Title: From Physics to Information Theory and Back

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Abstract: Inspired by the notion that the differences between quantum theory and classical physics are best expressed in terms of information theory, Hardy (2001) and Clifton, Bub, and Halvorson (2003) have constructed frameworks general enough to embrace both quantum and classical physics, within which one can invoke principles that distinguish the classical from the quantum. Independently of any view that quantum theory is essentially about quantum information, such frameworks provide a useful tool for exploring the differences between classical and quantum physics, and the relations between the various properties of quantum mechanics that distinguish it from the classical. In particular, we can ask: on which features of quantum physics do our familiar possibility/impossibility theorems depend? It turns out that it is possible to extend the no-cloning theorem and other results, such as the Holevo bound on acquisition of information by a single measurement, beyond the quantum setting.

From Physics to Information Theory and Back

or, How I Learned to Stop Worrying and Love Alice and Bob

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PIQuDOS
May 3, 2006

Characterizing Quantum Mechanics

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- Lucien Hardy (2001) and Clifton, Bub & Halvorson (2003) have taken on the task of characterizing quantum theory.
- Strategy: construct a framework broad enough to encompass both classical and quantum theory, and invoke principles that, within such a framework, distinguish the classical from the quantum.

An Interesting Alternate Project

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- If done with a reasonable degree of generality, this will be informative about possible successor theories.

Some Features of Quantum Mechanics

1. No quantum states that are dispersion-free for all observables.
2. Impossibility of ascertaining the state of a quantum system via a non-disturbing measurement.
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- Nothing particularly quantum about this!

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- Heisenberg lacked a theory-neutral framework embracing both classical and quantum states.
- Armed with such, can we extract a theorem relating dispersion to indistinguishability and/or impossibility of cloning?

The CBH framework

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- A state is a linear functional assigning expectation values to operators, such that the identity operator gets value 1, and a self-adjoint operator with spectrum in $\mathbb{R}^+$ gets values in $\mathbb{R}^+$.
- General evolution a *non-selective operation* on states: a linear, completely positive map that preserves norm.

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- *Comment*: The C^* assumption imposes nontrivial constraints on the state space. Nevertheless, these constraints are satisfied by both classical and quantum theories, so, as long as we are interested in exploring the differences between classical and quantum, the assumption is not problematic.
- Nevertheless, it is worth considering what can be done with weaker assumptions.
- Tradeoff between: richness of mathematical structure presumed and generality.

What is a C^*-algebra? I

- A Banach space is a normed linear vector space that is complete with respect to the norm. That is, every Cauchy sequence converges to a limit.
- A Banach algebra $\mathfrak{A}$ is a Banach space that is also an algebra with identity I, such that the operation of multiplication is separately continuous. That is,
  - for each $B \in \mathfrak{A}$, if $A_n \to A$, then $A_n B \to AB$,
  - and, for each $A \in \mathfrak{A}$
- An involution is a mapping $A \to A^*$ such that
  - $(aA + bB)^* = \bar{a}A^* + \bar{b}B^*$
  - $(AB)^* = B^* A^*$
  - $(A^*)^* = A$
- A C^*-algebra is a complex Banach algebra with an involution that satisfies $\| A^* A \| = \| A \|^2$.

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- Any subalgebra of $\mathfrak{B}(\mathcal{H})$ that is closed under adjoints and complete in operator norm is a C^*-algebra.
- For any state ρ of a C^*-algebra $\mathfrak{A}$, there is a representation of $\mathfrak{A}$ an algebra of bounded operators on a Hilbert space, with ρ a vector state.

Generalizing orthogonality

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- If there exists an observable A that has distinct definite values in the states ρ, ω, then $\| \rho - \omega \| = 2$.
- Let us say that ρ, ω are *orthogonal* iff $\| \rho - \omega \| = 2$.

Generalized No-Cloning Theorem

- A pair $\{\rho, \omega\}$ of pure states of a C^*-algebra is clonable only if the states are orthogonal.

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- This generalizes: a pair of states (pure or mixed) is clonable iff they are orthogonal.
- Applies to classical algebras of observables as well as quantum!

