Title: Higher-Dimensional Algebra: A Language for Quantum Spacetime

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Abstract: Category theory is a general language for describing things and processes - called "objects" and "morphisms". In this language, many counterintuitive features of quantum theory turn out to be properties shared by the category of Hilbert spaces and the category of cobordisms, in which objects are choices of "space" and emorphisms are choices of "spacetime". This striking fact suggests that "n-categories with duals" are a promising language for a quantum theory of spacetime. We sketch the historical development of these ideas from Feynman diagrams to string theory, topological quantum field theory, spin networks and spin foams, and especially recent work on open-closed string theory, 3d quantum gravity coupled to point particles, and 4d BF theory coupled to strings.

Higher-Dimensional Algebra:
A Language For Quantum Spacetime

John C. Baez

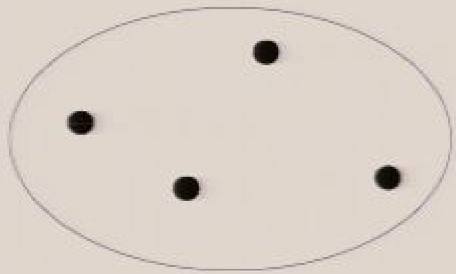
Perimeter Institute
May 31, 2006

figures by Aaron Lauda

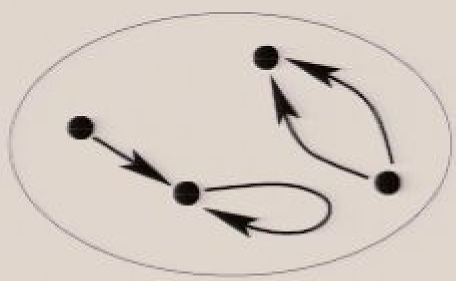
for more, see
http://math.ucr.edu/home/baez/quantum_spacetime

The Big Idea

Once upon a time, mathematics was all about *sets*:

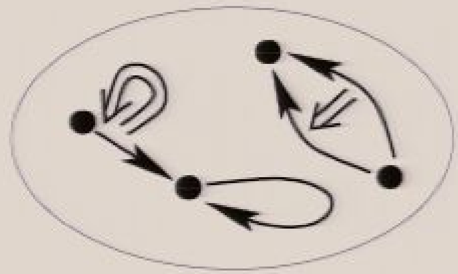


In 1945, Eilenberg and Mac Lane introduced *categories*:



These put *processes* on an equal footing with *things*.

In 1967 Bénabou introduced *weak 2-categories*:



These include *processes between processes.*

In 1995, Gordon, Power and Street introduced *weak 3-categories.*

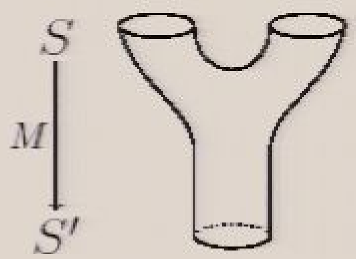
Since then we have been developing a general theory of *weak n-categories*, which is starting to have a big impact on math.

What about physics?

First, how do categories impact physics? I claim:

Quantum theory makes more sense when seen as part of a theory of spacetime — but this can only be understood using categories.

Why? The 'weird' features of quantum theory come from the ways that Hilb is less like Set than nCob — the category where objects are choices of 'space' and morphisms are choices of 'spacetime':



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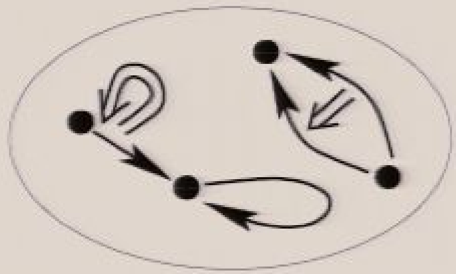

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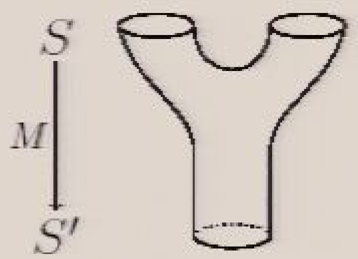
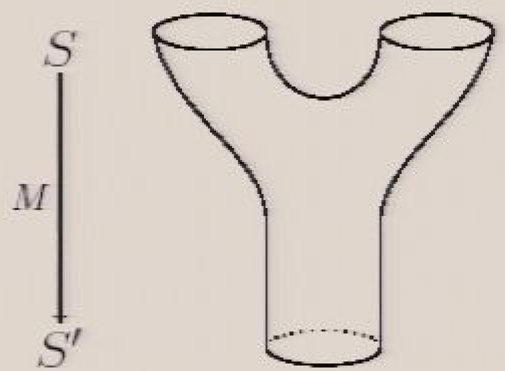


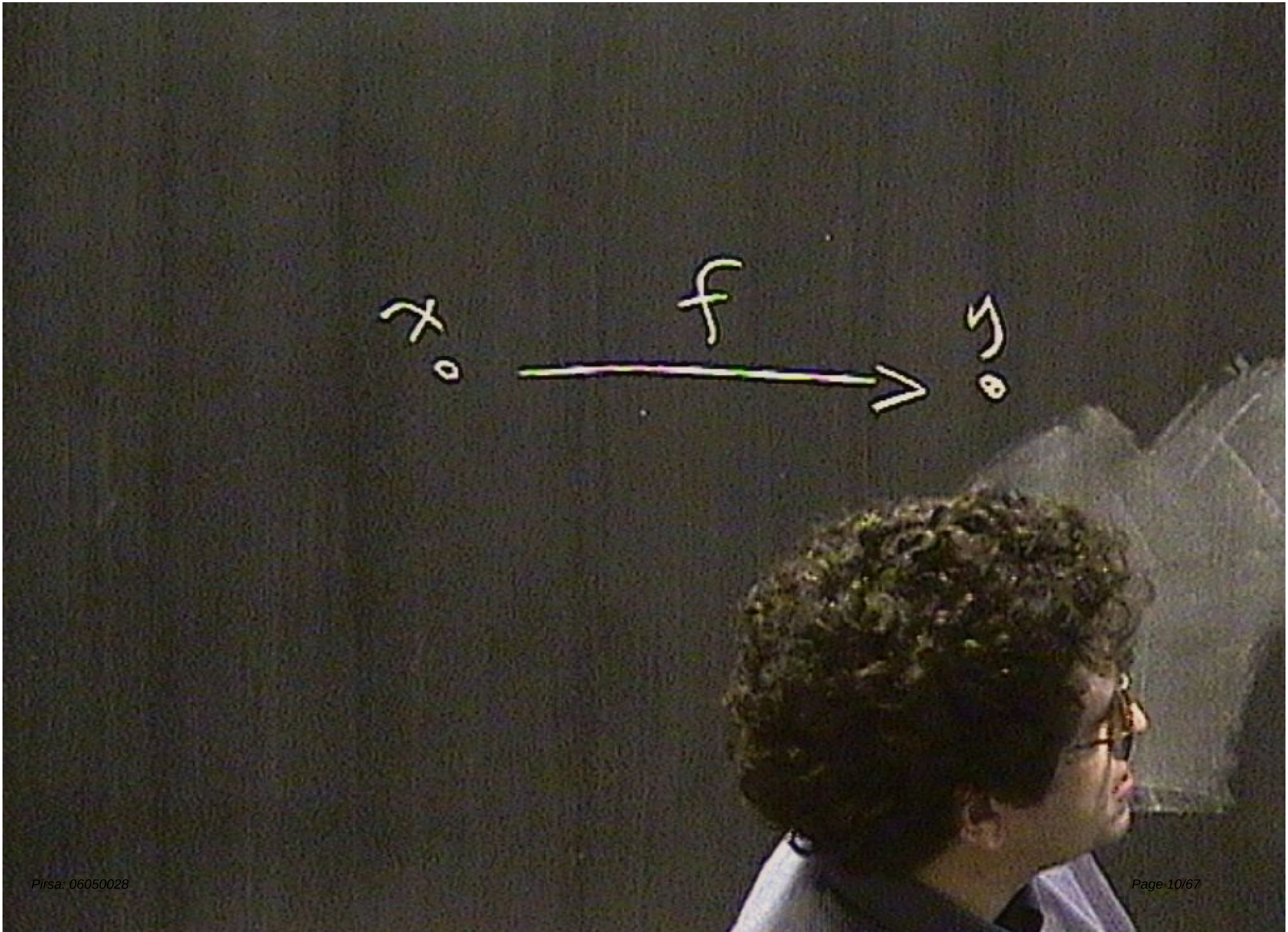
	object $\bullet$	morphism $\bullet \rightarrow \bullet$
SET THEORY	set	function between sets
QUANTUM THEORY	Hilbert space (state)	operator between Hilbert spaces (process)
GENERAL RELATIVITY	manifold (space)	cobordism between manifolds (spacetime)

Objects and Morphisms

Every category has *objects* and *morphisms*:

- In Set an object is a set, and a morphism is a function.
- In Hilb an object is a Hilbert space, and a morphism is a linear operator.
- In nCob an object is an $(n-1)$-dim manifold, and a morphism is a cobordism between such manifolds:

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f
y

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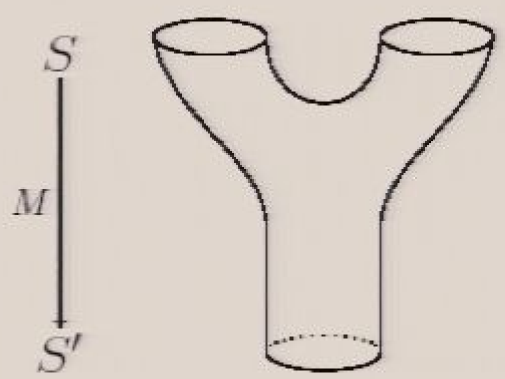


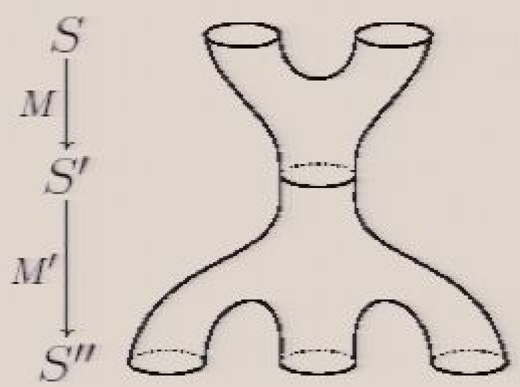
Composition

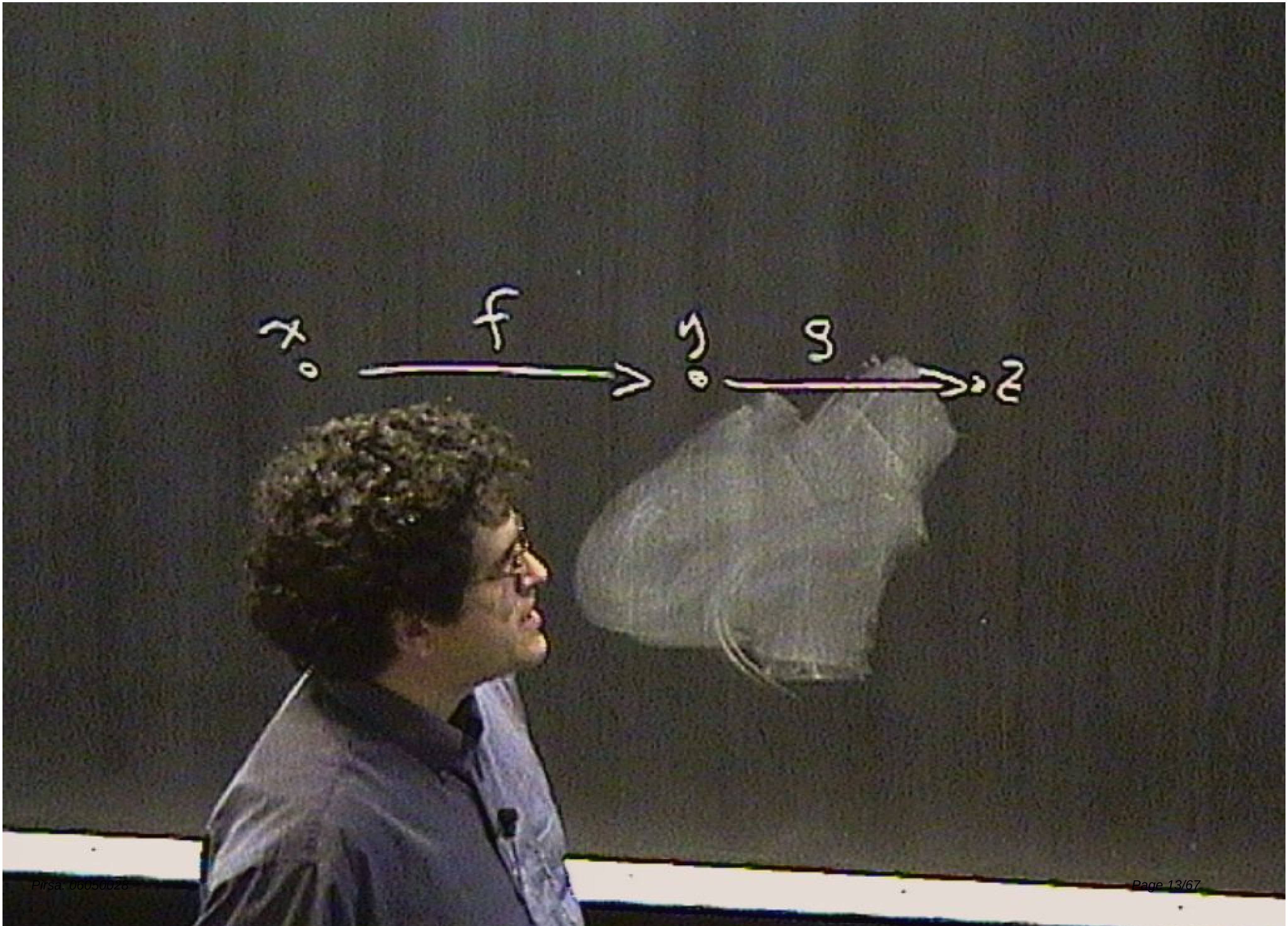
Every category lets us *compose* morphisms in an associative way:

- In Set, we compose functions as usual.
- In Hilb, we compose operators as usual:

$$(T'T)\psi = T'(T\psi).$$

- In nCob, we compose cobordisms like this:

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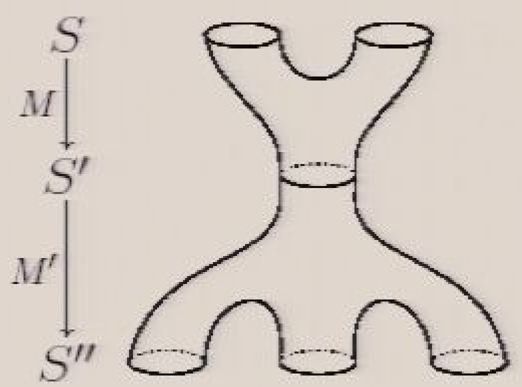

Monoidal Categories

In fact, all our examples are *monoidal* categories — they have a *tensor product* and *unit object*:

- In Set, the tensor product is $\times$, and the unit object is the 1-element set.
- In Hilb, the tensor product is $\otimes$, and the unit object is $\mathbb{C}$.
- In nCob, the tensor product looks like this:

and the unit object is the empty manifold.

