Title: Quantum Information Theory 7

Date: Jun 09, 2006 04:24 PM

URL: http://pirsa.org/06060032

Abstract:

Institute for Quantum Information Science at the University of Calgary

Friday 9th July 200

Physically Accessible Non-Completely Positive Maps

Hilary Carteret

hcartere@qis.ucalgary.ca

joint work with

Daniel R. Terno & Karol Życzkowski

quant-ph/0512167

Theory 2

UNIVERSITY OF CALGARY

MITACS

CIAR

The Canadian Institute for Advanced Research

Introduction and Motivation

When a system interacts with its environment, its dynamics (as seen locally) are no longer unitary.

If the system is initially uncorrelated with its environment, we can use the Kraus representation theorem.

This states that the most general dynamics such a system can display is a completely positive map (CP-map):

positivity: $$\Lambda(\rho) \geq 0, \ \forall \rho \geq 0$$

complete positivity: $$(\mathbb{1} \otimes \Lambda)(\rho) \geq 0, \ \forall \rho \geq 0$$

where the identity operator can have support on an auxiliary system of arbitrary dimension.

[open systems, channel capacities, process tomography]

The Kraus representation theorem

Elegant 3-way iff equivalence result.

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

"Stinespring" form

$$\rho^A_{m:n} = D^{i:j}_{m:n}\rho^A_{i:j}$$

dynamical matrix

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

CP dynamical map

[see Nielsen & Chuang]

The Kraus representation theorem

Elegant 3-way iff equivalence result.

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

"Stinespring" form

$$\rho^A_{m:n} = D^{i:j}_{m:n}\rho^A_{i:j}$$

dynamical matrix

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

CP dynamical map

superoperator representation (~lower dual index on ρ)

[see Nielsen & Chuang]

The Kraus representation theorem

Elegant 3-way iff equivalence result.

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

"Stinespring" form

$$\rho^A_{m:n} = D^{i:j}_{m:n}\rho^A_{i:j}$$

dynamical matrix

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

CP dynamical map

[see Nielsen & Chuang]

The Kraus representation theorem

Elegant 3-way iff equivalence result.

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

"Stinespring" form

$$\rho^A_{m:n} = D^{i:j}_{m:n}\rho^A_{i:j}$$

dynamical matrix

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

CP dynamical map

superoperator representation (~lower dual index on ρ)

[see Nielsen & Chuang]

The Kraus representation theorem

Elegant 3-way iff equivalence result.

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

"Stinespring" form

$$\rho^A_{m:n} = D^{i:j}_{m:n}\rho^A_{i:j}$$

dynamical matrix

more on this form later

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

CP dynamical map

superoperator representation (~lower dual index on ρ)

[see Nielsen & Chuang]

What can go wrong...?

The proof really does need the initial condition:

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

No initial correlations between the system & environment.

Even classical correlations are enough to break the strict equivalence proof -- never mind entanglement!

What can go wrong...?

The proof really does need the initial condition:

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

No initial correlations between the system & environment.

Even classical correlations are enough to break the strict equivalence proof -- never mind entanglement!

Why haven't we noticed this before...?

What can go wrong...?

The proof really does need the initial condition:

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

No initial correlations between the system & environment.

Even classical correlations are enough to break the strict equivalence proof -- never mind entanglement!

Why haven't we noticed this before...?

If the environment is a macroscopic piece of apparatus, the Stinespring form is a good enough approximation:

$$\sum_i \rho_i^A \otimes \rho_i^B \approx \left(\sum_i \rho_i^A\right) \otimes \rho^B$$

What can go wrong...?

The proof really does need the initial condition:

$$\Lambda(\rho^A) = \mathrm{tr}[U(\rho^A \otimes \rho^B)U^\dagger]$$

No initial correlations between the system & environment.

Even classical correlations are enough to break the strict equivalence proof -- never mind entanglement!

Why haven't we noticed this before...?

