Title: What to do if quantum channels are not noiseless enough

Date: May 17, 2006 04:00 PM

URL: http://pirsa.org/06050008

Abstract: This talk is concerned with the noise-insensitive transmission of quantum information. For this purpose, the sender incorporates redundancy by mapping a given initial quantum state to a messenger state on a larger-dimensional Hilbert space. This encoding scheme allows the receiver to recover part of the initial information if the messenger system is corrupted by interaction with its environment. Our noise model for the transmission leaves a part of the quantum information unchanged, that is, we assume the presence of a noiseless subsystem or of a decoherence-free subspace. We address the case when the noiseless component cannot contain all the quantum information to be transmitted, and investigate how to best spread the information in a quantum state across the noise-susceptible components. (Joint work with David Kribs and Vern Paulsen.)

What to do if quantum channels are not noiseless enough

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Joint Seminar of the Department of Applied Mathematics, University of Waterloo and the Perimeter Institute

May 17, 2006

Collaborators

David W. Kribs

UNIVERSITY of GUELPH

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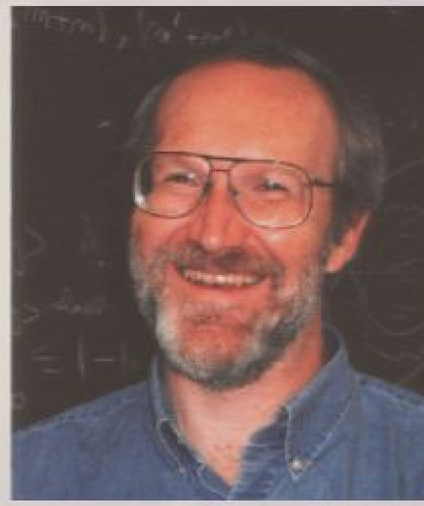
UNIVERSITY OF HOUSTON

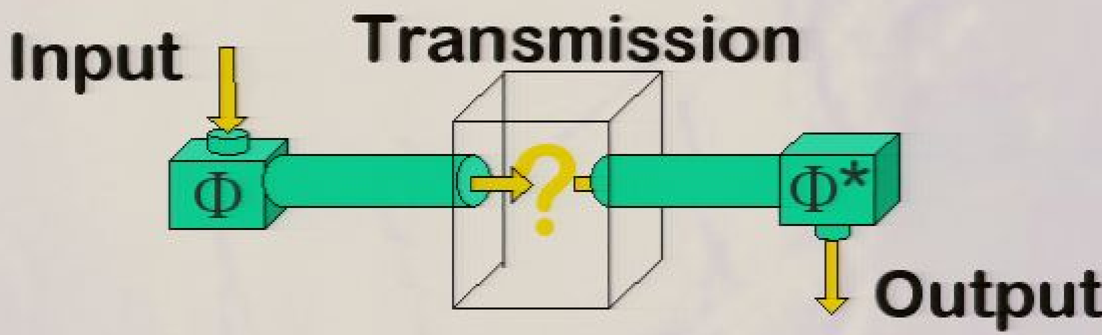


Acknowledgments

- Research funded in part by NSERC and NSF.
- Thanks to Mary Beth Ruskai and Cédric Bény for helpful discussions,
- Joseph Emerson for initiating and organizing, and Perimeter Institute for hosting this joint seminar.

Introduction

- Quantum communication: encoding, transmission (errors), decoding.



- If noise algebra leaves components of messenger state unchanged: noiseless subsystems (or subspaces).
- Study dynamics of typical interactions with environment, find encoding for noiseless transmission.

From the A to Z of Quantum Information and Error Correction

D. Aharonov, R. Alicki, K. Audenaert, H. Barnum, T. Beth, C. Bennett, A. Calderbank, M. Choi, I. Chuang, I. Devetak, D. DiVincenzo, M. Effros, E. Fortunato, D. Gottesman, M. Grassl, P. Hayden, P. Horodecki, H. Inamori, R. Jozsa, A. Kitaev, A. Klappenecker, M. Keyl, C. King, E. Knill, D. Kribs, R. Laflamme, D. Leung, D. Lidar, S. Lloyd, F. Markopoulou, M. Nielsen, M. Ohya, D. Petz, J. Preskill, B. Ruskai, D. Schlingemann, B. Schumacher, P. Shor, J. Smolin, R. Spekkens, A. Steane, T. Toffoli, A. Uhlmann, L. Viola, R. Werner, A. Winter, P. Zanardi, W. Zurek, K. Zyczkowski

Many results focus on qubits as basic units of quantum information, and correction of qubit errors.

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A Different Error Model

- Arena: Any given Hilbert space used for transmission.
- Problem: Instead of qubit errors, consider decohe-rence by phase damping (loss of off-diagonal entries) in transmitted state due to yes/no measurements. Protect against errors in order of sparseness.
- Goal: Optimal encoding by spreading information from input state across larger-dimensional Hilbert space. Blind reconstruction (passive error correction).
- Strategy: Somewhat analogous to classical linear signal transmission. See Holmes+Paulsen (2004) and BGB+Paulsen (2005).

