Title: Quantum Information Theory 2

Date: Jun 09, 2006 02:00 PM

URL: http://pirsa.org/06060027

Abstract:

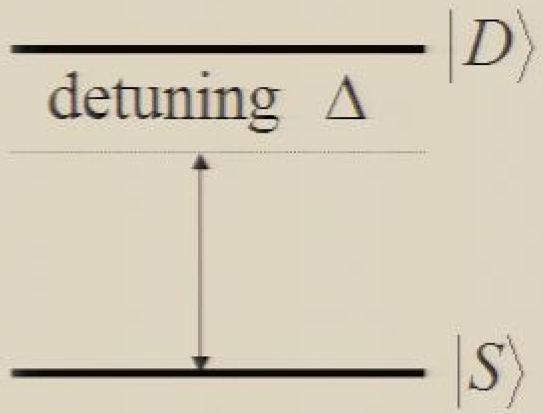
Theory CANADA 2
Perimeter Institute, 9 June 2006

The Theory of Effective Hamiltonians for Detuned Systems

Daniel F. V. JAMES
Department of Physics, University of Toronto

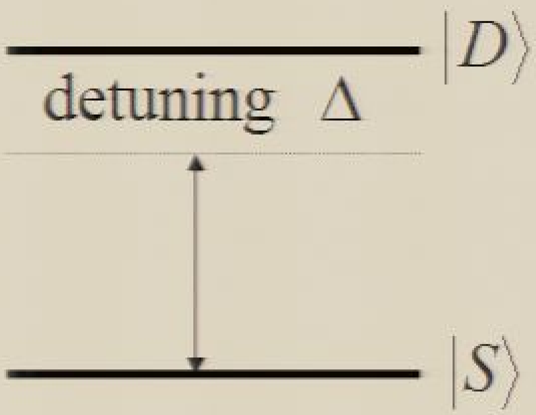
Detuned Systems

- Example: Two level system, detuned field



Detuned Systems

- Example: Two level system, detuned field

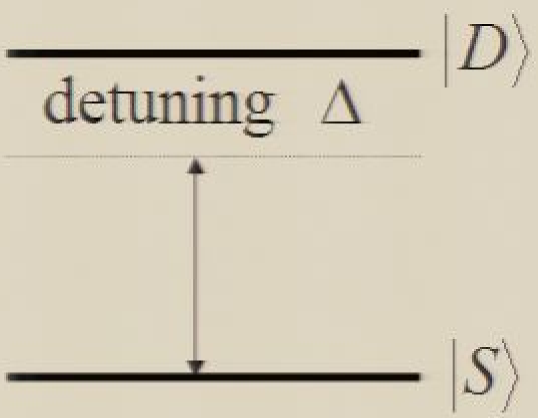


- Interaction Picture Hamiltonian:

$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

Detuned Systems

- Example: Two level system, detuned field



- Interaction Picture Hamiltonian:

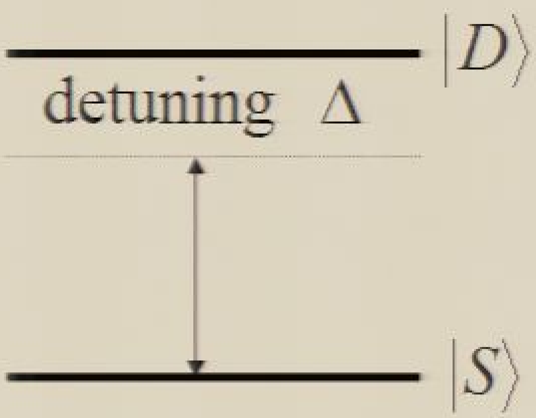
$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

- BUT: we <u>*know*</u> what really happens is the A.C. Stark shift, i.e.:

$$\hat{H}_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(|D\rangle\langle D| - |S\rangle\langle S|\right)$$

Detuned Systems

- Example: Two level system, detuned field



- Interaction Picture Hamiltonian:

$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

- BUT: we *know* what really happens is the A.C. Stark shift, i.e.:

$$\hat{H}_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(|D\rangle\langle D| - |S\rangle\langle S|\right)$$

- Is there a systematic way to get H_{eff} from H_I (preferably without all that tedious mucking about with adiabatic elimination)?

Time Averaged Dynamics: Definitions

- Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

Time Averaged Dynamics: Definitions

- Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0) \qquad (1)$$

Time Averaged Dynamics: Definitions

- Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0) \qquad (1)$$

- Time-Averaged evolution operator

$$\overline{\hat{U}(t,t_0)} = \int_{-\infty}^{\infty} f(t-t')\hat{U}(t',t_0)dt' \qquad (2)$$

Time Averaged Dynamics: Definitions

- Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0) \qquad (1)$$

- Time-Averaged evolution operator

$$\overline{\hat{U}(t,t_0)} = \int_{-\infty}^{\infty} f(t-t')\hat{U}(t',t_0)dt' \qquad (2)$$

Filter Function (real valued)

Time Averaged Dynamics: Definitions

- Unitary time evolution operator

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle$$

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_I(t)\hat{U}(t,t_0) \qquad (1)$$

- Time-Averaged evolution operator

$$\overline{\hat{U}(t,t_0)} = \int_{-\infty}^{\infty} f(t-t')\hat{U}(t',t_0)dt' \qquad (2)$$

Filter Function (real valued)

- Define the effective Hamiltonian by:

$$i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)} = \hat{H}_{eff}(t)\overline{\hat{U}(t,t_0)} \qquad (3)$$

General Expression I

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}_I(t)\hat{U}(t,t_0) \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}$$

General Expression I

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}_I(t)\hat{U}(t,t_0) \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}$$

$$\Rightarrow \boxed{\hat{H}_{eff}(t)\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}} \qquad (4)$$

General Expression I

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}_I(t)\hat{U}(t,t_0) \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}$$

$$\Rightarrow \boxed{\hat{H}_{eff}(t)\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}} \qquad (4)$$

- Use a perturbative series for U and H_{eff}:

$$\hat{U}(t,t_0)=\sum_{n=0}^{\infty}\lambda^n\hat{V}_n(t); \quad \hat{V}_{n+1}(t)=\frac{1}{i\hbar}\int^{t}\hat{H}_I(t')\hat{V}_n(t')dt'; \quad \hat{V}_0(t)=\hat{I}$$

General Expression I

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}_I(t)\hat{U}(t,t_0) \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}$$

$$\Rightarrow \boxed{\hat{H}_{eff}(t)\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}} \qquad (4)$$

- Use a perturbative series for U and H_{eff}:

$$\hat{U}(t,t_0)=\sum_{n=0}^{\infty}\lambda^n\hat{V}_n(t); \quad \hat{V}_{n+1}(t)=\frac{1}{i\hbar}\int^{t}\hat{H}_I(t')\hat{V}_n(t')dt'; \quad \hat{V}_0(t)=\hat{I}$$

$$\hat{H}_{eff}(t)=\sum_{n=0}^{\infty}\lambda^n\hat{W}_n(t)$$

General Expression I

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}_I(t)\hat{U}(t,t_0) \quad\Rightarrow\quad i\hbar\frac{\partial}{\partial t}\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}$$

$$\Rightarrow \boxed{\hat{H}_{eff}(t)\overline{\hat{U}(t,t_0)}=\overline{\hat{H}_I(t)\hat{U}(t,t_0)}} \qquad (4)$$

- Use a perturbative series for U and H_{eff}:

$$\hat{U}(t,t_0)=\sum_{n=0}^{\infty}\lambda^n\hat{V}_n(t); \quad \hat{V}_{n+1}(t)=\frac{1}{i\hbar}\int^{t}\hat{H}_I(t')\hat{V}_n(t')dt'; \quad \hat{V}_0(t)=\hat{I}$$

$$\hat{H}_{eff}(t)=\sum_{n=0}^{\infty}\lambda^n\hat{W}_n(t)$$

$$\boxed{\overline{\hat{H}_I(t)\hat{V}_n(t)}=\sum_{p=0}^{n}\hat{W}_{n-p}(t)\overline{\hat{V}_p(t)}} \qquad (5)$$

