Title: Nuclear Theory/Heavy Ions 6

Date: Jun 10, 2006 11:30 AM

URL: http://pirsa.org/06060039

Abstract:

Simplifying Calculations in Real-Time Finite-Temperature Field Theory

Todd Fugleberg

Brandon University & Winnipeg Institute for Theoretical Physics

Collaborators: Margaret Carrington, Doug Pickering, Darren Irvine (Undergrad),

OUTLINE

- Review Real-Time Finite-Temperature Field Theory
  - 1/2 Representation
- Keldysh Representation
- KMS conditions
  - RA Representation
- Mathematica program
- Verifying some Ward Identities

Finite Temperature Field Theory (FTTF)

Some applications:

- heavy ion collisions
- physics of the very early Universe, $\cdots$

As in zero temperature field theory:

- start with Feynman diagram
- apply FT Feynman rules
- integrate over momenta in loops
  - contour integration

Two Formalisms:

1. Imaginary Time (IT) Formalism
   - Green functions with imaginary time arguments
     - integration contour on imaginary axis
   - multiple analytic continuations required for physical results
     - for higher n-point functions $\Rightarrow$ increasingly complex (Guerin, 1994)
2. Real Time (RT) Formalism
   - Green functions with real time arguments
     - integration contour along real axis
   - straightforward to obtain physical Green functions
   - generalizable to non-equilibrium situations

Real Time Formalism

(References: Evans '91, van Eijck & van Weert '92, Guerin '94, Carrington & Heinz '96, Wang & Heinz '98, Defu, Carrington, Kobes & Heinz '99, Carrington, Defou & Sowiak '00)

Closed Time Path (CTP) formalism: time integration contour with two branches

- one "above" the real axis running from $-\infty$ to ∞
- another "below" the real axis running from ∞ to $-\infty$

$\Longrightarrow$ doubling of degrees of freedom

n-point function

$$G^{(n)}\left(x_1, x_2 \ldots, x_n\right)_{b_1 \cdots b_n} = (-i)^{-n-1} \left\langle \mathcal{P}\left[\phi(x_1)_{b_1} \ldots \phi(x_2)_{b_n}\right]\right\rangle$$

- $b_i = 1, 2$ indicates branch of time contour
- $\mathcal{P}$ path ordering on CTP
- $\#$ of indices indicates order of Green function

2-point function: matrix

$$\mathrm{G} = G_{1/2} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

- G_{11}: propagator for fields on upper contour
- G_{12}: propagator for fields moving from the top to bottom branch, *etc.*

Real Time Formalism

(References: Evans '91, van Eijck & van Weert '92, Guerin '94, Carrington & Heinz '96, Wang & Heinz '98, Defu, Carrington, Kobes & Heinz '99, Carrington, Defou & Sowiak '00)

Closed Time Path (CTP) formalism: time integration contour with two branches

- one "above" the real axis running from $-\infty$ to ∞
- another "below" the real axis running from ∞ to $-\infty$

$\Longrightarrow$ doubling of degrees of freedom

n-point function

$$G^{(n)}(x_1, x_2 \ldots, x_n)_{b_1 \cdots b_n} = (-i)^{-n-1} \langle \mathcal{P} [\phi(x_1)_{b_1} \ldots \phi(x_2)_{b_n}] \rangle$$

- $b_i = 1, 2$ indicates branch of time contour
- $\mathcal{P}$ path ordering on CTP
- # of indices indicates order of Green function

2-point function: matrix

$$\mathrm{G} = G_{1/2} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

- G_{11}: propagator for fields on upper contour
- G_{12}: propagator for fields moving from the top to bottom branch, *etc.*

Once thought that:

- type 1 fields were "physical"
- type 2 fields were "ghost" fields

But:

$$G_{11}(k) = \theta(k_0)\Delta_F(k) + \theta(-k_0)\Delta_F^*(k) + \eta\, n(k_0)\,(\theta(k_0) - \theta(-k_0))\,[\Delta_F(k) - \Delta_F^*(k)]$$

- where $\eta = \pm 1$ for bosons and fermions
- $n(k_0) = \frac{1}{e^{-\beta k_0} - \eta}$ are thermal distribution functions (van Eijck & van Weert 1992; Aurenche & Becherrawy 1992)

Comment: only 3 independent components because:

$$\sum_{a,b=1}^{2} (-1)^{a+b} G_{ab} = G_{11} - G_{12} - G_{21} + G_{22} = 0$$

3-pt functions: $2 \times 2 \times 2$ tensor:

- 7 independent components: $G_{111}, G_{112}, G_{122}, \ldots G_{11}$

n-pt functions: $\otimes^n 2$ tensor with $2^n - 1$ independent components

Truncated Green functions: $\Gamma^{b_1 b_2 \cdots b_n}$ (vertex):

$$G_{b_1 \cdots b_n} = G_{b_1 \bar{b}_1} \cdots G_{b_n \bar{b}_n} \Gamma^{\bar{b}_1 \cdots \bar{b}_n}$$

- where $\bar{1} = 2$ and $\bar{2} = 1$

Main Point: Propagators and vertices are tensors

Real Time Formalism

(References: Evans '91, van Eijck & van Weert '92, Guerin '94, Carrington & Heinz '96, Wang & Heinz '98, Defu, Carrington, Kobes & Heinz '99, Carrington, Defou & Sowiak '00)

Closed Time Path (CTP) formalism: time integration contour with two branches

- one "above" the real axis running from $-\infty$ to ∞
- another "below" the real axis running from ∞ to $-\infty$

$\Longrightarrow$ doubling of degrees of freedom

n-point function

$$G^{(n)}(x_1, x_2 \ldots, x_n)_{b_1 \cdots b_n} = (-i)^{-n-1} \langle \mathcal{P} [\phi(x_1)_{b_1} \ldots \phi(x_2)_{b_n}] \rangle$$

- $b_i = 1, 2$ indicates branch of time contour
- $\mathcal{P}$ path ordering on CTP
- $\#$ of indices indicates order of Green function

2-point function: matrix

$$\mathrm{G} = G_{1/2} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

- G_{11}: propagator for fields on upper contour
- G_{12}: propagator for fields moving from the top to bottom branch, *etc.*

Once thought that:

- type 1 fields were "physical"
- type 2 fields were "ghost" fields

But:

$$G_{11}(k) = \theta(k_0)\Delta_F(k) + \theta(-k_0)\Delta_F^*(k) + \eta\, n(k_0)\left(\theta(k_0) - \theta(-k_0)\right)\left[\Delta_F(k) - \Delta_F^*(k)\right]$$

- where $\eta = \pm 1$ for bosons and fermions
- $n(k_0) = \frac{1}{e^{-\beta k_0} - \eta}$ are thermal distribution functions (van Eijck & van Weert 1992; Aurenche & Becherrawy 1992)

Comment: only 3 independent components because:

$$\sum_{a,b=1}^{2} (-1)^{a+b} G_{ab} = G_{11} - G_{12} - G_{21} + G_{22} = 0$$

3-pt functions: $2 \times 2 \times 2$ tensor:

- 7 independent components: $G_{111}, G_{112}, G_{122}, \ldots G_{11}$

n-pt functions: $\otimes^n 2$ tensor with $2^n - 1$ independent components

Truncated Green functions: $\Gamma^{b_1 b_2 \cdots b_n}$ (vertex):

$$G_{b_1 \cdots b_n} = G_{b_1 \bar{b}_1} \cdots G_{b_n \bar{b}_n} \Gamma^{\bar{b}_1 \cdots \bar{b}_n}$$

– where $\bar{1} = 2$ and $\bar{2} = 1$

Main Point: Propagators and vertices are tensors

Real Time Formalism

(References: Evans '91, van Eijck & van Weert '92, Guerin '94, Carrington & Heinz '96, Wang & Heinz '98, Defu, Carrington, Kobes & Heinz '99, Carrington, Defou & Sowiak '00)

Closed Time Path (CTP) formalism: time integration contour with two branches

- one "above" the real axis running from $-\infty$ to ∞
- another "below" the real axis running from ∞ to $-\infty$