An example

- Alice writes down a number from 1 to 10, via one of two procedures:
  1. ρ_0: She picks an even number at random, with all even numbers equiprobable.
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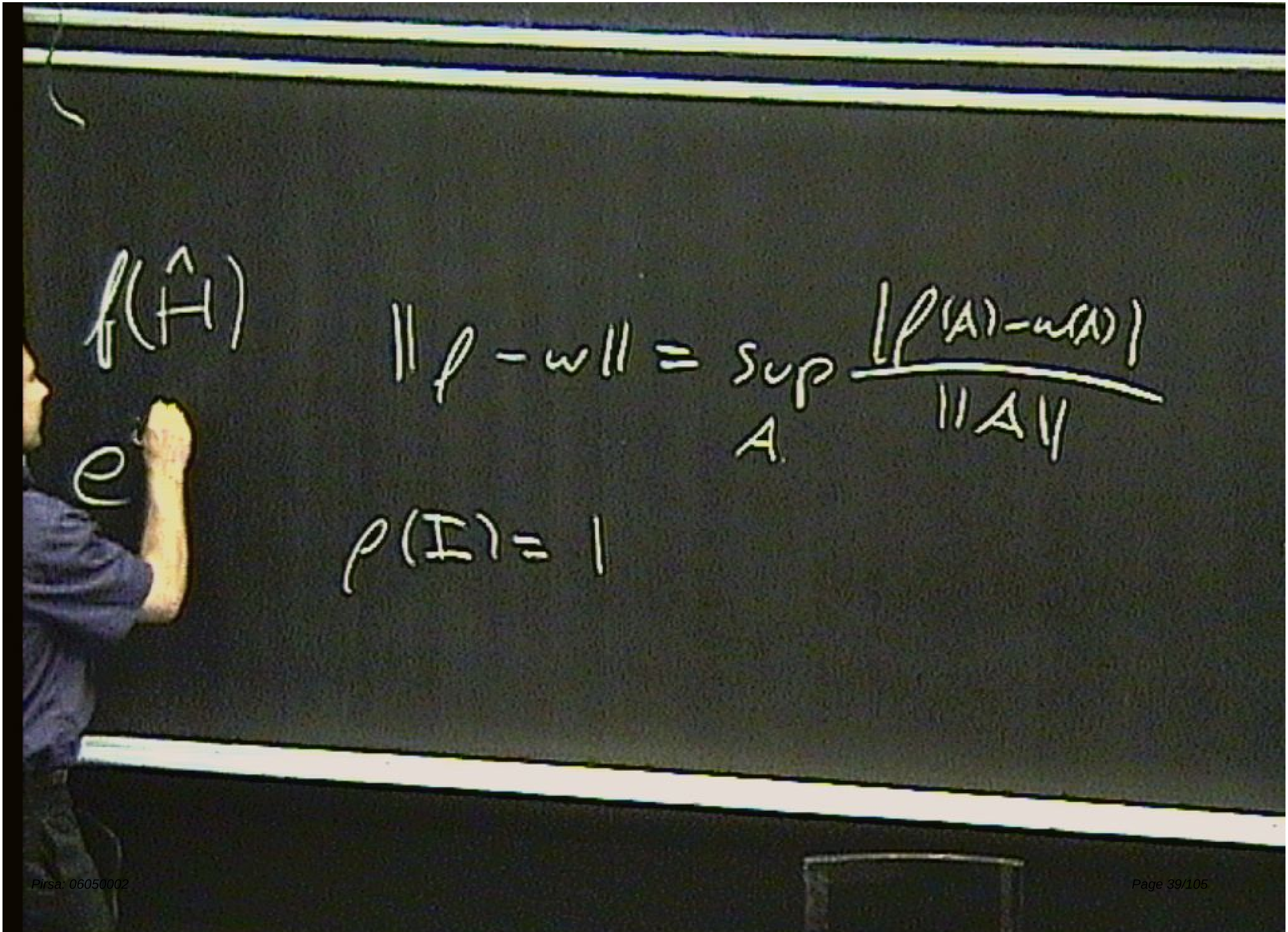
- Unless the supports of the two states are disjoint, he can't do this!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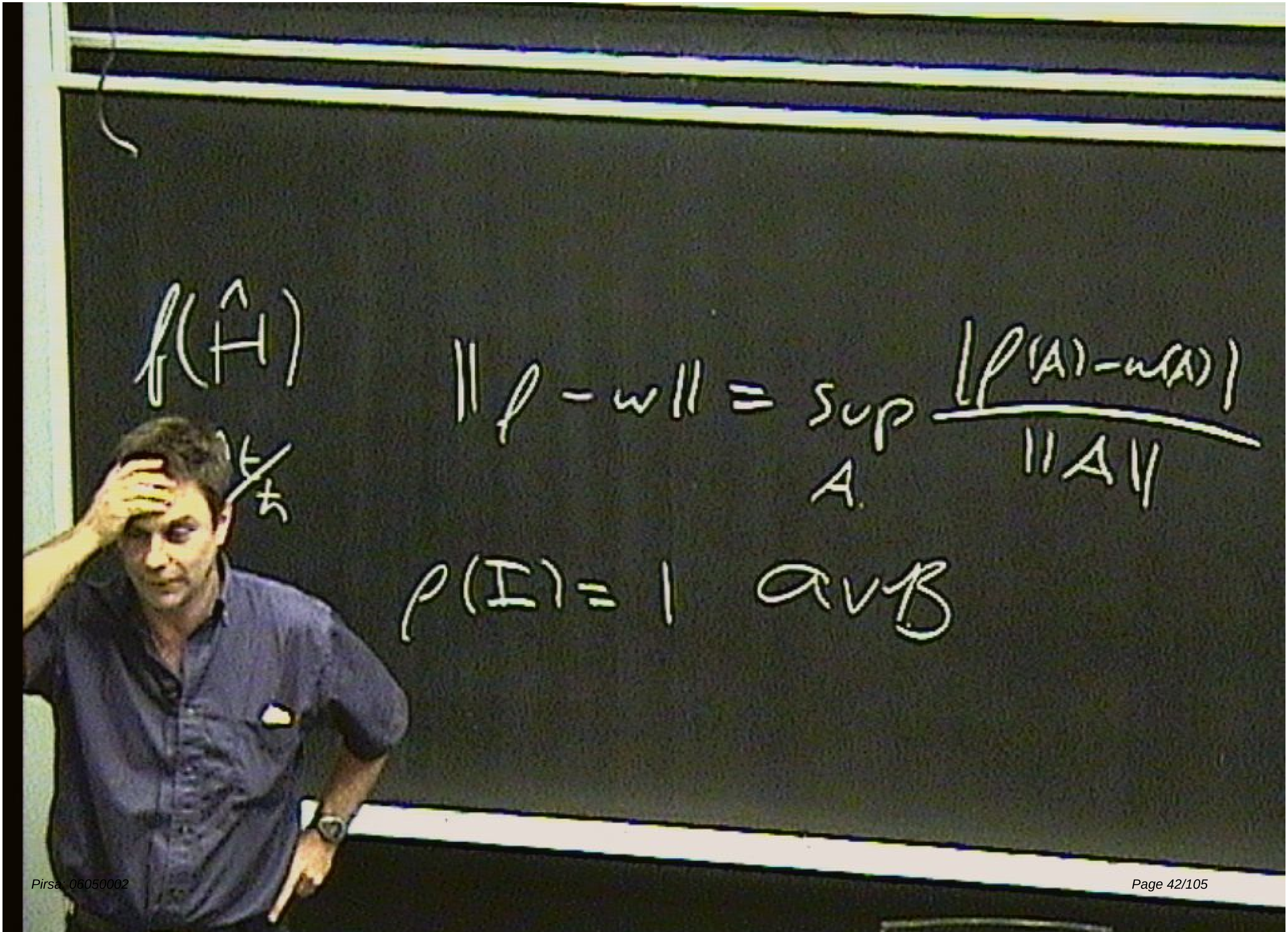
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- A dispersion-free state is pure; an algebra $\mathfrak{A}$ is abelian iff all pure states are dispersion-free.

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$$s(\rho, \omega) = \inf \left[\frac{(\Delta_\rho A)^2 + (\Delta_\omega A)^2}{(\rho(A) - \omega(A))^2} \right].$$

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- When ρ, ω are pure states, the equality holds.

