Title: QCD recursion relations from the largest time equation

Date: May 30, 2006 02:00 PM

URL: http://pirsa.org/06050026

Abstract:

QCD recursion relations and the largest time equation

Diana Vaman
MCTP, University of Michigan

Based on: hep-th/0512031(JHEP) by D.V. and Ed Yao

Perimeter Institute, Waterloo, May 2006

Outline

- Introduction
- The space-cone gauge
- QCD and the recursion relations: examples
- Tree level recursion relations and the largest time equation
- The general proof
- Including fermions and tree level recursion relations
- Conclusions and future directions

Introduction

Over the last couple of years great strides have been made to simplify the standard perturbative approach to QCD calculations

- Witten's formulation of perturbative N=4 QCD as a string theory on twistor space
- MHV rules as a way to expedite perturbative calculations
- BCFW *on-shell* recursion relations

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It is obvious that the on-shell recursion relations bear on the cutting rules in field theory.

In fact, cut-constructibility has been reliably used to compute loop amplitudes over the last decade. The unitarity of the S-matrix and the existence of an ordering among a sequence of points (largest time equation) are intimately related.

The question is: which particular formulation of QCD and which particular ordering are the most suited for a field theoretical derivation of the on=shell recursion relations?

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The question is: which particular formulation of QCD and which particular ordering are the most suited for a field theoretical derivation of the on=shell recursion relations?

The answer is: the space-cone gauge QCD, and a light-like ordering.

...and Summary

- We offer a purely quantum field theoretical proof (at the level of Feynman diagrams) of the BCFW recursion relations.
- A key ingredient is the use of space-cone gauge.
- The tree level recursion relations emerge at a more fundamental level from the largest time equation.
- Our results lend themselves to natural generalizations to include massive scalars and fermions.

Setting notation

A 4-component Lorentz vector can be written as a bispinor as

$$V_{a\dot{a}} = V_\mu \sigma^\mu_{a\dot{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix} = \begin{pmatrix} v^+ & \bar{v} \\ v & v^- \end{pmatrix}$$

The inner product of 2 vectors is given by

$$V \cdot U = V^{a\dot{a}} U_{a\dot{a}}$$

and the norm is $-\det(V)$.

The basic principle of twistors is that a null vector $\equiv$ the square of a commuting spinor

$$V^2 = 0 \Longrightarrow V_{a\dot{a}} = v_a v_{\dot{a}}$$

Notation: $< vu >= v^a u_a, [vu] = v^{\dot{a}} u_{\dot{a}}, UV =< vu > [vu]$.

The space-cone gauge

The Yang-Mills gauge field can be decomposed in a light-cone basis as

$$A^{\mu} = (a, \bar{a}, a^{+}, a^{-})$$

In a twistor basis $|+\rangle, |+], |-\rangle, |-]$, these components read

$$A = a^{+}|+\rangle[+| + a^{-}|-\rangle[-| + a|-\rangle[+| + \bar{a}|+\rangle[-|$$

The Yang-Mills theory Lagrangean

$$L = -\frac{1}{8}Tr(\partial^{\mu}A^{\nu} - \partial^{\mu}A^{\nu} + i[A^{\mu}, A^{\nu}])^{2} .$$

yields the simplest Feynman diagrams in the space-cone gauge

$$N \cdot A = 0, \text{ with } N = |+\rangle[-| \Rightarrow a = 0$$

Chalmers, Siegel '98

The space cone gauge-fixed Yang-Mills Lagrangean is given by

$$\begin{aligned}\mathcal{L} &= Tr\Big[\frac{1}{2}a^+\Box a^- - i\left(\frac{\partial^-}{\partial}a^+\right)[a^+,\partial a^-] - i\left(\frac{\partial^+}{\partial}a^-\right)[a^-\partial a^+] \\ &+ [a^+,\partial a^-]\frac{1}{\partial^2}[a^-,\partial a^+]\Big].\end{aligned}$$

where a^+ corresponds to a positive helicity gluon and a^- to a negative helicity one.

The twistor basis is selected such that an external positive helicity gluon has momentum

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$$P = \frac{1}{\partial}\left(e^{iPx} \cdots\right) =$$

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$$P = (p, \bar{p}, p^+, p^-)$$

$$\frac{1}{\partial}\left(e^{iP\cdot x} \ldots\right) = \frac{1}{\bar{p}}$$

$$A + \frac{c}{r^n}$$

$$ds^2 = -dt^2 + g \, dx + e^{2b} dy^2$$

$$V = A(\ldots)$$

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We have:

$$\epsilon_+ = \frac{|+\rangle[p|}{\langle p+\rangle}, \qquad \epsilon_- = \frac{|p\rangle[-|}{[-p]}$$

This implies $(\epsilon_+)^+ = \frac{[-p]}{\langle +p\rangle}$ and $(\epsilon_-)^- = \frac{\langle +p\rangle}{[-p]}$.

Notice that for the reference gluons

$$p^- = 1, \qquad q^+ = 1, \qquad (\epsilon_p)^+ = [-p] = 0, \qquad (\epsilon_q)^- = \langle +q\rangle = 0.$$

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However, in the vertices with a reference gluon we also need to evaluate $(\epsilon_p)^+ \times p^-/p = \frac{[-p]}{[-p]} = 1$.

The lesson: the reference gluons (+) participate only in (++K) 3-point vertices and the vertex is equal to k, where K is a negative helicity gluon.

Example: A 3-point function

Consider $(++-) = (123)$. Select the reference null vectors the momenta P_1 and some arbitrary $P_4 = |+\rangle[+|$. The answer is

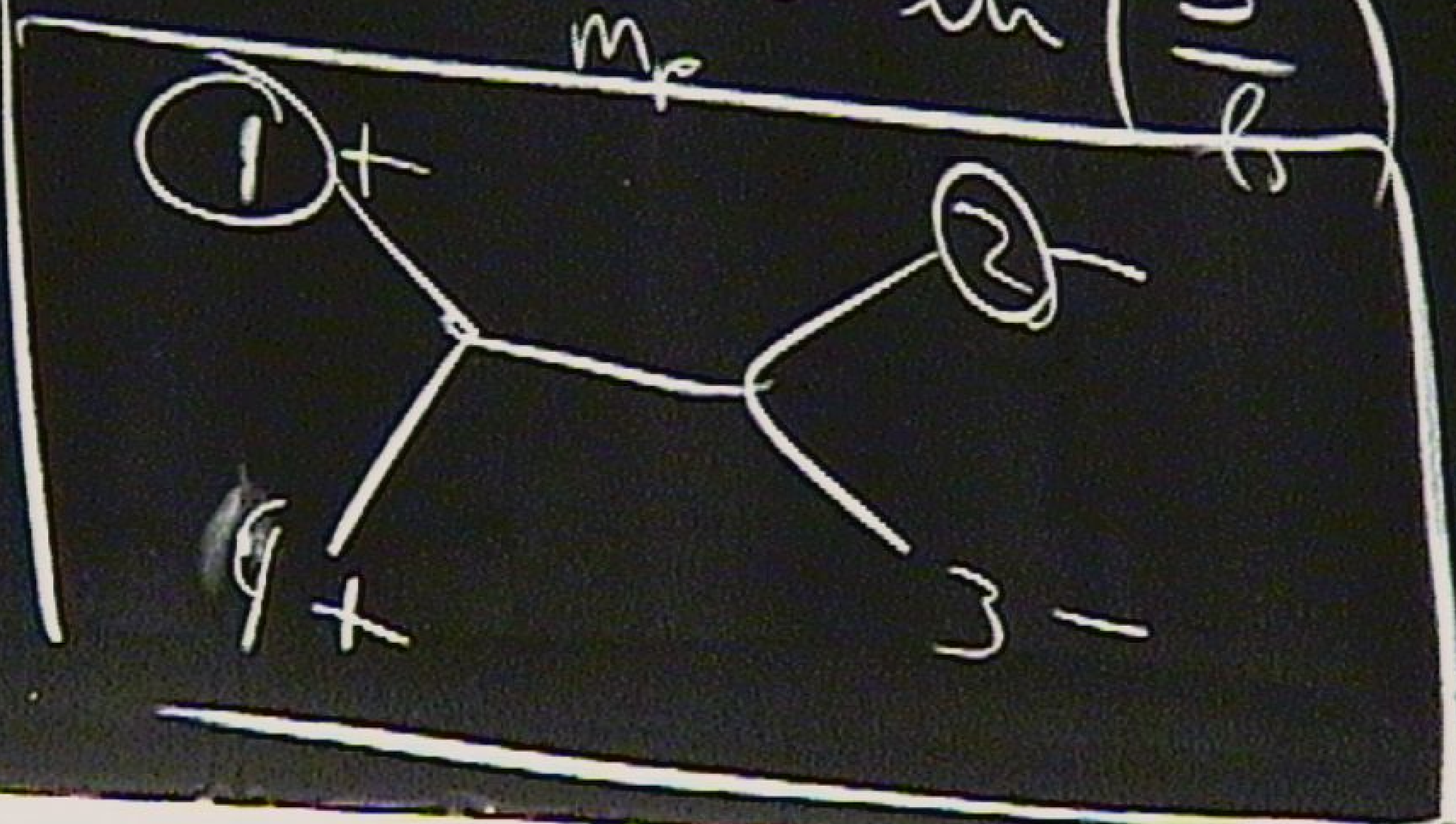
$$(++-) = p_3 \epsilon_2^+ \epsilon_3^- = \frac{[12]^3}{[23][31]}$$

