Title: Quantum Observables as Semispectral Measures - new problems with old questions

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Abstract: The modern view of representing a quantum observable as a semispectral measure as opposed to the traditional approach of using only spectral measures has added a great deal to our understanding of the mathematical structures and conceptual foundations of quantum mechanics. The old questions of 1) how to determine a quantum observable from its classical counter-part (if any), 2) how much statistical information is needed to determine an observable, 3) which observables can be measured together, and 4) are there noiseless measurements, all appear in a new perspective, calling for a study of problems such as: 1) how to obtain a semispectral measure by a quantization map, 2) do the moment operators of a semispectral measure determine the operator measure, 3) are coexistent observables jointly measurable, and 4) does minimal variance occur only in the case of a spectral measure? In my talk I will survey some of the recent developments concerning these questions and problems.

Quantum Observables as Semispectral Measures

– new problems with old questions –

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Perimeter Institute, 14.6.2006

Outline

Old questions

- How to determine a quantum observable from its classical counter-part (if any)?
- How much statistical information is needed to determine an observable?
- Which observables can be measured together?
- Are there noiseless measurements?

New problems

- **Quantization**:
  how to obtain semispectral measures by a quantization map?
- **Moment problem**:
  do the moment operators of a semispectral measure determine the operator measure?
- **Coexistence**:
  are coexistent observables jointly measurable?
- **Minimal variance**:
  does minimal variance occur only in the case of spectral measures?

Collaborators

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A semispectral measure

$\Omega \neq \emptyset$, $\mathcal{A} \subset 2^{\Omega}$ σ-algebra,

A semispectral measure is a map $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that for each unit vector $\varphi \in \mathcal{H}$ the set function

$$\mathcal{A} \ni X \mapsto p_{\varphi}^{E}(X) := \langle \varphi \,|\, E(X)\varphi \rangle \equiv E_{\varphi,\varphi}(X) \in \mathbb{C}$$

is a probability measure.

We assume that quantum observables are described by semispectral measures through their measurement statistics.

Measurement statistics

For a state T (positive trace one operator), the probability measure

$$X \mapsto p_T^E(X) = \mathrm{tr}\,[TE(X)]$$

describes the distribution of measurement outcomes of E in the state T.

The full statistics $\{p_T^E \mid T \text{ a state}\}$ determines E.
(The full statistics $\{p_T^E \mid E \text{ an observable}\}$ determines T.)

Less is enough, but how much of that is needed?

The basic tool

$$D(f,E) = \{\varphi \in \mathcal{H} \mid f \text{ is } E_{\psi,\varphi} - \text{integrable for each } \psi \in \mathcal{H}\},$$
$$\widetilde{D}(f,E) = \{\varphi \in \mathcal{H} \mid |f|^2 \text{ is } E_{\varphi,\varphi} - \text{integrable}\}.$$

Theorem (LMY ROMP98)

(a) *The set $D(f,E)$ is a linear (not necessarily dense) subspace of $\mathcal{H}$, and there is a unique linear operator $L(f,E) = \int f dE$ on the domain $D(f,E)$, which satisfies*

$$\langle \psi \mid L(f,E)\varphi \rangle = \int_\Omega f \, dE_{\psi,\varphi}$$

for all $\psi \in \mathcal{H}$ and $\varphi \in D(f,E)$.

(b) *The set $\widetilde{D}(f,E)$ is a subspace of $D(f,E)$.*

(c) *If f is real valued, $L(f,E)$ is a symmetric operator.*

(d) *While the inclusion $\widetilde{D}(f,E) \subset D(f,E)$ may in general be proper, $\widetilde{D}(f,E) = D(f,E)$ in the case where E is a spectral measure.*

The uniqueness part of the moment problem

The moment operators of $E:\mathcal{B}(\mathbb{R})\to\mathcal{L}(\mathcal{H})$ are $E[k]:=L(x^k,E)$, $k\in\mathbb{N}$.

Question: $(E[k])_{k\in\mathbb{N}}\Rightarrow E$?

- If E is multiplicative, then $\widetilde{\mathcal{D}}(E[k])=\mathcal{D}\big(E[k]\big)$, and all $E[k]$ are densely defined, selfadjoint, and $E[k]=E[1]^k$.

  The operator $E[1]$ determines its spectral measure and there is only one semispectral measure with the moment sequence $E[1]^k$, $k\in\mathbb{N}$ (see below).

A characterization of the noisless case

If $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ is a spectral measure, then $E[1]$ is selfadjoint, and $\int x^2 dE_{\varphi,\varphi} = \|E[1]\varphi\|^2$ for all $\varphi \in \mathcal{D}(E[1])$.

In the case of a general semispectral measure, this need not be true. It turns out that this condition is essential for a positive operator measure E to be a spectral measure.