General Expression II

$$\overline{\hat{H}_I(t)\hat{V}_n(t)} = \sum_{p=0}^{n} \hat{W}_{n-p}(t)\overline{\hat{V}_p(t)} \qquad (5)$$

General Expression II

$$\overline{\hat{H}_I(t)\hat{V}_n(t)} = \sum_{p=0}^{n} \hat{W}_{n-p}(t)\overline{\hat{V}_p(t)} \qquad (5)$$

$$n = 0: \quad \boxed{\hat{W}_0(t) = \overline{\hat{H}_I(t)}} \qquad (6a)$$

General Expression II

$$\overline{\hat{H}_I(t)\hat{V}_n(t)} = \sum_{p=0}^{n} \hat{W}_{n-p}(t)\overline{\hat{V}_p(t)} \qquad (5)$$

$$n=0: \quad \boxed{\hat{W}_0(t) = \overline{\hat{H}_I(t)}} \qquad (6a)$$

$$n=1: \quad \overline{\hat{H}_I(t)\hat{V}_1(t)} = \hat{W}_1(t) + \hat{W}_0(t)\overline{\hat{V}_1(t)}$$

General Expression II

$$\overline{\hat{H}_I(t)\hat{V}_n(t)} = \sum_{p=0}^{n} \hat{W}_{n-p}(t)\overline{\hat{V}_p(t)} \qquad (5)$$

$$n=0: \quad \boxed{\hat{W}_0(t) = \overline{\hat{H}_I(t)}} \qquad (6a)$$

$$n=1: \quad \overline{\hat{H}_I(t)\hat{V}_1(t)} = \hat{W}_1(t) + \hat{W}_0(t)\overline{\hat{V}_1(t)}$$

$$\Rightarrow \quad \boxed{\hat{W}_1(t) = \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)}} \qquad (6b)$$

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \ldots$$

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \ldots$$

What’s wrong with this result?

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\;\overline{\hat{V}_1(t)} + \ldots$$

What's wrong with this result?

- Hamiltonians have to be Hermitian!

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \ldots$$

What's wrong with this result?

- Hamiltonians have to be Hermitian!
- This is easy to fix:

$$\hat{H}_{eff}(t) = \mathrm{HP}\left\{\overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \ldots\right\} \quad (7)$$

where $\mathrm{HP}\left\{\hat{A}\right\} = \frac{1}{2}\left(\hat{A} + \hat{A}^{\dagger}\right)$

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \dots$$

What's wrong with this result?

- Hamiltonians have to be Hermitian!
- This is easy to fix:

$$\hat{H}_{eff}(t) = \mathrm{HP}\ \left\{\overline{\hat{H}_I(t)} + \overline{\hat{H}_I(t)\hat{V}_1(t)} - \overline{\hat{H}_I(t)}\ \overline{\hat{V}_1(t)} + \dots\right\} \quad (7)$$

where $\mathrm{HP}\ \{\hat{A}\} = \frac{1}{2}\left(\hat{A} + \hat{A}^{\dagger}\right)$

- This <u>can</u> be justified by deriving a master equation:
  - *excluded part of the frequency domain takes role of reservoir;*
  - *Lindblat equation with unitary part given by* (7);
  - *Neglect dephasing effects.*

General Expression III

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^\dagger = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^\dagger = -\overline{\hat{V}_1(t)}$$

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^{\dagger} = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^{t} \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^{\dagger} = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\boxed{\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \ldots\right)} \quad (8)$$

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^\dagger = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^\dagger = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t),\hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)},\overline{\hat{V}_1(t)}\right] + \ldots\right) \qquad (8)$$

- Result is independent of lower limit in integral for $V_1(t)$.

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^\dagger = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^\dagger = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\boxed{\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t),\hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)},\overline{\hat{V}_1(t)}\right] + \ldots\right)} \quad (8)$$

- Result is independent of lower limit in integral for $V_1(t)$.
- Also applies statistical averages over a stationary ensemble.

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^\dagger = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^\dagger = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\boxed{\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t),\hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)},\overline{\hat{V}_1(t)}\right] + \ldots\right)} \quad (8)$$

- Result is independent of lower limit in integral for $V_1(t)$.
- Also applies statistical averages over a stationary ensemble.
- This is NOT a perturabtive theory.

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^{\dagger} = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^{t} \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^{\dagger} = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\boxed{\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t),\hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)},\overline{\hat{V}_1(t)}\right] + \ldots\right)} \quad (8)$$

- Result is independent of lower limit in integral for $V_1(t)$.
- Also applies statistical averages over a stationary ensemble.
- This is NOT a perturabtive theory.

  -YES, we have used perturbation theory with reckless abandon, BUT

General Expression III

- Definition of a real averaging process implies:

$$\overline{\hat{H}_I(t)}^\dagger = \overline{\hat{H}_I(t)} \qquad \hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' \;\Rightarrow\; \overline{\hat{V}_1(t)}^\dagger = -\overline{\hat{V}_1(t)}$$

and so, (AT LAST):

$$\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \ldots\right) \qquad (8)$$

- Result is independent of lower limit in integral for $V_1(t)$.
- Also applies statistical averages over a stationary ensemble.
- This is NOT a perturabtive theory.

  -YES, we have used perturbation theory with reckless abandon, BUT

  -Solving Schrödinger's equation with this Hamiltonian gives a result that involves all orders of the perturbation parameter

important special case:

Harmonic Hamiltonians + Low Pass Filter

- Suppose we have a Hamiltonian made up of a sum of harmonic terms:

$$\hat{H}_I(t) = \sum_m \hat{h}_m e^{-i\omega_m t} + h.a. \qquad (\omega_m > 0) \qquad (9a)$$

important special case:

Harmonic Hamiltonians + Low Pass Filter

- Suppose we have a Hamiltonian made up of a sum of harmonic terms:

$$\hat{H}_I(t) = \sum_m \hat{h}_m e^{-i\omega_m t} + h.a. \qquad (\omega_m > 0) \qquad (9a)$$

$$\hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' = \sum_m \frac{1}{\hbar\omega_m}\left(\hat{h}_m e^{-i\omega_m t} - \hat{h}_m^\dagger e^{i\omega_m t}\right) \qquad (9b)$$

important special case:

Harmonic Hamiltonians + Low Pass Filter

- Suppose we have a Hamiltonian made up of a sum of harmonic terms:

$$\hat{H}_I(t) = \sum_m \hat{h}_m e^{-i\omega_m t} + h.a. \qquad (\omega_m > 0) \qquad (9a)$$

$$\hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' = \sum_m \frac{1}{\hbar\omega_m}\left(\hat{h}_m e^{-i\omega_m t} - \hat{h}_m^\dagger e^{i\omega_m t}\right) \qquad (9b)$$

- And the time averaging has the effect of removing all frequencies ≥ min{ω_m}, so that

$$\overline{\hat{H}_I(t)} = 0$$

$$\overline{\hat{V}_1(t)} = 0$$

important special case:

Harmonic Hamiltonians + Low Pass Filter

- Suppose we have a Hamiltonian made up of a sum of harmonic terms:

$$\hat{H}_I(t) = \sum_m \hat{h}_m e^{-i\omega_m t} + h.a. \qquad (\omega_m > 0) \qquad (9a)$$

$$\hat{V}_1(t) = \frac{1}{i\hbar}\int^t \hat{H}_I(t')dt' = \sum_m \frac{1}{\hbar\omega_m}\left(\hat{h}_m e^{-i\omega_m t} - \hat{h}_m^\dagger e^{i\omega_m t}\right) \qquad (9b)$$

- And the time averaging has the effect of removing all frequencies ≥ min{ω_m}, so that

$$\left.\begin{aligned}\overline{\hat{H}_I(t)} &= 0 \\ \overline{\hat{V}_1(t)} &= 0\end{aligned}\right\}$$ on the whole, looks rather boring