If the environment is a macroscopic piece of apparatus, the Stinespring form is a good enough approximation:

$$\sum_i \rho_i^A \otimes \rho_i^B \approx \left(\sum_i \rho_i^A\right) \otimes \rho^B$$

because the overlaps between the various ρ^B_i s are very large, so the mismatches can be neglected.

When is this (potentially) important...?

Whenever we have a system that has significant prior correlations with the other systems that we would like to use to interact with it, or "measure" it, then the mismatch between the Stinespring form initial condition and the actual initial condition may cause us problems.

Most notoriously, this can be a problem when we try to do process tomography in the laboratory; a lot of very careful experimenters keep getting non-CP maps!

Non-CP maps occur naturally in dynamical decoupling and spin-echo experiments.

This is also an issue when analysing channel capacities.

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

* a notoriously non-CP map!

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

1. Take the description of ρ^A and calculate $(\rho^A)^T$.

* a notoriously non-CP map!

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

1. Take the description of ρ^A and calculate $(\rho^A)^T$.
2. Throw away your original state ρ^A.

* a notoriously non-CP map!

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

1. Take the description of ρ^A and calculate $(\rho^A)^T$.
2. Throw away your original state ρ^A.
3. Prepare the state $(\rho^A)^T$ directly from the description.

* a notoriously non-CP map!

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

1. Take the description of ρ^A and calculate $(\rho^A)^T$.
2. Throw away your original state ρ^A.
3. Prepare the state $(\rho^A)^T$ directly from the description.

Thus we can implement any map pointwise, given complete knowledge of the state. But if someone swapped our original ρ^A for a different one, $\rho^{A'}$, this wouldn't work!

* a notoriously non-CP map!

Ignorance, robustness and realizability

If I have a complete description of the density matrix ρ^A, as well as the state ρ^A itself, then I can implement any map I like on it.

Here's the recipe for the transpose*:

1. Take the description of ρ^A and calculate $(\rho^A)^T$.
2. Throw away your original state ρ^A.
3. Prepare the state $(\rho^A)^T$ directly from the description.

Thus we can implement any map pointwise, given complete knowledge of the state. But if someone swapped our original ρ^A for a different one, $\rho^{A'}$, this wouldn't work!

So, to analyse practical situations, we need recipes that will work for non-zero volumes of states in Hilbert space.

* a notoriously non-CP map!

Prior correlations and Extension maps

For a given density matrix ρ^A there are a variety of ways to extend it to a larger system, and the choice of extension map E affects the induced dynamics.

The differences between extension maps lie in the type and degree of the correlations between the original system and the environment:

uncorrelated extension:
$$E(\rho^A) \rightarrow \rho^A \otimes \rho^B$$

classically correlated:
$$E(\rho^A) \rightarrow \sum_i p_i \rho_i^A \otimes \rho_i^B$$

entangled extension:
$$E(\rho^A) \rightarrow \rho^{AB}$$

where

$$\rho^{AB} \neq \sum_i p_i \rho_i^A \otimes \rho_i^B$$

Extensions and positivity

Once we allow prior correlations, the dynamical map Ξ may not even be positive, let alone CP.

So, we should restrict the domain of Ξ to those states for which Ξ is positive (domain of positivity) but insist that this domain must have non-zero volume, so Ξ is practical.

Furthermore, the extension map E should produce an extended state that is also positive, such that

$$\rho^A = \mathrm{tr}_B(\rho^{AB})$$

The set of states ρ^A for which E satisfies this condition is called the consistency domain for the extension map E.

The consistency condition is more restrictive, so the consistency domain is a subset of the positivity domain.

Positivity vs. Linearity of Extension Maps

Call the set of all density matrices $\mathcal{M}$, and the positivity domain of Ξ in $\mathcal{M}$ to be $\mathcal{W}$, and the consistency domain of E in $\mathcal{M}$ to be $\mathcal{V}$.

Positivity vs. Linearity of Extension Maps

Call the set of all density matrices $\mathcal{M}$, and the positivity domain of Ξ in $\mathcal{M}$ to be $\mathcal{W}$, and the consistency domain of E in $\mathcal{M}$ to be $\mathcal{V}$.

