Title: Field Theory 2

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Abstract:

The Spectrum of Yang-Mills Theory in 2+1 Dimensions

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with D. Minic and
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Theory CANADA 2
June 2006

Yang-Mills and QCD

- 't Hooft's solution in the 1970's of 1+1 QCD at large N has been tremendously important conceptually
  - followed by work of Witten in Euclidean theory and string interpretation by Gross & Taylor
  - collective field theory, matrix models, etc.
- in 2+1 dimensions, there have also been a number of 'toy models', such as lattice compact QED (Polyakov '75) and the Georgi-Glashow model (P '77)
  - explicit demonstration of confinement, condensation of magnetic monopoles
- pure Yang-Mills in 2+1
  - Feynman ('81) argued that theory <u>should</u> confine, with mass gap generated because configuration space is compact
- 3+1: similar expectations

QCD Basics

- pure Yang-Mills theory is given by the path integral

$$Z = \int \frac{[dA^a_\mu]}{Vol\ G} e^{iS_{YM}[A]}$$

with

$$S_{YM}[A] = -\frac{1}{2g^2_{YM}} \int d^{D+1}x\ tr\ F^2_{\mu\nu}$$

gauge group $SU(N)$

$$A_\mu = A^a_\mu t^a$$

$$tr\ t^a t^b = \frac{1}{2}\delta^{ab}$$

- in D=2, g^2_{YM} has units of mass, and we define

$$m = \frac{g^2_{YM} N}{2\pi}$$

't Hooft coupling

- this is the basic (bare) mass scale in the theory.
  - conceptually different than D=3, where the bare YM coupling is dimensionless and the physical mass scale is generated dynamically
    - *D=2 is simpler in this regard (and also has fewer microscopic degrees of freedom)*
  - nevertheless, D=2 is otherwise quite similar to D=3
    - *asymptotically free*
    - *believed to confine at long distances*

Phases of YM/QCD

- Short distance:
  - free theory at arbitrarily high energies
    - *perturbative regime of free massless gluons*

asymptotic freedom

- Long distance:
  - *confinement* of colour charges
    - *generation of a mass gap (no massless excitations in spectrum)*
    - *has a spectrum of gauge invariant states*
      - *in pure Yang-Mills: glueballs (~ "closed strings"?)*
      - *in QCD: glueballs, mesons, baryons*
  - Phenomenology:
    - *expect some effective* QCD *string picture*

- *this is not expected to be a "fundamental string theory" but should have features in common*

What do we want?

- the 'solution' of the theory
  - in pure Yang-Mills, compute the spectrum of glueball states and their masses
  - display important observable consequences of confinement (mass gap, area law, string tension)
- a basic problem that one has is to identify the relevant (constituent) degrees of freedom, and tractably rewrite the theory in their terms
  - gluons are appropriate near the UV fixed point, but are inconvenient elsewhere
- the physically propagating modes generically are not point-like and interact in complicated ways
  - the large N limit simplifies these properties drastically, and probably is required
- if we want to understand the spectrum of excitations, then the most fundamental object to elucidate is the *vacuum state*
  - we will discuss this in the Schrödinger picture
  - this must know about both asymptotic freedom as well as low energy confining physics

Experiment

- in 2+1 Yang-Mills, the 'experimental data' consists of a number of lattice simulations, largely by M. Teper, et al

Teper: hep-lat/9804008
Lucini & Teper: hep-lat/0206027

$m_G/\sqrt{\sigma}$						
state	SU(2)	SU(3)	SU(4)	SU(5)	SU(4)	SU(6)
0^{++}	4.716(21)	4.330(24)	4.239(34)	4.180(39)	4.235(25)	4.196(27)
0^{++*}	6.78(7)	6.485(55)	6.383(77)	6.22(8)	6.376(45)	6.20(7)
0^{++**}	8.07(10)	8.21(10)	8.12(13)	7.87(18)	7.93(7)	8.22(12)
0^{--}		6.464(48)	6.27(6)	6.06(11)	6.230(44)	6.097(80)
0^{--*}		8.14(8)	7.84(13)	7.85(15)	8.20(15)*	7.98(15)
2^{++}	7.81(6)	7.12(7)	7.14(8)	7.15(12)	7.17(8)	6.67(18)
2^{++*}			8.50(17)	8.56(15)	8.06(22)	8.89(20)
2^{--}		8.73(10)	8.25(21)	8.25(18)	8.49(13)	8.52(20)

Table 4: Glueball masses in units of the string tension, in the continuum limit. Reanalysis of [2] on left; new calculations on right. *from Lucini & Teper '02*

