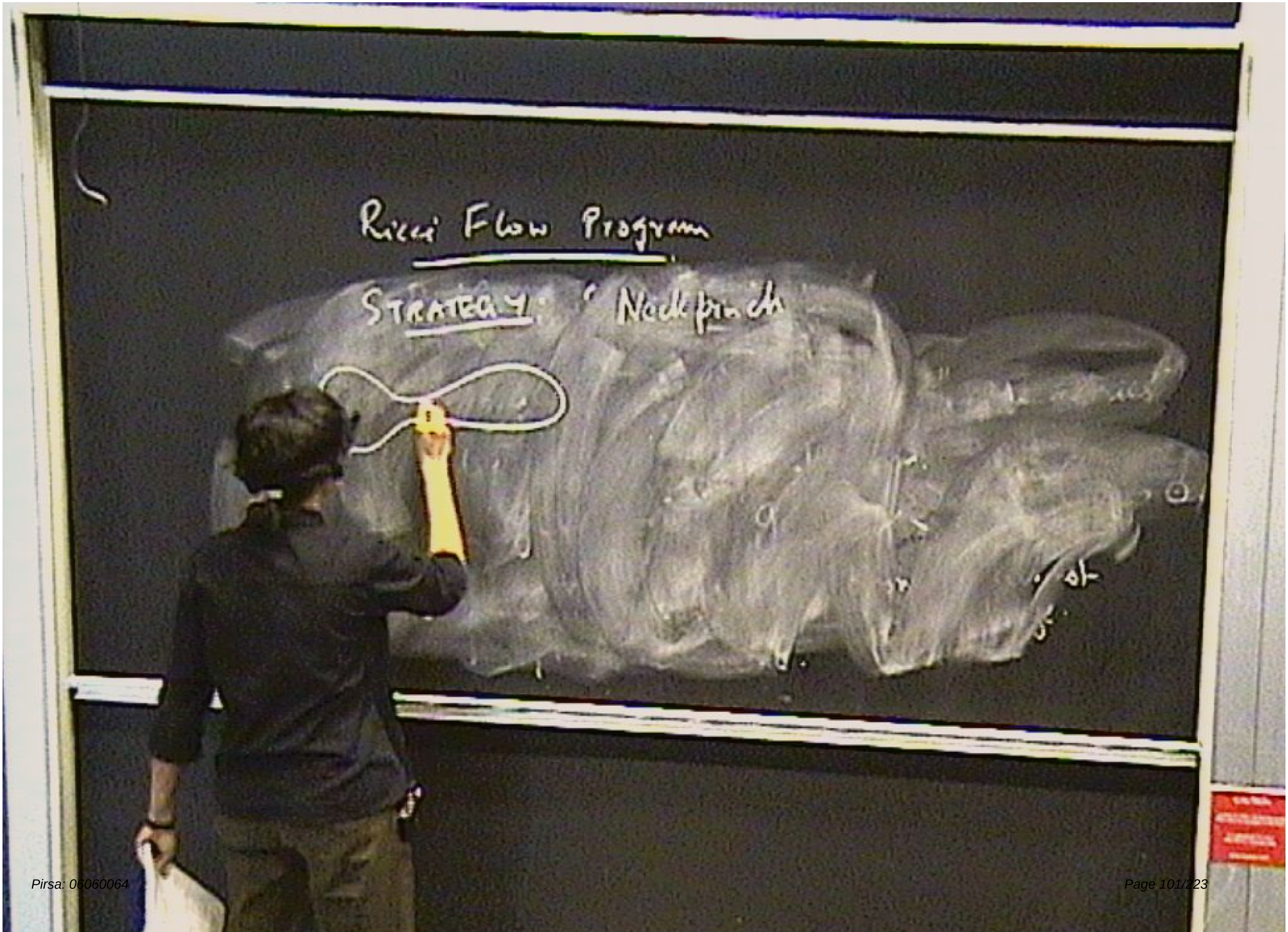
S^2
T^2 incompressible.

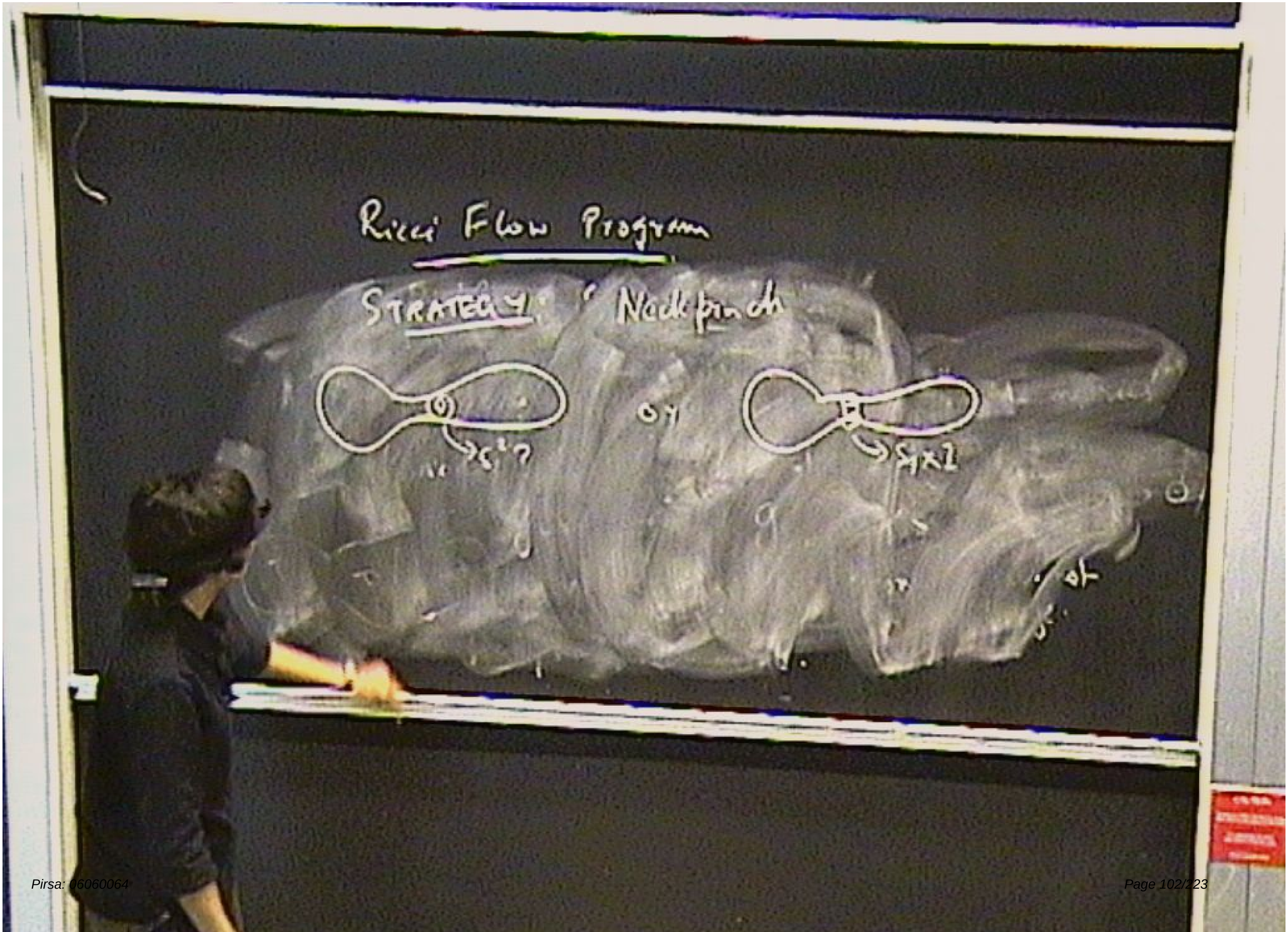

S^2
T^2 incompressible
$X_1 \times I$

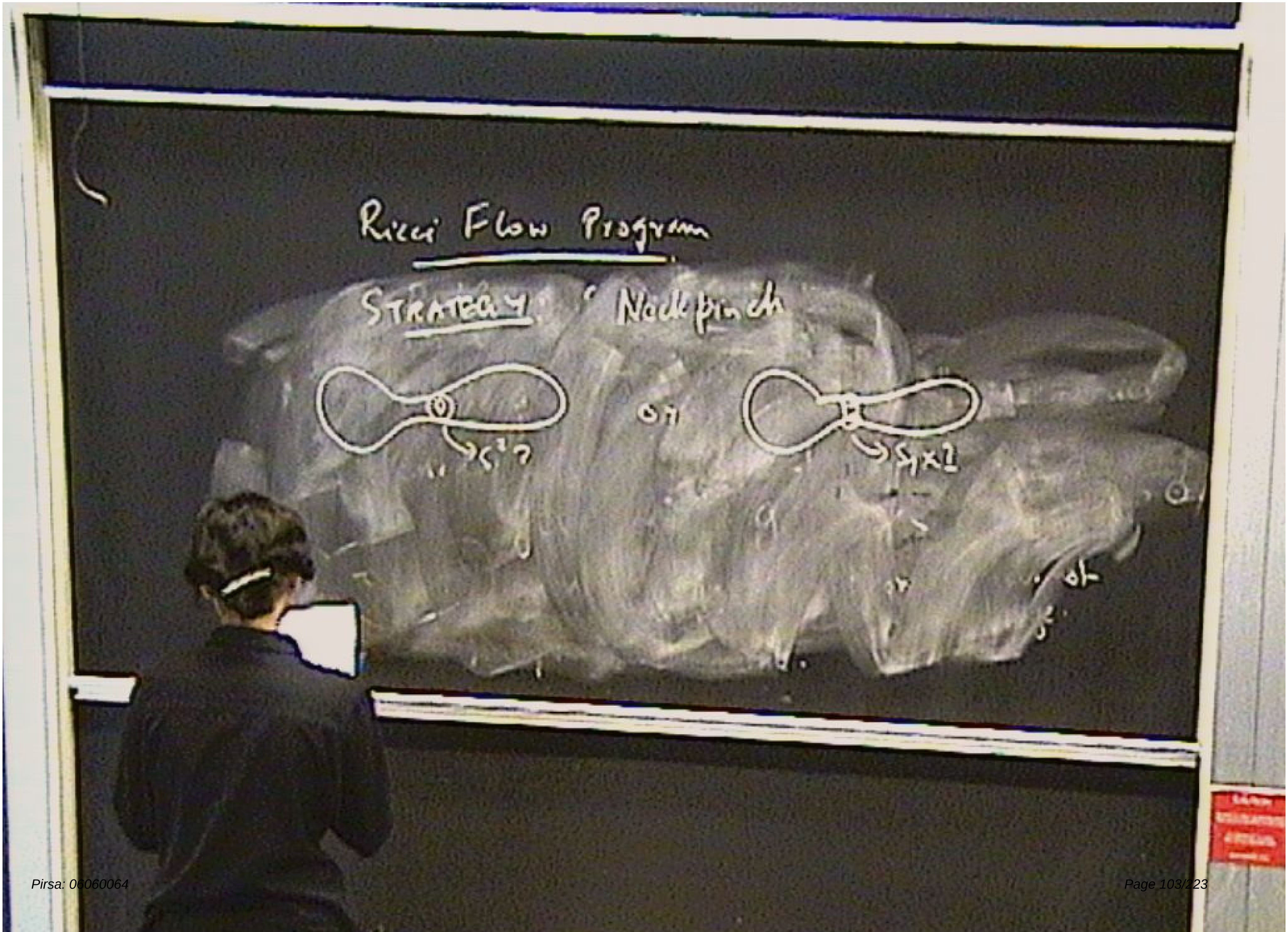

incompressible

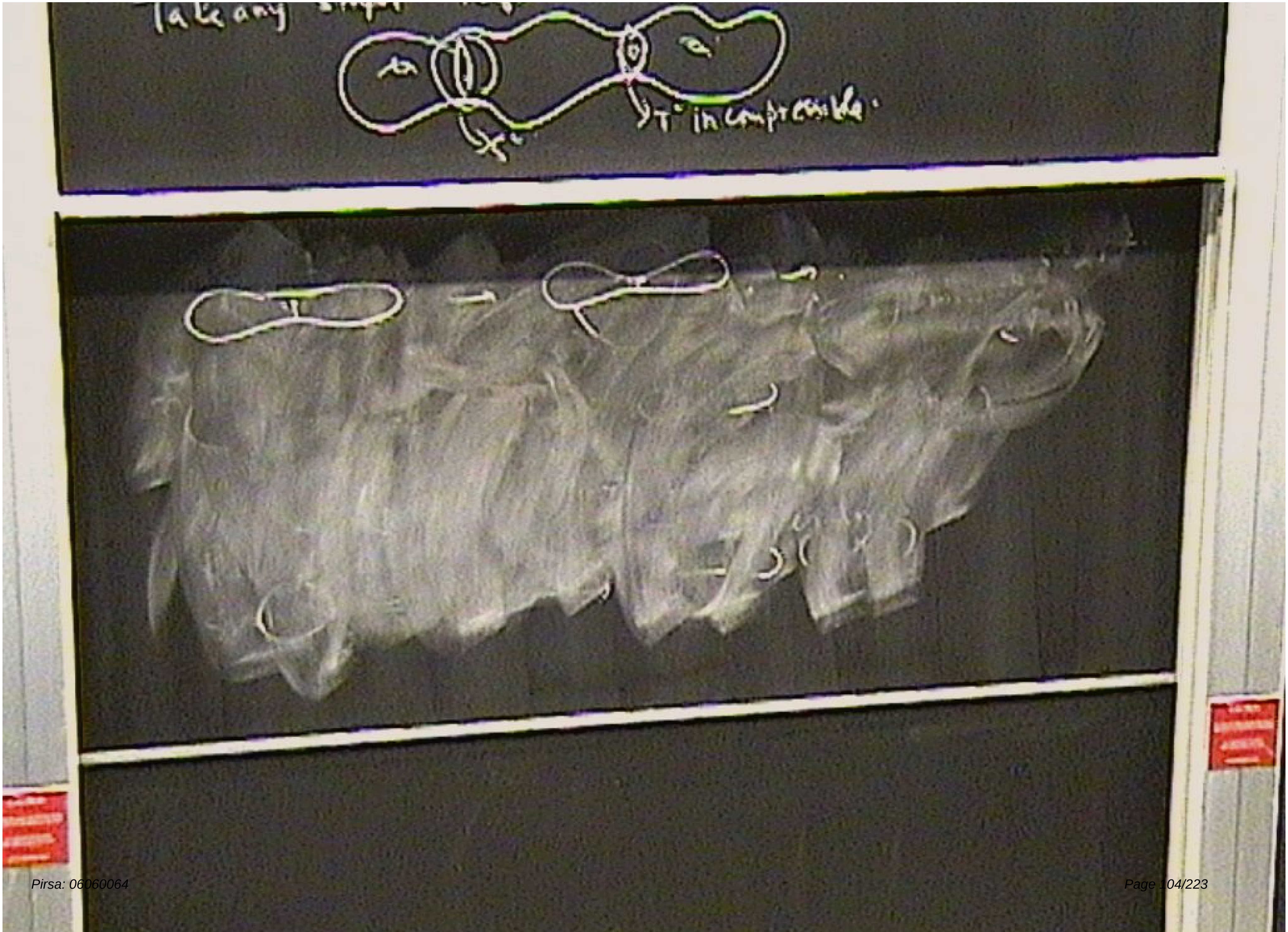

S^1
T^2 incompressible
$S^1 \times I$

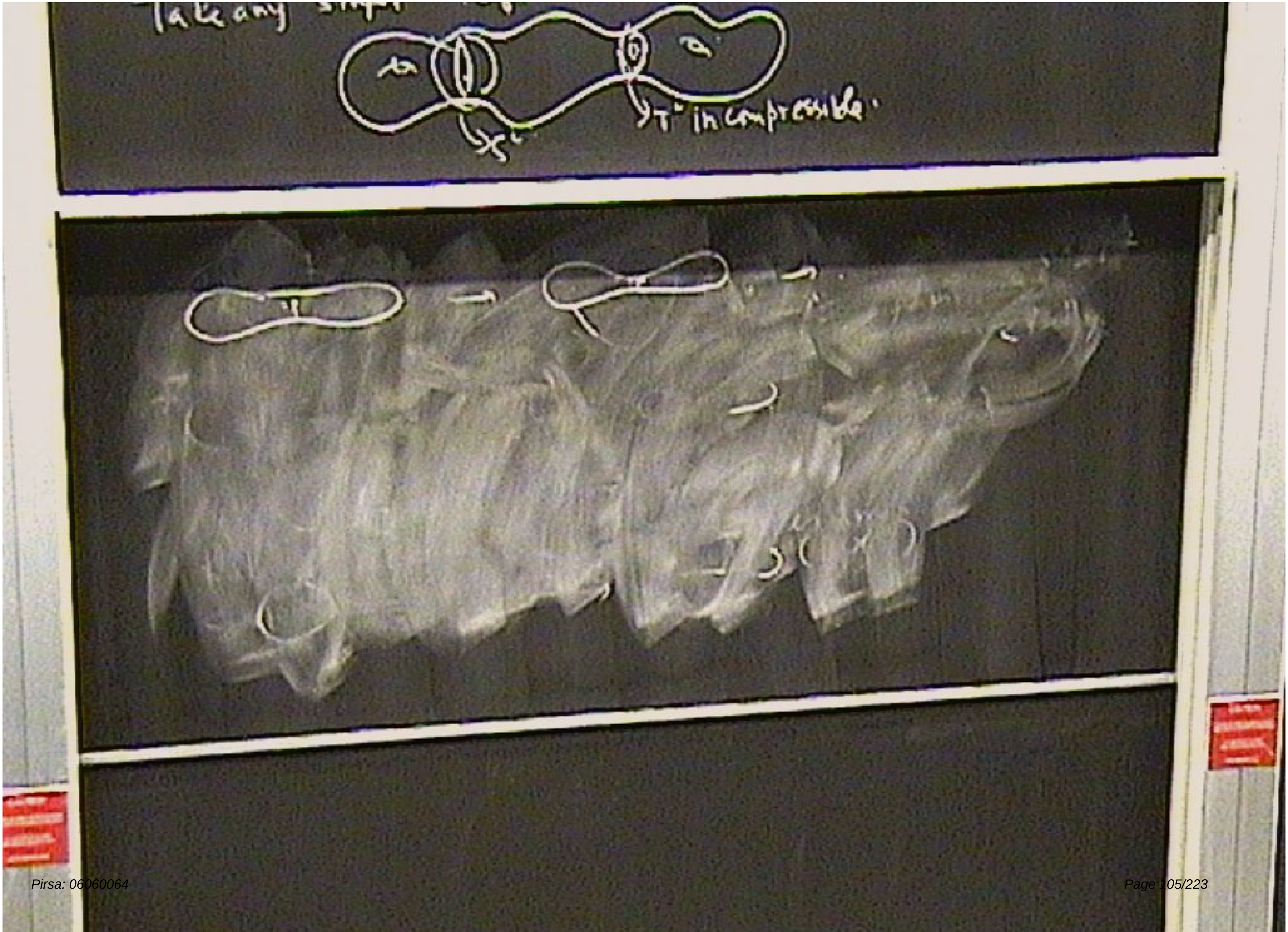

Ricci Flow Program
STRATEGY:

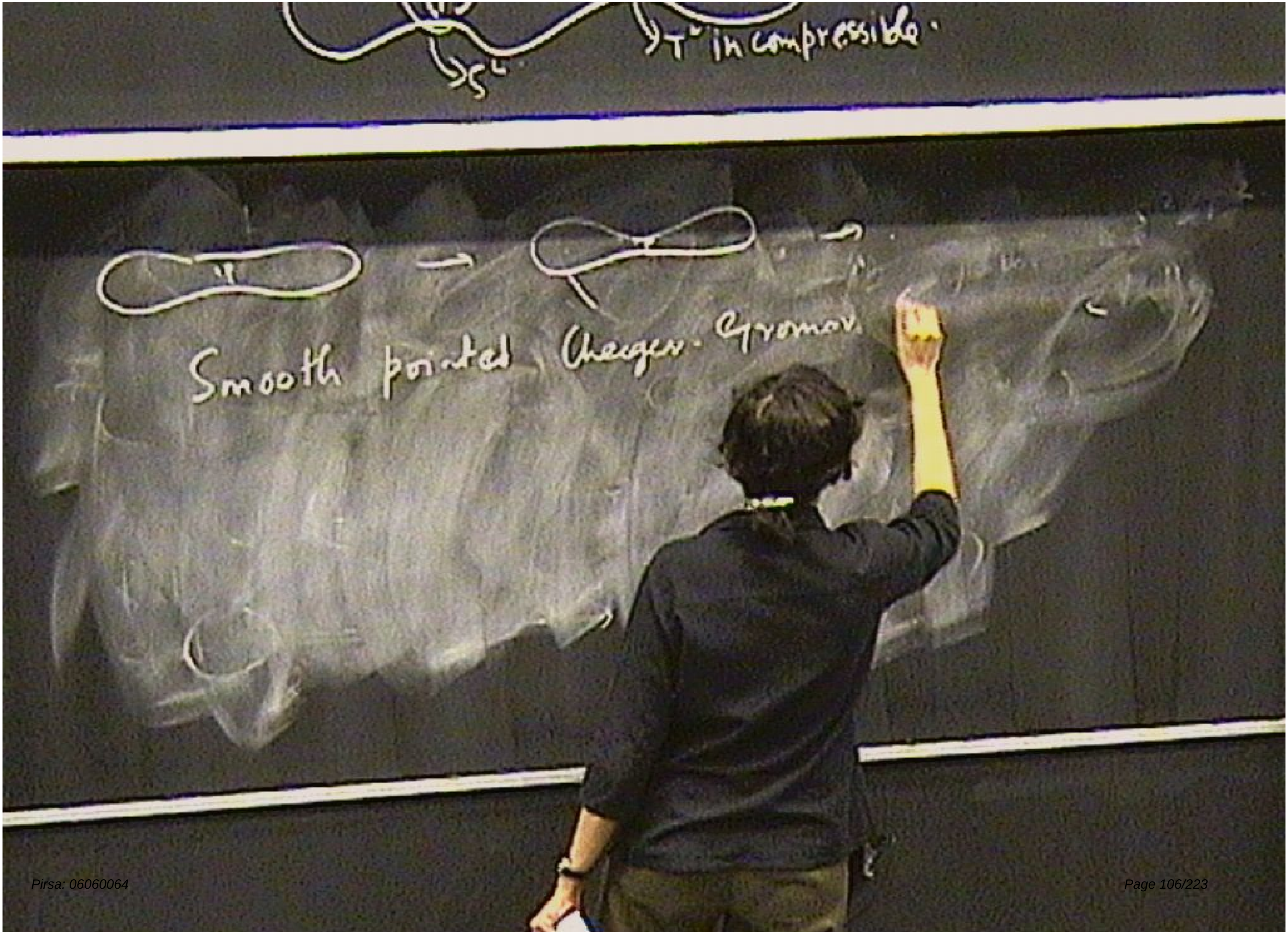

Ricci Flow Program
STRATEGY: ' Neckpinch

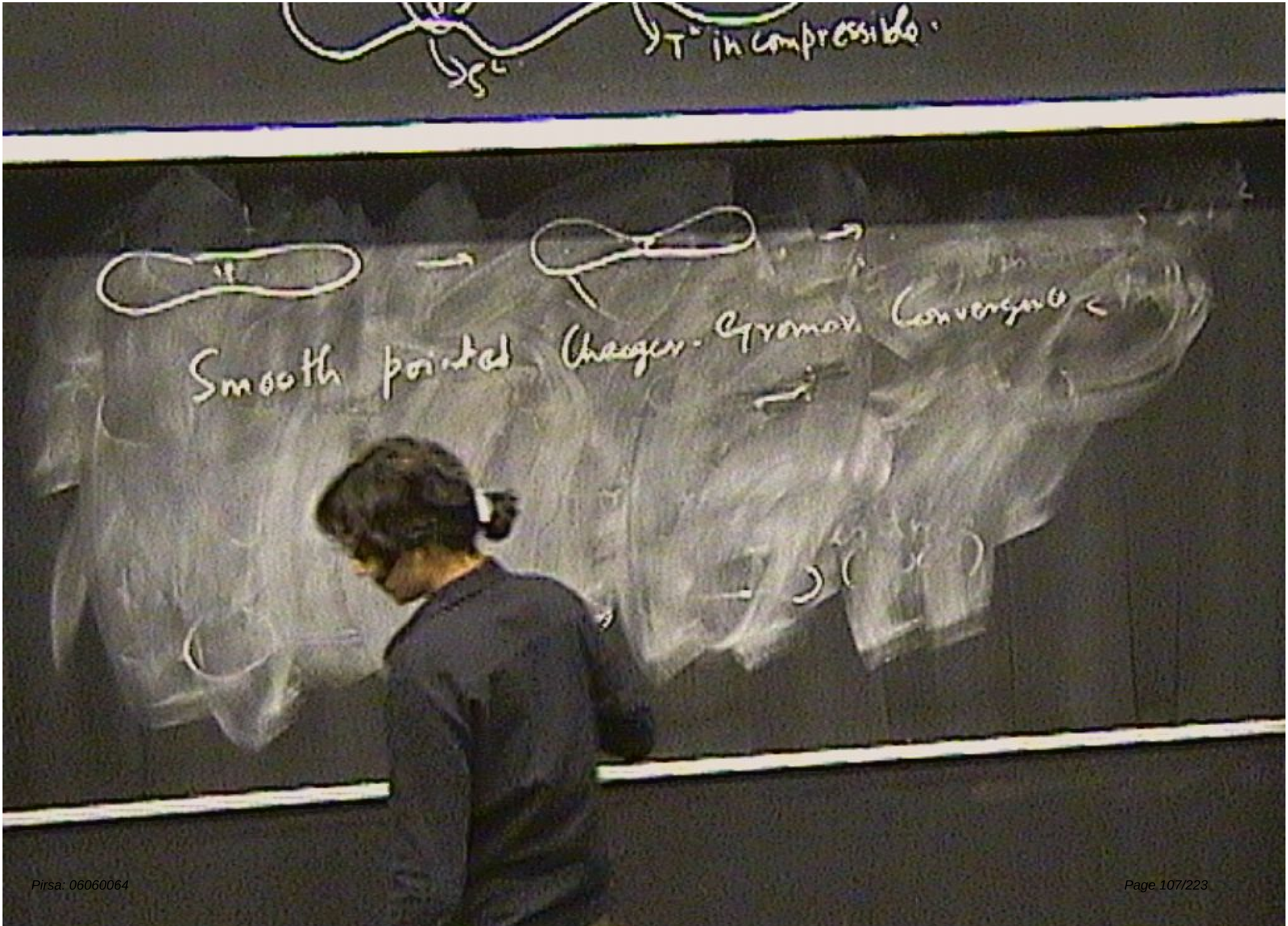

Ricci Flow Program
STRATEGY:
Neckpinch


Ricci Flow Program
STRATEGY:
Neckpinch
S^2?
$S^2 \times I$


T' incompressible


Take any simple
T' incompressible
T

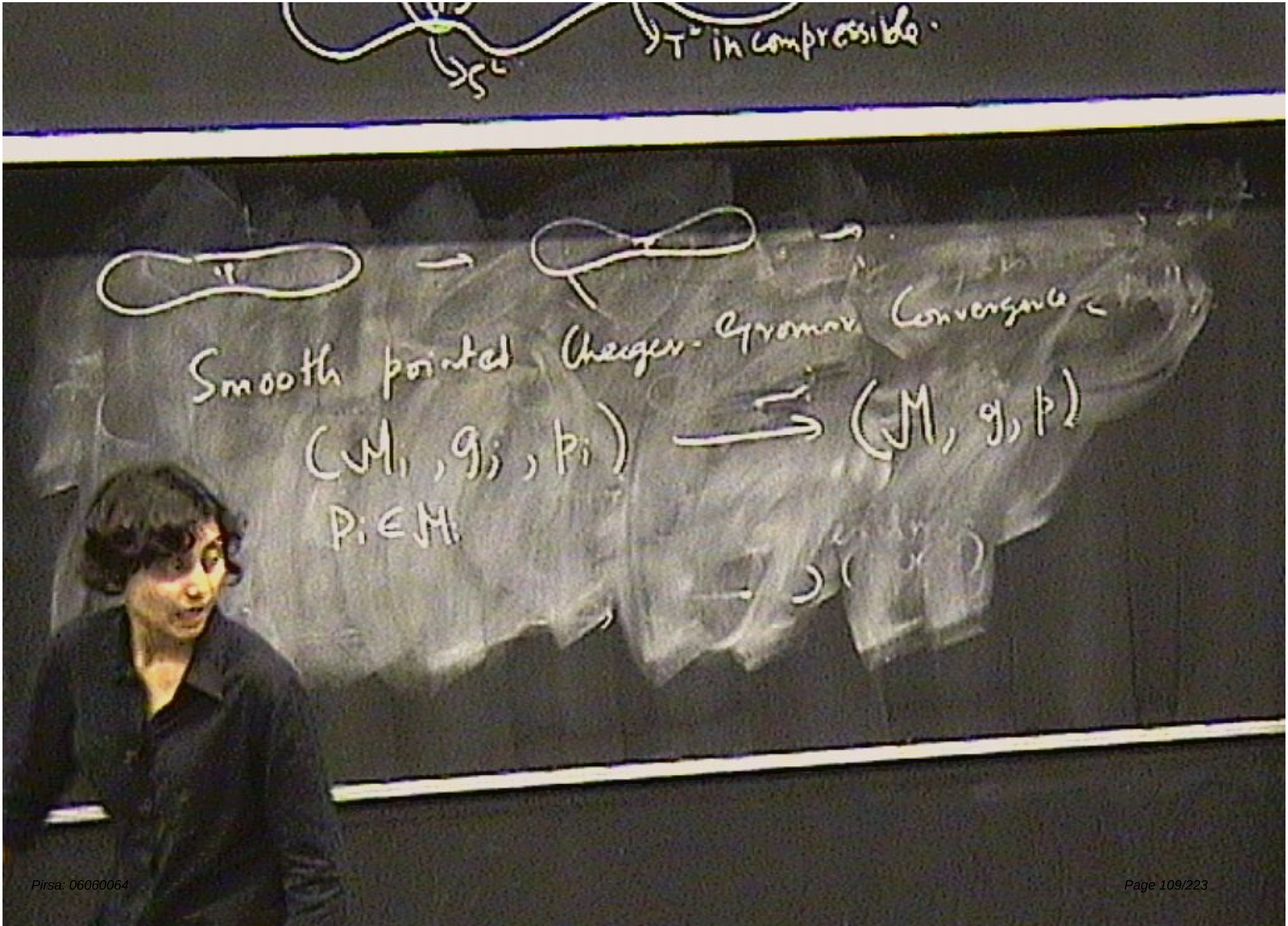

T^2 incompressible.
S^2
Smooth pointed Cheeger-Gromov


T^2 incompressible.
S^2
Smooth pointed Cheeger-Gromov Convergence

Smooth pointed Cheeger-Gromov Convergence

(M_i, g_i, p_i)

$p_i \in M_i$


S²
T² incompressible.
Smooth pointed Cheeger-Gromov Convergence
(Mi , gi , pi) ⟶ (M, g, p)
pi ∈ Mi

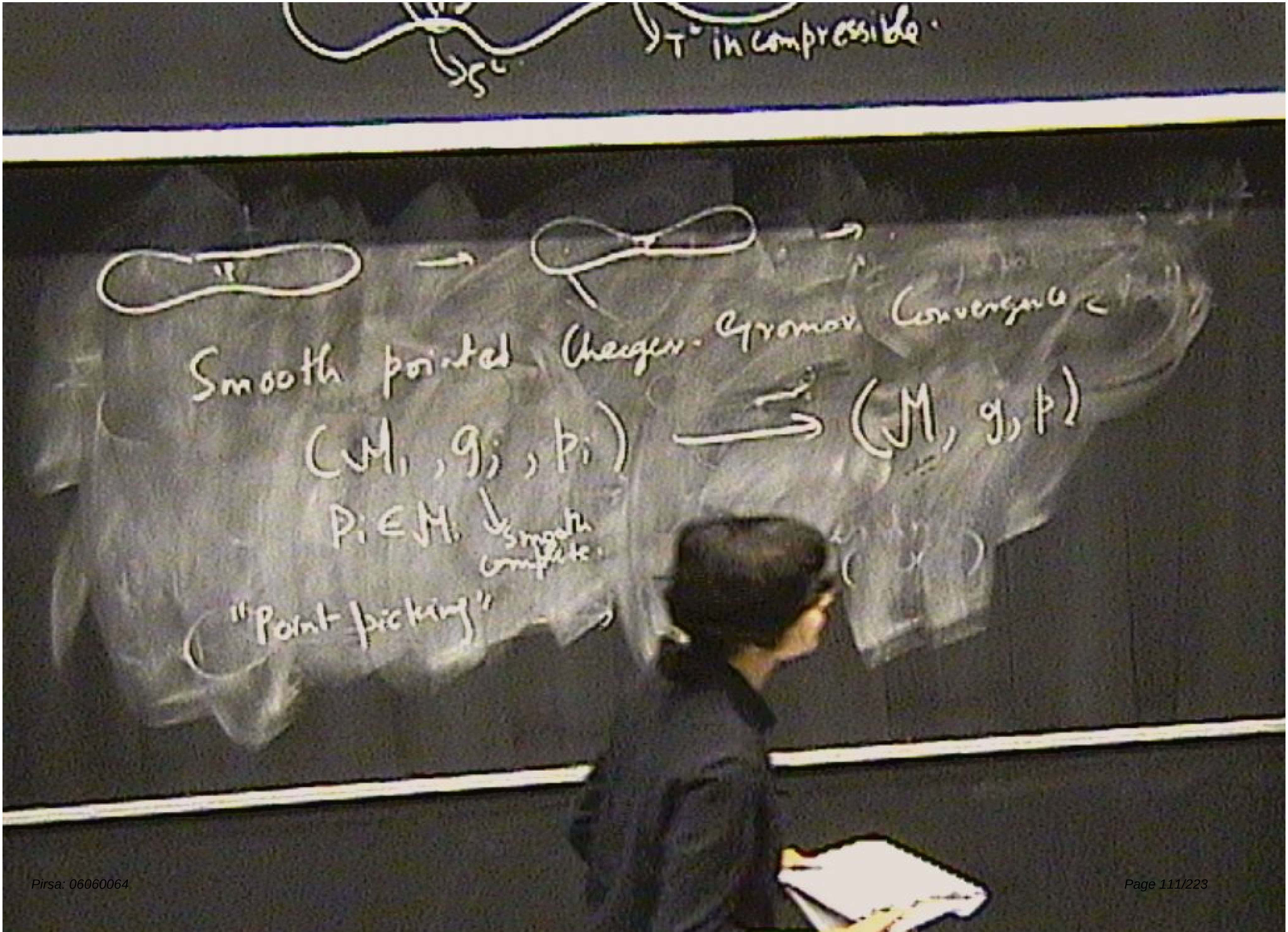
T^2 incompressible.

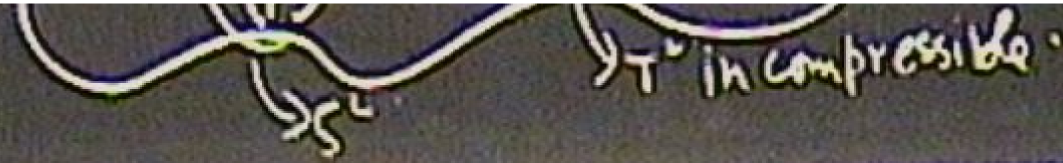
S^2

Smooth pointed Cheeger-Gromov Convergence

$$(M_i, g_i, p_i) \xrightarrow{S} (M, g, p)$$

$p_i \in M_i$ smooth complete.