(A bunch of axioms must hold, and they do....)

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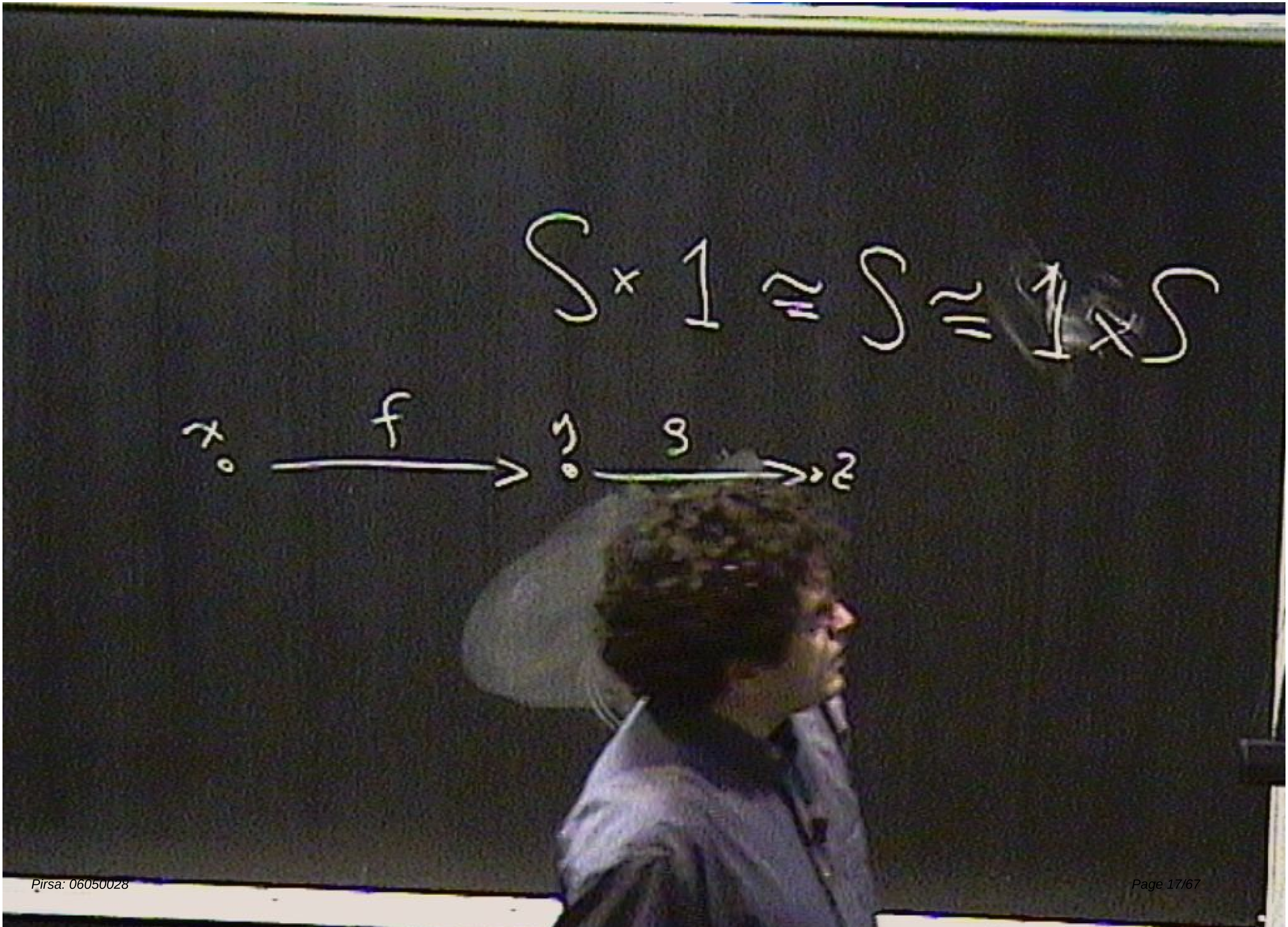
S × 1 ≅ S ≅ 1 × S
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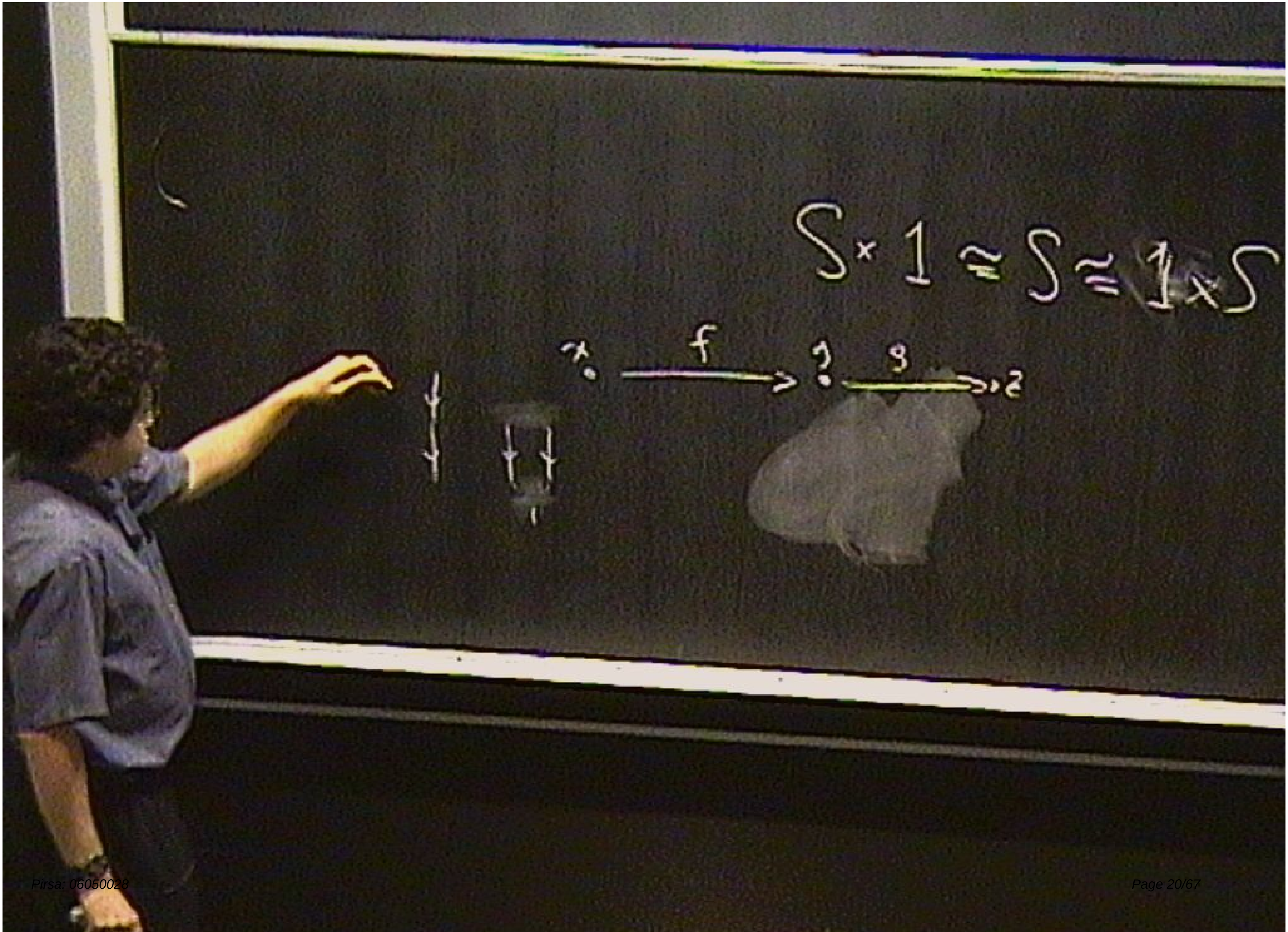
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Now for the first big difference: the tensor product in Set is 'cartesian', while those in nCob and Hilb are not!

A monoidal category is *cartesian* when you can duplicate data:

$$\Delta\colon x \to x \otimes x$$

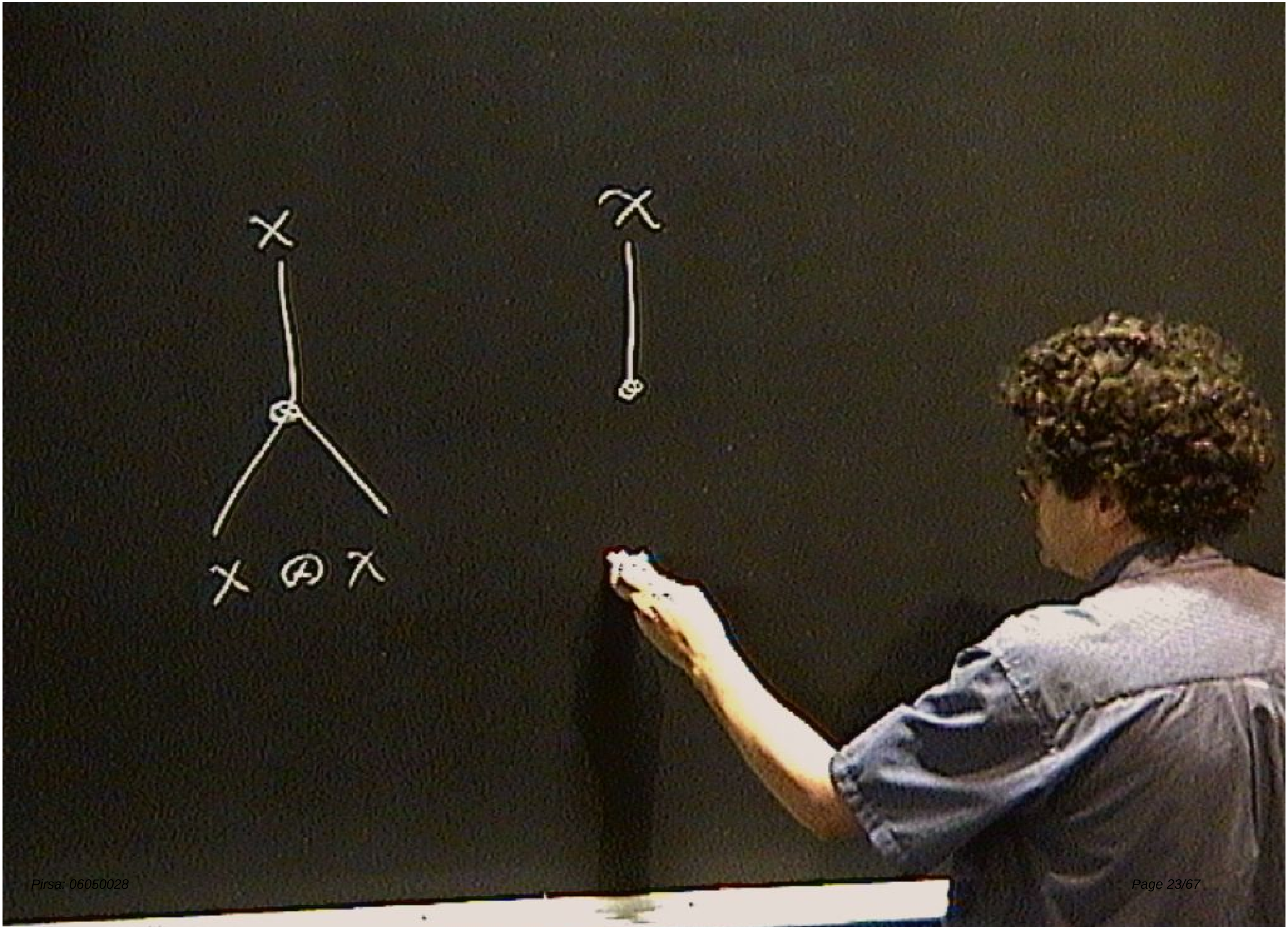
and delete it:

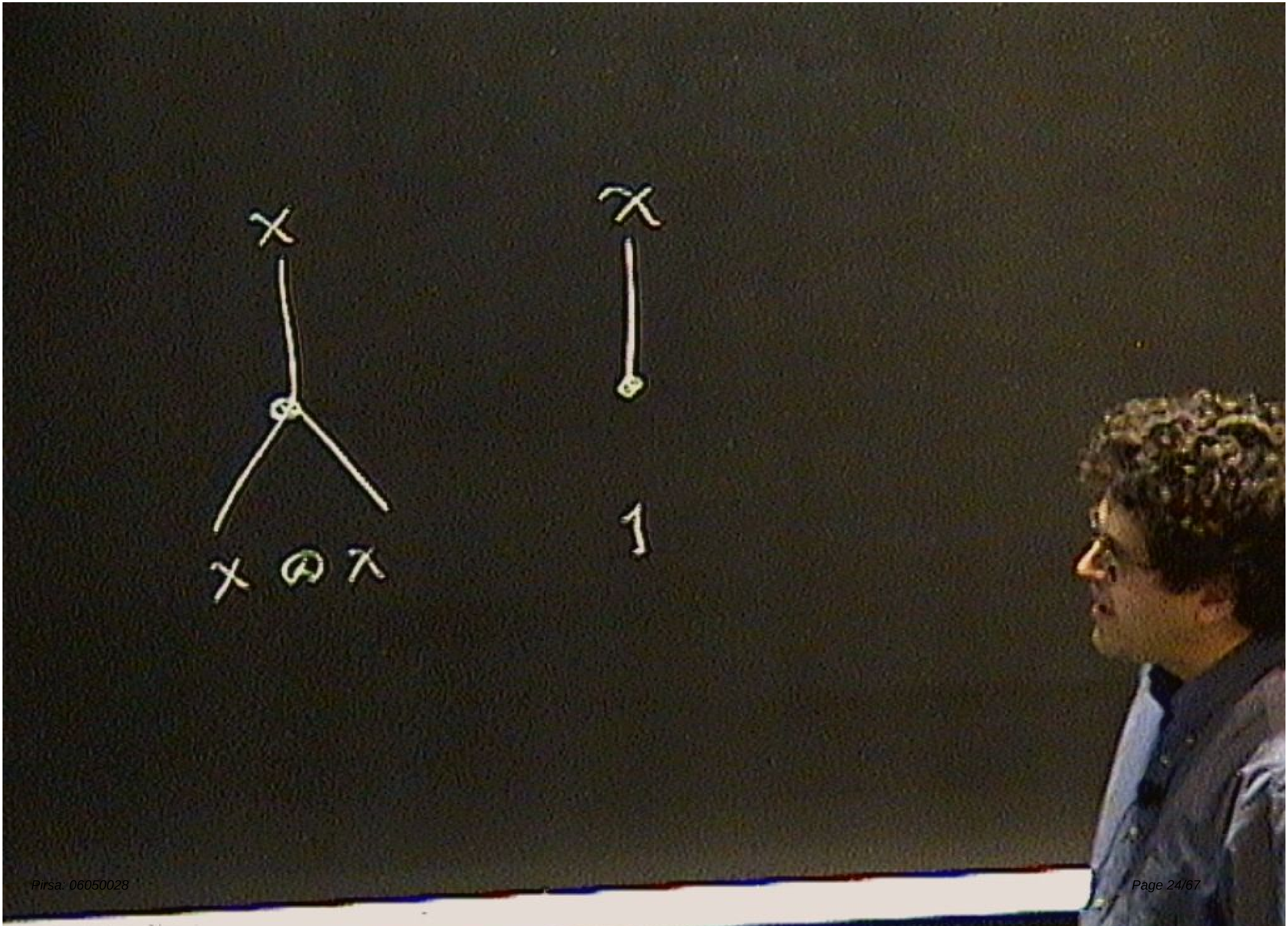
$$e\colon x \to 1$$

so that these diagrams commute:

$$\begin{array}{ccc} x & \xrightarrow{\Delta} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle e \otimes 1_x} \\ x & \xleftarrow{\sim} & 1 \otimes x \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\Delta} & x \otimes x \\ {\scriptstyle 1_x}\downarrow & & \downarrow{\scriptstyle 1_x \otimes e} \\ x & \xleftarrow{\sim} & x \otimes 1 \end{array}$$

In Set, you can do this. In Hilb you can't: you can neither clone a quantum, nor cleanly delete quantum information. *Nor can you do this in* nCob!