Eq.(8): $\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \ldots\right)$

Eq.(8): $\hat{H}_{eff}(t) = \underbrace{\overline{\hat{H}_I(t)}}_{0} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\underbrace{\overline{\hat{H}_I(t)}}_{0}, \underbrace{\overline{\hat{V}_1(t)}}_{0}\right] + \dots\right)$

Eq.(8): $\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \ldots \right)$

$\qquad\qquad\qquad\quad 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \qquad\quad 0$

$$\sum_{m,n} \frac{1}{2\hbar\omega_n} \left\{ \left[\hat{h}_m, \hat{h}_n\right] \overline{e^{-i(\omega_m+\omega_n)t}} - \left[\hat{h}_m, \hat{h}_n^\dagger\right] \overline{e^{-i(\omega_m-\omega_n)t}} \right.$$

$$\left. + \left[\hat{h}_m^\dagger, \hat{h}_n\right] \overline{e^{i(\omega_m-\omega_n)t}} - \left[\hat{h}_m^\dagger, \hat{h}_n^\dagger\right] \overline{e^{i(\omega_m+\omega_n)t}} \right\}$$

$$\text{Eq.(8): } \hat{H}_{eff}(t) = \underbrace{\overline{\hat{H}_I(t)}}_{0} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\underbrace{\overline{\hat{H}_I(t)}}_{0}, \underbrace{\overline{\hat{V}_1(t)}}_{0}\right] + \dots\right)$$

$$\sum_{m,n} \frac{1}{2\hbar\omega_n} \left\{ \left[\hat{h}_m, \hat{h}_n\right] \underbrace{\overline{e^{-i(\omega_m+\omega_n)t}}}_{0} - \left[\hat{h}_m, \hat{h}_n^\dagger\right] \overline{e^{-i(\omega_m-\omega_n)t}} + \left[\hat{h}_m^\dagger, \hat{h}_n\right] \overline{e^{i(\omega_m-\omega_n)t}} - \left[\hat{h}_m^\dagger, \hat{h}_n^\dagger\right] \underbrace{\overline{e^{i(\omega_m+\omega_n)t}}}_{0} \right\}$$

Eq.(8): $\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \dots \right)$

0 0 0

$$\sum_{m,n} \frac{1}{2\hbar\omega_n} \left\{ \left[\hat{h}_m, \hat{h}_n\right] \overline{e^{-i(\omega_m+\omega_n)t}} - \left[\hat{h}_m, \hat{h}_n^\dagger\right] \overline{e^{-i(\omega_m-\omega_n)t}} \right.$$

0

$$\left. + \left[\hat{h}_m^\dagger, \hat{h}_n\right] \overline{e^{i(\omega_m-\omega_n)t}} - \left[\hat{h}_m^\dagger, \hat{h}_n^\dagger\right] \overline{e^{i(\omega_m+\omega_n)t}} \right\}$$

0

$$\hat{H}_{eff}(t) = \sum_{m,n} \frac{1}{\hbar \overline{\overline{\omega}}_{mn}} \left[\hat{h}_m^\dagger, \hat{h}_n\right] e^{i(\omega_m-\omega_n)t} \qquad (10)$$

where: $\frac{1}{\overline{\overline{\omega}}_{mn}} = \frac{1}{2}\left(\frac{1}{\omega_m} + \frac{1}{\omega_n} \right)$

Eq.(8): $\hat{H}_{eff}(t) = \overline{\hat{H}_I(t)} + \frac{1}{2}\left(\overline{\left[\hat{H}_I(t), \hat{V}_1(t)\right]} - \left[\overline{\hat{H}_I(t)}, \overline{\hat{V}_1(t)}\right] + \ldots \right)$

0 0 0

$$\sum_{m,n} \frac{1}{2\hbar\omega_n} \left\{ \left[\hat{h}_m, \hat{h}_n\right] \overline{e^{-i(\omega_m+\omega_n)t}} - \left[\hat{h}_m, \hat{h}_n^\dagger\right] \overline{e^{-i(\omega_m-\omega_n)t}} \right.$$

0

$$\left. + \left[\hat{h}_m^\dagger, \hat{h}_n\right] \overline{e^{i(\omega_m-\omega_n)t}} - \left[\hat{h}_m^\dagger, \hat{h}_n^\dagger\right] \overline{e^{i(\omega_m+\omega_n)t}} \right\}$$

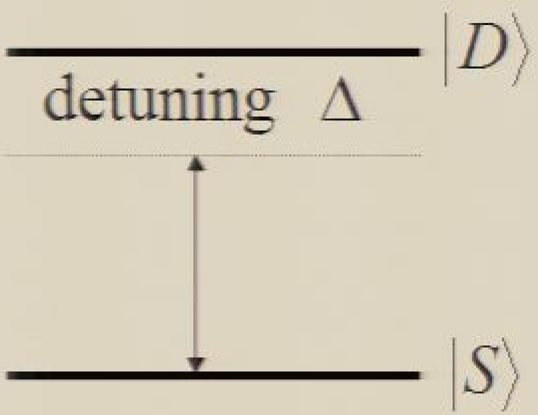
0

$$\hat{H}_{eff}(t) = \sum_{m,n} \frac{1}{\hbar\overline{\overline{\omega}}_{mn}} \left[\hat{h}_m^\dagger, \hat{h}_n\right] e^{i(\omega_m-\omega_n)t} \qquad (10)$$

where: $\frac{1}{\overline{\overline{\omega}}_{mn}} = \frac{1}{2}\left(\frac{1}{\omega_m} + \frac{1}{\omega_n} \right)$

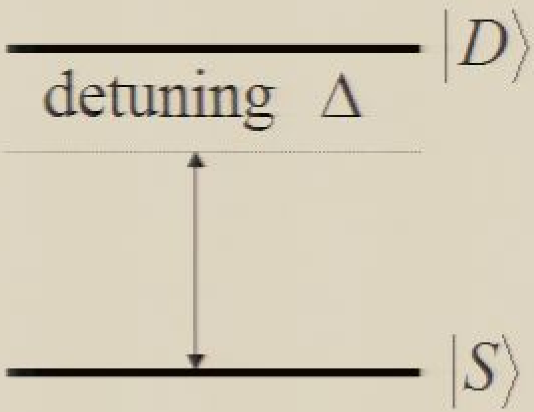
Ref: D. F. V. James, *Fortschritte der Physik* **48**, 823-837 (2000); Related results: Average Hamiltonians (NMR); C. Cohen-Tannoudji J Dupont-Roc and G. Grynberg, *Atom-Photon Interactions* (Wiley, 1992), pp. 38-48.

Example 1: AC Stark Shifts



$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

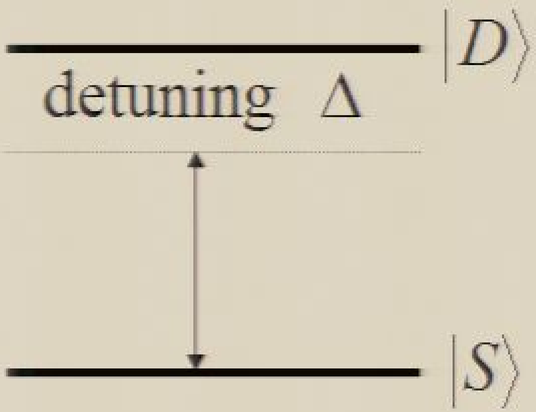
Example 1: AC Stark Shifts



$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

i.e.: $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$; $\omega_1 = \Delta$

Example 1: AC Stark Shifts

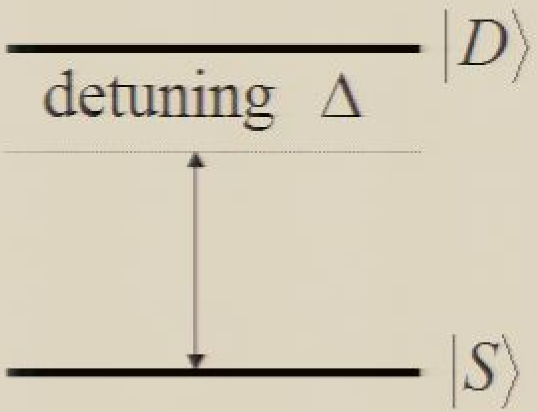


$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

i.e.: $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$; $\omega_1 = \Delta$

$$\left[\hat{h}_1^\dagger, \hat{h}_1\right] = \frac{\hbar^2|\Omega|^2}{4}\left[|S\rangle\langle D|, |D\rangle\langle S|\right] \; ; \qquad \overline{\omega_{11}} \equiv \omega_1 = \Delta$$