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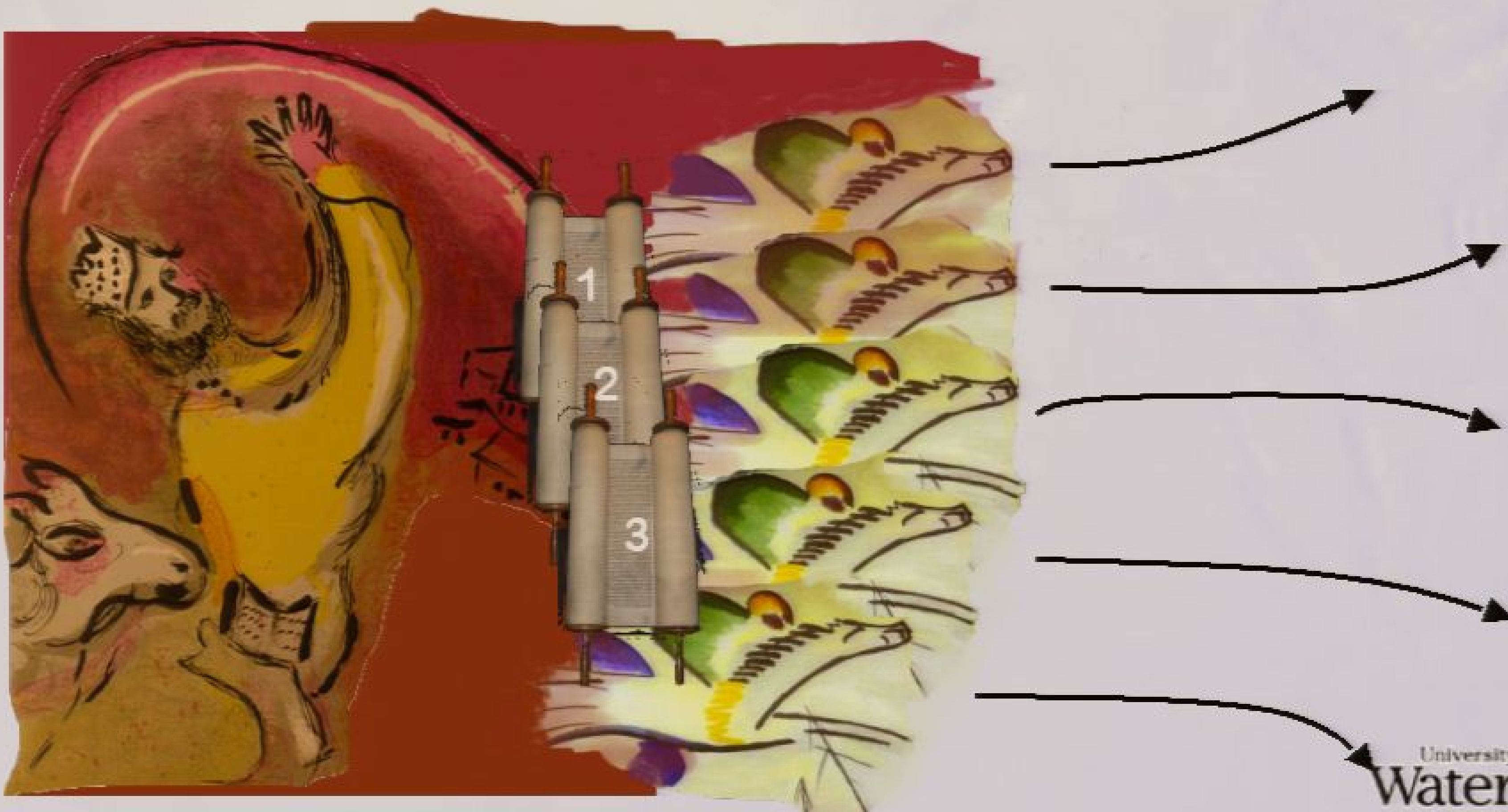
A Classical Tale – King Solomon's Messengers

Problem: How to transmit three scrolls as safely as possible…



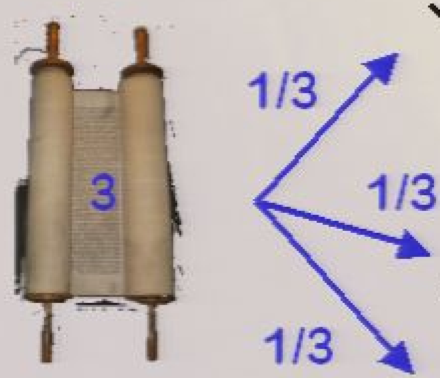
How to transmit three scrolls as safely as possible across partially unsafe territory

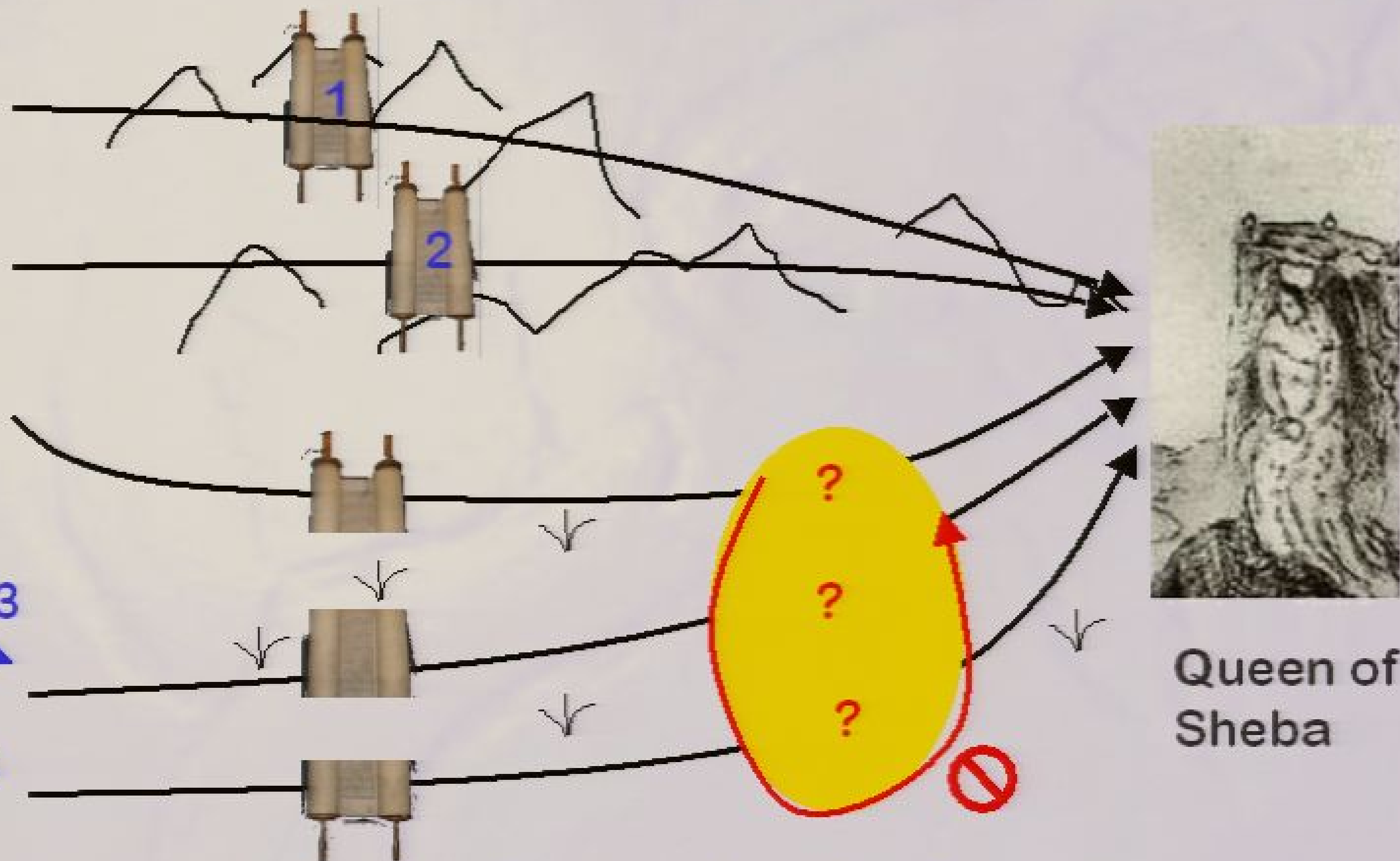
Optimal Strategy?