x
x ⊗ x
x


x
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x
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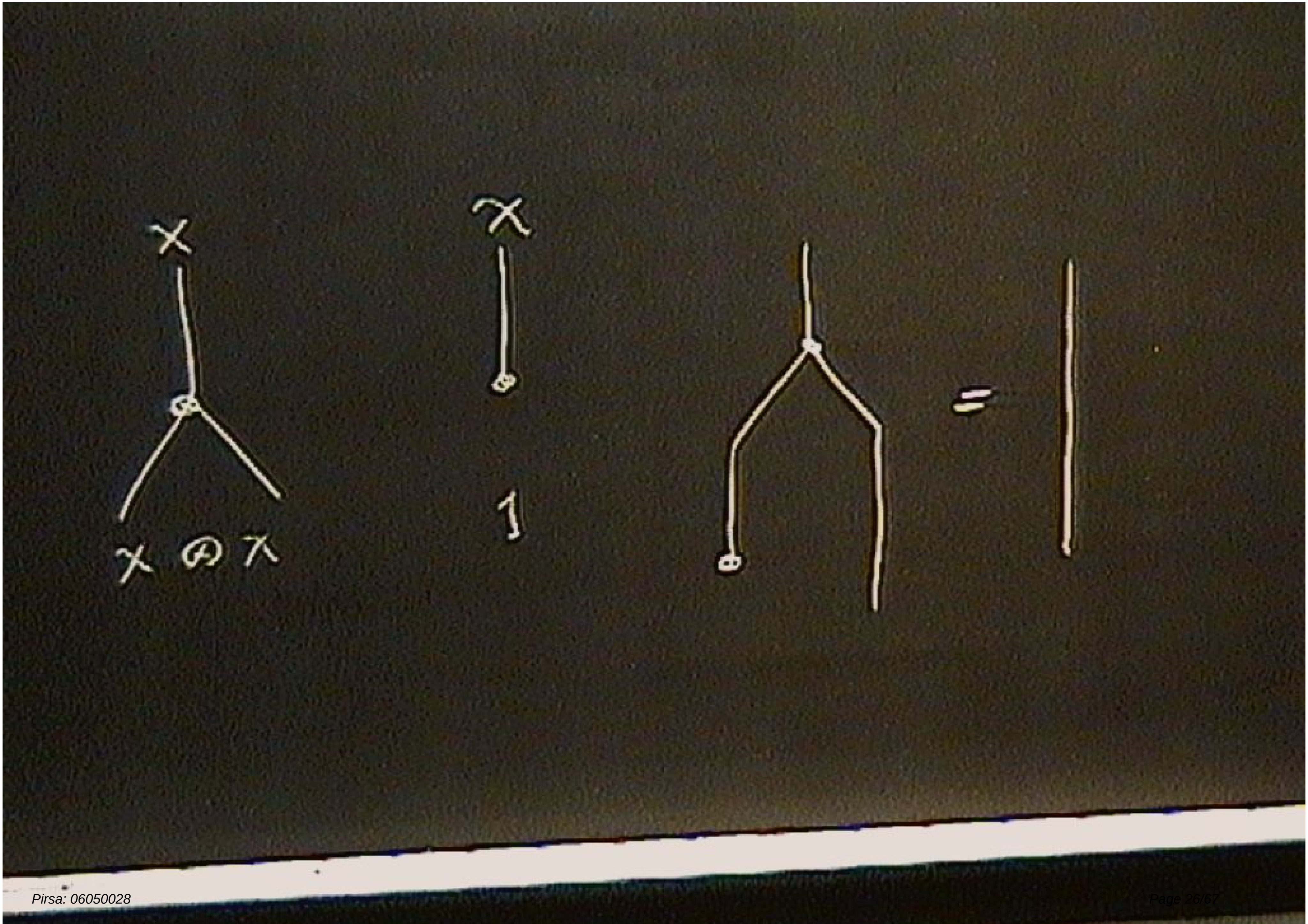
X
X ⊗ X
X
1
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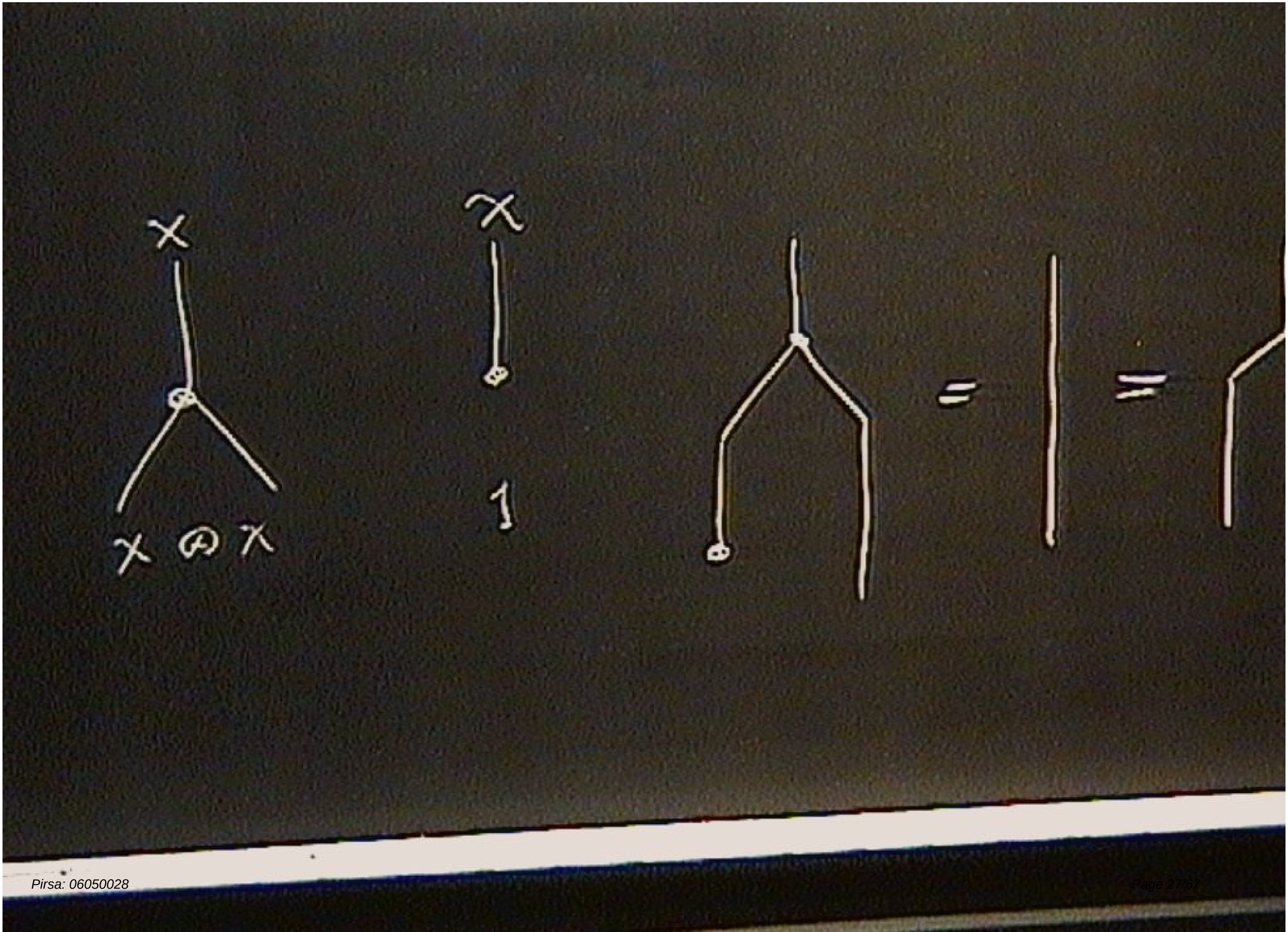
x
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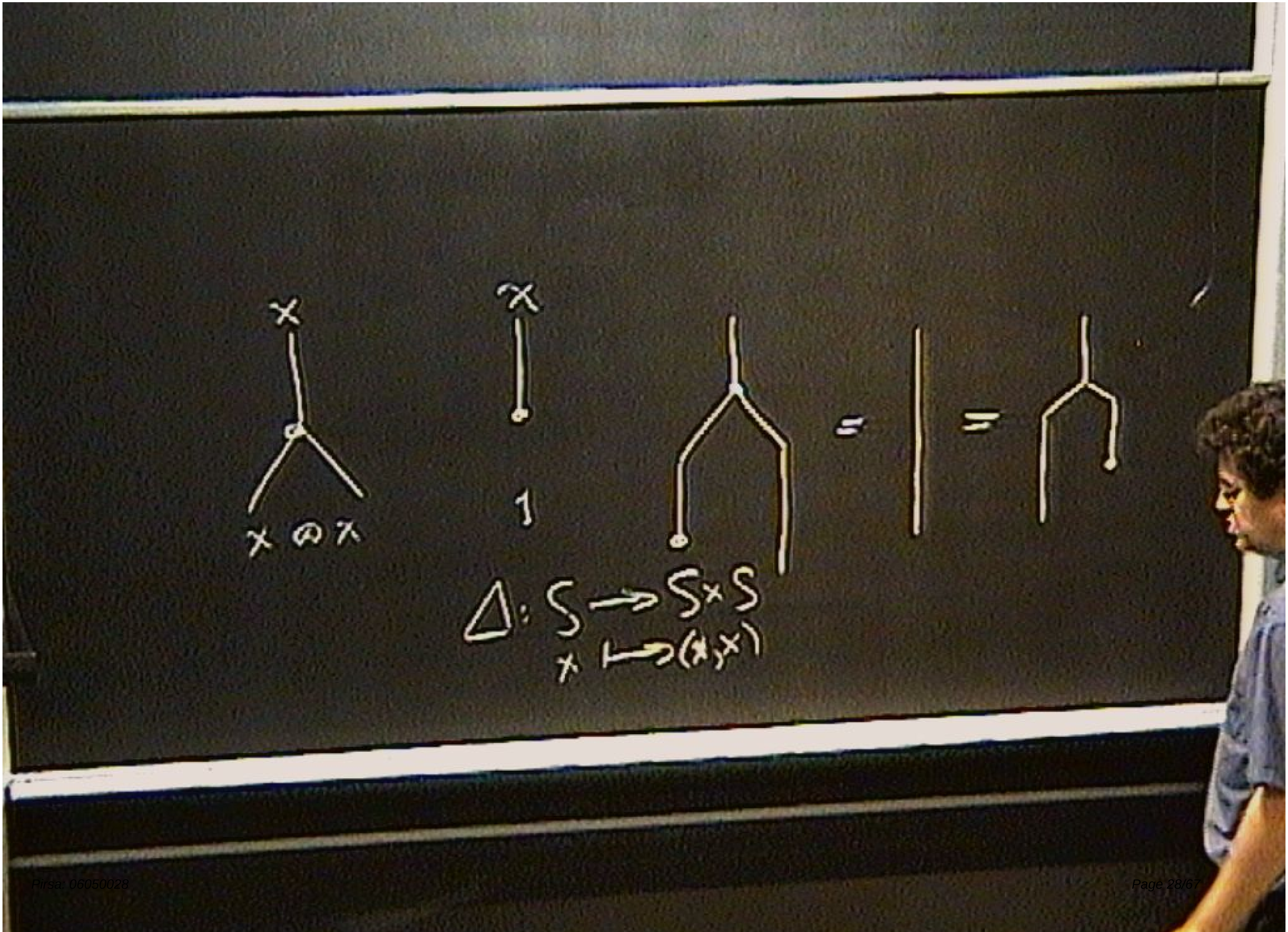
x
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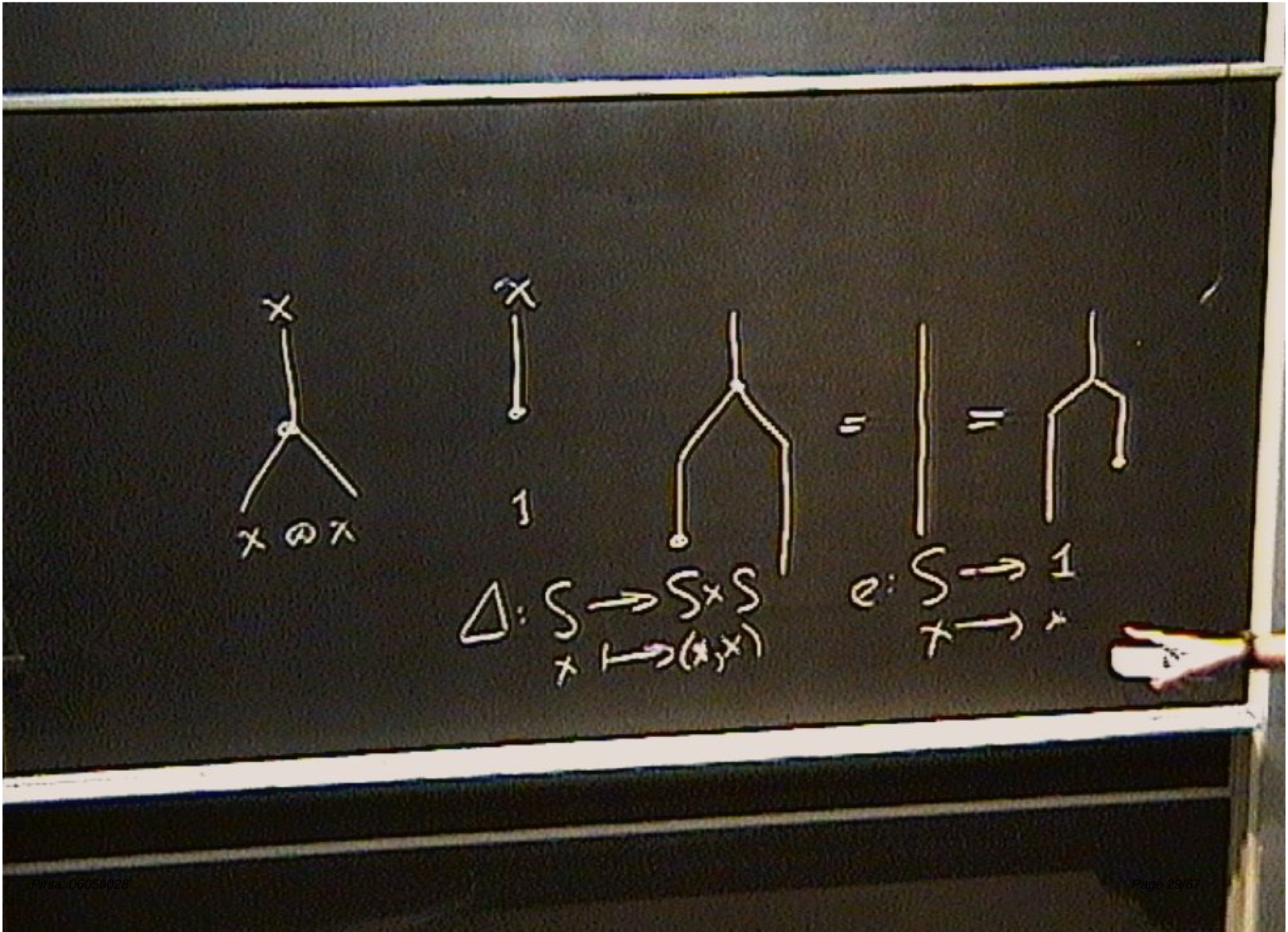
Δ: S ⟶ S×S
x ⟼ (x,x)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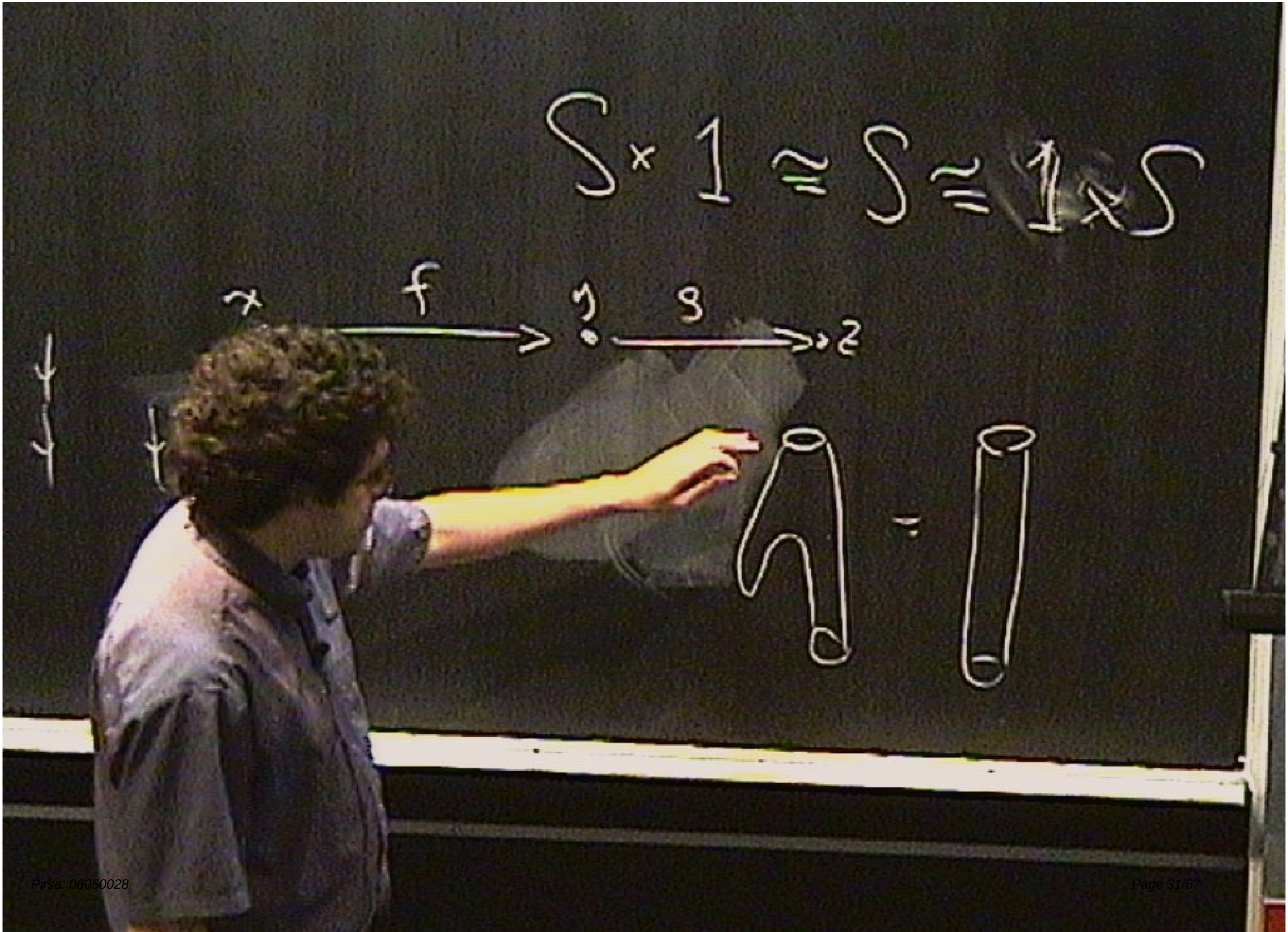