Example 1: AC Stark Shifts



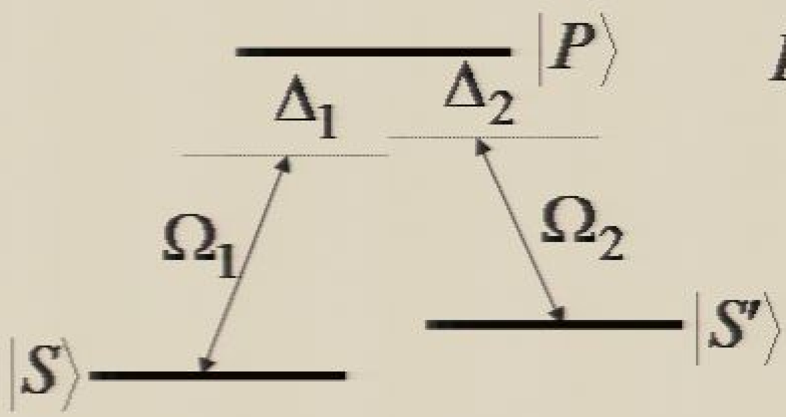
$$\hat{H}_I = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{-i\Delta t} + h.a.$$

i.e.: $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$; $\omega_1 = \Delta$

$$\left[\hat{h}_1^\dagger, \hat{h}_1\right] = \frac{\hbar^2|\Omega|^2}{4}\left[|S\rangle\langle D|, |D\rangle\langle S|\right] \; ; \qquad \overline{\omega_{11}} \equiv \omega_1 = \Delta$$

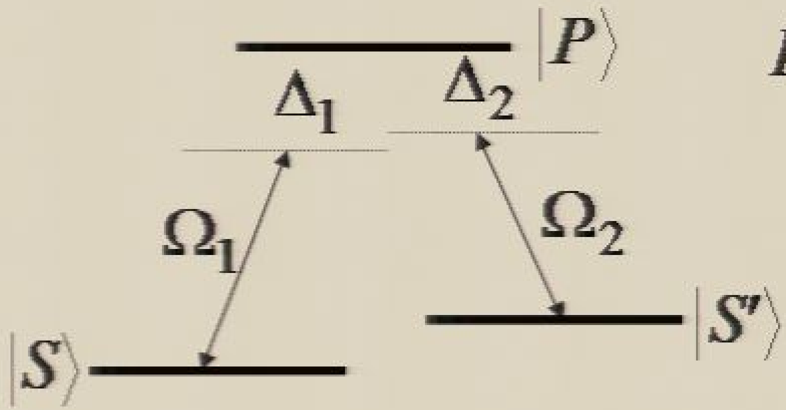
$$\hat{H}_{eff} = -\frac{\hbar|\Omega_0|^2}{4\Delta}\left(|D\rangle\langle D| - |S\rangle\langle S|\right)$$

Example 2: Raman Processes



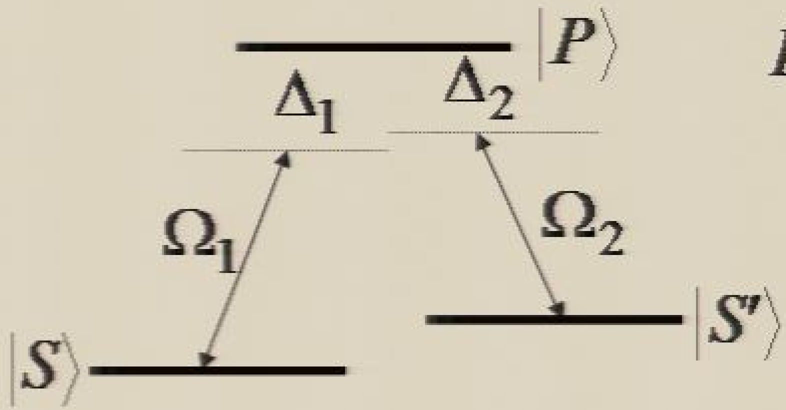
$$\hat{H}_I = \frac{\hbar\Omega_1}{2}|P\rangle\langle S|e^{-i\Delta_1 t} + \frac{\hbar\Omega_2}{2}|P\rangle\langle S'|e^{-i\Delta_2 t} + h.a.$$

Example 2: Raman Processes



$$\hat{H}_I = \underbrace{\frac{\hbar\Omega_1}{2}|P\rangle\langle S|e^{-i\Delta_1 t}}_{\hat{h}_1} + \underbrace{\frac{\hbar\Omega_2}{2}|P\rangle\langle S'|e^{-i\Delta_2 t}}_{\hat{h}_2} + h.a.$$

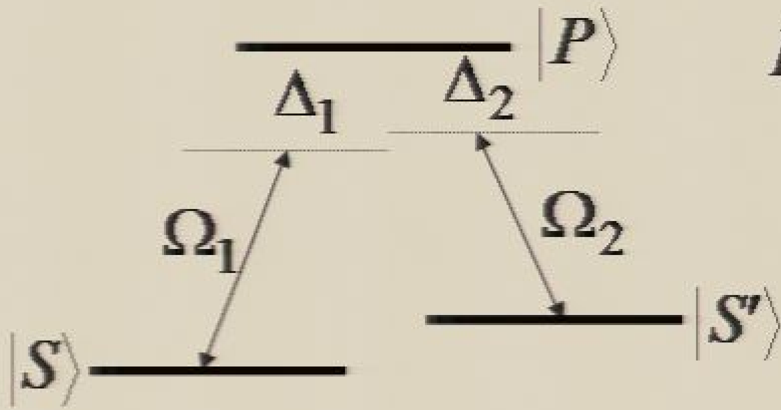
Example 2: Raman Processes



$$\hat{H}_I = \underbrace{\frac{\hbar\Omega_1}{2}|P\rangle\langle S|e^{-i\Delta_1 t}}_{\hat{h}_1} + \underbrace{\frac{\hbar\Omega_2}{2}|P\rangle\langle S'|e^{-i\Delta_2 t}}_{\hat{h}_2} + h.a.$$

$$\hat{H}_{eff} = \frac{1}{\hbar\omega_1}\left[\hat{h}_1^\dagger, \hat{h}_1\right] + \frac{1}{\hbar\omega_2}\left[\hat{h}_2^\dagger, \hat{h}_2\right] + \left(\frac{1}{\hbar\omega_{12}}\left[\hat{h}_1^\dagger, \hat{h}_2\right]e^{i(\omega_1-\omega_2)t} + h.a.\right)$$