S²
T² incompressible
Smooth pointed Cheeger-Gromov Convergence
(Mᵢ , gᵢ , pᵢ) → (M, g, p)
pᵢ ∈ Mᵢ
smooth complete
"Point picking"

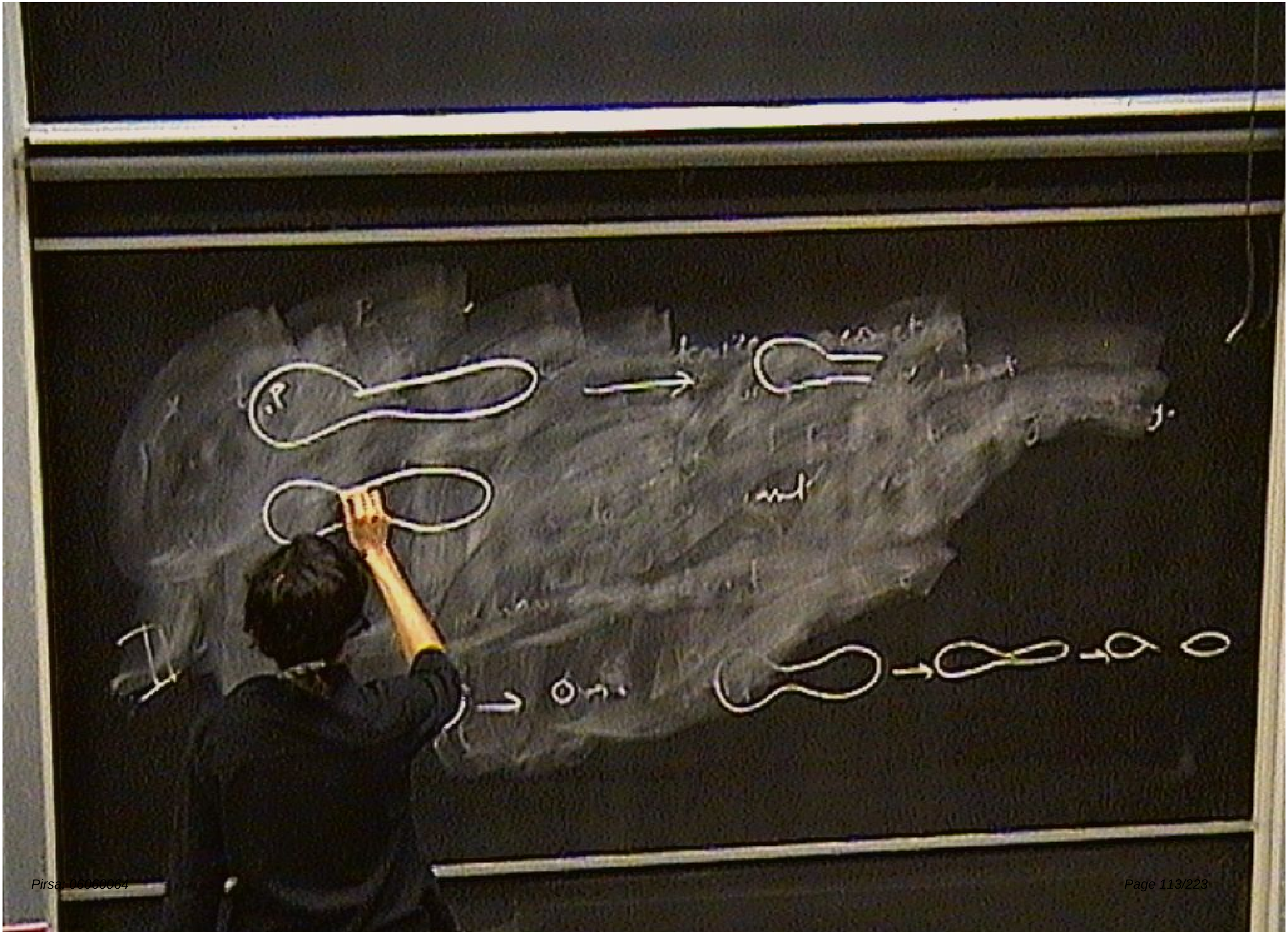


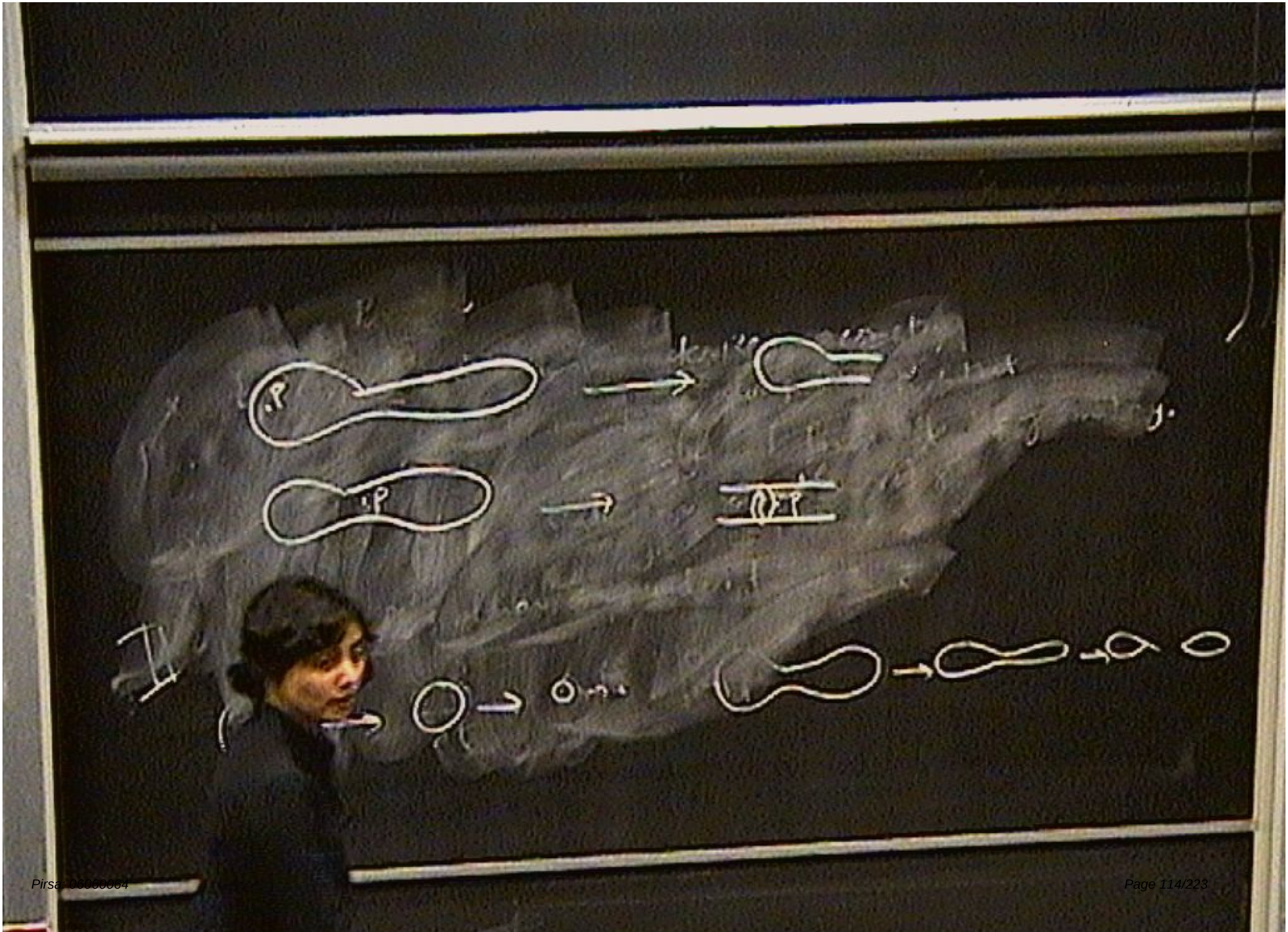
Smooth pointed Cheeger-Gromov Convergence

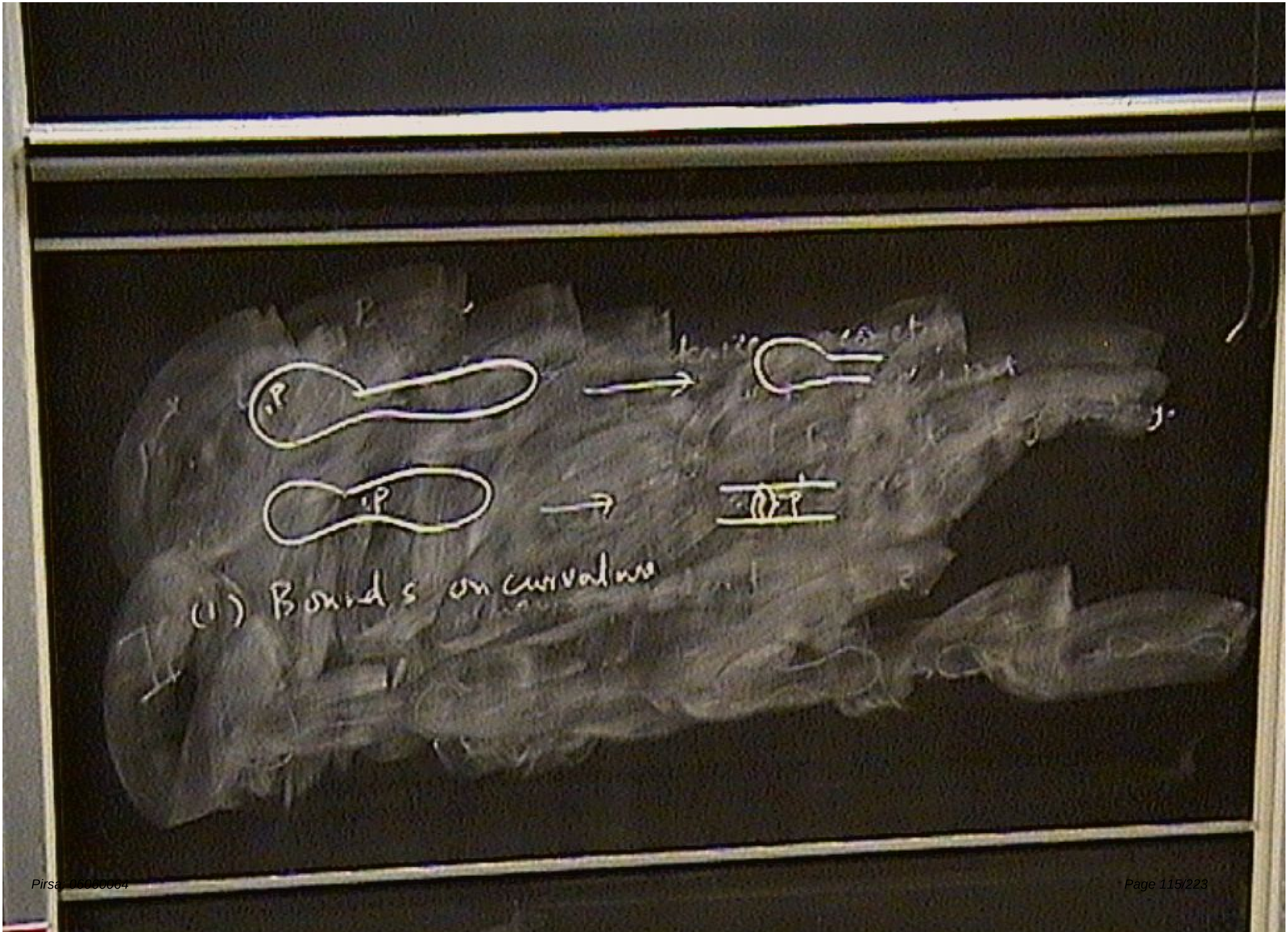
$$(M_i, g_i, p_i) \longrightarrow (M, g, p)$$

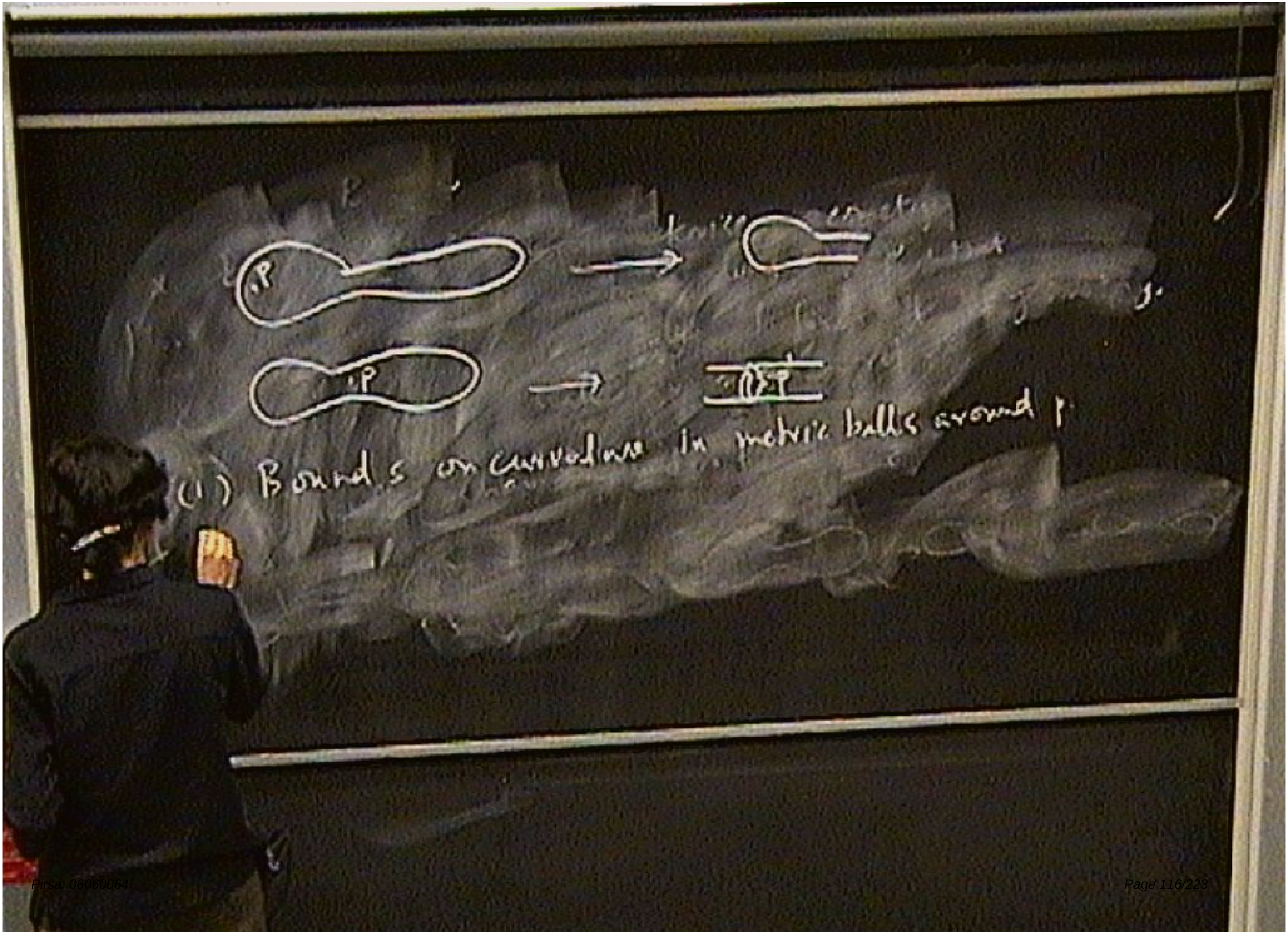
$p_i \in M_i$ smooth complete.

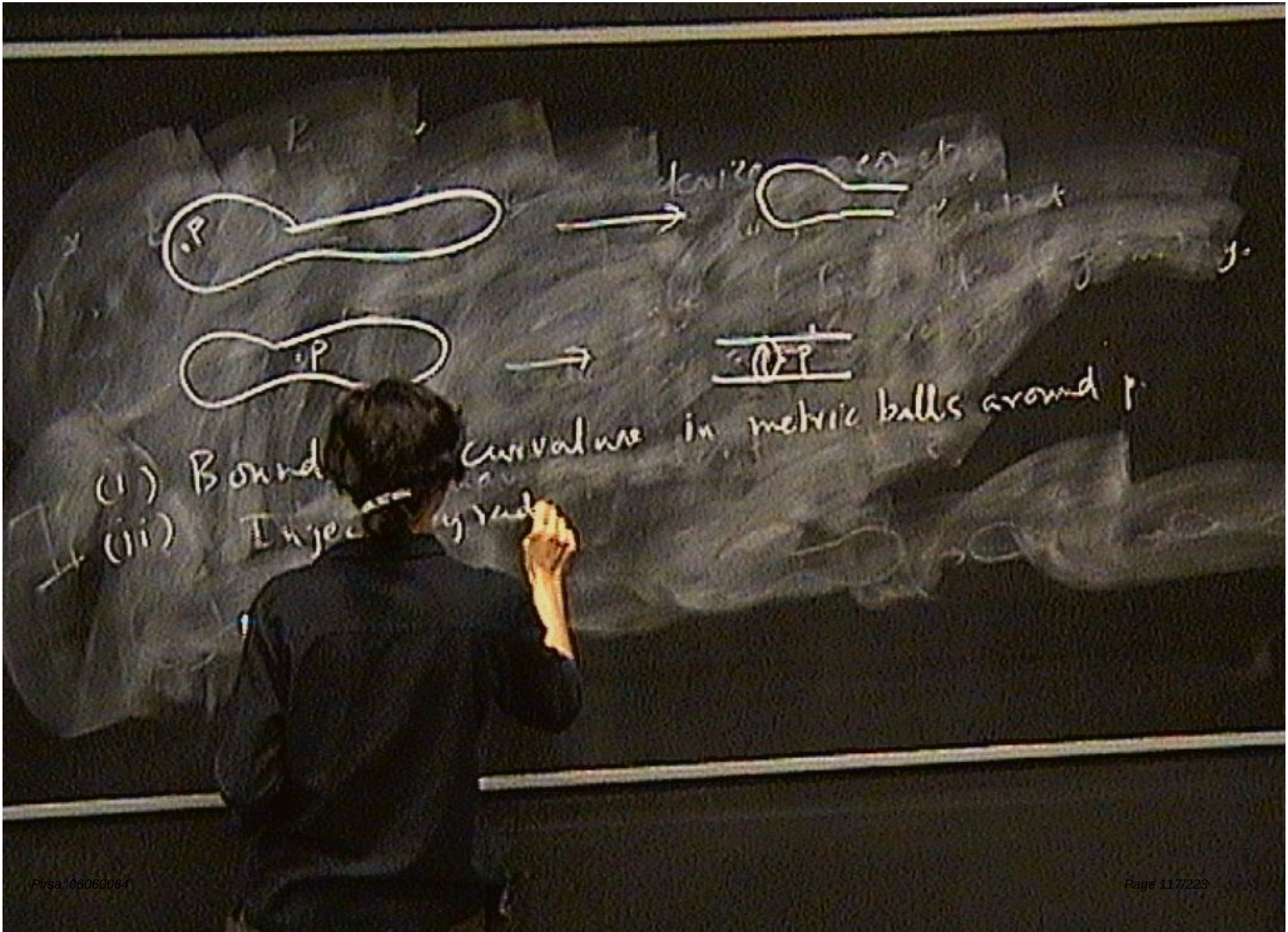
"Point picking"

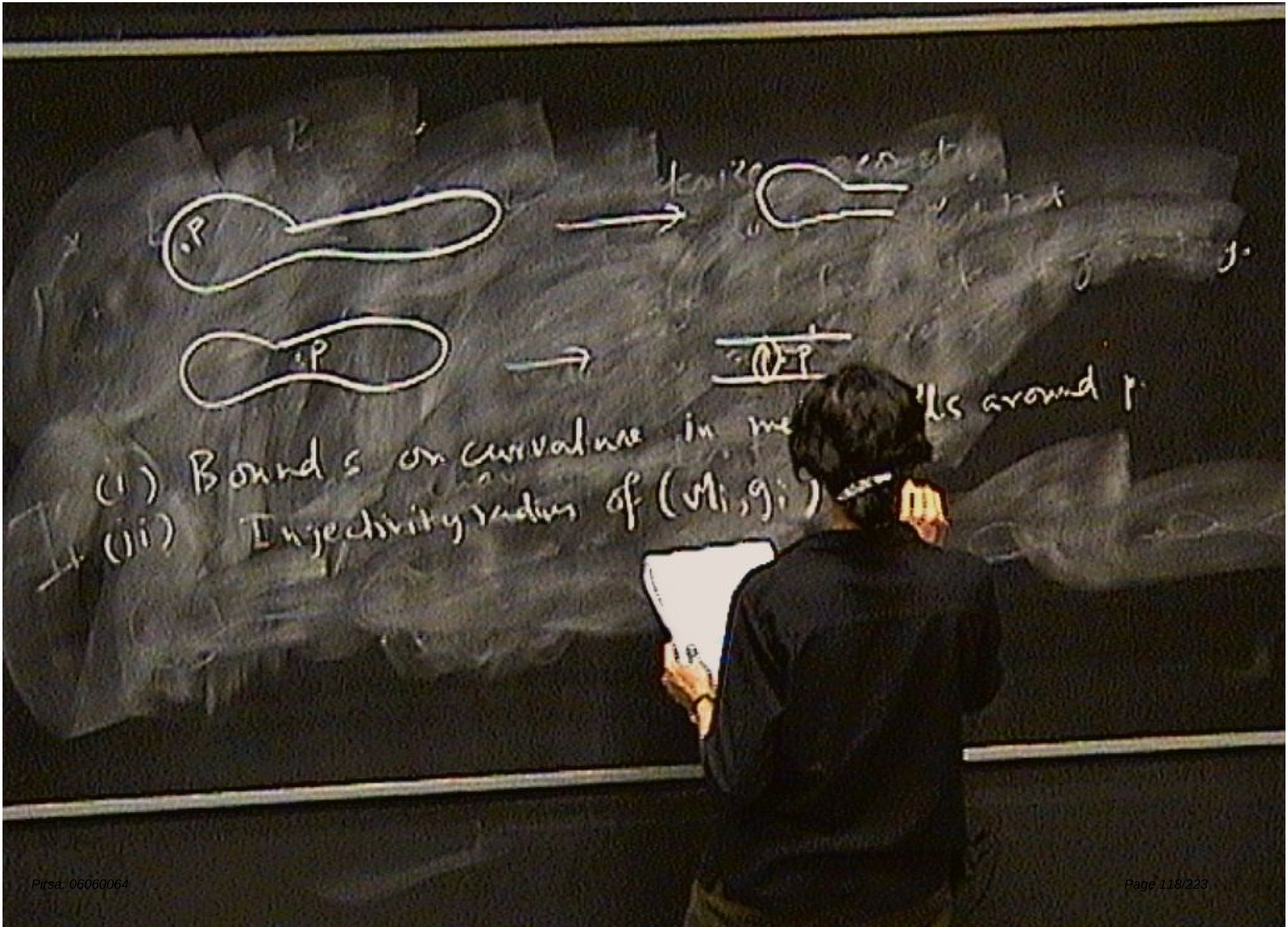

and

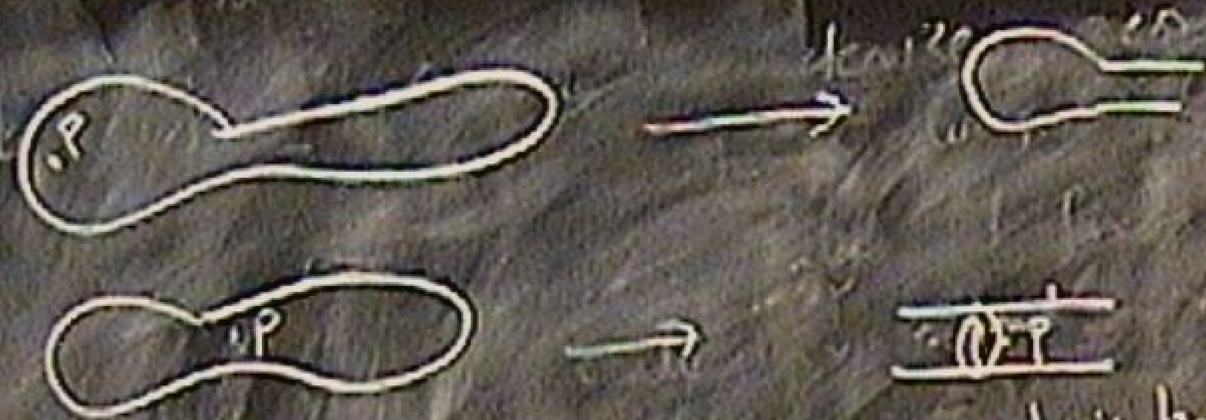



(1) Bounds on curvature


(1) Bounds on curvature in metric balls around p

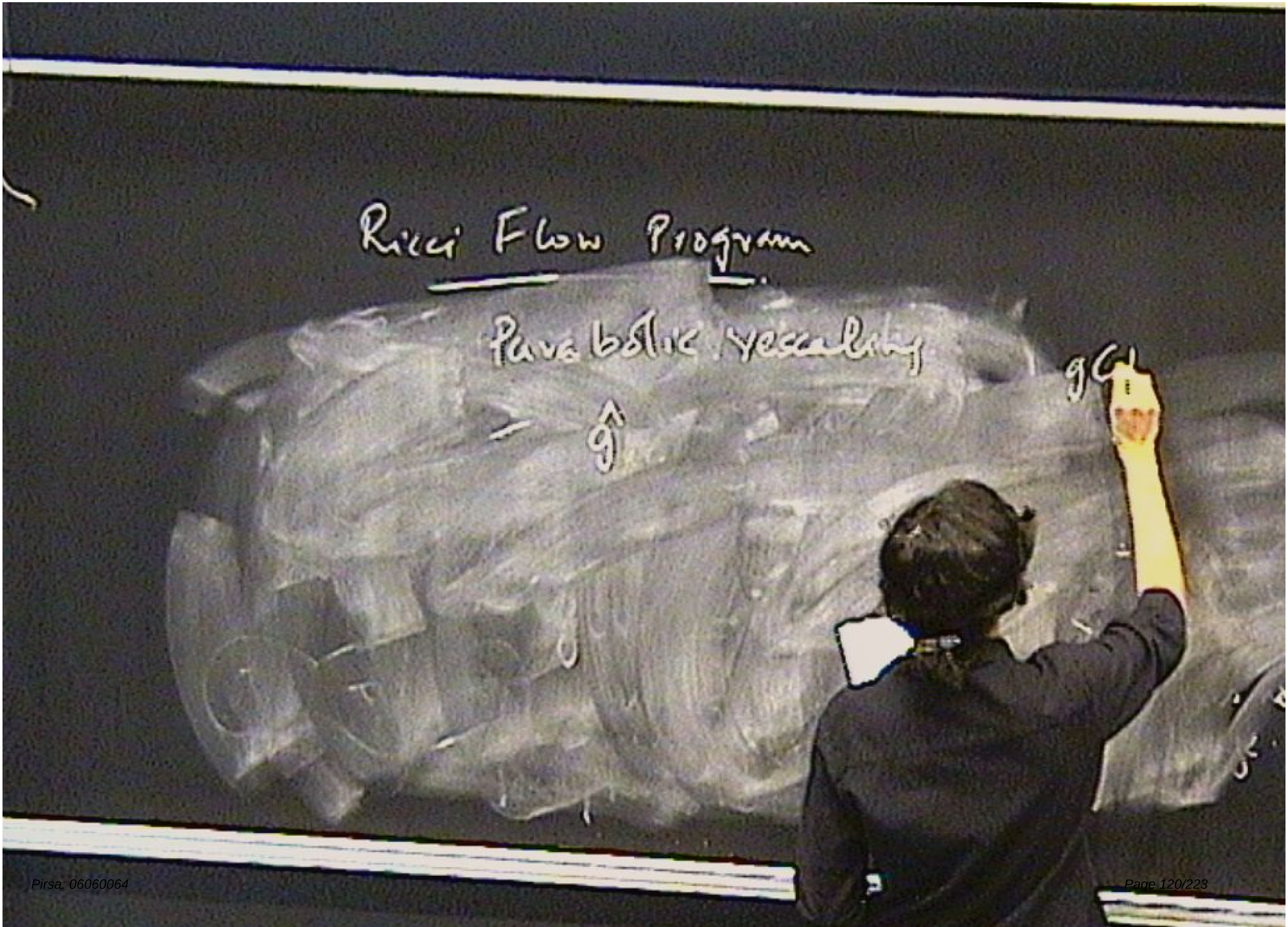

curvature in metric balls around p
(i) Bound
(ii) Injec

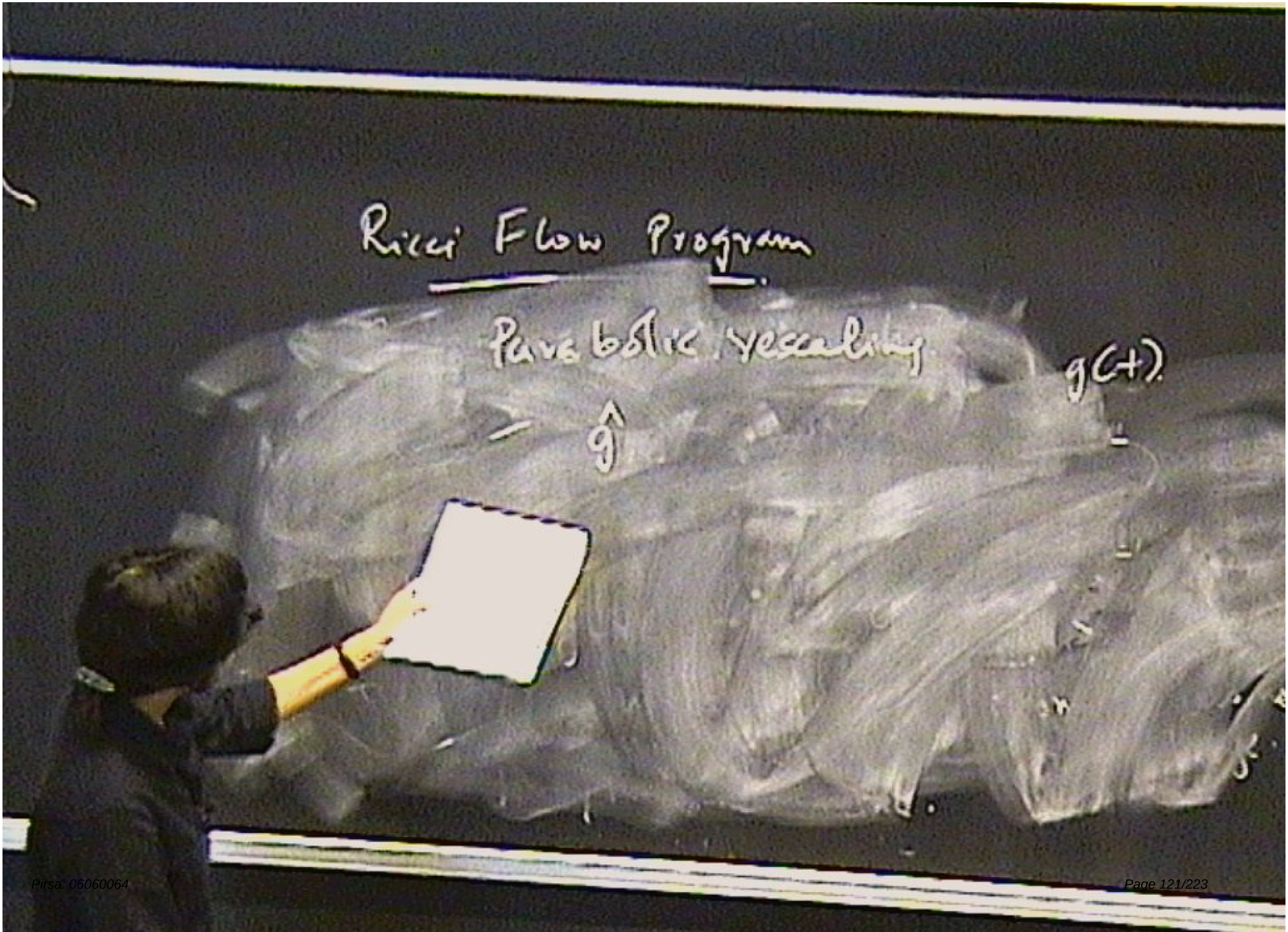

(i) Bounds on curvature in me
lls around p
(ii) Injectivity radius of (Mi, gi)

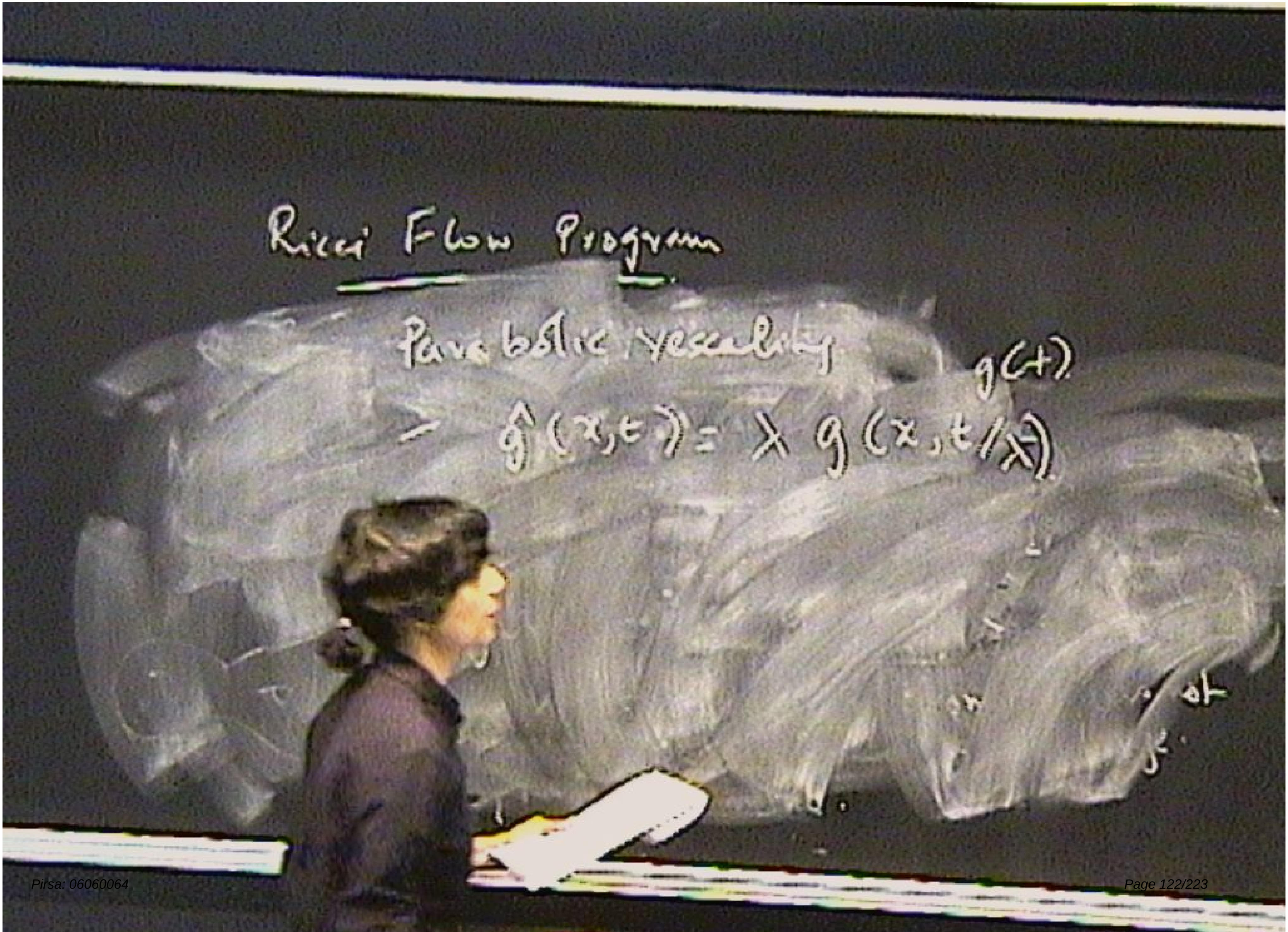


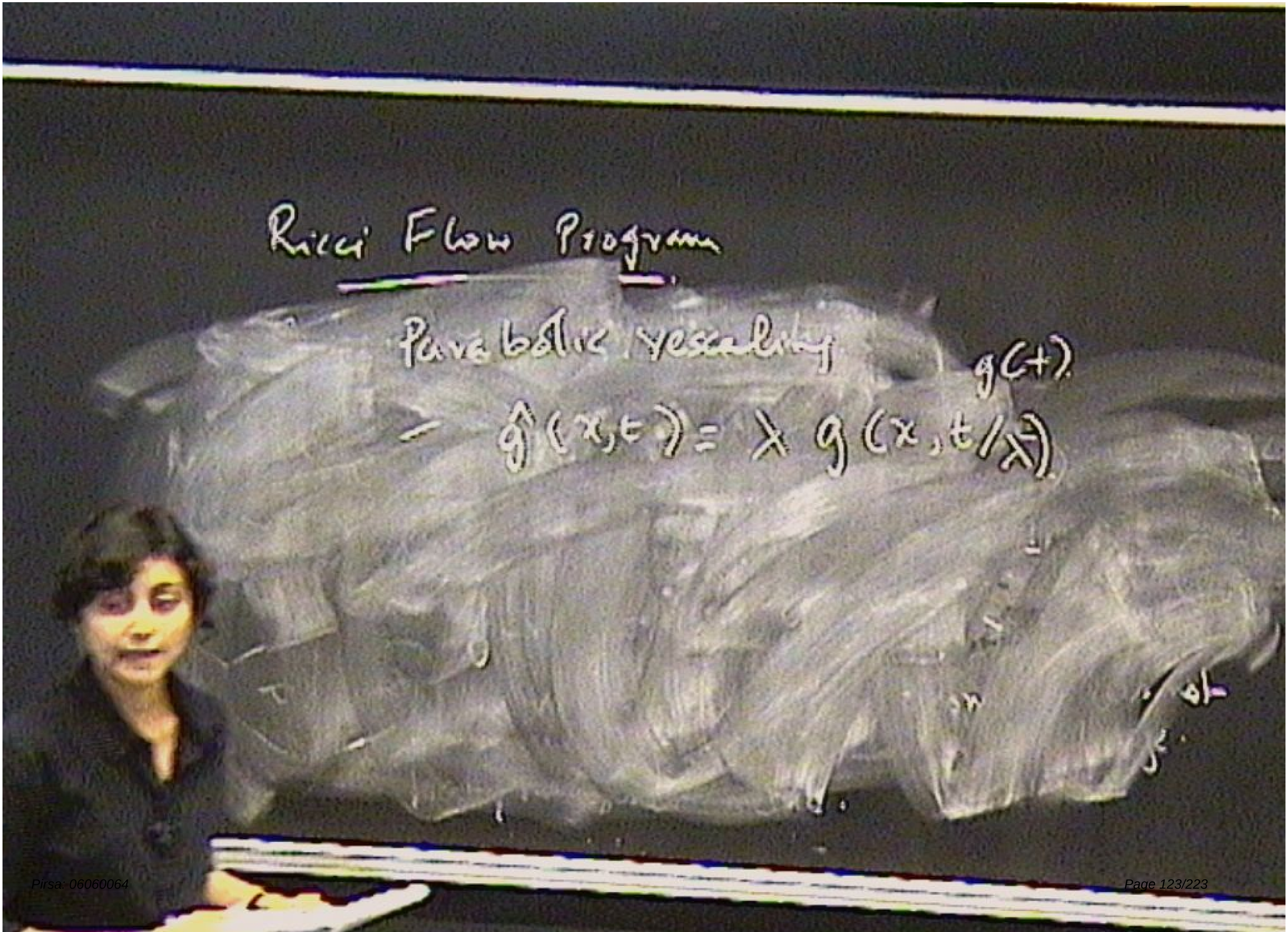
(i) Bounds on curvature in metric balls around p

(ii) Injectivity radius of (M_i, g_i) at p_i has a lower bound.


