

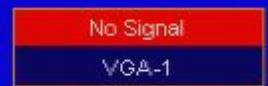
Title: Field Theory 3

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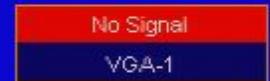
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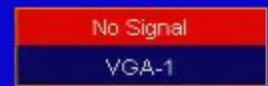


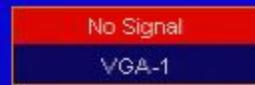


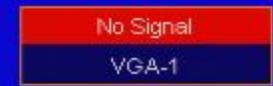


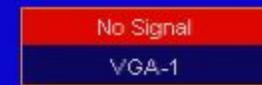




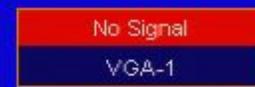


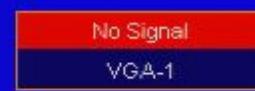




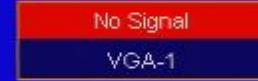




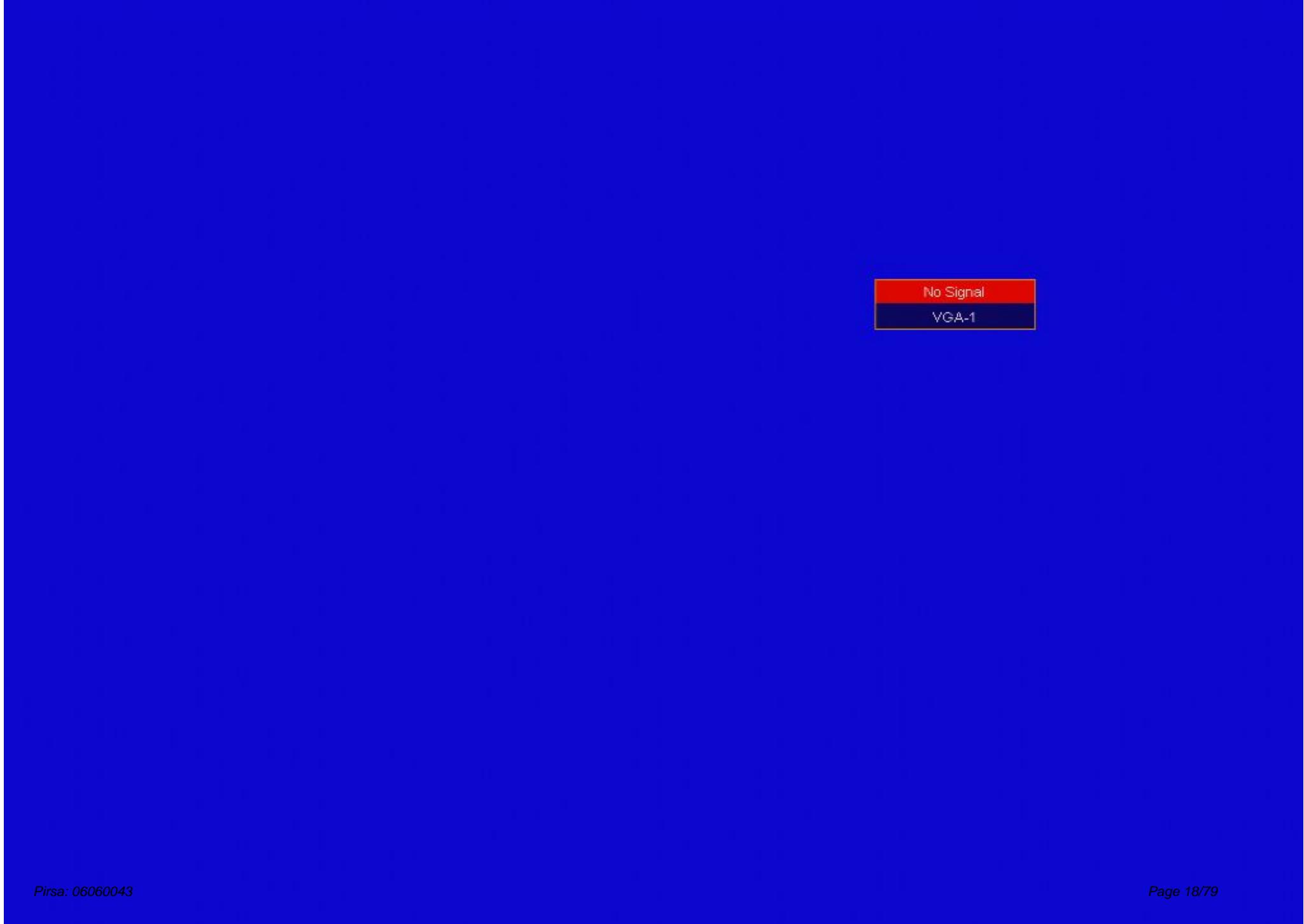




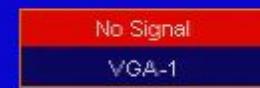


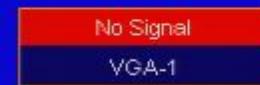


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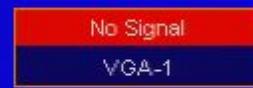


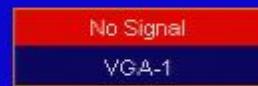
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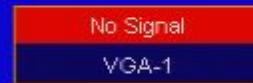
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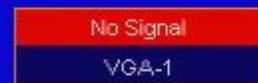
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# Multisoliton configurations in Skyrme-like Models

L. Marleau<sup>1</sup>

<sup>1</sup>Département de physique, génie physique et d'optique,  
Université Laval, Québec, Canada.