  - extract masses of some low lying states for smallish values of N, and extrapolate to large N

- limited data is also available for 3+1 Yang-Mills

Glueball Masses: analytic results

- we have computed these masses in 2+1 using an analytic technique, with the following results

TABLE I: 0^{++} glueball masses in QCD_3. All masses are in units of the square root of the string tension. Results of AdS/CFT computations in the supergravity limit are also given for comparison. The percent difference between our prediction and lattice data is given in the last column.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff. %
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0^{++**}	7.99 ± 0.22	9.92	6.72	16
0^{++***}	9.44 ± 0.38^a	12.80	7.99	15

[a] Mass of 0^{++***} state was computed on the lattice for $SU(2)$ only [9]. The number quoted here was obtained by a simple rescaling of $SU(2)$ result.

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from hep-th/0512111

- the results agree extremely well with the lattice data
  - analytic methods make use of a re-parameterization of the gauge fields within a Hamiltonian framework, pioneered by Karabali and Nair and Karabali, Kim and Nair
  - we have new results for the ground-state wavefunctional and simple correlators, for large N

YM in the Hamiltonian Formalism

- we consider 2+1 SU(N) Yang-Mills theory with Hamiltonian

$$\mathcal{H}_{YM} = \frac{1}{2}\int Tr\ \left(g_{YM}^2 \Pi_i^2 + \frac{1}{g_{YM}^2} B^2\right)$$

- we choose the temporal or Hamiltonian gauge, $A_0 = 0$, leaving the dynamical gauge fields A_i

$A_i = -it^a A_i^a$

- $\Pi_i \sim E_i$ is the momentum conjugate to A_i
  - $\text{quantize}: \Pi_i^a(x) \to i\frac{\delta}{\delta A_i^a(x)}, \quad \text{'position representation'}: \psi[A_i^a(x)]$
- time-independent gauge transformations preserve the gauge condition, and the spatial gauge fields transform as a connection
- Gauss' law implies that observables and physical states are gauge invariant
- hard to deal with gauge-fixing, so we would like to perform a field redefinition to gauge-invariant variables
  - traditionally, this is taken to mean Wilson loops $W_R(C) = tr_R P e^{i\oint_C A}$

A variables do not create physical excitations

Gauge Invariant Formalism

- would like to transform to gauge invariant variables $\{\Phi\}$
- path integral would transform $\rightarrow \int [d\Phi] \frac{1}{\det \frac{d\Phi}{dA}} e^{iS}$
  - the Jacobian is typically hard to compute
- a natural choice is to take variables to be Wilson loops
  - expectation value is order parameter for confinement $\langle W_R(C)\rangle \sim e^{-\sigma A + \dots}$
  - Wilson loops are a complete set of operators but are over-complete and constrained
    - *at large N, they become independent, due to factorization* $\langle \Phi\Phi \dots\rangle \longrightarrow \langle\Phi\rangle\langle\Phi\rangle \dots$
- equation of motion $\leftrightarrow$ loop equation

(Makeenko & Migdal)

  - hard to proceed
- can compute (formally!!) in Hamiltonian formalism (Sakita '80; Jevicki & Sakita '81)
  - Hamiltonian has "collective field form"
  - formally, if one knew the Jacobian, one could do a saddle point approximation, and compute
    - *validity is equivalent to large N*
  - this is essentially what we will do, in a more convenient parameterization

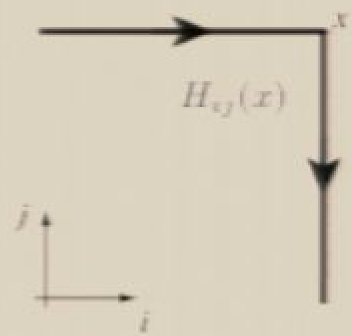
'Corner' Variables

- another possibility, initially introduced by Bars ('78), are quasi-local variables

$$A_i = -\partial_i M_i M_i^{-1} \quad \text{(no sum on } i\text{)} \qquad M_i(x) = Pexp[-\int_{-\infty}^{x} A]$$

- a gauge transformation acts as $M_i \mapsto g M_i$
- and then, local gauge-invariant variables are

$$H_{ij} = M_i^{-1} M_j$$



- (these are generally constrained)

- there is a new 'holomorphic' invariance which does not act on A_i

$$M_i \mapsto M_i\, h_i^{-1}(x^j), \quad j \neq i$$

- a regulator may be introduced which preserves this symmetry, and the Jacobian of the reparameterization computed

Karabali-Nair Formalism

- in 2+1, this was developed by Karabali and Nair, and there are some simplifications in complex spatial coordinates