Example 2: Raman Processes



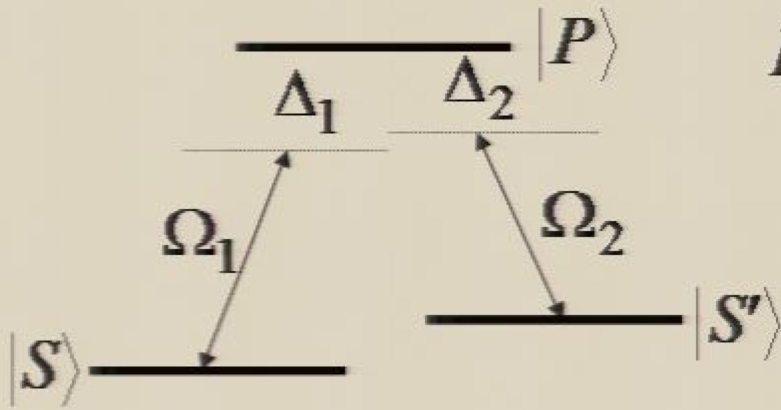
$$\hat{H}_I = \underbrace{\frac{\hbar\Omega_1}{2}|P\rangle\langle S|e^{-i\Delta_1 t}}_{\hat{h}_1} + \underbrace{\frac{\hbar\Omega_2}{2}|P\rangle\langle S'|e^{-i\Delta_2 t}}_{\hat{h}_2} + h.a.$$

$$\hat{H}_{eff} = \frac{1}{\hbar\omega_1}\left[\hat{h}_1^\dagger, \hat{h}_1\right] + \frac{1}{\hbar\omega_2}\left[\hat{h}_2^\dagger, \hat{h}_2\right] + \left(\frac{1}{\hbar\bar{\omega}_{12}}\left[\hat{h}_1^\dagger, \hat{h}_2\right]e^{i(\omega_1-\omega_2)t} + h.a.\right)$$

$$= -\frac{\hbar|\Omega_1|^2}{4\Delta_1}\left(|P\rangle\langle P| - |S\rangle\langle S|\right) - \frac{\hbar|\Omega_2|^2}{4\Delta_2}\left(|P\rangle\langle P| - |S'\rangle\langle S'|\right)$$

$$+\left(\frac{\hbar\Omega_1^*\Omega_2}{4\bar{\Delta}}|S\rangle\langle S'|e^{i(\Delta_1-\Delta_2)t} + h.a.\right)$$

Example 2: Raman Processes



$$\hat{H}_I = \underbrace{\frac{\hbar\Omega_1}{2}|P\rangle\langle S|}_{\hat{h}_1} e^{-i\Delta_1 t} + \underbrace{\frac{\hbar\Omega_2}{2}|P\rangle\langle S'|}_{\hat{h}_2} e^{-i\Delta_2 t} + h.a.$$

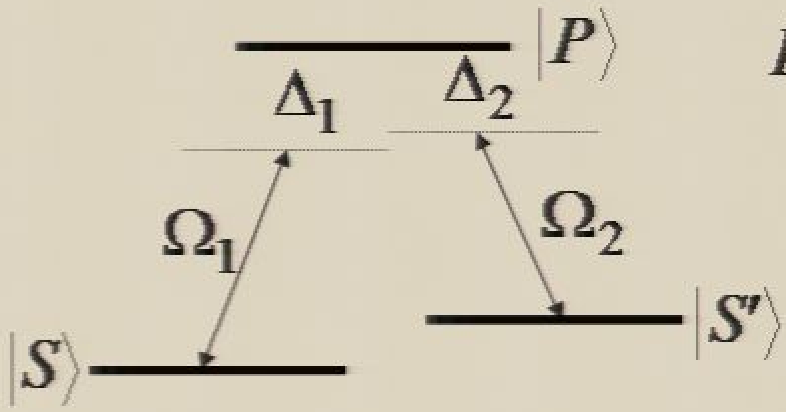
$$\hat{H}_{eff} = \frac{1}{\hbar\omega_1}\left[\hat{h}_1^\dagger, \hat{h}_1\right] + \frac{1}{\hbar\omega_2}\left[\hat{h}_2^\dagger, \hat{h}_2\right] + \left(\frac{1}{\hbar\bar{\omega}_{12}}\left[\hat{h}_1^\dagger, \hat{h}_2\right] e^{i(\omega_1-\omega_2)t} + h.a.\right)$$

$$= -\frac{\hbar|\Omega_1|^2}{4\Delta_1}\left(|P\rangle\langle P| - |S\rangle\langle S|\right) - \frac{\hbar|\Omega_2|^2}{4\Delta_2}\left(|P\rangle\langle P| - |S'\rangle\langle S'|\right)$$

$$+ \left(\frac{\hbar\Omega_1^*\Omega_2}{4\bar{\Delta}}|S\rangle\langle S'| e^{i(\Delta_1-\Delta_2)t} + h.a.\right)$$

A.C. Stark shifts (again!)

Example 2: Raman Processes



$$\hat{H}_I = \underbrace{\frac{\hbar\Omega_1}{2}|P\rangle\langle S|e^{-i\Delta_1 t}}_{\hat{h}_1} + \underbrace{\frac{\hbar\Omega_2}{2}|P\rangle\langle S'|e^{-i\Delta_2 t}}_{\hat{h}_2} + h.a.$$

$$\hat{H}_{eff} = \frac{1}{\hbar\omega_1}\left[\hat{h}_1^\dagger, \hat{h}_1\right] + \frac{1}{\hbar\omega_2}\left[\hat{h}_2^\dagger, \hat{h}_2\right] + \left(\frac{1}{\hbar\bar{\omega}_{12}}\left[\hat{h}_1^\dagger, \hat{h}_2\right]e^{i(\omega_1-\omega_2)t} + h.a.\right)$$

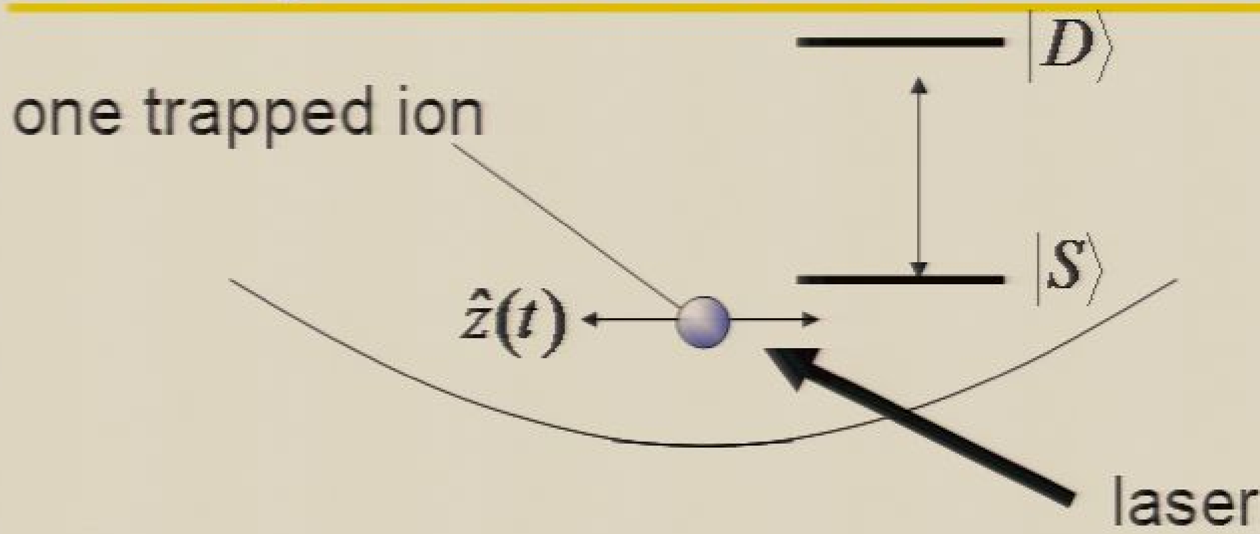
$$= -\frac{\hbar|\Omega_1|^2}{4\Delta_1}\left(|P\rangle\langle P| - |S\rangle\langle S|\right) - \frac{\hbar|\Omega_2|^2}{4\Delta_2}\left(|P\rangle\langle P| - |S'\rangle\langle S'|\right)$$

$$+ \frac{\hbar\Omega_1^*\Omega_2}{4\bar{\Delta}}|S\rangle\langle S'|e^{i(\Delta_1-\Delta_2)t} + h.a.$$

A.C. Stark shifts (again!)