For each $k \in \mathbb{N}$, let $\widetilde{E}[k] := E[k]|_{\widetilde{\mathcal{D}}(E[k])}$.

Proposition (KLY quant-ph/0601009)

Let $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a semispectral measure, such that

$$\int x^2 dE_{\varphi,\varphi} = \|E[1]\varphi\|^2 \quad (1)$$

for all $\varphi \in \widetilde{\mathcal{D}}(E[1])$.

(a) $\widetilde{E}[k] = \widetilde{E}[1]^k$ *for all* $k \in \mathbb{N}$.

(b) *If $\widetilde{E}[1]$ is assumed to be selfadjoint, then E is projection valued.*

Theorem

Let $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a semispectral measure, such that $\widetilde{E}[1]$ is selfadjoint. Then the following conditions are equivalent.

(i) *E is a spectral measure;*

(ii) *$E[2] = E[1]^2$;*

(iii) *$\int x^2 dE_{\varphi,\varphi} = \|E[1]\varphi\|^2$ for all $\varphi \in \widetilde{\mathcal{D}}(E[1])$.*

This result is well-known (at least) for bounded E. A general proof is given in [KLY quant-ph/0601009]

The operator $R(E) = E[2] - E[1]^2$ is the noise operator of (the observable) E.
A measurement is noisless if and only if it constitutes a spectral measure.

Uniqueness question

Two other known cases for

$$\{E[k], k \in \mathbb{N}\} \Rightarrow E$$

are:

Uniqueness: bounded support

If the measure p^E_φ has a bounded support, then by the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem the distribution p^E_φ is completely determined by its moments $\langle \varphi \,|\, E[k]\varphi \rangle, k \in \mathbb{N}$.

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If the measure p_φ^E has a bounded support, then by the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem the distribution p_φ^E is completely determined by its moments $\langle \varphi \,|\, E[k]\varphi \rangle$, $k \in \mathbb{N}$.

An operationally meaningful formulation is as follows:

Proposition

The probability measure p_φ^E has a bounded support if and only if there are positive constants C and R such that $|\langle \varphi \,|\, E[k]\varphi \rangle| \leq CR^k$ for all $k \in \mathbb{N}$.

Uniqueness: exponential boundedness

If the measure p_φ^E is exponentially bounded, that is, there is an $a > 0$ such that $\int e^{a|x|}\, dp_\varphi^E < \infty$, then p_φ^E is uniquely determined by its moments $\langle\, \varphi \,|\, E[k]\varphi \,\rangle$, $k \in \mathbb{N}$.

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If the measure p^E_φ is exponentially bounded, that is, there is an $a > 0$ such that $\int e^{a|x|}\, dp^E_\varphi < \infty$, then p^E_φ is uniquely determined by its moments $\langle\, \varphi \,|\, E[k]\varphi \,\rangle\,, k \in \mathbb{N}$.

Exponential boundedness of p^E_φ admits a simple operational characterization.

Proposition

The probability measure p^E_φ is exponentially bounded if and only if for some $C, R > 0$ the moment sequence $(\langle\, \varphi \,|\, E[k]\varphi \,\rangle)_{k\in\mathbb{N}}$ satisfies

$$|\,\langle\, \varphi \,|\, E[k]\varphi \,\rangle\,| \leq CR^k\, k! \qquad (2)$$

for all $k \in \mathbb{N}$.

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In the noisy case *the latter three* properties remain equivalent.