Ricci Flow Program
Parabolic rescaling
ĝ


Ricci Flow Program
Parabolic rescaling
g(t)
ĝ


Ricci Flow Program
Parabolic rescaling
g(t)
→ ĝ(x,t) = λ g(x, t/λ)


Ricci Flow Program
Parabolic rescaling
g(t)
$\hat{g}(x,t) = \lambda\, g(x, t/\lambda)$

Ricci Flow Program

Parabolic rescaling

$g(t) \rightarrow t \in [0, T]$

$$\hat{g}(x,t) = \lambda\, g(x, t/\lambda)$$

for $t \in [0, \lambda T]$

Ricci Flow Program

Parabolic rescaling

$g(t) \to t \in [0, T)$

$$\hat{g}(x,t) = \lambda\, g(x, t/\lambda) \quad \text{for } t \in [0, \lambda T]$$

$$\frac{\partial \hat{g}(x,t)}{\partial t}$$

Ricci Flow Program

Parabolic rescaling

$g(t) \to t \in [0, T]$

$$\hat{g}(x,t) = \lambda\, g(x, t/\lambda)$$

for $t \in [0, \lambda T]$

$$\frac{\partial \hat{g}(x,t)}{\partial t} = \frac{\partial g(x, t/\lambda)}{\partial t/\lambda} = -R_{ij}[g(t/\lambda)]$$

Ricci Flow Program

Parabolic rescaling $\quad g(t) \rightarrow t \in [0, T]$

$$\hat{g}(x,t) = \lambda\, g(x, t/\lambda)$$

$$\text{for } t \in [0, \lambda T]$$

$$\frac{\partial \hat{g}(x,t)}{\partial t} = \frac{\partial g(x,t/\lambda)}{\partial t/\lambda} = -R_{ij}[g(t/\lambda)]$$

$$= -R_{ij}[\hat{g}(t)]$$

Ricci Flow Program

Parabolic rescaling

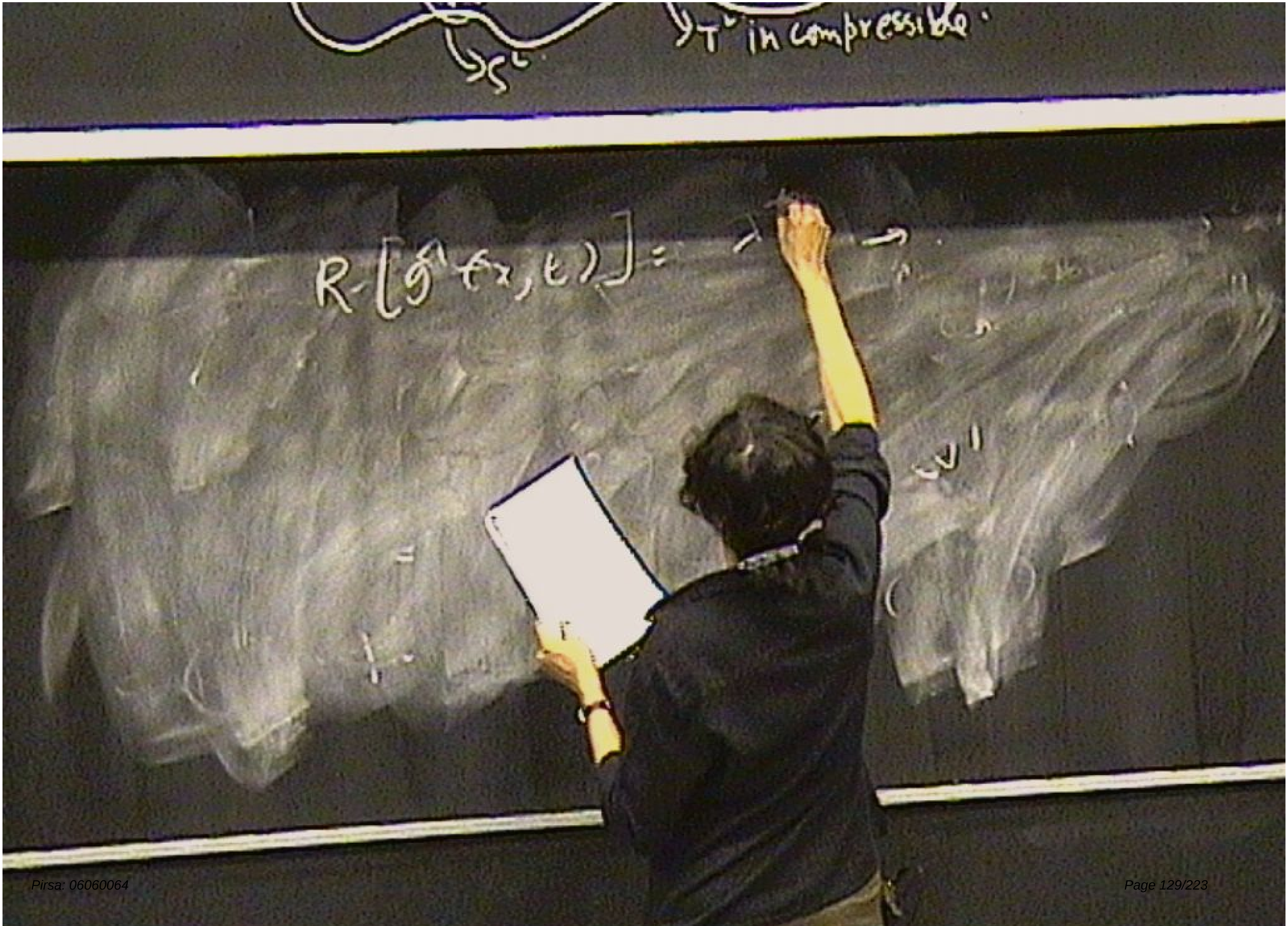
$g(t) \to t \in [0, T)$

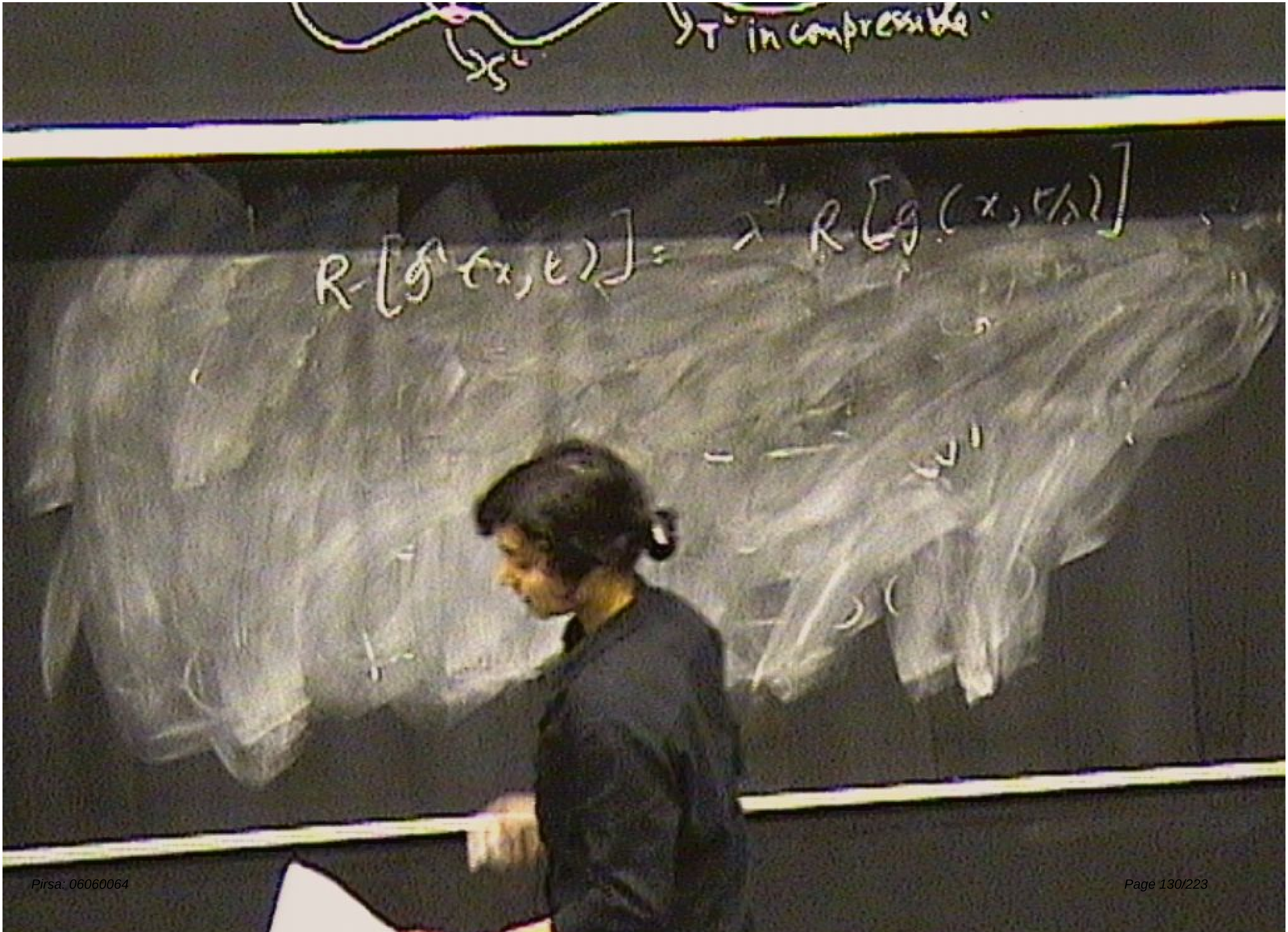
$$\hat{g}(x,t) = \lambda\, g(x, t/\lambda)$$

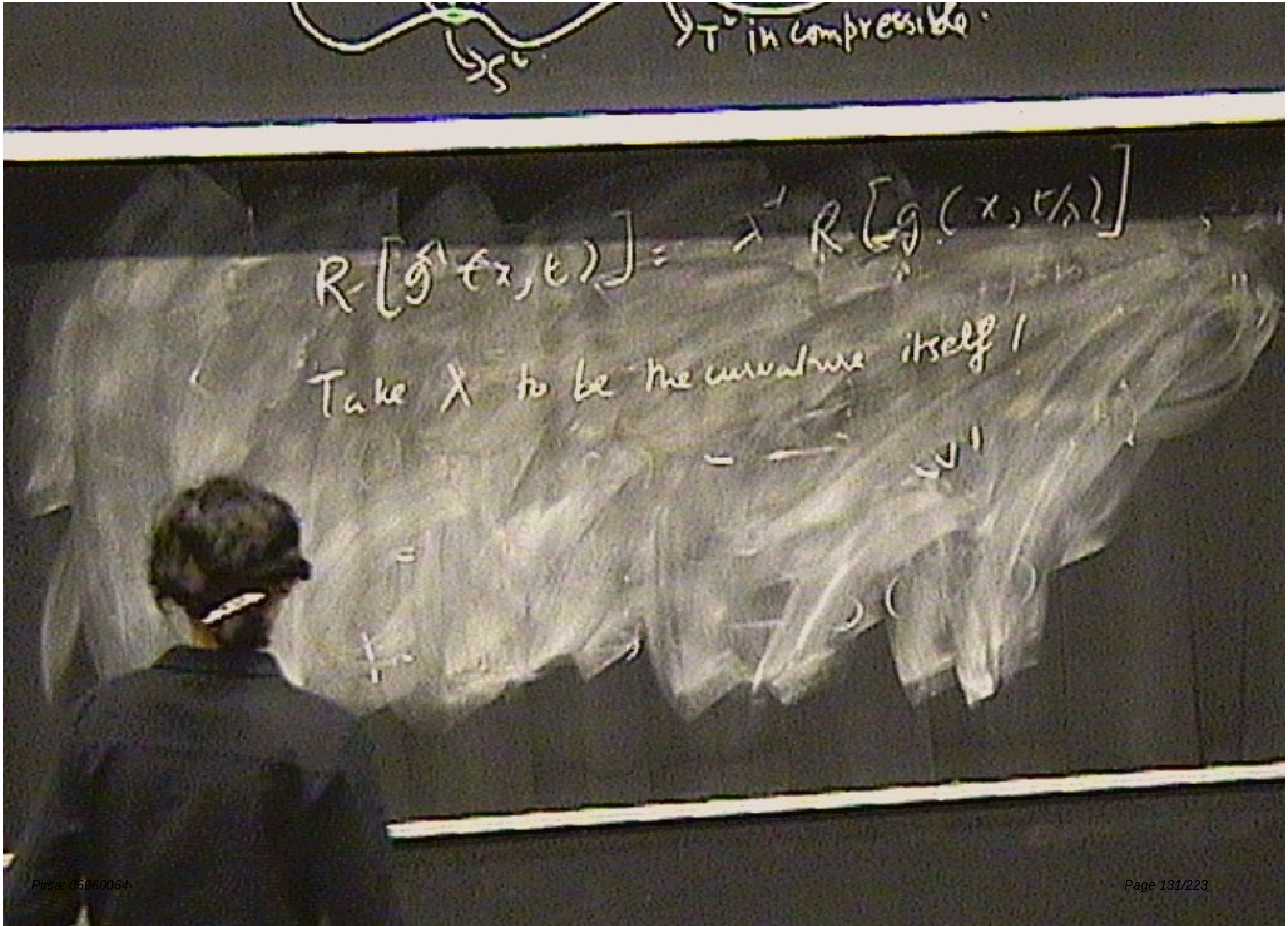
for $t \in [0, \lambda T)$

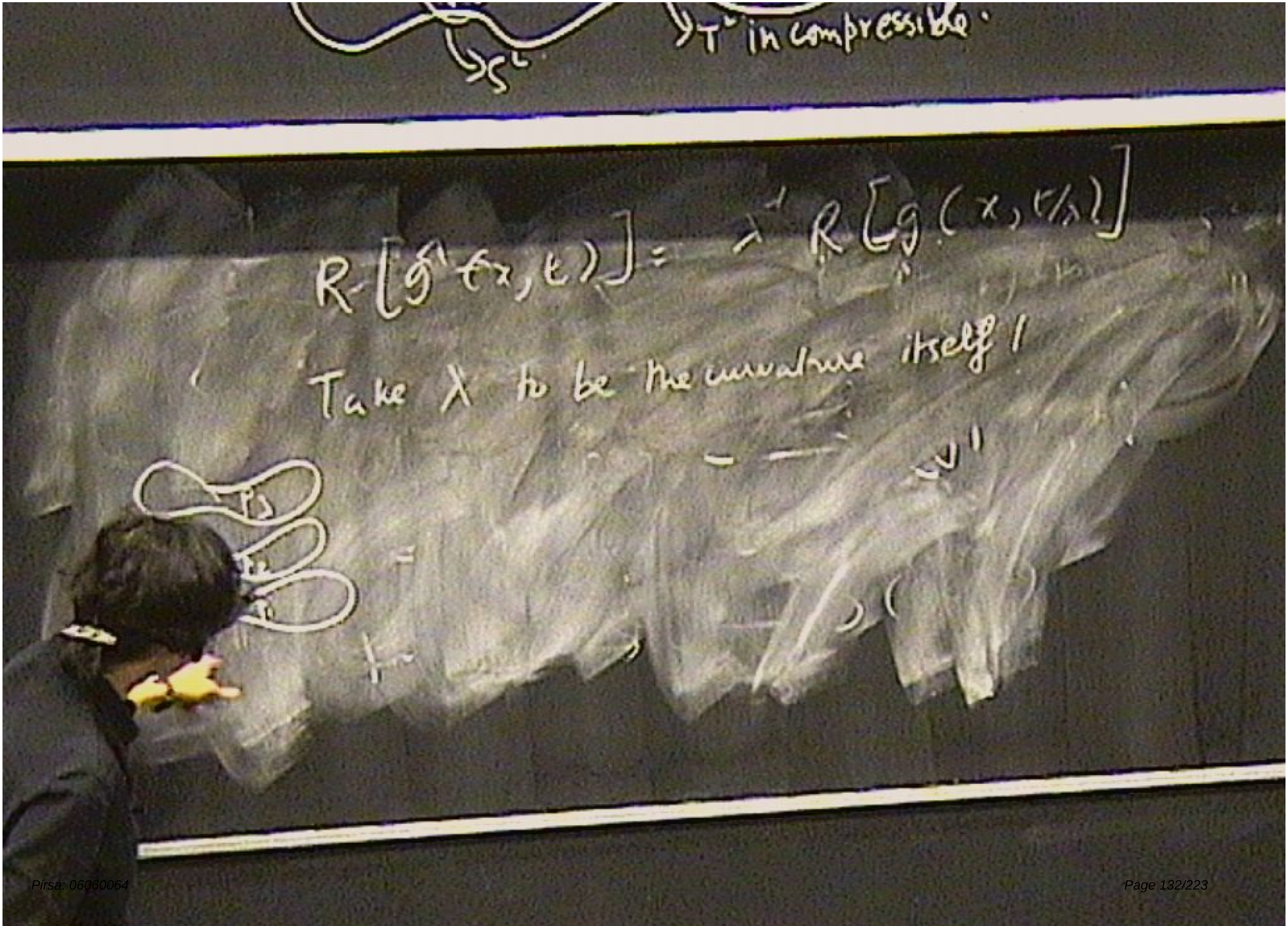
$$\frac{\partial \hat{g}(x,t)}{\partial t} = \frac{\partial g(x, t/\lambda)}{\partial t/\lambda} = -R_{ij}[g(t/\lambda)] = -R_{ij}[\hat{g}(t)]$$

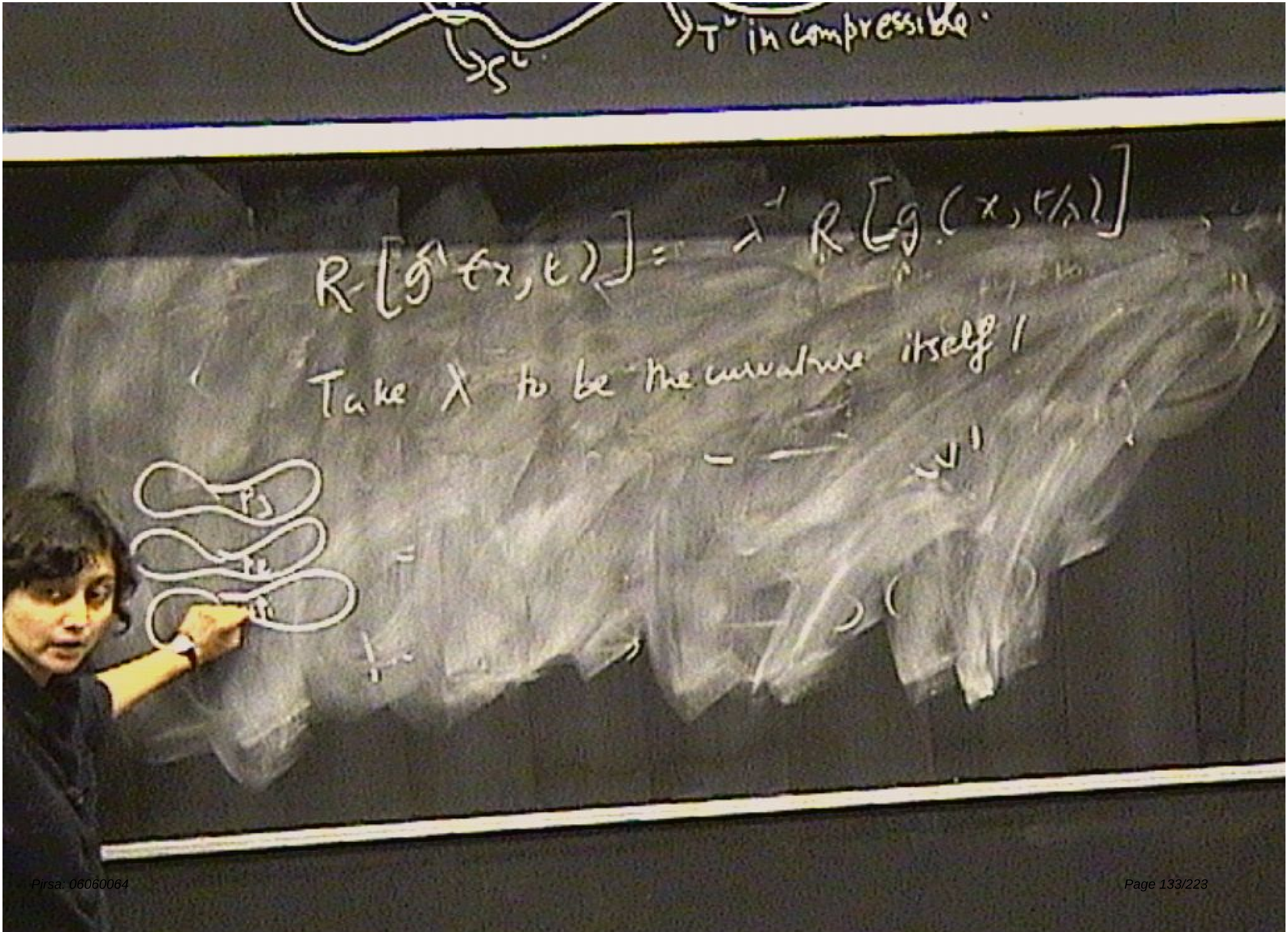
$\hat{g}(x,t)$ is a solution of Ricci flow.

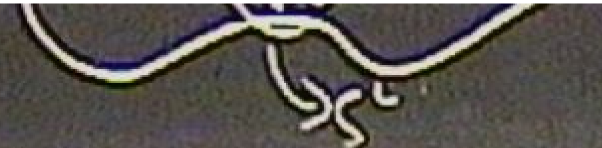

in compressible
R[g(x,t)] =


incompressible
R[g^λ(x,t)] = λ^d R[g(x,t/λ)]


T° incompressible.
S°
R[g^λ(x,t)] = λ^-1 R[g(x,t/λ)]
Take λ to be the curvature itself!


T^2 incompressible.
$R[g^\lambda(x,t)] = \lambda^{-1} R[g(x,t/\lambda)]$
Take λ to be the curvature itself!

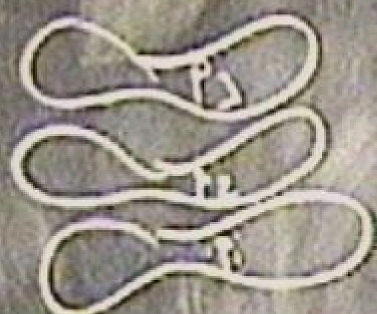

T^i incompressible.
Σ^i
$R[g^\lambda(x,t)] = \lambda^{-1} R[g(x,t/\lambda)]$
Take λ to be the curvature itself!



T^i incompressible.

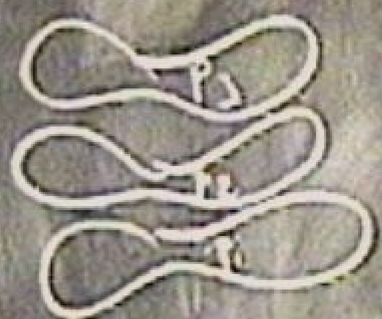
$$R[g'(x,t)] = \lambda^{-1} R[g(x, t/\lambda)]$$

Take λ to be the curvature itself!



$$R[g^{\lambda}(x,t)] = \lambda^{-1} R[g(x,t/\lambda)]$$

Take λ to be the curvature itself!

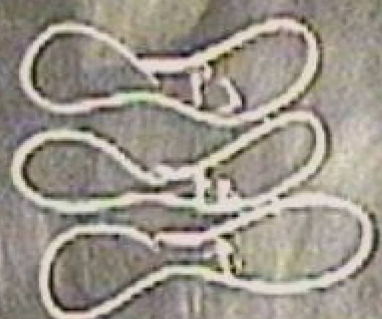


$$(M_i, \bar{g}_i, p_i)$$

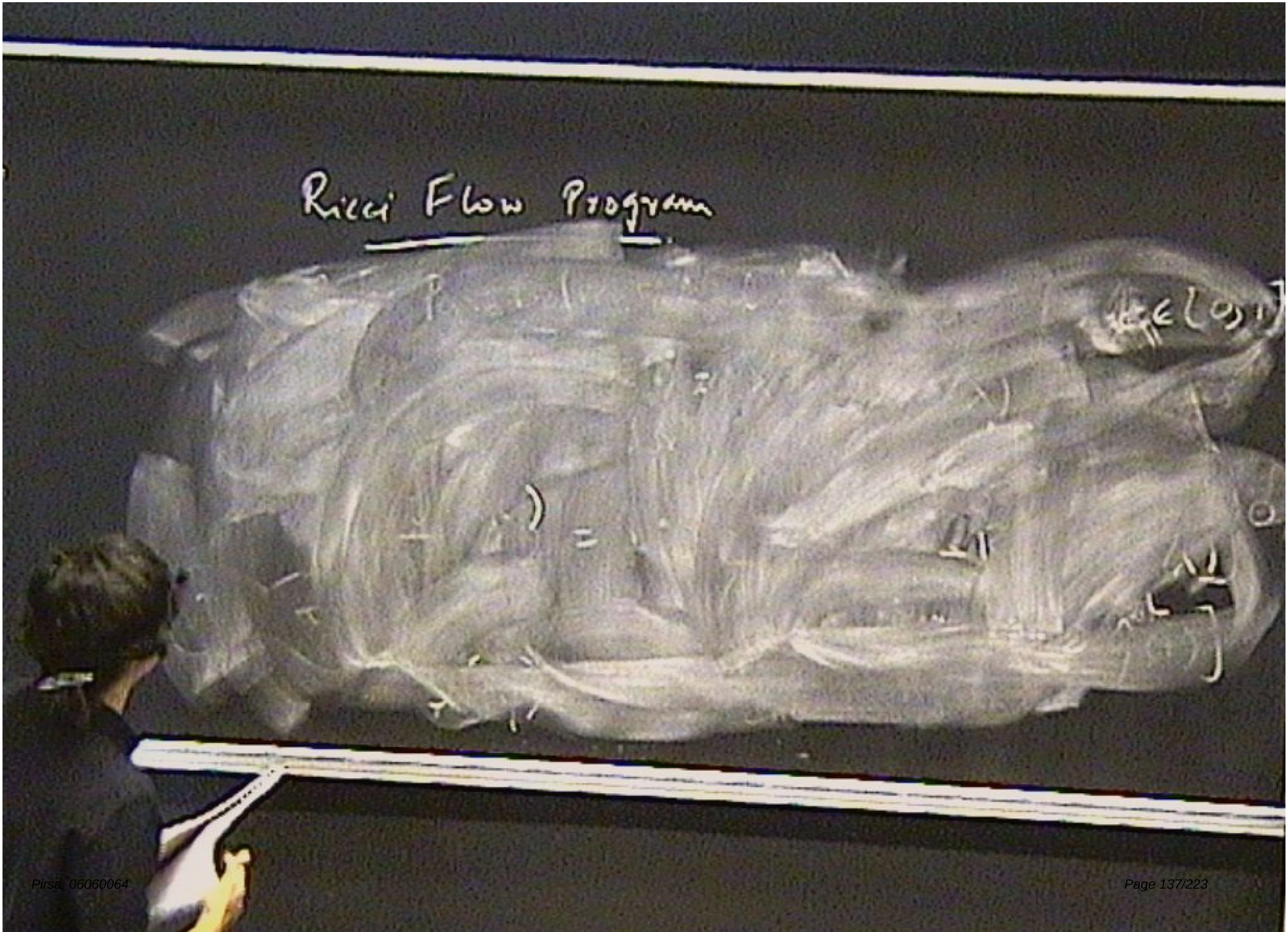
T^2 incompressible.