We could engineer correlated and consistent extension maps E to induce positive dynamical maps Ξ over the whole Hilbert space, but only by making E a function of ρ^A.

Positivity vs. Linearity of Extension Maps

Call the set of all density matrices $\mathcal{M}$, and the positivity domain of Ξ in $\mathcal{M}$ to be $\mathcal{W}$, and the consistency domain of E in $\mathcal{M}$ to be $\mathcal{V}$.

We could engineer correlated and consistent extension maps E to induce positive dynamical maps Ξ over the whole Hilbert space, but only by making E a function of ρ^A.

However, this would make E non-linear.

Positivity vs. Linearity of Extension Maps

Call the set of all density matrices $\mathscr{M}$, and the

positivity domain of Ξ in $\mathscr{M}$ to be $\mathscr{W}$, and the

consistency domain of E in $\mathscr{M}$ to be $\mathscr{V}$.

We could engineer correlated and consistent extension maps E to induce positive dynamical maps Ξ over the whole Hilbert space, but only by making E a function of ρ^A.

However, this would make E non-linear.

Unless the dynamical map Ξ is induced by an uncorrelated extension map E (which would make Ξ CP after all) we must choose between linearity of E and positivity of Ξ.

Definition of Accessibility*

A non-CP map Ξ is called **physically accessible** if there is an extension map $E_{\mathcal{V}}$, with domain $\mathcal{V}$ which is a subset of $\mathcal{W}$ and there is a unitary U such that

$$\Xi(\rho^A) = \mathrm{tr}_B[U E_{\mathcal{V}}(\rho^A) U^\dagger]$$

where U is independent of ρ^A.

(*This definition is different from that used by some other authors who looked at continuous time dynamics.)

Definition of Accessibility*

A non-CP map Ξ is called **physically accessible** if there is an extension map $E_\mathcal{V}$, with domain $\mathcal{V}$ which is a subset of $\mathcal{W}$ and there is a unitary U such that

$$\Xi(\rho^A) = \mathrm{tr}_B[U E_\mathcal{V}(\rho^A) U^\dagger]$$

where U is independent of ρ^A.

Note the importance of restricting the domain to a subset $\mathcal{V}$ of $\mathcal{M}$, but that $\mathcal{V}$ must still have a non-zero volume. If we impose that $\mathcal{V} = \mathcal{M}$, then Ξ can only be CP.

(*This definition is different from that used by some other authors who looked at continuous time dynamics.)

Superoperators and Dynamical Matrices

This is awkward to work with:

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

Superoperators and Dynamical Matrices

This is awkward to work with:

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

Instead, move to the superoperator Hilbert space, in which density matrices become vectors, and Λ is a matrix, D(Λ):

$$\rho^A_{m:n} = D^{i:j}_{m:n} \rho^A_{i:j}$$

Superoperators and Dynamical Matrices

This is awkward to work with:

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

Instead, move to the superoperator Hilbert space, in which density matrices become vectors, and Λ is a matrix, D(Λ):

$$\rho^A_{m:n} = D^{i:j}_{m:n} \rho^A_{i:j}$$

Moving to superoperator space is like lowering indices in tensors with a trivial metric tensor, but the colon is not a derivative - instead it indicates which were the vector and dual indices in the original Hilbert space.

Superoperators and Dynamical Matrices

This is awkward to work with:

$$\Lambda(\rho^A) = \sum_a M_a \rho^A M_a^\dagger$$

Instead, move to the superoperator Hilbert space, in which density matrices become vectors, and Λ is a matrix, D(Λ):

$$\rho^A_{m:n} = D^{i:j}_{m:n} \rho^A_{i:j}$$

Moving to superoperator space is like lowering indices in tensors with a trivial metric tensor, but the colon is not a derivative - instead it indicates which were the vector and dual indices in the original Hilbert space.