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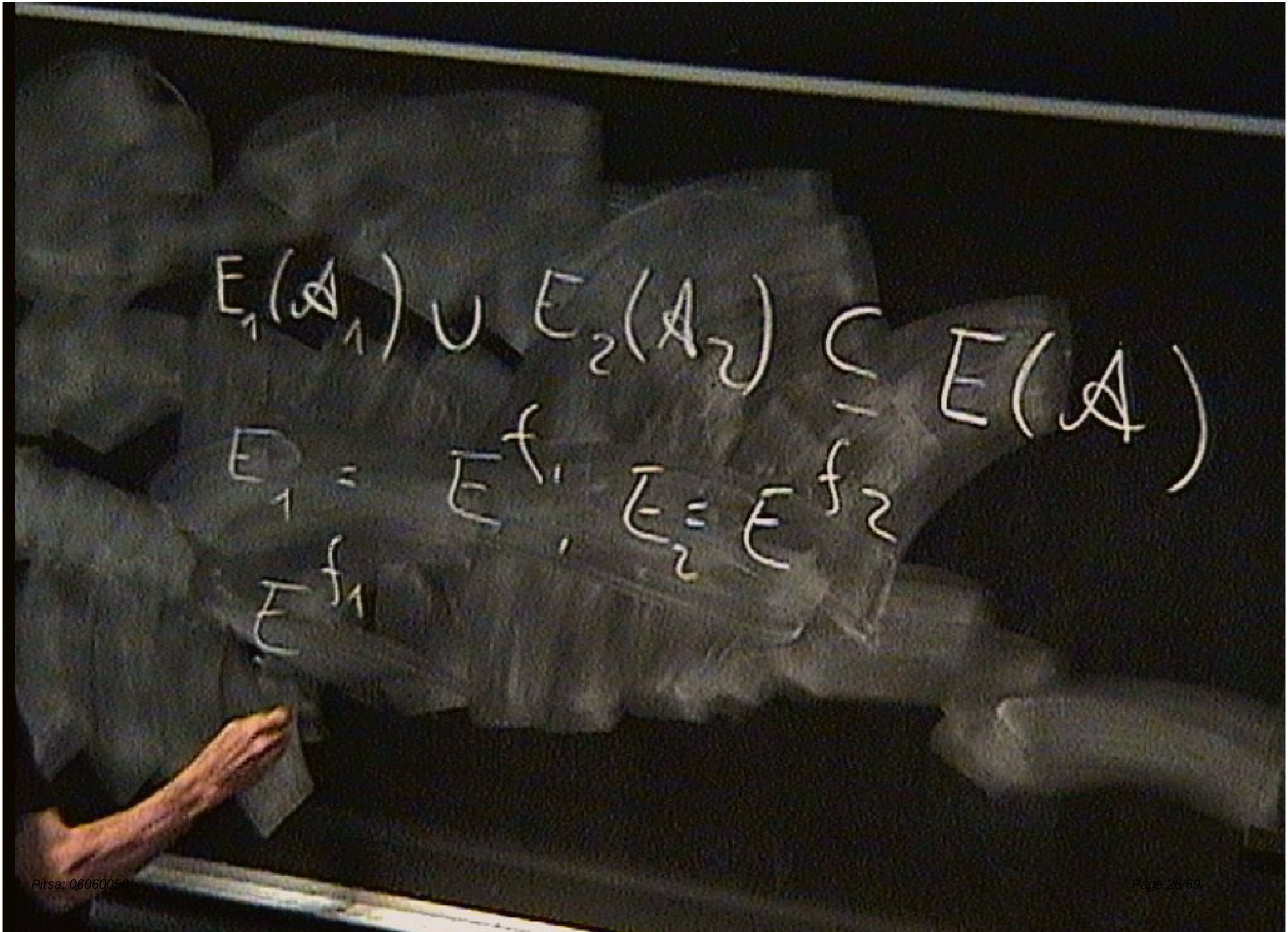
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$$X \times Y \mapsto G_{\gamma}(X \times Y)$$

$$E_1(\mathcal{A}_1) \cup E_2(\mathcal{A}_2) \subseteq E(\ldots$$

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$$(X, Y) \mapsto B(X, Y) = B(X, \Omega_2) = E\ldots \qquad (Y = \Omega_2)$$

$$X \times Y \mapsto G(X \times Y), \quad G(X \times \Omega_2) = E_1(X)$$

$$E_1(A_1) \cup E_2(A_2) \subseteq E(A)$$

$$E_1 = E^{f_1}, \quad E_2 = E^{f_2}$$

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Quantization maps

A quantization map Γ is a function which assingns a linear operator $\Gamma(f)$ to a dynamical variable $f : \Omega \to \mathbb{R}$.

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To obtain a semispectral measure from f we consider the collection of operators $\Gamma(f^k)$, $k \in \mathbb{N}$, and search for a semispectral measure $E^f : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ with the property $\Gamma(f^k) = L(x^k, E^f)$.

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If Γ and f are such that there exists a unique solution E^f of the described moment problem, then the operators $\Gamma(f^k), k \in \mathbb{N}$, or, equivalently E^f, constitute a quantization of f.

An implementation by a POM

One may write down conditions under which a quantization map $f \mapsto \Gamma(f)$ coincides with the map $f \mapsto L(f, E)$ for a (unique) E. We assume that such conditions are fulfilled:

$$\Gamma(f) = L(f, E) = \int f \, dE.$$

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In that case for any f and k

$$\Gamma(f^k) = \int_\Omega f^k \, dE = \int_{\mathbb{R}} x^k \, dE^f,$$

where E^f is the induced measure $X \mapsto E(f^{-1}(X))$.

Immediate observations

In this scheme all observables obtained as quantizations are functionally coexistent, they can be measured jointly.

A nontrivial case requires E to be a semispectral measure and, preferably, even informationally complete.