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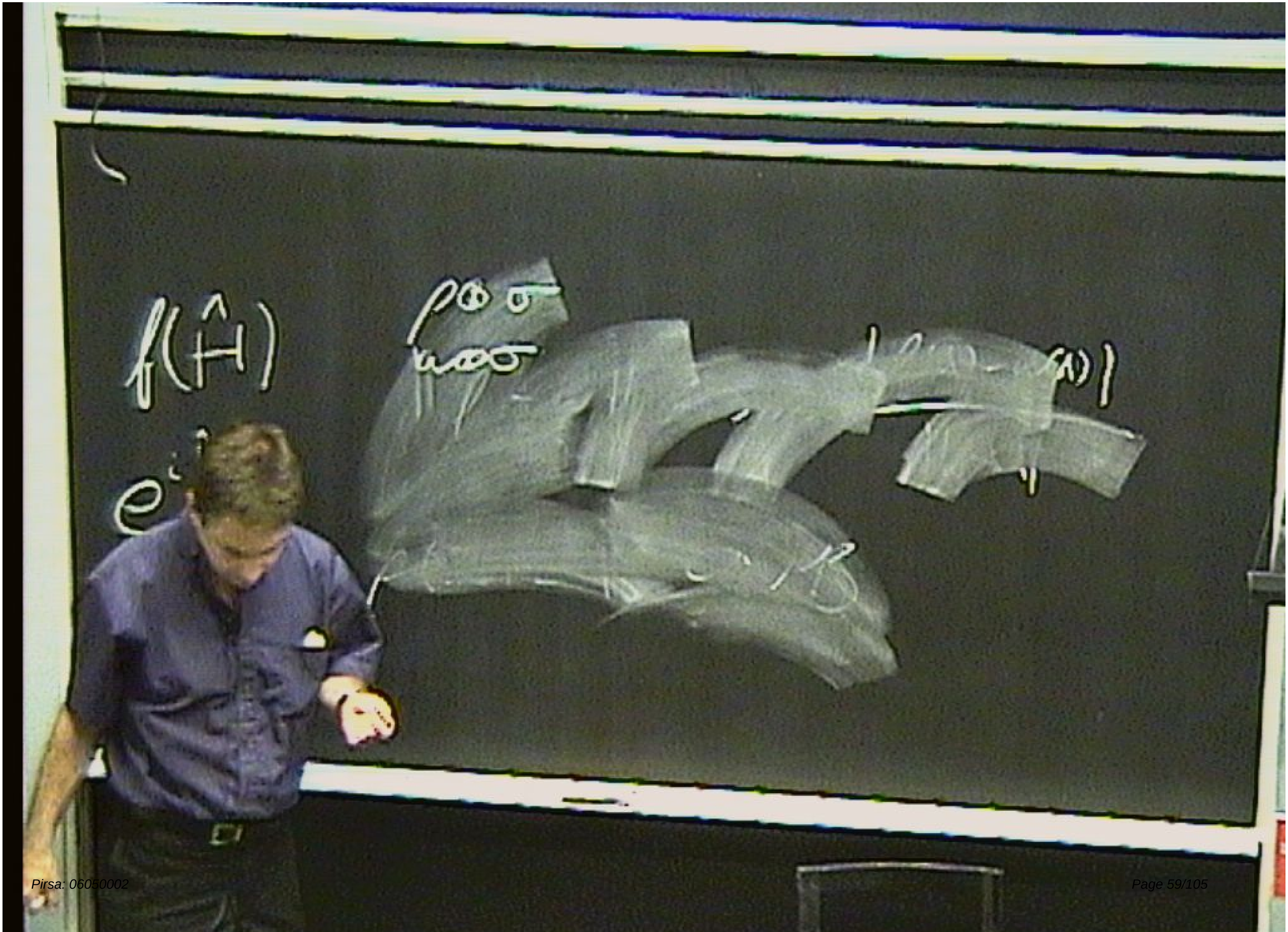
- Define
$$\delta(\rho,\omega) = \frac{1}{1 + 2s(\rho,\omega)}.$$

Cloning and dispersion

- A pair of states $\{\rho, \omega\}$ is clonable iff $s(\rho, \omega) = 0$, or, equivalently, $\delta(\rho, \omega) = 1$.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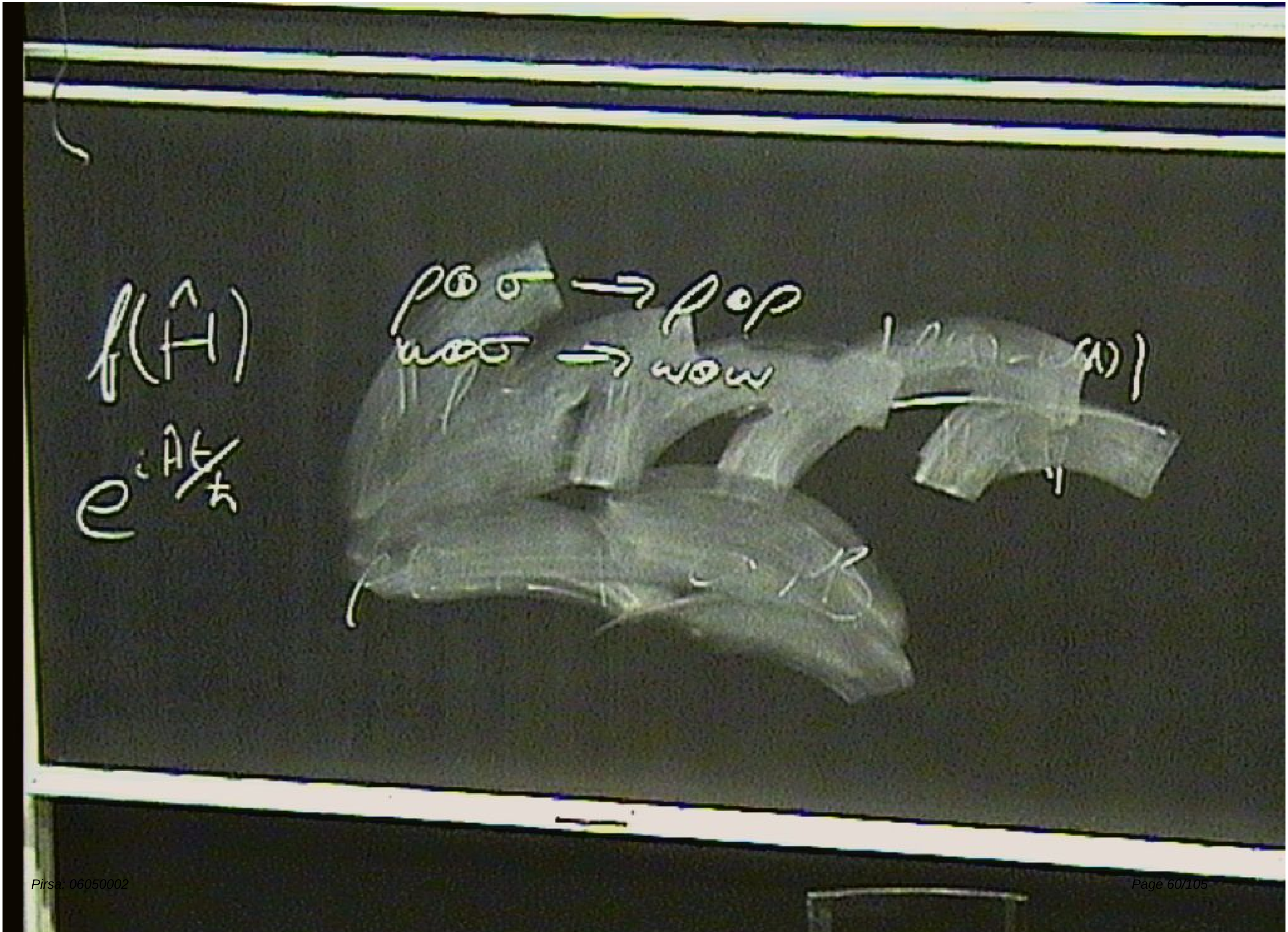
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- Moreover, we can obtain a bound relating maximum faithfulness of cloning to the value of $\delta(\rho, \omega)$.
- A bound that holds whether or not the algebra of observables is classical!