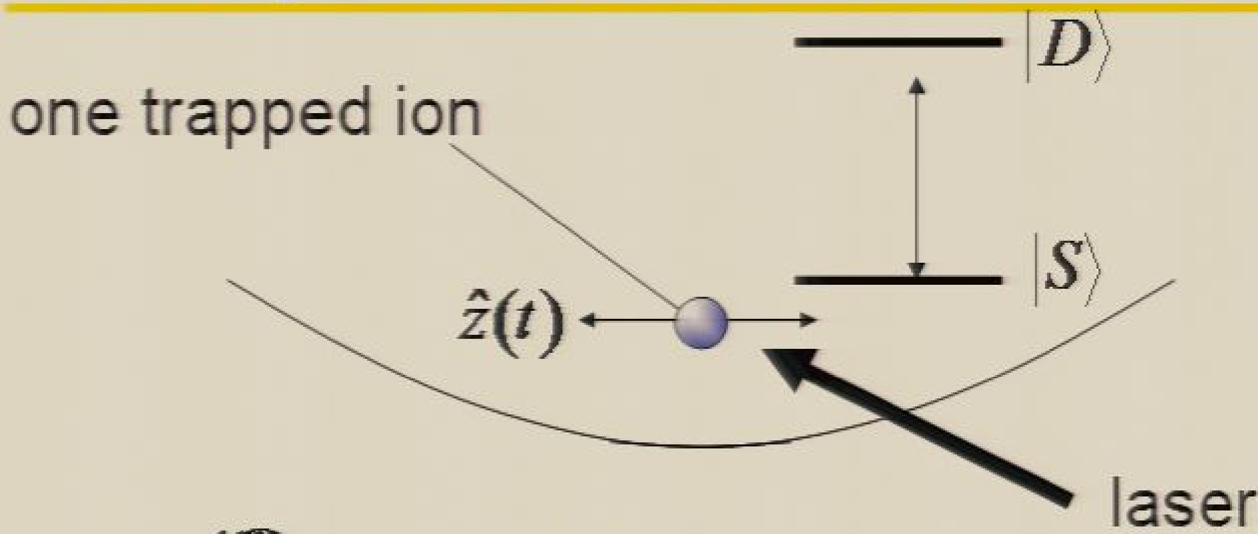
Raman Transitions

Example 3: Quantum A.C. Stark Shift



C. d'Helon and G. Milburn, *Phys. Rev. A* **54**, 5141 (1996); S. Schneider et al., *J. Mod Opt.* **47**, 499 (2000); F. Schmidt-Kaler et al, *Europhys. Lett.* **65**, 587 (2004).

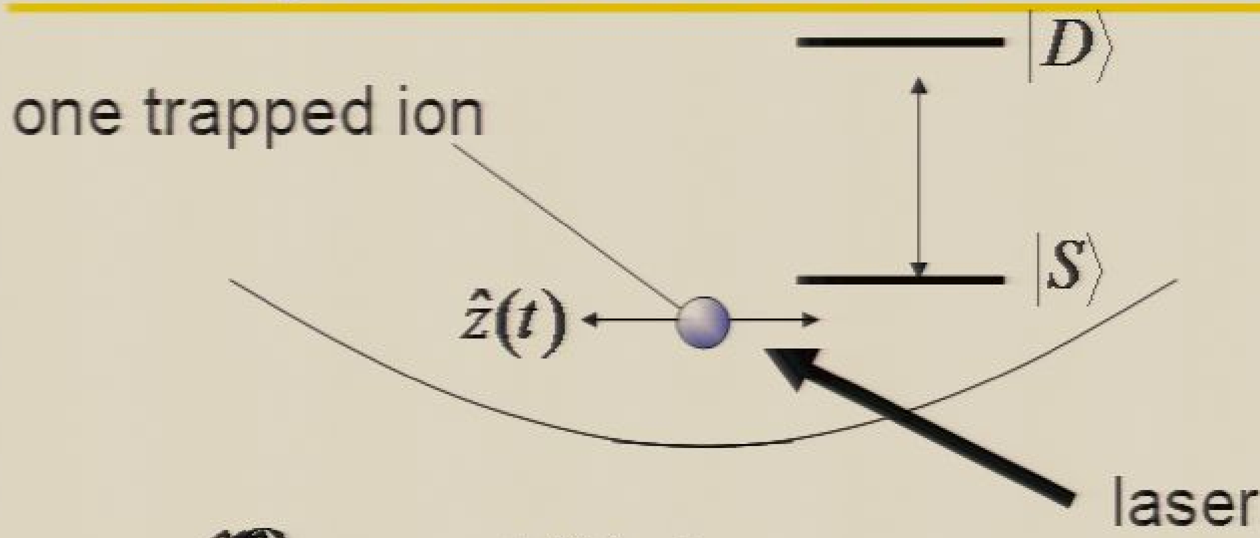
Example 3: Quantum A.C. Stark Shift



$$\hat{H}_I(t) = \frac{\hbar\Omega}{2}|D\rangle\langle S|e^{ik_z\hat{z}(t)-i\Delta t} + h.a.$$

C. d'Helon and G. Milburn, *Phys. Rev. A* **54**, 5141 (1996); S. Schneider et al., *J. Mod Opt.* **47**, 499 (2000); F. Schmidt-Kaler et al, *Europhys. Lett.* **65**, 587 (2004).

Example 3: Quantum A.C. Stark Shift

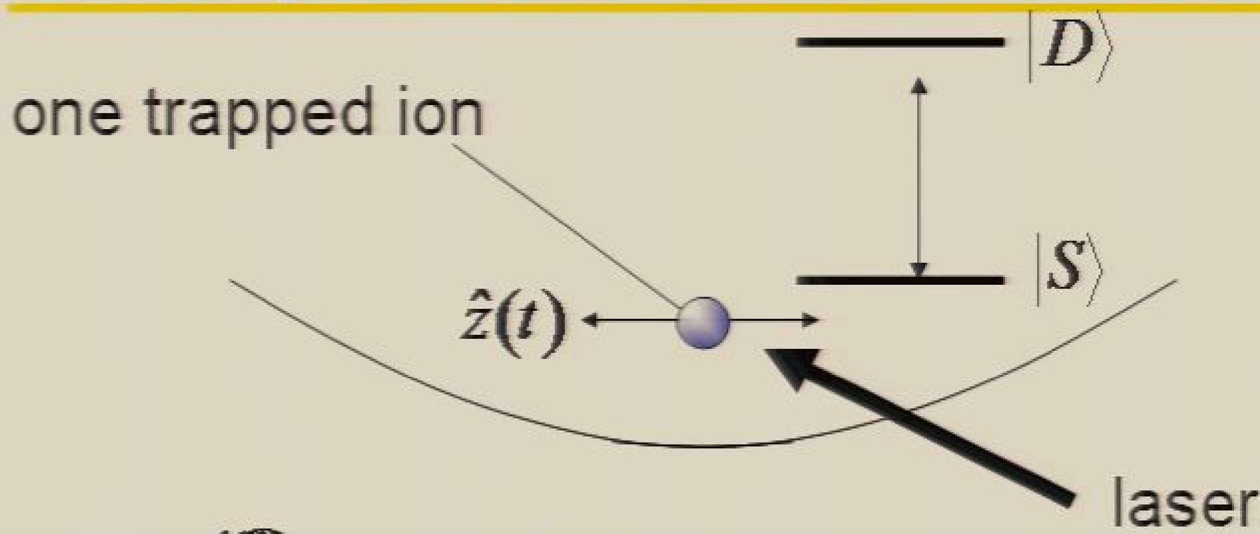


$$\hat{H}_I(t) = \frac{\hbar\Omega}{2}|D\rangle\langle S| e^{ik_z\hat{z}(t) - i\Delta t} + h.a.$$

$$k_z\hat{z}(t) = \frac{\eta}{\sqrt{N}} \sum_{\substack{p=1 \\ \text{(all Z modes)}}}^{N} s_m^p \left(\hat{a}_p e^{-i\omega_p t} + \hat{a}_p^\dagger e^{i\omega_p t}\right)$$

C. d'Helon and G. Milburn, *Phys. Rev. A* **54**, 5141 (1996); S. Schneider et al., *J. Mod Opt.* **47**, 499 (2000); F. Schmidt-Kaler et al, *Europhys. Lett.* **65**, 587 (2004).