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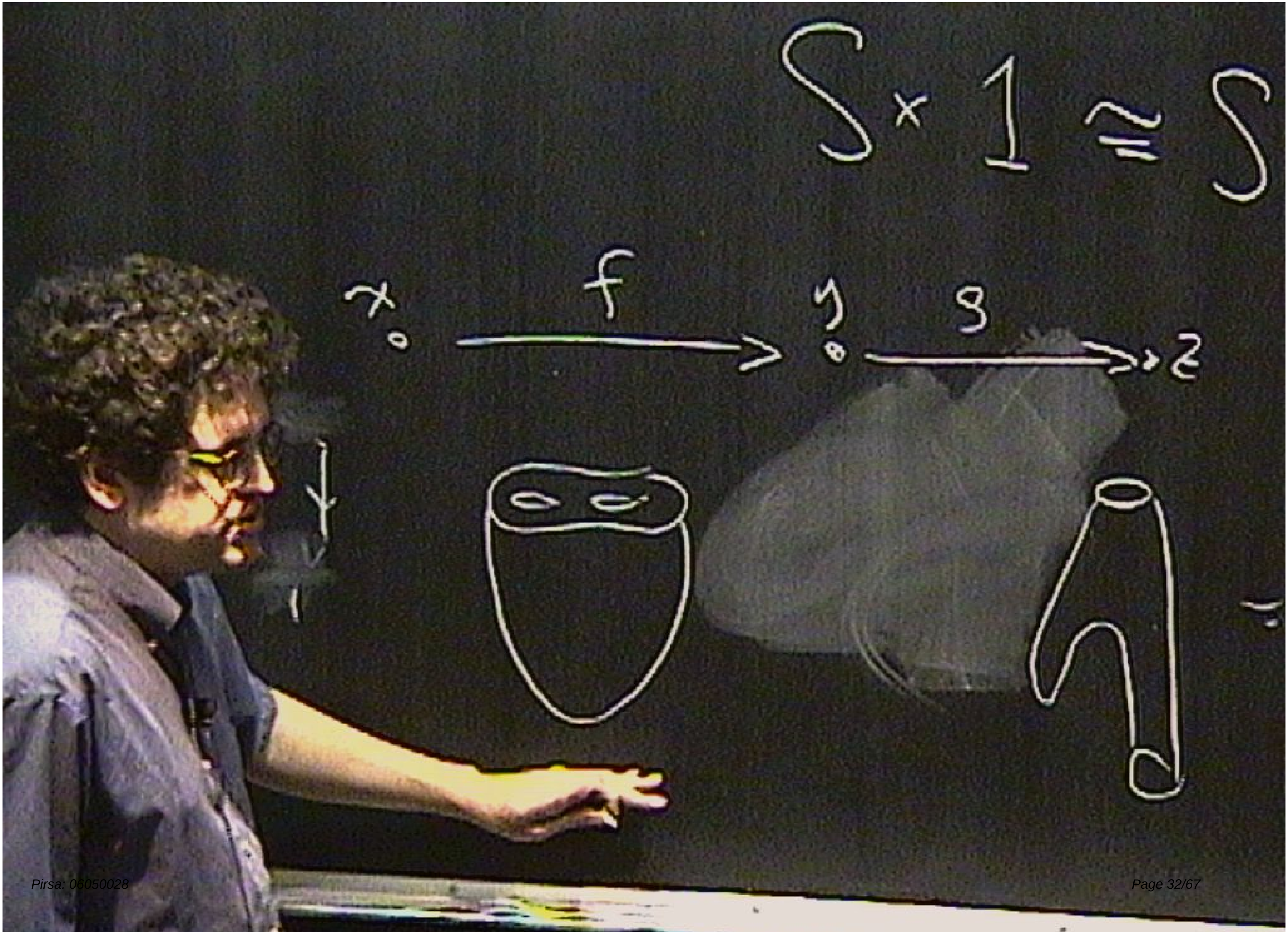
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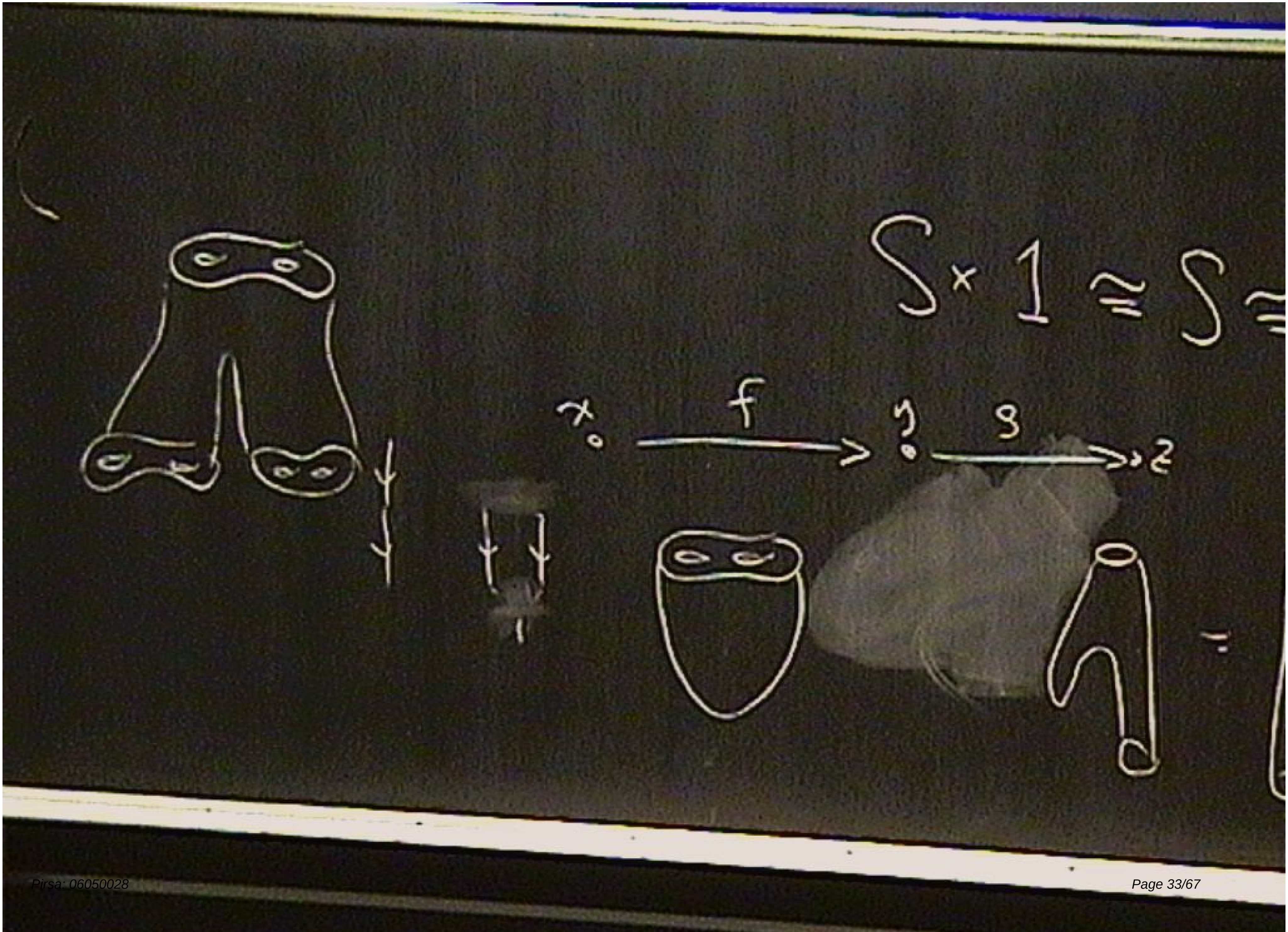
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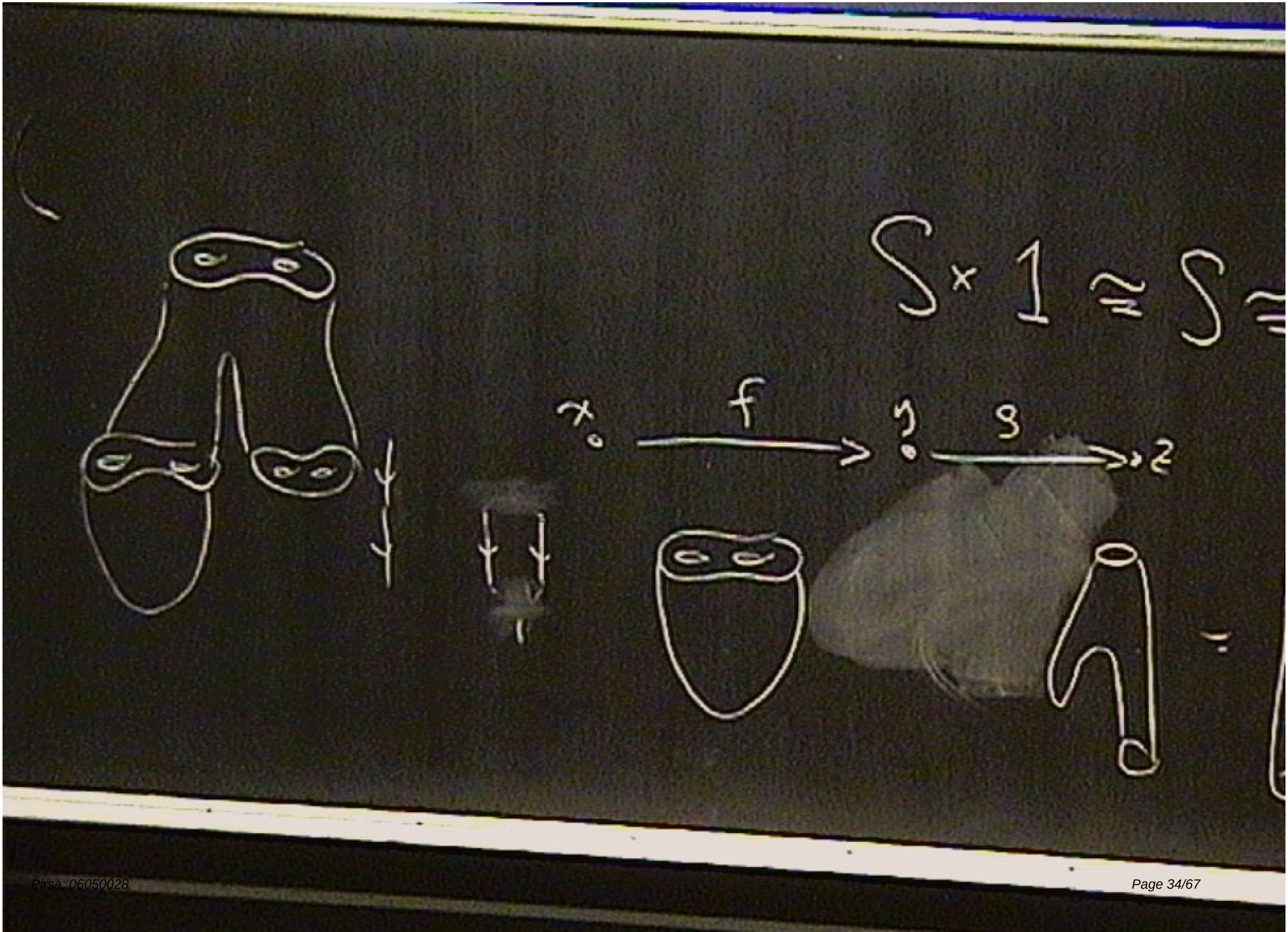
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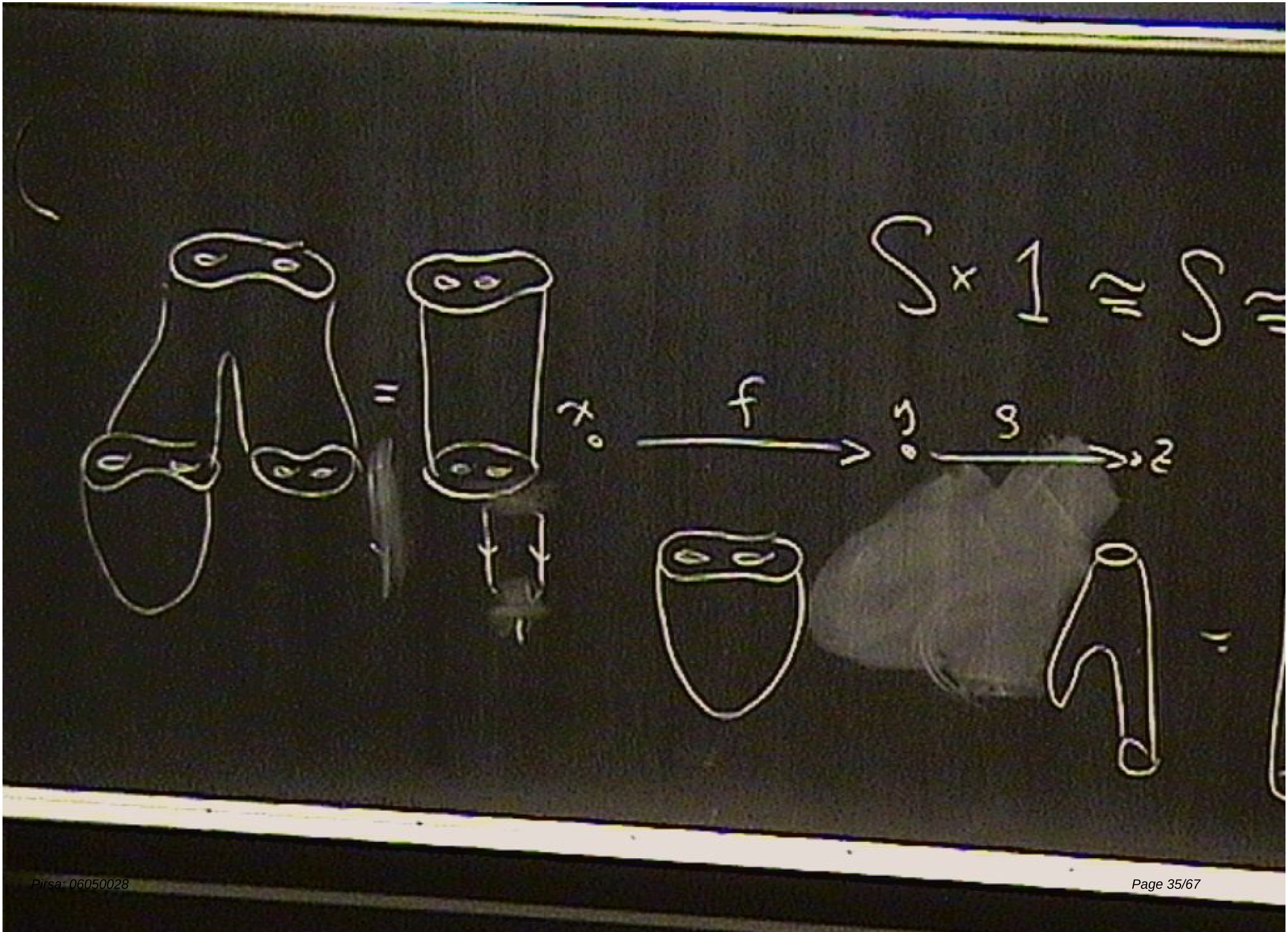
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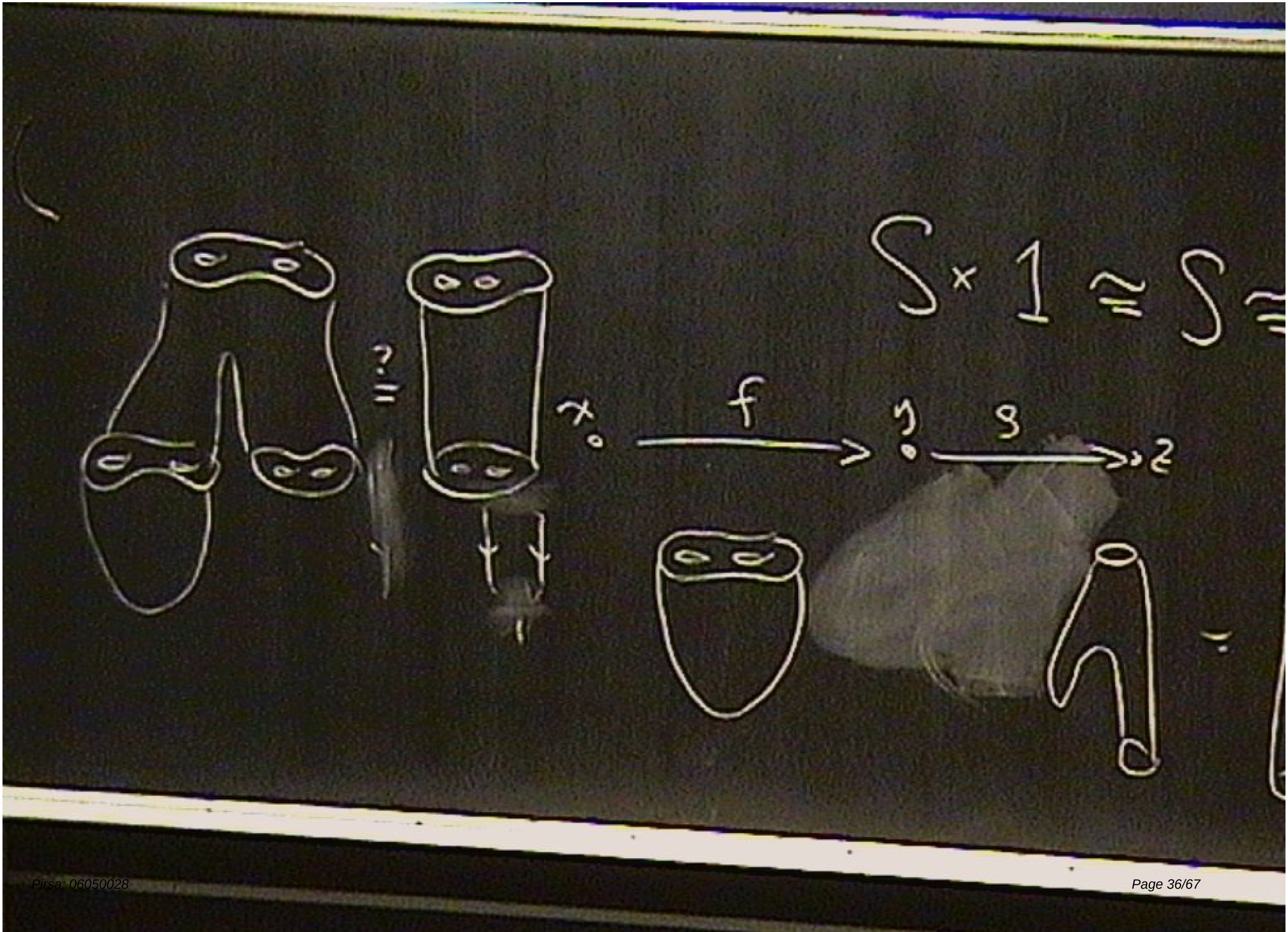
S × 1 ≅ S ≅
?=
x
f
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z