A Bound on Fidelity of Cloning

- Suppose we have a pair $\{\rho_0, \rho_1\}$ of states that we want to clone. That is, we want a ready state σ and an evolution T such that the action of T is

$$\begin{aligned} \rho_0 \otimes \sigma &\longrightarrow \rho_0 \otimes \rho_0 := \omega_0 \\ \rho_1 \otimes \sigma &\longrightarrow \rho_1 \otimes \rho_1 := \omega_1 \end{aligned}$$

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- Take as measure of degree of success, the quantity

$$\frac{\| \tau_0 - \omega_0 \|^2 + \| \tau_1 - \omega_1 \|^2}{\| \omega_0 - \omega_1 \|^2}.$$

- One can show that, as long as $\delta(\rho, \omega) \geq 1/9$,

$$\frac{\| \tau_0 - \omega_0 \|^2 + \| \tau_1 - \omega_1 \|^2}{\| \omega_0 - \omega_1 \|^2} \quad \geq \quad \frac{1}{2}\left(1 - \frac{1 + 3\,\delta(\rho, \omega}{4\sqrt{\delta(\rho, \omega)}}\right)^2 .$$

(This bound can be improved, I think.)

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(This bound can be improved, I think.)

- When the states to be cloned are pure we get a better bound:

$$\begin{aligned} \frac{\| \tau_0 - \omega_0 \|^2 + \| \tau_1 - \omega_1 \|^2}{\| \omega_0 - \omega_1 \|^2} & \geq \frac{1}{2}\left(1 - \frac{1}{\sqrt{2 - \delta(\rho, \omega)}}\right)^2 \\ & = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + p(\rho, \omega)}}\right)^2 . \end{aligned}$$

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  - Why is this a “natural” choice?
  - Can it be related to norm-distance $\| \rho - \omega \|$ or some other representation-independent concept?

Distinguishability and measurement

- Let $\mathcal{S}$ be a physical system, and let $\mathcal{A}$ be another physical system, regarded as a measurement apparatus. Let $\mathfrak{S}$ and $\mathfrak{A}$ be their respective algebras of observables. Let $\{\rho_0, \rho_1\}$ be (pure or mixed) states of $\mathfrak{S}$. We say that $\{\rho_0, \rho_1\}$ can be distinguished by a non-disturbing measurement iff there are:

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- More generally, we say that $\{\rho_0, \rho_1\}$ can be distinguished by measurement iff the pointer observable P has distinct definite

Measurement Distinguishability and Cloning

- *Theorem.* Two states ρ, ω are distinguishable by measurement iff $\delta(\rho, \omega) = 1$.

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- More generally, we say that $\{\rho_0, \rho_1\}$ can be distinguished by measurement iff the pointer observable P has distinct definite

Measurement Distinguishability and Cloning

- *Theorem.* Two states ρ, ω are distinguishable by measurement iff $\delta(\rho, \omega) = 1$.

Distinguishability and measurement

- Let $\mathcal{S}$ be a physical system, and let $\mathcal{A}$ be another physical system, regarded as a measurement apparatus. Let $\mathfrak{S}$ and $\mathfrak{A}$ be their respective algebras of observables. Let $\{\rho_0, \rho_1\}$ be (pure or mixed) states of $\mathfrak{S}$. We say that $\{\rho_0, \rho_1\}$ can be distinguished by a non-disturbing measurement iff there are:
  - a ready-state σ of $\mathfrak{A}$
  - pointer states π_0, π_1 of $\mathfrak{A}$,
  - a pointer observable $P \in \mathfrak{A}$ such that P has distinct definite values in π_0 and π_1, and
  - a non-selective operation T, such that the action of T is

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- Also, $\{\rho, \omega\}$ is a clonable set iff $\delta(\rho, \omega) = 1$.

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bits, where $\alpha(1-\alpha) = \frac{1}{4} p(\rho_0, \rho_1) \geq \frac{1}{4}(1 - \delta(\rho, \omega))$.
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- Distinguishing with certainty (1 bit of info) requires $s(\rho_0, \rho_1) = 0$.

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- *Proof.* Let ρ be a state of $\mathfrak{A} \vee \mathfrak{B}$. By CBH Lemma 3, if $\rho|_{\mathfrak{A}}$ or $\rho|_{\mathfrak{B}}$ is pure, then ρ is a product state. In other words: if ρ is entangled, then $\rho|_{\mathfrak{A}}$ and $\rho|_{\mathfrak{B}}$ are mixtures, hence not dispersion-free. Therefore, ρ is not dispersion-free.

No-broadcasting theorem

- Broadcasting:

$$\rho_0 \otimes \sigma \quad \longrightarrow \quad \tau_0$$
$$\rho_1 \otimes \sigma \longrightarrow \tau_1$$

such that

$$\tau_0|_{\mathfrak{A}} = \tau_0|_{\mathfrak{B}} = \rho_0$$
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- No-broadcasting theorem says that a pair of states is broadcastable iff the corresponding density operators commute.
- *Conjecture.* Let $\{\rho_0, \rho_1\}$ be a broadcastable pair of states of a C^*-algebra $\mathfrak{A}$. Then there exists a set $\mathcal{K} = \{\omega_i\}$ of mutually orthogonal pure states, such that ρ_0 and ρ_1 are both mixtures of elements of the set $\mathcal{K}$.

Cloning and indiscernible mixtures

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- In the C^* framework:
  Let $\{\rho_0, \rho_1, \omega_0, \omega_1\}$ be a clonable set of distinct states, and let

$$\begin{aligned}\rho &= w\rho_0 + (1-w)\rho_1 \\ \omega &= \lambda\omega_0 + (1-\lambda)\omega_1.\end{aligned}$$

  Then $\rho \neq \omega$.

And Beyond?