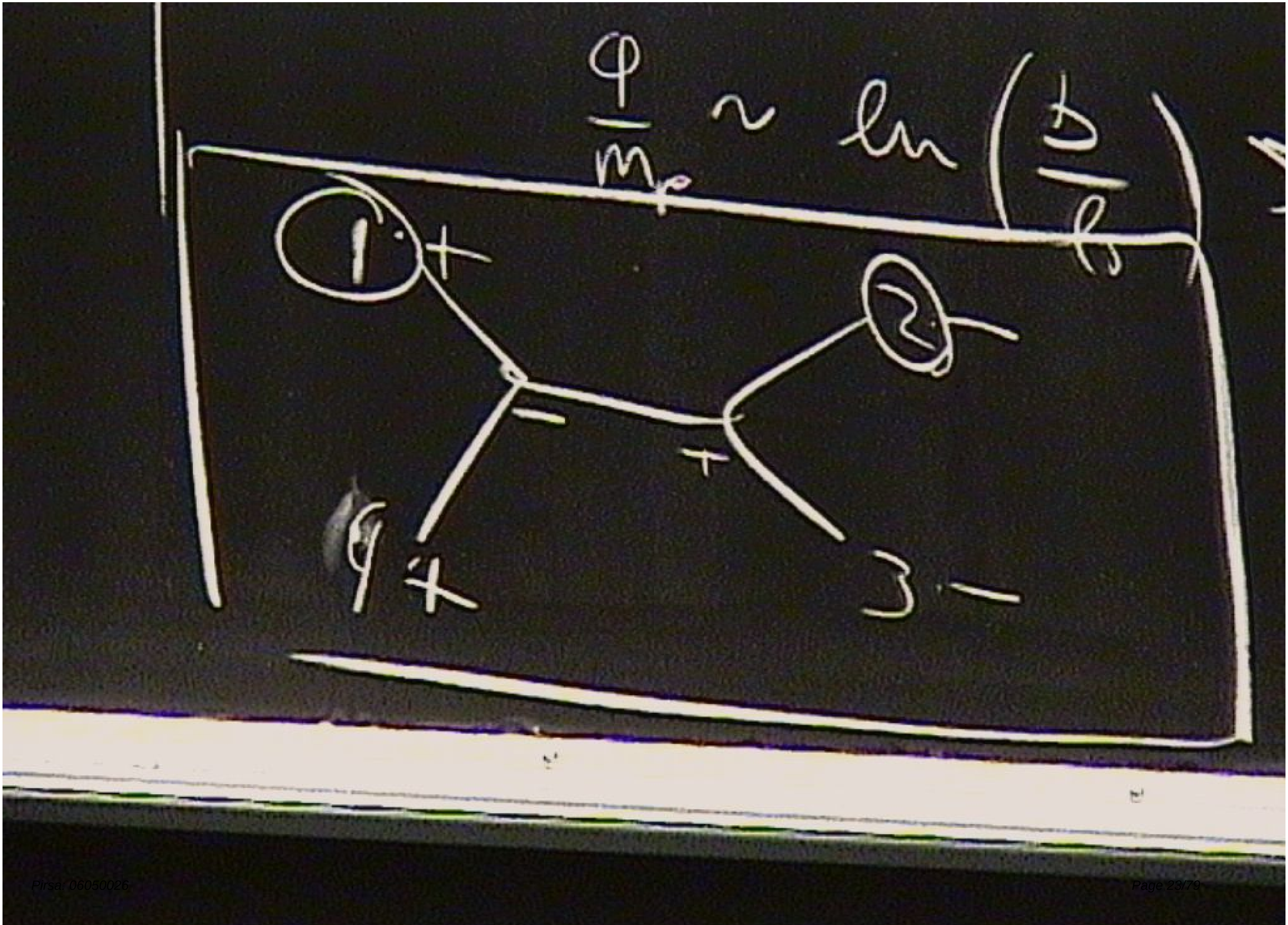
Example: A 4-point function

Consider $(+--+) = (1234)$. Select the reference vectors $P_1 = |-\rangle[-|$ and $P_2 = |+\rangle[+|$.

$$(++--) = \frac{1}{P_{14}^2} p_{14}^2 \epsilon_3^- \epsilon_4^+ = \frac{[12]^3}{[23][34][41]}$$

$$\frac{\phi}{M_P} \sim \ln\left(\frac{b}{b}\right)$$

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1 +
2 −
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QCD and tree level on-shell recursion relations

The on-shell recursion relations (BCFW) state:

$$\mathcal{A}(P, \{P_i\}, Q, \{Q_j\}) = \sum_{i,j} \mathcal{A}_L(\hat{P}, \{P_i\}) \frac{1}{(P + \sum_i P_i)^2} \mathcal{A}_R(\hat{Q}, \{Q_j\}) ,$$

where $\mathcal{A}_L, \mathcal{A}_R$ are lower n-point functions obtained by isolating two reference gluons with shifted momenta, $\hat{P} = P - z\eta$, $\hat{Q} = Q + z\eta$ with $\eta^2 = \eta \cdot P = \eta \cdot Q = 0$, on the two sides of the cut. The shifting of the two external momenta is necessary in order to preserve energy-momentum conservation.

Britto, Cachazo, Feng, Witten '05

$$\eta = |+\rangle_{C-1}$$

$$\phi = |-\rangle_{C-1}\,;\quad Q = |+\rangle_{C+1}$$

Conton

+ Quevedo

$$V = A + \frac{C}{r^n}$$

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Using the cutting rules which follow from the largest time equation and from the space-cone gauge QCD Lagrangean we provided a direct proof of the BCFW recursion relations.

DV, Ed Yao '05

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Consider first the effect of the $z\eta$ shift of the external momenta on the vertices. Take the shifted external gluons to coincide with the reference external gluons.

$$P \longrightarrow \hat{P} = |-\rangle[-| + z\eta, \qquad Q \longrightarrow \hat{Q} = |+\rangle[+| - z\eta$$
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The effect on components

$$\hat{p}^- = \langle +|\hat{P}|+] = 1, \qquad \bar{\hat{p}} = \langle -|\hat{P}|+] = -z$$
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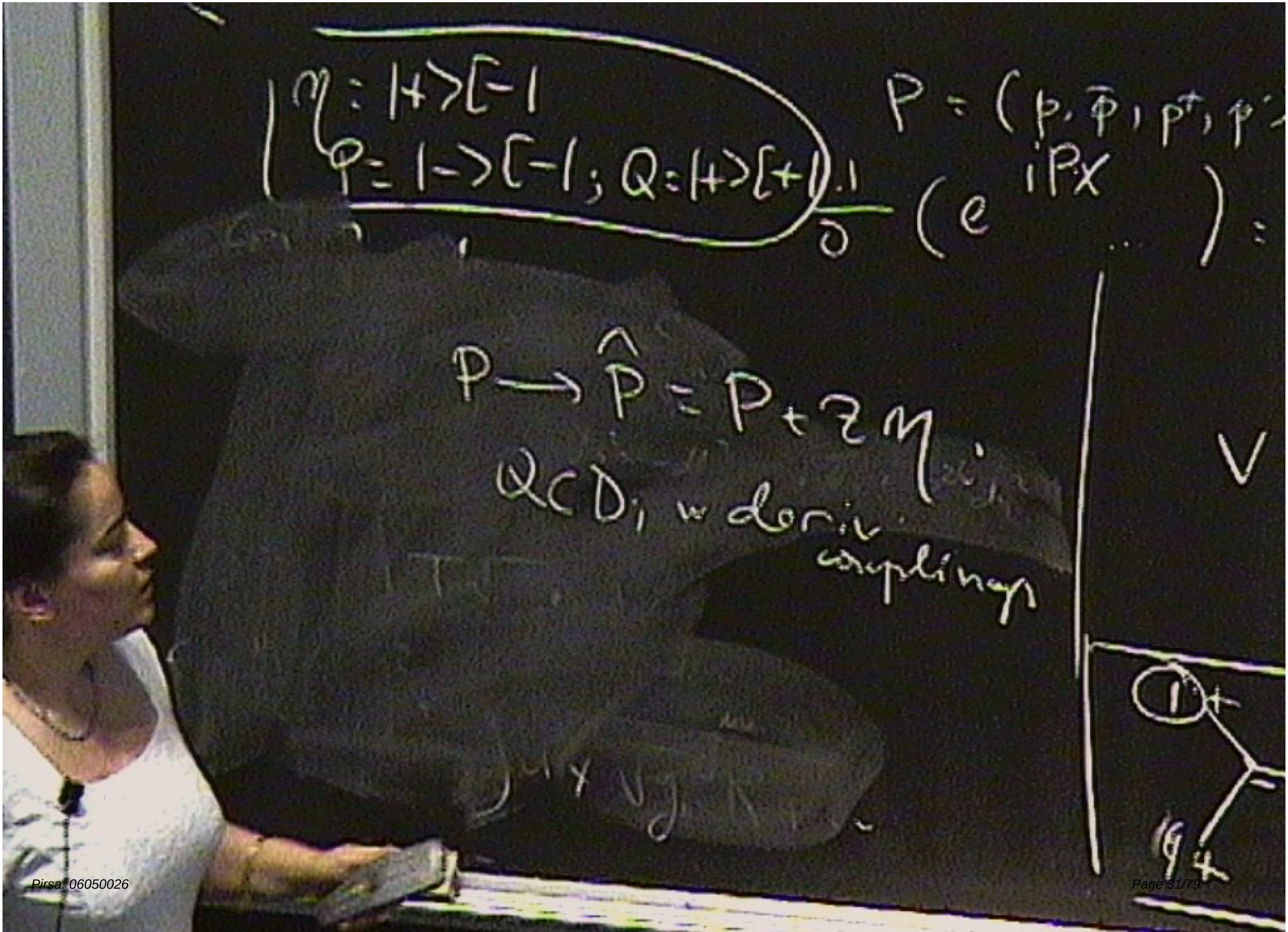
Do these shifts change the vertices?