$\Longrightarrow$ doubling of degrees of freedom

n-point function

$$G^{(n)}\left(x_1, x_2 \ldots, x_n\right)_{b_1 \cdots b_n} = (-i)^{-n-1} \left\langle \mathcal{P}\left[\phi(x_1)_{b_1} \ldots \phi(x_2)_{b_n}\right]\right\rangle$$

- $b_i = 1, 2$ indicates branch of time contour
- $\mathcal{P}$ path ordering on CTP
- # of indices indicates order of Green function

2-point function: matrix

$$\mathbf{G} = G_{1/2} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

- G_{11}: propagator for fields on upper contour
- G_{12}: propagator for fields moving from the top to bottom branch, *etc.*

Once thought that:

- type 1 fields were "physical"
- type 2 fields were "ghost" fields

But:

$$G_{11}(k) = \theta(k_0)\Delta_F(k) + \theta(-k_0)\Delta_F^*(k) + \eta\, n(k_0)\,(\theta(k_0) - \theta(-k_0))\,[\Delta_F(k) - \Delta_F^*(k)]$$

- where $\eta = \pm 1$ for bosons and fermions
- $n(k_0) = \frac{1}{e^{-\beta k_0} - \eta}$ are thermal distribution functions (van Eijck & van Weert 1992; Aurenche & Becherrawy 1992)

Comment: only 3 independent components because:

$$\sum_{a,b=1}^{2} (-1)^{a+b} G_{ab} = G_{11} - G_{12} - G_{21} + G_{22} = 0$$

3-pt functions: $2 \times 2 \times 2$ tensor:

- 7 independent components: $G_{111}, G_{112}, G_{122}, \ldots G_{11}$

n-pt functions: $\otimes^n 2$ tensor with $2^n - 1$ independent components

Truncated Green functions: $\Gamma^{b_1 b_2 \cdots b_n}$ (vertex):

$$G_{b_1 \cdots b_n} = G_{b_1 \bar{b}_1} \cdots G_{b_n \bar{b}_n} \Gamma^{\bar{b}_1 \cdots \bar{b}_n}$$

  - where $\bar{1} = 2$ and $\bar{2} = 1$

Main Point: Propagators and vertices are tensors

Calculating Diagrams

- tensor product of propagators and vertices

Ex) truncated 2-pt diagram:

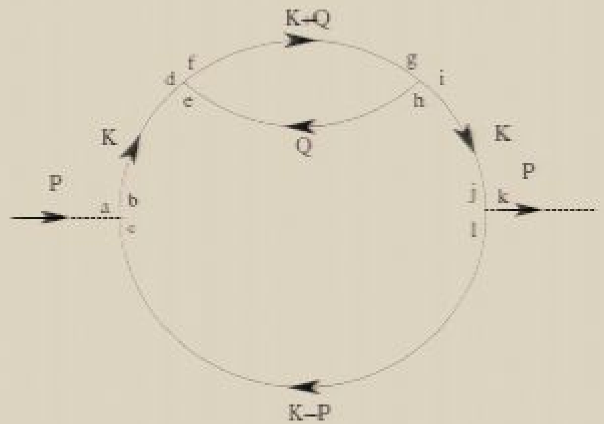


Figure 1: Typical Amplitude

can be expressed as:

$$\Gamma^{ak} \sim \int dK dQ \, \mathbf{g}^{abc} G_{bd}(K) \, \mathbf{g}^{def} D_{fg}(K+Q) \, \mathbf{g}^{ghi} D_{he}(Q) D_{ij}(k) \, \mathbf{g}^{jkl} D_{lc}(K-P).$$

where

- $\mathbf{g}^{abc}$: bare vertex
- Γ^{ak} components of a tensor
- raised-lowered pairs of indices summed over

Calculations quickly become very complicated:

- each component has $\left(2^{\sum_{vertices}(\#\, internal\ legs)}\right)$ terms
- however, actually only a small # contribute to final result

Keldysh Representation (Keldysh, 1965)

Transform Green functions to Keldysh basis using the tranformation matrix (Gelis, 2002):

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{1}$$

in the following way:

$$G_{\alpha_1 \cdots \alpha_n} = \sum_{\alpha_i = r}^{a} U_{\alpha_i}{}^{b_i} G_{b_1 \cdots b_n}$$

- where $\alpha_i = r, a$

Advantage #1

Components of propagator are familar retarded, advanced and symmetric Green functions:

$$\mathbf{G} = G_{Keldysh} = \begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & 0 \end{pmatrix} = \begin{pmatrix} \Delta_F & \Delta_R \\ \Delta_R & 0 \end{pmatrix}$$

Advantage #2

Only $2^n - 1$ explicit components

- $G_{a \cdots a} = 0$ for arbitary $n-$point functions
- $\Gamma^{r \cdots r} = 0$ for vertex functions
- fewer terms in Feynman diagram

Advantage #3

Any product of all retarded or all advanced propagators around a closed loop will vanish

- from contour integration
- eliminates large number of terms

Advantage #4: At thermal equilibrium:

- relations between RT Green functions in this basis and analytic continuations of corresponding IT amplitude (Evans 1991, Aurenche & Becherrawy 1992, Guerin 1994)

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

- first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

- $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta \, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned} G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\ G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\ G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa} \end{aligned}$$

Keldysh Representation (Keldysh, 1965)

Transform Green functions to Keldysh basis using the tranformation matrix (Gelis, 2002):

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (1)$$

in the following way:

$$G_{\alpha_1 \cdots \alpha_n} = \sum_{\alpha_i = r}^{a} U_{\alpha_i}{}^{b_i} G_{b_1 \cdots b_n}$$

– where $\alpha_i = r, a$

Advantage #1

Components of propagator are familar retarded, advanced and symmetric Green functions:

$$\mathbf{G} = G_{Keldysh} = \begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & 0 \end{pmatrix} = \begin{pmatrix} \Delta_F & \Delta_R \\ \Delta_R & 0 \end{pmatrix}$$

Advantage #2

Only $2^n - 1$ explicit components

– $G_{a \cdots a} = 0$ for arbitary $n-$point functions
– $\Gamma^{r \cdots r} = 0$ for vertex functions
– fewer terms in Feynman diagram

Advantage #3

Any product of all retarded or all advanced propagators around a closed loop will vanish

– from contour integration
– eliminates large number of terms

Advantage #4: At thermal equilibrium:

- relations between RT Green functions in this basis and analytic continuations of corresponding IT amplitude (Evans 1991, Aurenche & Becherrawy 1992, Guerin 1994)

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

– first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

– $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta \, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned} G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\ G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\ G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa} \end{aligned}$$

Keldysh Representation (Keldysh, 1965)

Transform Green functions to Keldysh basis using the tranformation matrix (Gelis, 2002):

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

in the following way:

$$G_{\alpha_1 \cdots \alpha_n} = \sum_{\alpha_i = r}^{a} U_{\alpha_i}{}^{b_i} G_{b_1 \cdots b_n}$$

- where $\alpha_i = r, a$

Advantage #1

Components of propagator are familar retarded, advanced and symmetric Green functions:

$$\mathbf{G} = G_{Keldysh} = \begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & 0 \end{pmatrix} = \begin{pmatrix} \Delta_F & \Delta_R \\ \Delta_R & 0 \end{pmatrix}$$

Advantage #2

Only $2^n - 1$ explicit components

- $G_{a \cdots a} = 0$ for arbitary $n-$point functions
- $\Gamma^{r \cdots r} = 0$ for vertex functions
- fewer terms in Feynman diagram

Advantage #3

Any product of all retarded or all advanced propagators around a closed loop will vanish

- from contour integration
- eliminates large number of terms

Advantage #4: At thermal equilibrium:

- relations between RT Green functions in this basis and analytic continuations of corresponding IT amplitude (Evans 1991, Aurenche & Becherrawy 1992, Guerin 1994)