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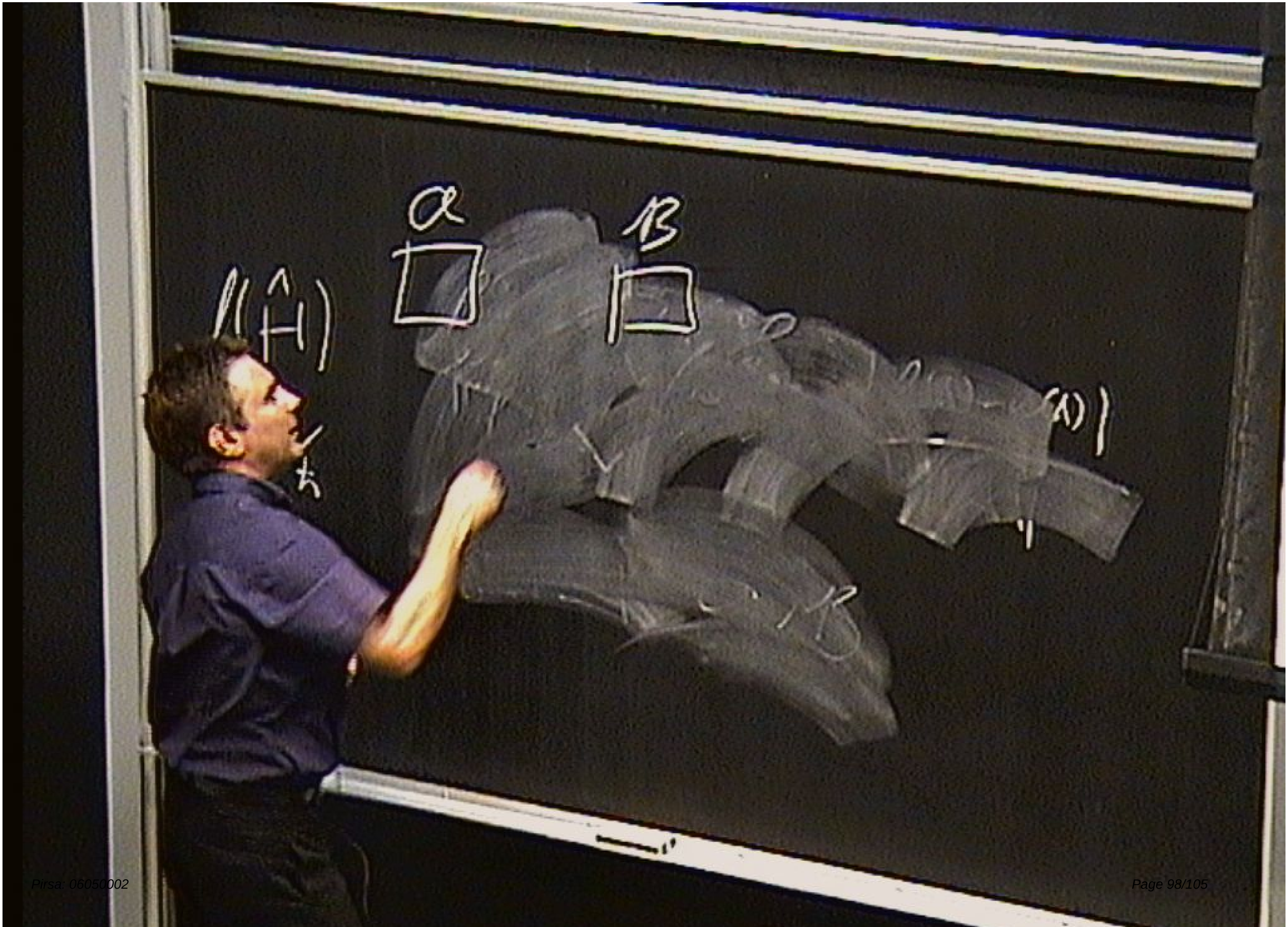
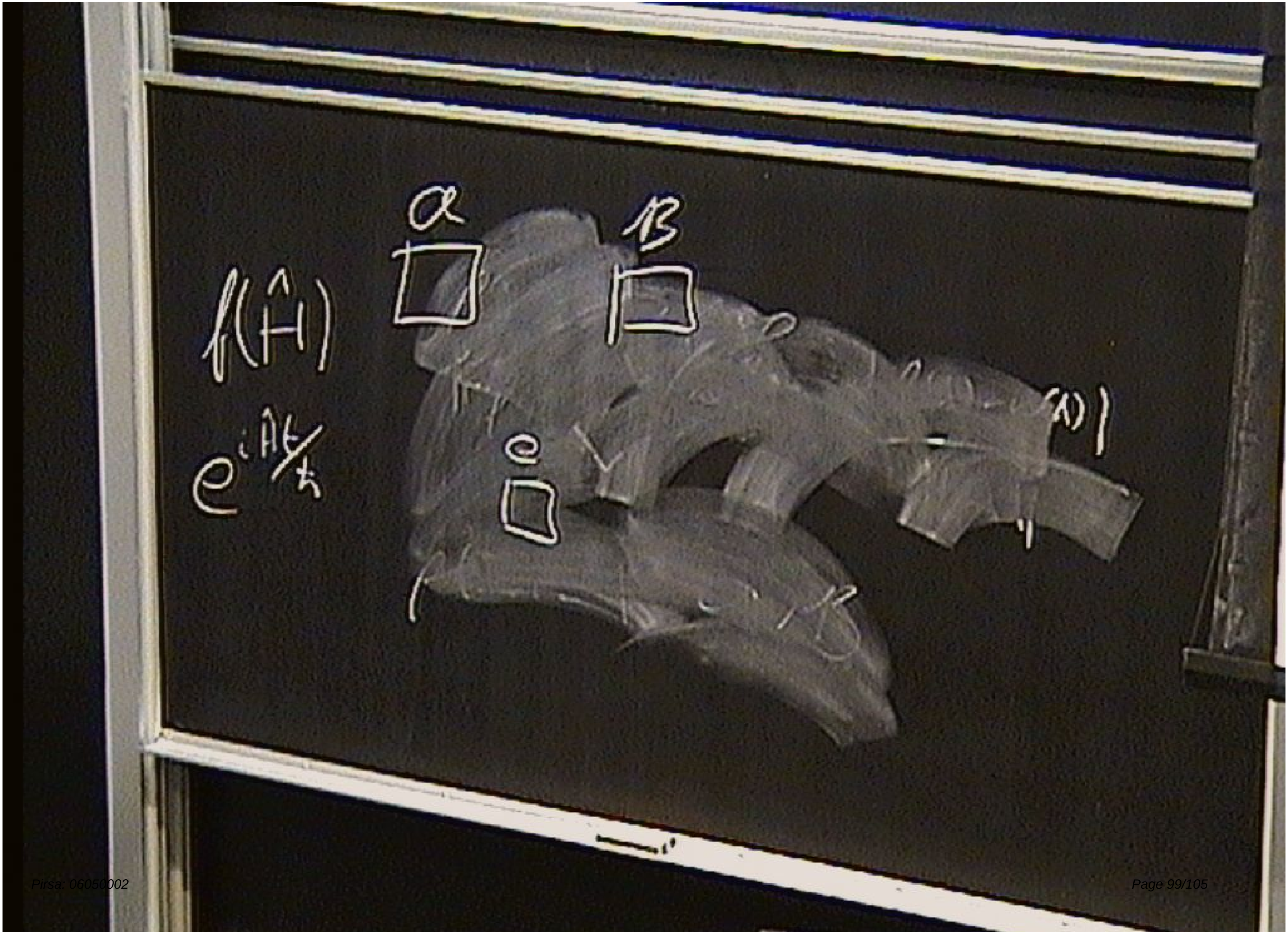
- Pick your favourite result, replace representation-dependent concepts (*e.g.* state vector, density operator, trace) with representation-independent concepts (*e.g.* $s(\rho, \omega)$, norm-distance between states), and ask whether there is a generalized analogue of the quantum theorem (may involve weakening of bounds).