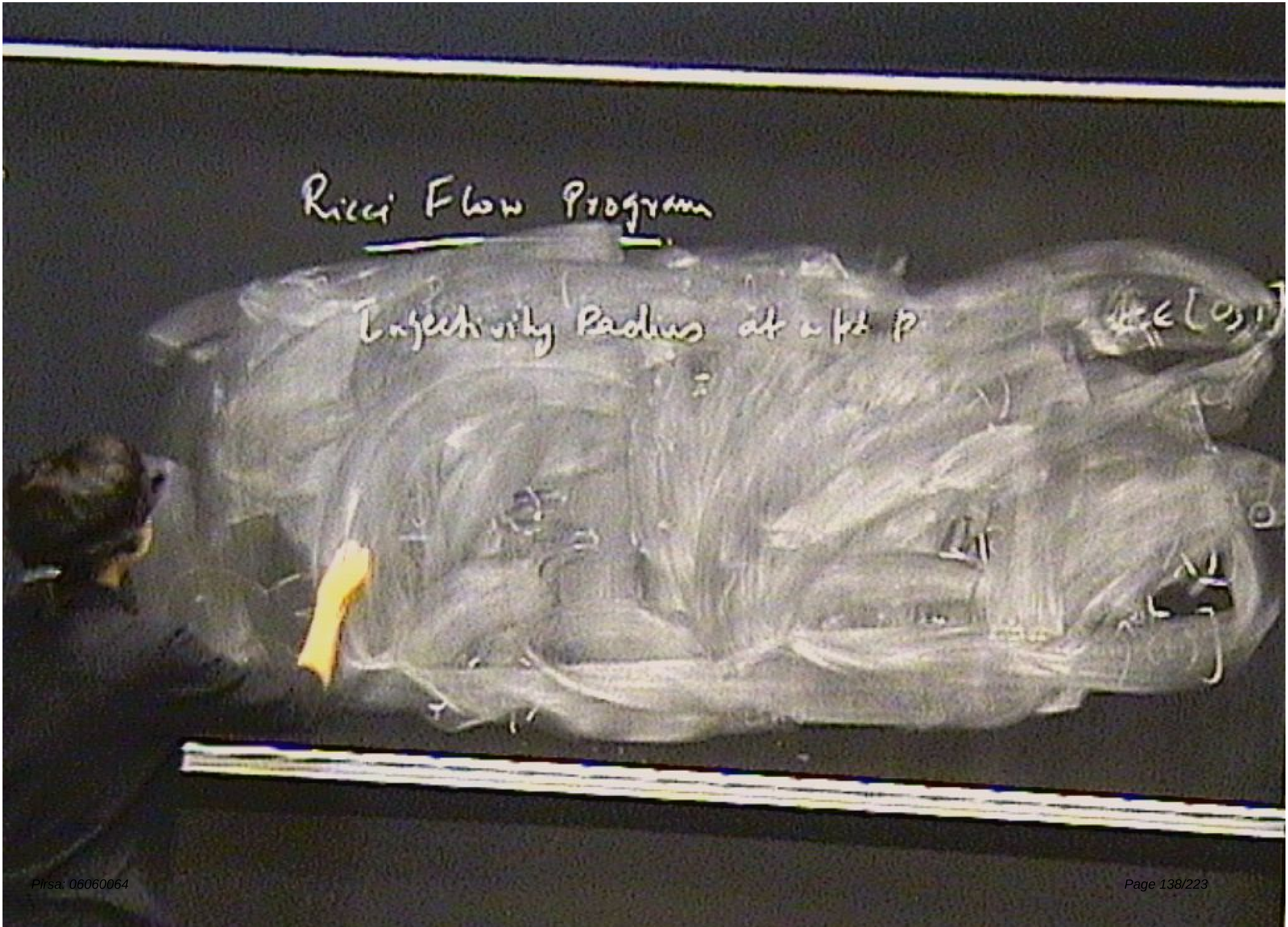
$$R[g^\lambda(x,t)] = \lambda^{-1} R[g(x, t/\lambda)]$$

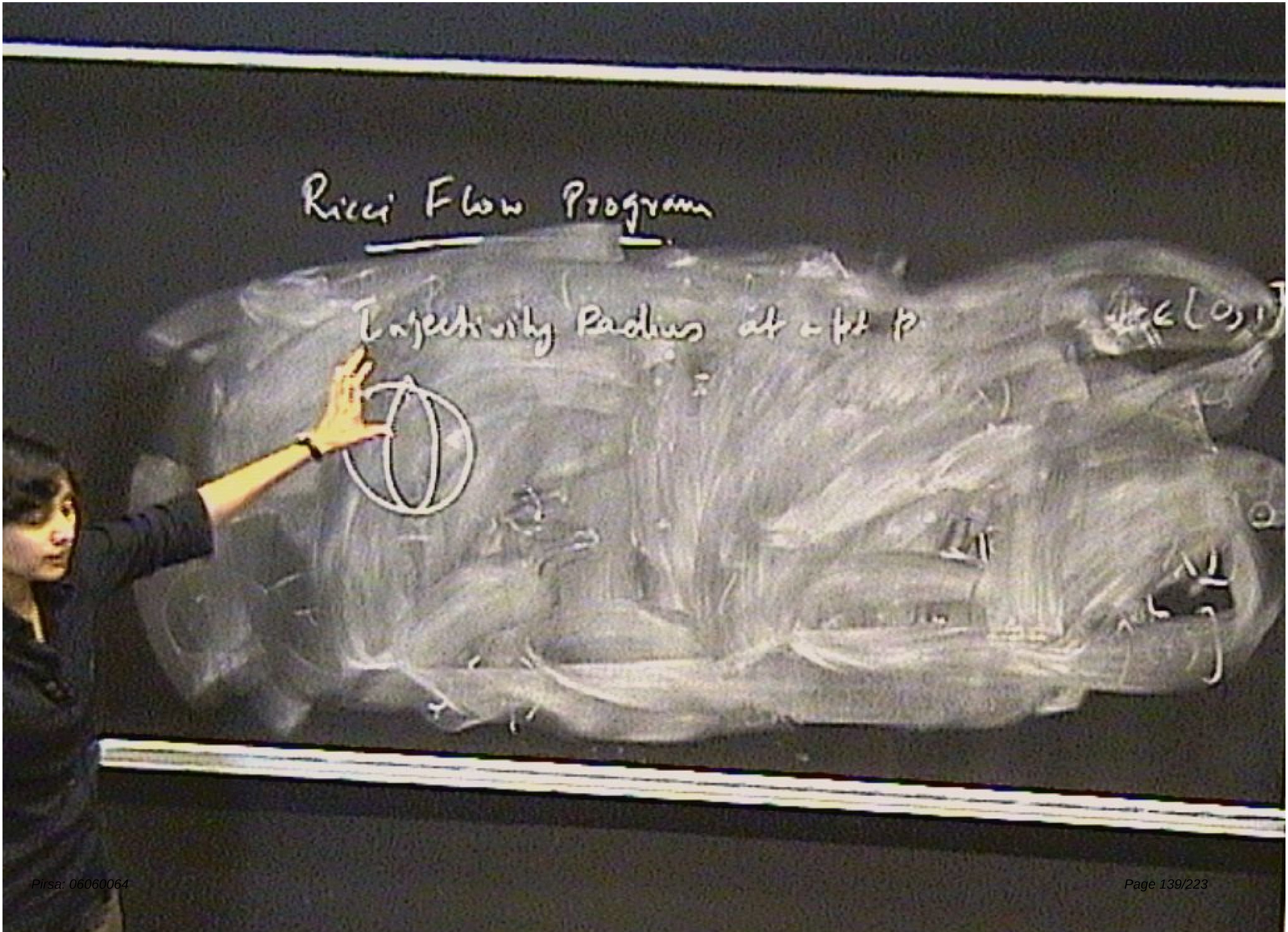
Take λ to be the curvature itself!

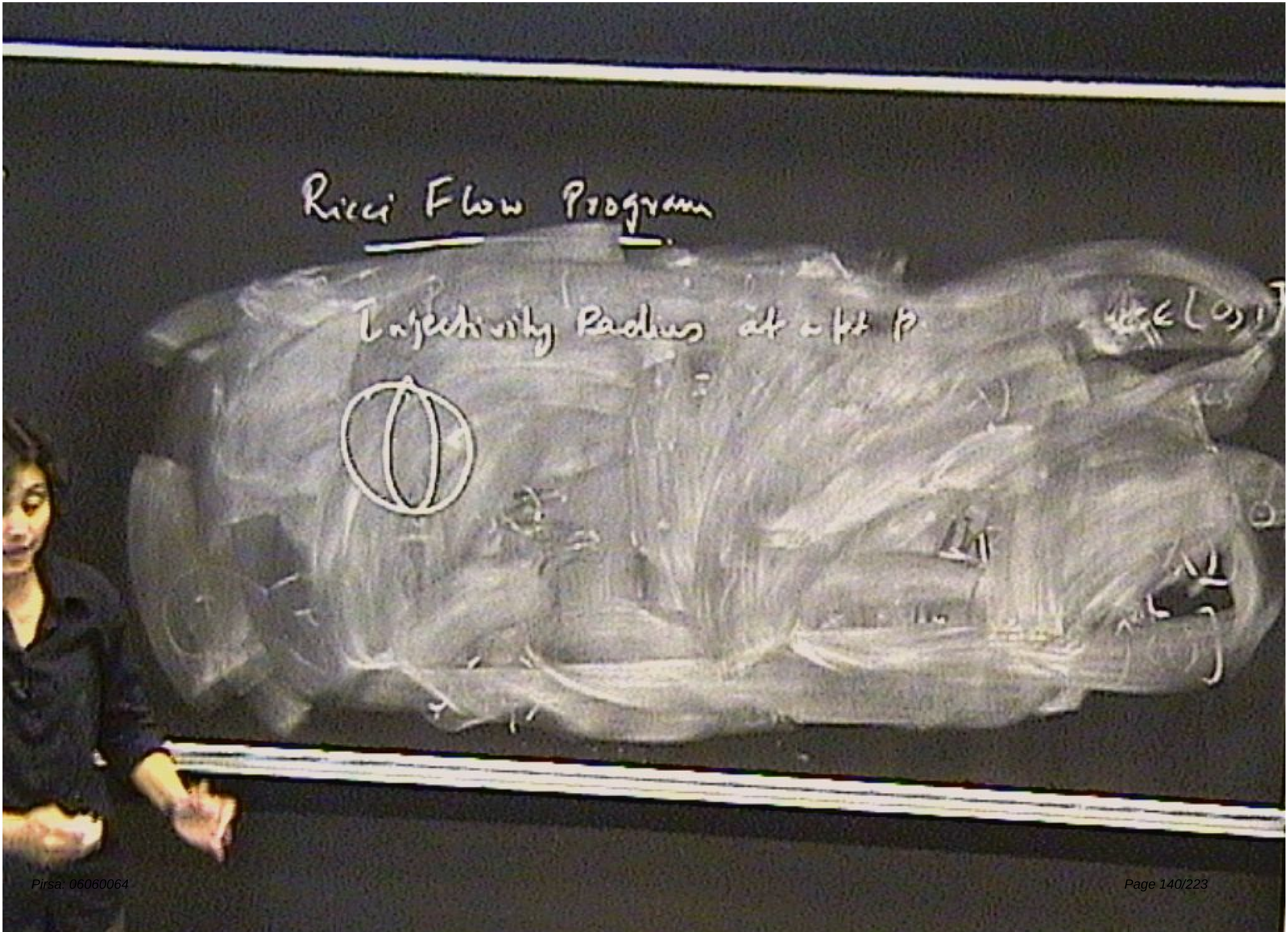


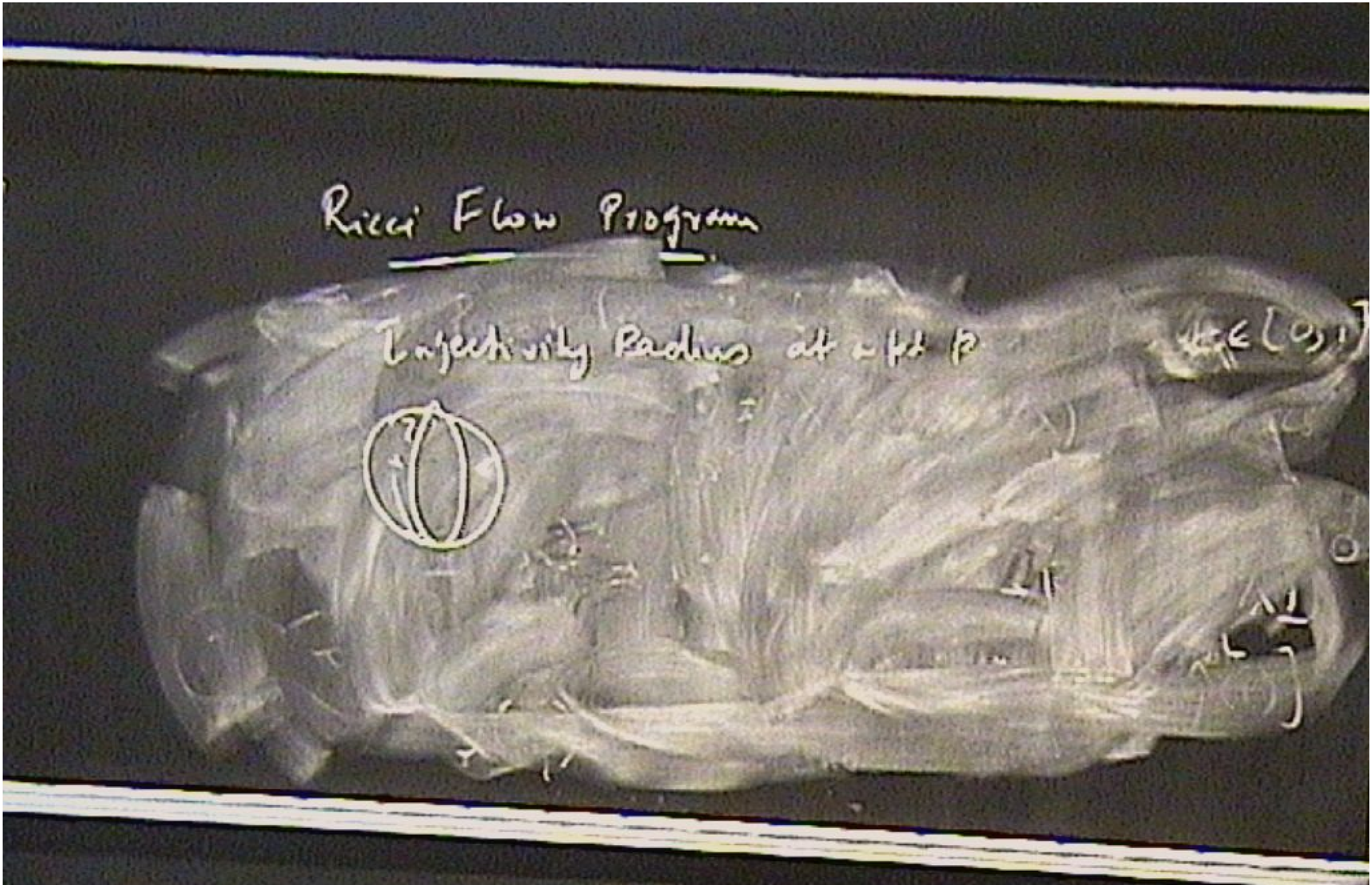
$(M_i, \bar{g}_i, p_i)$ res $\|R\| \le 1$

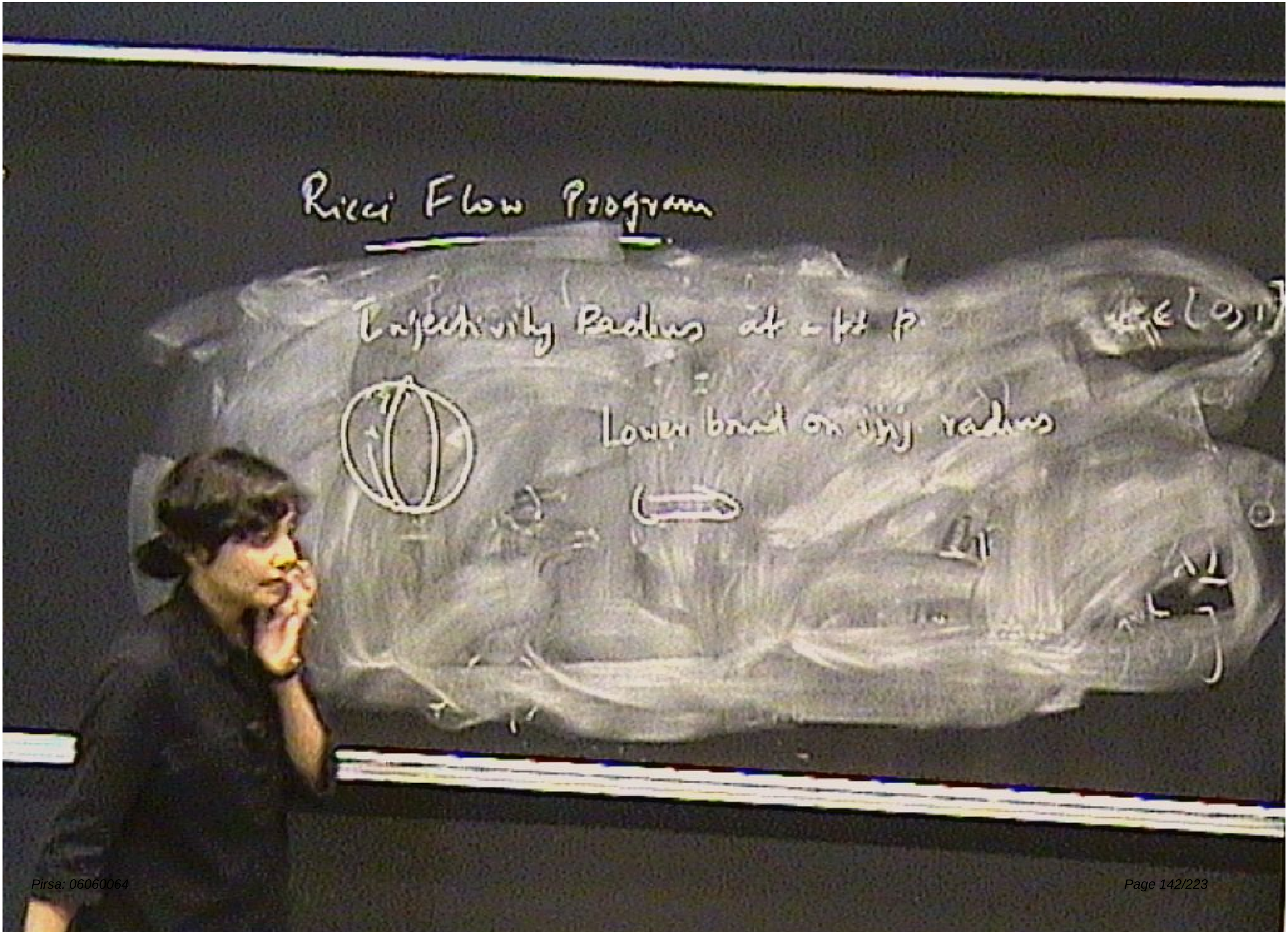

Ricci Flow Program

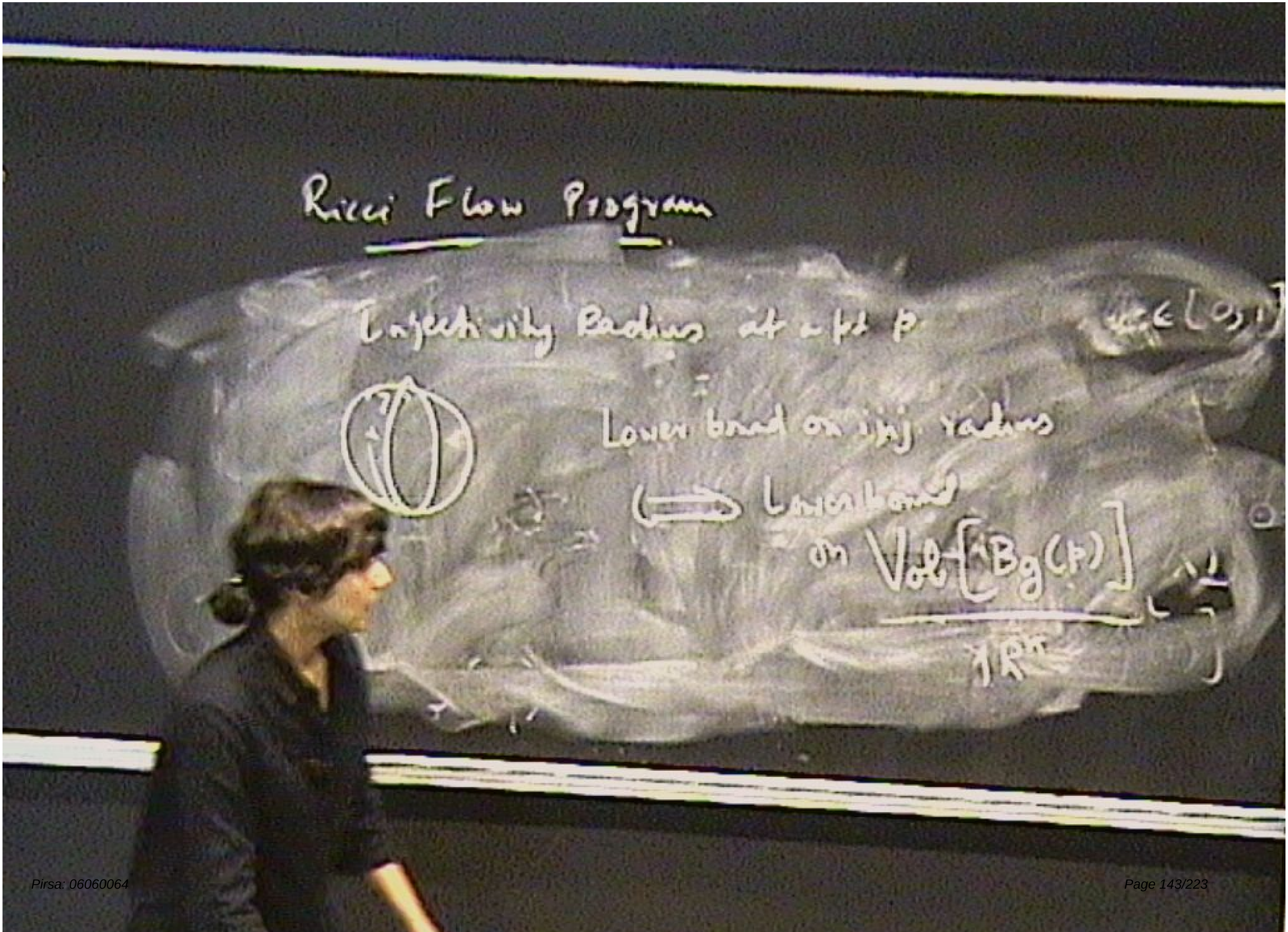

Ricci Flow Program
Injectivity Radius at a pt p
t ∈ [0,1]


Ricci Flow Program
Injectivity Radius at a pt p


Ricci Flow Program
Injectivity Radius at a pt p


Ricci Flow Program
Injectivity Radius at a pt p


Ricci Flow Program
Injectivity Radius at a pt p
Lower bound on inj radius

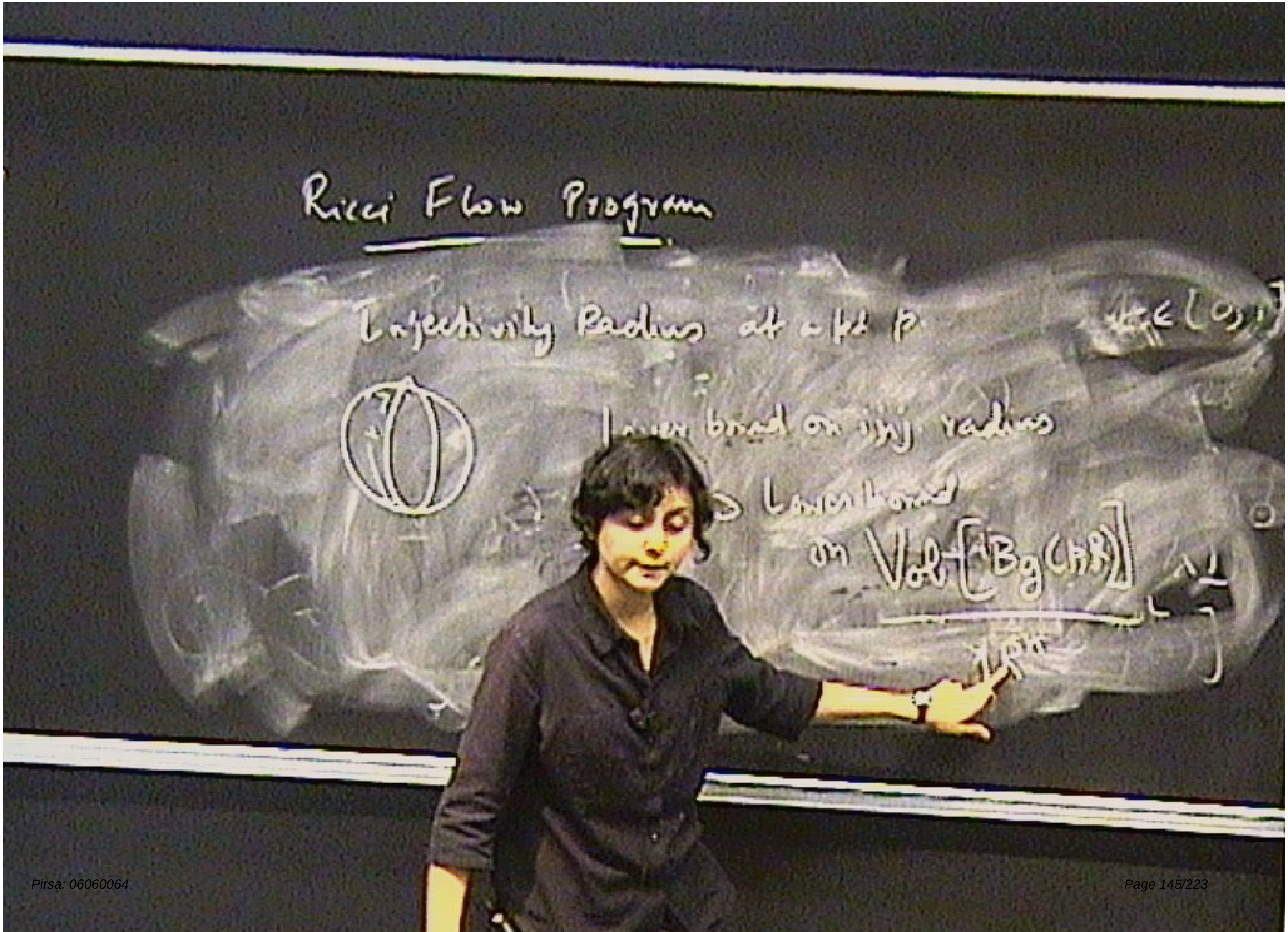

Ricci Flow Program
Injectivity Radius at a pt p
Lower bound on inj. radius
Lower bound
on Vol

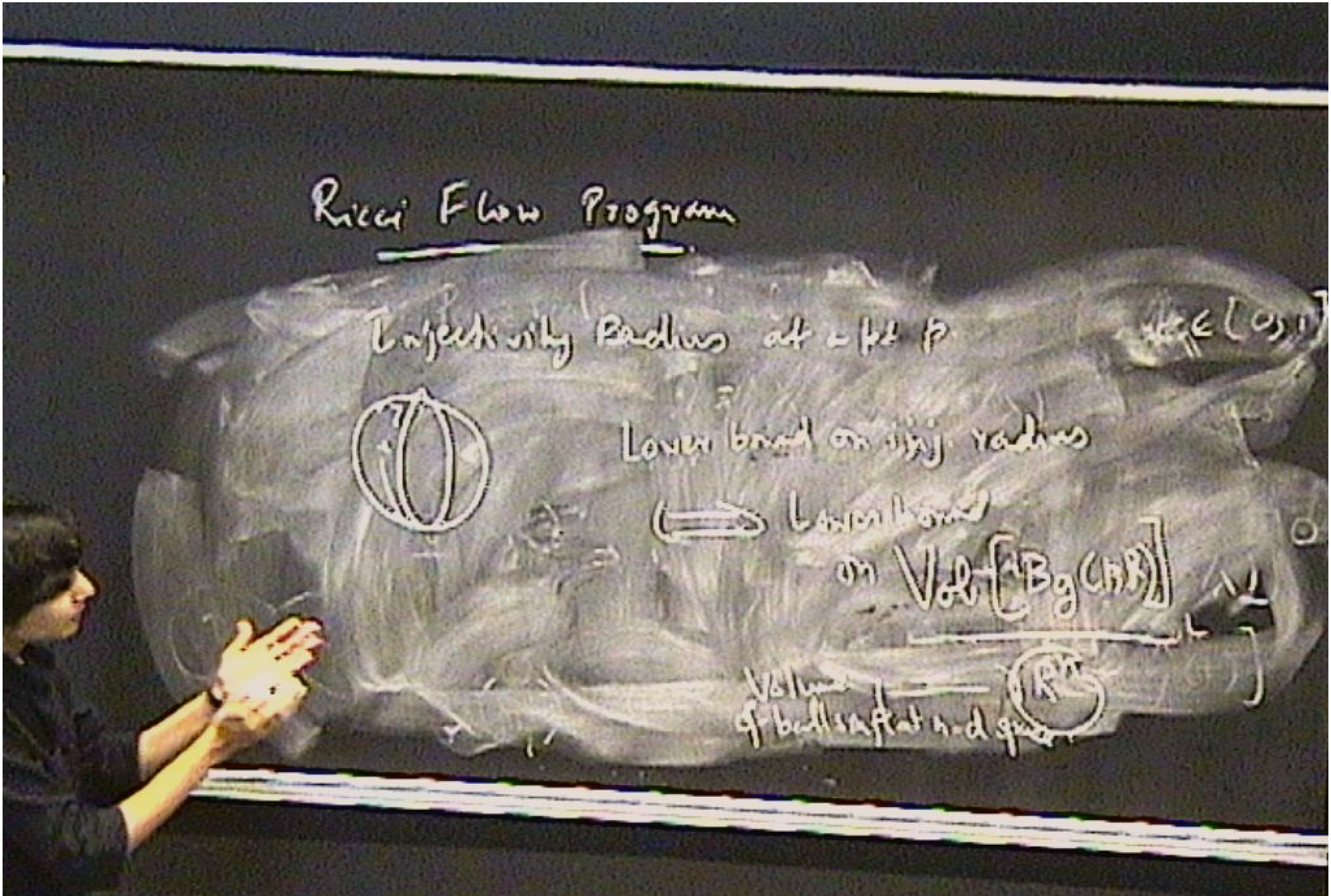
Ricci Flow Program

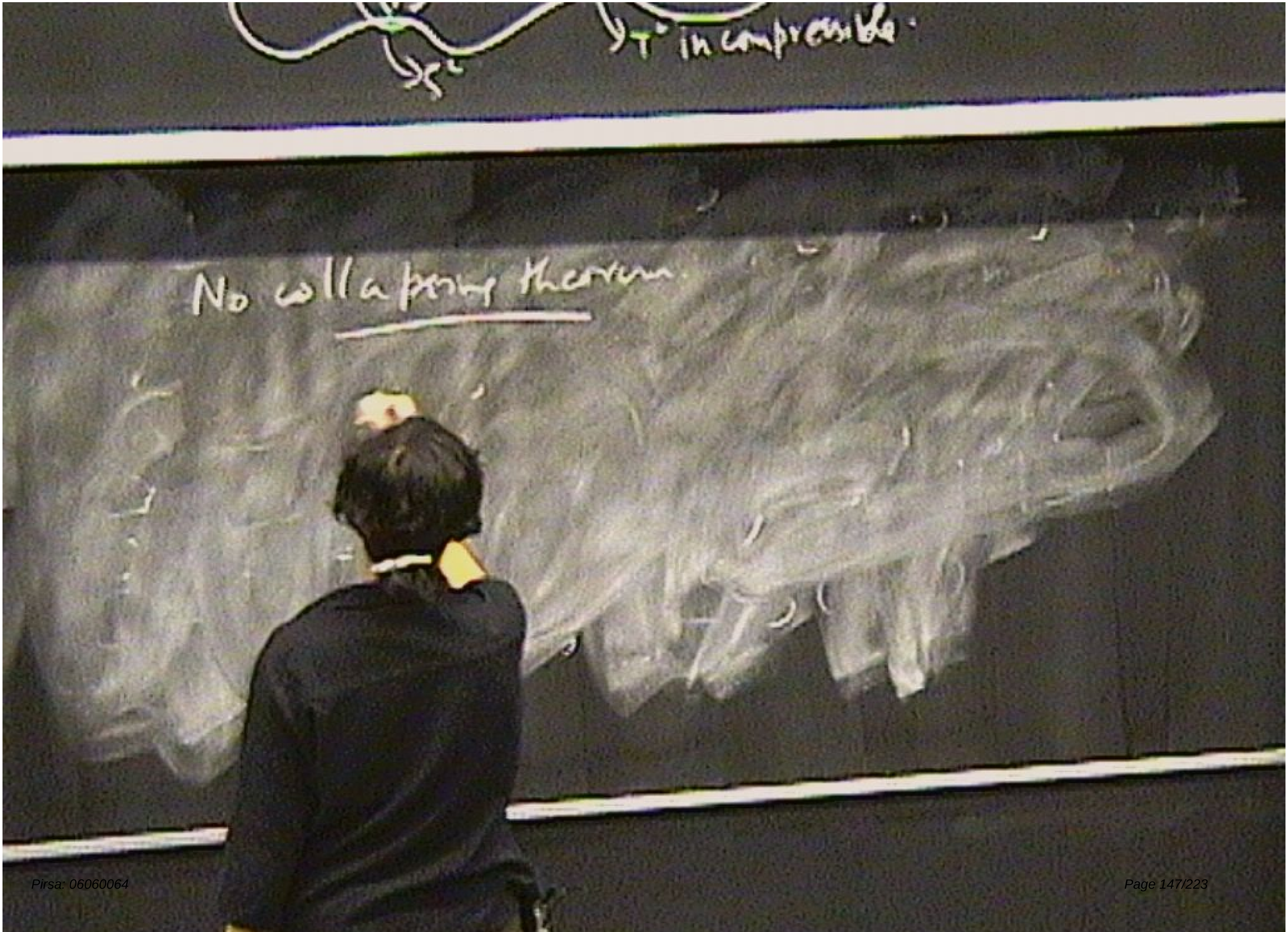
Injectivity Radius at a pt p $\quad t \in [0,1]$

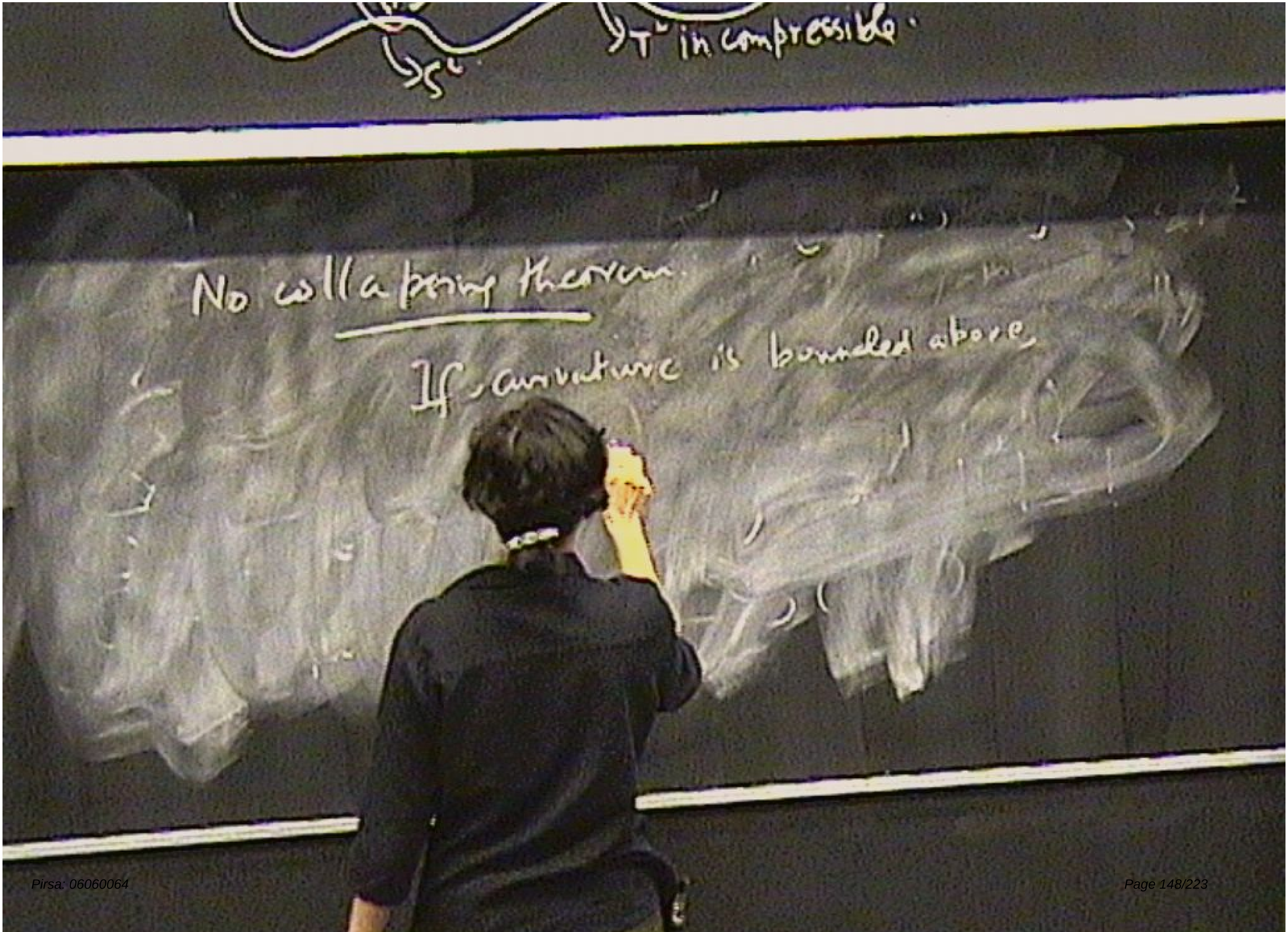
Lower bound on inj. radius

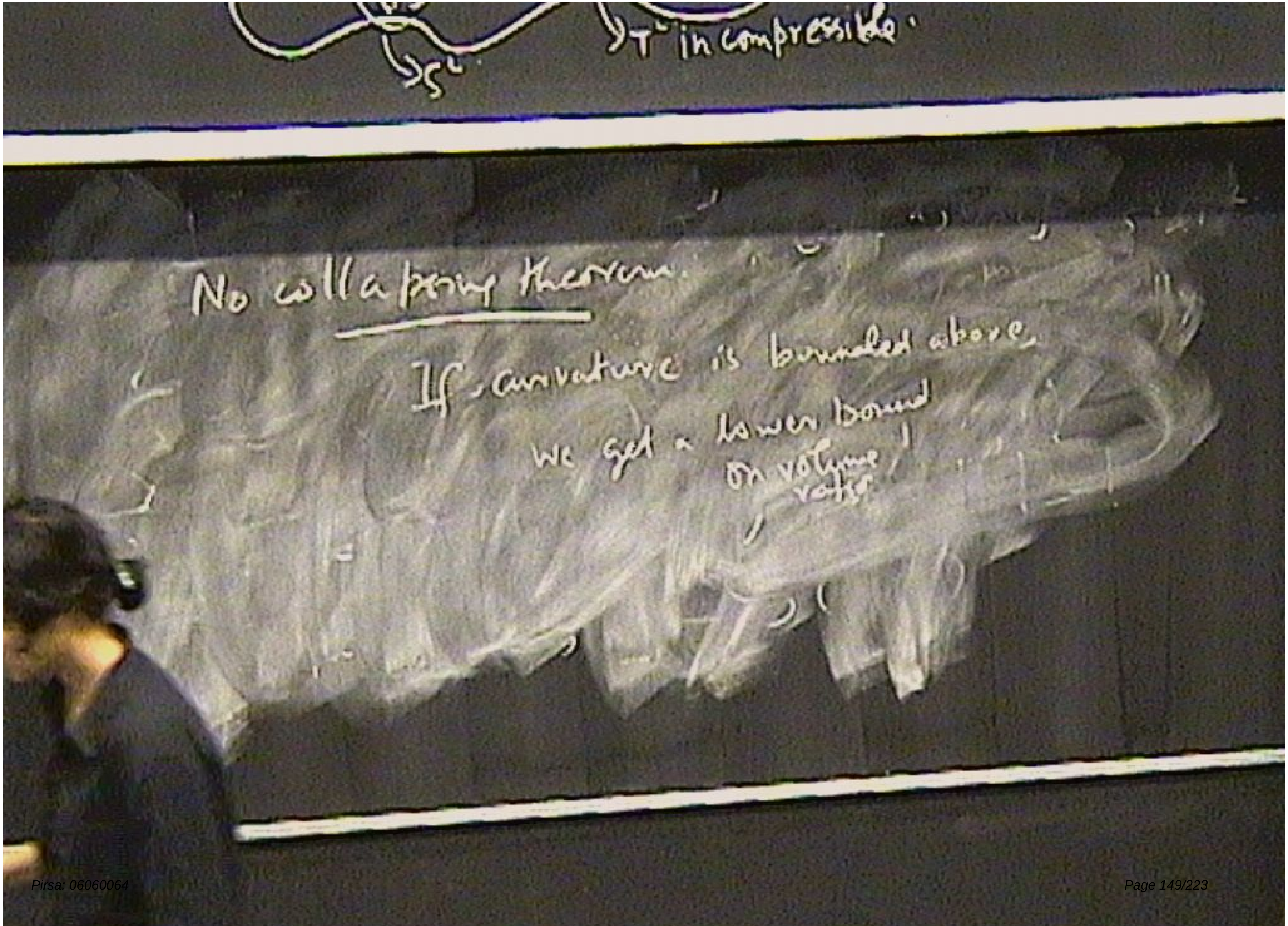
$\iff$ Lower bound on $\mathrm{Vol}\left[B_g(p)\right]$


Ricci Flow Program
Injectivity Radius at a pt p
Lower bound on inj radius
Lower bound

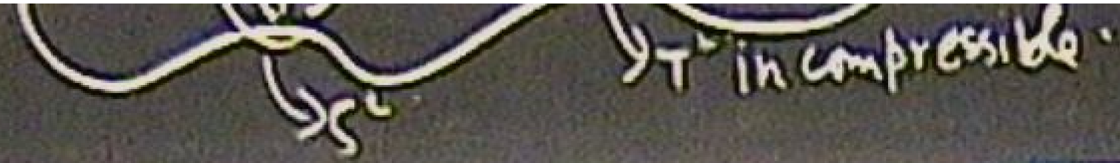

Ricci Flow Program
Injectivity Radius at a pt p
Lower bound on inj. radius
Lower bound
on Vol
Volume


incompressible
No collapsing theorem.