A roaming tribe could possibly intercept (at most) one of three messengers

How to transmit three scrolls as safely as possible across partially unsafe territory

Solution:
To minimize worst case error, split 3[rd] scroll





Queen gets either 3 scrolls (no interception) or 2 2/3 scrolls (one interception).

The Message of this Tale – Classical and Modern

Problem: Transmit information as safely as possible

Strategy: To minimize worst-case error, exhaust noiseless component of channel, then **spread** remaining information **evenly** across noise-susceptible component.

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Classical vs. Quantum Carriers of Information

- **Classical State**

A probability vector in a d-dimensional real vector space

$$w \in \mathbb{R}^d_+,$$

so $w_j \geq 0$ and $w_1 + w_2 + \ldots + w_d = 1$.

- **Quantum State**

A trace-normalized positive operator on a d-dimensional complex Hilbert space

$$W \in L_+(\mathbb{C}^d)\,,$$

so $W \geq 0$ and $\mathrm{tr}\, W = 1$.

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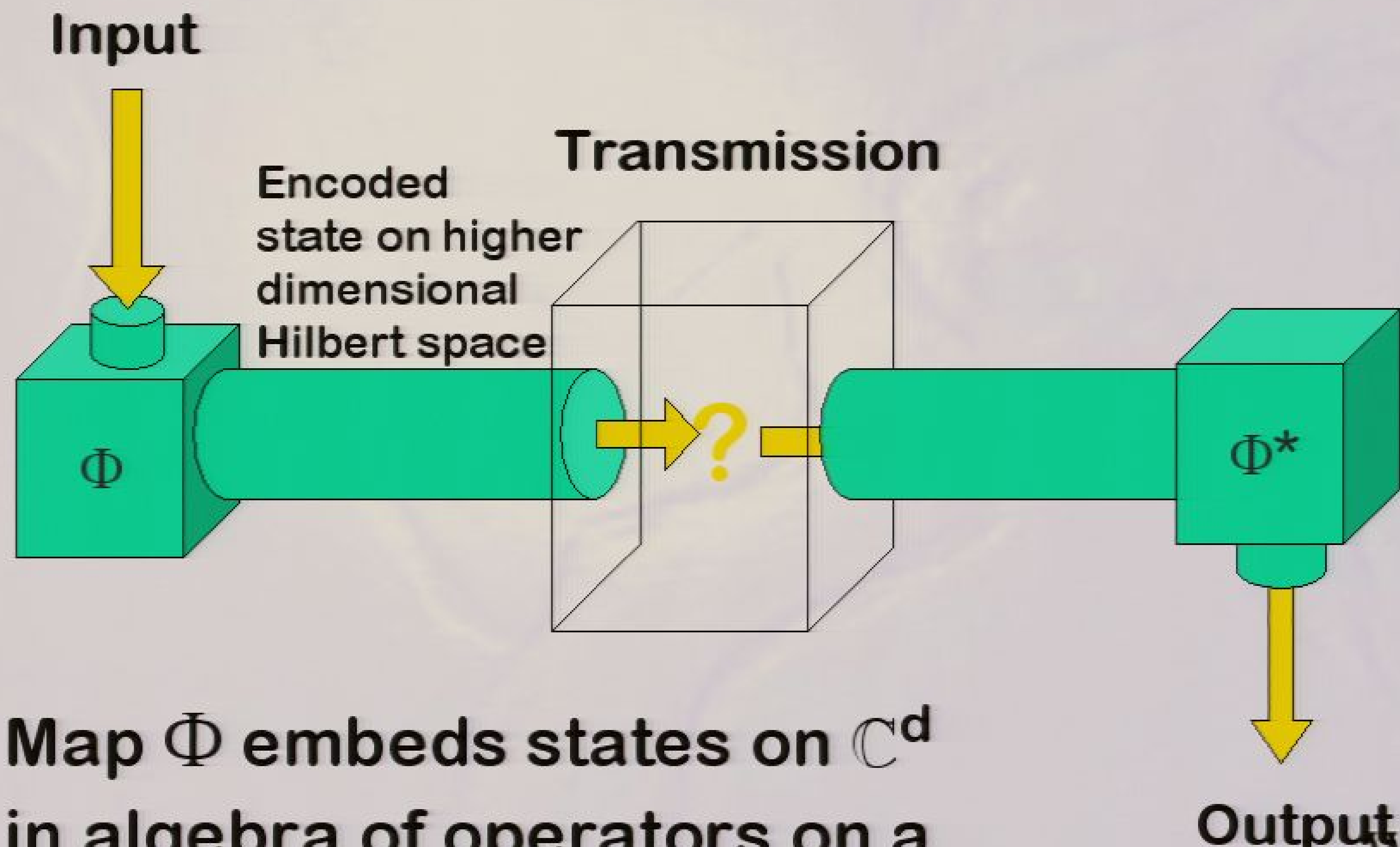
Encoding and Decoding of Quantum States



Map Φ embeds states on $\mathbb{C}^d$ in algebra of operators on a higher dimensional space $\mathbb{C}^N$.

Encoding by Algebra Embedding

- **Classical**

Algebra: Vectors in $\mathbb{R}^d$, multiplication is component-wise.

Encoding:

Probability preserving algebra embedding $\phi : \mathbb{R}^d \to \mathbb{R}^N$, $\phi(f)\phi(g) = \phi(fg)$ $\Rightarrow \phi$ preserves inner product.

- **Quantum**

Algebra: Operators on $\mathbb{C}^d$, multiplication is composition.

Encoding:

Trace preserving C^*-algebra monomorphism $\Phi : L(\mathbb{C}^d) \to L(\mathbb{C}^N)$, $\Phi(F)\Phi(G)^* = \Phi(FG^*)$, $\Rightarrow \Phi$ is Hilbert-Schmidt isometry.

Transmission Errors

- **Classical**

Given a stochastic matrix $(p_{i,j})$,

$$\sum_i p_{i,j} = 1\,,$$

we define the associated error as

$$\begin{aligned} \varepsilon : \mathbb{R}^N &\rightarrow \mathbb{R}^N \\ \varepsilon(m)_i &= \sum_j p_{i,j} m_j\,. \end{aligned}$$

- **Quantum**

Given a set of noise operators $\{E_a\}$ such that

$$\sum_a E_a^* E_a = I\,,$$

we define the associated error as

$$\begin{aligned} \mathcal{E} : L(\mathcal{H}) &\rightarrow L(\mathcal{H}) \\ \mathcal{E}(M) &= \sum_a E_a M E_a^*\,. \end{aligned}$$

Noiseless Subsystems

A quantum error $\mathcal{E}$ on $B(\mathcal{H})$ admits a noiseless subsystem if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and for any given state $M = M_1 \otimes M_2$ there is a state T_1 on $\mathcal{H}_1$ such that

$$\mathcal{E}(M_1 \otimes M_2) = T_1 \otimes M_2 \,.$$

Generalized theory of noiseless subsystems, "Operator Quantum Error Correction" D. Kribs et al. (2005).