η: |+>ℓ=-1
φ = |->ℓ=-1 ; Q: |+>ℓ=+1
P → P̂ = P + Zη
QCD, w deriv
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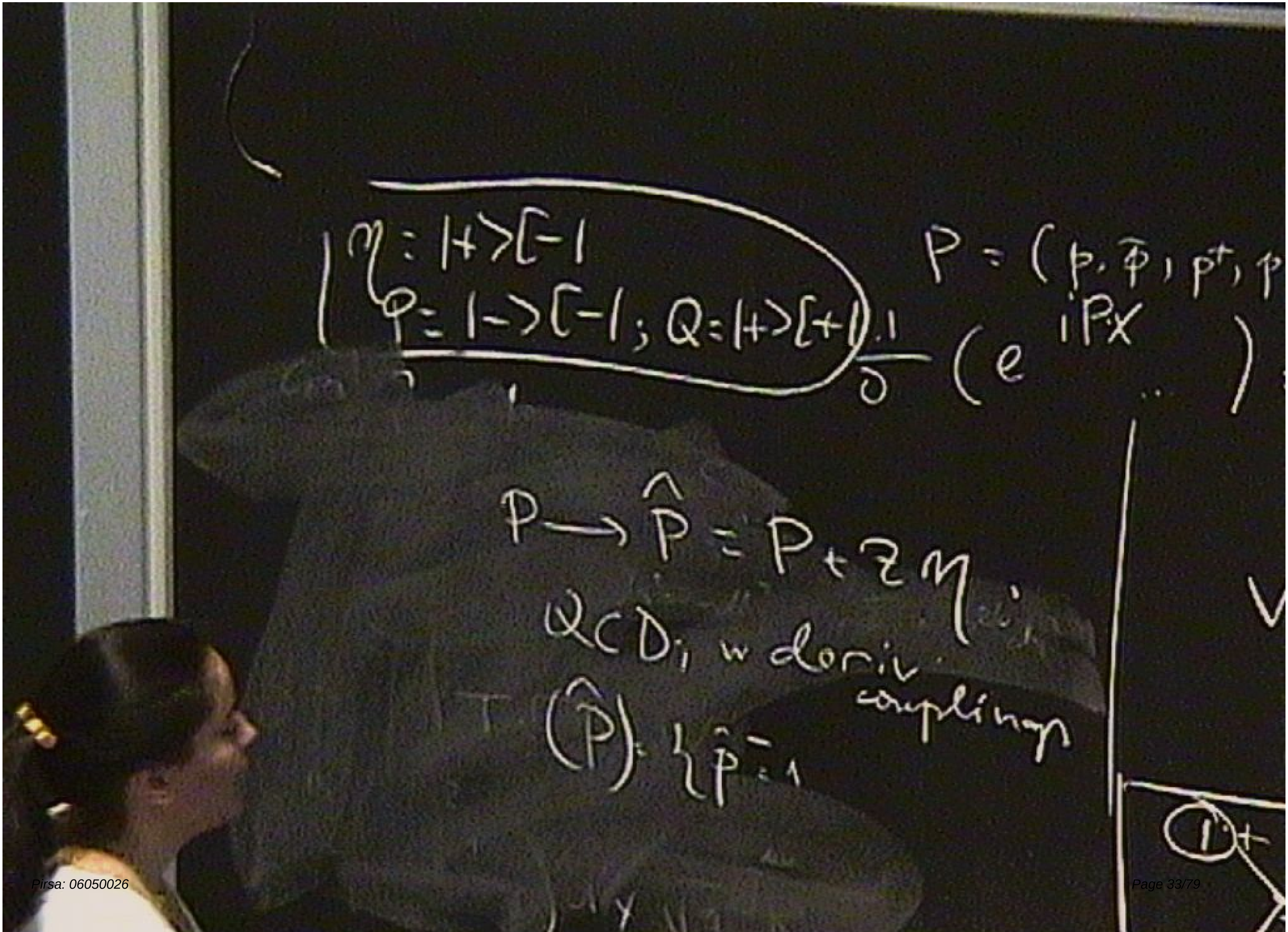
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Example: A 4-point function

Consider the same 4-point function $(+--+) = (1234)$ as before, and see what the recursion relation implies.

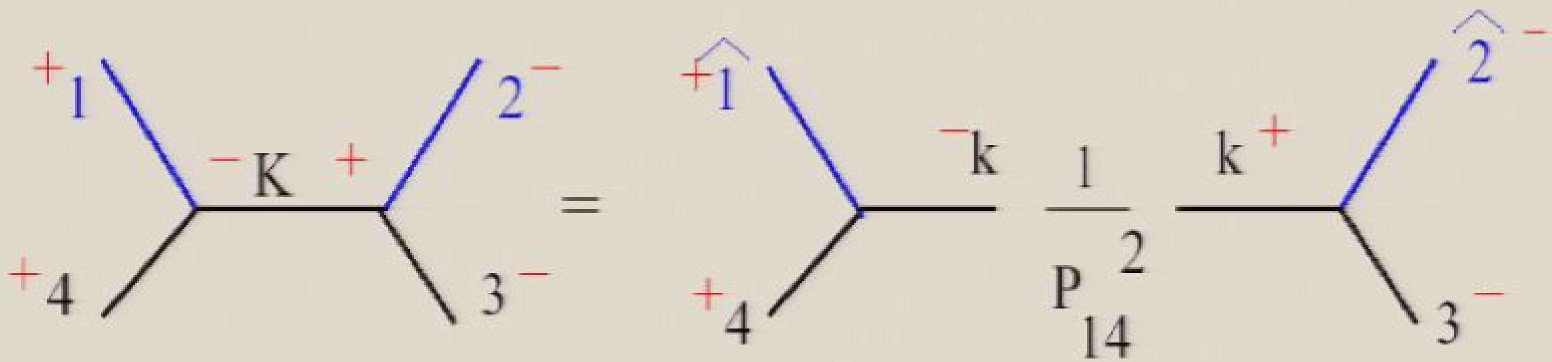

Figure 1: Factorization of the 4-point function

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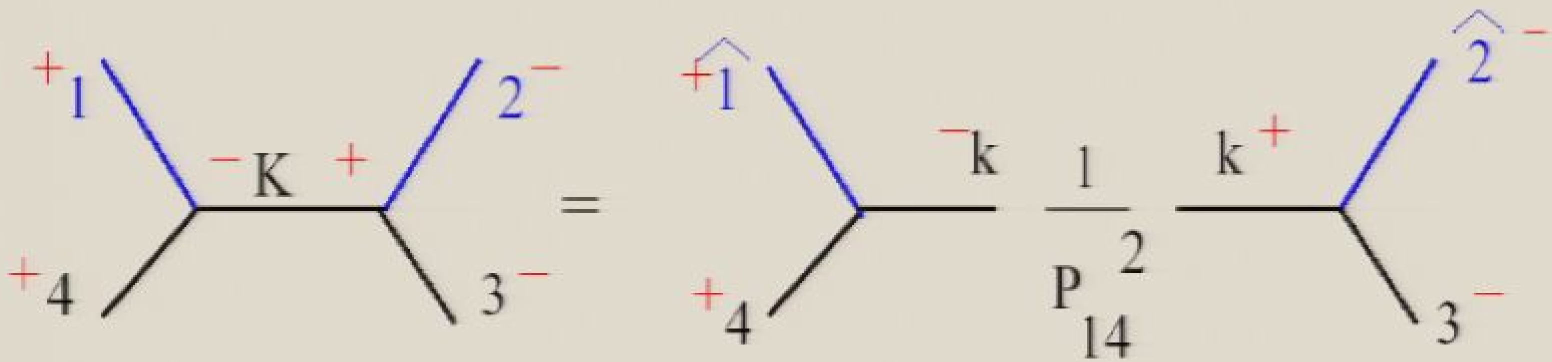

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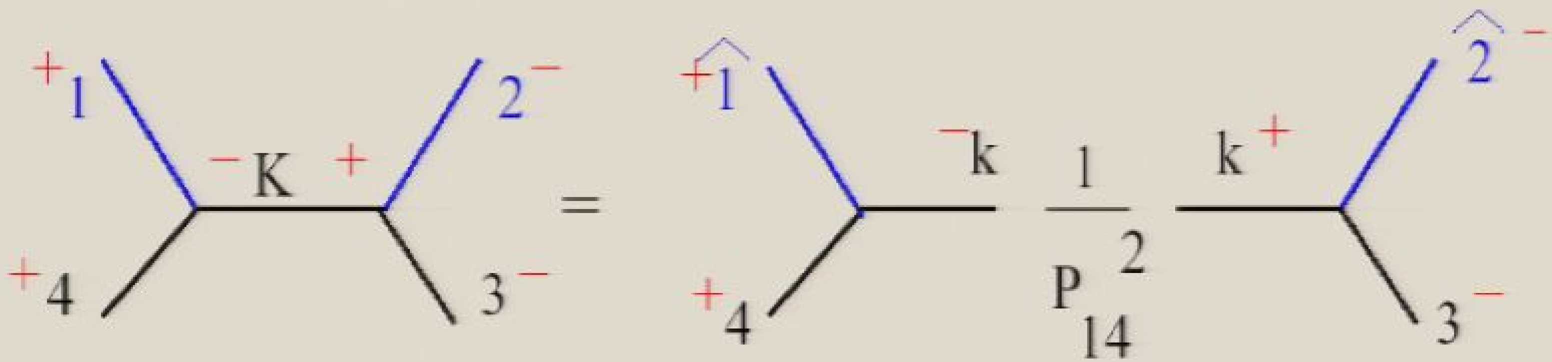

Figure 1: Factorization of the 4-point function

Notice: the vertices are the same, multiplication with external line factors is the same, and on the internal line, now on-shell, $\epsilon_K^+\epsilon_K^- = 1$. The factorization works trivially in this case.

Example: A 5-point function

Consider this time the 5-point function $(+++--) = (12345)$.

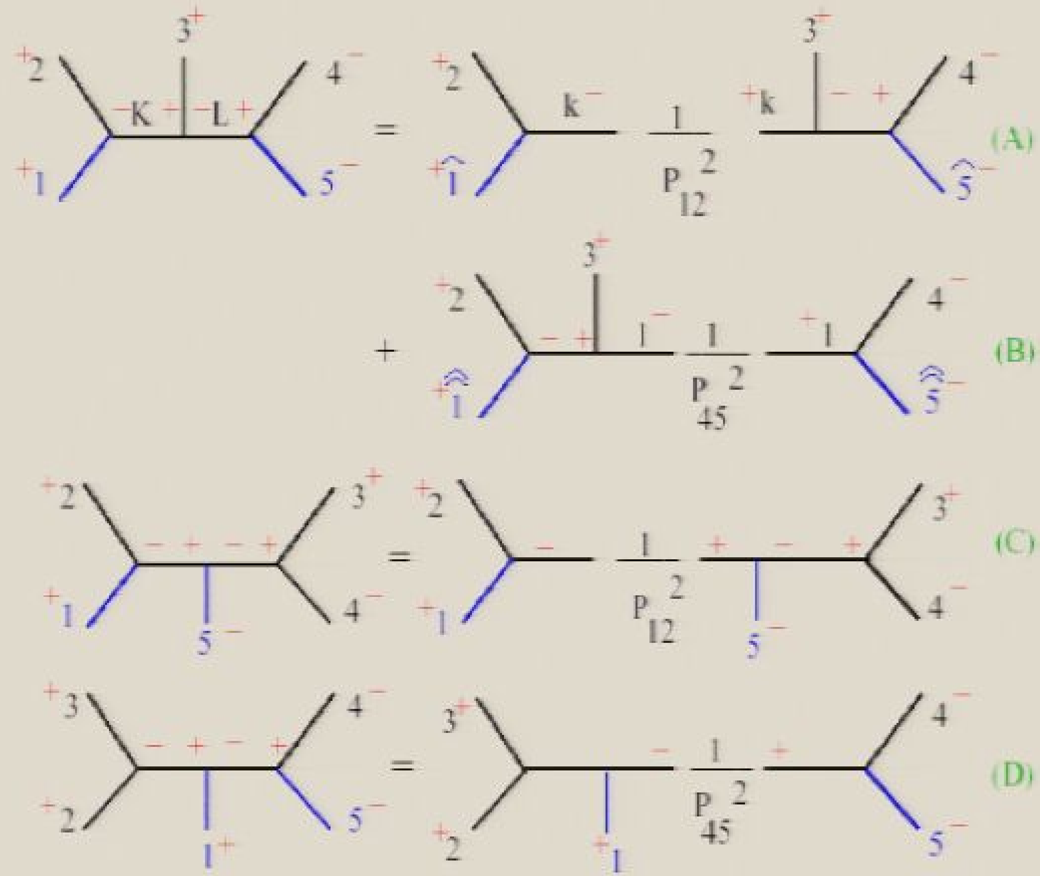

Figure 1: Factorization of the 5-point function

Comments:

- Each time there is only one internal line between the shifted/external gluons, the factorization is trivial.