$z = x_1 - ix_2, \bar{z} = x_1 - ix_2$

$A = (A_1 + iA_2)/2, A = (A_1 - iA_2)/$

- we parameterize the gauge fields as

$$A = -\partial M M^{-1}, \qquad \bar{A} = {M^\dagger}^{-1} \bar{\partial} M^\dagger$$

A traceless $\leftarrow \det M = 1$

$M \in SL(N, \mathbb{C})$

where M is complex, invertible, unimodular

- M transforms linearly under gauge transformations $M \mapsto gM$ and under holomorphic transformations $M(z, \bar{z}) \mapsto M(z, \bar{z}) h^\dagger(\bar{z})$
- there is a single unitary corner variable $H = M^\dagger M$
- the Wilson loop evaluates to

$$\Phi(C) = TrPe^{i \oint_C (A dz + \bar{A} d\bar{z})} = TrPe^{-i \oint_C dz \; \partial H H^{-1}}$$

- dependence on C is by choice; one can use the local H variables instead.
    - although Wilson loop retains its usefulness as an order parameter for confinement

The Jacobian

- now, a change of variables is not too remarkable, classically. However, in this particular case, the path integral Jacobian of the transformation can be worked out -- in fact it is given in terms of the level $-2c_A$ hermitian Wess-Zumino-Witten model

$$d\mu[C] = \sigma\, d\mu[H] e^{2c_A S_{WZW}[H]}$$

$$d\mu[H] \leftrightarrow ds^2_H = \int Tr\,(\delta H H^{-1}$$

$$S_{WZW}[H] = -\frac{1}{2\pi}\int d^2z\, Tr\, H^{-1}\partial H H^{-1}\bar{\partial} H + \frac{i}{12\pi}\int d^3x \epsilon^{\mu\nu\lambda} Tr\, H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\lambda H$$

Polyakov & Weigmann

- this is both gauge and holomorphic invariant
- thus the inner product on states can be written in the position representation as an overlap integral of gauge and holomorphic invariant wave functionals with non-trivial measure

$$\langle 1|2\rangle = \int d\mu[H] e^{2c_A S_{WZW}[H]} \Psi_1^* \Psi_2$$

- this non-trivial measure has important consequences -- e.g., $\Psi = 1$ is normalizable! (but not a solution)
  - in fact, this is an approximation to the ground-state wavefunctional

The Hamiltonian

- it is natural to introduce the 'current'

$$J = \frac{c_A}{\pi} \partial H H^{-1}$$

J is a connection for holomorphic invariance:

$$J \to hJh^{-1} + \frac{\pi}{c_A} \partial h h^{-1}$$

- the YM Hamiltonian can then be rewritten in terms of J

$$\mathcal{H}_{KN}[J] = \underbrace{m \left(\int_x J^a(x) \frac{\delta}{\delta J^a(x)} + \int_{x,y} \Omega_{ab}(x,y) \frac{\delta}{\delta J^a(x)} \frac{\delta}{\delta J^b(y)} \right)}_{=T} + \underbrace{\frac{\pi}{m c_A} \int_x \bar{\partial} J^a \bar{\partial} J^a}_{=V}$$

Karabali & Na

- at strong coupling, T dominates, whereas at weak coupling, V dominates

- the derivation of the Hamiltonian has involved a careful gauge-invariant regularization
  - this is true of all computations that we will discuss, but the details will be (mostly) suppressed

Vacuum Wavefunctional

- the vacuum wavefunctional should satisfy the Schrödinger equation

$$\mathcal{H}_{KN}\Psi_0 = E_0\Psi_0$$

- a wavefunctional in position representation may be regarded as a functional of H, or as a functional of J
  - we should also require that it be holomorphic invariant, as well as invariant under spacetime symmetries (J,P,C)
  - such a wavefunctional can be built from ∂J and $D = \partial - \frac{\pi}{c_A} J$
- if the KN Hamiltonian contained just the kinetic part, then $\Psi = 1$ would be a suitable *normalizable* solution (because of the non-trivial measure)
  - note the potential term vanishes in the limit of large g^2_{YM}
- more generally, the potential term will make a contribution
  - we will take as ansatz

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int tr\ \bar{\partial}JK(L)\bar{\partial}J + \ldots\right).$$

$L = (\partial D + D\partial)/m^2$

Schrödinger

- the Schrödinger equation takes the form

$$\mathcal{H}_{KN}\Psi_0 = \left[\ldots + \frac{\pi}{mc_A}\int tr\ \bar{\partial}J(\mathcal{R})\bar{\partial}J + \ldots\right]\Psi_0$$

(divergent) vacuum energy

- by careful computation (regularization required!) we find

$$\boxed{\mathcal{R} = -K(L) - \frac{L}{2}\frac{d}{dL}[K(L)] + LK(L)^2 + 1 = 0}$$

"Riccati diff. eq."