Duality for Objects

Both nCob and Hilb have 'duals for objects', but Set does not. This is why quantum teleportation seems odd.

A monoidal category has *duals for objects* if every object x has an object x^* with morphisms

$$e_x \colon x^* \otimes x \to 1, \quad i_x \colon 1 \to x \otimes x^*$$

satisfying the *zig-zag identities.*

In nCob, S^* is S with its orientation reversed. We have

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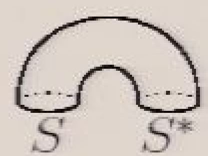
$i_S =$ 

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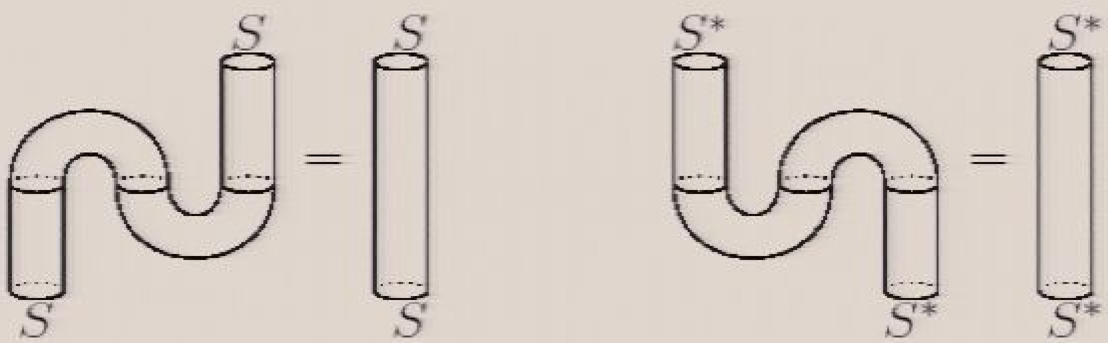


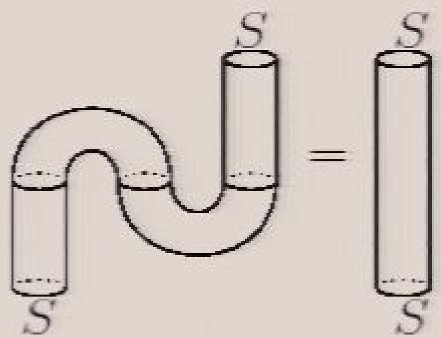
In Hilb, H^* is the dual Hilbert space. We have

$$e_H\colon\ H^* \otimes H \to \mathbb{C} \qquad i_H\colon\ \mathbb{C} \to H \otimes H^*$$
$$\ell \otimes \psi \mapsto \ell(\psi) \qquad c \mapsto c1_H$$

and the zig-zag identities say familiar things about linear algebra.

But... *there is no 'dual' of a set!*

Abramsky and Coecke have shown that quantum teleportation relies on the zig-zag axiom:



A particle interacts with one of a pair of particles prepared in the Bell state. Its quantum state gets transferred to the other member of the pair!

Read their paper *A Categorical Semantics of Quantum Protocols* for details.

In summary:

Quantum theory seems counterintuitive if we expect Hilb to act like Set, since it acts more like nCob. Superficially, Hilbert spaces and operators seem like sets and functions. But, they're really more like *spaces* and *spacetimes!*

This is a clue.

Perhaps Feynman was the first to get it...

...or maybe Penrose, with his spin networks.

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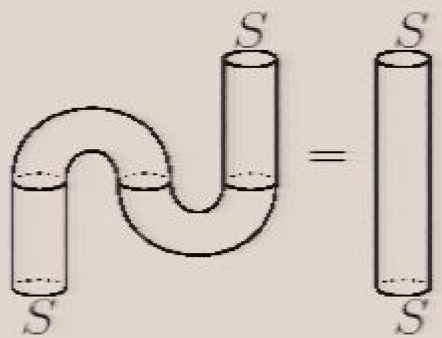

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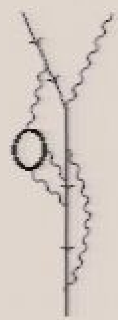
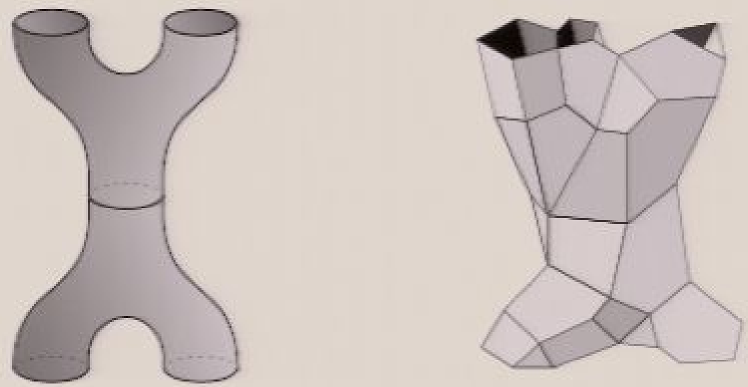


...or maybe Penrose, with his spin networks.

Both string theory and spin foam models are trying to exploit this clue. They are groping towards a language for quantum spacetime that will usefully blur the distinction between *pieces of spacetime geometry*:



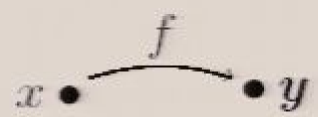
and *quantum processes.*

At this point we should think of them, not as predictive theories, but as explorations of the mathematical possibilities!

Since strings and spin foams are both 2d generalizations of Feynman diagrams, it's natural to use 2-categories to describe the ways of 'composing' them.

A *(strict) 2-category* has objects:

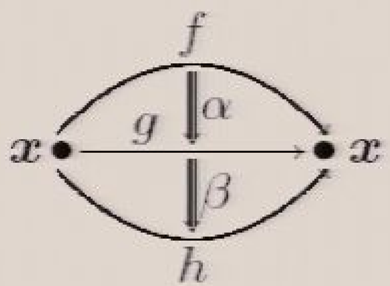
morphisms:

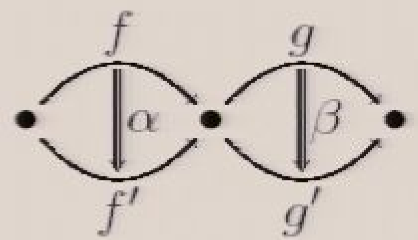


and also 2-morphisms:

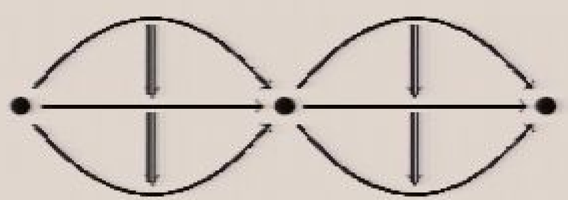
We can compose morphisms as before:

We can compose 2-morphisms vertically and horizontally:



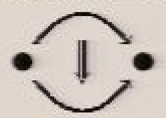


Each composition is associative and has identities. Lastly we have the interchange law, saying this diagram gives a well-defined 2-morphism:



So far we see 2-categories playing four distinct but closely related roles:

1. In string theory — more precisely, in any *conformal field theory* — we have a 2-category where:

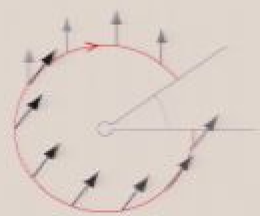
- objects are *D*-branes: •
- morphisms are string states: • ⟶ •
- 2-morphisms are evolution operators corresponding to string worldsheets: 

For details see *Categorification and Correlation Functions in Conformal Field Theory* by Runkel, Fuchs, and Schweigert.

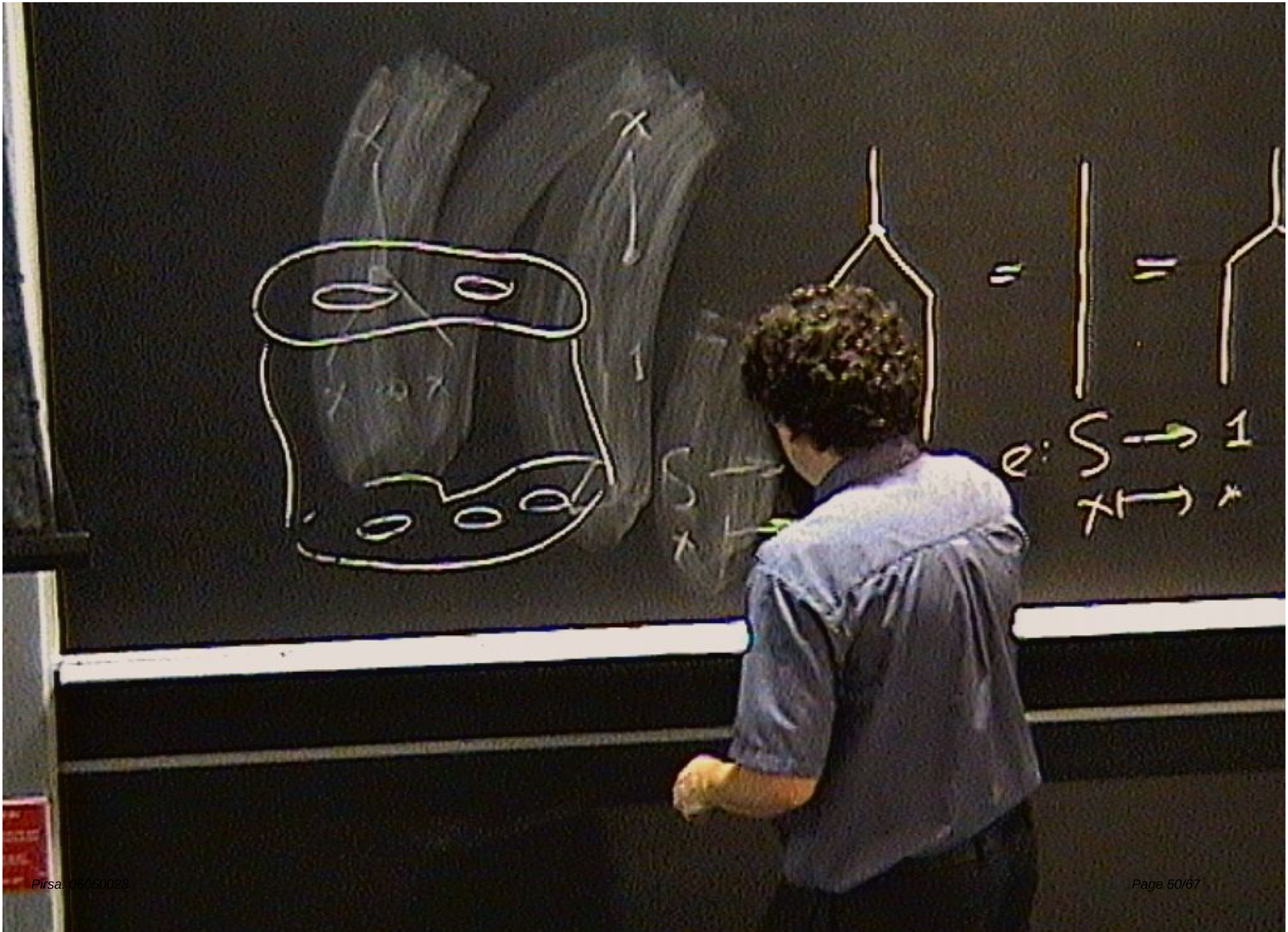
2. In 3d quantum gravity — more generally, in any *extended topological quantum field theory* — we have a 2-category where:

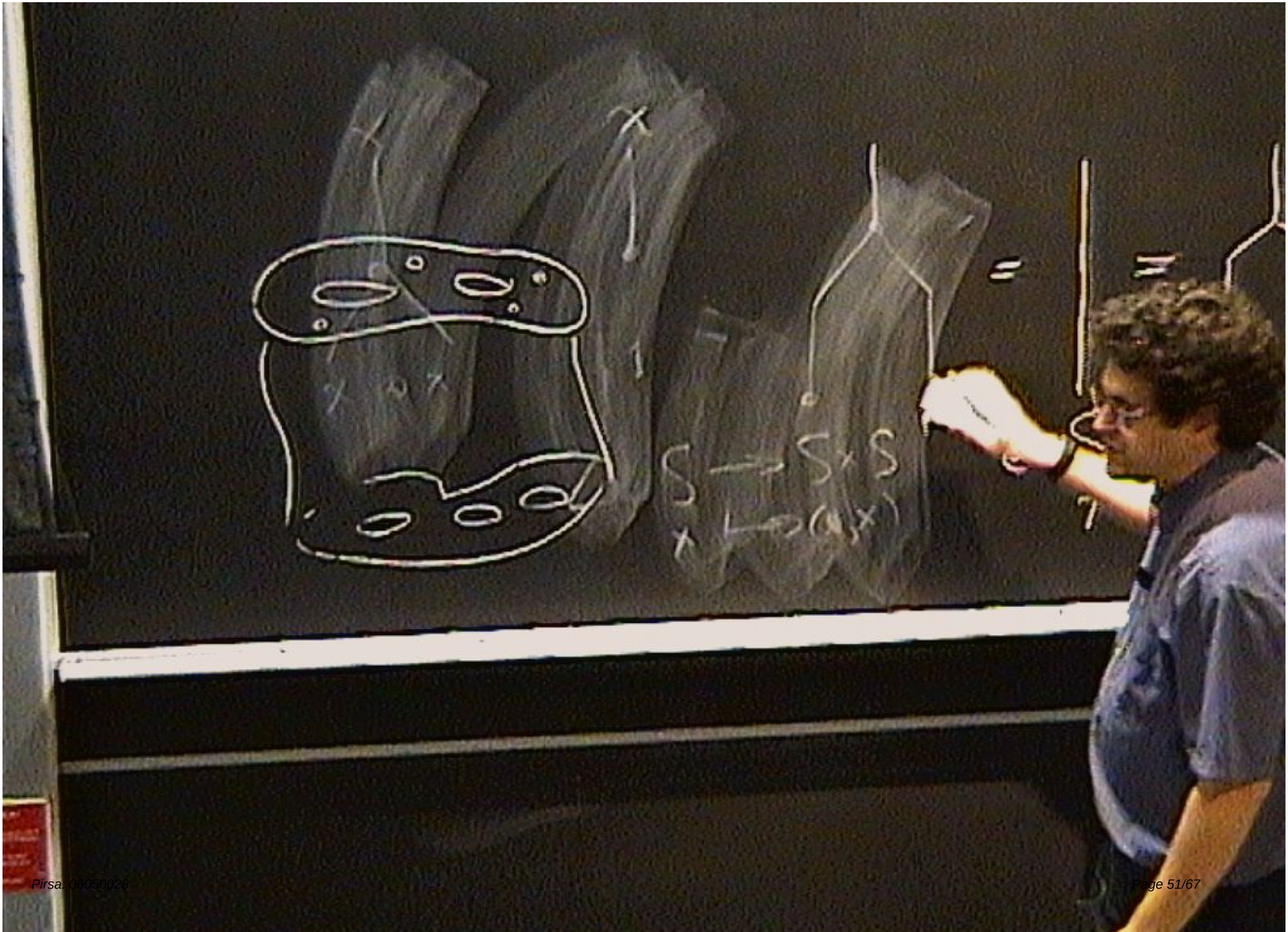
- objects describe kinds of *matter*
- morphisms describe choices of *space*
- 2-morphisms describe choices of *spacetime*

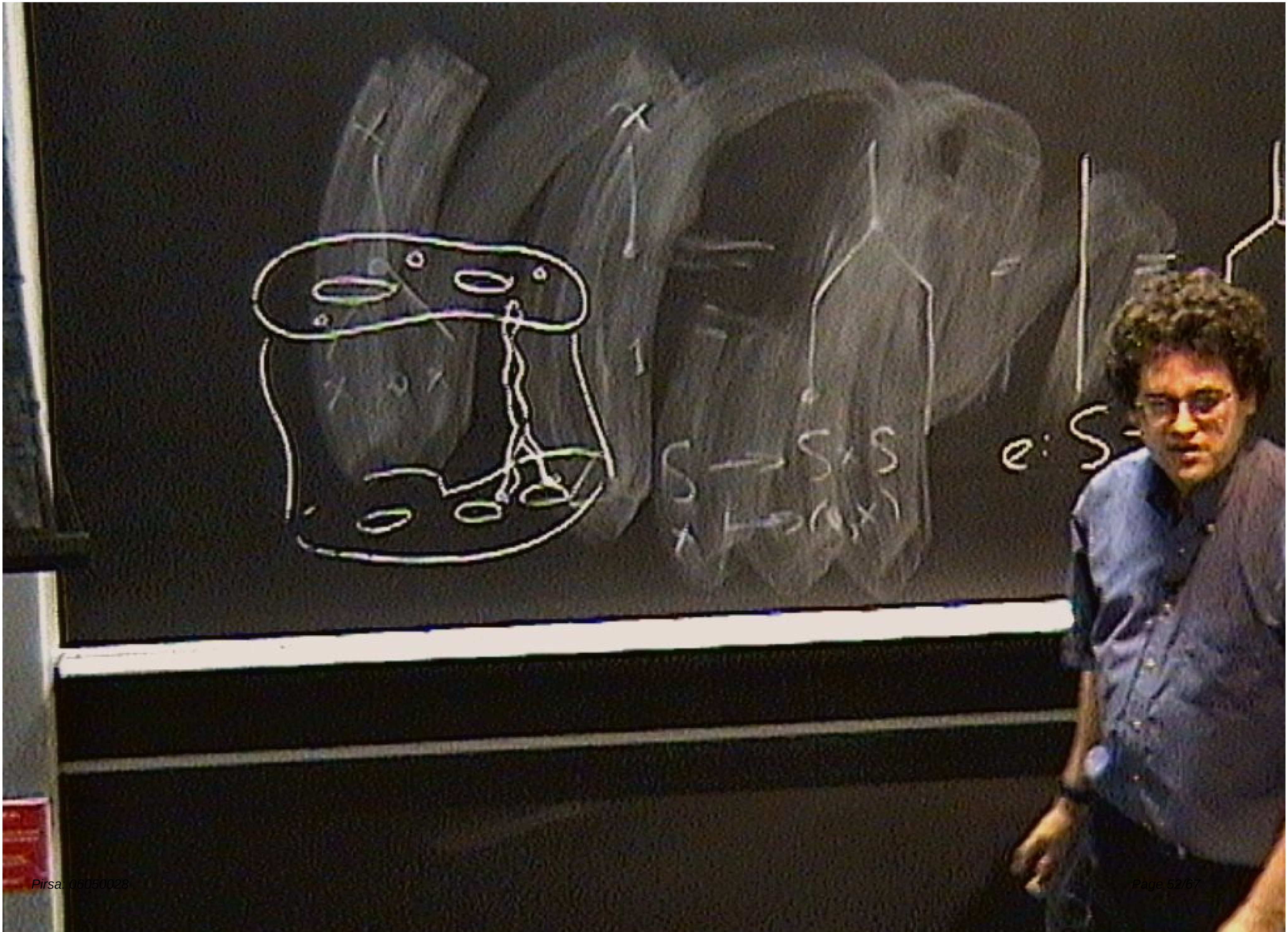
In 3d quantum gravity this matter consists of *point particles* — see the work of Freidel et al:



In 4d topological gravity this matter consists of *strings* — see my papers with Crans, Wise and Perez.


e: S → 1
x ↦ *

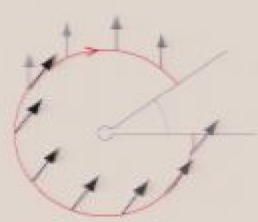

S ⟶ S×S
x ⟼ Δ(x,x)


S → S × S
x ↦ (x, x)
e: S

2. In 3d quantum gravity — more generally, in any *extended topological quantum field theory* — we have a 2-category where:

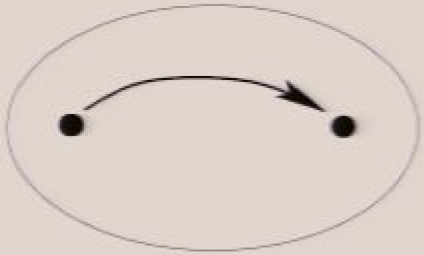
- objects describe kinds of *matter*
- morphisms describe choices of *space*
- 2-morphisms describe choices of *spacetime*

In 3d quantum gravity this matter consists of *point particles* — see the work of Freidel et al:

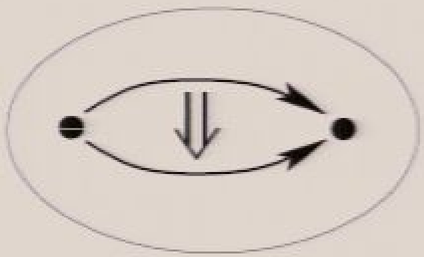


In 4d topological gravity this matter consists of *strings* — see my papers with Crans, Wise and Perez.

3. In *higher gauge theory* we have fields describing parallel transport not just for *point particles* moving along *paths*:



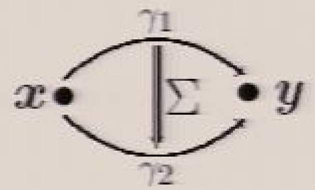
but also for *strings* tracing out *surfaces*:



I've developed this in papers with Bartels, Crans, Lauda, Schreiber and Stevenson.

Indeed, every manifold gives a 2-category where:

- objects are points: $\bullet x$
- morphisms are paths: 

- 2-morphisms are surfaces like this: 

Ordinary gauge theory uses *groups*, which are special categories: 

Higher gauge theory uses *2-groups*, which are special 2-categories: 

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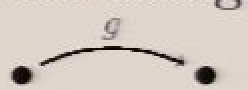
Ordinary gauge theory uses *groups*, which are special categories: 

Higher gauge theory uses *2-groups*, which are special 2-categories: 

In practice a 2-group consists of two groups, G and H, related by various operations. A *2-connection* consists of:

- a $\mathfrak{g}$-valued 1-form A
- an $\mathfrak{h}$-valued 2-form B

Parallel transport along a path should give an element of G: 

Parallel transport along a surface should give an element of H: 

But, A and B must satisfy an *equation* for this to work. And in 4d spacetime, this equation has the BF theory equation $B \propto F$ as a special case.

So, we get *parallel transport for particles and strings in 4d topological gravity!*

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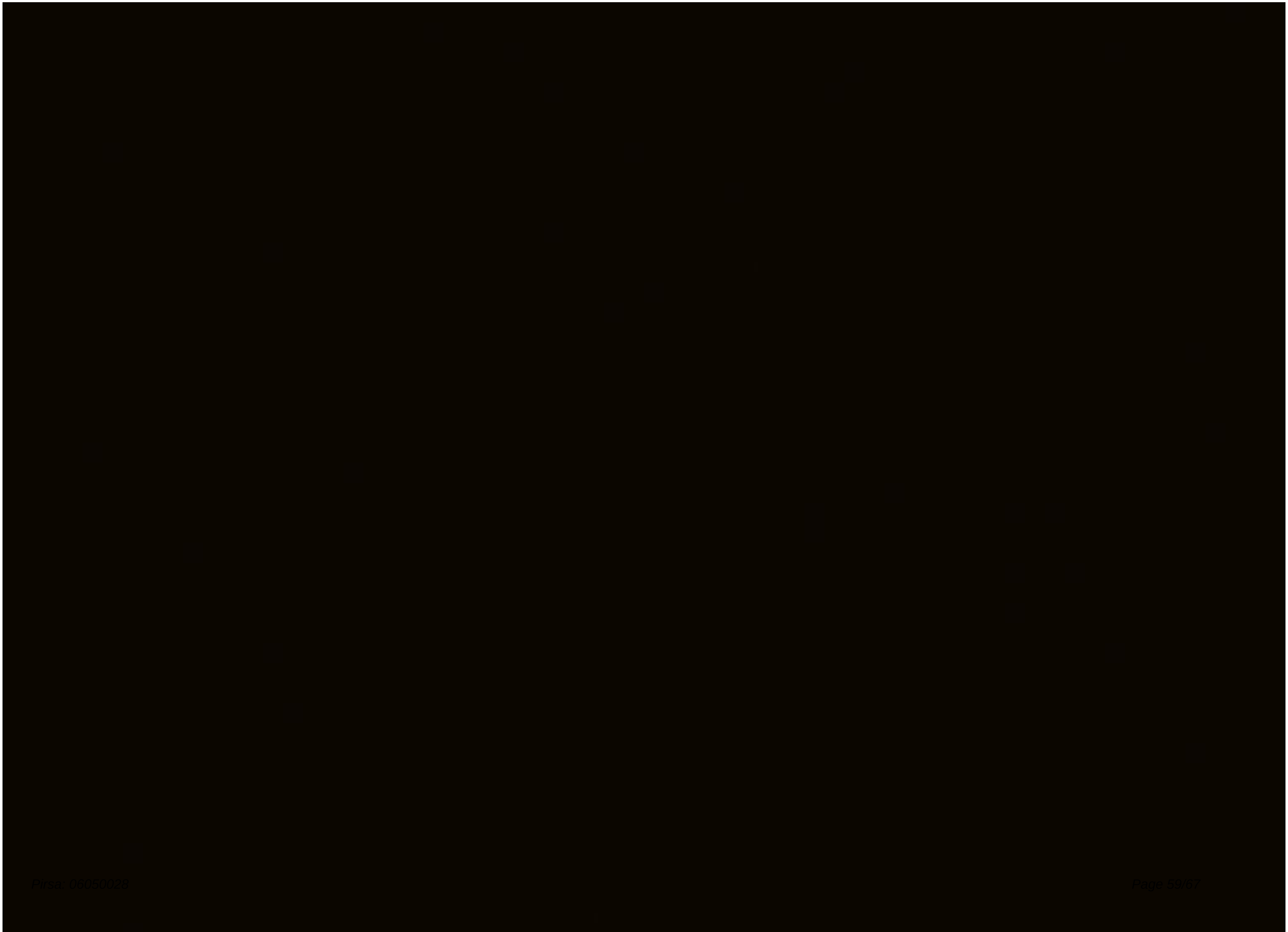
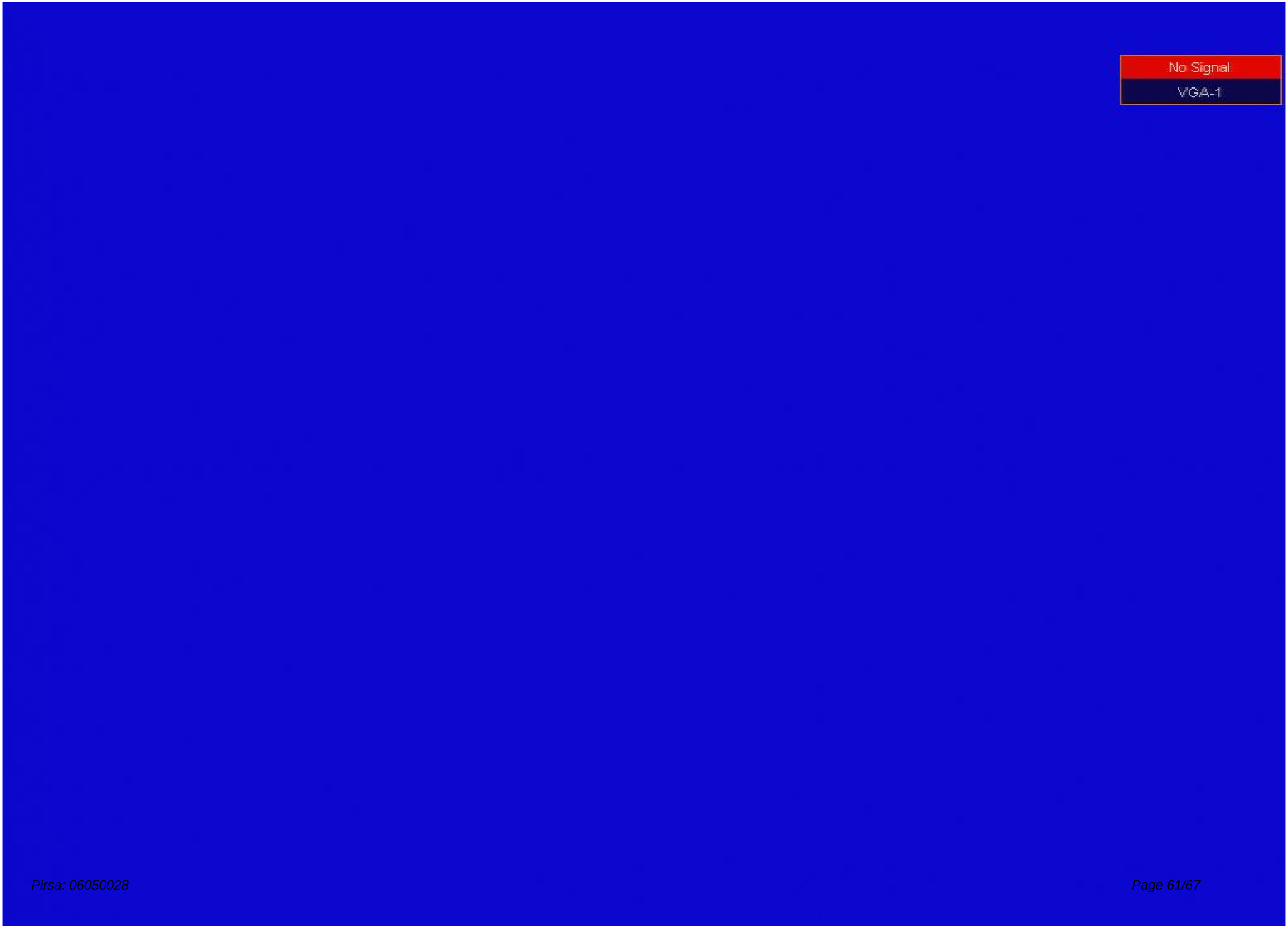