Covariant phase space quantizations on $\mathbb{R}^2$

Covariant quantization maps Γ on $\mathbb{R}^2$ are obtained (under appropriate assumptions) from the covariant phase space observables which are known to be of the form

$$E^T(Z) = \frac{1}{2\pi}\int_Z W(q,p)TW(q,p)^*\, dqdp, \tag{3}$$

where $W(q,p)$ are the Weyl operators and the generating operator T is positive and of trace one.

For each generating T we denote

$$\Gamma^T(f) = L(f, E^T).$$

The problems / observations

- Is there a T such that Γ^T provides a quantization for all f?

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- The only projections in the range of any E^T are O and I so that no observables as spectral measures are obtained.
- All observables (semispectral measures) obtained by a quantization map Γ^T are functionally coexistent.
- If $\mathrm{tr}\,[TW(q,p)] \neq 0$ for a.e. $(q,p) \in \mathbb{R}^2$, then Γ^T / E^T is info complete.

What about $L(f, E^T)$?

Proposition (KLY, quant-ph/0601009)

For any covariant phase space observable E^T the following are obtained.

(a) *Assume that f is such that $\widetilde{D}(f, E^T) \subset \widetilde{D}(f(g\cdot), E^T)$ for all $g = (q, p) \in \mathbb{R}^2$. Then $U(g)\widetilde{D}(f, E^T) = \widetilde{D}(f, E^T)$ for all $g \in \mathbb{R}^2$, and either $\widetilde{D}(f, E^T) = \{0\}$ or $\widetilde{D}(f, E^T)$ is dense.*

(b) *Assume that f is such that $D(f, E^T) \subset D(f(g\cdot), E^T)$ for all $g = (q, p) \in \mathbb{R}^2$. Then $U(g)D(f, E^T) = D(f, E^T)$ for all $g \in \mathbb{R}^2$, and either $D(f, E^T) = \{0\}$ or $D(f, E^T)$ is dense. Moreover,*

$$U(g)^* L(f, E^T) U(g) \subset L(f(g\cdot), E^T) \tag{4}$$

for all $g \in \mathbb{R}^2$.

$$x^k : (q,p) \mapsto q^k,\ y^k(q,p) = p^k.$$

Proposition (KLY, quant-ph/0601009)

(a) *Let* $k \in \mathbb{N}$*. Then* $\widetilde{D}(x^k, E^T) \neq \{0\}$ *if and only if* $Q^k\sqrt{T}$ *is a Hilbert-Schmidt operator, and in that case,* $\widetilde{D}(x^k, E^T) = D(Q^k)$.

(b) *The statement in (a) holds true, if "x" and "Q" are replaced by "y" and "P".*

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Theorem (KLY, quant-ph/0601009)

(a) *Assume that T satisfies the condition of Proposition (a). Then $L(x^k, E^T) = \sum_{l=0}^{k} s_{kl}^{Q} Q^l$, where $s_{kl}^{Q} = \binom{k}{l}(-1)^{k-l} tr\,[Q^{k-l}T]$, with each $Q^{k-l}T$ a trace class operator.*

(b) *The statement in (a) holds true, if "(a)", "x" and "Q" are replaced by "(b)", "y" and "P".*

The case $T = |n\rangle\langle n|$.

Let $\{|n\rangle\}_{n\in\mathbb{N}}$ be the number basis associated with the Weyl operators $W(q,p), (q,p) \in \mathbb{R}^2$.

Quantization maps $\Gamma^{|n\rangle}$ and the phase space observables $E^{|n\rangle}$ defined by the number basis are distinguished by the property of being phase shift covariant, informationally complete, and giving a unique solution to the moment problem for some (basic) f.

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Theorem (DLY ROMP 2000)

The operator measures $X \mapsto E^{|n\rangle}(X \times \mathbb{R})$ *and* $Y \mapsto E^{|n\rangle}(\mathbb{R} \times Y)$ *are uniquely determined by their respective moment operators* $\{\Gamma^{|n\rangle}(x^k) \,|\, k \in \mathbb{N}\}$ *and* $\{\Gamma^{|n\rangle}(y^k) \,|\, k \in \mathbb{N}\}$.

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The sets $\{\Gamma^{|n\rangle}(x^k) \,|\, k \in \mathbb{N}\}$ and $\{\Gamma^{|n\rangle}(y^k) \,|\, k \in \mathbb{N}\}$ constitute quantizations of x and y.

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Also: the operators $\{\Gamma^{|n\rangle}(h^k) \mid k \in \mathbb{N}\}$ quantize $h(q,p) = \frac{1}{2}(q^2 + p^2)$, and they constitute the radial marginal measure of $E^{|n\rangle}$.