The original notation in our paper was: $$\rho^A_{mn} = D_{mi;nj} \rho^A_{ij}$$

Some useful facts:

Theorem [Choi]:

$$D \geq 0 \iff \Lambda \text{ is CP}$$

Some useful facts:

Theorem [Choi]:

$$D \geq 0 \iff \Lambda \text{ is CP}$$

i.e., we can test for positivity via the eigendecomposition:

$$D^{i:j}_{m:n} = \sum_a \lambda_a (M_a)^i_m (M^\dagger_a)^j_n$$

If D is not positive, we can partition the spectrum into positive and negative parts (D is Hermitian) and write:

$$\rho' = \Lambda_1(\rho) - \Lambda_2(\rho)$$

where Λ_1 and Λ_2 are both CP maps.

Some useful facts:

Theorem [Choi]:

$$D \geq 0 \iff \Lambda \text{ is CP}$$

i.e., we can test for positivity via the eigendecomposition:

$$D^{i:j}_{m:n} = \sum_a \lambda_a (M_a)^i_m (M^\dagger_a)^j_n$$

If D is not positive, we can partition the spectrum into positive and negative parts (D is Hermitian) and write:

$$\rho' = \Lambda_1(\rho) - \Lambda_2(\rho)$$

where Λ_1 and Λ_2 are both CP maps.

Unfortunately, not all maps that can be written like this are physically accessible - need additional conditions.

in superoperator notation

If we look at the generalised dynamical matrix, we see

$$D_{m:n}^{i:j} = G_{m:n}^{i:j} + \vec{\xi} \cdot \vec{\sigma}_{m:n} \delta^{i:j}$$

where G is the dynamical matrix of a CP map.

in superoperator notation

If we look at the generalised dynamical matrix, we see

$$D^{i:j}_{m:n} = G^{i:j}_{m:n} + \vec{\xi} \cdot \vec{\sigma}_{m:n} \delta^{i:j}$$

where G is the dynamical matrix of a CP map.

Note that the correction term $\neq \Lambda_2$ in general.

in superoperator notation

If we look at the generalised dynamical matrix, we see

$$D^{i:j}_{m:n} = G^{i:j}_{m:n} + \vec{\xi} \cdot \vec{\sigma}_{m:n} \delta^{i:j}$$

where G is the dynamical matrix of a CP map.

Note that the correction term ≠ Λ_2 in general.

In the original H[illegible]space, this is

$$\Xi(\rho^A) = [illegible]_B[U(E_{\mathcal{V}}(\rho^A))U^\dagger]$$

in superoperator notation

If we look at the generalised dynamical matrix, we see

$$D^{i:j}_{m:n} = G^{i:j}_{m:n} + \boxed{\vec{\xi} \cdot \vec{\sigma}_{m:n} \delta^{i:j}}$$

where G is the dynamical matrix of a CP map.

Note that the correction term $\boxed{}$ $\neq \Lambda_2$ in general.

In the original Hilbert space, this is

$$\Xi(\rho^A) = \mathrm{tr}_B[U(E_{\mathcal{V}}(\rho^A))U^\dagger]$$

in superoperator notation

If we look at the generalised dynamical matrix, we see

$$D^{i:j}_{m:n} = G^{i:j}_{m:n} + \vec{\xi} \cdot \vec{\sigma}_{m:n} \delta^{i:j}$$

where G is the dynamical matrix of a CP map.

Note that the correction term $\neq \Lambda_2$ in general.

In the original Hilbert space, this is

$$\Xi(\rho^A) = \mathrm{tr}_B[U(E_\mathcal{V}(\rho^A))U^\dagger]$$
$$= \sum_{\mu,\nu} M_{\mu\nu}\rho^A M^\dagger_{\mu\nu}$$
$$+ \sum_\mu \langle \mu | U \Gamma_{ij} \sigma^A_i \otimes \sigma^B_j U^\dagger | \mu \rangle$$

Γ_{ij} is the correlation tensor, and the σs are generators.