In particular, C and D are trivial.

- When there is more than one internal line that can be set on-shell, the proof of factorization rests on a simple algebraic identity involving the propagators.

  In particular, the sum A+B equals

$$\frac{1}{P_{12}^2}\frac{1}{P_{45}^2} = \frac{1}{P_{12}^2}\frac{1}{P_{4\hat{5}}^2} + \frac{1}{P_{\hat{\hat{1}}2}^2}\frac{1}{P_{45}^2},$$

  where we defined the shifted reference gluon external momenta

$$P_{\hat{1}} = P_1 + \hat{z}\eta, \qquad P_{\hat{5}} = P_5 - \hat{z}\eta$$
$$P_{\hat{\hat{1}}} = P_1 + \hat{\hat{z}}\eta, \qquad P_{\hat{\hat{5}}} = P_5 - \hat{\hat{z}}\eta\,.$$

  $\hat{z}, \hat{\hat{z}}$ are such that we put the internal lines K, L, respectively, on-shell

$$\hat{z} = -\frac{P_{12}^2}{2\eta\cdot P_{12}} = -\frac{\langle 12\rangle}{\langle 52\rangle}, \qquad \hat{\hat{z}} = \frac{P_{45}^2}{2\eta\cdot P_{45}} = \frac{\langle 45\rangle}{\langle 15\rangle}.$$

Massaging "A+B":

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- Recall $\eta = |+\rangle[-| = |5\rangle[1|$.
- Use that

$$P_{12}^2 = -2\hat{z}\eta\cdot P_{12}, \qquad P_{45}^2 = 2\hat{\hat{z}}\eta\cdot P_{45}$$

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- then "A+B" becomes a trivial algebraic identity

$$\frac{1}{\hat{z}\hat{\hat{z}}} = \frac{1}{(\hat{\hat{z}} - \hat{z})\hat{z}} - \frac{1}{(\hat{\hat{z}} - \hat{z})\hat{\hat{z}}}\,.$$

This completes the proof of the BCFW recursion relation from Feynman diagrams for the 5-point function.

The largest time equation

Consider the scalar propagator:

$$\Delta(x-y) \quad = \quad \frac{1}{i}\int \frac{d^4L}{(2\pi)^4}\frac{1}{L^2 - i\epsilon}e^{iL\cdot(x-y)}\,.$$

We begin by explicitly construct a representation of the Feynman propagator, wherein a light-like four-vector is introduced as a parameter.

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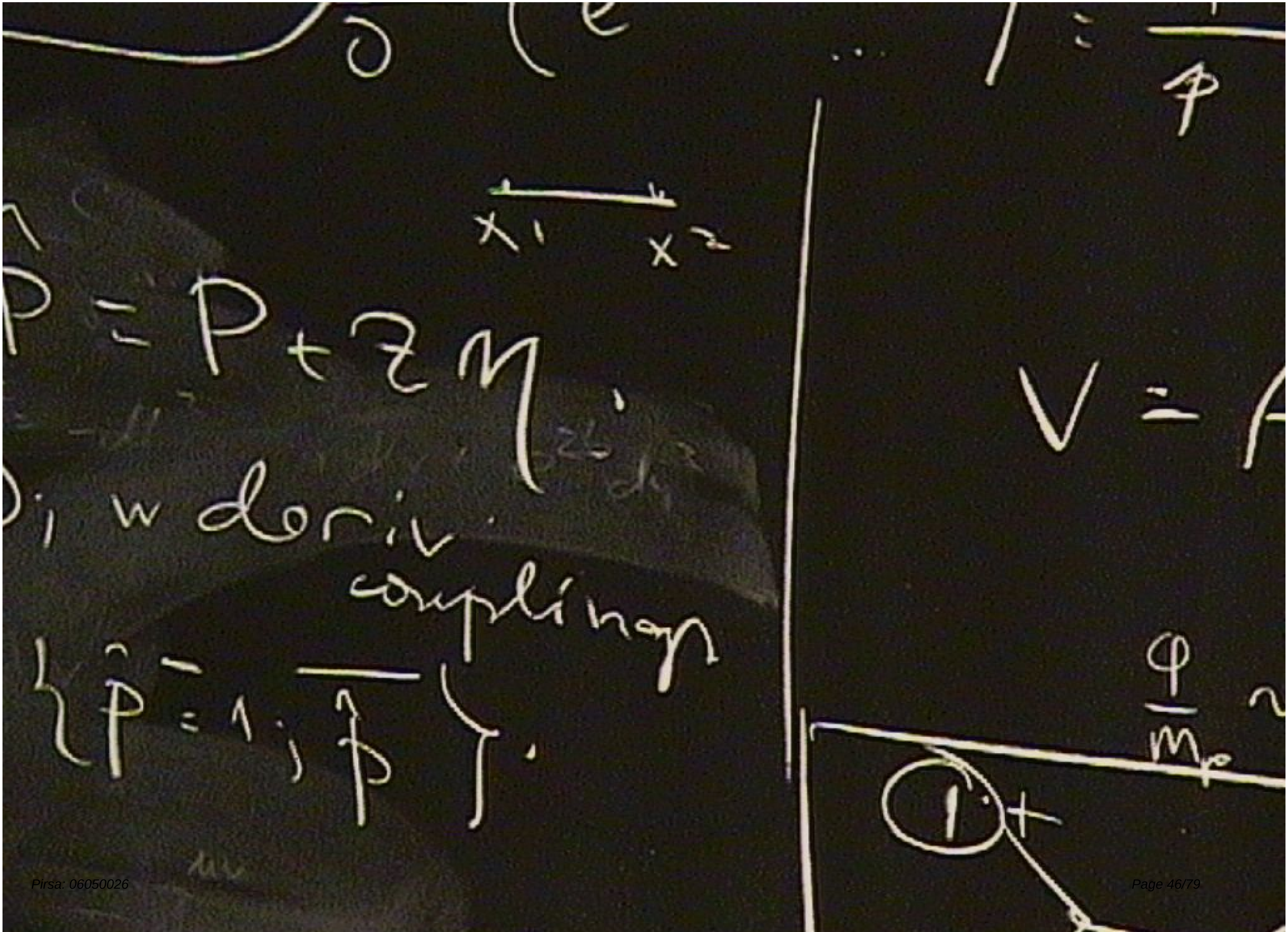
$$\begin{aligned}\Delta(x-y) \quad &= \quad \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\int \frac{dz}{z-i\epsilon}\theta(p^+)\delta(p^2)e^{i(p-z\eta)\cdot(x-y)} \\ &- \quad \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\int \frac{dz}{z+i\epsilon}\theta(-p^+)\delta(p^2)e^{i(p-z\eta)\cdot(x-y)}\,,\end{aligned}$$

with η an arbitrary null vector. Appropriating some common notations to the light-cone frame context, we can rewrite the position space propagator

$$\Delta(x-y) \quad = \quad \theta((x-y)^+)\Delta^+(x-y)+\theta(-(x-y)^+)\Delta^-(x-y)\,.$$


x1 x2
w deriv.
couplings
V = A

The largest time equation

Consider the scalar propagator:

$$\Delta(x-y) \quad = \quad \frac{1}{i}\int \frac{d^4L}{(2\pi)^4}\frac{1}{L^2-i\epsilon}e^{iL\cdot(x-y)}\,.$$

We begin by explicitly construct a representation of the Feynman propagator, wherein a light-like four-vector is introduced as a parameter.

$$\begin{aligned}\Delta(x-y) \quad &= \quad \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\int \frac{dz}{z-i\epsilon}\theta(p^+)\delta(p^2)e^{i(p-z\eta)\cdot(x-y)} \\ &- \quad \frac{1}{i}\int \frac{d^4p}{(2\pi)^4}\int \frac{dz}{z+i\epsilon}\theta(-p^+)\delta(p^2)e^{i(p-z\eta)\cdot(x-y)}\,,\end{aligned}$$

with η an arbitrary null vector. Appropriating some common notations to the light-cone frame context, we can rewrite the position space propagator

$$\Delta(x-y) \quad = \quad \theta((x-y)^+)\Delta^+(x-y)+\theta(-(x-y)^+)\Delta^-(x-y)\,.$$

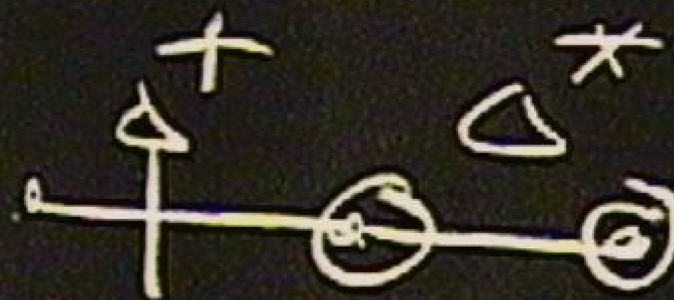
The algebraic identity involving propagators, which sits at the core of our proof of factorization, is a consequence of the largest time equation. Following Veltman '63, we begin with the following set of rules:

- for any Feynman diagram, construct all copies s.t. the vertices can be circled or un-circled;
- any circled vertex brings i, and each un-circled vertex a factor $(-i)$
- the propagator between to uncircled vertices is $\Delta(x-y)$, and the one between circled vertices is $\Delta^*(x-y)$
- the propagator between circled and un-circled vertices is $\Delta^+(x-y)$ and between un-circled and circled is $\Delta^+(x-y)$

$$F(x_i) + F^*(x_i) + \mathbf{F}(x_i) = 0$$

$p^{-})$

$$= \frac{1}{p}$$



$$V = A(\cdots) + B e^{-\varphi/M_P} + e^{2\varphi/\cdots}$$

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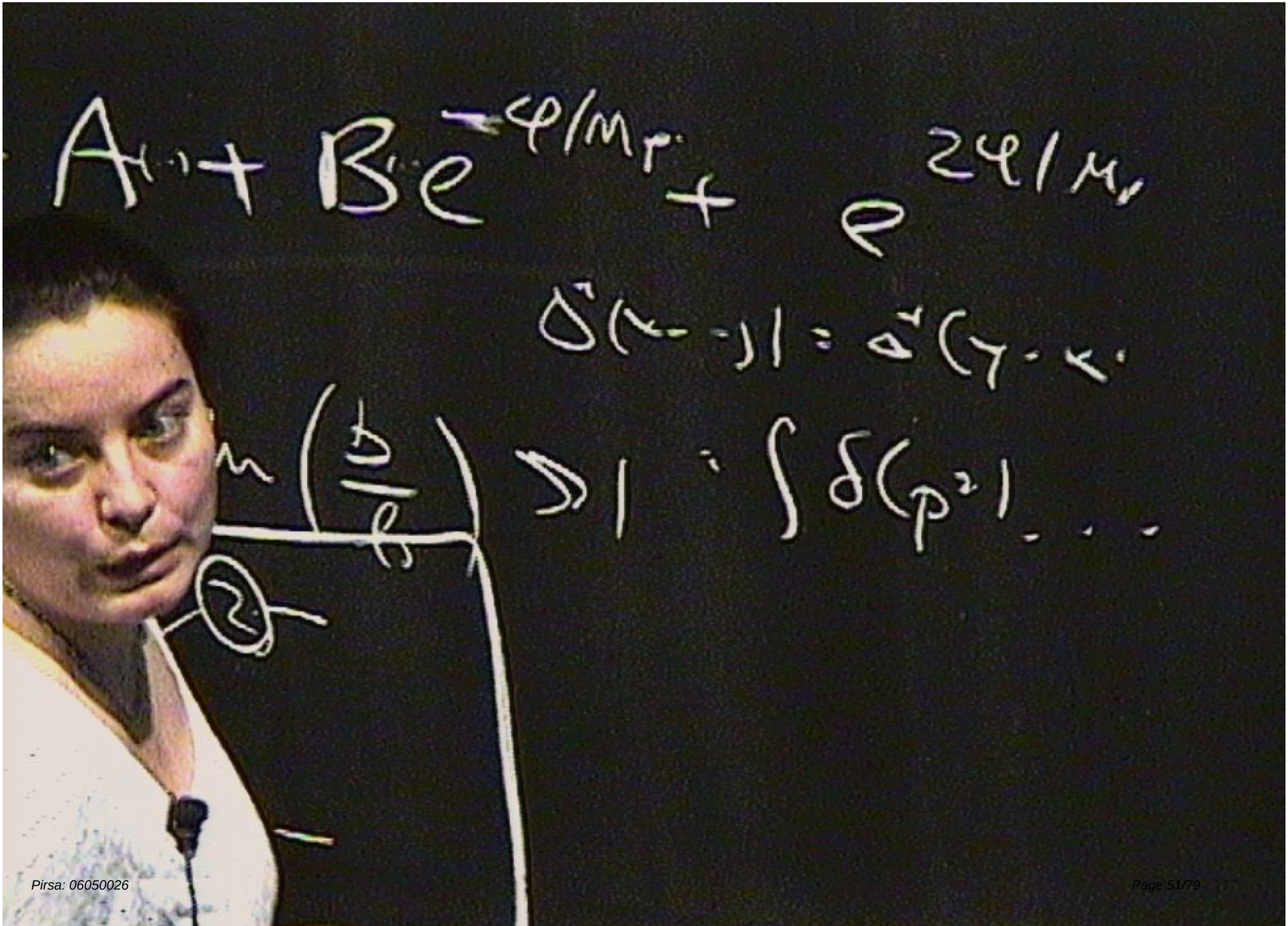

Another way to phrase the largest time equation is to select x_k and x_l. Assume $x_k^+ < x_l^+$. Then, one has

$$\theta((x_l - x_k)^+)(F(x_i) + \mathbf{F}(k, x_i)) = 0\,,$$

where $\mathbf{F}(k, x_i)$ is the sum of all diagrams with k uncircled, but at least one other vertex circled. Similarly, one has

$$\theta((x_k - x_l)^+)(F(x_i) + \mathbf{F}(l, x_i)) = 0\,,$$

By adding these two equations one finds

$$\mathsf{F}(x_i) = -\mathbf{F}(k, l, x_i) - \theta((x_l - x_k)^+)\mathbf{F}(k, \mathbf{l}, x_i) - \theta((x_k - x_l)^+)\mathbf{F}(\mathbf{k}, l, x_i))\,,$$

where $\mathbf{F}(k, l, x_i)$ is the sum of all diagrams with neither k, l circled, but at least one other vertex circled, $\mathbf{F}(k, \mathbf{l}, x_i)$ is the sum of all amplitudes with k uncircled , but l circled.

The shift of two external momenta by $\pm z\eta$ in the on-shell recursion relations is nothing but the result of the implementation of the step functions that appear in the largest time equation.

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Consider the propagator identity we found for the 5-point function $(+++--)$. Reinstating the usual $i\epsilon$ prescription in the momentum space propagators, the left-hand-side of the identity becomes:

$$\frac{\delta(P_1+\cdots+P_5)}{P_{12}^2}\frac{1}{P_{45}^2}$$
$$= i^2\int dx_1dx_2dx_3\Delta(x_1-x_2)\Delta(x_2-x_3)e^{i(P_1+P_2)x_1+iP_3x_2+i(P_4+P_5)x_3}.$$

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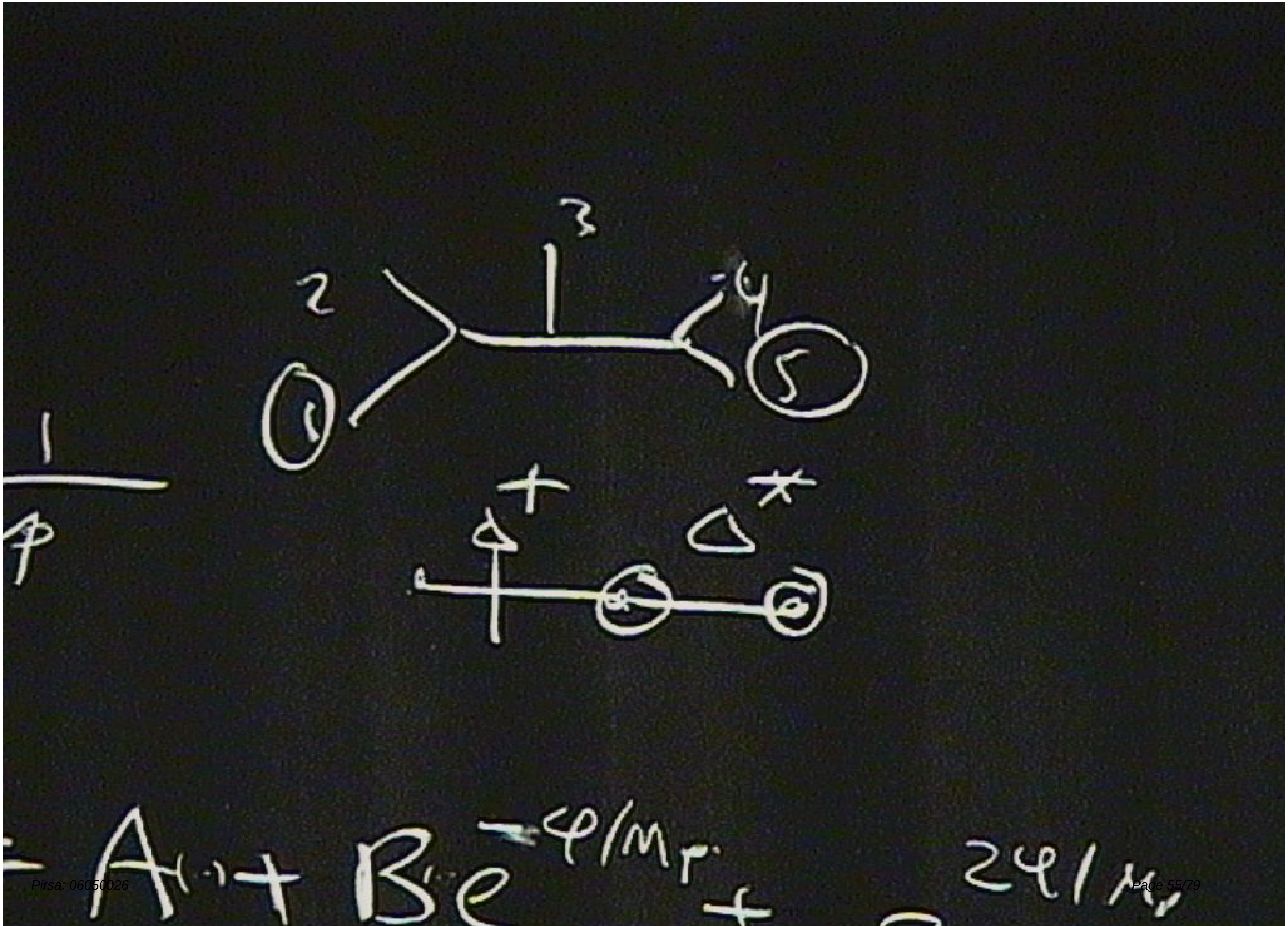
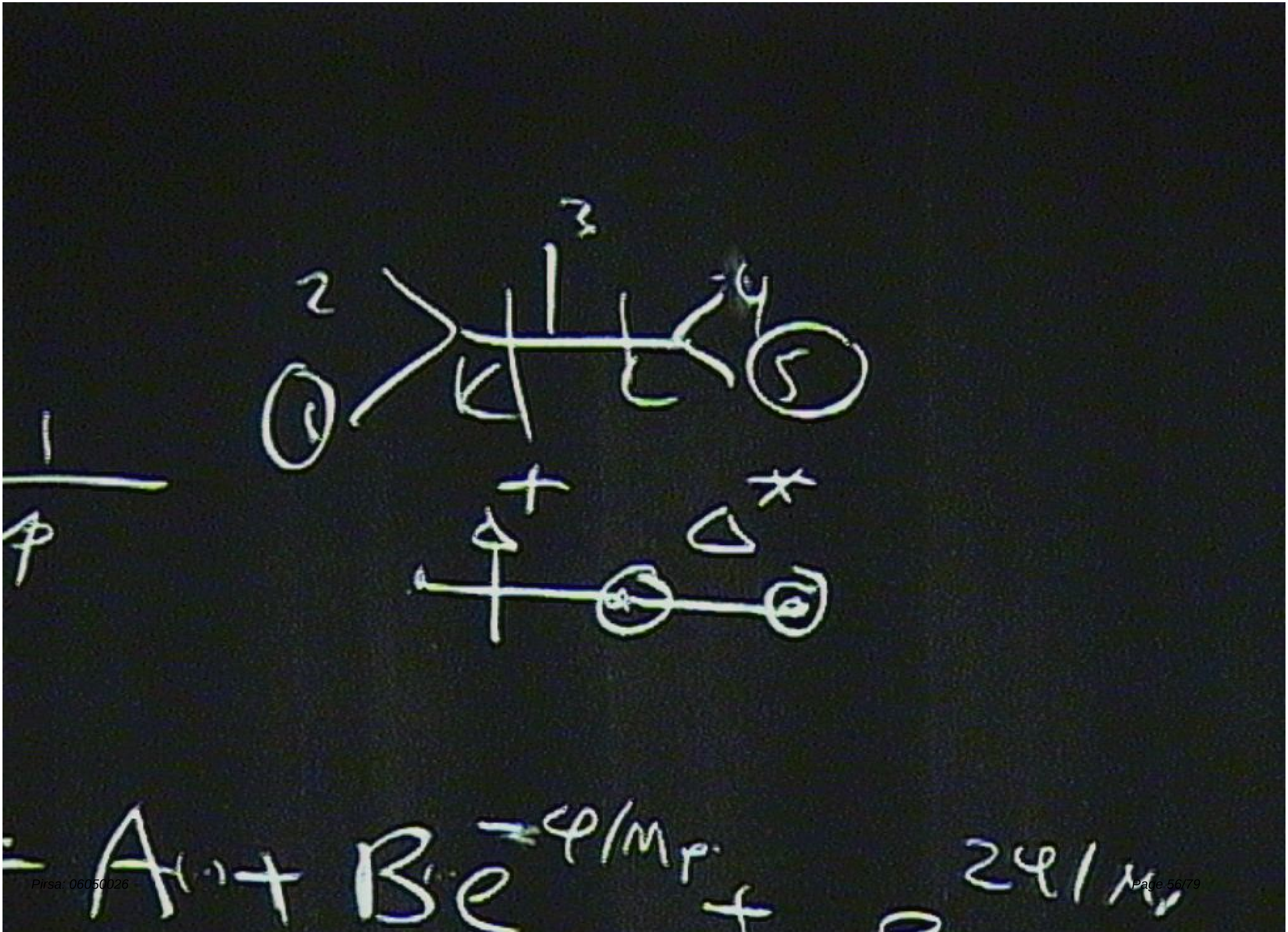
$$\left(\theta((x_1-x_3)^+)\Delta^+(x_1-x_2)+\theta((x_3-x_1)^+)\Delta^-(x_1-x_2)\right)\Delta(x_2-x_3)$$