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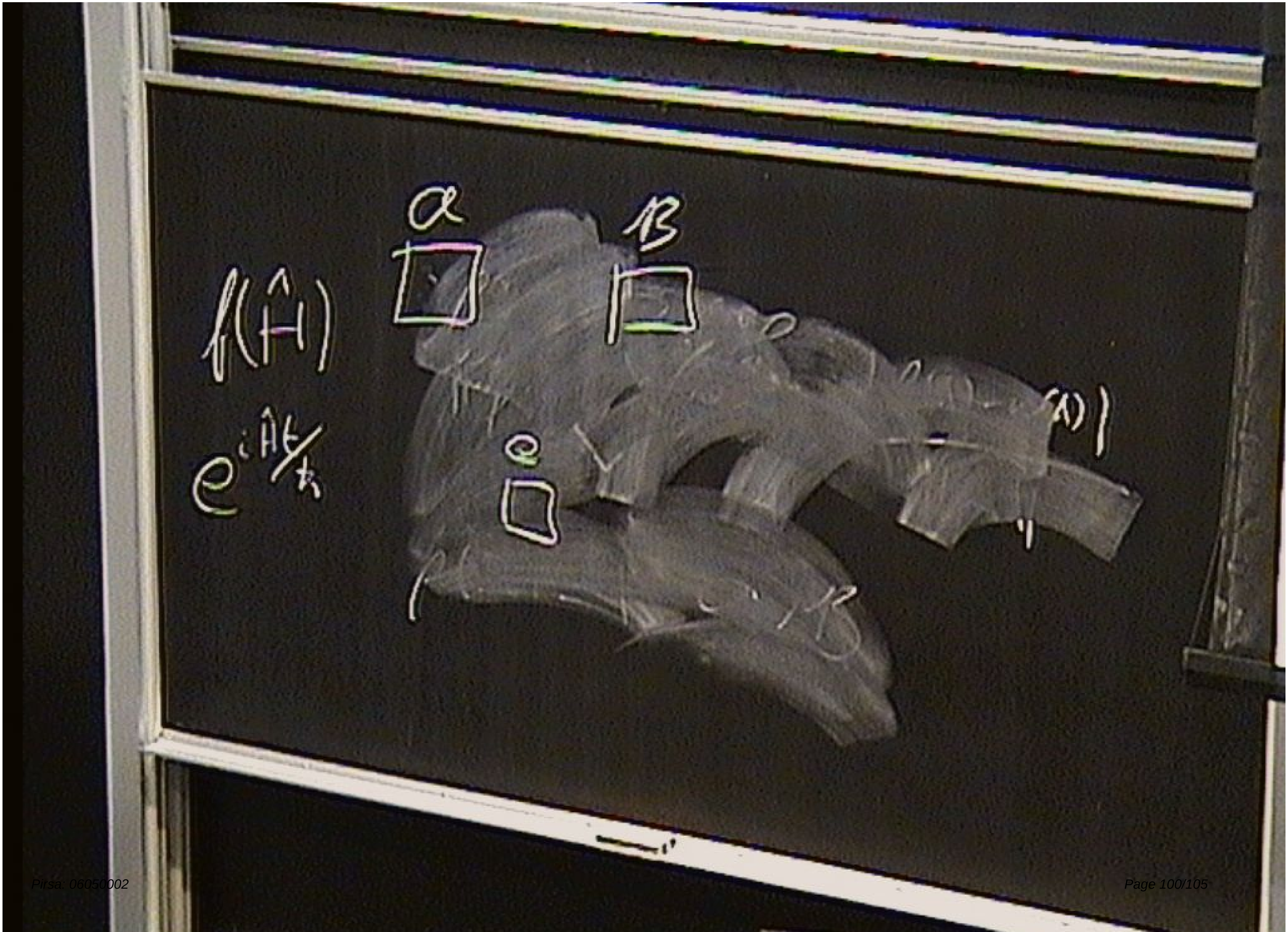
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