Theory CANADA 2, PI— June 8-10, 2006

## Brief introduction

The starting point : The Skyrme Model

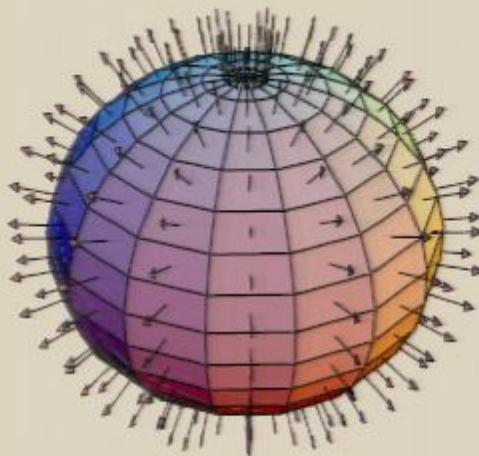
How to extend it

How to compute and what we get

What's next

Concluding remarks

# Brief introduction



- 1 Brief introduction
- 2 The starting point : The Skyrme Model
  - $B = 1$  Skyrmions
  - Multiskyrmions
- 3 How to extend it
- 4 How to compute and what we get
- 5 What's next
- 6 Concluding remarks

## /Motivations

- General motivation : study Skyrme Model and extensions as prototype of effective theory with 3D solitons
- Skyrme Model is a non-linear effective theory of pions with chiral symmetry → in low energy hadron physics, nucleon emerges as a  $N = B = 1$  soliton with spherical symmetry
- But multiskyrmion solutions ( $N = B \geq 2$ ) exhibit symmetries not expected for nuclei

$N = 2$  : toroidal  
 $N = 3$  : tetrahedral  
 $N = 4$  : cubic....

What if the Skyrme model is modified ???

Is it possible to get  $n$ -sphere configurations ???

**Brief introduction**

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How to extend it

How to compute and what we get

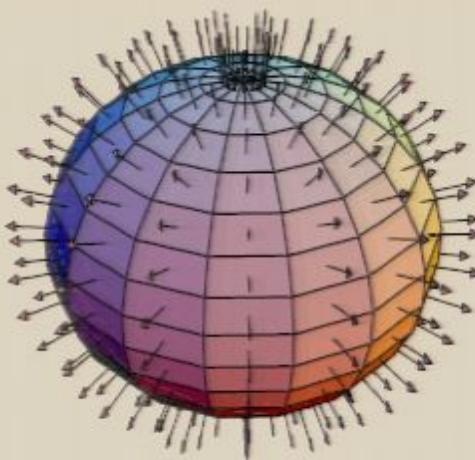
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## /Solitons



Solitons : Solutions of nonlinear differential equations :



- Localized finite energy (lumps, strings, walls,...)
- Move without distortion or dissipation
- Stable under perturbations and collisions with other solitons

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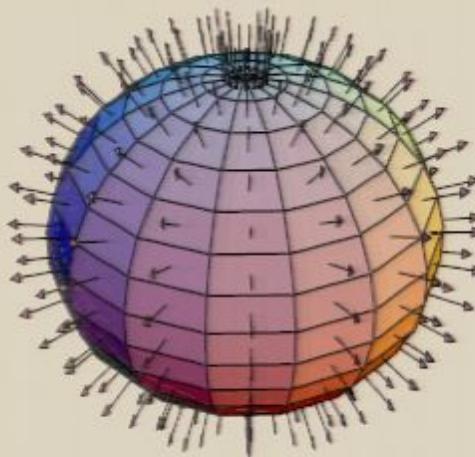
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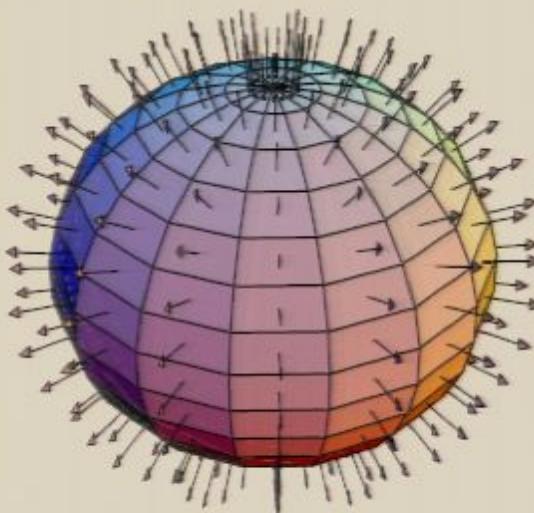
Solitons : Solutions of nonlinear differential equations :