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

– first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

– $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta \, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned}
G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\
G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\
G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa}
\end{aligned}$$

n-pt KMS Conditions

Defining:

$$
\begin{aligned}
C(k_1) &= 1 \\
C(k_1, k_2) &= N_1 + N_2 \\
C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\
&\vdots
\end{aligned}
$$

General expression for KMS conditions:

$$
C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}
$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \leftrightarrow r$
  - d) add in all terms with r's $\rightarrow$ a's together with factors of N_{k_i} for each α_i flipped from r to a

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

– first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

– $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta\, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned}
G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\
G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\
G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa}
\end{aligned}$$

n-pt KMS Conditions

Defining:

$$\begin{aligned} C(k_1) &= 1 \\ C(k_1, k_2) &= N_1 + N_2 \\ C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\ &\vdots \end{aligned}$$

General expression for KMS conditions:

$$C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^*_{\alpha_1 \cdots \alpha_n}$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \to r$
  - d) add in all terms with r's $\to$ a's together with factors of N_{k_i} for each α_i flipped from r to a

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

- first expression simplifies to:

$$G_{rr} = N_1 \ [G_{ra} - G_{ar}]$$

- $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta\, n(p_i)$$

> Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned} G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\ G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\ G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa} \end{aligned}$$

n-pt KMS Conditions

Defining:

$$\begin{aligned} C(k_1) &= 1 \\ C(k_1, k_2) &= N_1 + N_2 \\ C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\ &\vdots \end{aligned}$$

General expression for KMS conditions:

$$C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \longrightarrow r$
  - d) add in all terms with r's $\longrightarrow$ a's together with factors of N_{k_i} for each α_i flipped from r to a

Analogous conditions for vertex functions with $r \to a$ and $a \to r$

Ex)

$$\Gamma^{aaa} + N_1\Gamma^{raa} + N_2\Gamma^{ara} + N_3\Gamma^{aar} + N_1N_2\Gamma^{rra} + N_1N_3\Gamma^{rar} + N_2N_3\Gamma^{arr} = 0$$

$$\begin{aligned}
\Gamma^{aar} + N_1\Gamma^{rar} + N_2\Gamma^{arr} &= (N_1 + N_2)\,(\Gamma^{rra})^* \\
\Gamma^{ara} + N_1\Gamma^{rra} + N_3\Gamma^{arr} &= (N_1 + N_3)\,(\Gamma^{rar})^* \\
\Gamma^{raa} + N_2\Gamma^{rra} + N_3\Gamma^{rar} &= (N_2 + N_3)\,(\Gamma^{arr})^*
\end{aligned}$$

n-pt KMS Conditions

Defining:

$$
\begin{aligned}
C(k_1) &= 1 \\
C(k_1, k_2) &= N_1 + N_2 \\
C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\
&\vdots
\end{aligned}
$$

General expression for KMS conditions:

$$
C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}
$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \rightarrow r$
  - d) add in all terms with r's $\rightarrow$ a's together with factors of N_{k_i} for each α_i flipped from r to a

Analogous conditions for vertex functions with $r \to a$ and $a \to r$

Ex)

$$\Gamma^{aaa} + N_1\Gamma^{raa} + N_2\Gamma^{ara} + N_3\Gamma^{aar} + N_1N_2\Gamma^{rra} + N_1N_3\Gamma^{rar} + N_2N_3\Gamma^{arr} = 0$$

$$\begin{aligned}
\Gamma^{aar} + N_1\Gamma^{rar} + N_2\Gamma^{arr} &= (N_1 + N_2)\,(\Gamma^{rra})^* \\
\Gamma^{ara} + N_1\Gamma^{rra} + N_3\Gamma^{arr} &= (N_1 + N_3)\,(\Gamma^{rar})^* \\
\Gamma^{raa} + N_2\Gamma^{rra} + N_3\Gamma^{rar} &= (N_2 + N_3)\,(\Gamma^{arr})^*
\end{aligned}$$

n-pt KMS Conditions

Defining:

$$
\begin{aligned}
C(k_1) &= 1 \\
C(k_1, k_2) &= N_1 + N_2 \\
C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\
&\vdots
\end{aligned}
$$

General expression for KMS conditions:

$$
C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}
$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \leftrightarrow r$
  - d) add in all terms with r's $\rightarrow$ a's together with factors of N_{k_i} for each α_i flipped from r to a

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

- first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

  - $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta\, n(p_i)$$

> Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned} G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\ G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\ G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa} \end{aligned}$$

n-pt KMS Conditions

Defining:

$$\begin{aligned} C(k_1) &= 1 \\ C(k_1, k_2) &= N_1 + N_2 \\ C(k_1, k_2, k_2) &= 1 + N_1 N_2 + N_1 N_3 + N_2 N_3 \\ &\vdots \end{aligned}$$

General expression for KMS conditions:

$$C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k r} + \delta_{X_k a} \delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k) \delta_{X_k a} + \delta_{X_k r} \delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \leftrightarrow r$
  - d) add in all terms with r's $\rightarrow$ a's together with factors of N_{k_i} for each α_i flipped from r to a

Analogous conditions for vertex functions with $r \to a$ and $a \to r$

Ex)

$$\Gamma^{aaa} + N_1\Gamma^{raa} + N_2\Gamma^{ara} + N_3\Gamma^{aar} + N_1N_2\Gamma^{rra} + N_1N_3\Gamma^{rar} + N_2N_3\Gamma^{arr} = 0$$

$$\begin{aligned}
\Gamma^{aar} + N_1\Gamma^{rar} + N_2\Gamma^{arr} &= (N_1 + N_2)\,(\Gamma^{rra})^* \\
\Gamma^{ara} + N_1\Gamma^{rra} + N_3\Gamma^{arr} &= (N_1 + N_3)\,(\Gamma^{rar})^* \\
\Gamma^{raa} + N_2\Gamma^{rra} + N_3\Gamma^{rar} &= (N_2 + N_3)\,(\Gamma^{arr})^*
\end{aligned}$$

R/A Basis

Rotate from Keldysh to R/A basis using transformation matrix (Gelis 2003):

$$U(k_i) = \begin{pmatrix} 0 & -\sqrt{2}\, n(k_i) \\ -\frac{1}{\sqrt{2}\, n(-k_i)} & -\frac{N_i}{\sqrt{2}\, n_B(-k_i)}. \end{pmatrix}$$

- at thermal equilibrium propagator becomes:

$$\mathbf{G} = G_{R/A} = \begin{pmatrix} 0 & G_{RA} \\ G_{AR} & 0 \end{pmatrix}$$

$$G_{RA} = G_{ar} = G_{ra}^{*} = G_{AR}^{*}$$

3-pt functions

$$\begin{aligned}
G_{RRR} &= 0 \\
G_{RRA} &= -\frac{(N_1+N_2)}{\sqrt{2}}\, G_{aar} \\
G_{RAR} &= -\frac{(N_1+N_3)}{\sqrt{2}}\, G_{ara} \\
G_{ARR} &= -\frac{(N_2+N_3)}{\sqrt{2}}\, G_{raa} \\
G_{AAR} &= \frac{\sqrt{2}}{(N_1+N_2)}\,(G_{rra} + N_1 G_{ara} + N_2 G_{raa}) \\
G_{ARA} &= \frac{\sqrt{2}}{(N_1+N_3)}\,(G_{rar} + N_1 G_{aar} + N_3 G_{raa}) \\
G_{RAA} &= \frac{\sqrt{2}}{(N_2+N_3)}\,(G_{arr} + N_2 G_{aar} + N_3 G_{ara}) \\
G_{AAA} &= 0
\end{aligned}$$

Advantages

1) Simplifies the diagram still further
2) KMS conditions are particularly simple