S²
T² incompressible.
No collapsing theorem.
If curvature is bounded above



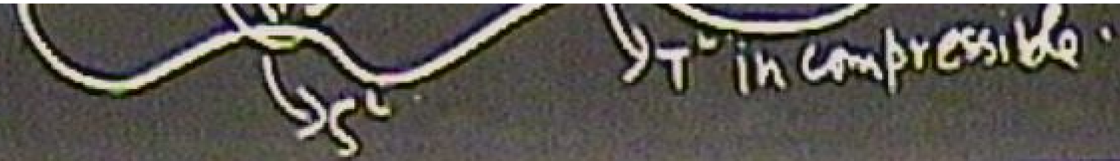
S²
T² incompressible.
No collapsing theorem.
If curvature is bounded above,
we get a lower bound
on volume ratio



No collapsing theorem.

If curvature is bounded above, we get a lower bound on volume ratio

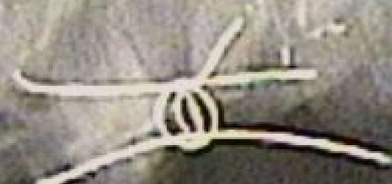
$$\Rightarrow \frac{\mathrm{Vol}\left[B_g(p,R)\right]}{R^n} \geq c > 0.$$

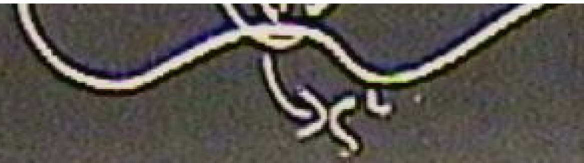


No collapsing theorem.

If curvature is bounded above, we get a lower bound on volume ratio!

$$\Rightarrow \frac{Vol[B_g(p,R)]}{\gamma R^n} \geq c > 0.$$





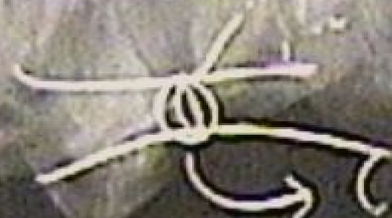
S^2

T^2 in compressible.

No collapsing theorem.

If curvature is bounded above, we get a lower bound on volume ratio

$$\Rightarrow \frac{\mathrm{Vol}\,[B_g(p,R)]}{R^n} \geq c > 0.$$



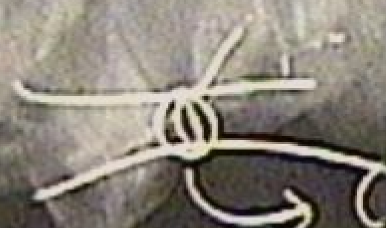
Crossection of a neck is an S^2

T^2 incompressible.

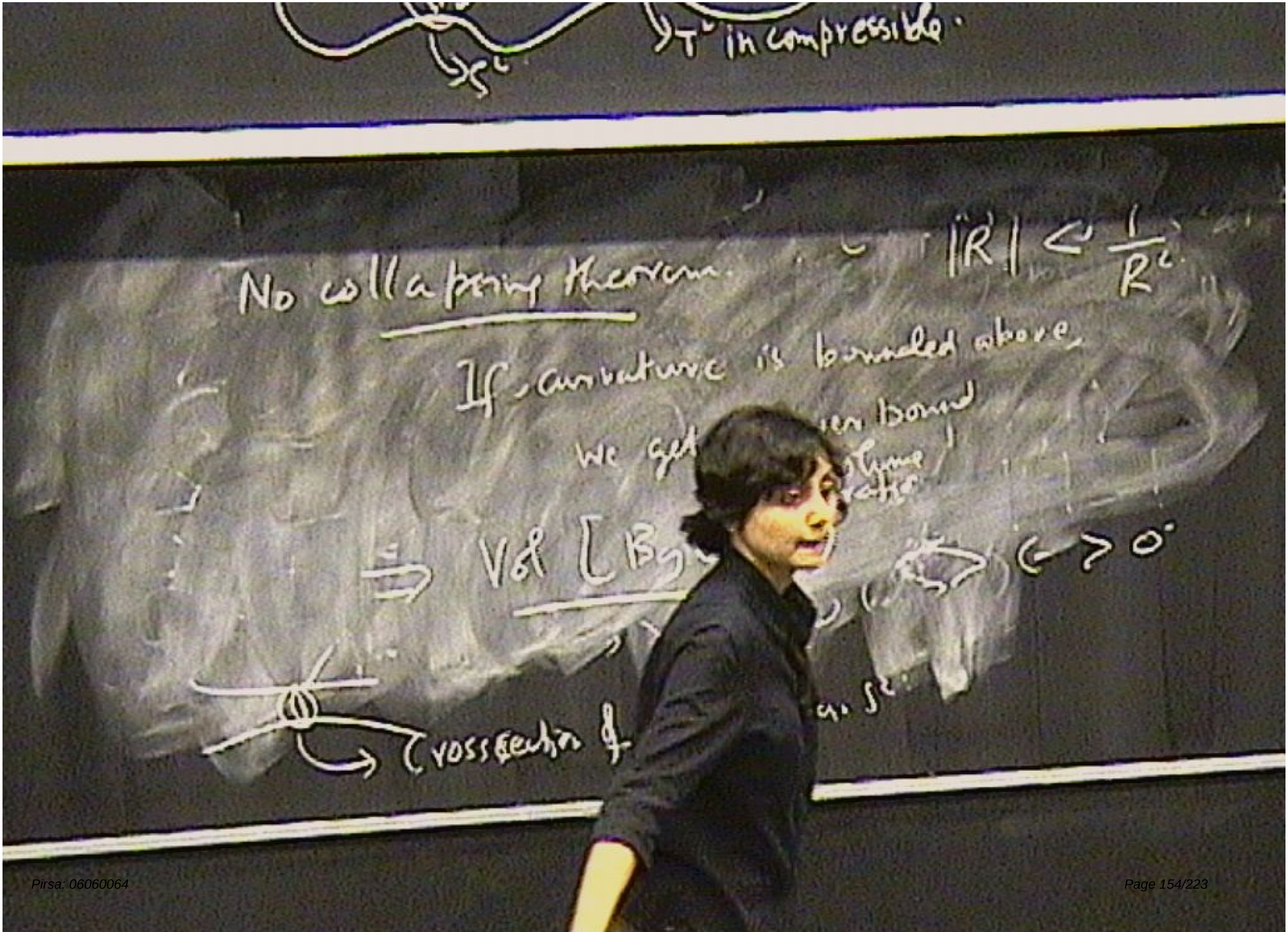
No collapsing theorem.

If curvature is bounded above we get a lower bound on volume ratio

$$\Rightarrow \frac{Vol\left[B_g(p,R)\right]}{R^n} \geq c > 0.$$



Crosssection of a neck is an S^2.



in compressible.
No collapsing theorem.
|R| < 1/R²
If curvature is bounded above
we get
ver bound
Vol [B
c > 0.
Crosssection of

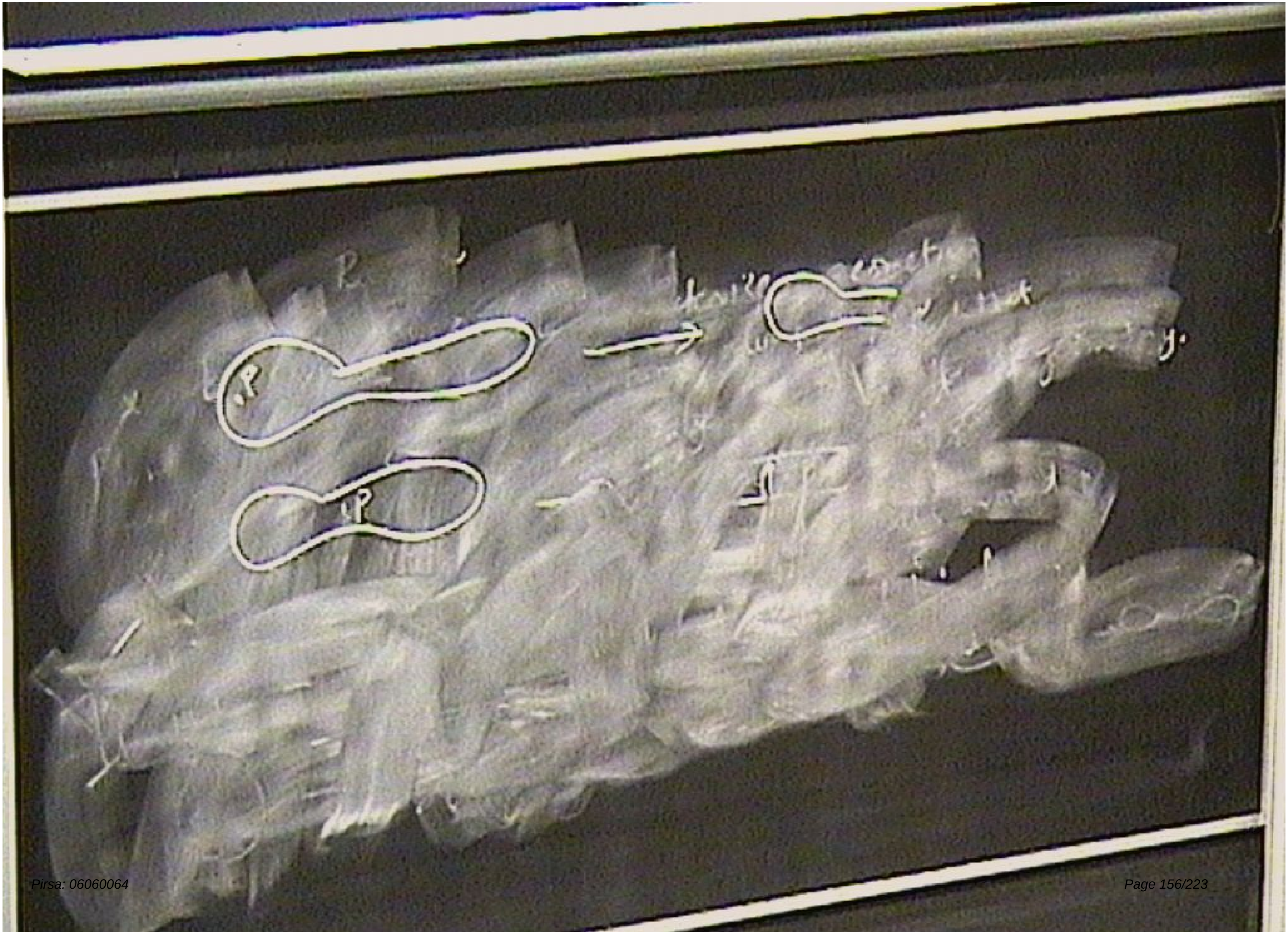
S^2 · T^2 incompressible.

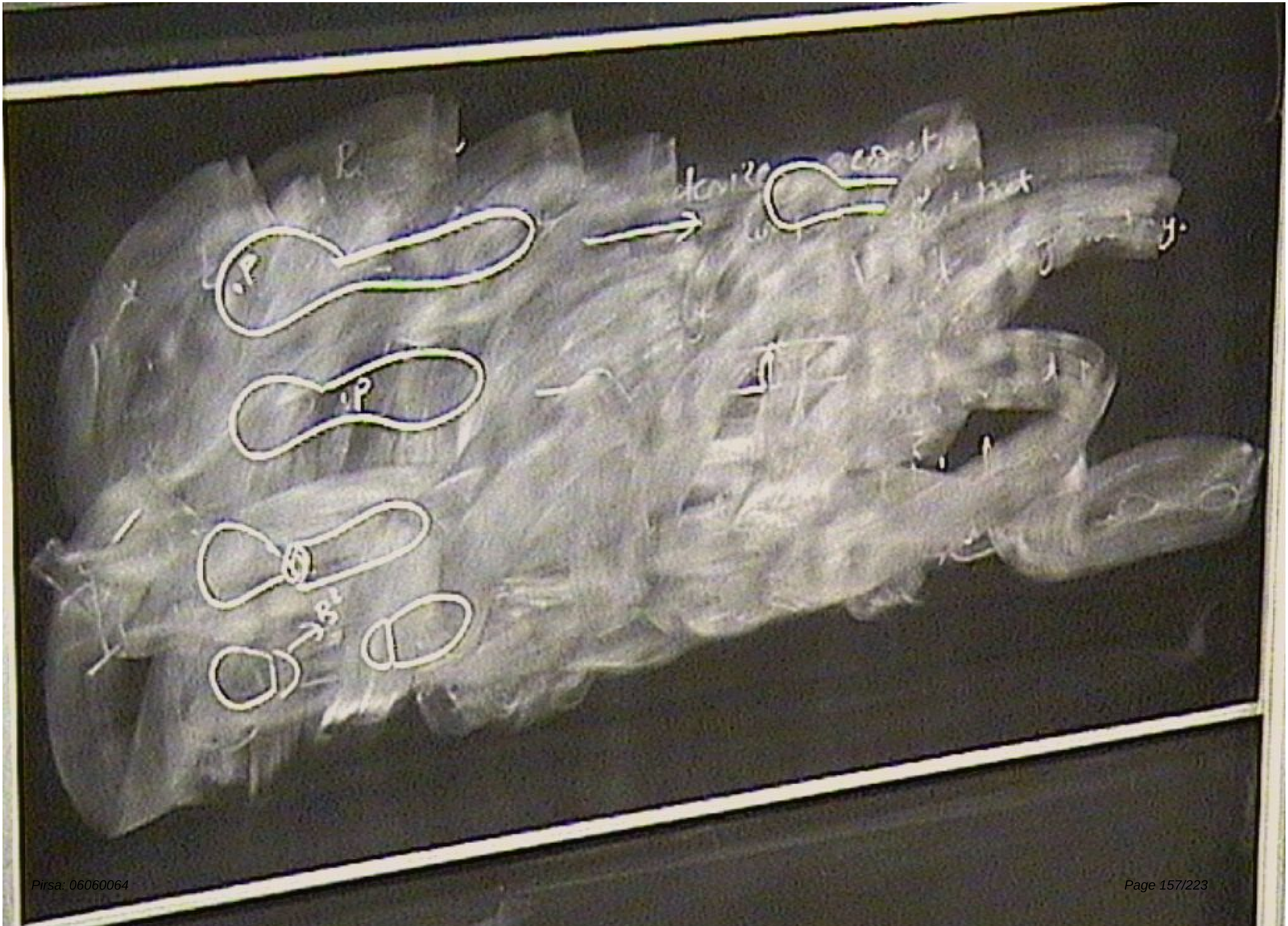
No collapsing theorem. $|Rm| \leq \frac{1}{R^2}$

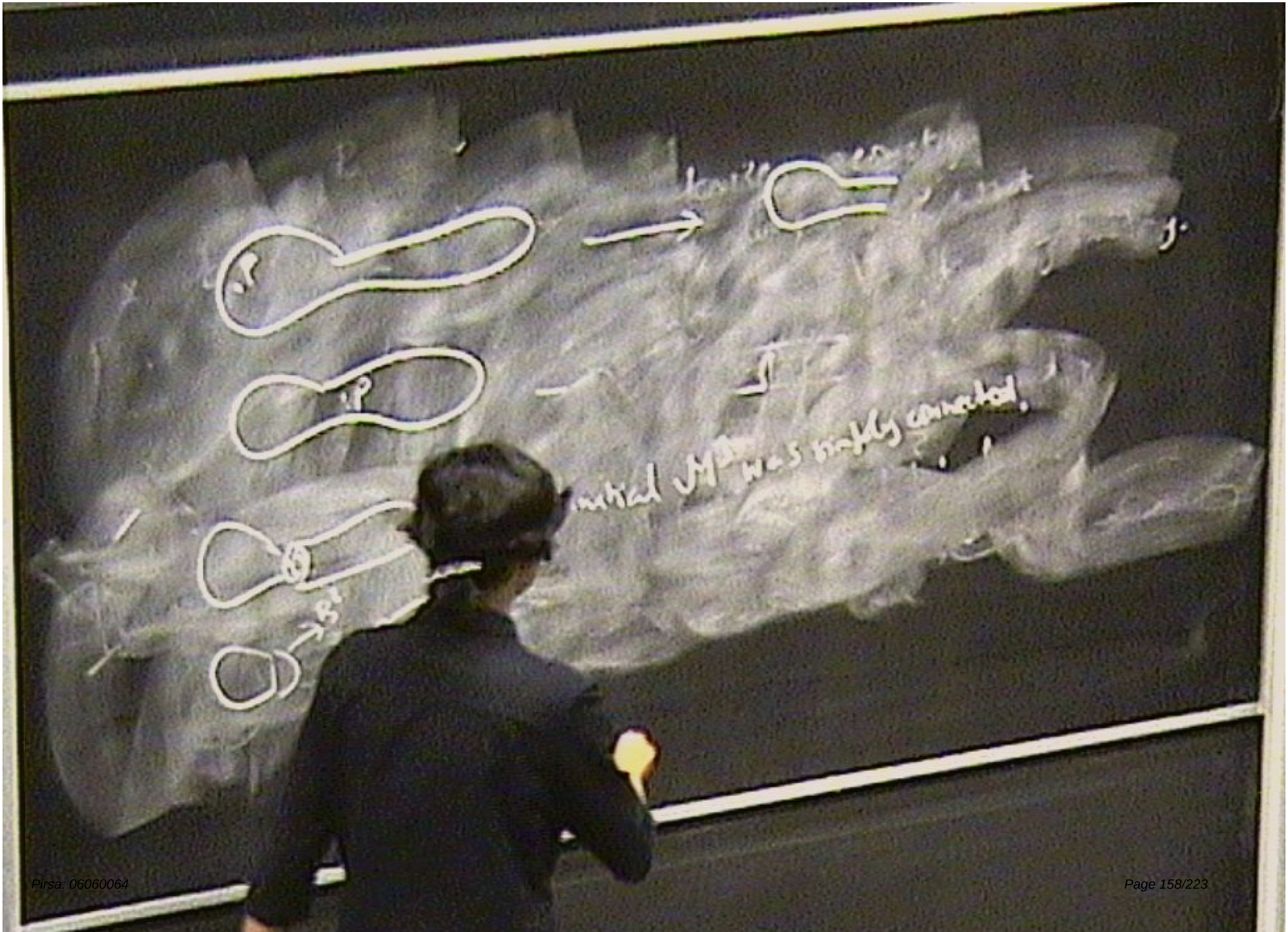
If curvature is bounded above, we get a lower bound on volume ratio

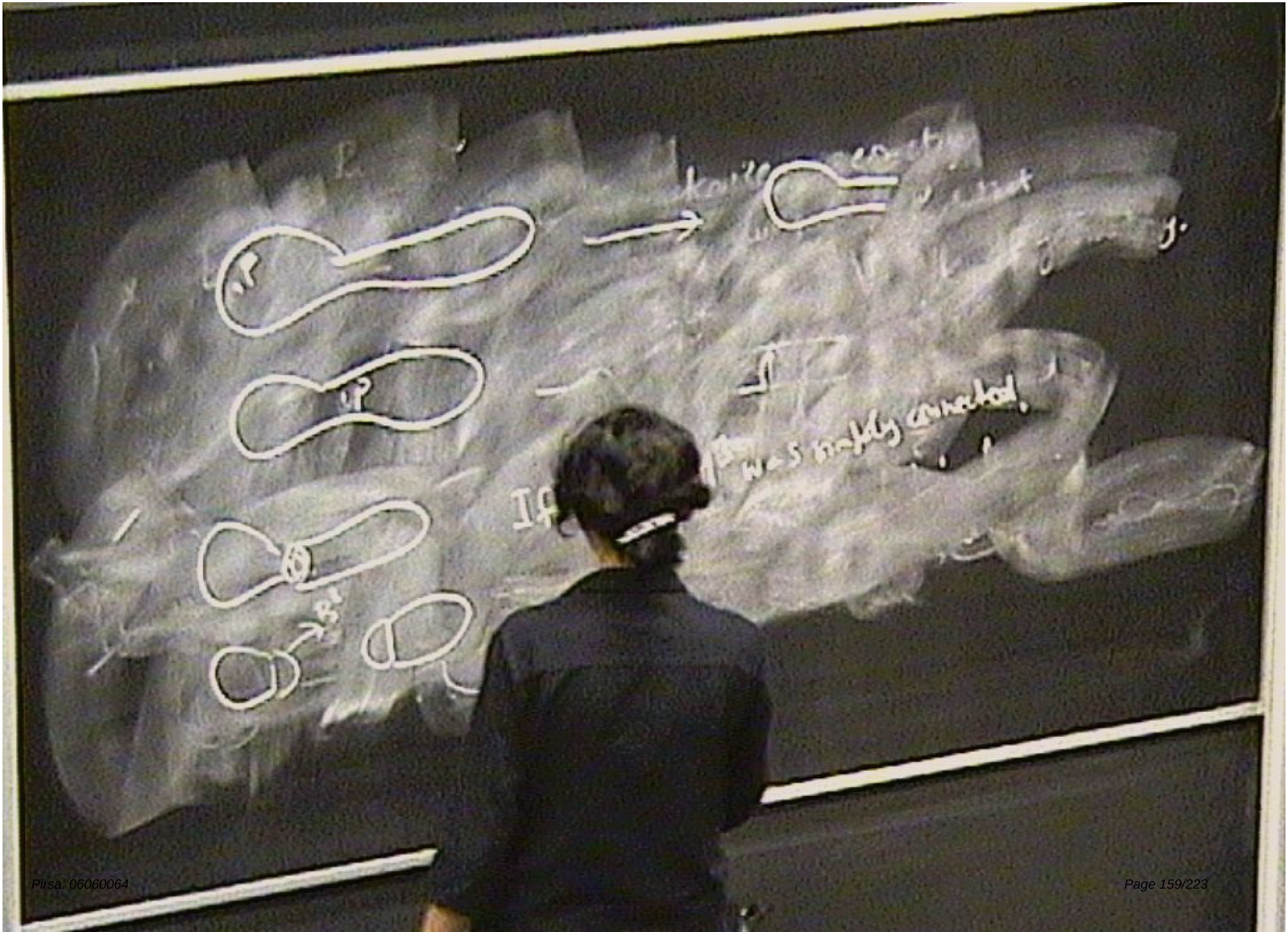
$$\Rightarrow \frac{\mathrm{Vol}\,[B_g(p,R)]}{R^n} > c > 0.$$

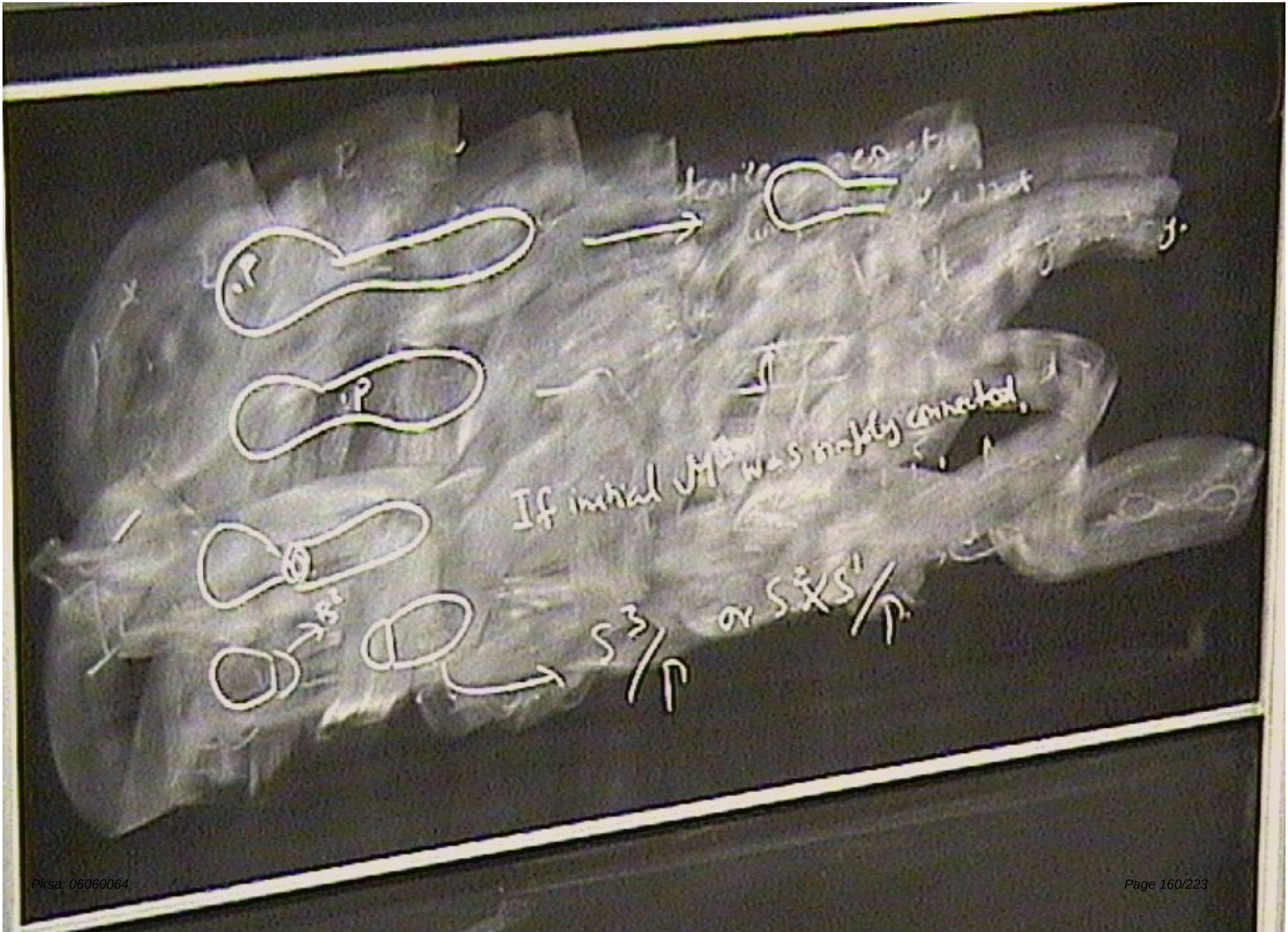
Crossection of a neck is an S^2.