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What if the noiseless component is too small?

- Initial state W on $\mathbb{C}^d$
- Messenger state M on $\mathbb{C}^m \otimes \mathbb{C}^l$
- Dimension of noiseless component $l < d$.
- Need to entrust some information to noisy component. How?

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Encoding...

Given trace-preserving C^*-monomorphism

$$\Phi : L(\mathbb{C}^d) \to L(\mathbb{C}^m \otimes \mathbb{C}^l) = L(\oplus_{j=1}^m \mathbb{C}^l)$$

then there are maps $V_j : \mathbb{C}^d \to \mathbb{C}^l$ such that

$$\Phi(W) = (M_{ij})_{i,j=1}^m, M_{ij} = V_i W V_j^*$$

and $\sum_{j=1}^m V_j^* V_j = I$.

In other words, there is POVM $\{A_j\}_{j=1}^m$, $A_j = V_j^* V_j$ and $\sum_j A_j = I$ on $\mathbb{C}^d$, with maximal rank $\mathrm{rk}(A_j) \leq l$.

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Example d=3, m=3, l=2

Let $\{A_j\} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \right\}$

State $W = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}$ maps to

$$M = \frac{1}{2} \left(\begin{array}{cc:cc:cc} W_{11} & W_{12} & W_{12} & W_{13} & W_{11} & W_{13} \\ W_{21} & W_{22} & W_{22} & W_{23} & W_{21} & W_{23} \\ \hdashline W_{21} & W_{22} & W_{22} & W_{23} & W_{21} & W_{23} \\ W_{31} & W_{32} & W_{32} & W_{33} & W_{31} & W_{33} \\ \hdashline W_{11} & W_{12} & W_{12} & W_{13} & W_{11} & W_{13} \\ W_{31} & W_{32} & W_{32} & W_{33} & W_{31} & W_{33} \end{array}\right)$$

…by Copying Blocks,

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Decoding...

The map Φ is an isometry with respect to the Hilbert-Schmidt inner products.

Consequently, its adjoint is a left inverse,

$$\begin{aligned} \Phi^* : L(\mathbb{C}^m \otimes \mathbb{C}^l) &\rightarrow L(\mathbb{C}^d) \\ M &\mapsto \sum_{i,j} V_i^* M_{ij} V_j \,. \end{aligned}$$

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...by Summing Blocks

$$\text{Let } \{A_j\} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \right\}$$

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Error Model with Noiseless Subsystem

Commutative set of projections

$$\mathcal{Q} = \{Q_j = E_{jj} \otimes I\}_{j=1}^{m},$$

with diagonal matrix units

j-th position

$$E_{jj} = \mathrm{diag}(0, 0, \ldots 0, 1, 0, \ldots 0).$$

"Environment performs dephasing on j-th row/column of first component."

$$\mathcal{E}_j : M \mapsto Q_j M Q_j + Q_j^{\perp} M Q_j^{\perp}$$

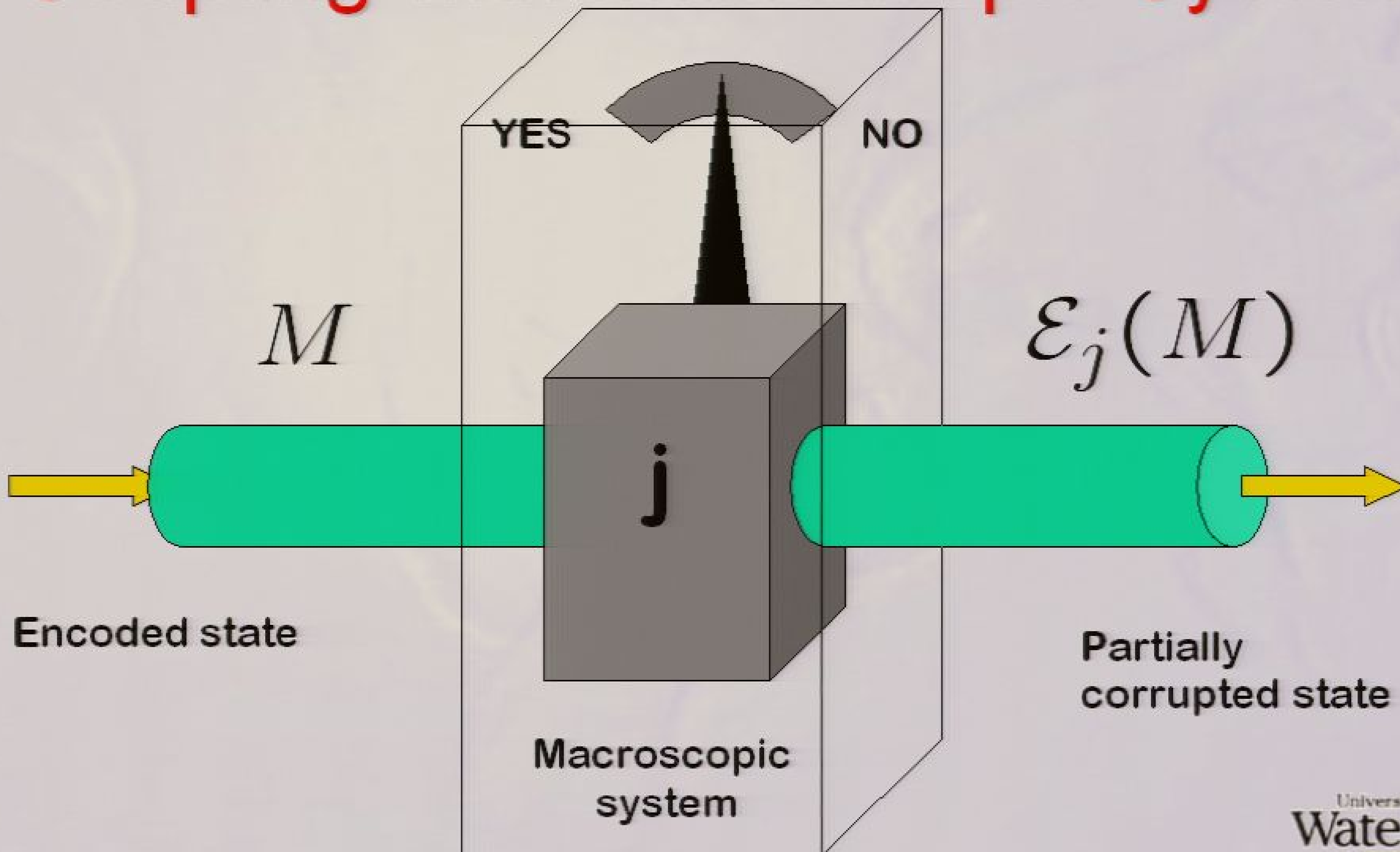
Example: Choose j=1

Let $Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Error: $\mathcal{E}(M) = Q_1 M Q_1 + Q_1^{\perp} M Q_1^{\perp}$