- Localized finite energy (lumps, strings, walls,...)
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## $B = 1$ Skyrmions/Hadrons physics

- In hadron physics :
  - ① Nucleon emerges as a topological soliton : i.e. extended object, topological number, fermion, etc....
  - ② 2 parameters → reasonable but not excellent agreement with data.
- Other implementations :
  - ① Electroweak physics (hidden gauge symmetry)
  - ② Baby skyrmions (1+2D, e.g. condensed matter physics)
  - ③ Higher dimensional theories (e.g. locally wrapped brane, Born-Infeld-Skyrme...)
  - ④ Faddeev-Skyrme Model

## The starting point : The Skyrme Model/ $B = 1$ Skyrmions



- Non-linear effective theory of pions with chiral symmetry (Skyrme)
- $1/N_c$  expansion  $\rightarrow$  low-energy limit of QCD (Witten).
- Allows for non-perturbative soliton solutions (extended objects characterized by a winding number)  $\rightarrow$  identified with baryons
- Solitons in scalar field theory  $\implies$  fermions

## $B = 1$ Skyrmions/Hadrons physics

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## $B = 1$ Skyrmions/Lagrangian

- The  $SU(2)$  Skyrme Lagrangian :

$$\mathcal{L} = -\frac{F_\pi^2}{16} \text{Tr } L_\mu L^\mu + \frac{1}{32e^2} \text{Tr } f_{\mu\nu} f^{\mu\nu}$$

where

$$L_\mu = U^\dagger \partial_\mu U$$

$$f_{\mu\nu} \equiv [L_\mu, L_\nu]$$

with  $U = U(x) \in SU(2)$ .

- ①  $\mathcal{L}_1 \sim \text{Tr } L_\mu L^\mu$  : Lagrangian of the non-linear  $\sigma$ -model, with  $\sigma$ - and  $\pi$ -fields replaced by

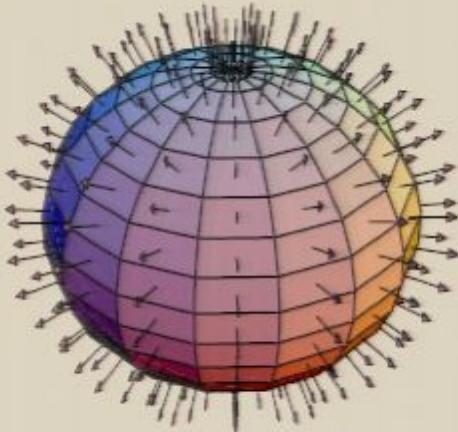
$$U = \frac{2}{F_\pi} (\sigma + i\tau \cdot \pi)$$

- ②  $\mathcal{L}_2 \sim \text{Tr } f_{\mu\nu} f^{\mu\nu}$  : 4-derivatives terms introduced to stabilize the soliton against shrinking to zero size. (“dynamical balancing”)

## $B = 1$ Skyrmions/Hedgehog Solution

- Solutions are maps from physical space  $\mathbb{R}^3$  onto the group manifold  $SU(2)$
- $\pi_3(SU(2)) = \mathbb{Z} \implies$  Winding number  $N \in \mathbb{Z}$  = baryon number  $B$ ,

$$N = B = \frac{1}{24\pi^2} \int d^3 r \epsilon_{0\nu\rho\sigma} L^\nu L^\rho L^\sigma$$



- For  $B = 1$  : Spherically symmetric static "hedgehog" solution

$$U(\mathbf{r}) = \exp [i\boldsymbol{\tau} \cdot \hat{\mathbf{r}} F(r)]$$

where  $F(r)$  = chiral angle or profile function.

## $B = 1$ Skyrmions/Hedgehog Solution

What about  $B > 1$  solution (multiskyrmions) ?

Should one expect  $n-$  sphere configuration or else ? ! ? !

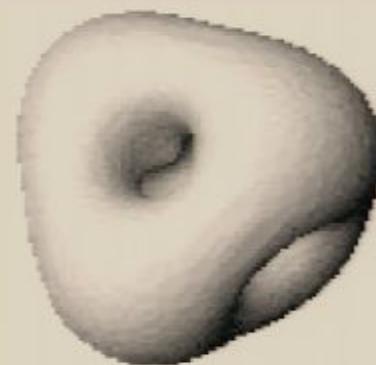
## Multiskyrmions/Multiskyrmions

- For the Skyrme model (no pion mass term), multiskyrmion solutions exhibit particular symmetries
  - 1  $B = 1$  : spherical
  - 2  $B = 2$  : toroidal
  - 3  $B = 3 \approx$  tetrahedral
  - 4  $B = 4 \approx$  cubic....