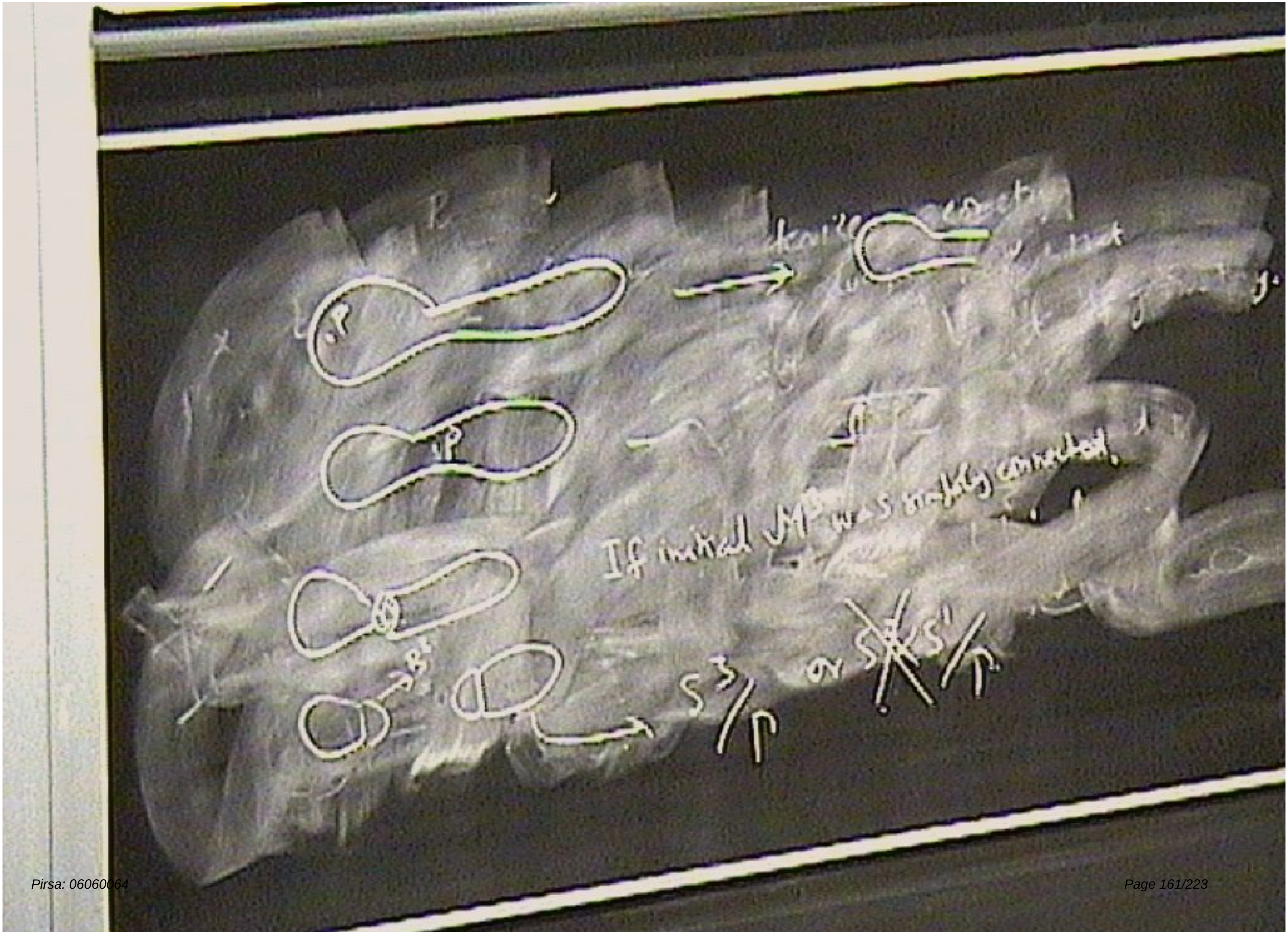


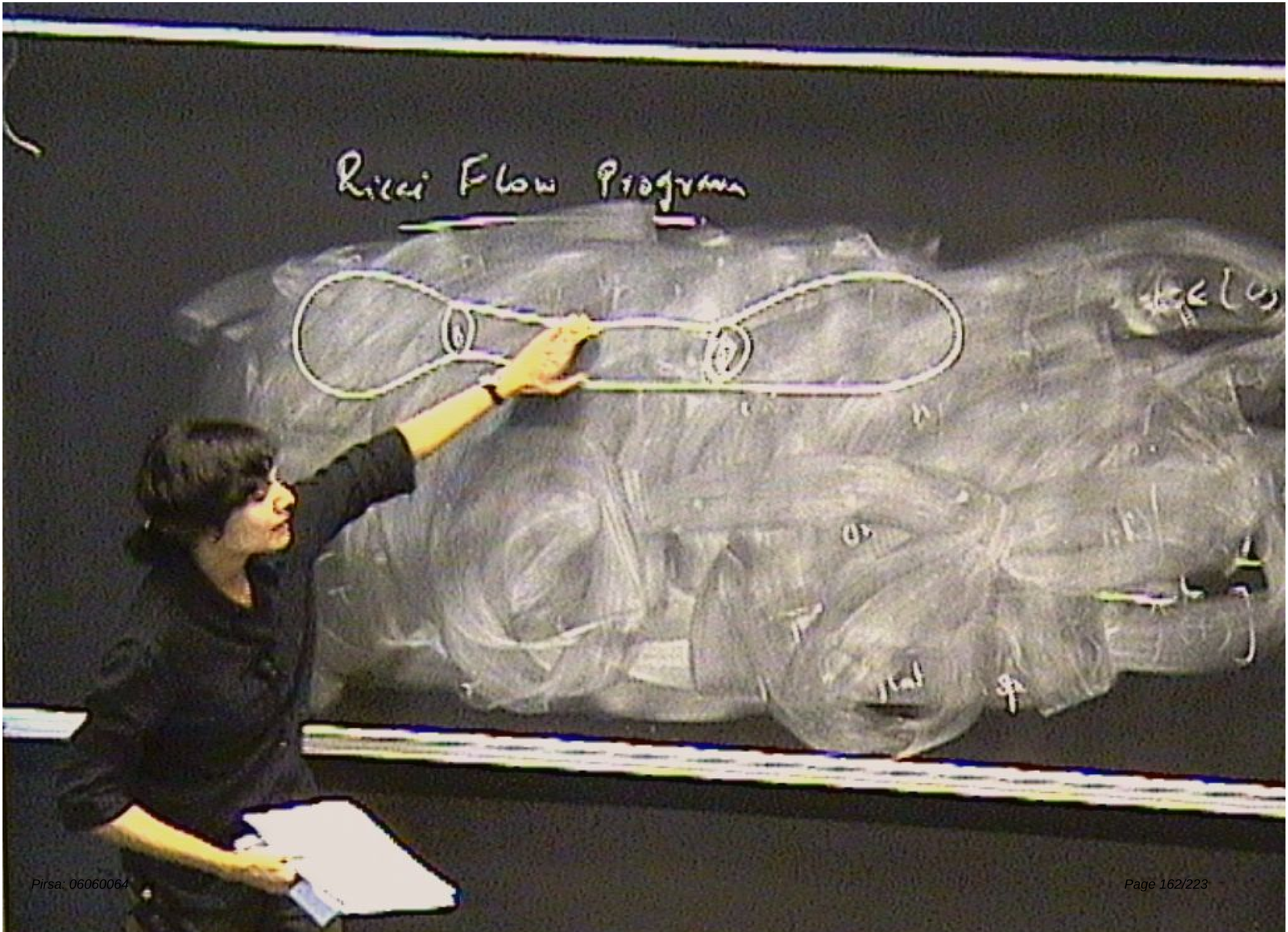


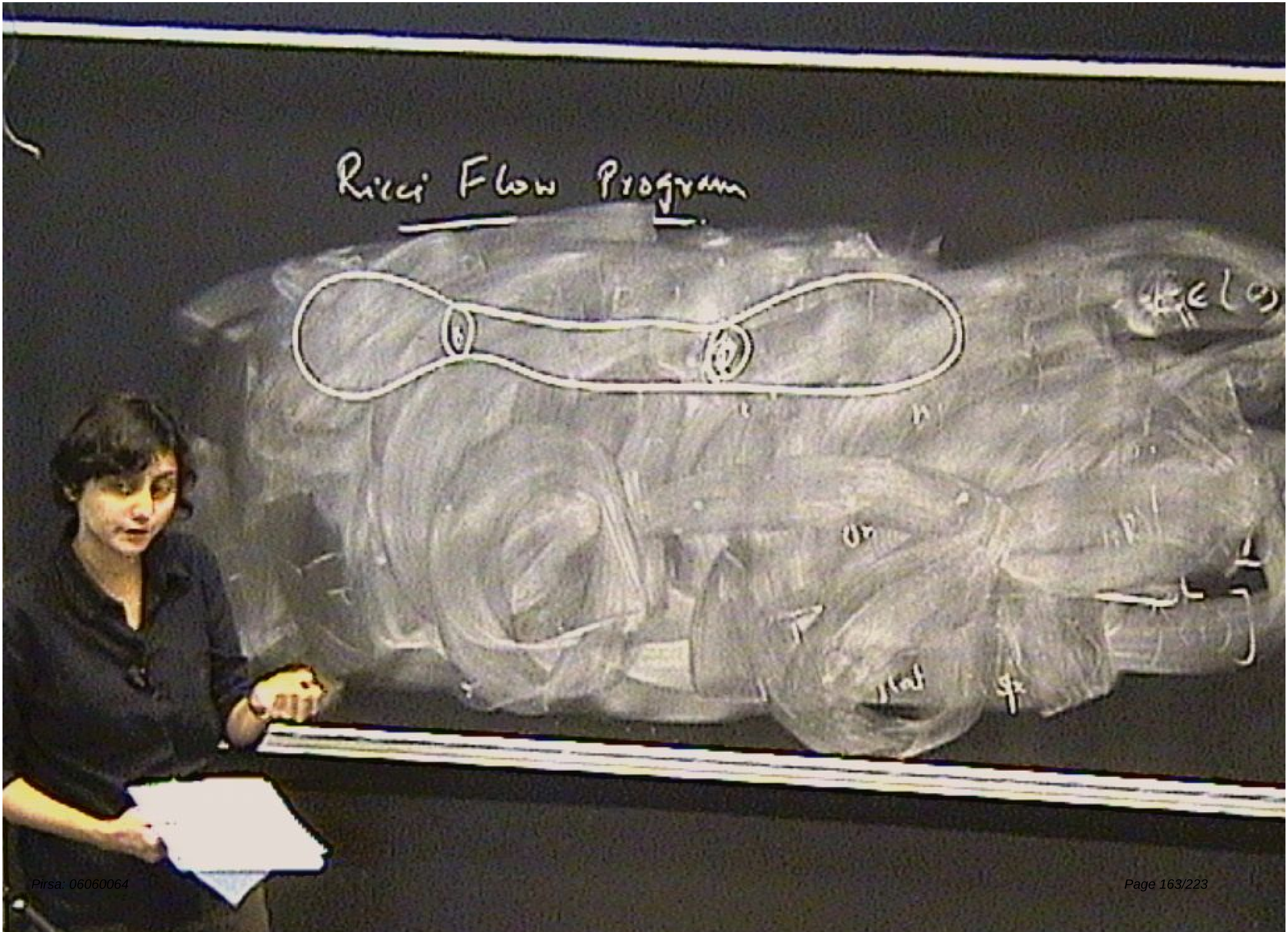

was simply connected.

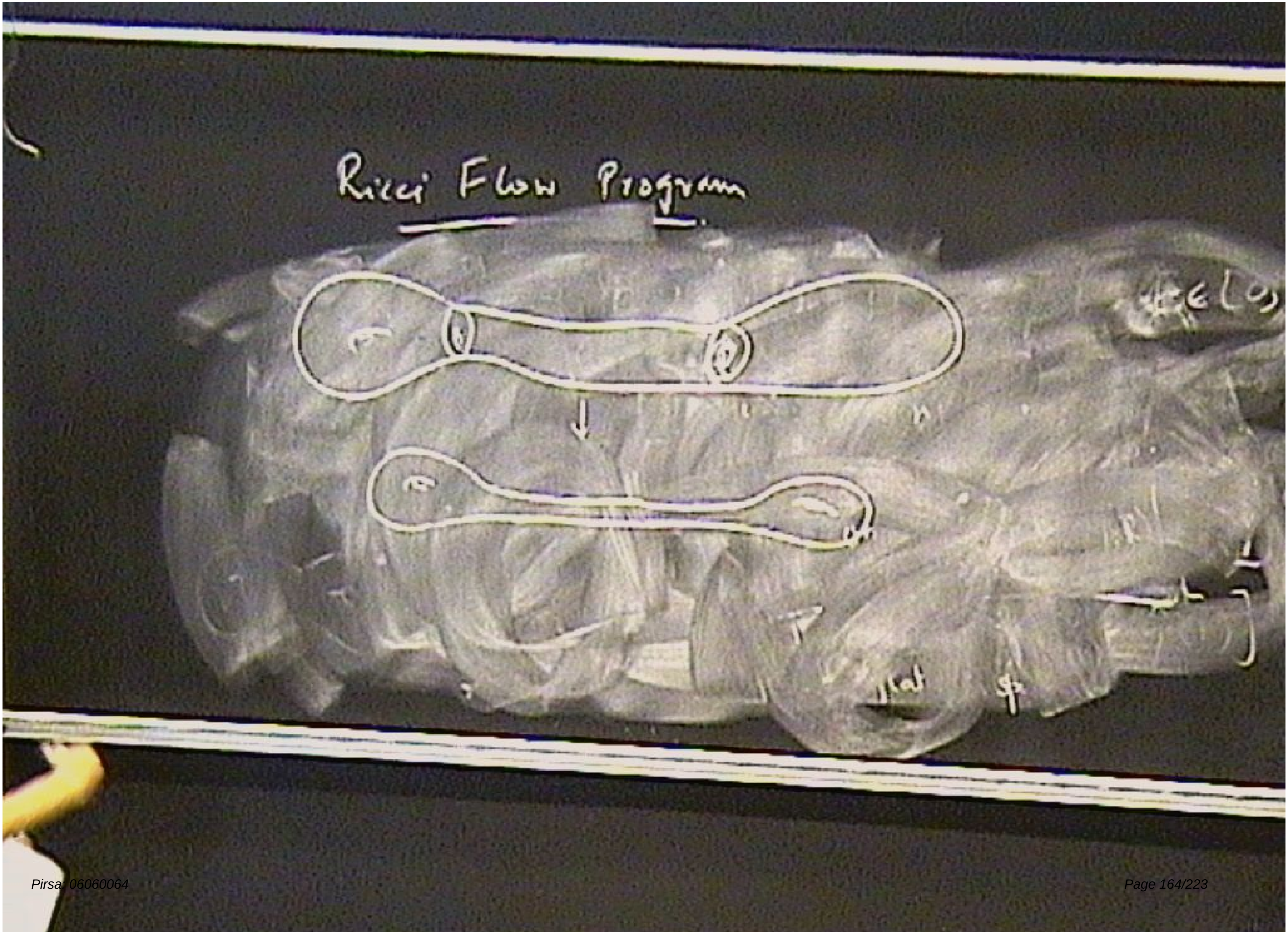

If
simply connected.

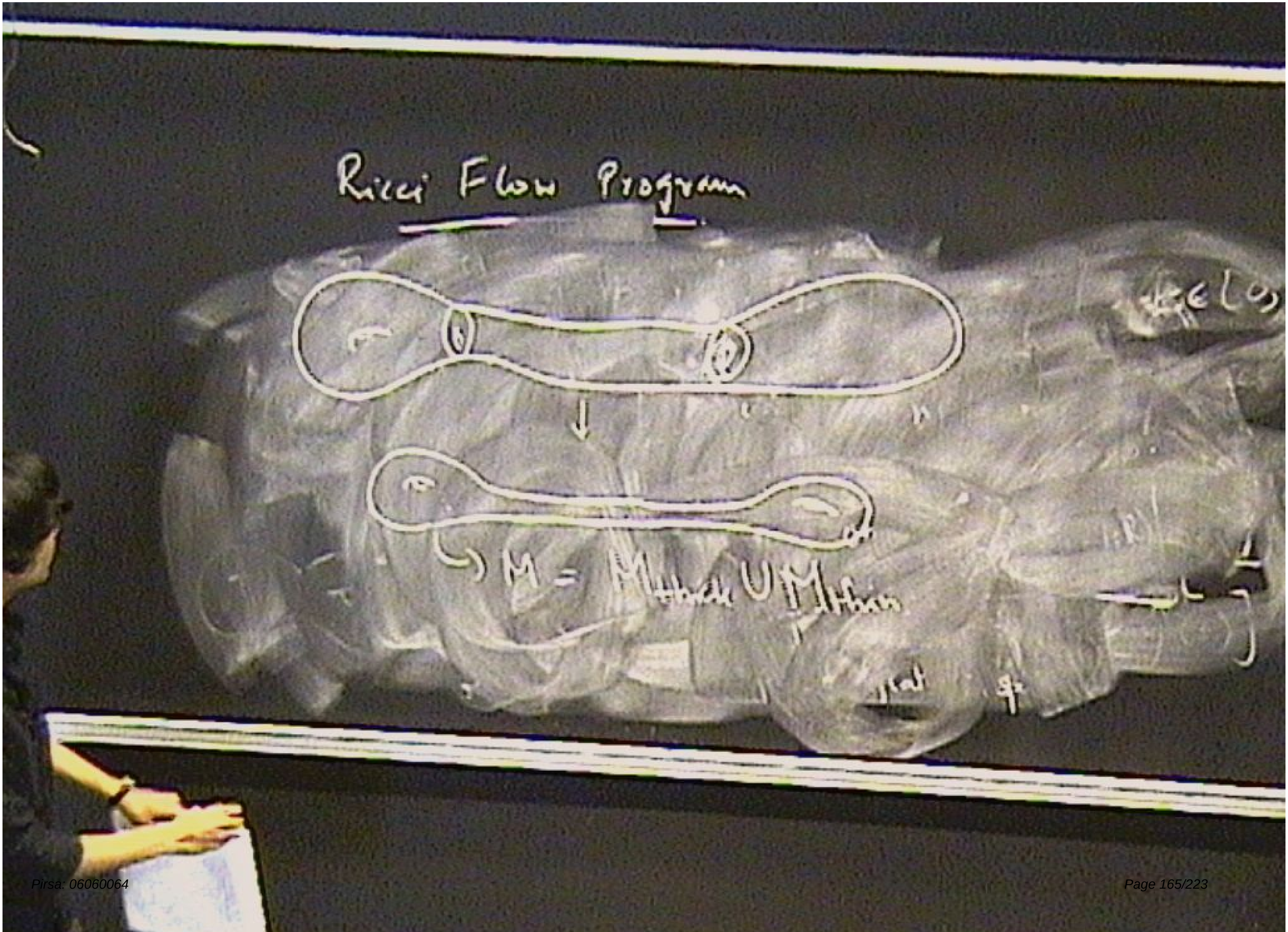

If initial M was simply connected,
S³/Γ
or S²×S¹/Γ

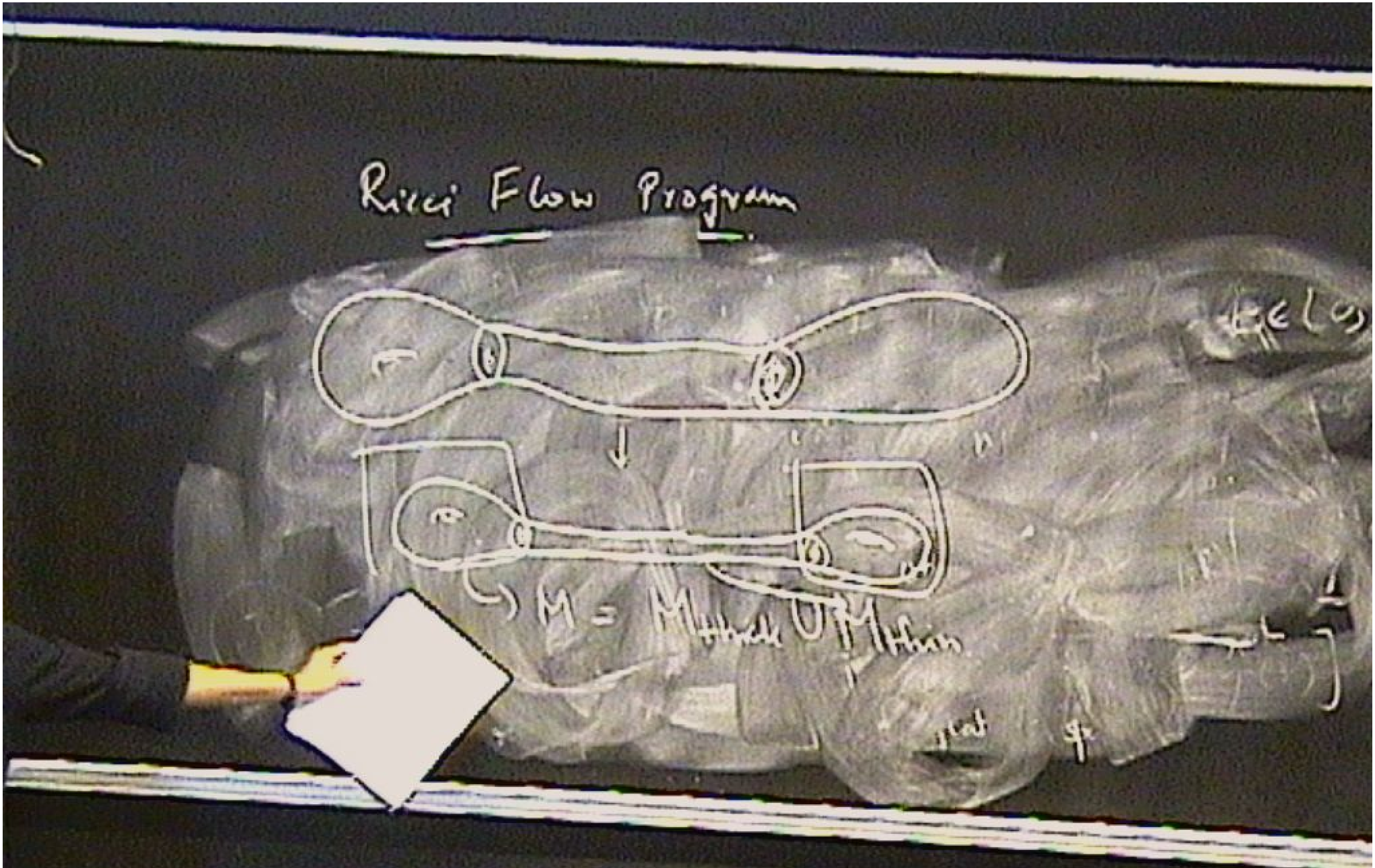

If initial M³ was simply connected,
S³/Γ
or S²×S¹/Γ

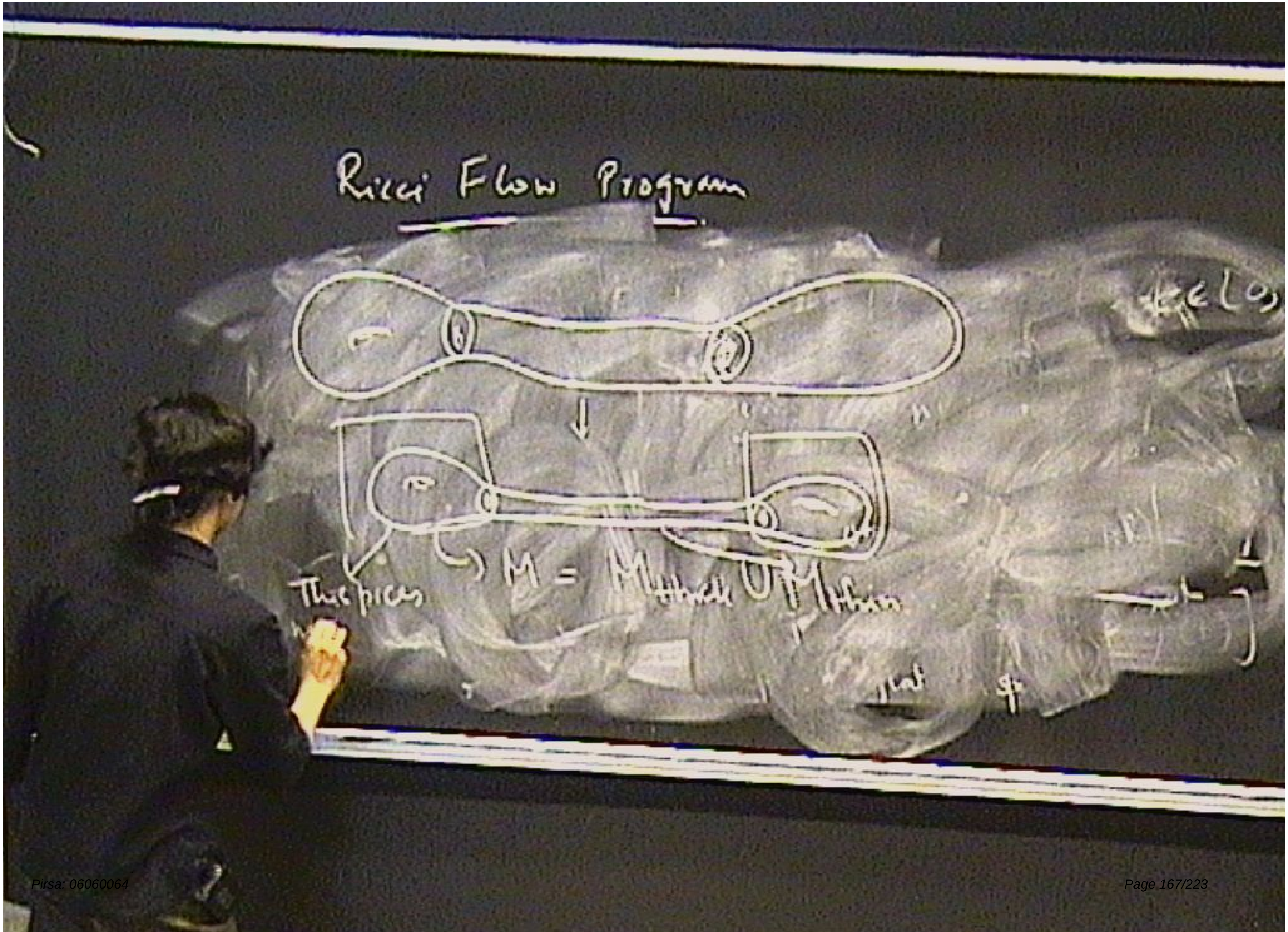

Ricci Flow Program

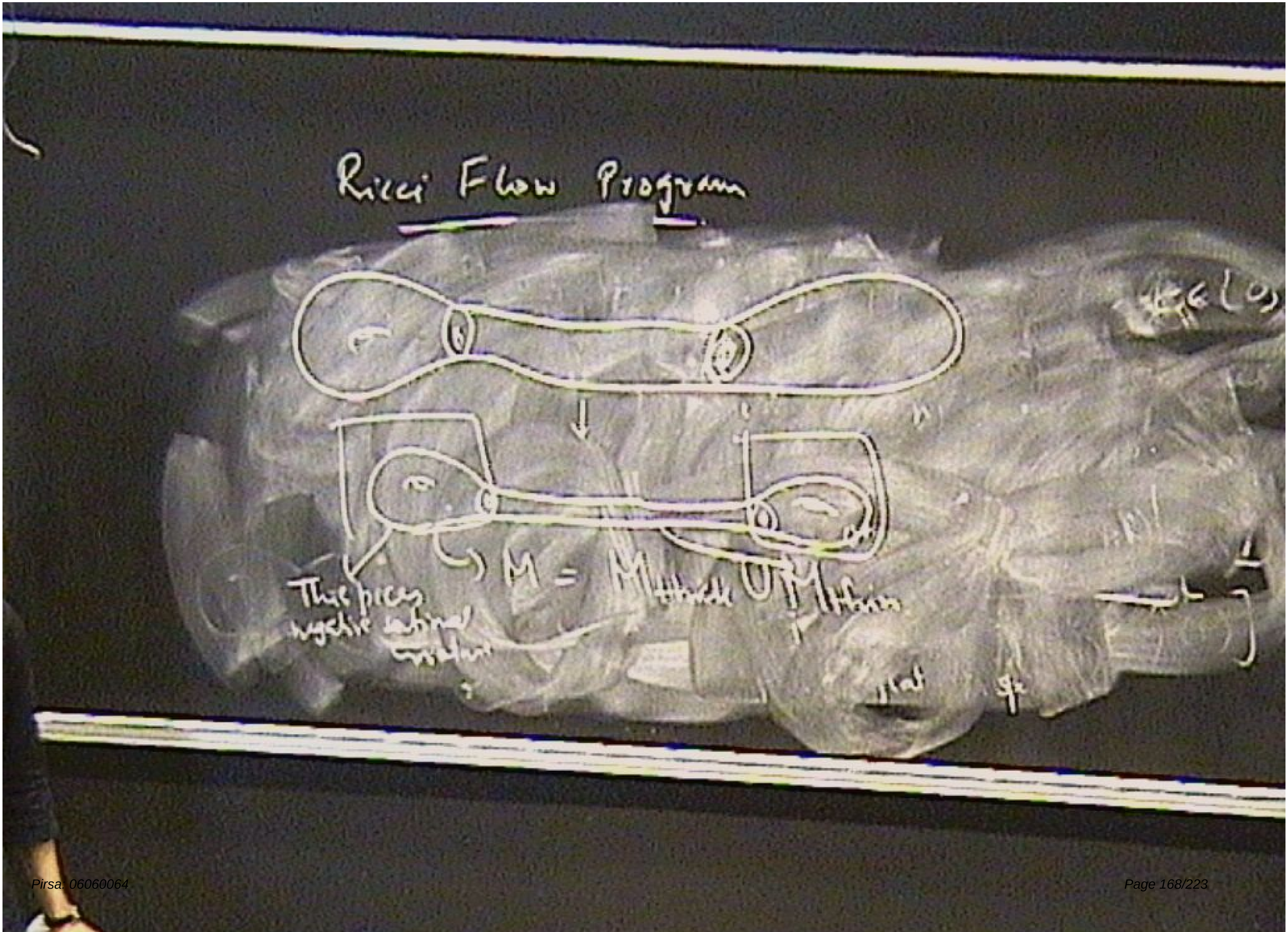

Ricci Flow Program

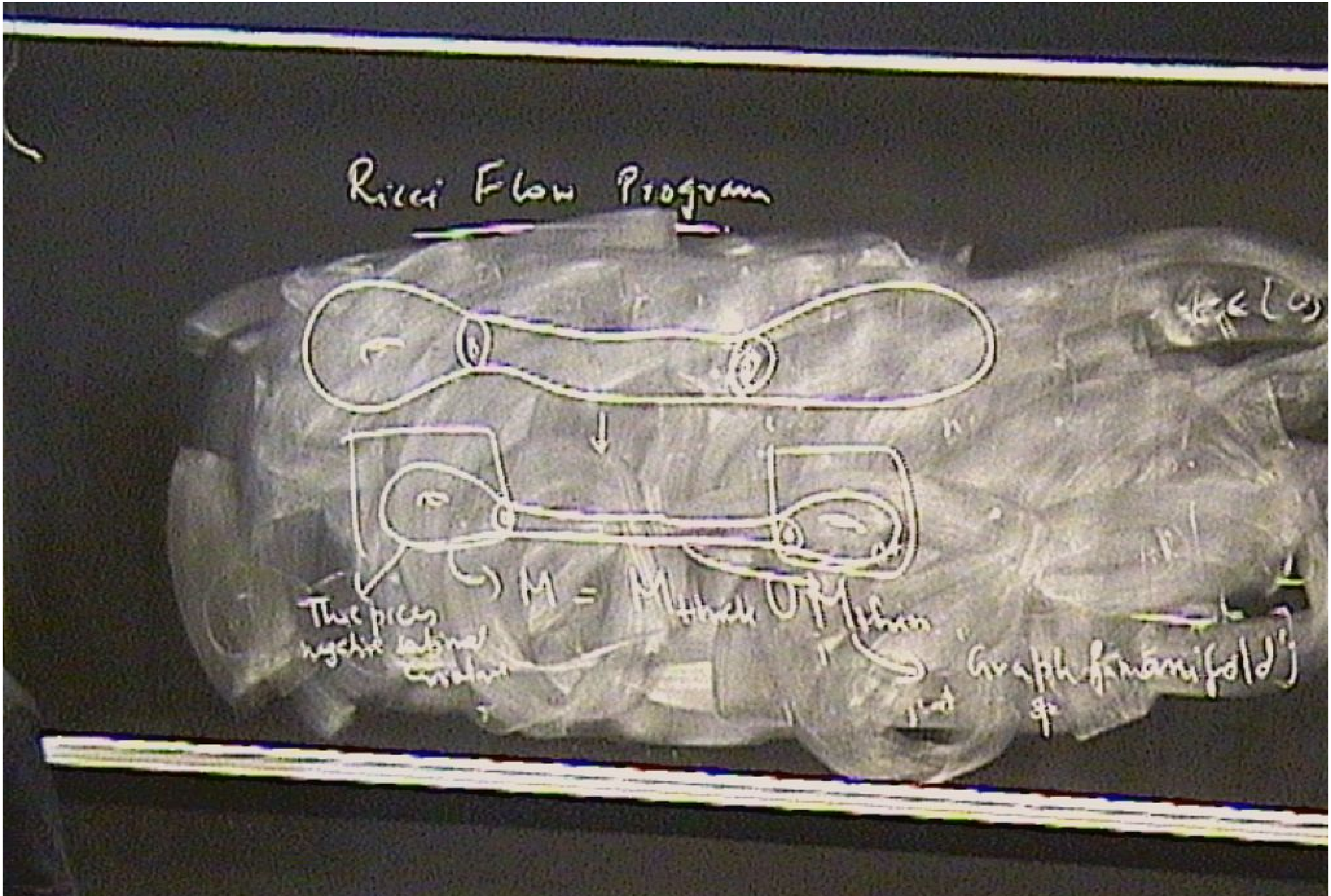

Ricci Flow Program

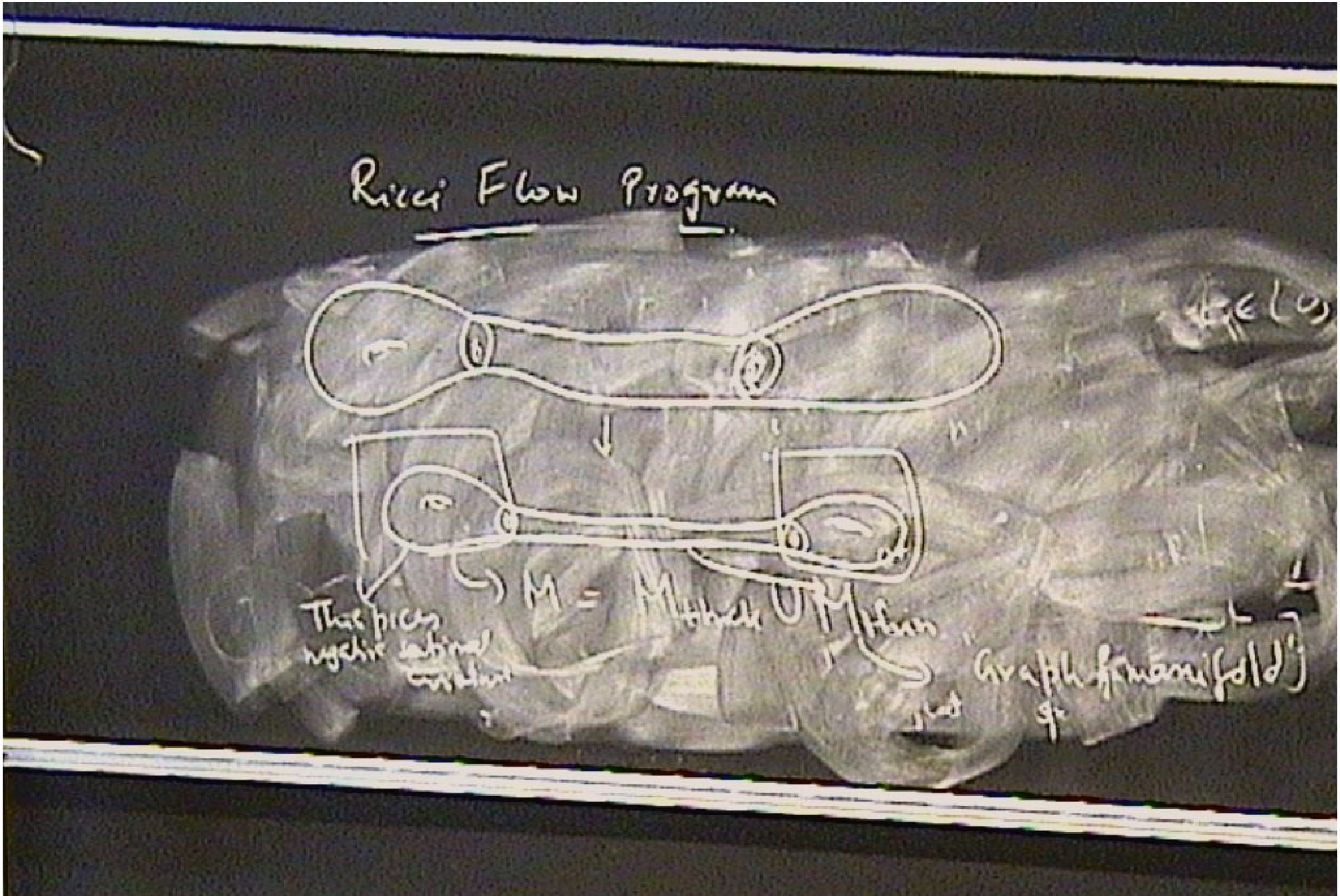

Ricci Flow Program
M = M_thick ∪ M_thin

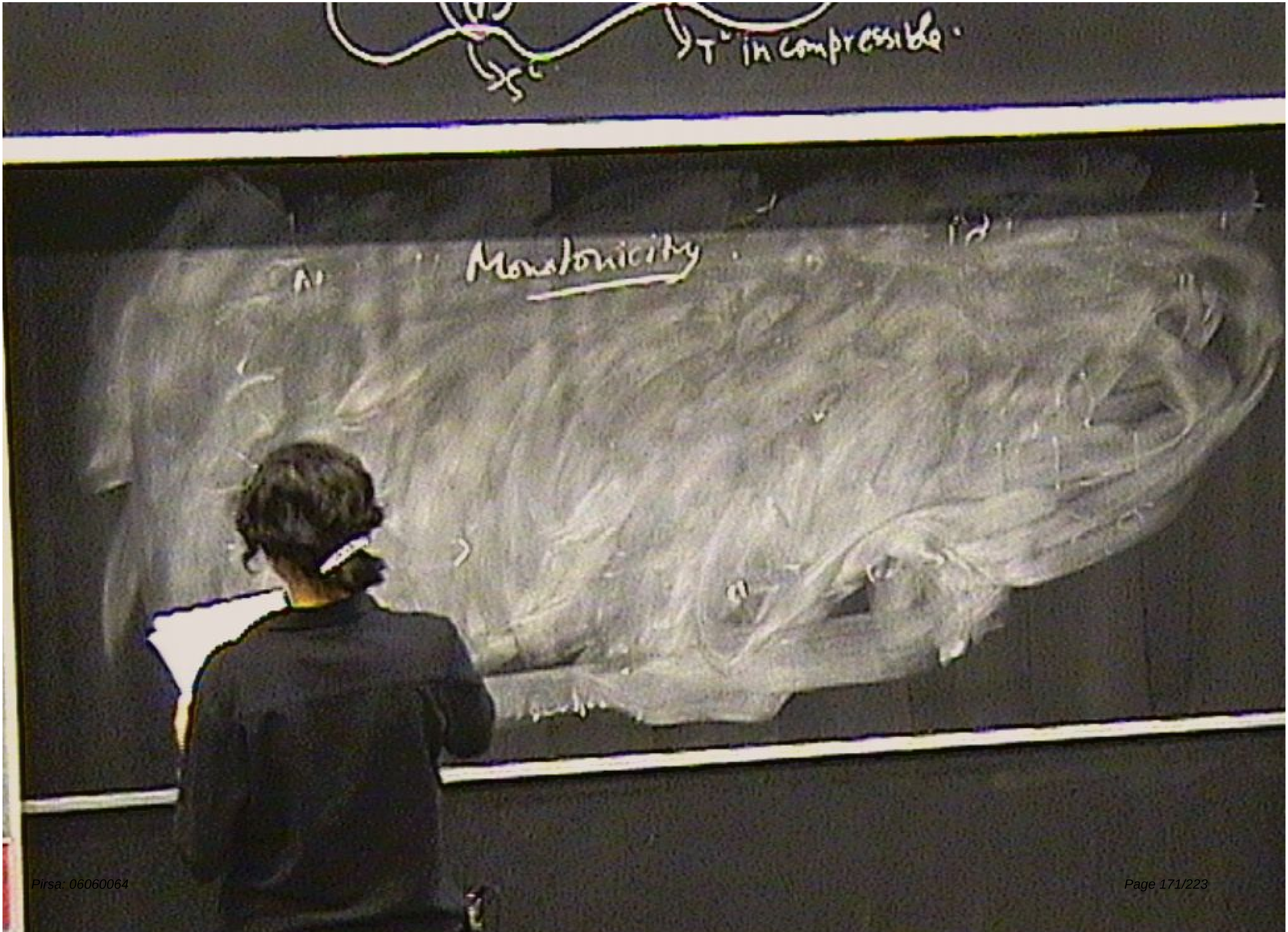

Ricci Flow Program
$M = M_{thick} \cup M_{thin}$


Ricci Flow Program
The pieces
M = M thick ∪ M thin


Ricci Flow Program
M = M thick ∪ M thin
This piece


Ricci Flow Program
M = M thick ∪ M thin
"Graph manifold"


Ricci Flow Program
M = Mthick ∪ Mthin
Graph manifold


"T" incompressible.
Monotonicity

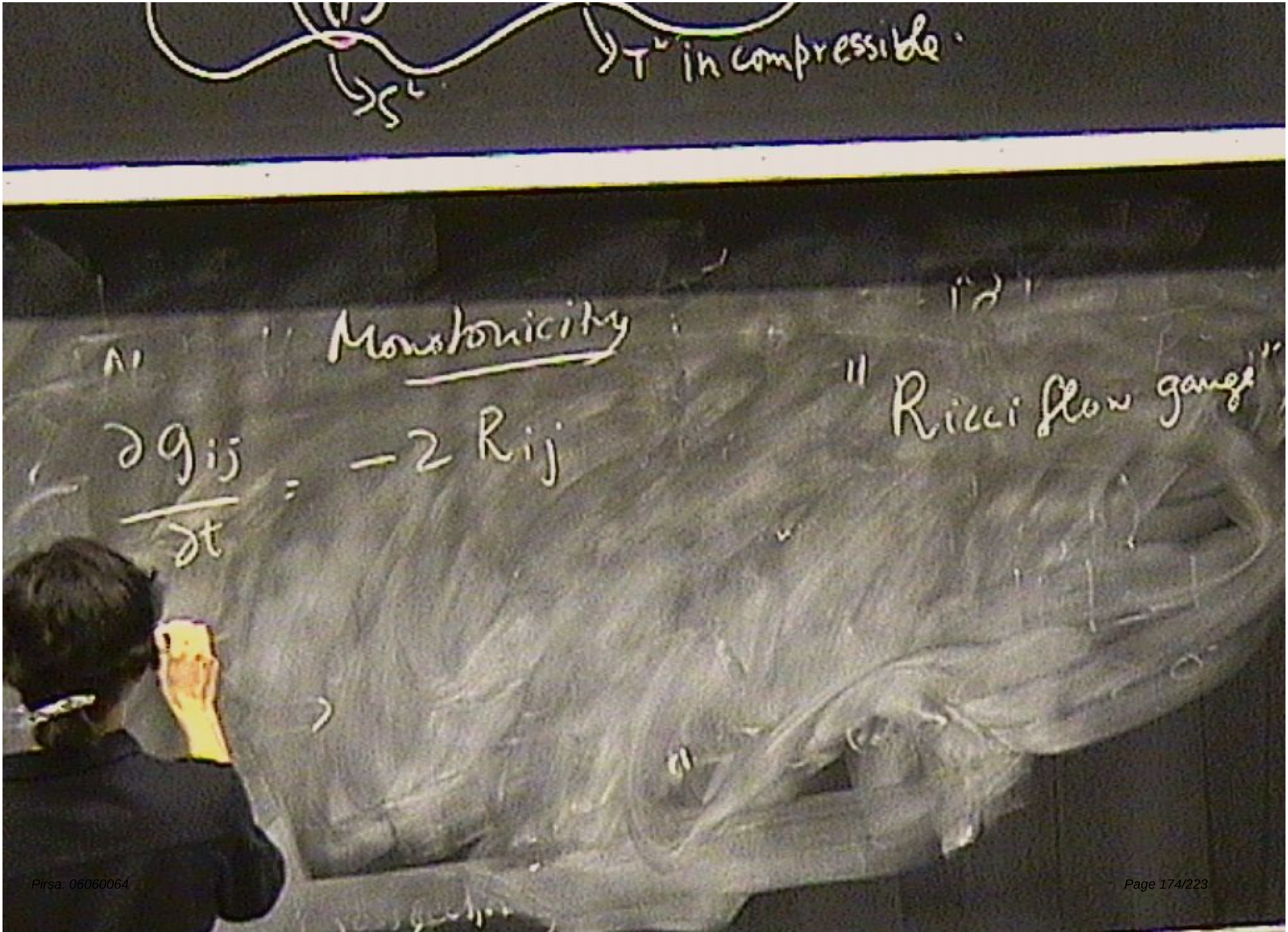
x^i T^* incompressible.