$$= \frac{1}{2} \begin{pmatrix} W_{11} & W_{12} & 0 & 0 & 0 & 0 \\ W_{21} & W_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & W_{22} & W_{23} & W_{21} & W_{23} \\ 0 & 0 & W_{32} & W_{33} & W_{31} & W_{33} \\ 0 & 0 & W_{12} & W_{13} & W_{11} & W_{13} \\ 0 & 0 & W_{32} & W_{33} & W_{31} & W_{33} \end{pmatrix}$$

Error Model – Phase Damping by Coupling with Macroscopic System



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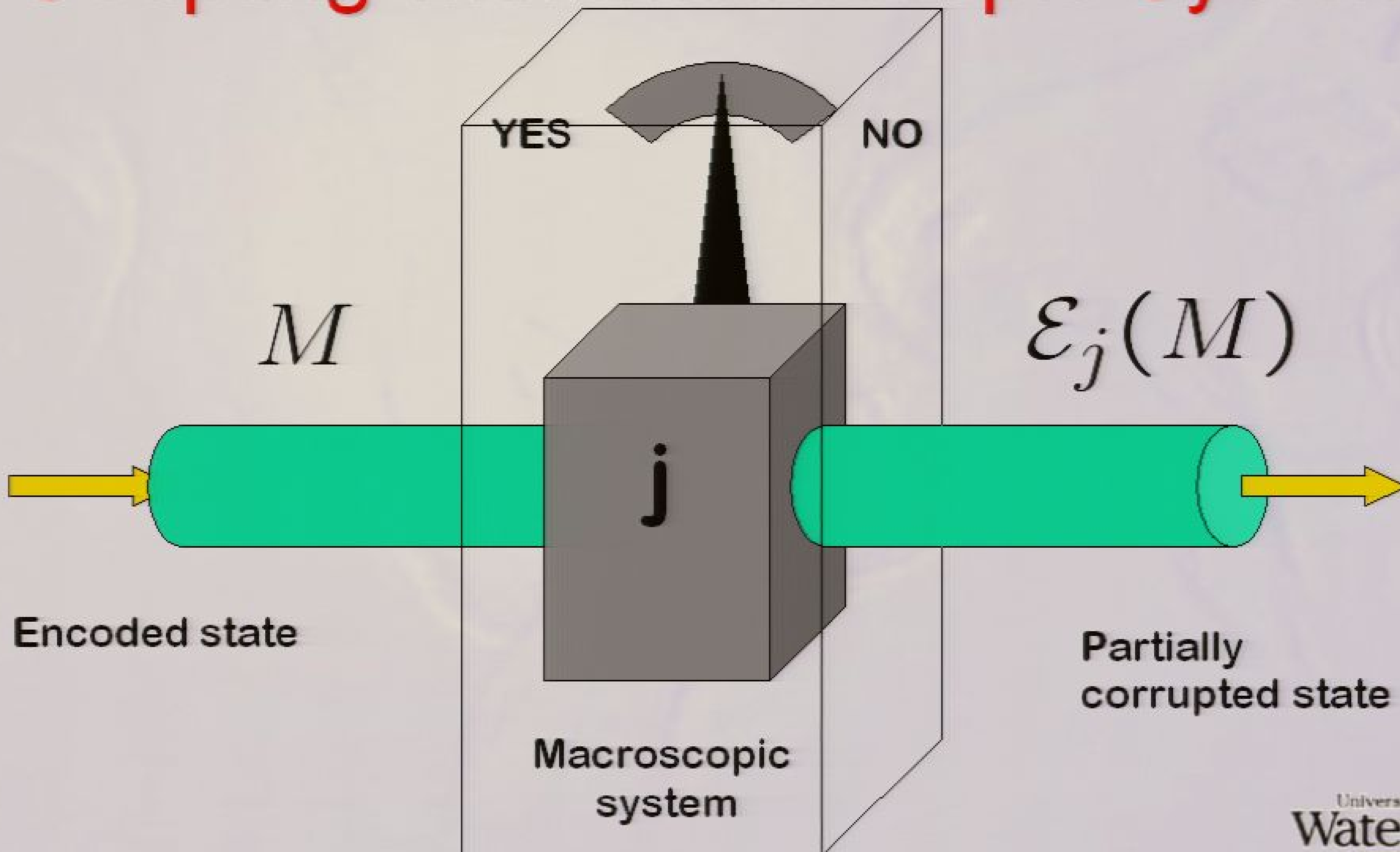


Worst-Case Input State

Given encoding map Φ and phase damping channel $\mathcal{E}_j$, the reconstruction error for a state W is

$$Y_j \;=\; W - \Phi^* \circ \mathcal{E}_j \circ \Phi(W)\,.$$

Maximize the Hilbert-Schmidt norm of Y_j over all states W,

$$e(\Phi, \mathcal{E}_j) = \max_{W \geq 0, \operatorname{tr} W = 1} (\operatorname{tr} Y_j Y_j^*)^{1/2}\,.$$