A simple example

Consider the state: $\rho^A = \frac{1}{2}(\mathbb{1} + \vec{\alpha}.\vec{\sigma})$

and the extension map

$$E(\rho^A) = \frac{1}{4}(\mathbb{1}_{AB} + \alpha_i \sigma_i^A \otimes \mathbb{1}_B + a\sigma_i^A \otimes \sigma_i^B)$$

and the unitary:

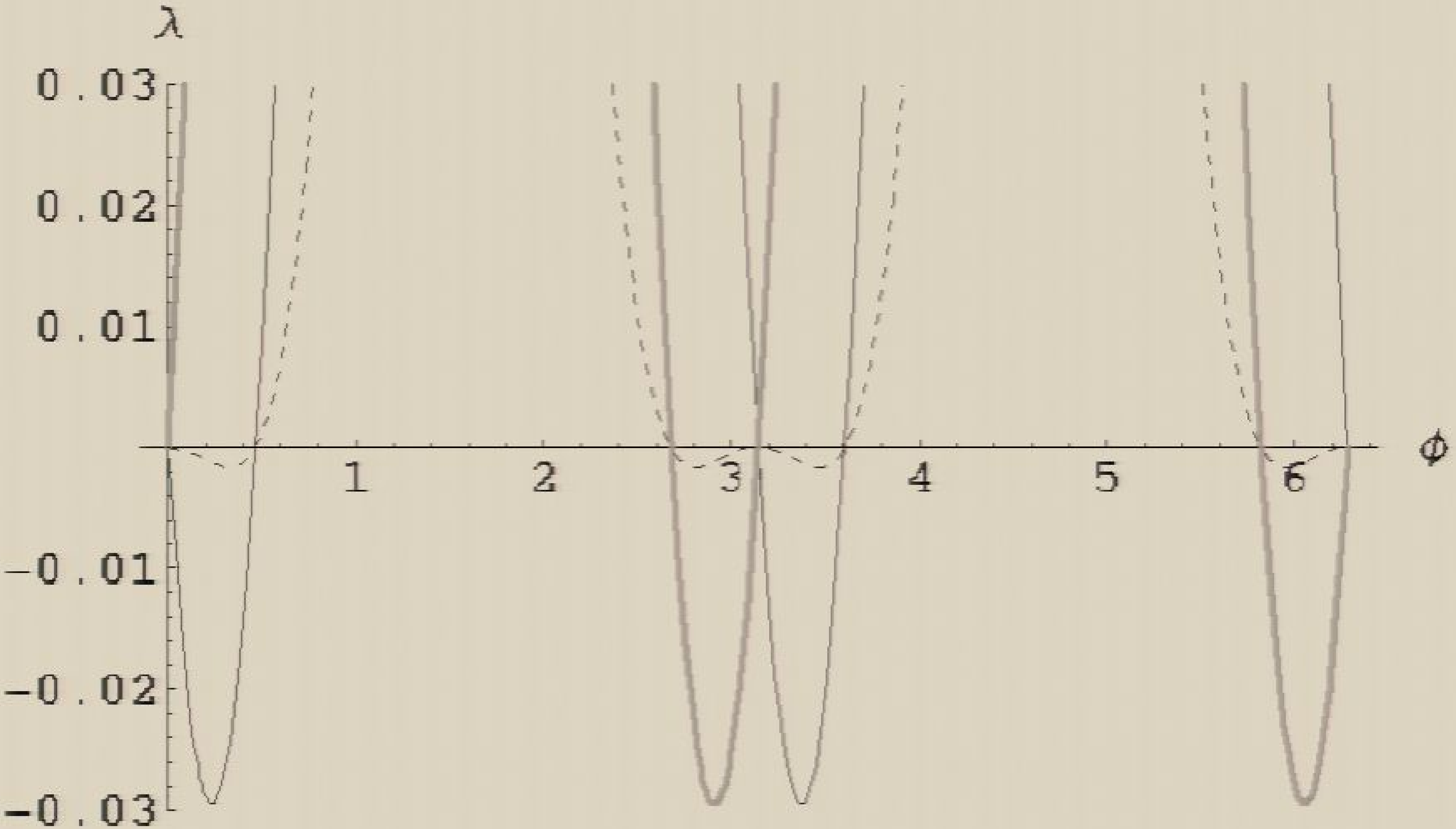
$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The correction term (in superoperator space) is

$$\frac{1}{2}\begin{pmatrix} a\sin 2\phi & 0 \\ 0 & -a\sin 2\phi \end{pmatrix}$$

A simple example, continued.