- this is a formal expression, obtained by regarding K as a power series in L, and computing term by term
- the boxed equation is a differential equation for K, which can be solved formally -- in fact, by a series of redefinitions, it can be cast as a Bessel eq.
    - *at small L, we should have* $K(L) \to 1$ *(confining regime)*
    - *will also obtain correct large L behaviour (asymptotic freedom)*
- in fact, *the only normalizable solution has these asymptotics*

Vacuum Wavefunctional

- the *normalizable* solution with the correct asymptotics is

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int tr\ \bar{\partial}JK(L)\bar{\partial}J + \ldots\right).$$

$p \to 0, \quad K \to 1$

$p \to \infty, \quad K \to 2m/$

$$K(L) = \frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})}$$

- this very non-trivial function interpolates between UV and IR
- the small L limit contains information about the string tension
  - indeed, because ∂J is similar to the Yang-Mills magnetic field B, and the computation of the expectation value of a spatial Wilson loop may be regarded as a computation in 2-dimensional Yang-Mills
  - one finds (correctly) $$\sqrt{\sigma} \simeq \frac{g_{YM}^2 N}{\sqrt{8\pi}}$$ $$\langle\Phi\rangle \sim \exp(-\sigma A)$$
- in the large L limit, the wavefunctional goes over to a form consistent with free gluons, with coupling g_{YM}^2

Correlation Functions

- we would like now to use this result to compute correlation functions of products of invariant operators $\langle \mathcal{O}_{-J,P,C}(\vec{x},t)\mathcal{O}_{J,P,C}(\vec{y},t)\rangle$
- at large distance, we will find contributions of single particle poles of the correct quantum numbers

$$\langle \mathcal{O}_{-J,P,C}(\vec{x},t)\mathcal{O}_{J,P,C}(\vec{y},t)\rangle \sim \frac{\#}{|x-y|}\sum_j e^{-m_j|x-y|}$$

- to find particle states of given spacetime quantum numbers, we consider operators of a suitable form

$$e.g.,\ \mathcal{O}_{0++} = tr : \bar{\partial}J\bar{\partial}J :$$

- one then finds

$$\langle tr\ \bar{\partial}J\bar{\partial}J(x)\ tr\ \bar{\partial}J\bar{\partial}J(y)\rangle \simeq K^{-2}(|x-y|)$$

- the kernel K determines physical correlators (but is not itself a propagator)
  - (we have Fourier transformed K)

0^{++} Glueballs

- using a product form of the Bessel function $J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} (1 - \frac{z^2}{\gamma_{\nu,n}^2})$

  we find
$$K^{-1}(\vec{k}) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{M_n^2}{M_n^2 + \vec{k}^2} \qquad M_n \equiv \gamma_{2,n} m/2$$

- Fourier transforming, we find a result which at long distance behaves as

$$K^{-1}(|x-y|) = -\frac{1}{4\sqrt{2\pi|x-y|}} \sum_{n=1}^{\infty} (M_n)^{3/2} e^{-M_n|x-y|}$$

- thus, we find the remarkable formula

$$\langle tr\ \bar{\partial}J\bar{\partial}J(x)\ tr\ \bar{\partial}J\bar{\partial}J(y)\rangle \simeq \sum_{m,n} \frac{\#}{|x-y|} e^{-(M_n+M_m)|x-y|}$$

  - with masses determined by the zeros of the second Bessel function

$$m_{m,n} = (\gamma_{2,m} + \gamma_{2,n})\frac{m}{2} = (\gamma_{2,m} + \gamma_{2,n})\frac{\sqrt{\sigma}}{\sqrt{2\pi}}$$