Analogous conditions for vertex functions with $r \to a$ and $a \to r$

Ex)

$$\Gamma^{aaa} + N_1 \Gamma^{raa} + N_2 \Gamma^{ara} + N_3 \Gamma^{aar} + N_1 N_2 \Gamma^{rra} + N_1 N_3 \Gamma^{rar} + N_2 N_3 \Gamma^{arr} = 0$$

$$\begin{aligned}
\Gamma^{aar} + N_1 \Gamma^{rar} + N_2 \Gamma^{arr} &= (N_1 + N_2) \left(\Gamma^{rra}\right)^* \\
\Gamma^{ara} + N_1 \Gamma^{rra} + N_3 \Gamma^{arr} &= (N_1 + N_3) \left(\Gamma^{rar}\right)^* \\
\Gamma^{raa} + N_2 \Gamma^{rra} + N_3 \Gamma^{rar} &= (N_2 + N_3) \left(\Gamma^{arr}\right)^*
\end{aligned}$$

n-pt KMS Conditions

Defining:

$$\begin{aligned} C(k_1) &= 1 \\ C(k_1, k_2) &= N_1 + N_2 \\ C(k_1, k_2, k_2) &= 1 + N_1N_2 + N_1N_3 + N_2N_3 \\ &\vdots \end{aligned}$$

General expression for KMS conditions:

$$C(N_p \mid X_p = a) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k)\delta_{X_k r} + \delta_{X_k a}\delta_{\alpha_k a} \right] \right) G_{\alpha_1 \cdots \alpha_n} = C(N_p \mid X_p = r) \sum_{\alpha_j \in \{r,a\}} \left(\prod_{k=1}^{n} \left[\tilde{N}(\alpha_k)\delta_{X_k a} + \delta_{X_k r}\delta_{\alpha_k a} \right] \right) G^{*}_{\alpha_1 \cdots \alpha_n}$$

Main Points

- $2^{n-1} - 1$ independent components
- general formula lends itself to a easier rules of thumb
  - a) start with complex conjugate of $n-$point function with more a indices on RHS
  - b) include factor of $C(k_1, \cdots, k_m)$ on RHS with momenta corresponding to each a index
  - c) on LHS start with $n-$pt function with $a \leftrightarrow r$
  - d) add in all terms with r's $\to$ a's together with factors of N_{k_i} for each α_i flipped from r to a

KMS Conditions

At thermal equilibrium:

- components of n−point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

– first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

– $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta\, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned} G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\ G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\ G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa} \end{aligned}$$

Analogous conditions for vertex functions with $r \to a$ and $a \to r$

Ex)

$$\Gamma^{aaa} + N_1\Gamma^{raa} + N_2\Gamma^{ara} + N_3\Gamma^{aar} + N_1N_2\Gamma^{rra} + N_1N_3\Gamma^{rar} + N_2N_3\Gamma^{arr} = 0$$

$$\begin{aligned}
\Gamma^{aar} + N_1\Gamma^{rar} + N_2\Gamma^{arr} &= (N_1 + N_2)\,(\Gamma^{rra})^* \\
\Gamma^{ara} + N_1\Gamma^{rra} + N_3\Gamma^{arr} &= (N_1 + N_3)\,(\Gamma^{rar})^* \\
\Gamma^{raa} + N_2\Gamma^{rra} + N_3\Gamma^{rar} &= (N_2 + N_3)\,(\Gamma^{arr})^*
\end{aligned}$$

R/A Basis

Rotate from Keldysh to R/A basis using transformation matrix (Gelis 2003):

$$U(k_i) = \begin{pmatrix} 0 & -\sqrt{2}\, n(k_i) \\ -\frac{1}{\sqrt{2}\, n(-k_i)} & -\frac{N_i}{\sqrt{2}\, n_B(-k_i)}, \end{pmatrix}$$

- at thermal equilibrium propagator becomes:

$$\mathbf{G} = G_{R/A} = \begin{pmatrix} 0 & G_{RA} \\ G_{AR} & 0 \end{pmatrix}$$

$$G_{RA} = G_{ar} = G^*_{ra} = G^*_{AR}$$

3-pt functions

$$\begin{aligned}
G_{RRR} &= 0 \\
G_{RRA} &= -\frac{(N_1+N_2)}{\sqrt{2}}\, G_{aar} \\
G_{RAR} &= -\frac{(N_1+N_3)}{\sqrt{2}}\, G_{ara} \\
G_{ARR} &= -\frac{(N_2+N_3)}{\sqrt{2}}\, G_{raa} \\
G_{AAR} &= \frac{\sqrt{2}}{(N_1+N_2)} (G_{rra} + N_1 G_{ara} + N_2 G_{raa}) \\
G_{ARA} &= \frac{\sqrt{2}}{(N_1+N_3)} (G_{rar} + N_1 G_{aar} + N_3 G_{raa}) \\
G_{RAA} &= \frac{\sqrt{2}}{(N_2+N_3)} (G_{arr} + N_2 G_{aar} + N_3 G_{ara}) \\
G_{AAA} &= 0
\end{aligned}$$

Advantages

1) Simplifies the diagram still further
2) KMS conditions are particularly simple

KMS Conditions

At thermal equilibrium:

- components of $n-$point amplitudes must satisfy a set of relations $\Rightarrow$ KMS conditions (Kubo '57 '66, Martin & Schwinger '59)

2-pt KMS Conditions

$$(G_{rr} + N_1 G_{ar} + N_2 G_{ra}) = 0$$
$$G_{ra} = G^*_{ar}$$

- first expression simplifies to:

$$G_{rr} = N_1 \; [G_{ra} - G_{ar}]$$

  - $N_i = N(p_i)$ where p_i is incoming momentum on that leg and:

$$N(p_i) = 1 + 2\eta \, n(p_i)$$

Main Point: Only one independent component of propagator

3-pt KMS Conditions

$$G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa} = 0$$

$$\begin{aligned}
G_{rra} + N_1 G_{ara} + N_2 G_{raa} &= (N_1 + N_2) G^*_{aar} \\
G_{rar} + N_1 G_{aar} + N_3 G_{raa} &= (N_1 + N_3) G^*_{ara} \\
G_{arr} + N_2 G_{aar} + N_3 G_{ara} &= (N_2 + N_3) G^*_{raa}
\end{aligned}$$

R/A Basis

Rotate from Keldysh to R/A basis using transformation matrix (Gelis 2003):

$$U(k_i) = \begin{pmatrix} 0 & -\sqrt{2}\,n(k_i) \\ -\frac{1}{\sqrt{2}\,n(-k_i)} & -\frac{N_i}{\sqrt{2}\,n_B(-k_i)}. \end{pmatrix}$$

- at thermal equilibrium propagator becomes:

$$\mathbf{G} = G_{R/A} = \begin{pmatrix} 0 & G_{RA} \\ G_{AR} & 0 \end{pmatrix}$$

$$G_{RA} = G_{ar} = G^*_{ra} = G^*_{AR}$$

3-pt functions

$$\begin{aligned}
G_{RRR} &= 0 \\
G_{RRA} &= -\frac{(N_1+N_2)}{\sqrt{2}} G_{aar} \\
G_{RAR} &= -\frac{(N_1+N_3)}{\sqrt{2}} G_{ara} \\
G_{ARR} &= -\frac{(N_2+N_3)}{\sqrt{2}} G_{raa} \\
G_{AAR} &= \frac{\sqrt{2}}{(N_1+N_2)}(G_{rra} + N_1 G_{ara} + N_2 G_{raa}) \\
G_{ARA} &= \frac{\sqrt{2}}{(N_1+N_3)}(G_{rar} + N_1 G_{aar} + N_3 G_{raa}) \\
G_{RAA} &= \frac{\sqrt{2}}{(N_2+N_3)}(G_{arr} + N_2 G_{aar} + N_3 G_{ara}) \\
G_{AAA} &= 0
\end{aligned}$$

Advantages

1) Simplifies the diagram still further
2) KMS conditions are particularly simple

2-pt KMS conditions:

$$G_{AR} = G^*_{RA} \qquad G_{AA} = 0$$

3-pt KMS conditions:

$$\begin{aligned}
G_{AAA} &= 0 \\
G_{RRA} &= -\frac{(N_1 + N_2)}{2} G^*_{AAR} \\
G_{RAR} &= -\frac{(N_1 + N_3)}{2} G^*_{ARA} \\
G_{ARR} &= -\frac{(N_2 + N_3)}{2} G^*_{RAA}
\end{aligned}$$