Monotonicity

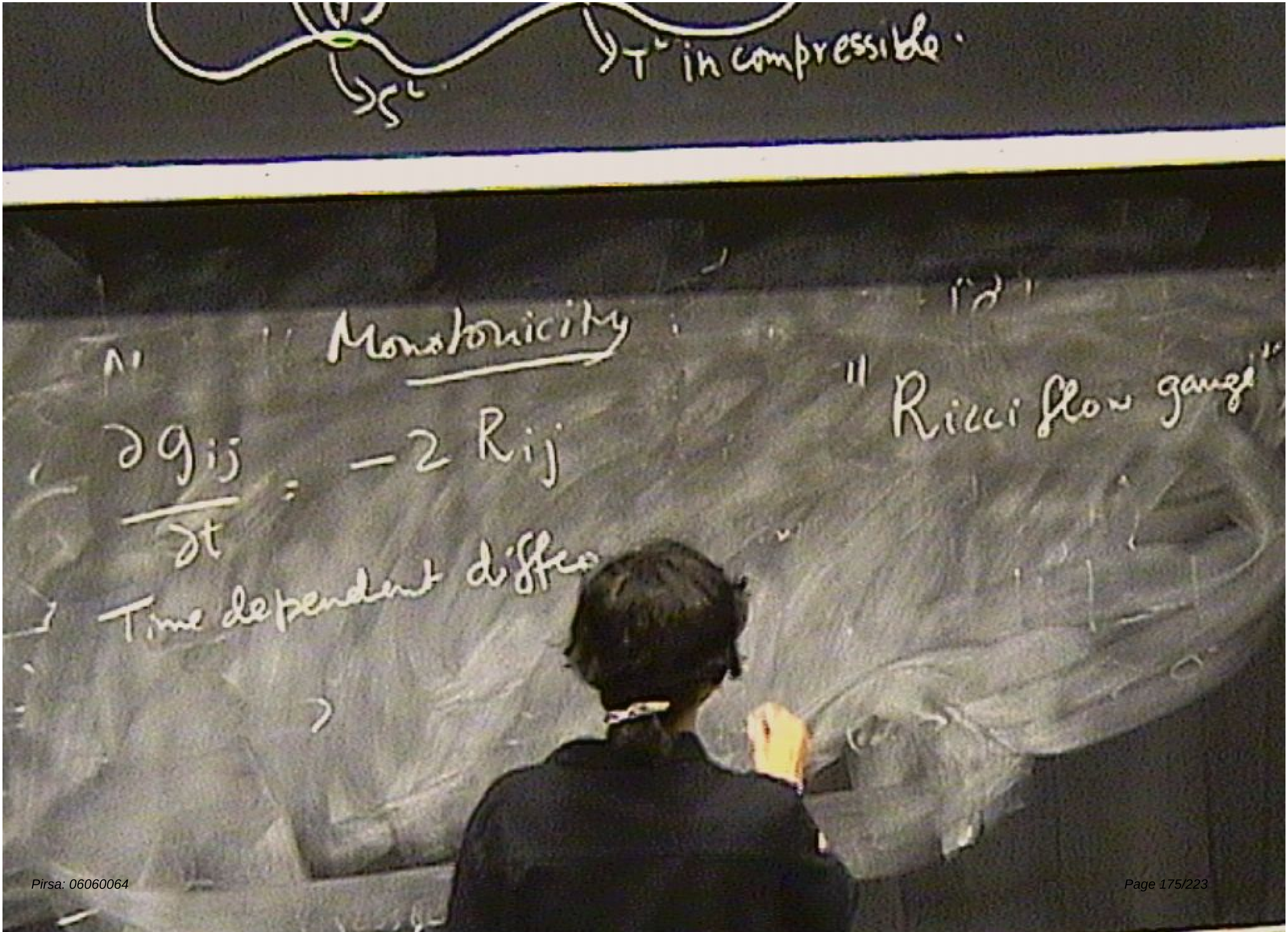
$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

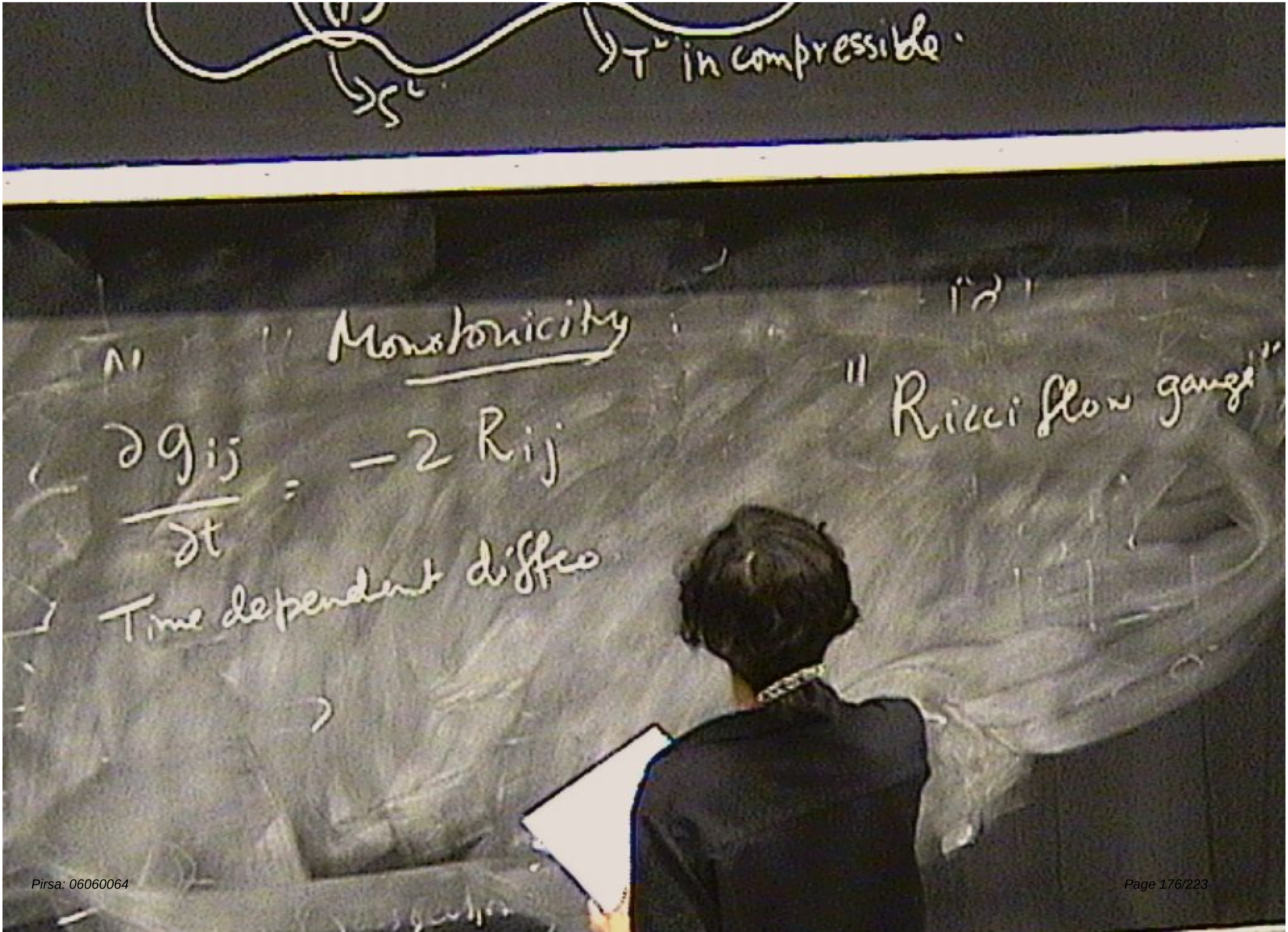
Monotonicity:

$$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$$



S²
T² incompressible.
Monotonicity
"Ricci flow gauge"
∂g_ij / ∂t = −2 R_ij


T^2 incompressible.
S^2
Monotonicity
"Ricci flow gauge"
$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$
Time dependent diffeo


T^2 incompressible.
S^2
Monotonicity
"Ricci flow gauge"
$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$
Time dependent diffeo

"Monotonicity"

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

"Ricci flow gauge"

Time dependent diffeo $\varphi_t^* g$

<u>Monotonicity</u>

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

"Ricci flow gauge"

Time dependent diffeo $\varphi_t^* g$

generator of diffeo. X^i

Monotonicity:

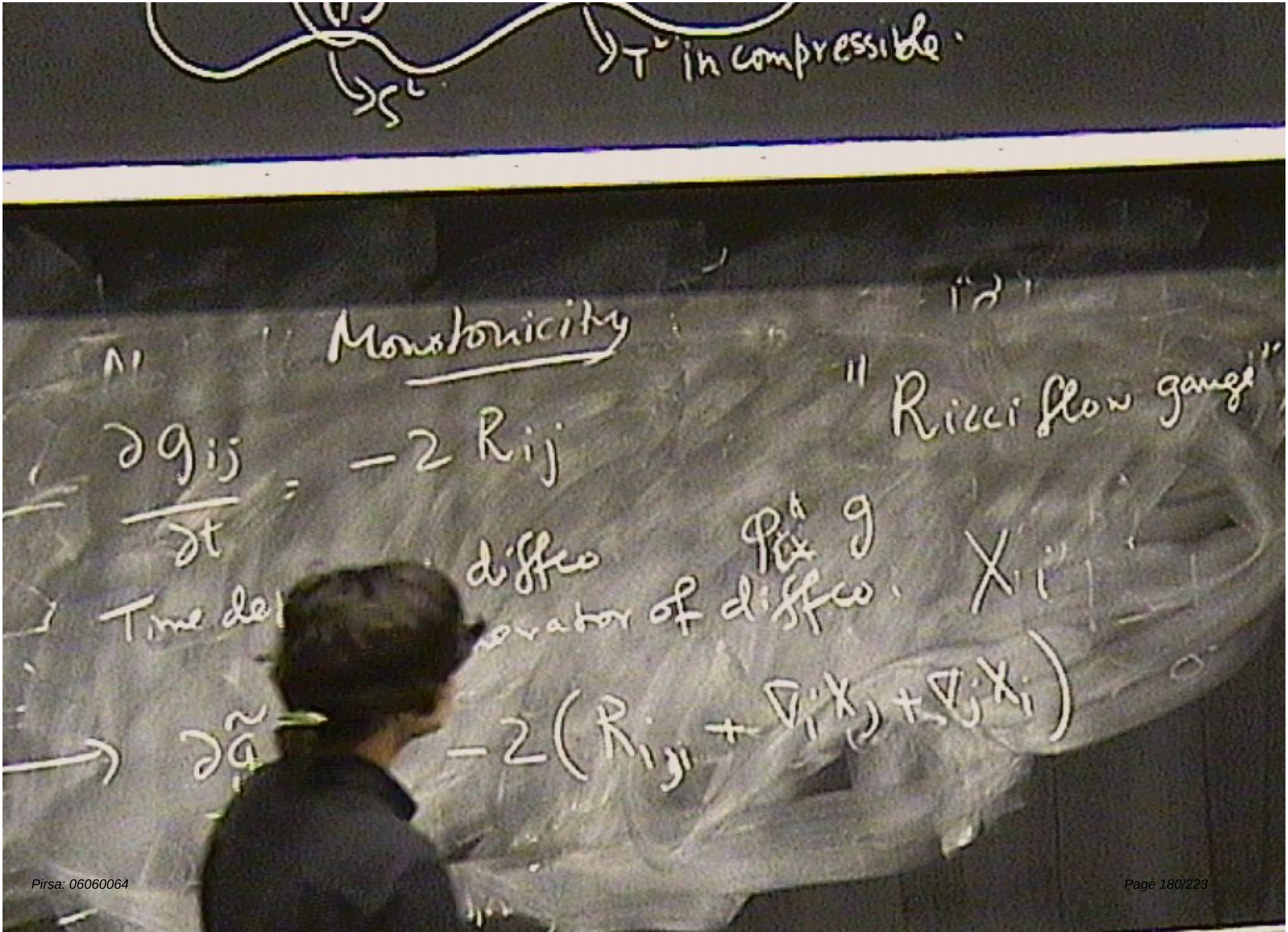
"Ricci flow gauge"

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Time dependent diffeo $\Phi_t^* g$

generator of diffeo: X_i

$$\rightarrow \frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$


T^2 incompressible.
S^2
Monotonicity
"Ricci flow gauge"
$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$
diffeo
Time de
nerator of diffeo.
X_i
$-2(R_{ij} + \nabla_i X_j + \nabla_j X_i)$

Monotonicity

"Ricci flow gauge"

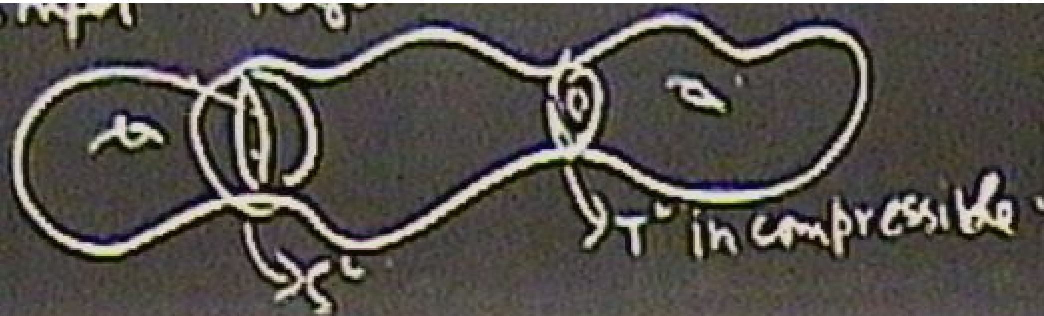
$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Time dependent diffeo $\varphi^*_{t} g$

Generator of diffeo. $X_i = -\nabla_i P$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

Take any simple ...



Monotonicity

"Ricci flow gauge"

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Time dependent diffeo $\phi_t^* g$

generator of diffeo. $X_i = -\nabla_i P$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j P\right)$$

Take any simpl...

Monotonicity :

"Ricci flow gauge"

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

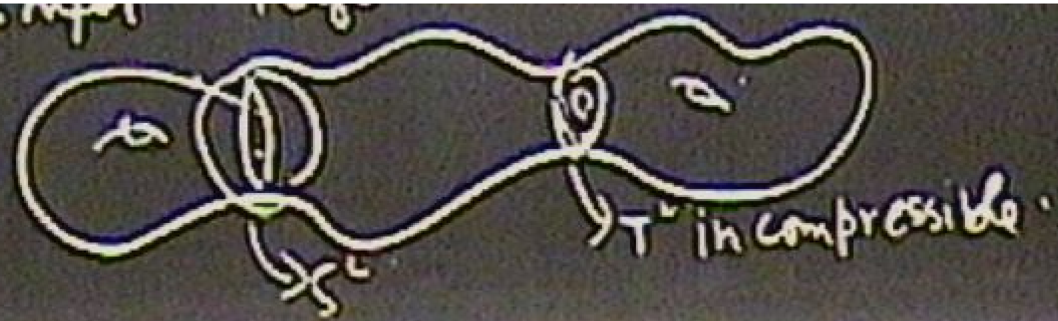
Time dependent diffeo $\varphi_t^* g$

generator of diffeo. $X_i = -\nabla_i P$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j P\right) \quad \text{"P gauge"}$$

Take any



Monotonicity

"Ricci flow gauge"

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

Time dependent diffeo $\Phi_t^* g$

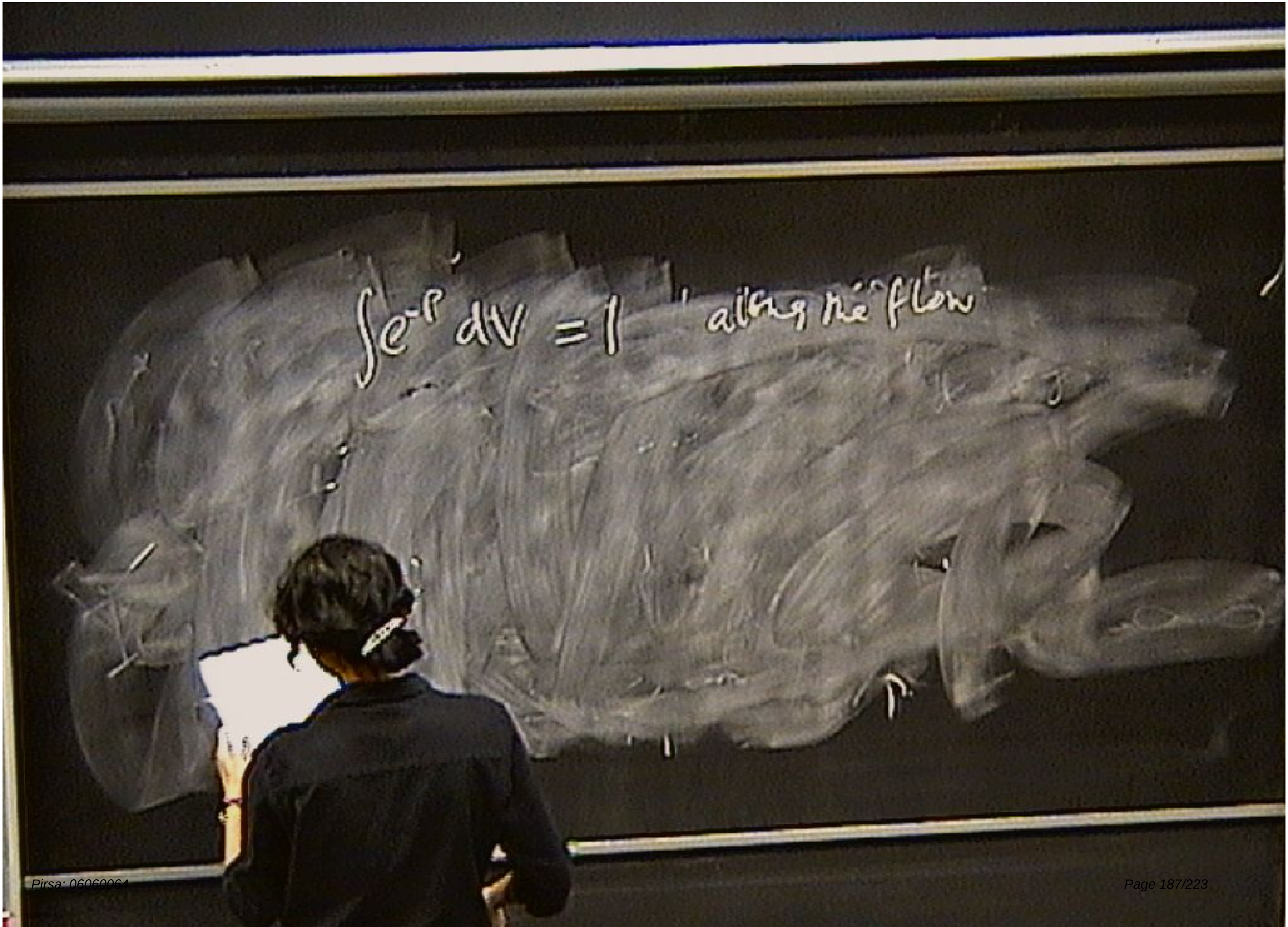
generator of diffeo. $X_i = -\nabla_i P$

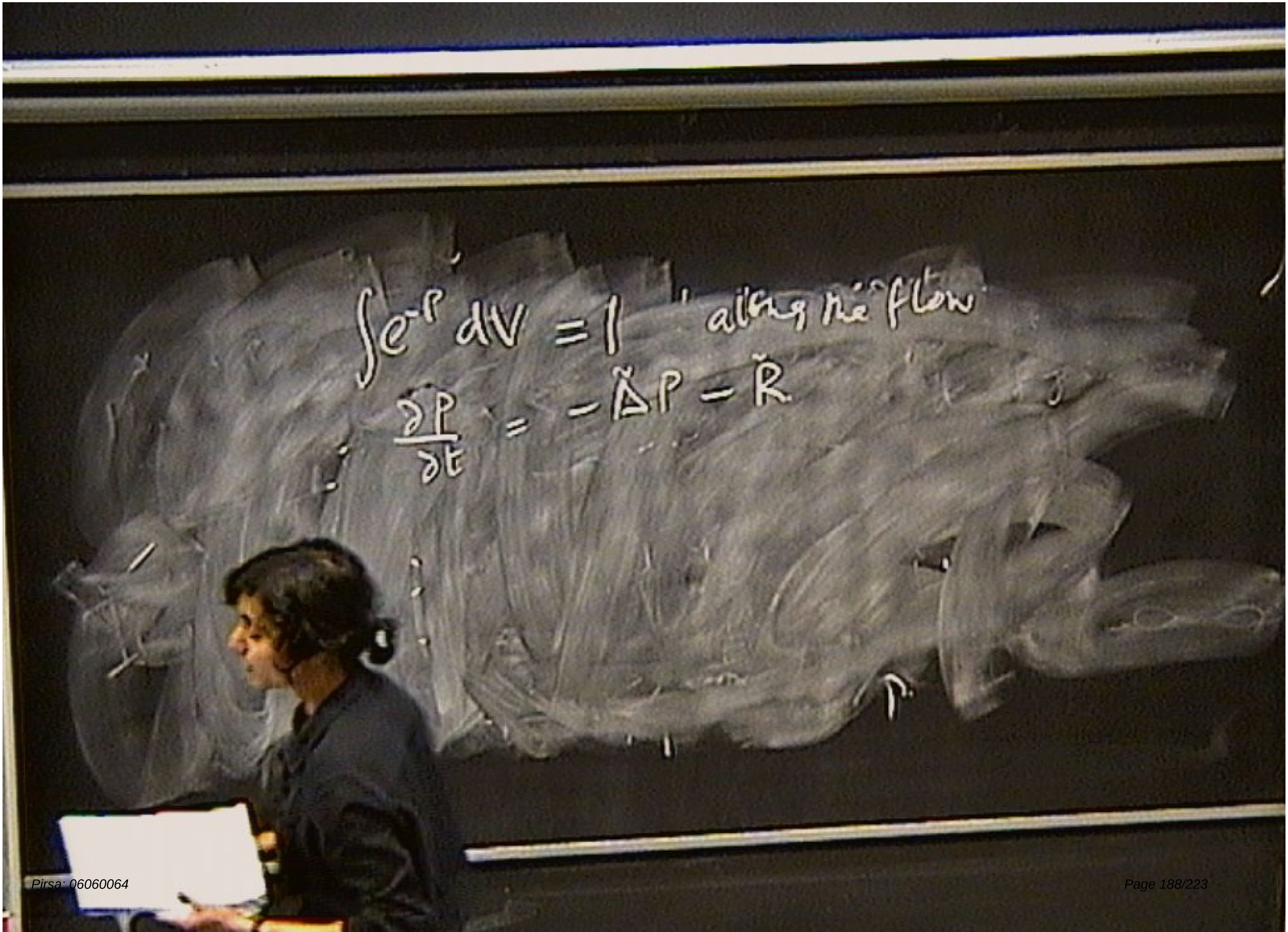
$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j P\right) \quad \text{"P gauge"}$$

$\int e^{-\rho}\, dV = 1$ along the flow

$$\int e^{-\rho}\, dV = 1$$ along the flow


$\int e^{-\rho}\, dV = 1$ along the flow


$\int e^{-\rho} dV = 1$ along the flow
$\frac{\partial \rho}{\partial t} = -\tilde{\Delta} \rho - \tilde{R}$

$$\int e^{-p}\, dV = 1 \quad \text{along the flow}$$

$$\frac{\partial p}{\partial t} = -\tilde{\Delta} p - \tilde{R}$$

Backward heat flow

$$\int e^{-f}\, dV = 1$$ along the flow

$$\frac{\partial f}{\partial t} = -\tilde{\Delta} f - \tilde{R}$$

Backward heat flow

Construct

$$\mathcal{F} = \int_M \left(R + |\nabla f|^2 \right) e^{-f}\, dV$$

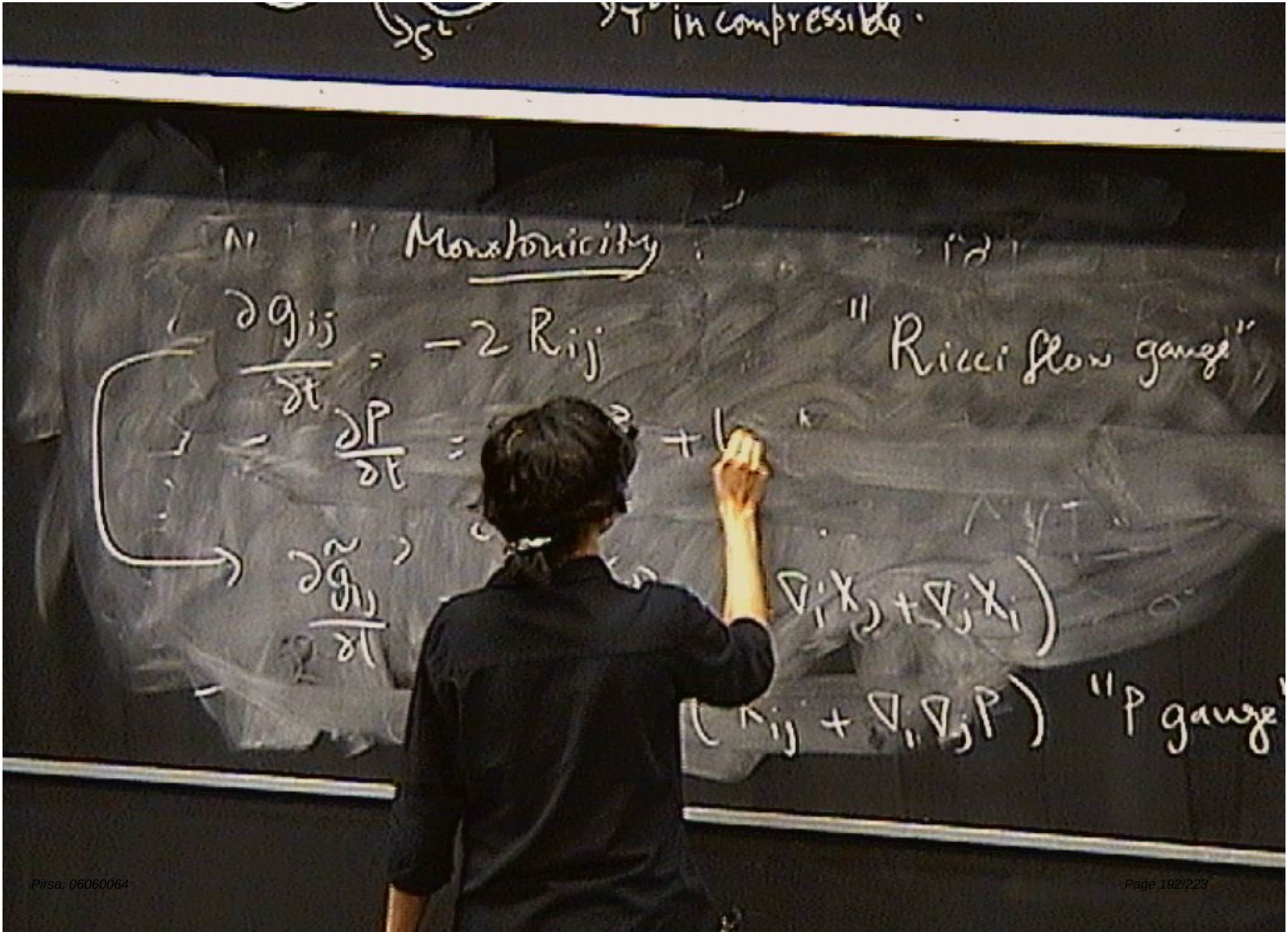
$$\int e^{-\rho}\, dV = 1 \quad \text{along the flow}$$

$$\frac{\partial \rho}{\partial t} = -\tilde{\Delta}\rho - \tilde{R}$$

Backward heat flow

Construct

$$\mathcal{F} = \int_M \left(R + |\nabla \rho|^2\right) e^{-\rho}\, dV = \mathcal{F}(t)$$


incompressible.
Monotonicity
$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$
"Ricci flow gauge"
$\nabla_i X_j + \nabla_j X_i$
$R_{ij} + \nabla_i \nabla_j f$
"f gauge"

T incompressible.