$\gamma_{2,1} = 5.14$
$\gamma_{2,2} = 8.42$
$\gamma_{2,3} = 11.62$

Comparison to Lattice

- using this result, we tabulate states

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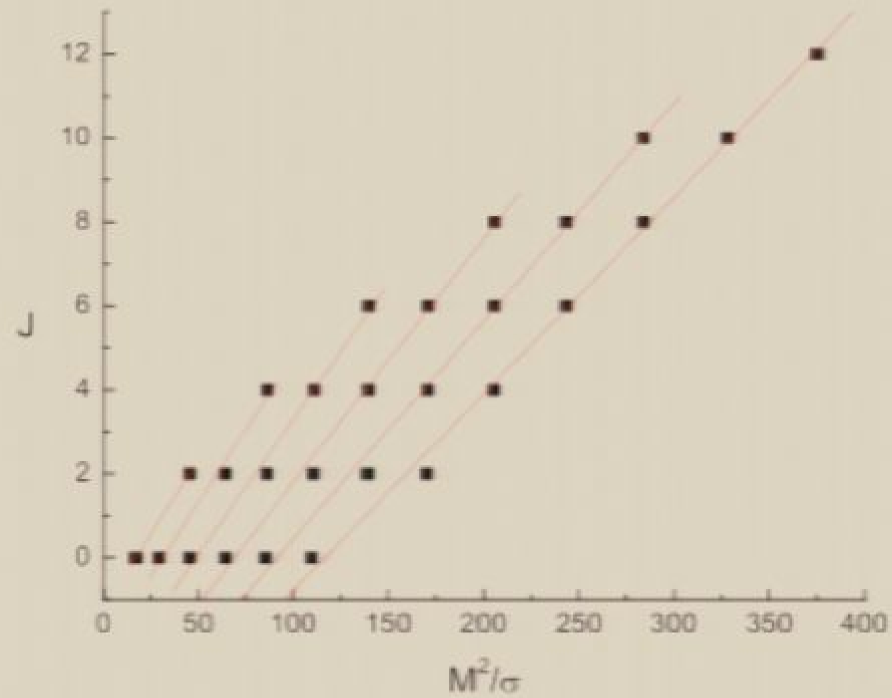
from hep-th/0512111

- the lowest lying 0^{++} state agrees very well with the lattice result
  - other 0^{++} states states fit reasonably well
    - *it is possible that the lattice results should either have larger error bars, and/or some misidentifcation has taken place.*
- results for other spin states come from correlation functions of operators with the appropriate quantum numbers

Comments on Regge Trajectories

- although there is little lattice data beyond spin 2, we can plot our states in the traditional way
  - and draw lines
  - (it's not clear whether or not these should be linear for glueballs)



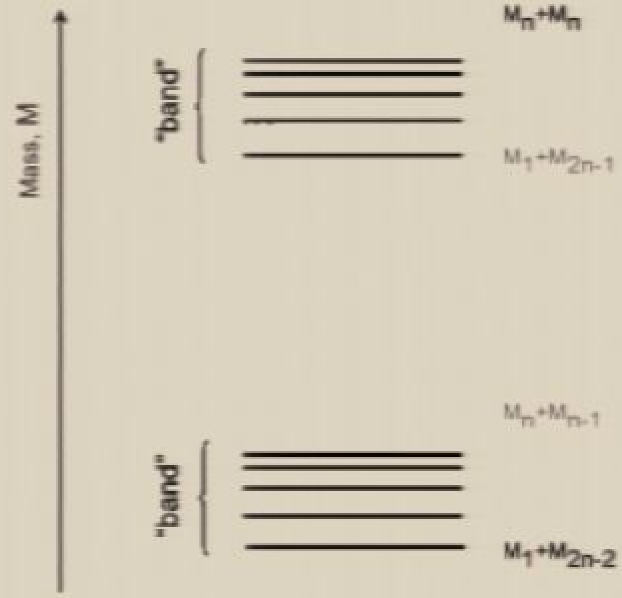
Comments on the QCD String

- the Bessel function is essentially sinusoidal, and so its zeros are evenly spaced (better for large *n*) $\gamma_{2,n} \sim \pi(n + 3/4) + \ldots$
- thus, the predicted spectrum has approximate degeneracies

$$e.g.,\ M_1 + M_5 \simeq M_2 + M_4 \simeq M_3 + M_3$$

and the spectrum is organized into bands concentrated around a given level (which are well separated)

- there is an approximate (in the sense that degeneracies are not exact) Hagedorn spectrum of states
    - degeneracies are more precise at high levels
- string-like, but not a free string
    - knows about both confinement *and* asymptotic freedom



QCD string

Final Remarks

- we have presented a vacuum wavefunctional for Yang-Mills theory which computes masses of glueball states which are tantalizingly close to large N lattice data
  - the approximations used include large N, but there is also an additional expansion (wavefunctional is quasi-Gaussian)
  - we believe this is something like the α' -expansion in string theory, or an expansion in the size of the glueballs
- the addition of quarks in the fundamental representation can be done
  - appears to follow 't Hooft's description of confinement in 1+1 fairly closely
- analogue variables are at hand for 3+1 dimensions
  - perhaps the same conceptual framework will hold
  - additional physics of dimensional transmutation, nature of 'continuum limit'

see
L. Freidel, RGL, D. Minic, hep-th/0604184
L. Freidel, hep-th/0604185