## Multiskyrmions/Multiskyrmions symmetries



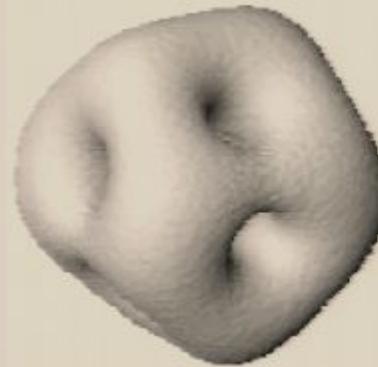
$B = 2$



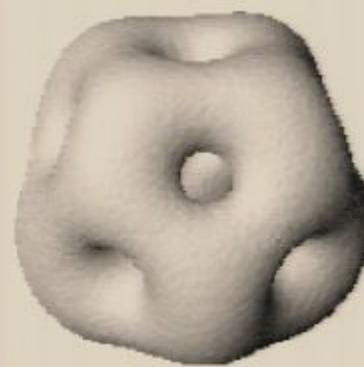
$B = 3$



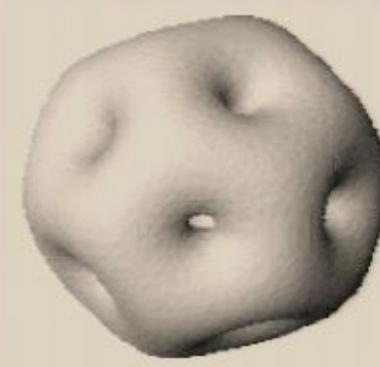
$B = 4$



$B = 5$



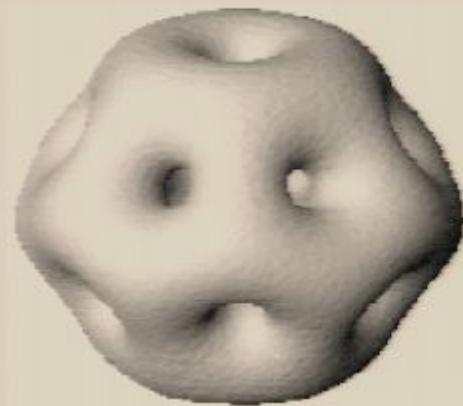
$B = 6$



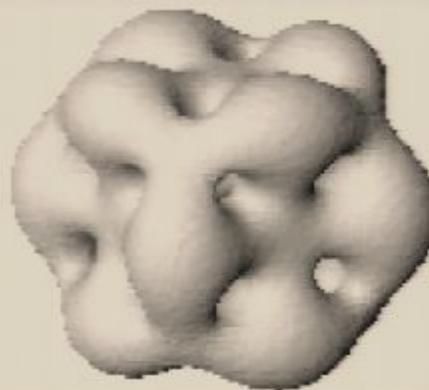
$B = 7$

(Sutcliffe et al.)

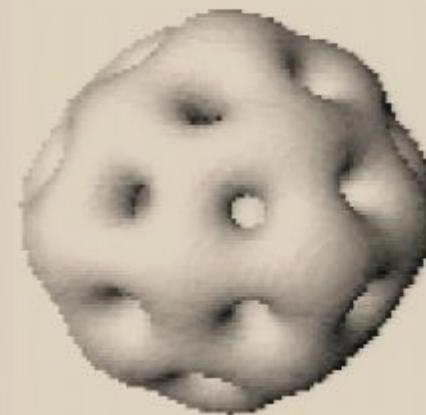
## Multiskyrmions/Multiskyrmions symmetries



$B = 8$



$B = 12$



$B = 13$

- These solutions and symmetries are approximately described by the rational map ansatz.

## Multiskyrmions/Rational Maps

- Manton et al. suggest that multiskyrmions are analogous to BPS solitons  $\Rightarrow$  rational map ansatz

$$U_R(r, z) = e^{i\hat{n}_R \cdot \vec{\tau} F(r)}$$

where

$$\begin{aligned}\hat{n}_R &= \frac{1}{1 + |R_N|^2} (R_N + \bar{R}_N, R_N - \bar{R}_N, 1 - |R_N|^2) \\ &= (\sin \Theta(\theta) \cos \Phi(\phi), \sin \Theta(\theta) \sin \Phi(\phi), \cos \Theta(\theta))\end{aligned}$$

- Here

$$R_N(z) = \frac{p(z)}{q(z)} \quad \text{with} \quad z = \tan\left(\frac{\theta}{2}\right) e^{i\phi}$$

where at least one of the polynomials  $p(z)$  or  $q(z)$  is of degree  $N$  (winding number).

## Multiskyrmions/Rational Maps

- One recovers  $\theta, \phi$ -angular dependance by inverting

$$\begin{aligned} z = \tan\left(\frac{\theta}{2}\right) e^{i\phi} &\implies R_N(z) = \tan\left(\frac{\Theta(\theta)}{2}\right) e^{i\Phi(\phi)} \\ \theta = \arctan |z| &\implies \Theta(\theta) = \arctan |R_N| \\ \phi = i \ln \frac{z}{|z|} &\implies \Phi(\phi) = i \ln \frac{R_N}{|R_N|} \end{aligned}$$

## Multiskyrmions/Rational Maps

What happens if the Skyrme Model is modified ?

Will multiskyrmions exhibit the same symmetries ?

More precisely, is there a model where multiskyrmions possess  $n$ -sphere configurations ?

- In order to provide an answer we need :
  - ➊ a systematic approach to construct other Skyrme-like models
  - ➋ a convenient way to find the **exact numerical** solutions for the multiskyrmions.