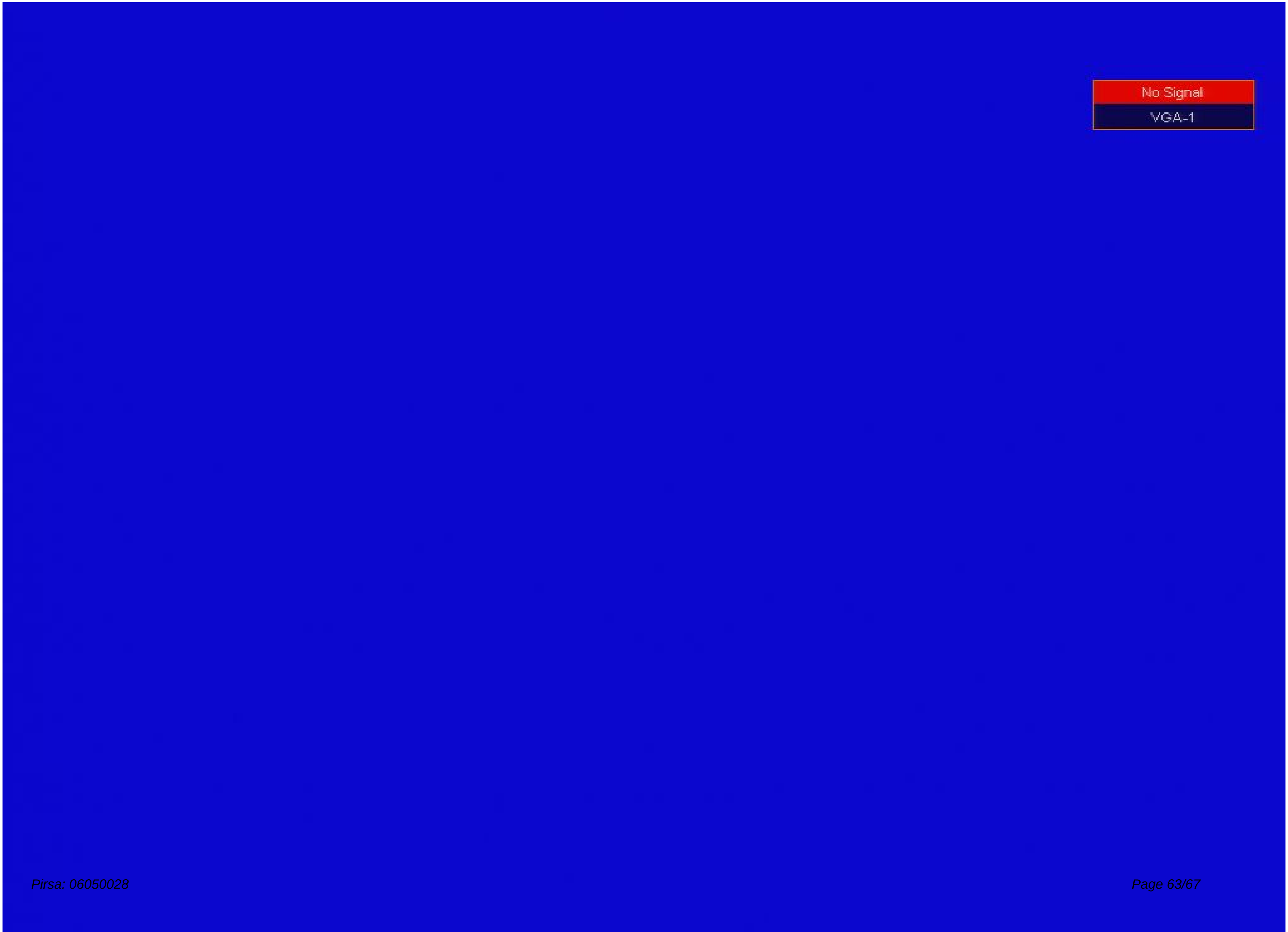
No Signal

VGA-1


No Signal
VGA-1

No Signal

VGA-1


No Signal
VGA-1

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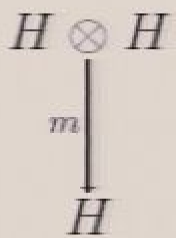
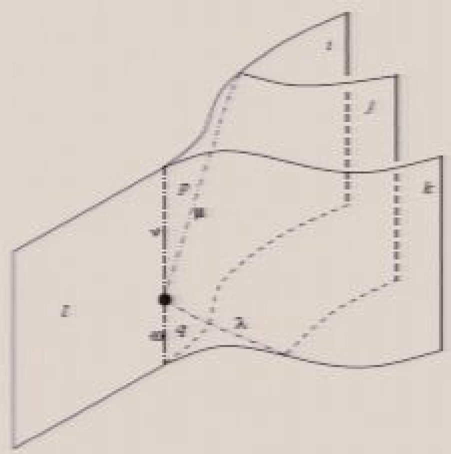
4. Feynman diagrams and spin networks are really just a way of reasoning diagrammatically with operators — morphisms in Hilb. All these:

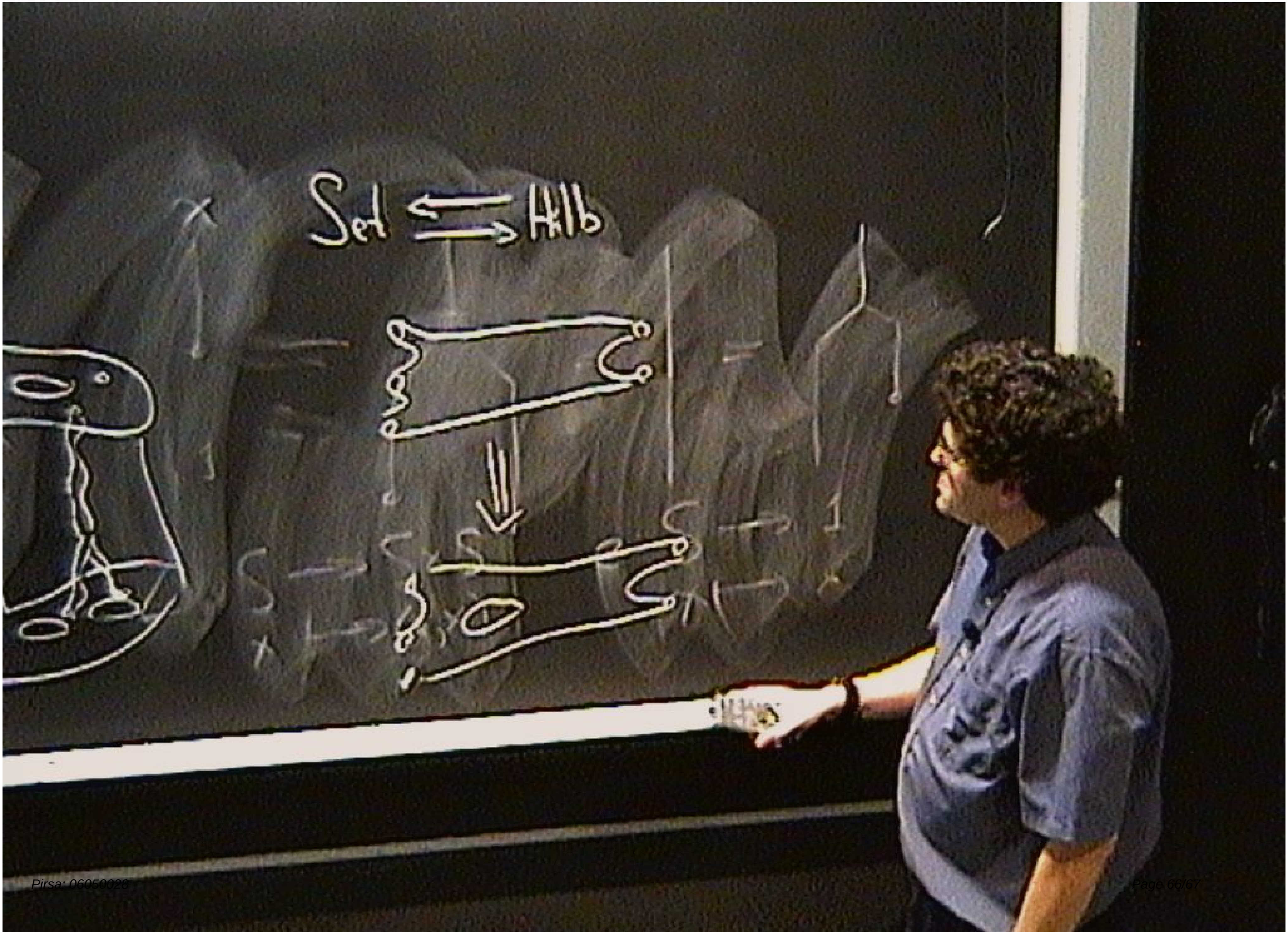


m_k^{ij}

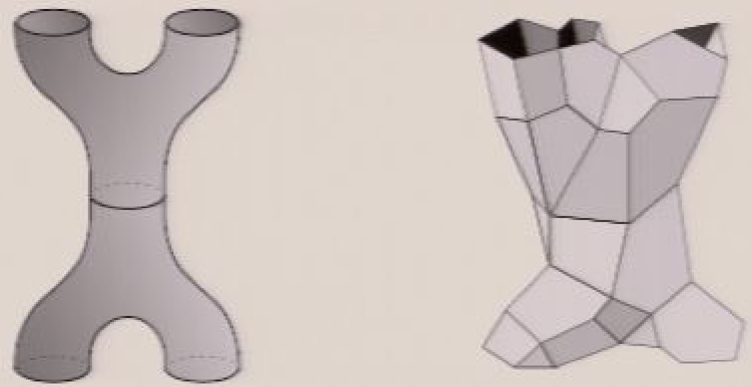
are just different ways of writing the same thing!

Similarly, spin foams are a way of reasoning with 2-morphisms in a 2-category of '2-Hilbert spaces':




Set ⇄ Hilb

Both string theory and spin foam models are trying to exploit this clue. They are groping towards a language for quantum spacetime that will usefully blur the distinction between *pieces of spacetime geometry*:



and *quantum processes*.

At this point we should think of them, not as predictive theories, but as explorations of the mathematical possibilities!