Optimal quantization

Choose T is such that

- $\Gamma^T(x^k) = \sum_{l=0}^{k} s_{kl}^{Q} Q^l$,
- $\Gamma^T(y^k) = \sum_{l=0}^{k} s_{kl}^{P} P^l$,

so that the noise operators are:

$$R^T(x) = L(x^2, E^T) - L(x, E^T)^2 = \mathrm{Var}(Q, T) I,$$
$$R^T(y) = L(y^2, E^T) - L(y, E^T)^2 = \mathrm{Var}(P, T) I.$$

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Choose T such that

- $L(x, E^T) = Q, \; L(y, E^T) = P$,

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Choose T such that

- $L(x, E^T) = Q,\ L(y, E^T) = P$,
- the operator product $R^T(x)\, R^T(y)$ is minimal,

Optimal quantization

Choose T is such that

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- $\Gamma^T(y^k) = \sum_{l=0}^{k} s_{kl}^P P^l$,

so that the noise operators are:

$$R^T(x) = L(x^2, E^T) - L(x, E^T)^2 = \mathrm{Var}(Q, T)I,$$
$$R^T(y) = L(y^2, E^T) - L(y, E^T)^2 = \mathrm{Var}(P, T)I.$$

Choose T such that

- $L(x, E^T) = Q,\ L(y, E^T) = P$,
- the operator product $R^T(x)\, R^T(y)$ is minimal,
- $R^T(x) = R^T(y)$,

then $T = |0\rangle\langle 0|$.

Are coexistent observables jointly measurable?

For any two noiseless observable E_1 and E_2 the following concepts coincide:

- E_1 and E_2 commute;
- E_1 and E_2 are commensurable;
- E_1 and E_2 are coexistent;
- E_1 and E_2 are functionally coexistent;
- E_1 and E_2 have a biobservable;
- E_1 and E_2 have a joint observable.

Theorem

Let $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a semispectral measure, such that $\widetilde{E}[1]$ is selfadjoint. Then the following conditions are equivalent.