Example 3: Quantum A.C. Stark Shift



$$\hat{H}_I(t) = \frac{\hbar\Omega}{2}|D\rangle\langle S| e^{ik_z\hat{z}(t) - i\Delta t} + h.a.$$

$$k_z\hat{z}(t) = \frac{\eta}{\sqrt{N}} \sum_{\substack{p=1 \\ \text{(all Z modes)}}}^{N} s_m^p \left(\hat{a}_p e^{-i\omega_p t} + \hat{a}_p^\dagger e^{i\omega_p t}\right)$$

ion-mode coupling factor: DFVJ, *Appl. Phys. B* **66**, 181 (1998).

C. d'Helon and G. Milburn, *Phys. Rev. A* **54**, 5141 (1996); S. Schneider et al., *J. Mod Opt.* **47**, 499 (2000); F. Schmidt-Kaler et al, *Europhys. Lett.* **65**, 587 (2004).

- Lamb-Dicke approximation: $e^{ik_z\hat{z}(t)} \approx 1 + ik_z\hat{z}(t)$

- Lamb-Dicke approximation: $e^{ik_z\hat{z}(t)} \approx 1 + ik_z\hat{z}(t)$
- “carrier” term $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$ $\omega_1 = \Delta$

- Lamb-Dicke approximation: $e^{ik_z\hat{z}(t)} \approx 1 + ik_z\hat{z}(t)$

- “carrier” term $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$ $\omega_1 = \Delta$

- red sideband: $\hat{h}_2 = \frac{i\eta\hbar\Omega}{2\sqrt{N}} s^{p}_{m}|D\rangle\langle S|\hat{a}_p$ $\omega_2 = \Delta + \omega_p$

- Lamb-Dicke approximation: $e^{ik_z\hat{z}(t)} \approx 1 + ik_z\hat{z}(t)$

- "carrier" term $\hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S|$ $\omega_1 = \Delta$

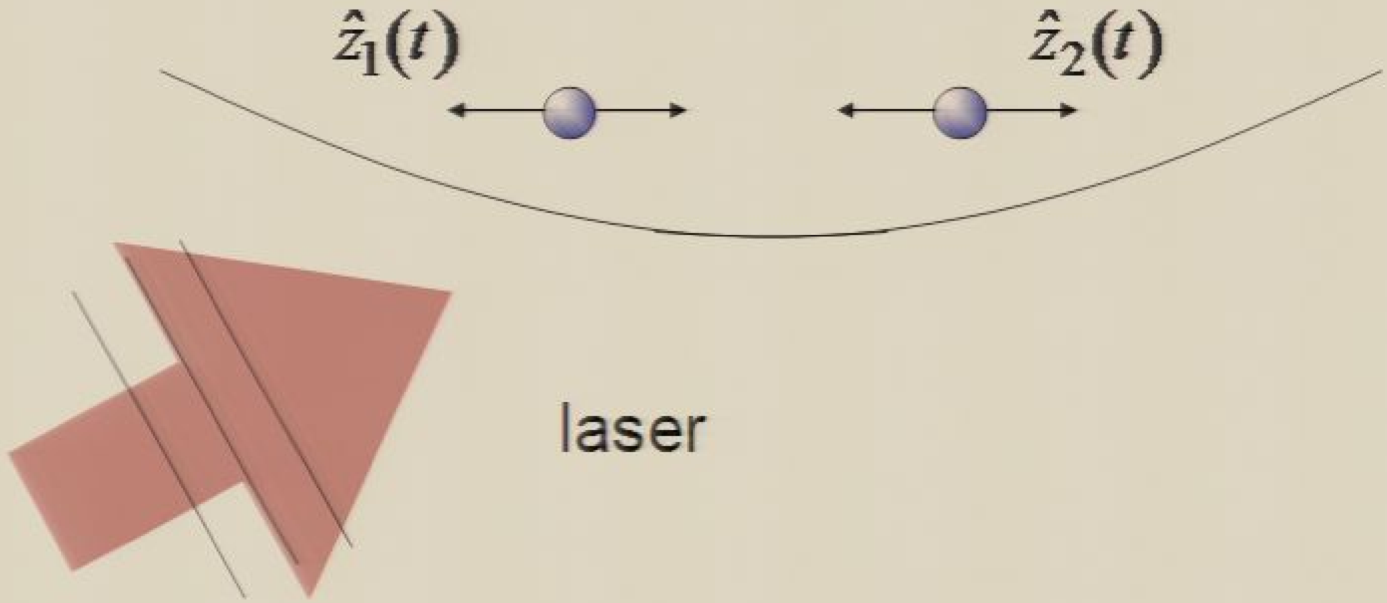
- red sideband: $\hat{h}_2 = \frac{i\eta\hbar\Omega}{2\sqrt{N}}\sum_p |D\rangle\langle S|\hat{a}_p$ $\omega_2 = \Delta + \omega_p$

- blue sideband: $\hat{h}_3 = \frac{i\eta\hbar\Omega}{2\sqrt{N}}\sum_p |D\rangle\langle S|\hat{a}_p^\dagger$ $\omega_3 = \Delta - \omega_p$

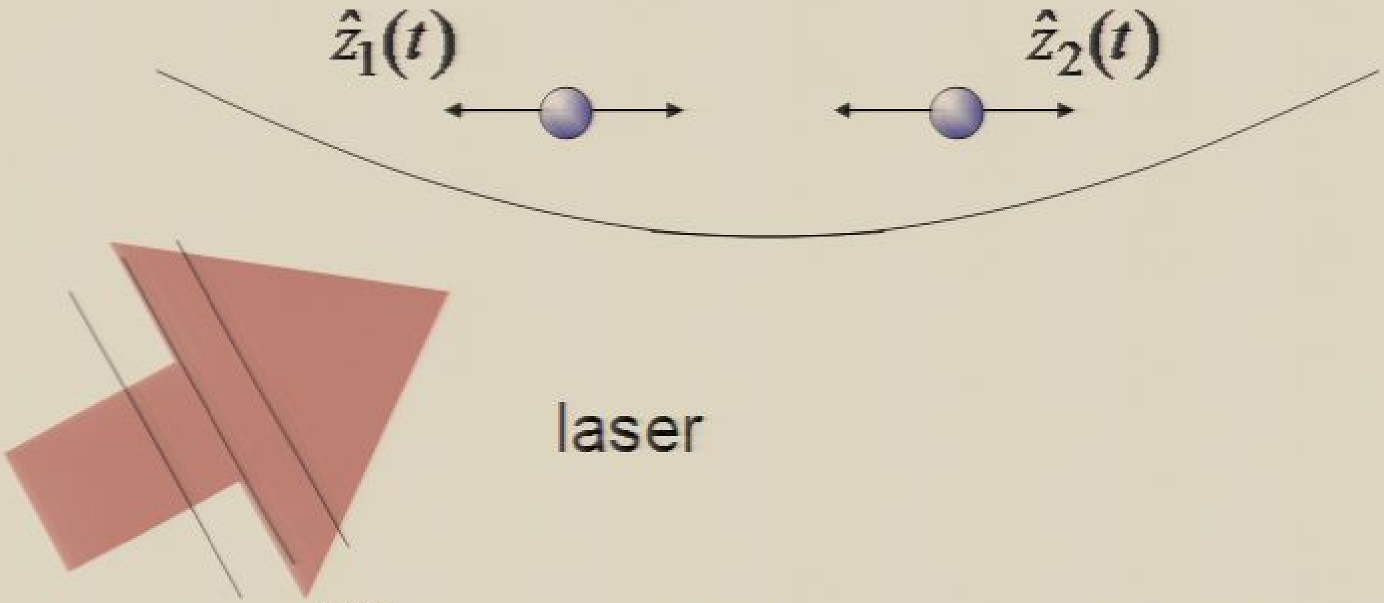
- Lamb-Dicke approximation: $e^{ik_z\hat{z}(t)} \approx 1 + ik_z\hat{z}(t)$
- "carrier" term $\quad \hat{h}_1 = \frac{\hbar\Omega}{2}|D\rangle\langle S| \quad\quad \omega_1 = \Delta$
- red sideband: $\quad \hat{h}_2 = \frac{i\eta\hbar\Omega}{2\sqrt{N}} s_m^p |D\rangle\langle S| \hat{a}_p \quad\quad \omega_2 = \Delta + \omega_p$
- blue sideband: $\quad \hat{h}_3 = \frac{i\eta\hbar\Omega}{2\sqrt{N}} s_m^p |D\rangle\langle S| \hat{a}_p^\dagger \quad\quad \omega_3 = \Delta - \omega_p$
- low pass filter excludes oscillations at ω_p, hence:

$$H_{eff} = -\frac{|\hbar\Omega|^2}{4\Delta}\left(1 + \frac{2\eta^2}{N}\left(s_m^p\right)^2 \frac{\Delta^2}{\Delta^2 - \omega_p^2}\left(n_p + 1/2\right)\right)\left(|D\rangle\langle D| - |S\rangle\langle S|\right)$$

What about two ions?