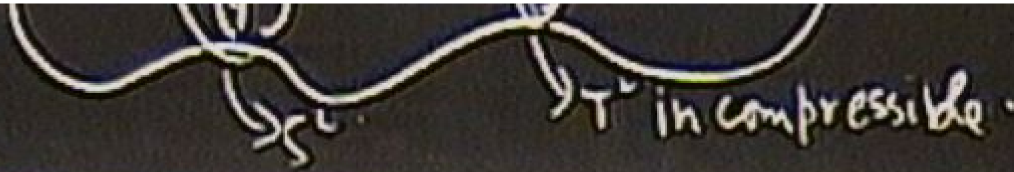
"Monotonicity":

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad \text{"Ricci flow gauge"}$$

$$\frac{\partial P}{\partial t} = -\Delta P + |\nabla P|^2 - R.$$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j P\right) \quad \text{"P gauge"}$$



Monotonicity

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \qquad \text{"Ricci flow gauge"}$$

$$-\frac{\partial p}{\partial t} = -\Delta p + |\nabla p|^2 - R.$$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j p\right) \qquad \text{"p gauge"}$$

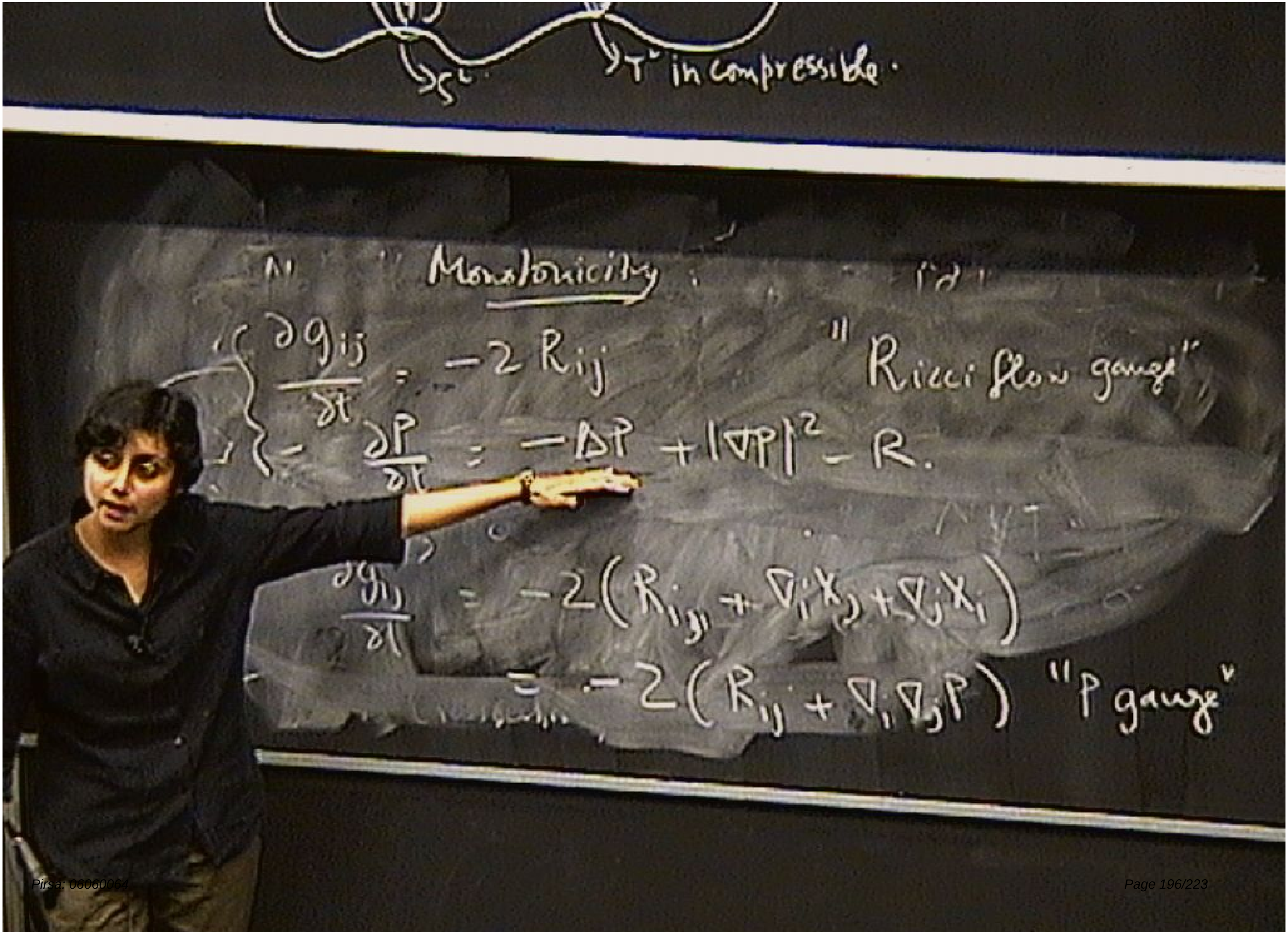
"Monotonicity":

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \qquad \text{"Ricci flow gauge"}$$

$$-\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R$$

$$\frac{\partial \tilde{g}_{ij}}{\partial t} = -2\left(R_{ij} + \nabla_i X_j + \nabla_j X_i\right)$$

$$= -2\left(R_{ij} + \nabla_i \nabla_j f\right) \qquad \text{"f gauge"}$$


T² incompressible.
Monotonicity
∂g_ij/∂t = −2 R_ij
"Ricci flow gauge"
−∂f/∂t = −Δf + |∇f|² − R.
∂g_ij/∂t = −2(R_ij + ∇_i X_j + ∇_j X_i)
= −2(R_ij + ∇_i ∇_j f)
"f gauge"

$$\int e^{-P}\, dV = 1 \quad \text{along the flow}$$

$$\frac{\partial P}{\partial t} = -\tilde{\Delta} P - \tilde{R}$$

Backward heat flow

Construct

$$\mathcal{F} = \int_M \left(R + |\nabla P|^2\right) e^{-P}\, dV = \mathcal{F}(t)$$

$$\int e^{-p}\, dV = 1 \quad \text{along the flow}$$

$$\frac{\partial p}{\partial t} = -\tilde{\Delta} p - \tilde{R}$$

Backward heat flow

Construct

$$\int_M (R + |\nabla p|^2)\, e^{-p}\, dV = \mathcal{F}(t)$$

$$\mathcal{F}' = \int_M \cdots$$

$q(t)$ $p(t)$ $t=T$ $\mathcal{F}(t)$ $t=0$

$$\int e^{-P}\, dV = 1 \quad \text{along the flow}$$

$$\frac{\partial P}{\partial t} = -\tilde{\Delta} P - \tilde{R}$$

Backward heat flow

Construct $\displaystyle \int_M \left(R + |\nabla P|^2 \right) e^{-P}\, dV = \mathcal{F}(t)$

$$\frac{d\mathcal{F}}{dt} \geq 0$$

$$g(t) = q_k^{*}\langle g(0)\rangle_{11}$$

$$g(1) = \varphi_t^* \langle g(0) \rangle_{11}$$

↪ would be ruled out if $\mathcal{F}$ depended only on geometry

$$g(t) = \varphi_t^* \, \tilde{g}(0)|_1$$

↳ would be ruled out if $\mathcal{F}$ depended only on geometry

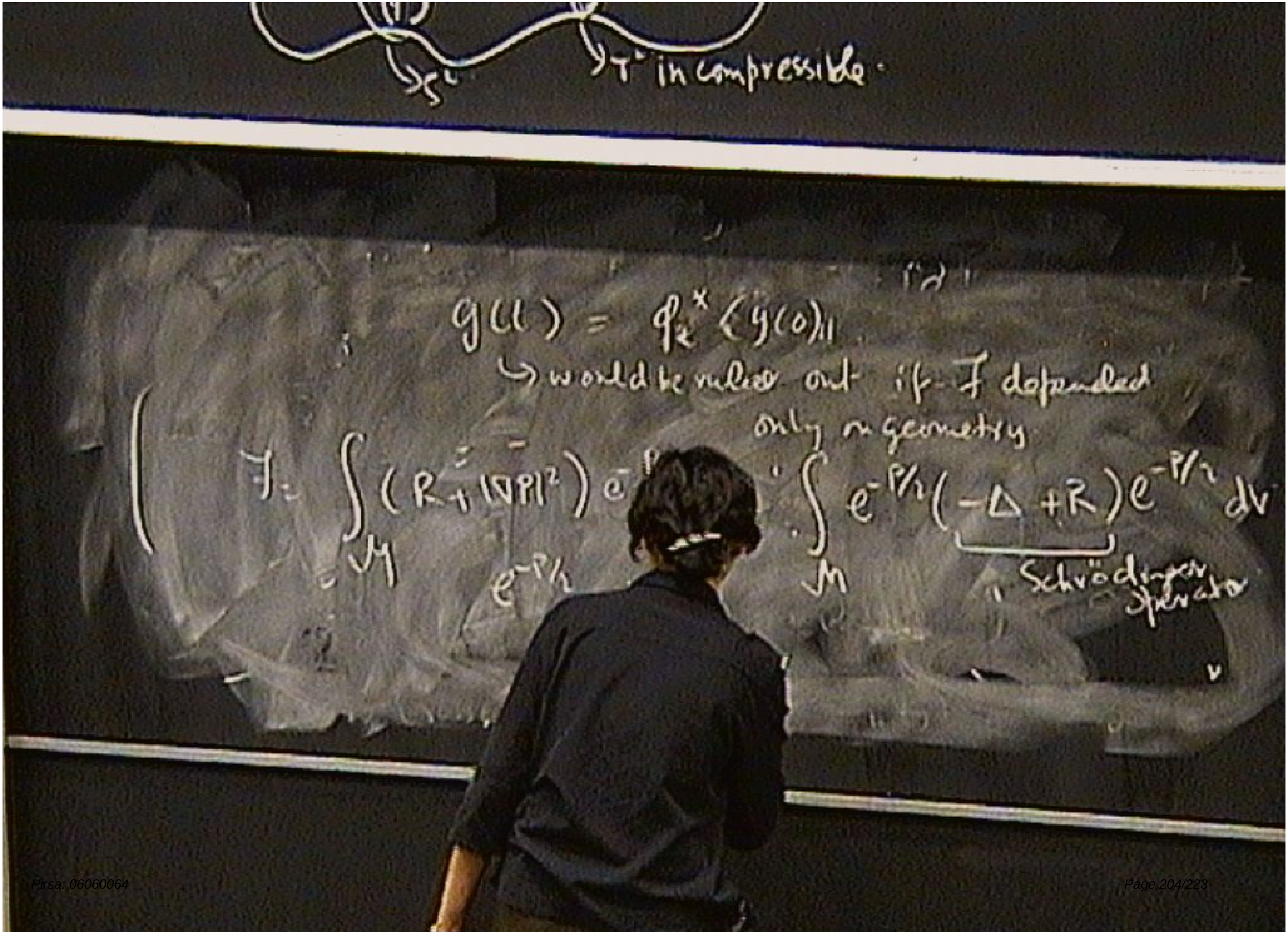
$$\mathcal{F} = \int_M (R + |\nabla f|^2)\, e^{-f} \, dV = \int_M e^{-f/2}(-\Delta + R)\, e^{-f/2} \, dV$$

T^2 incompressible.

$$g(t) = \phi_t^* \lambda(t)\, g(0)_{ij}$$

↳ would be ruled out if $\mathcal{F}$ depended only on geometry

$$\mathcal{F} = \int_M (R + |\nabla f|^2)\, e^{-f}\, dV = \int_M e^{-f/2} \underbrace{(-\Delta + R)}_{\text{Schrödinger operator}} e^{-f/2}\, dV$$


S^2
T^2 incompressible
$g(t) = \varphi_t^* (g(0))$
would be ruled out if $\mathcal{F}$ depended only on geometry
$\mathcal{F} = \int_M (R + |\nabla f|^2) e^{-f} \, dV = \int_M e^{-f/2} \underbrace{(-\Delta + R)}_{\text{Schrödinger operator}} e^{-f/2} \, dV$

$$g(t) = \varphi_t^* \{g(0)\}_{11}$$

↪ would be ruled out if $\mathcal{F}$ depended only on geometry

$$\mathcal{F} = \int_M (R + |\nabla f|^2)\, e^{-f}\, dV = \int_M e^{-f/2} \big(\underbrace{-\Delta + R}_{\text{Schrödinger operator}}\big) e^{-f/2}\, dV$$

$$e^{-f/2} = u$$

$$\int_M u^2\, dV = 1$$

$\hookrightarrow \Sigma^2$ $\hookrightarrow T^2$ incompressible.

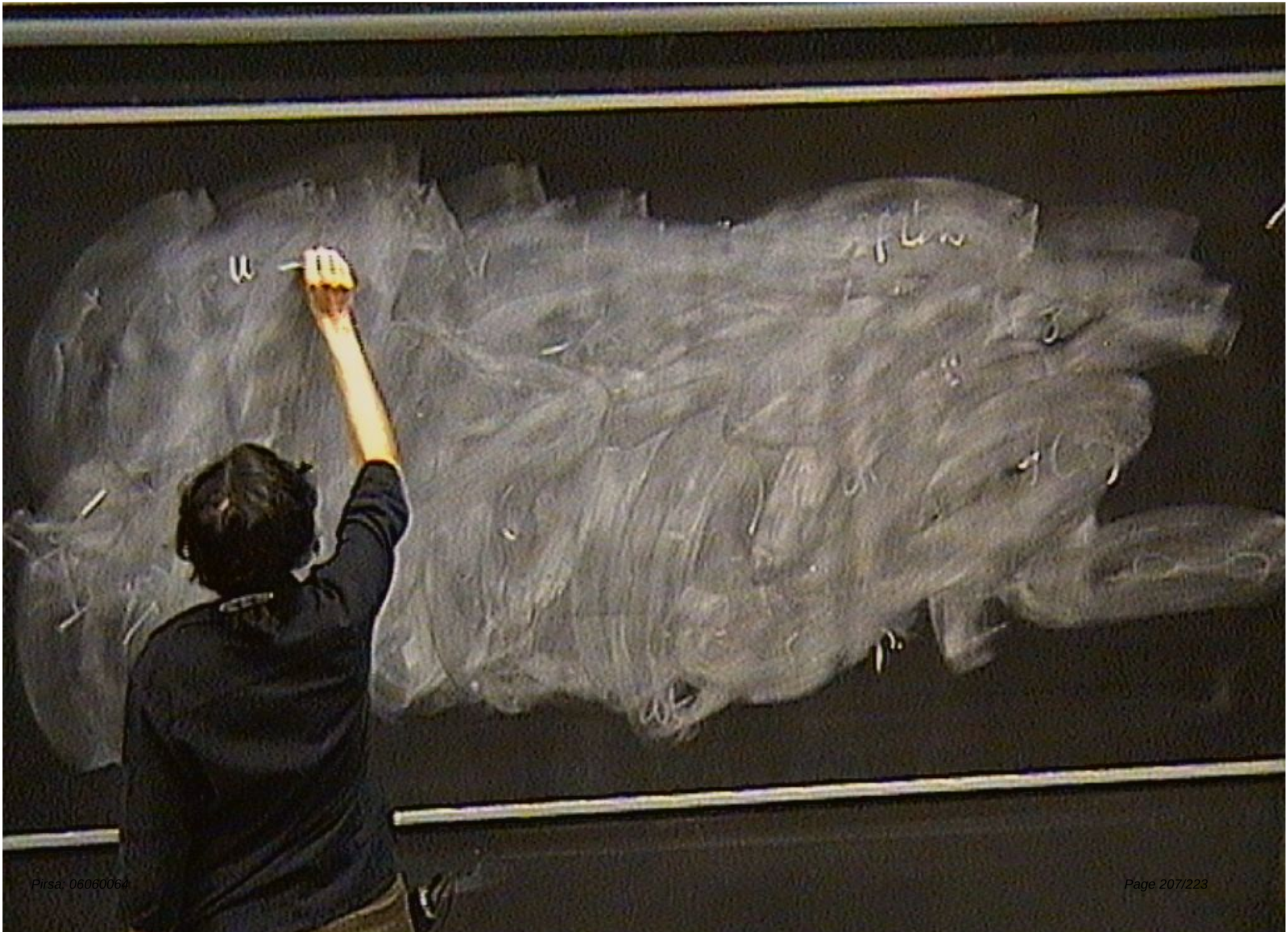
$$g(t) = \varphi_t^*\,(g(0))$$

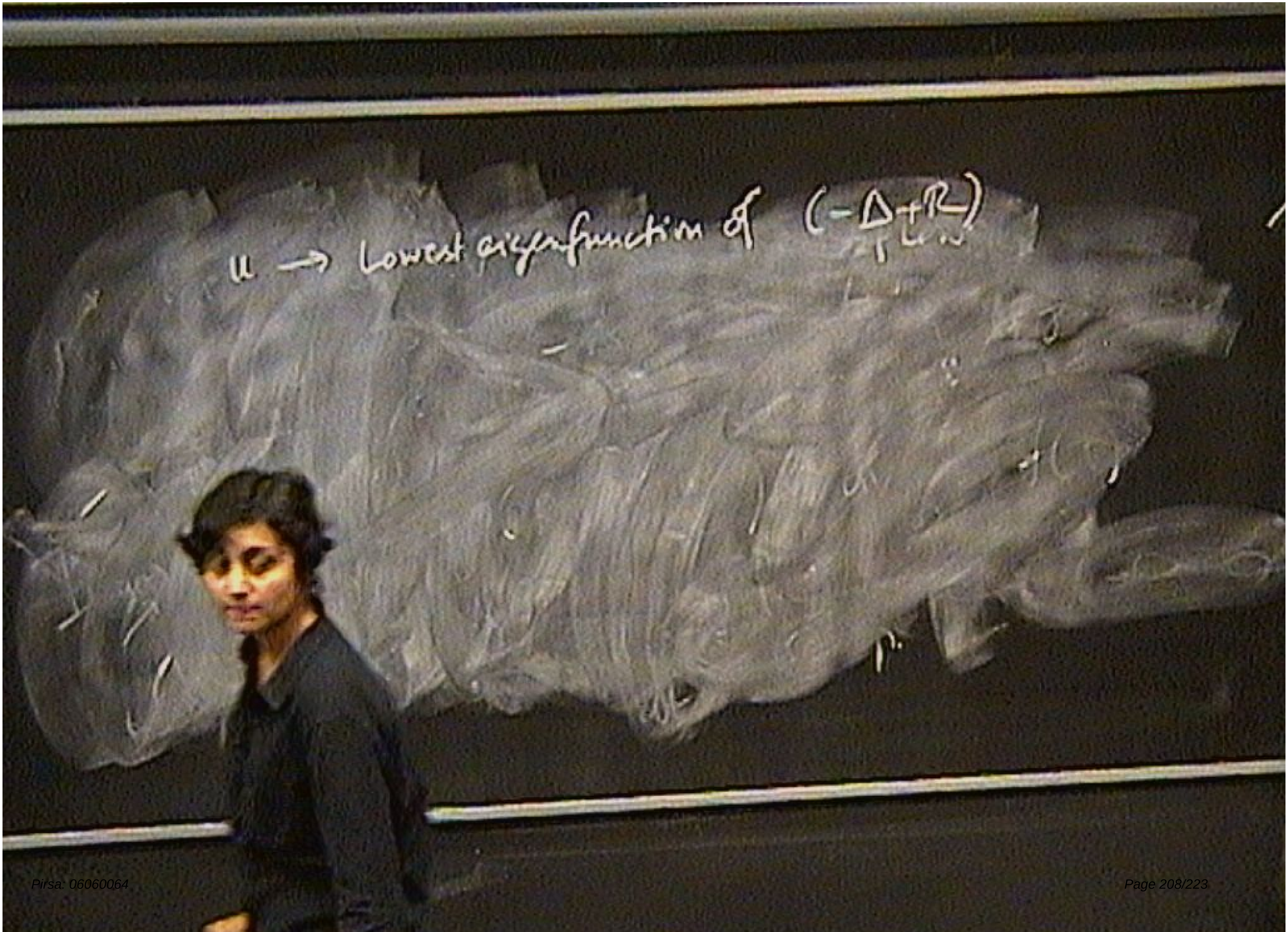
$\hookrightarrow$ would be ruled out if $\mathcal{F}$ depended only on geometry

$$\mathcal{F} = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV = \int_M e^{-f/2} \underbrace{\left(-\Delta + R\right)}_{\text{Schrödinger operator}} e^{-f/2}\, dV$$

$$e^{-f/2} = u$$

$$u \in C^2 \qquad \int_M u^2\, dV = 1$$




u → Lowest eigenfunction of (−Δ + R)

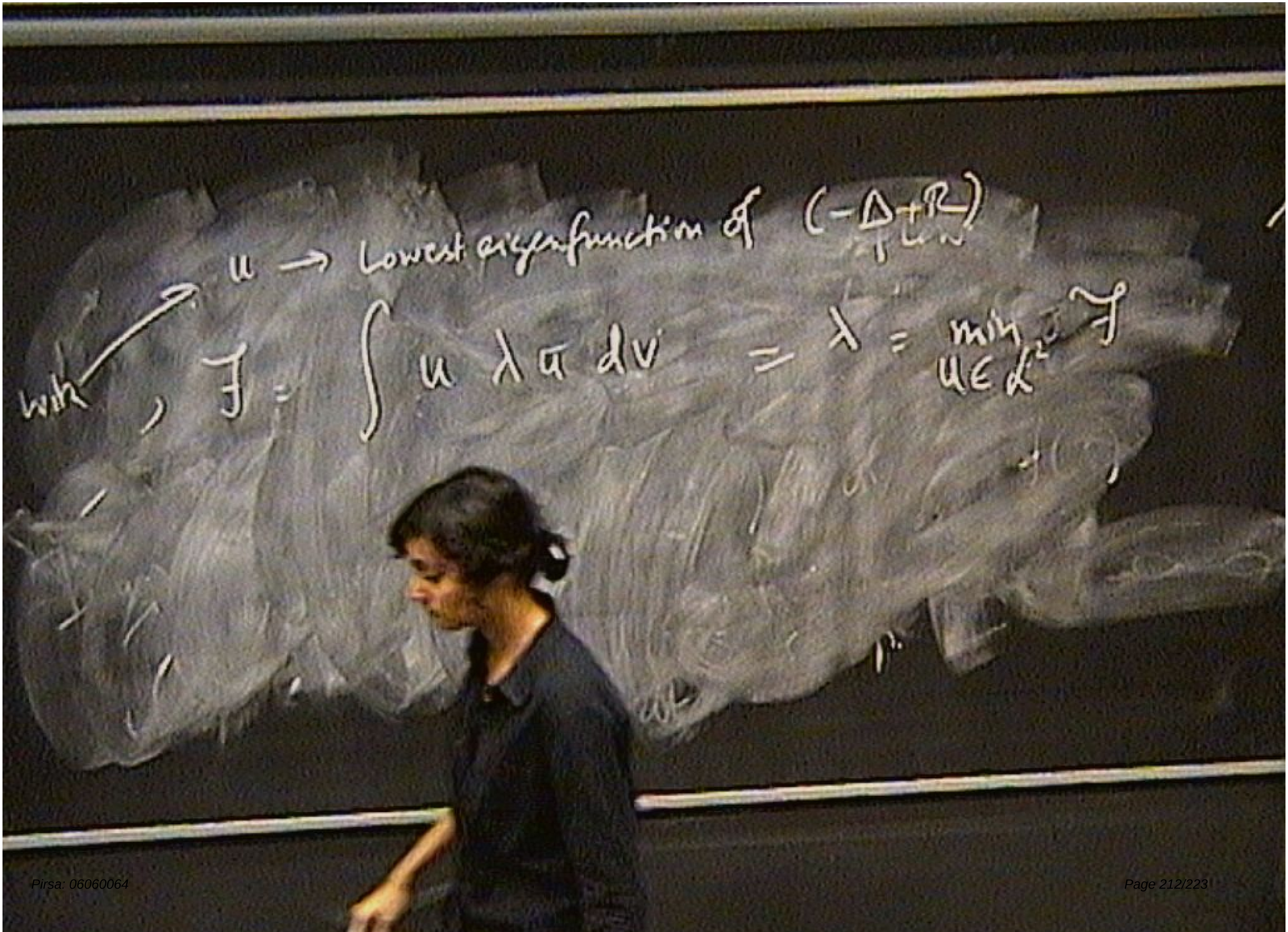
$u \rightarrow$ Lowest eigenfunction of $(-\Delta + R)$

$u \to$ Lowest eigenfunction of $\left(-\Delta + \frac{R}{4}\right)$

$$\mathcal{F} = \int u \, \lambda \, \bar{u} \, dV = \lambda$$

with $u \to$ Lowest eigenfunction of $(-\Delta + R)$

$\mathcal{F} = \int u \lambda \bar{u} \, dV = \lambda$


u → Lowest eigenfunction of (−Δ+R)
with , F = ∫ u λ u dV = λ = min F
u ∈ L²

$u \rightarrow$ Lowest eigenfunction of $(-\Delta + R)$

with $\mathcal{F} = \int u \, \lambda \, \bar{u} \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

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with $\mathcal{F} = \int u \lambda \bar{u} \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T$ $\mathcal{F}(t = T) = \lambda$

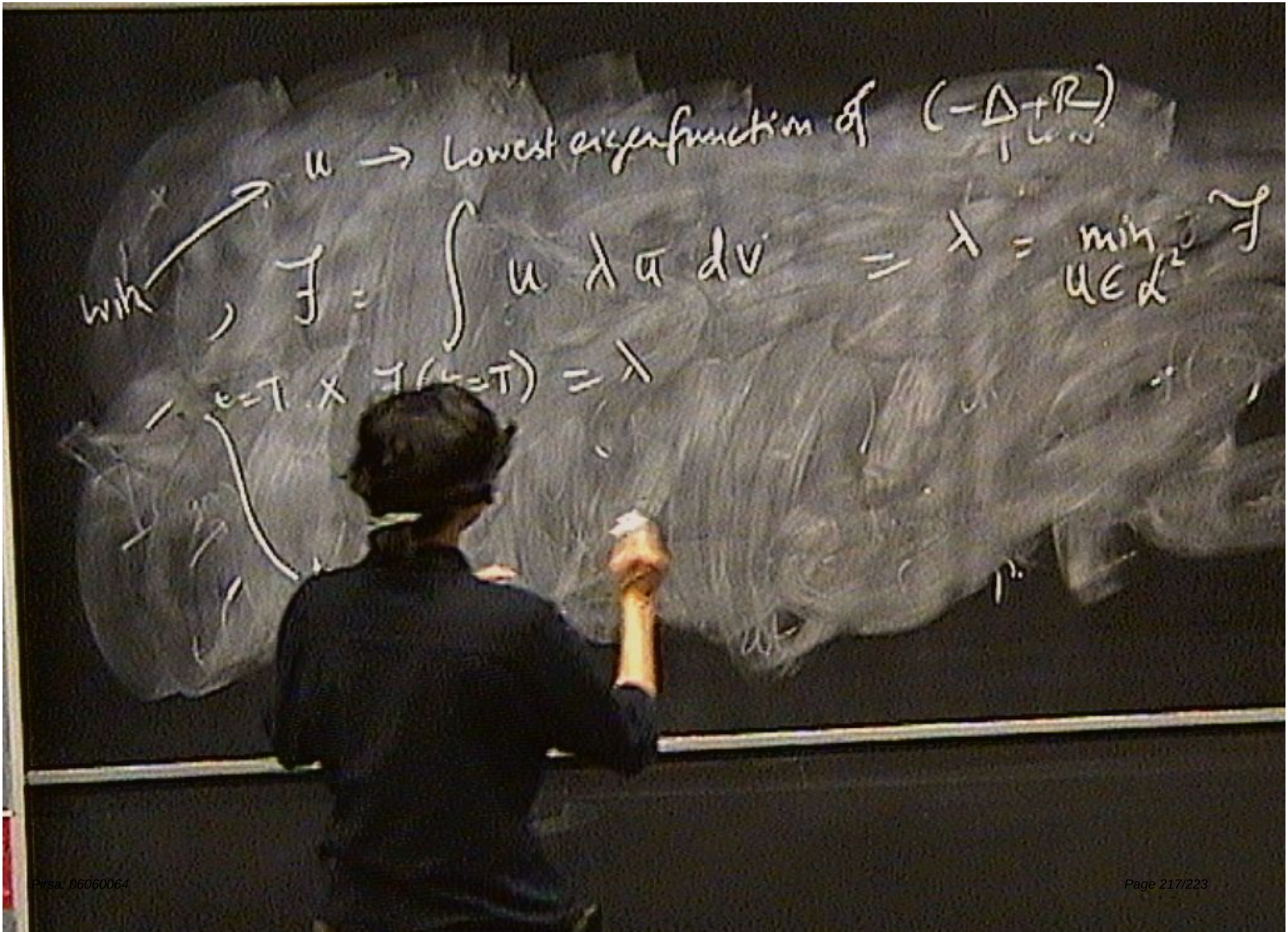
$t = 0$

$u \to$ Lowest eigenfunction of $(-\Delta + R)$

with $\mathcal{F} = \int u \lambda \bar{u} \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T \quad \mathcal{F}(t=T) = \lambda$

$t = 0$


u → Lowest eigenfunction of (−Δ + R)
∫ u λ u dV = λ = min ∫
u ∈ L²
= λ

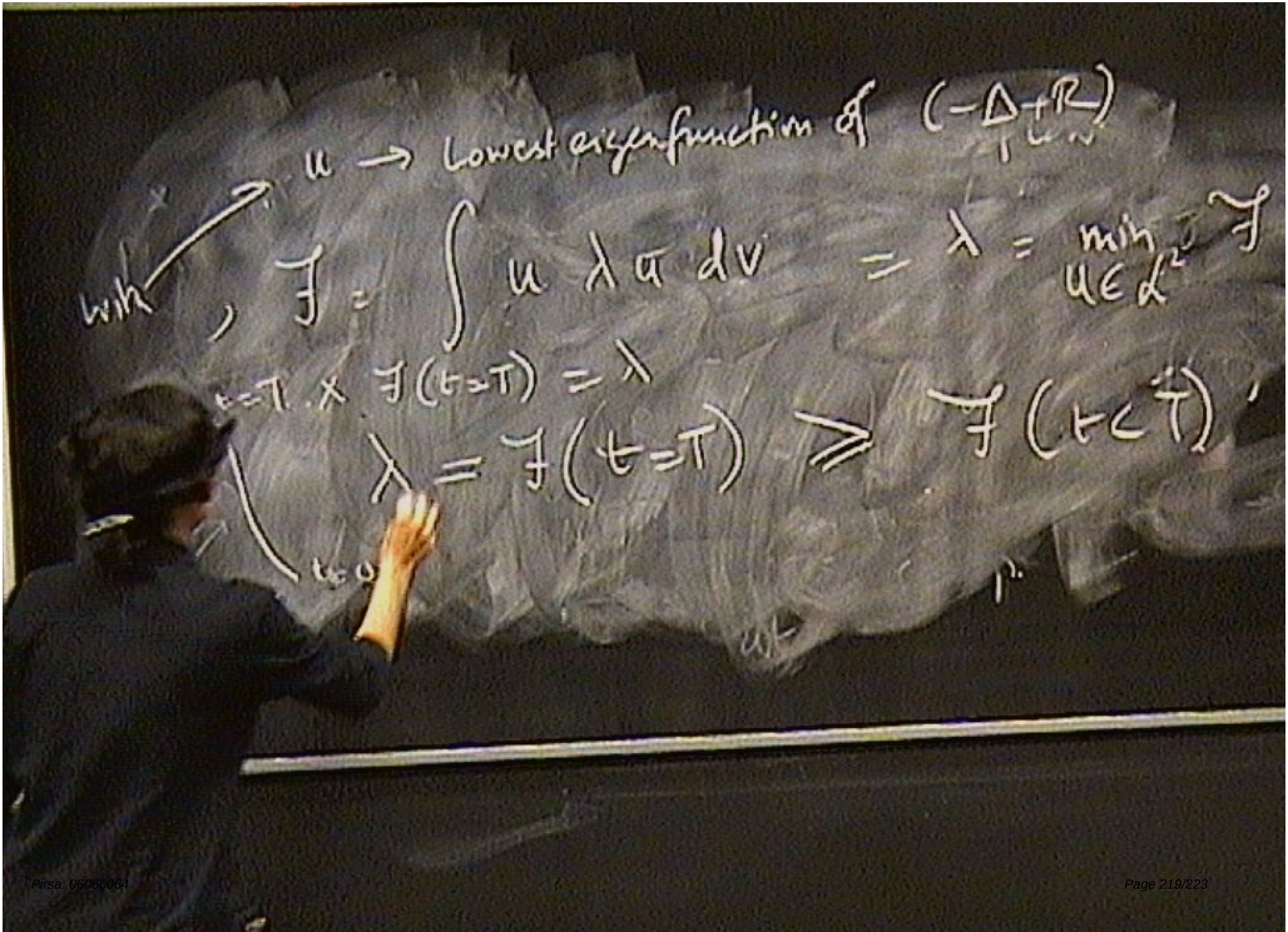
$u \to$ Lowest eigenfunction of $(-\Delta + R)$

with, $\mathcal{F} = \int u \, \lambda \, u \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T$. $\mathcal{F}(t=T) = \lambda$

$\mathcal{F}(t=T) \geq \mathcal{F}(t<T)$

$t=0$


u → Lowest eigenfunction of (−Δ+R)
with
F = ∫ u λ u dV = λ = min F
u∈L²
t=T × F(t=T) = λ
λ = F(t=T) ≥ F(t<T)
t=0

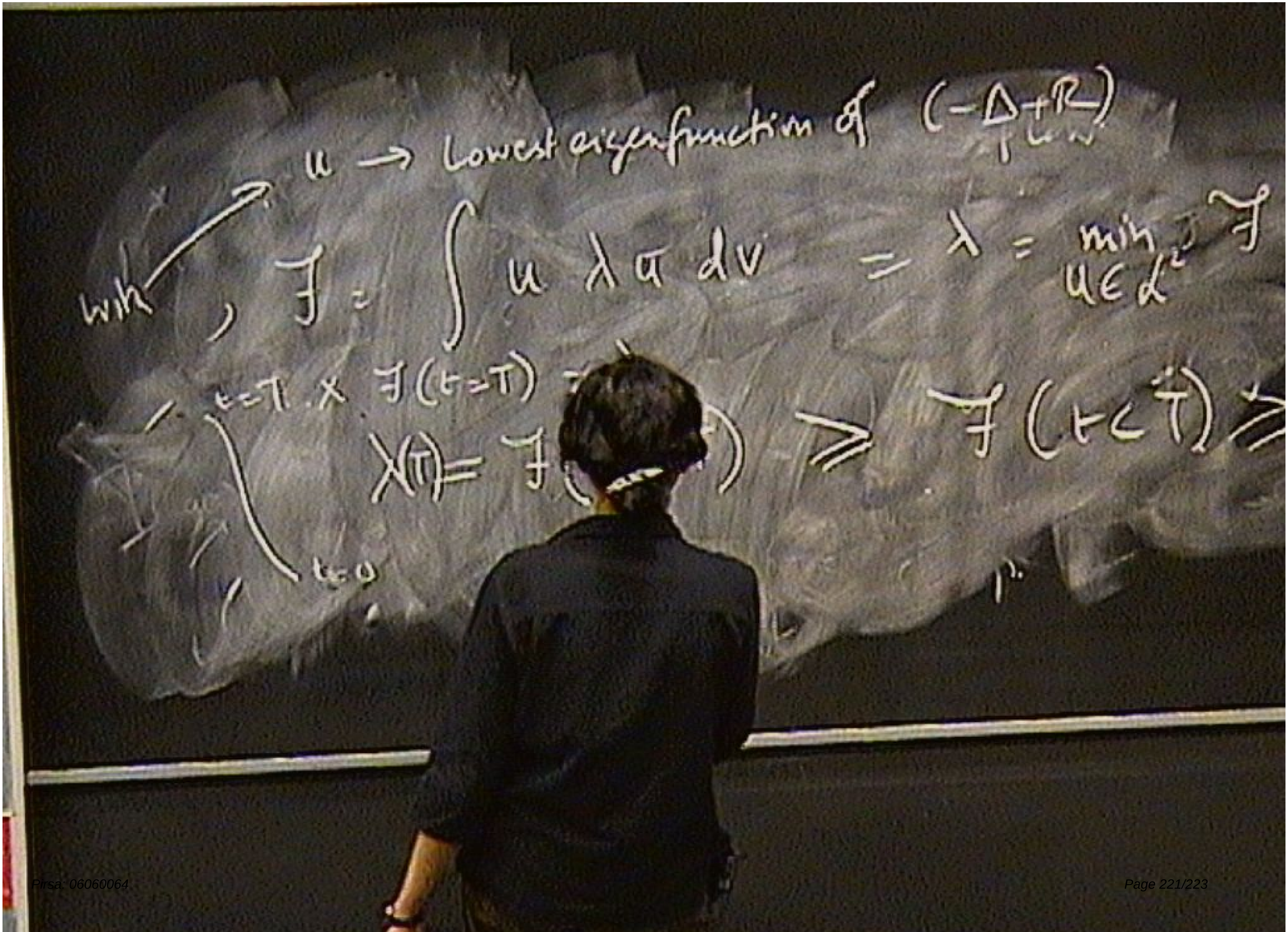
$u \to$ Lowest eigenfunction of $(-\Delta + R)$

with u, $\mathcal{F} = \int u \, \lambda \, u \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T$ $\mathcal{F}(t=T) = \lambda$

$\lambda(T) = \mathcal{F}(t=T) \geq \mathcal{F}(t<T)$

$t=0$


u → Lowest eigenfunction of (−Δ + R)
with
F = ∫ u λ u dV = λ = min F
u ∈ L²
t = T
λ(T) = F(
≥ F (t < T) ≥
t = 0

$u \rightarrow$ Lowest eigenfunction of $(-\Delta + R)$

with

$, \mathcal{F} = \int u \, \lambda \, u \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T, x \quad \mathcal{F}(t=T) = \lambda$

$\lambda(T) = \mathcal{F}(t=T) \geq \mathcal{F}(t<T) \geq \lambda(t)$

$\lambda(T) \geq \lambda(t)$

$t=0$

$u \rightarrow$ Lowest eigenfunction of $(-\Delta + R)$

with $\mathcal{F} = \int u \, \lambda \, \bar{u} \, dV = \lambda = \min_{u \in L^2} \mathcal{F}$

$t = T \quad \mathcal{F}(t = T) = \lambda$

$\lambda(T) = \mathcal{F}(t = T) \geq \mathcal{F}(t < T) \geq \lambda(t)$

$\lambda(T) \geq \lambda(t)$

λ is monotonic