Analogous rules for vertices:

- with $R \leftrightarrow A$

$$\begin{aligned}
\Gamma^{RRR} &= 0 \\
\Gamma^{AAR} &= -\frac{(N_1 + N_2)}{2} \left(\Gamma^{AAR}\right)^* \\
\Gamma^{ARA} &= -\frac{(N_1 + N_3)}{2} \left(\Gamma^{RAR}\right)^* \\
\Gamma^{RAA} &= -\frac{(N_2 + N_3)}{2} \left(\Gamma^{ARR}\right)^*
\end{aligned}$$

Mathematica Program

- Expressions for an arbitrary diagram at equilibrium simplest when written in the R/A basis
- Setting up calculation of an arbitrary diagram just a tensor product
  - done quickly and efficiently using Mathematica

Mathematica program will:

- set up calculation of diagrams in any basis (1/2, Keldysh, R/A)
- in or out of equilibrium
- bare vertices or dressed vertices of arbitary order
- can be used for QED diagrams
  - $n(k_0)$ — $n_B(k_0)$ or $n_F(k_0)$
  - Lorentz and Dirac structure is suppressed

Using the program:

1) Draw desired diagram
2) Label all momenta with upper case letters
3) Label all legs of every vertex with lower case indices
4) Enter the momenta and indices in appropiate place of an input cell
   - external indices and momenta
   - indices and momenta of bare 3-pt vertices
   - indices and momenta of dressed (3,4,5,n)-pt vertices
   - indices and momenta of propagators
5) Choose among output options the input cell
6) Evaluate the entire notebook

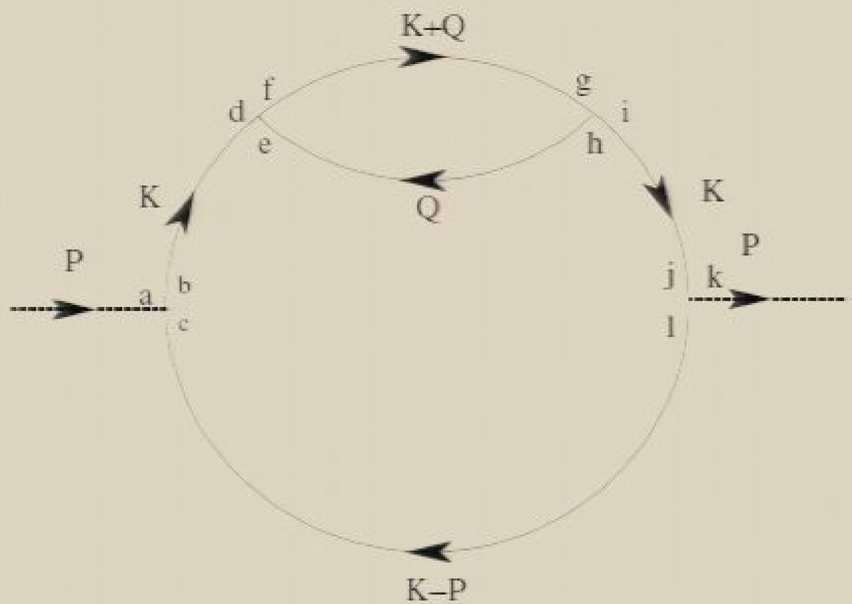


Figure 2: Diagram with indices

Three-point Ward Identity

A Ward Identity in QED for the three-point function to 1-loop order is:

$$Q^{\mu}\Gamma^{(1)}_{\mu,arr}\left[P,Q,-P-Q\right] = \Gamma^{(1)}_{ar}\left[P+Q\right] - \Gamma^{(1)}_{ar}\left[P\right]$$

where:

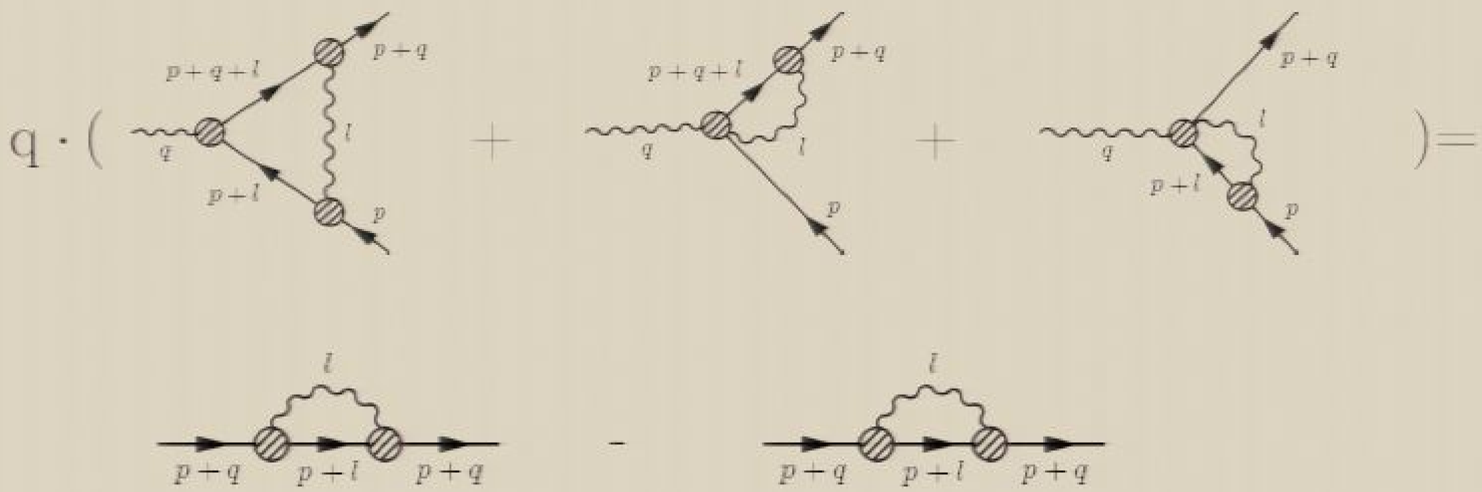
- $\Gamma^{(1)}_{\mu,arr}\left[P,Q,-P-Q\right]$ a three-point vertex function with:
  - incoming fermion fermion momentum P and incoming photon momentum Q^{μ}
- $\Gamma^{(1)}_{ar}\left[P\right]$ is a two-point vertex function (*ie.* inverse propagator)

Important Comment

The results of these calculations are expected and very well understood

- done for demonstration purposes and as a test of the program

Diagrammatically:

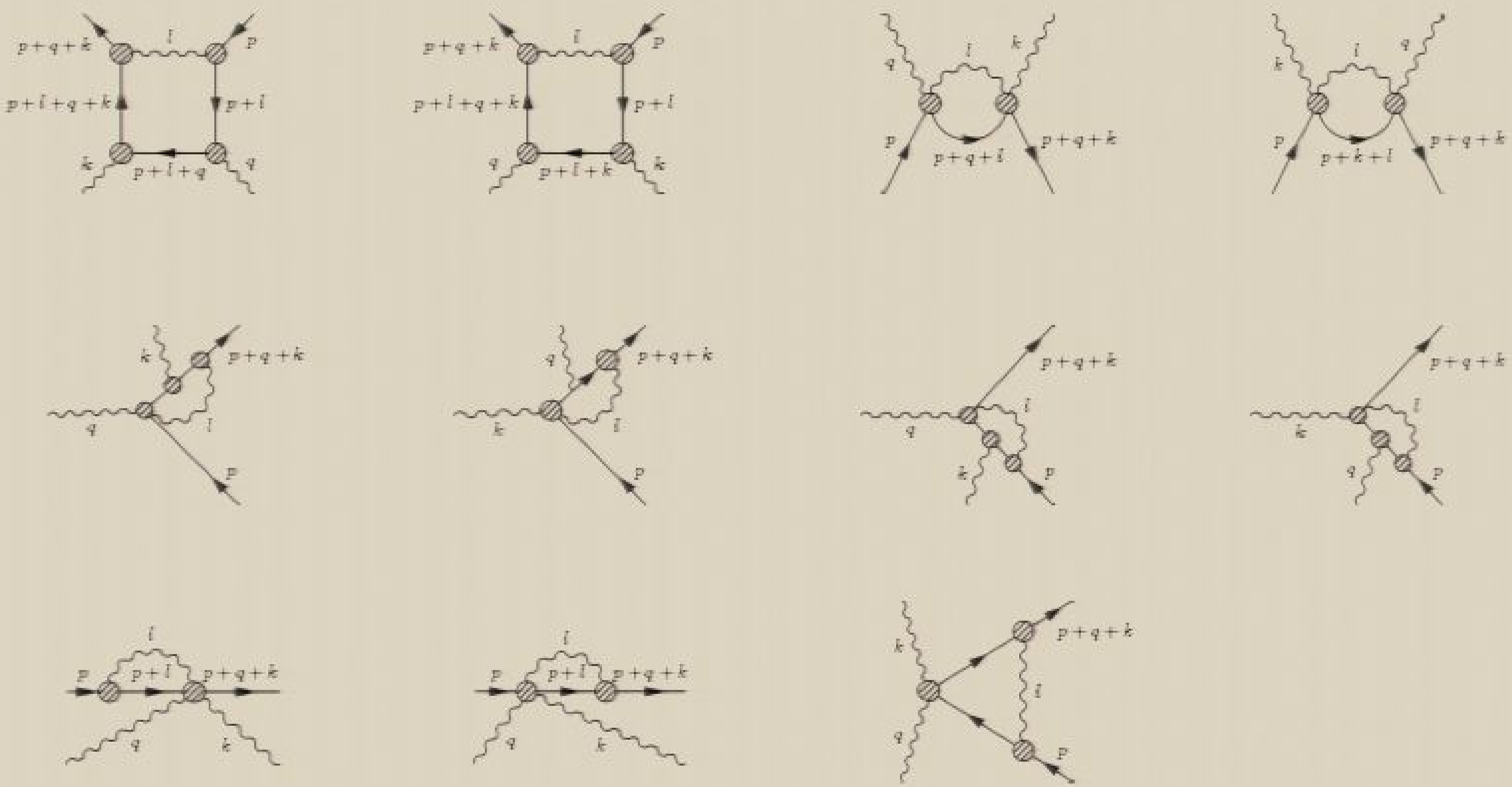


Four-point Ward Identity

A Ward Identity in QED for the 4-point function to 1-loop order is:

$$Q^{\mu}\Gamma^{(1)}_{\mu,rarr}\left[P,Q,K,-P-Q-K\right] = \Gamma^{(1)}_{rra}\left[P,K,P+K\right] - \Gamma^{(1)}_{arr}\left[P+Q,K,P+Q+K\right]$$

- 4-point diagrams to consider are:



Calculating these diagrams and contracting with q :

$$
\begin{aligned}
\Gamma_{\text{seagull-1}}[2,p,q,k,u] &= \alpha[2] + A[2] & \Gamma_{\text{seagull-2}}[2,p,q,k,u] &= \beta[2] + B[2] \\
\Gamma_{\text{leaftail-1}}[2,p,q,k,u] &= \gamma[2] + C[2] & \Gamma_{\text{leaftail-3}}[2,p,q,k,u] &= \delta[2] + D[2] \\
\Gamma_{\text{jellyfish-1}}[2,p,q,k,u] &= \epsilon[2] + E[2] & \Gamma_{\text{jellyfish-2}}[2,p,q,k,u] &= \phi[2] + F[2] \\
\Gamma_{\text{leaftail-2}}[2,p,q,k,u] &= -B[2] - F[2] & \Gamma_{\text{leaftail-4}}[2,p,q,k,u] &= -A[2] - E[2] \\
\Gamma_{\text{box}}[2,p,q,k,u] &= -G[2] - C[2] & \Gamma_{\text{crossed-box}}[2,p,q,k,u] &= -H[2] - D[2] \\
\Gamma_{\text{polywog}}[2,p,q,k,u] &= G[2] + H[2]
\end{aligned}
\tag{2}
$$

where:

$$
\begin{aligned}
\alpha[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[-a(l)r(l+s)\Gamma(2,s,l+s)(M(6)(l+s,k,-l,u) + M(3)(l+s,k,-l,u)N_B(l+s) - M(1)(l+s,k,-l,u)N_F(l))] \\
\beta[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+t)M(1)(p,k,l,l+t)(\Gamma(6,l+t,t) + \Gamma(1,l+t,t)N_B(l+t) - \Gamma(2,l+t,t)N_F(l)) \\
\gamma[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+s)\Gamma(2,s,l+s)\,(a(l+u)\Gamma(1,l+u,u)(-\Gamma(4,l+s,l+u) - \Gamma(3,l+s,l+u)N_B(l+s) + \Gamma(2,l+s,l+u)N_B(l+u) \\
&\quad -r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l))))] \\
\delta[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(a(l+t)\Gamma(1,l+t,t)(\Gamma(4,l+p,l+t) + \Gamma(3,l+p,l+t)N_B(l+p) - \Gamma(2,l+p,l+t)N_B(l+t)) \\
&\quad +r(l+t)\Gamma(2,l+p,l+t)(\Gamma(6,l+t,t) + \Gamma(1,l+t,t)N_B(l+t) - \Gamma(2,l+t,t)N_F(l)))\,] \\
\epsilon[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(M(6)(l+p,k,-l,t) + M(3)(l+p,k,-l,t)N_B(l+p) - M(1)(l+p,k,-l,t)N_F(l))\,] \\
\phi[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,-a(l)r(l+u)M(1)(s,k,l,l+u)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l))\,] \\
A[2] &= [\,a(l)r(l+s)\Gamma(2,p,l+p)(M(6)(l+s,k,-l,u) + M(3)(l+s,k,-l,u)N_B(l+s) - M(1)(l+s,k,-l,u)N_F(l))\,] \\
B[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,-a(l)r(l+t)M(1)(p,k,l,l+t)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+t) - \Gamma(2,l+u,u)N_F(l))\,] \\
C[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+s)\Gamma(2,p,l+p)(a(l+u)\Gamma(1,l+u,u)(\Gamma(4,l+s,l+u) + \Gamma(3,l+s,l+u)N_B(l+s) - \Gamma(2,l+s,l+u)N_B(l+u)) \\
&\quad +r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l)))\,] \\
D[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(-a(l+t)\Gamma(1,l+u,u)(\Gamma(4,l+p,l+t) + \Gamma(3,l+p,l+t)N_B(l+p) - \Gamma(2,l+p,l+t)N_B(l+t)) \\
&\quad -r(l+t)\Gamma(2,l+p,l+t)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+t) - \Gamma(2,l+u,u)N_F(l)))\,] \\
E[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(-M(6)(l+s,k,-l,u) - M(3)(l+s,k,-l,u)N_B(l+p) + M(1)(l+s,k,-l,u)N_F(l))\,] \\
F[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+u)M(1)(p,k,l,l+t)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l))\,]
\end{aligned}
$$