## Multiskyrmions/Higher order terms



- Higher order terms : First, let us write the general form for the solution

$$U(\mathbf{r}) = \exp [i\boldsymbol{\tau} \cdot \hat{\mathbf{n}} F]$$

with

$$\hat{\mathbf{n}} = (\cos \Theta, \sin \Theta \cos \Phi, \sin \Theta \sin \Phi)$$

and  $F, \Theta$  and  $\Phi$  are functions of  $(r, \theta, \phi)$

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## Multiskyrmions/Higher order terms

There are several ways to modify the Skyrme Model.

- If one considers Skyrme-like models as effective theory :
  - ① Add terms to account for  $\rho$ - or  $\omega$ - mesons or...
  - ② ....construct an effective theory which contains terms to arbitrary orders of derivative of the pion fields.

## Multiskyrmions/Higher order terms

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and  $F, \Theta$  and  $\Phi$  are functions of  $(r, \theta, \phi)$

## Multiskyrmions/Higher order terms

- Static energy densities are given by

$$\text{Tr } L_\mu L^\mu \rightarrow \mathcal{E}_1 = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$$

$$\text{Tr } f_{\mu\nu} f^{\mu\nu} \rightarrow \mathcal{E}_2 = (\mathbf{X} \times \mathbf{Y})^2 + (\mathbf{Y} \times \mathbf{Z})^2 + (\mathbf{Z} \times \mathbf{X})^2$$

where

$$\mathbf{X} = \nabla F \quad \mathbf{Y} = \frac{\sin F}{r} \nabla \Theta \quad \mathbf{Z} = \frac{\sin F}{r} \frac{\sin \Theta}{\sin \theta} \nabla \Phi$$

## Multiskyrmions/Higher order terms

- Manton has shown that these invariants have a simple geometrical interpretation.
- They correspond to the eigenvalues of the strain tensor in the theory of elasticity

$$D_{ij} = \partial_i \phi_m \partial_j \phi_m$$

$$\mathcal{E}_1 = \text{Tr } D = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2 \sim \sum (\text{length})^2$$

$$\mathcal{E}_2 = \frac{1}{2} \{ (\text{Tr } D)^2 - \text{Tr } D^2 \}$$

$$= (\mathbf{X} \times \mathbf{Y})^2 + (\mathbf{Y} \times \mathbf{Z})^2 + (\mathbf{Z} \times \mathbf{X})^2 \sim \sum (\text{surface})^2$$

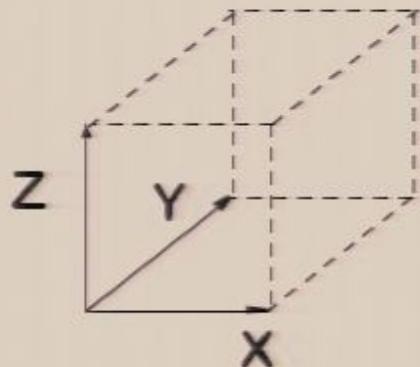
$$\mathcal{E}_3 = 3 \det D = 3 (\mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z}))^2 \sim \sum (\text{volume})^2$$

## Multiskyrmions/Hedgehog solution energy

For the hedgehog ansatz :

$\mathbf{X} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{Z} = \mathbf{X} \cdot \mathbf{Z} = 0$  and  $\mathbf{X}^2 = b$ ,  
 $\mathbf{Y}^2 = a$ , and  $\mathbf{Z}^2 = c = a$  with

$$a = c = \frac{\sin^2 F}{r^2} \quad b = F'^2$$



$$\mathcal{E}_1 = a + b + c = (2a + b)$$

$$\mathcal{E}_2 = ab + bc + ca = a(a + 2b)$$

$$\mathcal{E}_3 = 3abc = a^2(3b)$$

Baryonic density :  $\mathcal{B}_0 = \frac{1}{2\pi^2} \sqrt{3abc} = -\frac{a}{2\pi^2} \sqrt{3b}$

## Multiskyrmions/Special class of Skyrme-like models

- Higher order terms are arbitrary combinations of  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$
- Energy for order  $2m$  in derivatives of the field follows the same pattern as  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$  for hedgehog ansatz.

$$\mathcal{E}_1 = (3a + (b - a))$$

$$\mathcal{E}_2 = a(3a + 2(b - a))$$

$$\mathcal{E}_3 = a^2(3a + 3(b - a))$$

⋮

$$\mathcal{E}_m = a^{m-1}(3a + m(b - a))$$

## Multiskyrmions/Special class of Skyrme-like models

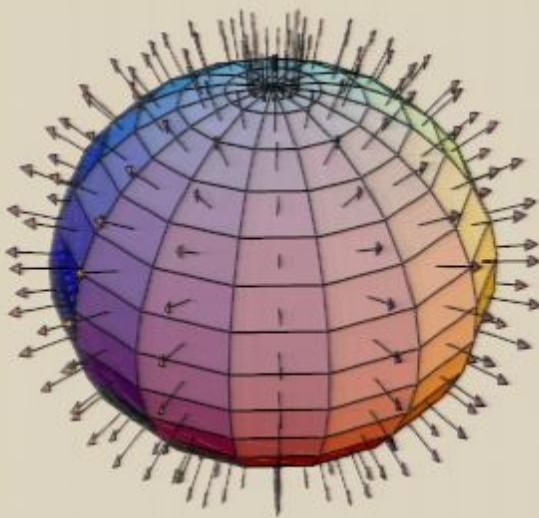
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- Full Lagrangian leads to

$$\mathcal{E} = \sum_{m=1}^{\infty} h_m \mathcal{E}_m = 3\chi(a) + (b-a)\chi'(a) \quad \text{with} \quad \chi(x) = \sum_{m=1}^{\infty} h_m x^m$$