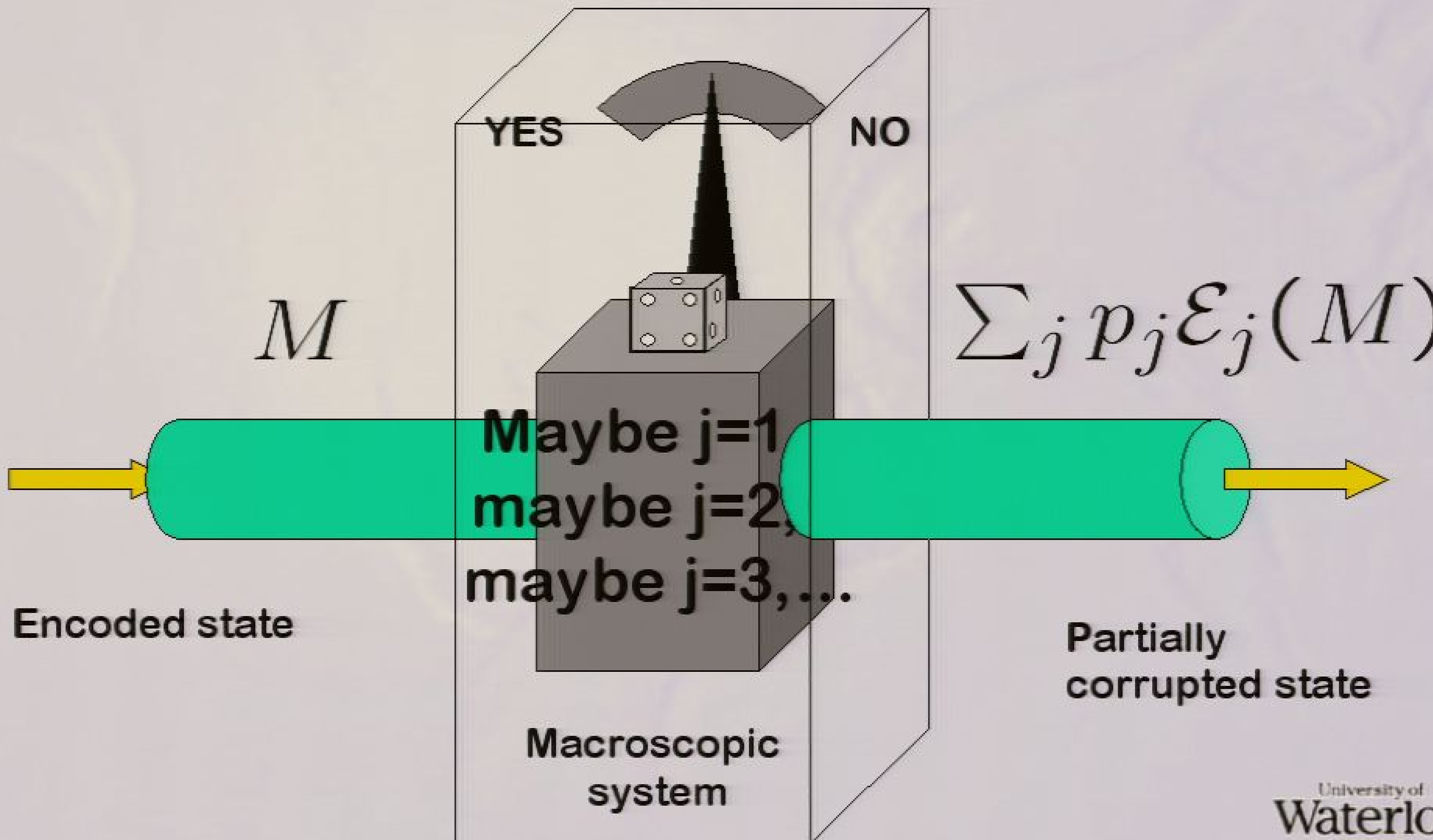
Why Hilbert-Schmidt Norm?

Quality measure: Hilbert-Schmidt norm of a state measures its purity.

For any phase damping error $\mathcal{E}$, Hilbert-Schmidt norm is non-increasing.

Same for $\Phi^* \circ \mathcal{E} \circ \Phi$. So worst-case error norm $e(\Phi, \mathcal{E})$ bounds loss of purity.

More General Phase Damping



Worst-Case Transmission

Pick probabilities p_j for each error $\mathcal{E}_j$. Then define transmission channel

$$\mathcal{E}(M) = \sum_j p_j \mathcal{E}_j(M)\,.$$

Maximize reconstruction error of $\Phi^* \circ \mathcal{E} \circ \Phi$ over all input states W and all probability distributions for dephasing of each individual row/column.

Optimal encoding minimizes worst case

$$\max_{p \geq 0, \sum_j p_j = 1} e(\Phi, \sum_j p_j \mathcal{E}_j)\,.$$

Optimal Encoding

Thm. Among all C^*-encodings $\Phi : L(\mathbb{C}^d) \to L(\mathbb{C}^m \otimes \mathbb{C}^l)$ and associated maps $\{V_j\}_{j=1}^m$ the optimal error suppression is achieved when $V_j^* V_j = k P_j$ with $\{P_j\}$ a set of orthogonal projections and $k = d/ml$.

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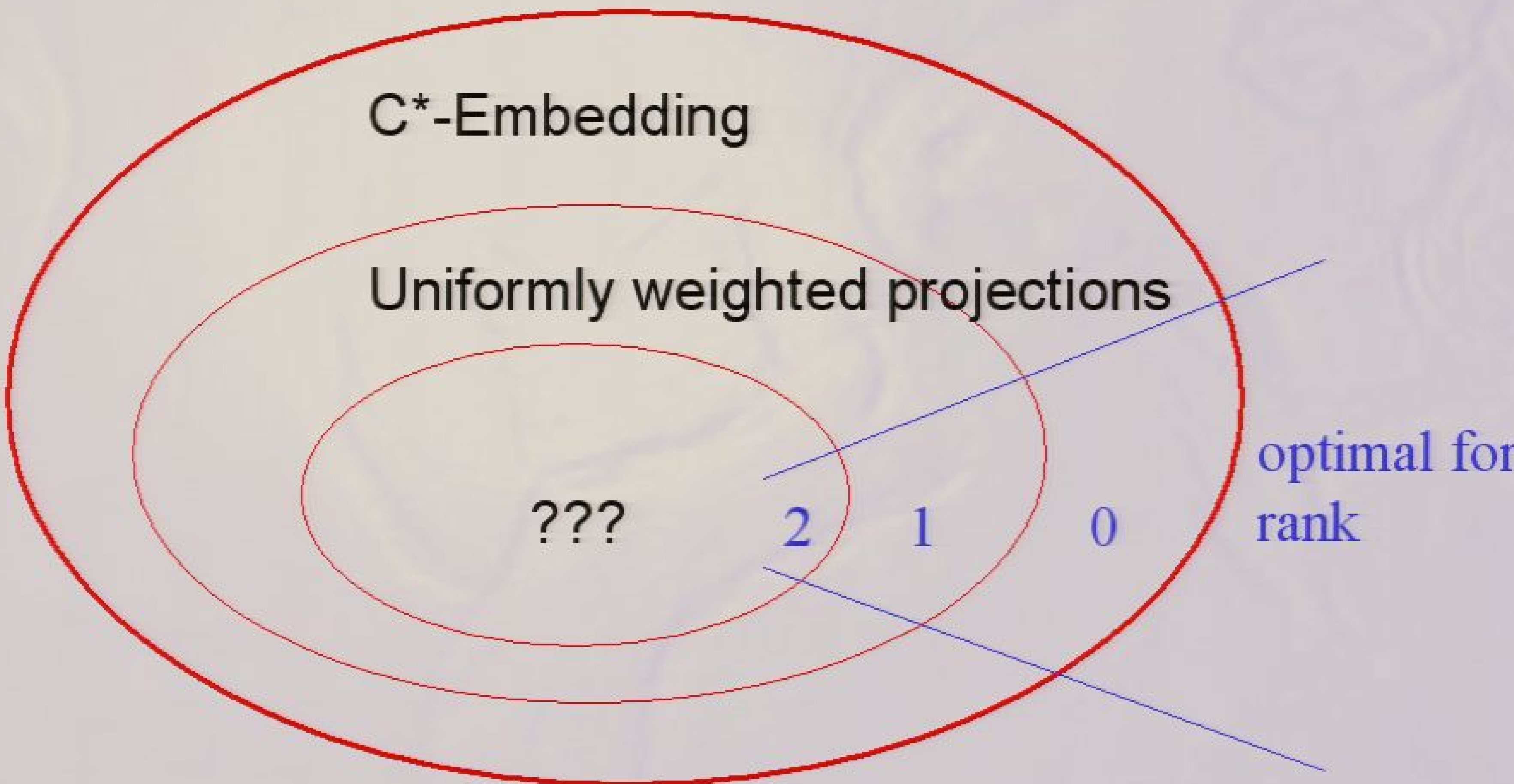
Hierarchy of Optimal Encoding