A non-zero correction term doesn't automatically* make Ξ non-CP, so we need to check the eigenspectrum of D:



(*since $\neq \Lambda_2$) 15

What kinds of correlations are present?

As this is a two-qubit system, we can test the state for entanglement using the Peres partial transpose test.

What kinds of correlations are present?

As this is a two-qubit system, we can test the state for entanglement using the Peres partial transpose test.

For *a* positive, the two qubits are never entangled, and yet for some values of φ we found Ξ was non-CP.

For some negative values of a, we do find entanglement and Ξ was non-CP, as expected...

...but for some other negative values of *a*, we still found entanglement, yet the dynamical map was still CP.

Therefore entanglement is not sufficient to induce non-CP maps. (This is less surprising; otherwise the issue of non-CP maps would be much more well-known!)

Conditions for Accessibility

Don't yet have a complete characterisation of accessibility. However, we have found two necessary conditions:

Conditions for Accessibility

Don't yet have a complete characterisation of accessibility. However, we have found two necessary conditions:

Condition 1

The existence of an affine form

$$\rho'_{m:n} = G^{i:j}_{m:n}\rho_{i:j} + \vec{\xi}\cdot\vec{\sigma}_{m:n}$$

with a trace-preserving G and a traceless $\xi\cdot\sigma$ term is a necessary condition for Ξ to be physically accessible.

Conditions for Accessibility

Don't yet have a complete characterisation of accessibility. However, we have found two necessary conditions:

Condition 1

The existence of an affine form

$$\rho'_{m:n} = G^{i:j}_{m:n} \rho_{i:j} + \vec{\xi} \cdot \vec{\sigma}_{m:n}$$

with a trace-preserving G and a traceless $\xi \cdot \sigma$ term is a necessary condition for Ξ to be physically accessible.

Condition 2

Any accessible unital map that has a state-independent affine form is completely positive.

Conditions for Accessibility

Don't yet have a complete characterisation of accessibility. However, we have found two necessary conditions:

Condition 1

The existence of an affine form

$$\rho'_{m:n} = G^{i:j}_{m:n}\rho_{i:j} + \vec{\xi}\cdot\vec{\sigma}_{m:n}$$

with a trace-preserving G and a traceless $\xi\cdot\sigma$ term is a necessary condition for Ξ to be physically accessible.

Condition 2

Any accessible unital map that has a state-independent affine form is completely positive.

In other words, a necessary condition for a non-CP map with a state-independent affine form to be accessible is that it must also be non-unital.

Conclusions and Open Questions

Entanglement is not required for a non-CP dynamical map. (Classical correlations can sometimes be enough.)

Entanglement is not sufficient either.

Non-CP maps have already been seen, in some process tomography experiments, so the correction terms are big enough to observe. See for example:

N Boulant, J Emerson, TF Havel, DG Cory & S Furuta, J. Chem. Phys. **121**, 2955 (2004),

YS Weinstein, TF Havel, J Emerson, N Boulant, M Saraceno, S Lloyd & DG Cory, J. Chem. Phys. 121, 6117 (2004)

A few references

HC, Daniel R Terno & Karol Życzkowski, quant-ph/0512167

K Życzkowski & I Bengtsson,
Open. Sys. Information Dyn. **11**, 3 (2004)

P Pechukas, PRL **73**, 1060 (1994)

P Štelmachovič & V Bužek, PRA **64**, 062106 (2001)

TF Jordan, A Shaji & ECG Sudarshan, PRA 70, 052110 (200

M-D Choi, Linear Algebra Appl. **10**, 285 (1975)

N Boulant, J Emerson, TF Havel, DG Cory & S Furuta,
J. Chem. Phys. **121**, 2955 (2004),

YS Weinstein, TF Havel, J Emerson, N Boulant,
M Saraceno, S Lloyd & DG Cory,
J. Chem. Phys. 121, 6117 (2004)