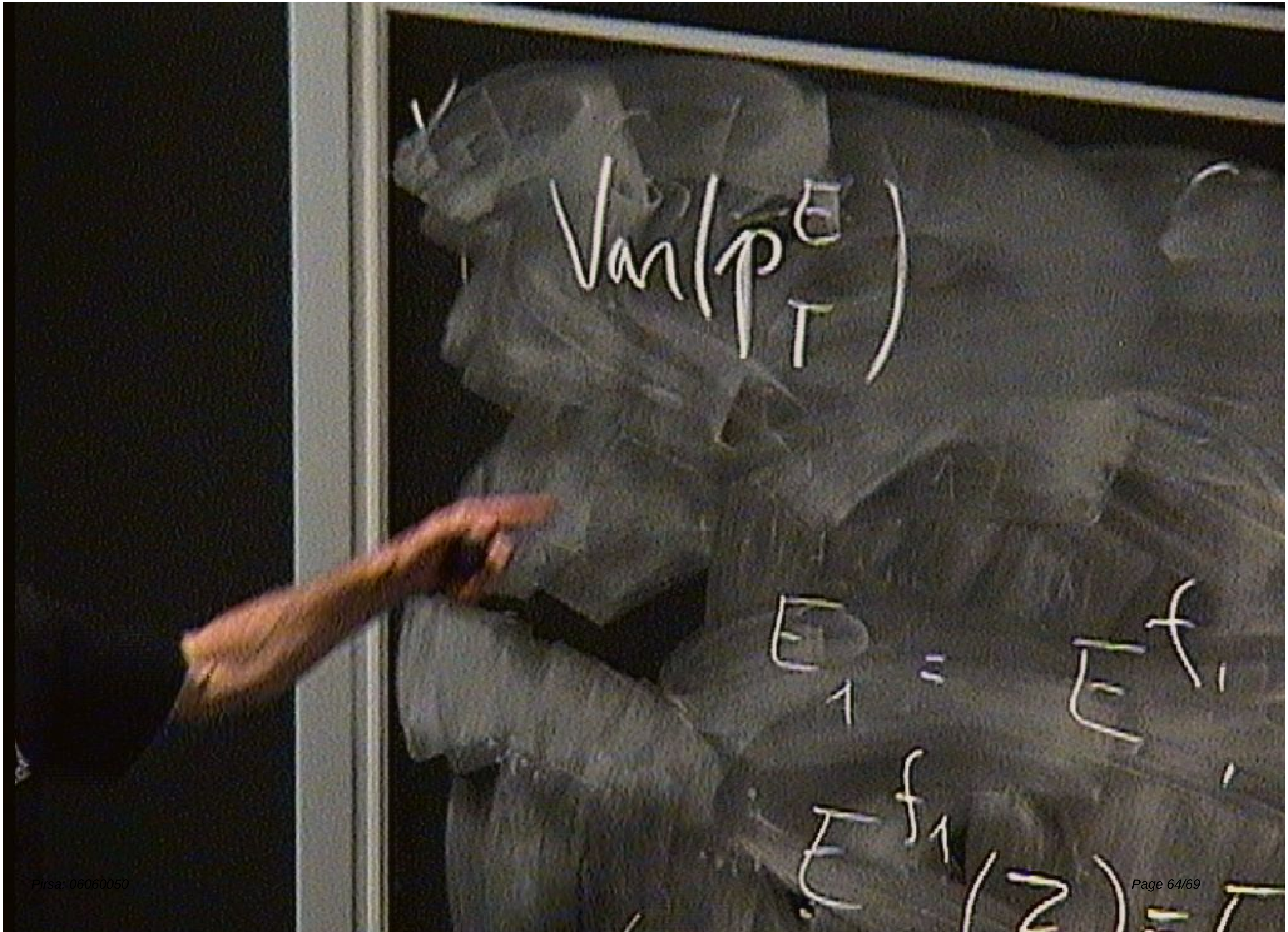
(i) *E is a spectral measure;*

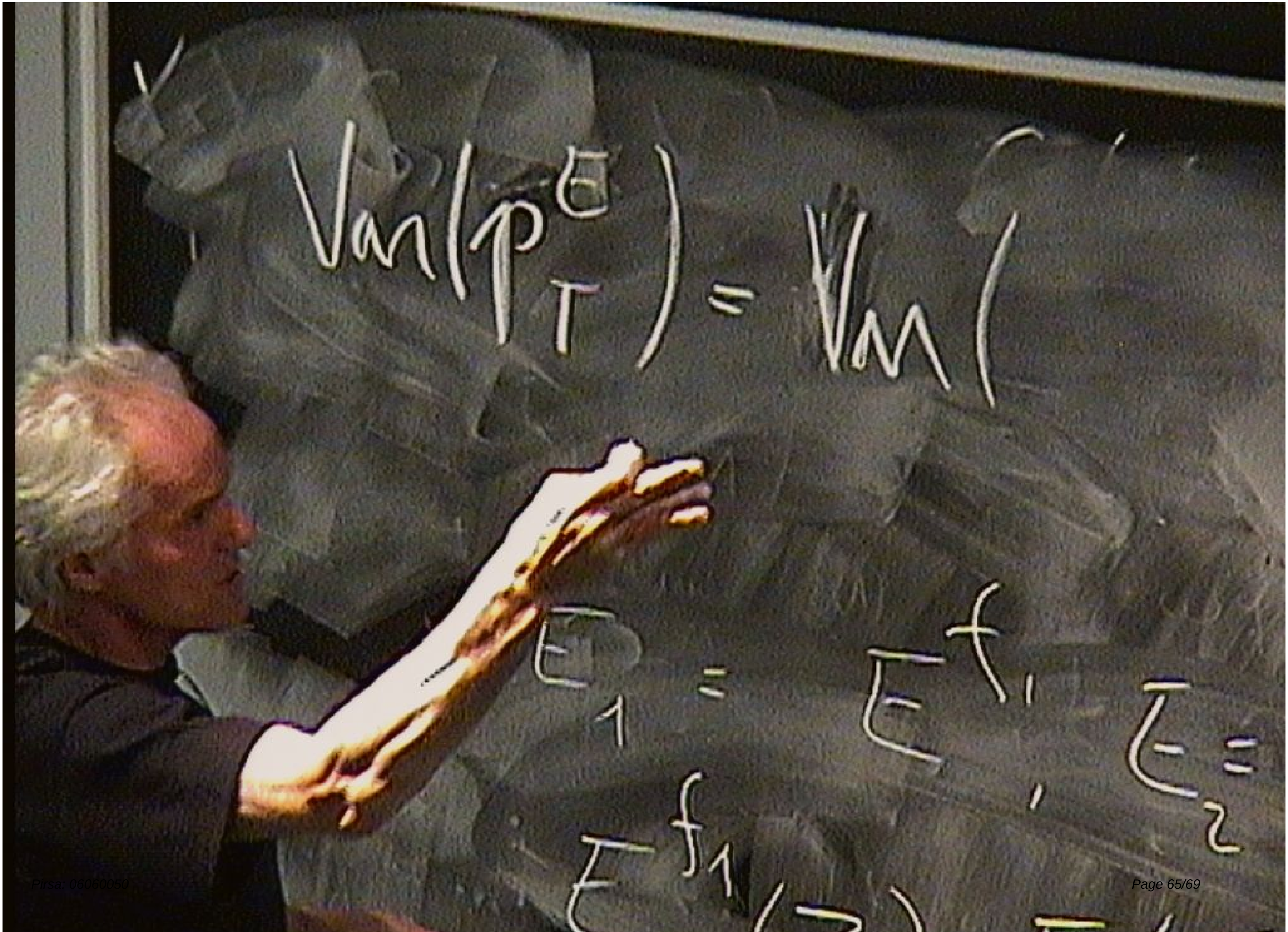
(ii) $E[2] = E[1]^2$;