Errors of Next Lowest Rank

Pick sets $K \subset \{1, 2, \ldots m\}$, $|K| = 2$. Define commutative set of projections

$$\mathcal{Q} = \{Q_K = \sum_{j \in K} E_{jj} \otimes I\}.$$

"Environment performs complete rank-two phase damping in the first component."

$$\mathcal{E}_K : M \mapsto Q_K M Q_K + Q_K^{\perp} M Q_K^{\perp}$$

Worst-Case Transmission

Pick probabilities p_K for each error $\mathcal{E}_K$. Then define transmission channel

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Maximize reconstruction error of $\Phi^* \circ \mathcal{E} \circ \Phi$ over all input states W and all probability distributions for dephasing of each individual row/column block.

Optimal encoding minimizes worst case

$$\max_{p \geq 0, \sum_K p_K = 1} e(\Phi, \sum_K p_K \mathcal{E}_K)\,.$$

Optimal Encoding for Errors of Next Lowest Rank

Thm. Among all C^*-encodings $\Phi : L(\mathbb{C}^d) \to L(\mathbb{C}^m \otimes \mathbb{C}^l)$ with associated $\{V_j\}$ that factor scaled rank l projections $kP_j = V_j^* V_j$ the optimal error suppression is achieved by $\{P_j\}$ which minimize $\max\{\|P_i + P_j\|\}_{i \neq j}$.

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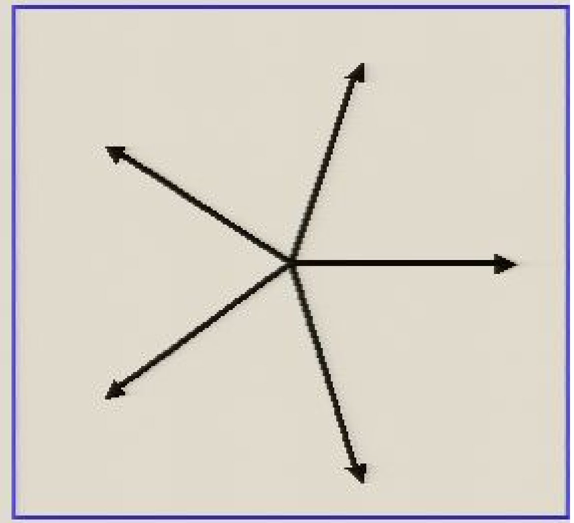
Special Case: Uniform Equiangular Frames

If $\{kP_j\}$ are all rank one, then optimality amounts to

$$\text{tr}\, P_i P_j = \frac{(m-d)}{d(m-1)}, i \neq j$$

'equiangular frames'.



m=5, d=2

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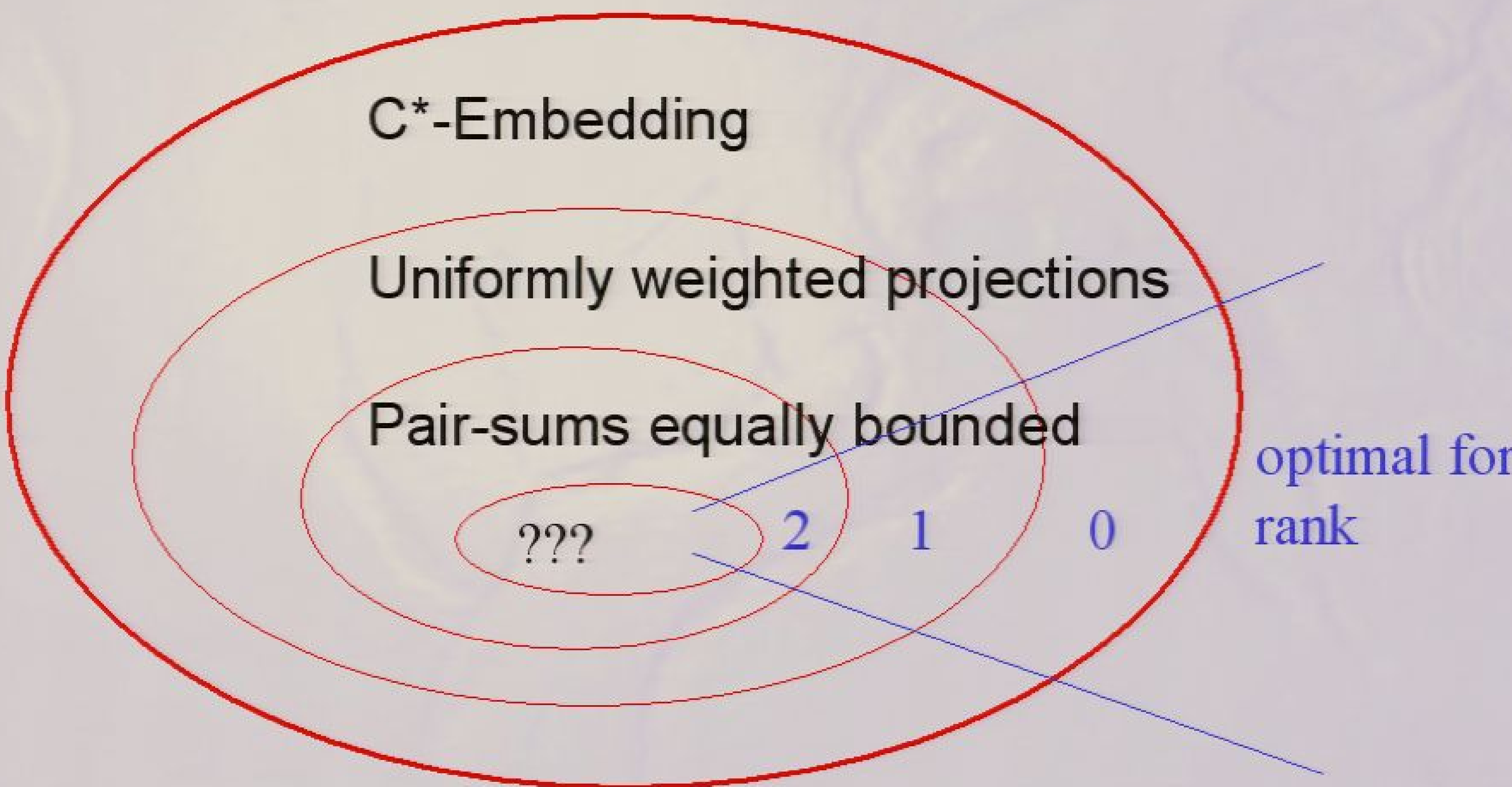
Hierarchy of Optimal Encoding