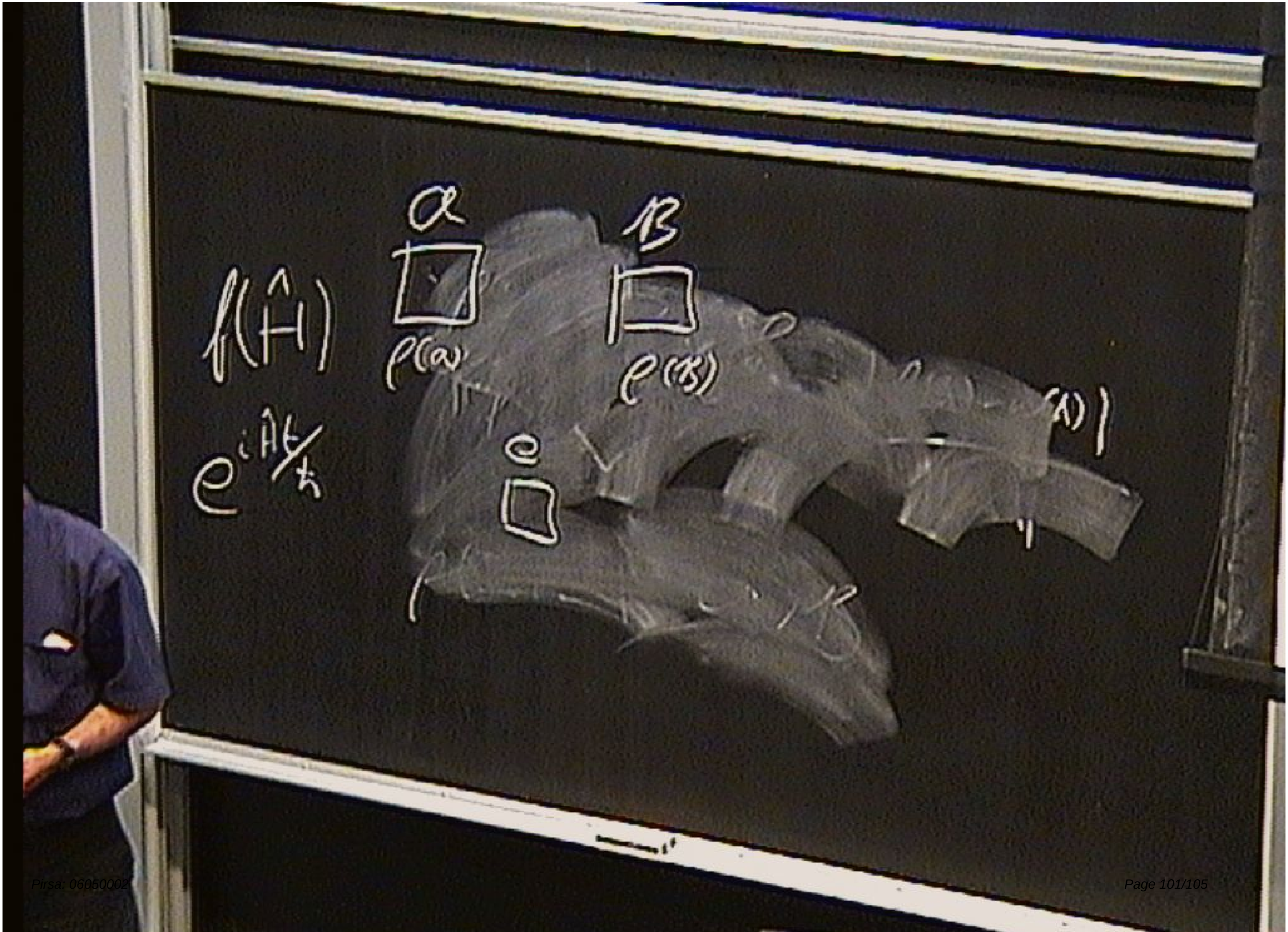
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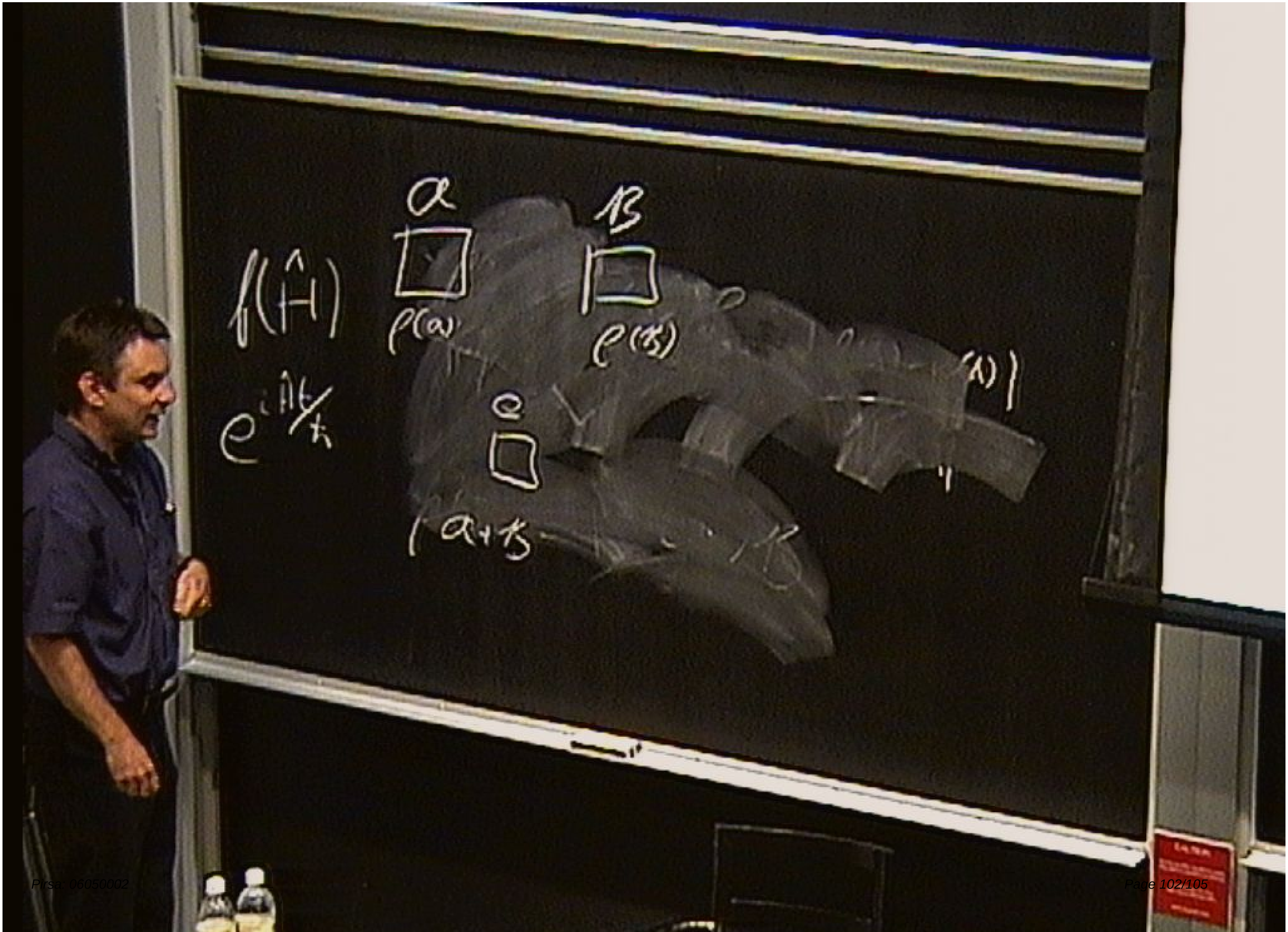