What about two ions?



$$\hat{H}_I(t) = \frac{\hbar\Omega}{2}\left(|D\rangle\langle S|_1 e^{ik_z \hat{z}_1(t)} + |D\rangle\langle S|_2 e^{ik_z \hat{z}_2(t)} \right) e^{-i\Delta t} + h.a.$$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1} = 1\right)$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1}=1\right)$

- Define a collective spin operator $\hat{J}^{(+)}=\left(|D\rangle\langle S|_1+|D\rangle\langle S|_2\right)$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1}=1\right)$
- Define a collective spin operator $\hat{J}^{(+)}=\left(|D\rangle\langle S|_1+|D\rangle\langle S|_2\right)$
- "carrier", red and blue sideband terms:

$$\hat{h}_1=\frac{\hbar\Omega}{2}\hat{J}^{(+)},\quad \hat{h}_2=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p,\quad \hat{h}_3=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p^{\dagger}$$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1}=1\right)$

- Define a collective spin operator $\hat{J}^{(+)}=\left(|D\rangle\langle S|_1+|D\rangle\langle S|_2\right)$

- "carrier", red and blue sideband terms:

$$\hat{h}_1=\frac{\hbar\Omega}{2}\hat{J}^{(+)},\quad \hat{h}_2=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p,\quad \hat{h}_3=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p^{\dagger}$$

$$\frac{1}{(\Delta+\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1^{\dagger},\hat{J}^{(+)}\hat{a}_1\right]+\frac{1}{(\Delta-\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1,\hat{J}^{(+)}\hat{a}_1^{\dagger}\right]=$$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1}=1\right)$

- Define a collective spin operator $\hat{J}^{(+)}=\left(|D\rangle\langle S|_1+|D\rangle\langle S|_2\right)$

- “carrier”, red and blue sideband terms:

$$\hat{h}_1=\frac{\hbar\Omega}{2}\hat{J}^{(+)},\quad \hat{h}_2=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p,\quad \hat{h}_3=\frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p^{\dagger}$$

$$\frac{1}{(\Delta+\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1^{\dagger},\hat{J}^{(+)}\hat{a}_1\right]+\frac{1}{(\Delta-\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1,\hat{J}^{(+)}\hat{a}_1^{\dagger}\right]=$$

$$\frac{2\Delta}{\left(\Delta^2-\omega_1^2\right)}\left(\hat{n}_1+\tfrac{1}{2}\right)\left[\hat{J}^{(-)},\hat{J}^{(+)}\right]-\frac{2\omega_1}{\left(\Delta^2-\omega_1^2\right)}\left\{\hat{J}^{(-)},\hat{J}^{(+)}\right\}$$

- nearly resonant with the CM (p=1) mode $\left(s_m^{p=1}=1\right)$
- Define a collective spin operator $\hat{J}^{(+)} = \left(|D\rangle\langle S|_1 + |D\rangle\langle S|_2\right)$
- "carrier", red and blue sideband terms:

$$\hat{h}_1 = \frac{\hbar\Omega}{2}\hat{J}^{(+)}, \quad \hat{h}_2 = \frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p, \quad \hat{h}_3 = \frac{i\eta\hbar\Omega}{2\sqrt{N}}\hat{J}^{(+)}\hat{a}_p^{\dagger}$$

$$\frac{1}{(\Delta+\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1^{\dagger}, \hat{J}^{(+)}\hat{a}_1\right] + \frac{1}{(\Delta-\omega_1)}\left[\hat{J}^{(-)}\hat{a}_1, \hat{J}^{(+)}\hat{a}_1^{\dagger}\right] =$$

$$\frac{2\Delta}{\left(\Delta^2-\omega_1^2\right)}\left(\hat{n}_1 + 1/2\right)\left[\hat{J}^{(-)}, \hat{J}^{(+)}\right] - \frac{2\omega_1}{\left(\Delta^2-\omega_1^2\right)}\left\{\hat{J}^{(-)}, \hat{J}^{(+)}\right\}$$

new term: wasn't there for single ion

Hence the effective Hamilton is

$$H_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(1+\frac{2\eta^2\Delta^2}{N\left(\Delta^2-\omega_1^2\right)}\left(n_1+\tfrac{1}{2}\right)\right)\sum_m\left(|D\rangle\langle D|_m - |S\rangle\langle S|_m\right)$$

$$-\frac{\hbar|\Omega|^2\eta^2\omega_1}{2\left(\Delta^2-\omega_1^2\right)}\left\{\hat{J}^{(-)},\hat{J}^{(+)}\right\}$$

Hence the effective Hamilton is

$$H_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(1+\frac{2\eta^2\Delta^2}{N\left(\Delta^2-\omega_1^2\right)}\left(n_1+\tfrac{1}{2}\right)\right)\sum_m\left(|D\rangle\langle D|_m - |S\rangle\langle S|_m\right)$$

Quantum A.C. Stark shift again: yawn

$$-\frac{\hbar|\Omega|^2\eta^2\omega_1}{2\left(\Delta^2-\omega_1^2\right)}\left\{\hat{j}^{(-)},\hat{j}^{(+)}\right\}$$

Hence the effective Hamilton is

$$H_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(1+\frac{2\eta^2\Delta^2}{N\left(\Delta^2-\omega_1^2\right)}\left(n_1+\tfrac{1}{2}\right)\right)\sum_m\left(|D\rangle\langle D|_m-|S\rangle\langle S|_m\right)$$

Quantum A.C. Stark shift again: yawn

$$-\frac{\hbar|\Omega|^2\eta^2\omega_1}{2\left(\Delta^2-\omega_1^2\right)}\left\{\hat{j}^{(-)},\hat{j}^{(+)}\right\}$$

Couples the two ions: VERY INTERESTING!

Hence the effective Hamilton is

$$H_{eff} = -\frac{\hbar|\Omega|^2}{4\Delta}\left(1+\frac{2\eta^2\Delta^2}{N\left(\Delta^2-\omega_1^2\right)}\left(n_1+\tfrac{1}{2}\right)\right)\sum_m\left(|D\rangle\langle D|_m - |S\rangle\langle S|_m\right)$$

Quantum A.C. Stark shift
again: yawn

$$-\frac{\hbar|\Omega|^2\eta^2\omega_1}{2\left(\Delta^2-\omega_1^2\right)}\left\{\hat{j}^{(-)},\hat{j}^{(+)}\right\}$$

Couples the two ions:
VERY INTERESTING!

- Add another laser (with negative detuning): Quantum A.C. Stark shifts cancel, but coupling term is doubled: **Mølmer-Sørensen gate**

Conclusions

• The time-averaged dynamics of a system with a harmonic Hamiltonian of the form:

$$\hat{H}_I(t) = \sum_m \hat{h}_m \mathrm{e}^{-i\omega_\mathrm{m} t} + h.a.$$

Is described by an effective Hamiltonian given by:

$$\hat{H}_{eff}(t) = \sum_{m,n} \frac{1}{\hbar \overline{\omega}_{mn}} \left[\hat{h}_m^\dagger , \hat{h}_n \right] e^{i(\omega_m - \omega_n)t}$$

where: $$\frac{1}{\overline{\omega}_{mn}} = \frac{1}{2}\left(\frac{1}{\omega_m} + \frac{1}{\omega_n} \right)$$