- Positive energy constraint :  $3\chi(x) - x\chi'(x) \geq 0$ ,  $\chi'(x) \geq 0$

## Multiskyrmions/Recursion relation



- Jackson et al. found a recursion relation for  $\mathcal{E}_m = a^{m-1}(3a + m(b - a))$

$$\mathcal{E}_m = \mathcal{E}_{m-1}\mathcal{E}_1 - \mathcal{E}_{m-2}\mathcal{E}_2 + \frac{1}{3}\mathcal{E}_{m-3}\mathcal{E}_3$$

or in terms of invariants  $a, b$  and  $c$ :

$$\mathcal{E}_m = \frac{(b - c)^3 a^m + (c - a)^3 b^m + (a - b)^3 c^m}{(a - b)(b - c)(c - a)}$$

Elegant for the static energy but not very practical to construct the full Lagrangian.

## Multiskyrmions/Generating function

- Need a closed form for higher-derivative Lagrangian  $\mathcal{L}_m$  written in terms

$$\mathcal{L}_1 = -\frac{1}{2} \text{Tr} (L_\mu L^\mu), \quad (1)$$

$$\mathcal{L}_2 = \frac{1}{16} \text{Tr} (f^{\mu\nu} f_{\mu\nu}), \quad (2)$$

$$\mathcal{L}_3 = -\frac{1}{32} \text{Tr} (f_{\mu\nu} f^{\nu\lambda} f_\lambda^\mu), \quad (3)$$

i.e.

$$\mathcal{L}_m = \sum_{n_1, n_2, n_3=0}^{\infty} C_{n_1, n_2, n_3} \mathcal{L}_1^{n_1} \mathcal{L}_2^{n_2} \mathcal{L}_3^{n_3}.$$

with  $m = n_1 + 2n_2 + 3n_3$ .

## Multiskyrmions/Generating function

(cont.)

- This is possible using the generating function

$$G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3; x) \equiv c_1 + \sum_{m=1}^{\infty} \mathcal{L}_m x^m$$

where

$$\mathcal{L}_m = \frac{1}{m!} \left. \frac{d^m}{dx^m} G(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3; x) \right|_{x=0}$$

## Multiskyrmions/Generating function

- Substitute  $\mathcal{L}_m$  in  $G$  and isolate terms of order  $x^m$

$$\mathcal{L}_m = \sum_{n_2=0}^{[\frac{m}{2}]} \sum_{n_3=0}^{[\frac{m-2n_2}{3}]} C_{m-2n_2-3n_3, n_2, n_3} \mathcal{L}_1^{m-2n_2-3n_3} \mathcal{L}_2^{n_2} \mathcal{L}_3^{n_3}$$

for  $m = n_1 + 2n_2 + 3n_3 \geq 4$  where  $[z] = \text{integer part of } z$

- We get

$$C_{n_1, n_2, n_3} = \frac{(n_1 + n_2 + n_3 - 2)!}{n_1! n_2! n_3!} \frac{(-)^{n_1+n_3}}{3^{n_3}} \cdot (-4n_1 n_3 + n_2^2 - n_2 - 2n_2 n_3 - 3n_3^2 + 3n_3)$$

- By inspection

$$C_{1,0,0} = C_{0,1,0} = C_{0,0,1} = 1,$$
$$C_{0,0,0} = C_{2,0,0} = C_{3,0,0} = 0$$

## Multiskyrmions/Generating function

- Full Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \sum_{m=1}^{\infty} h_m \mathcal{L}_m \\ &= \sum_{m=1}^{\infty} h_m \left( \sum_{n_2=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{n_3=0}^{\lfloor \frac{m-2n_2}{3} \rfloor} C_{m-2n_2-3n_3, n_2, n_3} \mathcal{L}_1^{m-2n_2-3n_3} \mathcal{L}_2^{n_2} \mathcal{L}_3^{n_3} \right).\end{aligned}$$

- Advantage of this approach :

- ① Need only to know  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$
- ② Easy to find Lagrangian and energy density for any given model defined by coefficients  $h_m$  or  $\chi(x)$ .