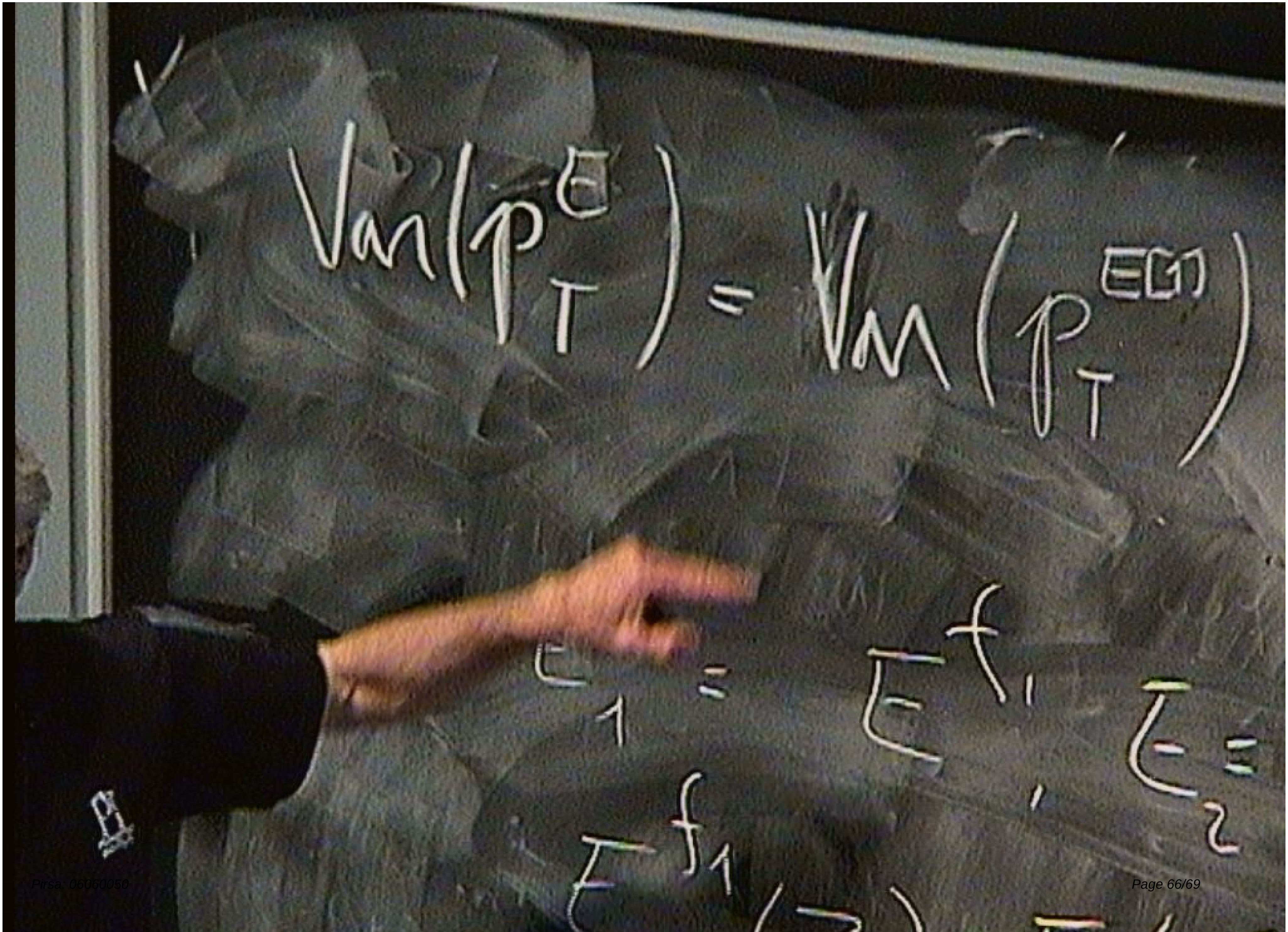
(iii) $\int x^2 dE_{\varphi,\varphi} = \|E[1]\varphi\|^2$ *for all* $\varphi \in \widetilde{\mathcal{D}}(E[1])$.