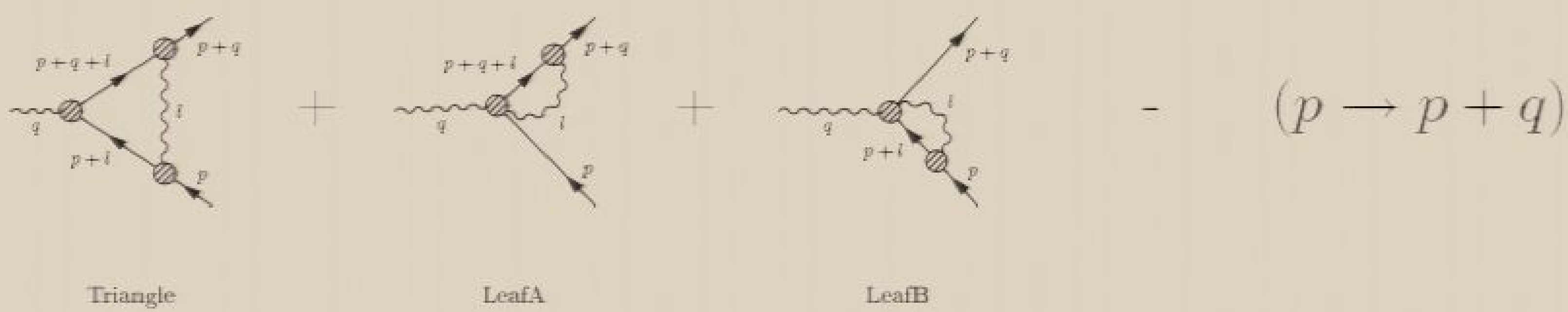
$$
\begin{aligned}
G[2] &= -\frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}\,[\, a(l) r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(-\Gamma(4,l+s,l+u) - \Gamma(3,l+s,l+u)N_B(l+p) + \Gamma(2,l+s,l+u)N_B(l+u)) \\
&\quad -r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l)))\,] \\
H[2] &= -\frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}\,[\, a(l) r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(\Gamma(4,l+p,l+t) + \Gamma(3,l+p,l+t)N_B(l+p) - \Gamma(2,l+p,l+t)N_B(l+u)) \\
&\quad +r(l+u)\Gamma(2,l+p,l+t)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l)))\,]
\end{aligned}
$$

Combining all these terms

- six terms left:

$$
\begin{aligned}
\delta[2] &= \Gamma_{\text{triangle}}(2,p,p+k) & \gamma[2] &= -\Gamma_{\text{triangle}}(2,p+q,p+q+k) \\
\beta[2] &= \Gamma_{\text{leafA}}(2,p,p+k) & \phi[2] &= -\Gamma_{\text{leafA}}(2,p+q,p+q+k) \\
\epsilon[2] &= \Gamma_{\text{leafB}}(2,p,p+k) & \alpha[2] &= -\Gamma_{\text{leafB}}(2,p+q,p+q+k)
\end{aligned}
$$

- which gives:



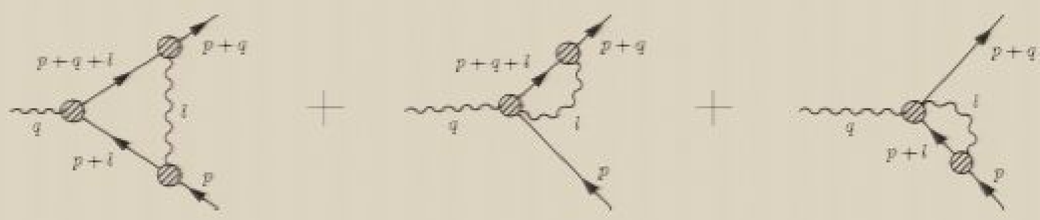
$$
\begin{aligned}
G[2] &= -\frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}\,[\; a(l) r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(-\Gamma(4,l+s,l+u) - \Gamma(3,l+s,l+u)N_B(l+p) + \Gamma(2,l+s,l+u)N_B(l+u)) \\
&\quad -r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l)))\;] \\
H[2] &= -\frac{i}{2}\int \frac{d^4 l}{(2\pi)^4}\,[\; a(l) r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(\Gamma(4,l+p,l+t) + \Gamma(3,l+p,l+t)N_B(l+p) - \Gamma(2,l+p,l+t)N_B(l+u)) \\
&\quad +r(l+u)\Gamma(2,l+p,l+t)(\Gamma(6,l+u,u) + \Gamma(1,l+u,u)N_B(l+u) - \Gamma(2,l+u,u)N_F(l)))\;]
\end{aligned}
$$

Combining all these terms

- six terms left:

$$
\begin{aligned}
\delta[2] &= \Gamma_{\text{triangle}}(2,p,p+k) & \gamma[2] &= -\Gamma_{\text{triangle}}(2,p+q,p+q+k) \\
\beta[2] &= \Gamma_{\text{leafA}}(2,p,p+k) & \phi[2] &= -\Gamma_{\text{leafA}}(2,p+q,p+q+k) \\
\epsilon[2] &= \Gamma_{\text{leafB}}(2,p,p+k) & \alpha[2] &= -\Gamma_{\text{leafB}}(2,p+q,p+q+k)
\end{aligned}
$$

- which gives:



- $(p \longrightarrow p+q)$

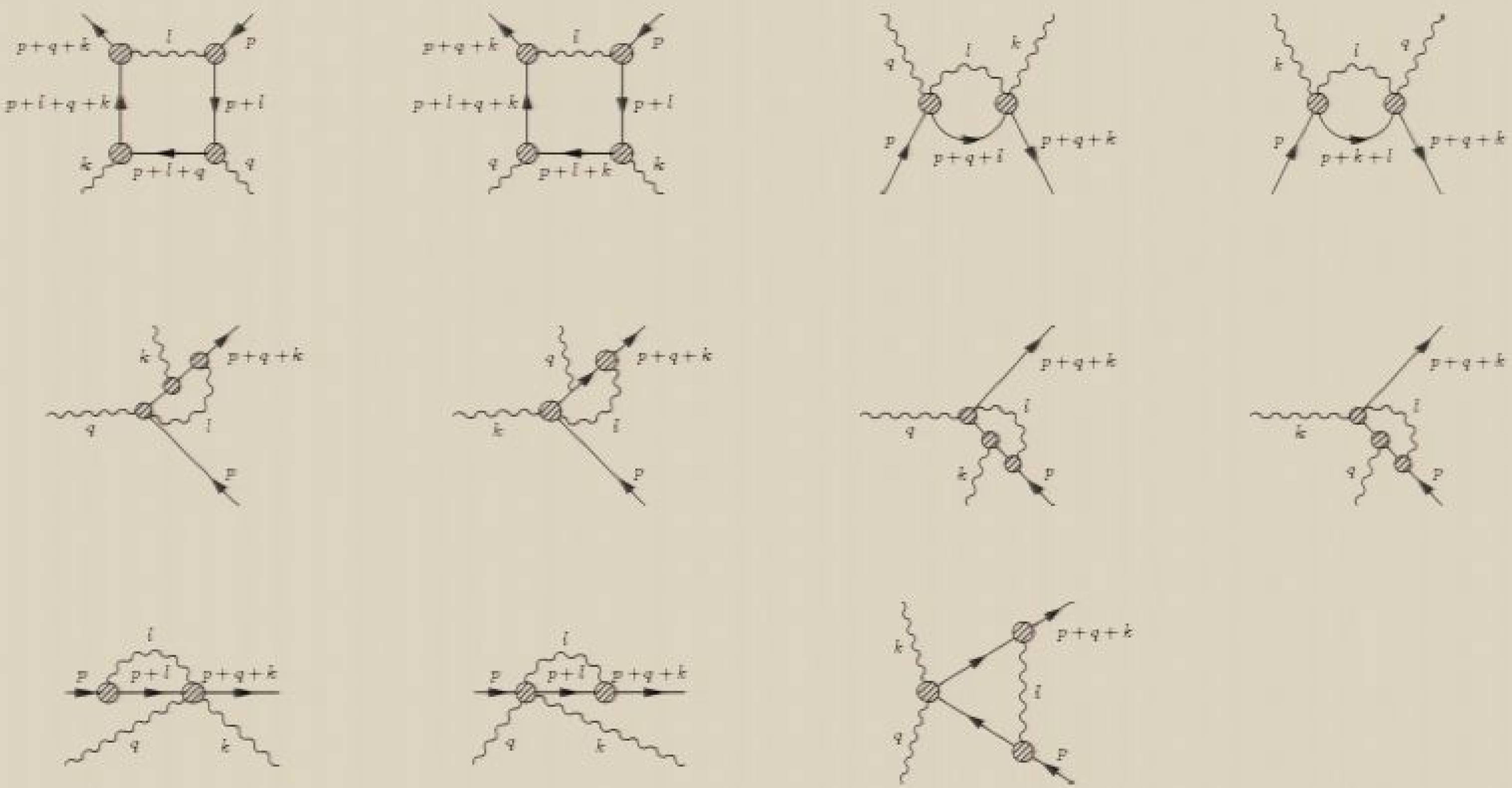
Triangle LeafA LeafB

Four-point Ward Identity

A Ward Identity in QED for the 4-point function to 1-loop order is:

$$Q^{\mu}\Gamma^{(1)}_{\mu,rarr}[P,Q,K,-P-Q-K] = \Gamma^{(1)}_{rra}[P,K,P+K] - \Gamma^{(1)}_{arr}[P+Q,K,P+Q+K]$$