1
2
3
4
5

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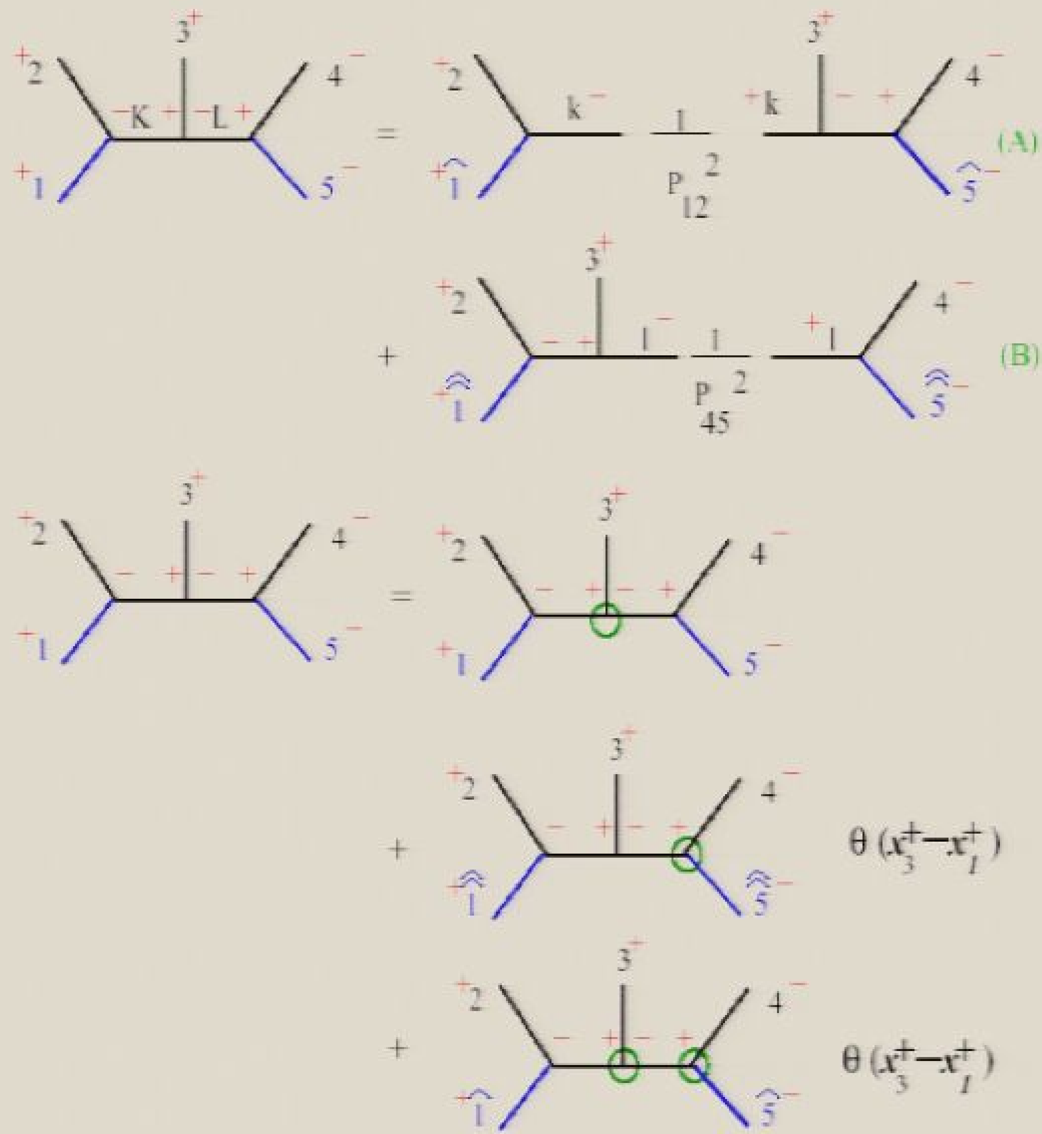
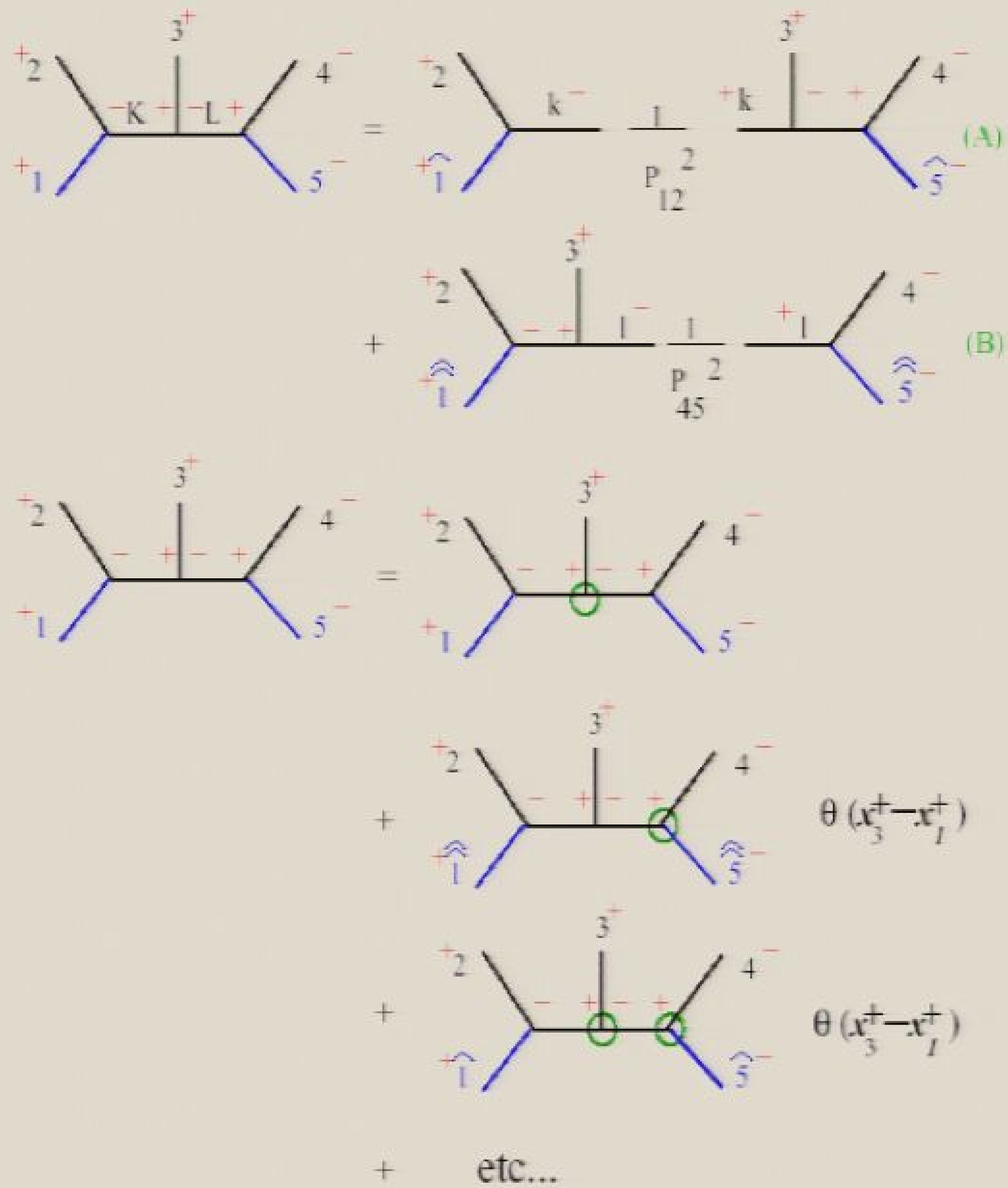


\+ etc...


(A)
(B)
θ (x₃⁺−x₁⁺)
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+ etc...