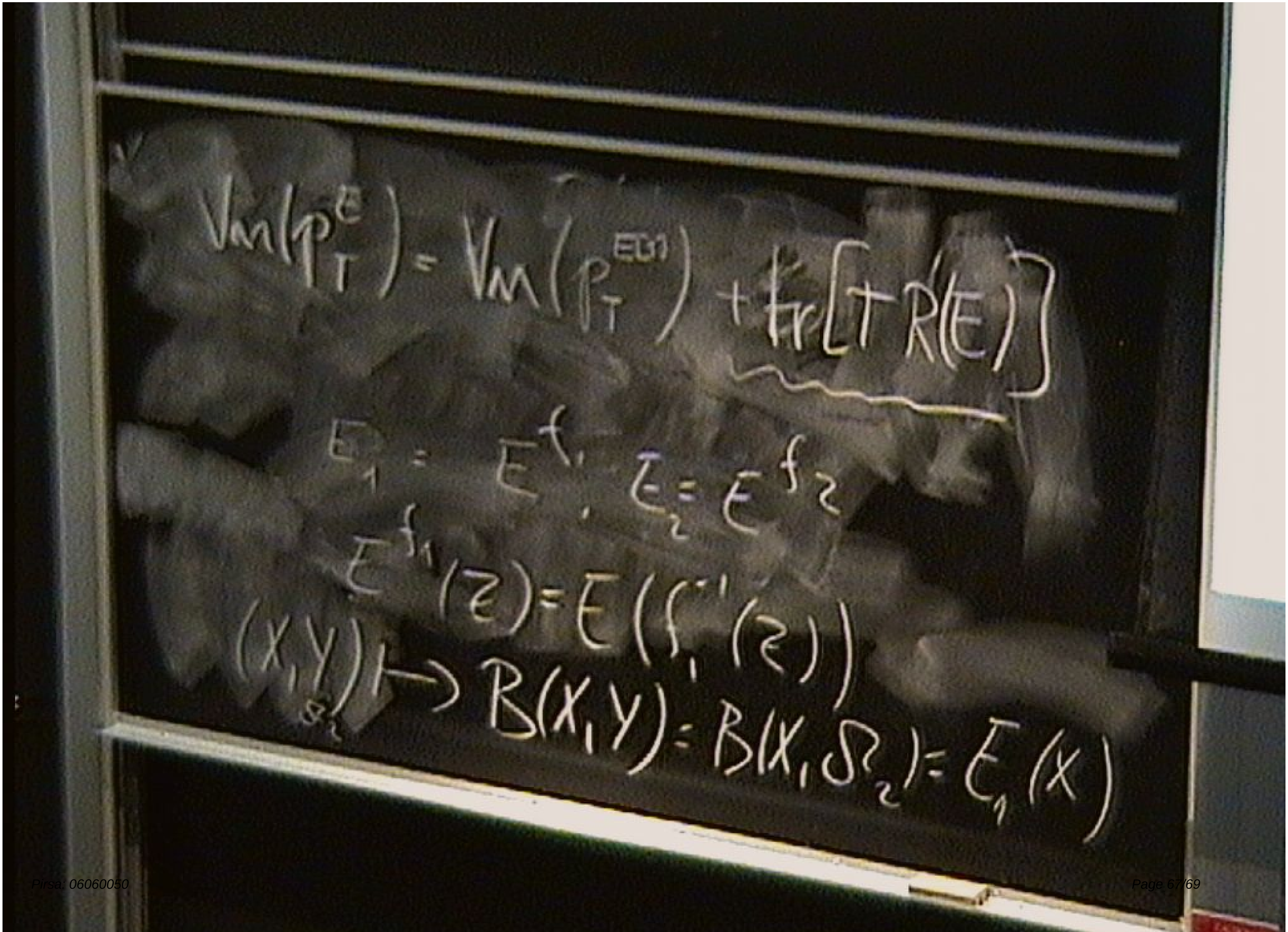
This result is well-known (at least) for bounded E. A general proof is given in [KLY quant-ph/0601009]

The operator $R(E) = E[2] - E[1]^2$ is the noise operator of (the observable) E.
A measurement is noisless if and only if it constitutes a spectral measure.


Var(p^E_T)


Var(p_T^E) = Var(
E_1 = E^{f_1}, E_2 =


Var(p_T^E) = Var(p_T^EGT)
E_1 = E^f1 , E_2 =


Vm(ρ_T^E) = Vm(ρ_T^{E⊗1}) + Tr[f R(E)]
E_1 = E^{f_1}, E_2 = E^{f_2}
E^{f_1}(z) = E(f_1^{-1}(z))
(x, y) ↦ B(x, y) = B(x, Ω_2) = E_1(x)

$$\mathrm{Vm}(\rho_T^{E}) = \mathrm{Vm}(\rho_T^{E[3]}) + \mathrm{Tr}[T\, R(E)]$$

$$E_1 = E^{f_1}, \quad E_2 = E^{f_2}$$

$$E^{f_1}(z) = E(f_1^{-1}(z))$$

$$(X, Y)_{\Omega_2} \mapsto B(X, Y) = B(X, \Omega_2) = E_1(X)$$

Theorem

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