## Multiskyrmions/Rational model

### Exemple

Consider a case where the energy density is given by

$$\chi(x) = \frac{x}{1-x} = x + x^2 + x^3 + \dots$$

In this case, the full Lagrangian has a compact form

$$\mathcal{L} = \frac{\mathcal{L}_1 + (\mathcal{L}_1^2 + \mathcal{L}_2) + \mathcal{L}_3}{(1 + \mathcal{L}_1 - \mathcal{L}_2 + \frac{1}{3}\mathcal{L}_3)}$$

## Multiskyrmions/Constraint on energy

- In general, in numerical solutions, arbitrary positive  $a, b, c$  can lead to negative values of energy density



$$\mathcal{E} = \frac{(c-a)^3 \chi(b) - (c-b)^3 \chi(a) - (b-a)^3 \chi(c)}{(b-a)(c-b)(c-a)}$$

- So one must find which models are viable : not easy since  $a, b, c$  are unknown.
- A (strong) sufficient condition for positive  $\mathcal{E}$  would be

$\chi^{\frac{1}{3}}(x)$  is a concave function on interval  $[0, \max(a, b, c)]$

for lower  $N$ 's.

## Multiskyrmions/Simulated annealing

Need a powerful numerical approach to find the stable configuration (lowest energy)  $\implies$  **Simulated annealing**

- Advantages
  - ① Ability to avoid becoming trapped at local minima.
  - ② Minimize the energy without differential equation.
  - ③ Can easily add rotational energy or any higher derivatives term to the energy
- Disadvantages :
  - ① Heavy computations.(for large 3D lattice)
  - ② Requires some parameter adjustments : e.g. rate of decrease and initial "temperature",...

## Multiskyrmions/The procedure

Procedure :

- ① Initialize the field configuration  $C$  on each point of a lattice
- ② Make a random perturbation on the configuration  $\implies$  change in energy  $E(C_{new}) = E(C) + \Delta E$
- ③ Accept the new configuration  $C = C_{new}$  if :
  - ①  $\Delta E \leq 0$  or
  - ②  $P(\Delta E) = \exp(-\Delta E/k_B T) > \text{Random}[0, 1]$  in the case where  $\Delta E > 0$  (possibility to escape from a local minimum)

## Multiskyrmions/Parameter adjustments

Ajustments :

- ① Thermal equilibrium must be reached for each temperature during cooling.
- ② Choice of initial configuration must be appropriate (topological sector).
- ③ Initial temperature must be large enough.
- ④ Cooling should not be too fast otherwise one might be trapped into a local minimum.
- ⑤ Random perturbations should refresh all points many times.

## Multiskyrmions/Case A (Model of order 6)

- Model of order 6 in derivatives with 3 terms :

$$\mathcal{L}_1 = \text{Tr } L_\mu L^\mu \rightarrow \mathcal{E}_1 = a + b + c$$

$$\mathcal{L}_2 = \text{Tr } f_{\mu\nu} f^{\mu\nu} \rightarrow \mathcal{E}_2 = ab + bc + ca$$

$$\mathcal{L}_3 = \text{Tr } f_{\mu\nu} f^{\nu\beta} f_\beta^\mu \rightarrow \mathcal{E}_3 = 3abc$$

- Model defined by simple choice of coefficients  $h_m \rightarrow$

$$\mathcal{L} = \mathcal{L}_1 + \frac{1}{2}\mathcal{L}_2 + \frac{1}{3!}\mathcal{L}_3 \equiv \sum_{m=1}^3 h_m \mathcal{L}_m$$

such that  $\chi^{\frac{1}{3}}(x)$  is a concave. Here :

$$\chi_{O6}(x) = x + \frac{x^2}{2} + \frac{x^3}{6}$$

## Multiskyrmions/Case A (Model of order 6)

Numerical results for  $B = 1, 2, 3, 4$

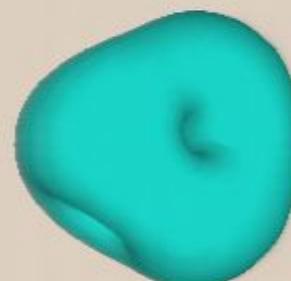
$$B_1 = 0.9984$$
$$E_1 = 130.83$$



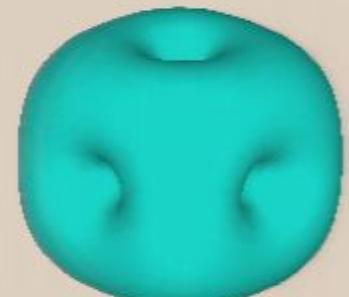
$$B_2 = 1.9995$$
$$E_2 = 245.57$$
$$E_2/E_1 = 1.8770$$



$$B_3 = 2.9997$$
$$E_3 = 354.41$$
$$E_3/E_1 = 2.7089$$



$$B_4 = 3.9997$$
$$E_4 = 455.81$$
$$E_4/E_1 = 3.4840$$



## Multiskyrmions/Case B (Model of order 8)

- Model of order 8 in derivatives with 4 terms :

$$\begin{aligned}\mathcal{E}_1 &= a + b + c & \mathcal{E}_2 &= ab + bc + ca & \mathcal{E}_3 &= 3abc \\ \mathcal{E}_4 &= \frac{2}{3}\mathcal{E}_1\mathcal{E}_3 - \mathcal{E}_2^2 = - (a^2b^2 + b^2c^2 + c^2a^2)\end{aligned}$$