… higher rank errors?

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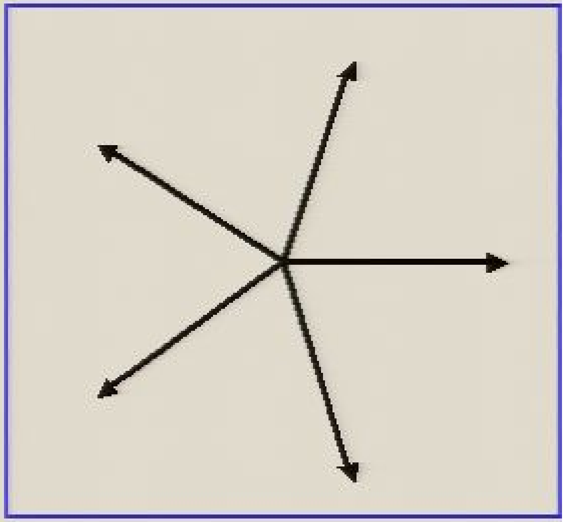

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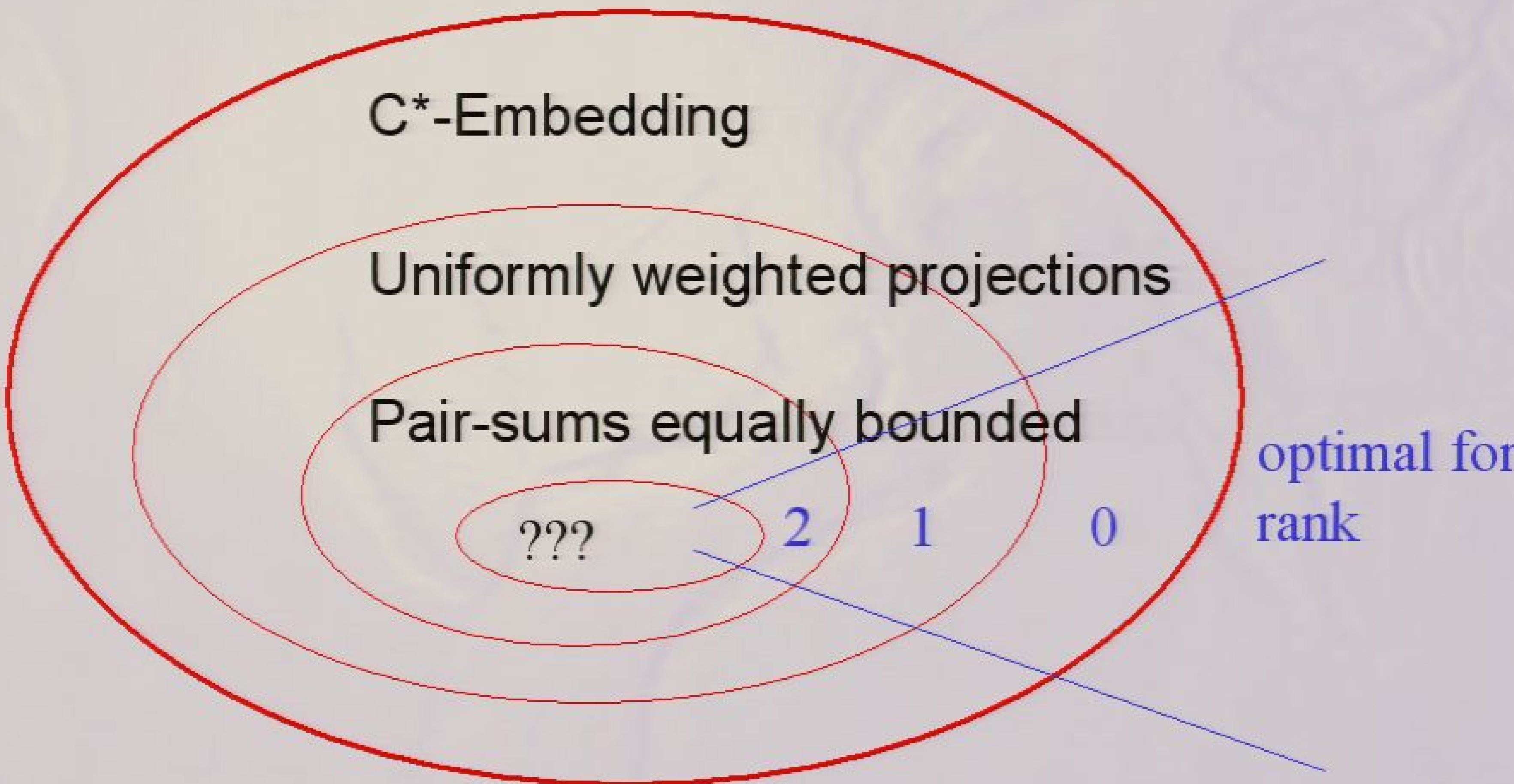

… higher rank errors?

Summary and Outlook

- Similar results for decoherence-free/noiseless subspaces.
- Encoding of algebraic structure of operators (C*-algebra) allows unrestricted operations in coded form. Application to quantum computing?
- Optimality result for up to rank-two dephasing. Combinatorial criteria for higher rank errors?
- Generalization to non-commutative interaction algebras? Work with D. Kribs and B. Ruskai.

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