- 4-point diagrams to consider are:



Calculating these diagrams and contracting with q :

$$
\begin{aligned}
\Gamma_{\text{seagull-1}}[2,p,q,k,u] &= \alpha[2]+A[2] & \Gamma_{\text{seagull-2}}[2,p,q,k,u] &= \beta[2]+B[2] \\
\Gamma_{\text{leaftail-1}}[2,p,q,k,u] &= \gamma[2]+C[2] & \Gamma_{\text{leaftail-3}}[2,p,q,k,u] &= \delta[2]+D[2] \\
\Gamma_{\text{jellyfish-1}}[2,p,q,k,u] &= \epsilon[2]+E[2] & \Gamma_{\text{jellyfish-2}}[2,p,q,k,u] &= \phi[2]+F[2] \\
\Gamma_{\text{leaftail-2}}[2,p,q,k,u] &= -B[2]-F[2] & \Gamma_{\text{leaftail-4}}[2,p,q,k,u] &= -A[2]-E[2] \\
\Gamma_{\text{box}}[2,p,q,k,u] &= -G[2]-C[2] & \Gamma_{\text{crossed-box}}[2,p,q,k,u] &= -H[2]-D[2] \\
\Gamma_{\text{polywog}}[2,p,q,k,u] &= G[2]+H[2]
\end{aligned} \tag{2}
$$

where:

$$
\begin{aligned}
\alpha[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[-a(l)r(l+s)\Gamma(2,s,l+s)(M(6)(l+s,k,-l,u)+M(3)(l+s,k,-l,u)N_B(l+s)-M(1)(l+s,k,-l,u)N_F(l))] \\
\beta[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+t)M(1)(p,k,l,l+t)(\Gamma(6,l+t,t)+\Gamma(1,l+t,t)N_B(l+t)-\Gamma(2,l+t,t)N_F(l)) \\
\gamma[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+s)\Gamma(2,s,l+s)\,(a(l+u)\Gamma(1,l+u,u)(-\Gamma(4,l+s,l+u)-\Gamma(3,l+s,l+u)N_B(l+s)+\Gamma(2,l+s,l+u)N_B(l+u) \\
&\quad -r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l))))] \\
\delta[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(a(l+t)\Gamma(1,l+t,t)(\Gamma(4,l+p,l+t)+\Gamma(3,l+p,l+t)N_B(l+p)-\Gamma(2,l+p,l+t)N_B(l+t)) \\
&\quad +r(l+t)\Gamma(2,l+p,l+t)(\Gamma(6,l+t,t)+\Gamma(1,l+t,t)N_B(l+t)-\Gamma(2,l+t,t)N_F(l)))\,] \\
\epsilon[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(M(6)(l+p,k,-l,t)+M(3)(l+p,k,-l,t)N_B(l+p)-M(1)(l+p,k,-l,t)N_F(l))\,] \\
\phi[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,-a(l)r(l+u)M(1)(s,k,l,l+u)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l))\,] \\
A[2] &= [\,a(l)r(l+s)\Gamma(2,p,l+p)(M(6)(l+s,k,-l,u)+M(3)(l+s,k,-l,u)N_B(l+s)-M(1)(l+s,k,-l,u)N_F(l))\,] \\
B[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,-a(l)r(l+t)M(1)(p,k,l,l+t)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+t)-\Gamma(2,l+u,u)N_F(l))\,] \\
C[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+s)\Gamma(2,p,l+p)(a(l+u)\Gamma(1,l+u,u)(\Gamma(4,l+s,l+u)+\Gamma(3,l+s,l+u)N_B(l+s)-\Gamma(2,l+s,l+u)N_B(l+u)) \\
&\quad +r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l)))\,] \\
D[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(-a(l+t)\Gamma(1,l+u,u)(\Gamma(4,l+p,l+t)+\Gamma(3,l+p,l+t)N_B(l+p)-\Gamma(2,l+p,l+t)N_B(l+t)) \\
&\quad -r(l+t)\Gamma(2,l+p,l+t)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+t)-\Gamma(2,l+u,u)N_F(l)))\,] \\
E[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+p)\Gamma(2,p,l+p)(-M(6)(l+s,k,-l,u)-M(3)(l+s,k,-l,u)N_B(l+p)+M(1)(l+s,k,-l,u)N_F(l))\,] \\
F[2] &= -\frac{i}{2}\int\frac{d^4l}{(2\pi)^4}[\,a(l)r(l+u)M(1)(p,k,l,l+t)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l))\,]
\end{aligned}
$$

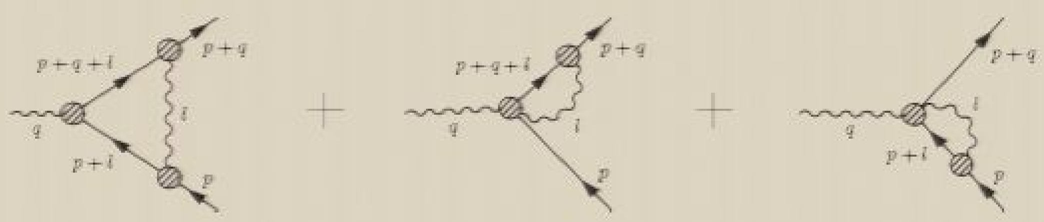
$$
\begin{aligned}
G[2] &= -\frac{i}{2}\int \frac{d^4l}{(2\pi)^4}\,[\, a(l)r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(-\Gamma(4,l+s,l+u)-\Gamma(3,l+s,l+u)N_B(l+p)+\Gamma(2,l+s,l+u)N_B(l+u)) \\
&\quad -r(l+u)\Gamma(2,l+s,l+u)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l)))\,] \\
H[2] &= -\frac{i}{2}\int \frac{d^4l}{(2\pi)^4}\,[\, a(l)r(l+p)\Gamma(2,p,l+p) \\
&\quad (a(l+u)\Gamma(1,l+u,u)(\Gamma(4,l+p,l+t)+\Gamma(3,l+p,l+t)N_B(l+p)-\Gamma(2,l+p,l+t)N_B(l+u)) \\
&\quad +r(l+u)\Gamma(2,l+p,l+t)(\Gamma(6,l+u,u)+\Gamma(1,l+u,u)N_B(l+u)-\Gamma(2,l+u,u)N_F(l)))\,]
\end{aligned}
$$

Combining all these terms

- six terms left:

$$
\begin{aligned}
\delta[2] &= \Gamma_{\text{triangle}}(2,p,p+k) & \gamma[2] &= -\Gamma_{\text{triangle}}(2,p+q,p+q+k) \\
\beta[2] &= \Gamma_{\text{leafA}}(2,p,p+k) & \phi[2] &= -\Gamma_{\text{leafA}}(2,p+q,p+q+k) \\
\epsilon[2] &= \Gamma_{\text{leafB}}(2,p,p+k) & \alpha[2] &= -\Gamma_{\text{leafB}}(2,p+q,p+q+k)
\end{aligned}
$$

- which gives:



Triangle LeafA LeafB

$$- \quad (p \rightarrow p+q)$$

Summary

- Real-Time Finite-Temperature Field theory greatly simplified judicious choice of basis
- Computing amplitudes for physical processes simplified by use of Mathematica program implementing Feynman diagrams as tensor products
  - still have to evaluate integrals
  - but only evaluate those that will actually contribute
- also useful for testing relations that hold at the level of Feynman diagrams
  - Ward Identities
  - cutting rules (?) (Kobes and Semenoff 1985,1986)

This work was supported by NSERC