After a similar manipulation of the second term on the rhs, the identity becomes

$$\int dx_1 dx_2 dx_3 e^{i(P_1+P_2)x_1+iP_3x_2+i(P_4+P_5)x_3}\Big[\Delta(x_1-x_2)\Delta(x_2-x_3)$$
$$-\ \Big(\theta((x_1-x_3)^+)\Delta^+(x_1-x_2)+\theta((x_3-x_1)^+)\Delta^-(x_1-x_2)\Big)\Delta(x_2-x_3)$$
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There is one more step that is needed in order to show the relationship with the largest time equation with x_1, x_3 the two vertices that are singled out

$$\Delta(x_1-x_2)\Delta(x_2-x_3) = \Delta^-(x_1-x_2)\Delta^+(x_2-x_3)$$
$$+\ \theta((x_1-x_3)^+)\Big(\Delta^+(x_1-x_2)\Delta(x_2-x_3)-\Delta^*(x_1-x_2)\Delta^+(x_2-x_3)\Big)$$
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The Fourier-transform of our identity equals the largest time eqution up to the following extra terms: $\theta((x_1-x_3)^+)\Delta^+(x_1-x_2)\Delta^+(x_2-x_3), \theta((x_3-x_1)^+)\Delta^-(x_1-x_2)\Delta^-(x_2-x_3)$.

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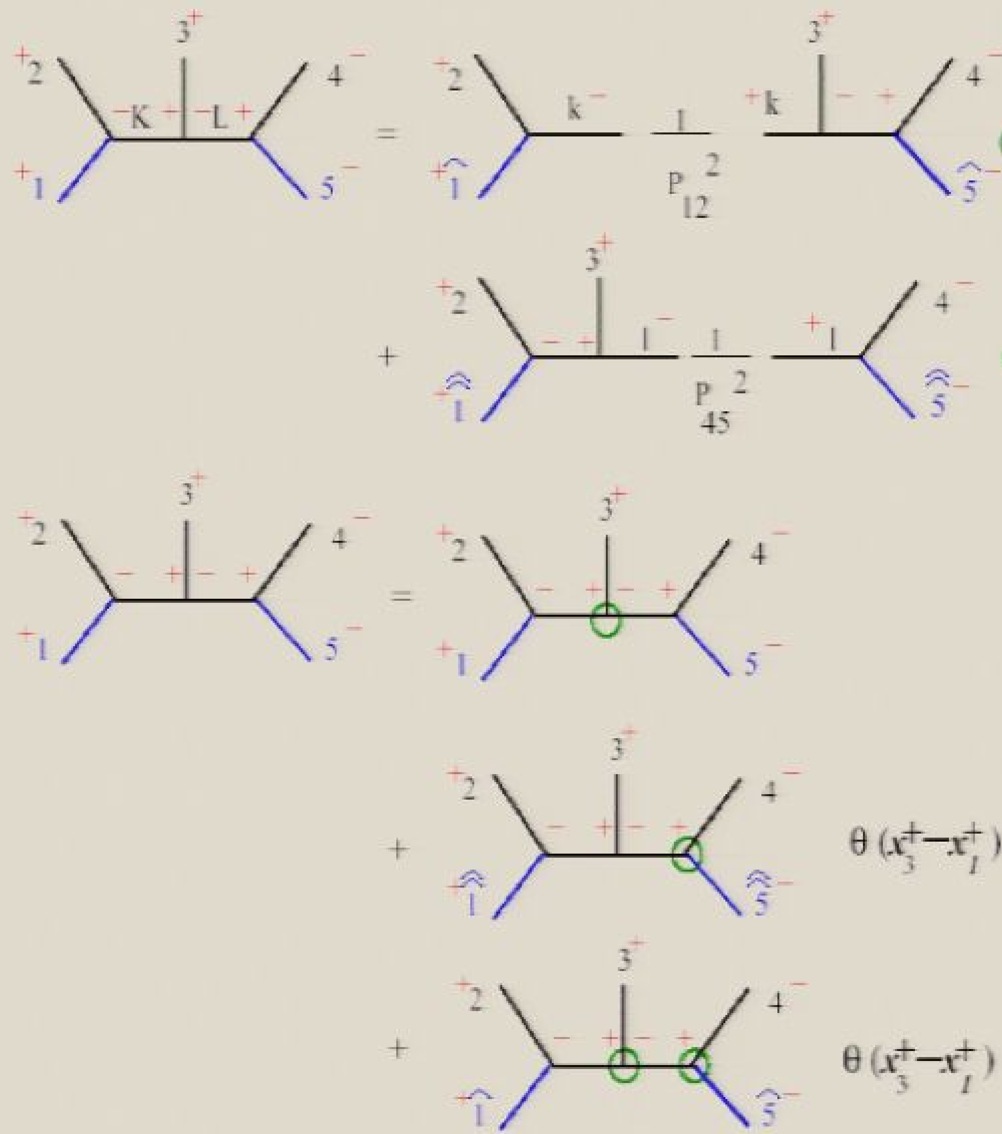

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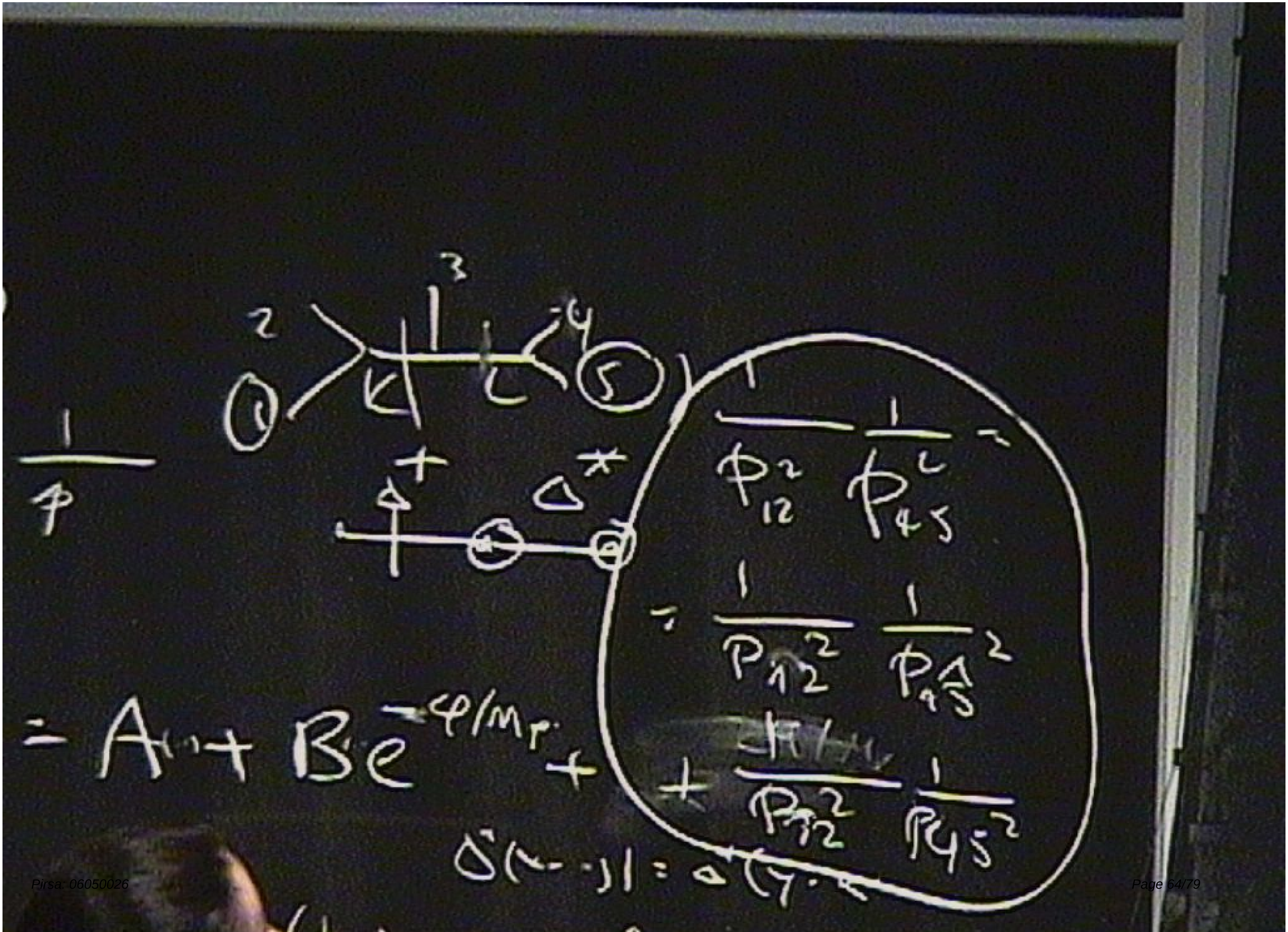

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by rewriting the step function as a z-integral, followed by the integration over x_1, x_2, x_3, to arrive at

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We have shown that the algebraic identity which was found by reassembling the Feynman diagrams into the BCFW recursion relations arises from the more fundamental largest time equation.

Massaging "A+B":

$$\frac{1}{P_{12}^2}\frac{1}{P_{45}^2} = \frac{1}{P_{12}^2}\frac{1}{P_{4\hat{5}}^2} + \frac{1}{P_{\hat{1}2}^2}\frac{1}{P_{45}^2}$$

- Recall $\eta = |+\rangle[-| = |5\rangle[1|$.
- Use that

$$P_{12}^2 = -2\hat{z}\eta \cdot P_{12}, \qquad P_{45}^2 = 2\hat{\hat{z}}\eta \cdot P_{45}$$
$$P_{\hat{\hat{1}}2}^2 = 2(\hat{\hat{z}} - \hat{z})\eta \cdot P_{12}, \qquad P_{4\hat{\hat{5}}}^2 = 2(\hat{\hat{z}} - \hat{z})\eta \cdot P_{45}$$

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- then "A+B" becomes a trivial algebraic identity

$$\frac{1}{\hat{z}\hat{\hat{z}}} = \frac{1}{(\hat{\hat{z}} - \hat{z})\hat{z}} - \frac{1}{(\hat{\hat{z}} - \hat{z})\hat{\hat{z}}}\,.$$

This completes the proof of the BCFW recursion relation from Feynman diagrams for the 5-point function.

Consider the propagator identity we found for the 5-point function $(+++--)$. Reinstating the usual $i\epsilon$ prescription in the momentum space propagators, the left-hand-side of the identity becomes:

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We have shown that the algebraic identity which was found by reassembling the Feynman diagrams into the BCFW recursion relations arises from the more fundamental largest time equation.

Comments: For the largest time equation η is real. For its Fourier-transform version into momentum space, appropriate for a physical process under consideration we keep η as a variable.

We then analytically complexify $\eta \longrightarrow |+\rangle[-|$.

A general proof which can be extended to massless/massive scalar particles coupled to Yang-Mills gauge bosons is based on partial fractioning.

The factorization procedure amounts to splicing the graph into a sum of products of two on-shell graphs with shifted momenta $\{p_a - z_i\eta, \ldots, \bar{q}_i\}$ and $\{-\bar{q}_i, \cdots, p_b + z_i\eta\}$, multiplying the propagator $\frac{1}{q_i^2+m_i^2}$.