- Model defined by coefficients adding a term : Static energy is

$$\mathcal{L} = \mathcal{L}_1 + \frac{1}{2}\mathcal{L}_2 + \frac{1}{3!}\mathcal{L}_3 - \frac{1}{240}\mathcal{L}_4 = \sum_{m=1}^4 h_m \mathcal{L}_m$$

such that  $\chi^{\frac{1}{3}}(x)$  is a concave. Here

$$\chi_{O8}(x) = x + \frac{x^2}{2} + \frac{x^2}{6} - \frac{x^4}{240}$$

## Multiskyrmions/Case B (Model of order 8)

Numerical results for  $B = 1, 2, 3, 4$

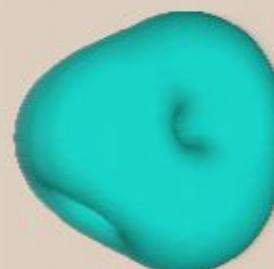
$$B_1 = 0.9995$$
$$E_1 = 130.24$$



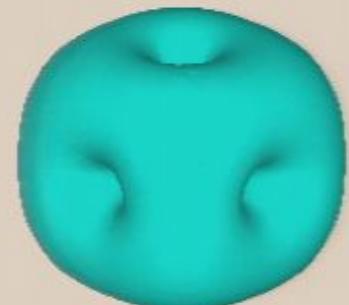
$$B_2 = 1.9998$$
$$E_2 = 244.54$$
$$E_2/E_1 = 1.8770$$



$$B_3 = 2.9999$$
$$E_3 = 352.94$$
$$E_3/E_1 = 2.7089$$



$$B_4 = 3.9997$$
$$E_4 = 454.21$$
$$E_4/E_1 = 3.4840$$



## Multiskyrmions/Case C (Rational Model)

- Consider the Rational Model (coincides with Case B up to order 8 in derivatives) :

$$\chi_R(x) = x + \frac{x^2}{6} \frac{120 + 43x}{40 + x} = \chi_{O8}(x) + O(x^{10})$$

- Using the generating function  $\mathcal{L}$  can be written in a rational form

$$\mathcal{L} = -\frac{797}{3}\mathcal{L}_1 + \frac{43}{3}\mathcal{L}_2 + 32000 \frac{1600\mathcal{L}_1 + 40(\mathcal{L}_1^2 + \mathcal{L}_2) + \mathcal{L}_3}{-192000 + 4800\mathcal{L}_1 + 120\mathcal{L}_2 + \mathcal{L}_3}$$

- Rational maps approximation cannot be used for comparison because it would require to compute an infinite number of angular integrals but



- ① consistent with energy constraint
- ② not very different from  $\chi_{O8}(x)$  so one compare with configurations and symmetries of  $\chi_{O8}(x)$ .

## Multiskyrmions/Case C (Rational Model)

Numerical results for  $B = 1, 2, 3, 4$

$$B_1 = 0.9994$$

$$E_1 = 130.16$$



$$B_2 = 1.9996$$

$$E_2 = 244.61$$

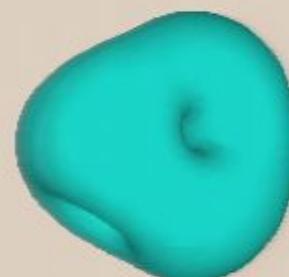
$$E_2/E_1 = 1.8793$$



$$B_3 = 2.9998$$

$$E_3 = 352.91$$

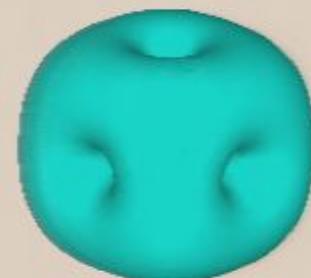
$$E_3/E_1 = 2.7114$$



$$B_4 = 3.9997$$

$$E_4 = 454.19$$

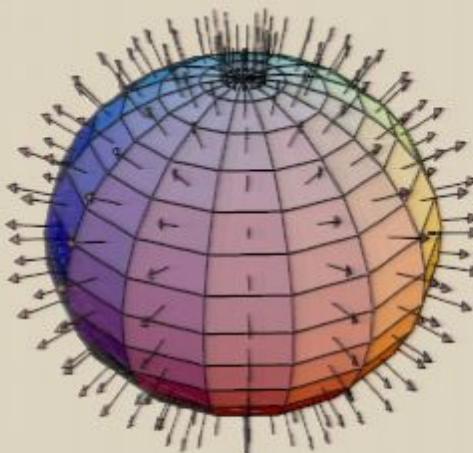
$$E_4/E_1 = 3.4895$$



## Multiskyrmions/Further investigations

- Next :
  - ① Continue search for models with solutions exhibiting different symmetries
    - ① Special class of Lagrangian may be too restrictive
    - ② Positivity constraint may be too strong
    - ③  $B \geq 5$  solutions unknown.
  - ② Include quantum effects directly in the energy and verify deformation (e.g. centrifugal effect due to rotational energy).

## Multiskyrmions/Concluding remarks



- Multiskyrmions in some extended models preserve rational-map type configuration for low  $B$ .
- Search for different configurations continues for other exotic models and deformed skyrmions
- SA should also useful for deformed Skyrmions, Skyrme-Faddeev model,...