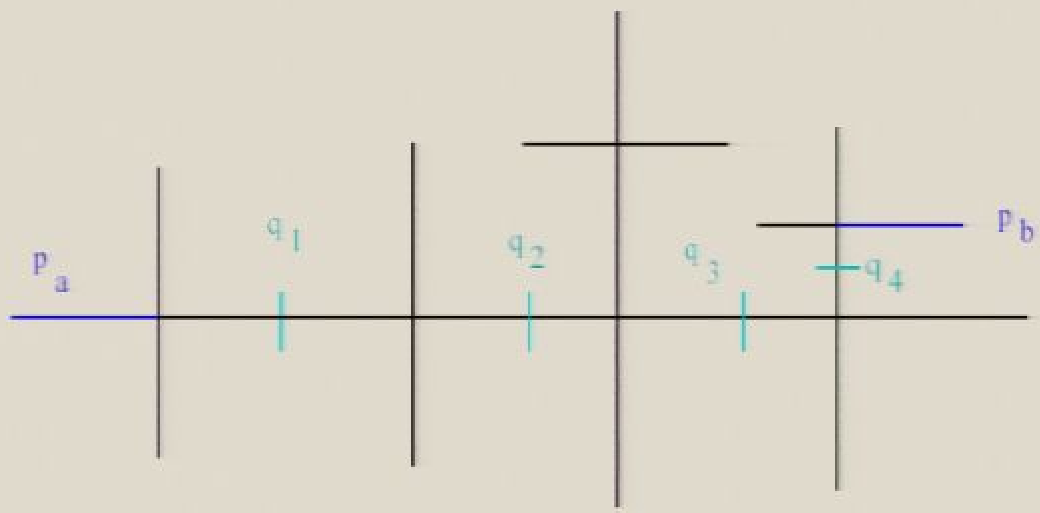


Figure 1: Factorization of tree level amplitudes

$$
\begin{aligned}
\frac{(-1)^n}{z_1 z_2 \cdots z_{n-1}} &= \frac{1}{z_1(z_1-z_2)(z_1-z_3)\cdots(z_1-z_{n-1})} \\
&+ \frac{1}{(z_2-z_1)z_2(z_2-z_3)\cdots(z_2-z_{n-1})} \\
& \cdots\cdots\cdots \\
&+ \frac{1}{(z_{n-1}-z_1)(z_{n-1}-z_2)\cdots(z_{n-1}-z_{n-2})z_{n-1}}.
\end{aligned}
$$

This partial fractioning formula emerges from $\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_{n-1})} = 0.$

Adding quarks

Tree level recursion relations with fermions:

Consider the Lagrangian of minimally coupled massive fermions

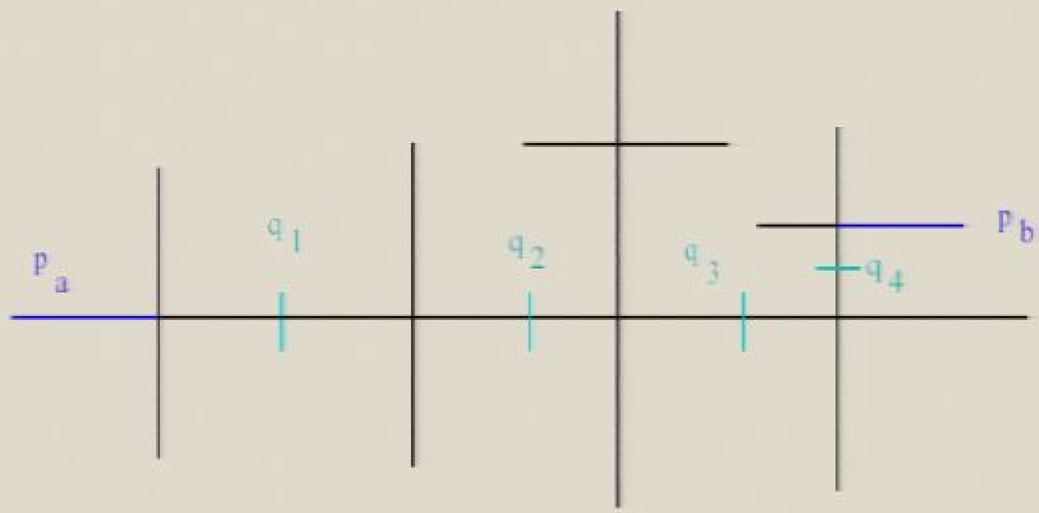
$$\mathcal{L}_f = \sum_i \bar{\Psi}_i (i\nabla\!\!\!/ - m_i)\Psi_i \,,$$

The recursion relations are formulated with the two reference gluons connected by a path which includes a fermionic line are based on the another identity involving momentum space fermion propagators

$$
\begin{aligned}
& \frac{1}{\not{q}_1 + m_1}\gamma^{\mu_1}\frac{1}{\not{q}_2 + m_2}\gamma^{\mu_2}\dots\gamma^{\mu_{n-2}}\frac{1}{\not{q}_{n-1} + m_{n-1}} \\
= \; & \frac{m_1 - \not{q}_1 + z_1 \not{\eta}}{(q_1^2 + m_1^2)}\gamma^{\mu_1}\frac{1}{\not{q}_2 - z_1\not{\eta} + m_2}\gamma^{\mu_2}\dots\gamma^{\mu_{n-2}}\frac{1}{\not{q}_{n-1} - z_1\not{\eta} + m_{n-1}} \\
+ \; & \frac{1}{\not{q}_1 - z_2\not{\eta} + m_1}\gamma^{\mu_1}\frac{m_2 - \not{q}_2 + z_2\not{\eta}}{(q_2^2 + m_2^2)}\gamma^{\mu_2}\dots\gamma^{\mu_{n-2}}\frac{1}{\not{q}_{n-1} - z_2\not{\eta} + m_{n-1}} \\
+ \; & \dots\dots\dots \\
+ \; & \frac{1}{\not{q}_1 - z_{n-1}\not{\eta} + m_1}\gamma^{\mu_1}\dots\gamma^{\mu_{n-3}}\frac{1}{\not{q}_{n-2} - z_{n-1}\not{\eta} + m_{n-2}}\gamma^{\mu_{n-2}}\frac{m_{n-1} - \not{q}_{n-1} + z_{n-1}\not{\eta}}{(q_{n-1}^2 + m_{n-1}^2)} .
\end{aligned}
$$

A general proof which can be extended to massless/massive scalar particles coupled to Yang-Mills gauge bosons is based on partial fractioning.

The factorization procedure amounts to splicing the graph into a sum of products of two on-shell graphs with shifted momenta $\{p_a - z_i\eta, \ldots, \bar{q}_i\}$ and $\{-\bar{q}_i, \cdots, p_b + z_i\eta\}$, multiplying the propagator $\frac{1}{q_i^2+m_i^2}$.



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&+ \frac{1}{(z_2-z_1)z_2(z_2-z_3)\cdots(z_2-z_{n-1})} \\
&\cdots\cdots\cdots \\
&+ \frac{1}{(z_{n-1}-z_1)(z_{n-1}-z_2)\cdots(z_{n-1}-z_{n-2})z_{n-1}}.
\end{aligned}
$$

This partial fractioning formula emerges from $\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_{n-1})} = 0.$

Adding quarks

Tree level recursion relations with fermions:

Consider the Lagrangian of minimally coupled massive fermions

$$\mathcal{L}_f = \sum_i \bar{\Psi}_i (i \not{\nabla} - m_i) \Psi_i \,,$$

The recursion relations are formulated with the two reference gluons connected by a path which includes a fermionic line are based on the another identity involving momentum space fermion propagators

$$\begin{aligned}
& \frac{1}{\not{q}_1 + m_1} \gamma^{\mu_1} \frac{1}{\not{q}_2 + m_2} \gamma^{\mu_2} \cdots \gamma^{\mu_{n-2}} \frac{1}{\not{q}_{n-1} + m_{n-1}} \\
= \; & \frac{m_1 - \not{q}_1 + z_1 \not{\eta}}{(q_1^2 + m_1^2)} \gamma^{\mu_1} \frac{1}{\not{q}_2 - z_1 \not{\eta} + m_2} \gamma^{\mu_2} \cdots \gamma^{\mu_{n-2}} \frac{1}{\not{q}_{n-1} - z_1 \not{\eta} + m_{n-1}} \\
+ \; & \frac{1}{\not{q}_1 - z_2 \not{\eta} + m_1} \gamma^{\mu_1} \frac{m_2 - \not{q}_2 + z_2 \not{\eta}}{(q_2^2 + m_2^2)} \gamma^{\mu_2} \cdots \gamma^{\mu_{n-2}} \frac{1}{\not{q}_{n-1} - z_2 \not{\eta} + m_{n-1}} \\
+ \; & \cdots\cdots\cdots \\
+ \; & \frac{1}{\not{q}_1 - z_{n-1} \not{\eta} + m_1} \gamma^{\mu_1} \cdots \gamma^{\mu_{n-3}} \frac{1}{\not{q}_{n-2} - z_{n-1} \not{\eta} + m_{n-2}} \gamma^{\mu_{n-2}} \frac{m_{n-1} - \not{q}_{n-1} + z_{n-1} \not{\eta}}{(q_{n-1}^2 + m_{n-1}^2)} .
\end{aligned}$$

For the proof, first rewrite the fermionic propagators such that all denominators will correspond to scalar propagators. We end up with a set of partial fractioning identities which arise from

$$\int dz \frac{z^m}{(z-z_1)(z-z_2)\ldots(z-z_{n-1})} = 0, \text{ for } m < n-2,$$

where the integral is evaluated over a contour which encircles all the poles.

Conclusions and future directions

- We offered a purely quantum field theoretical proof of the BCFW recursion relations.
- A key ingredient was the use of space-cone gauge.
- The tree level recursion relations emerge at a more fundamental level from the largest time equation.
- Our results lend themselves to natural generalizations to include massive scalars and fermions.
- We are currently investigating the extension of our methods at the loop level.
- A few preliminary results:
  - The dispersive integrals are a hallmark of the largest time equation.
  - We have computed one-loop 3-point functions. For same helicity gluons the amplitude is given by

$$(+++) = (12k) = \frac{\epsilon_1^+ \epsilon_2^+ (p_1^- p_2 - p_2^- p_1)(p_2^- p_k - p_k^- p_2)(p_k^- p_1 - p_1^- p_k)}{p_1 p_2 p_k} \frac{1}{P_1 \cdot P_2}$$

where $P_1^2 = P_2^2 = 0$.