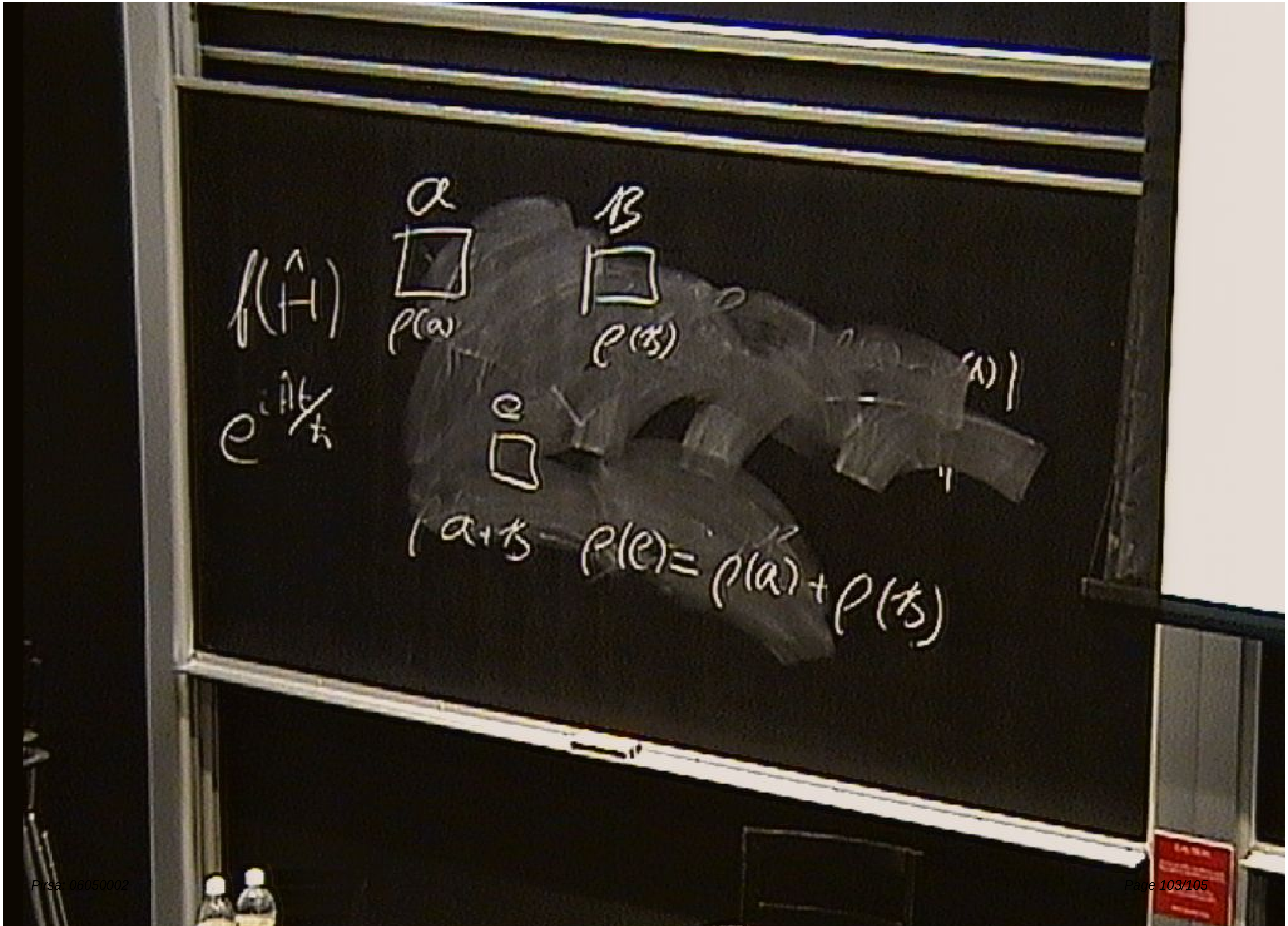
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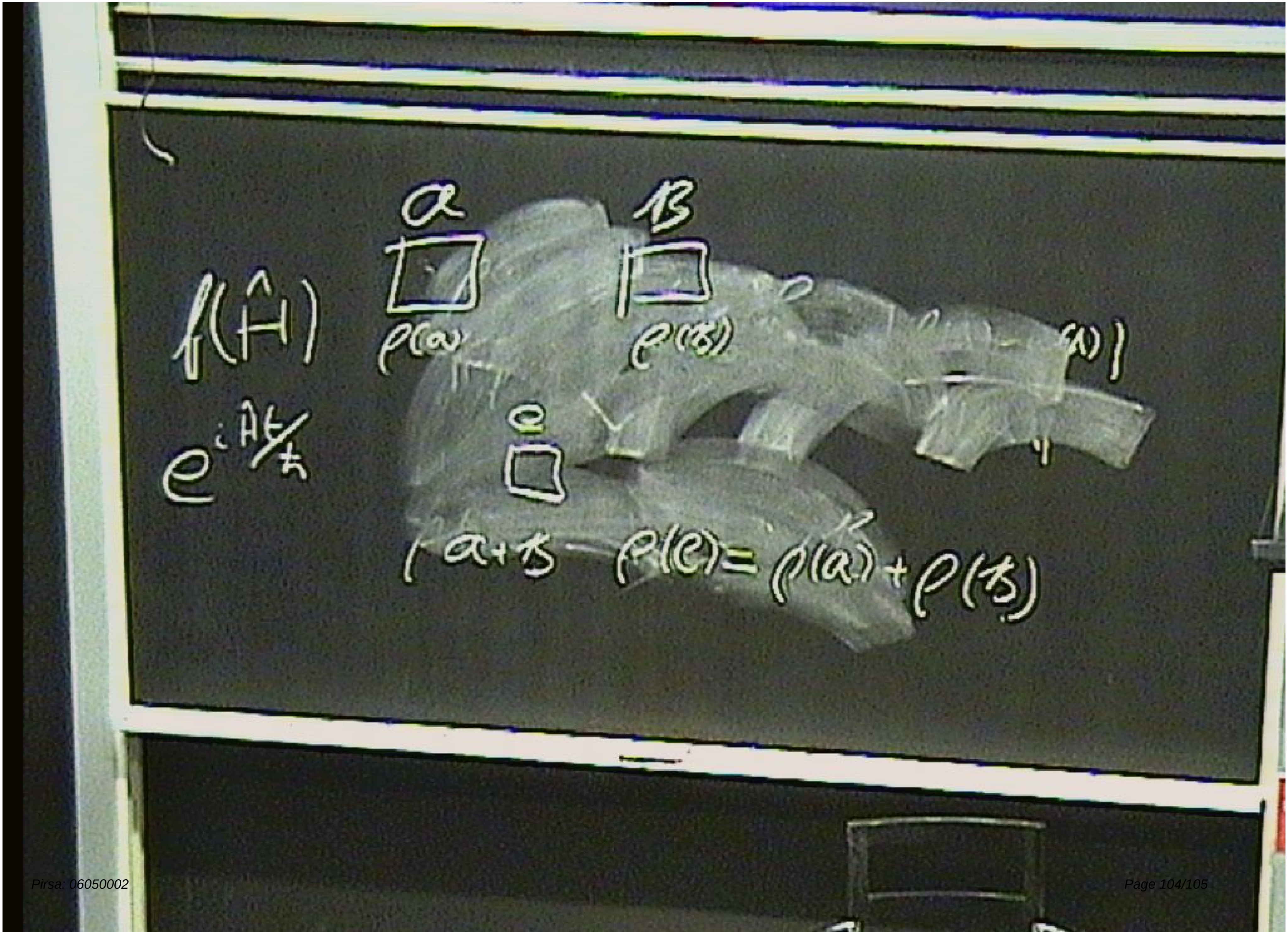
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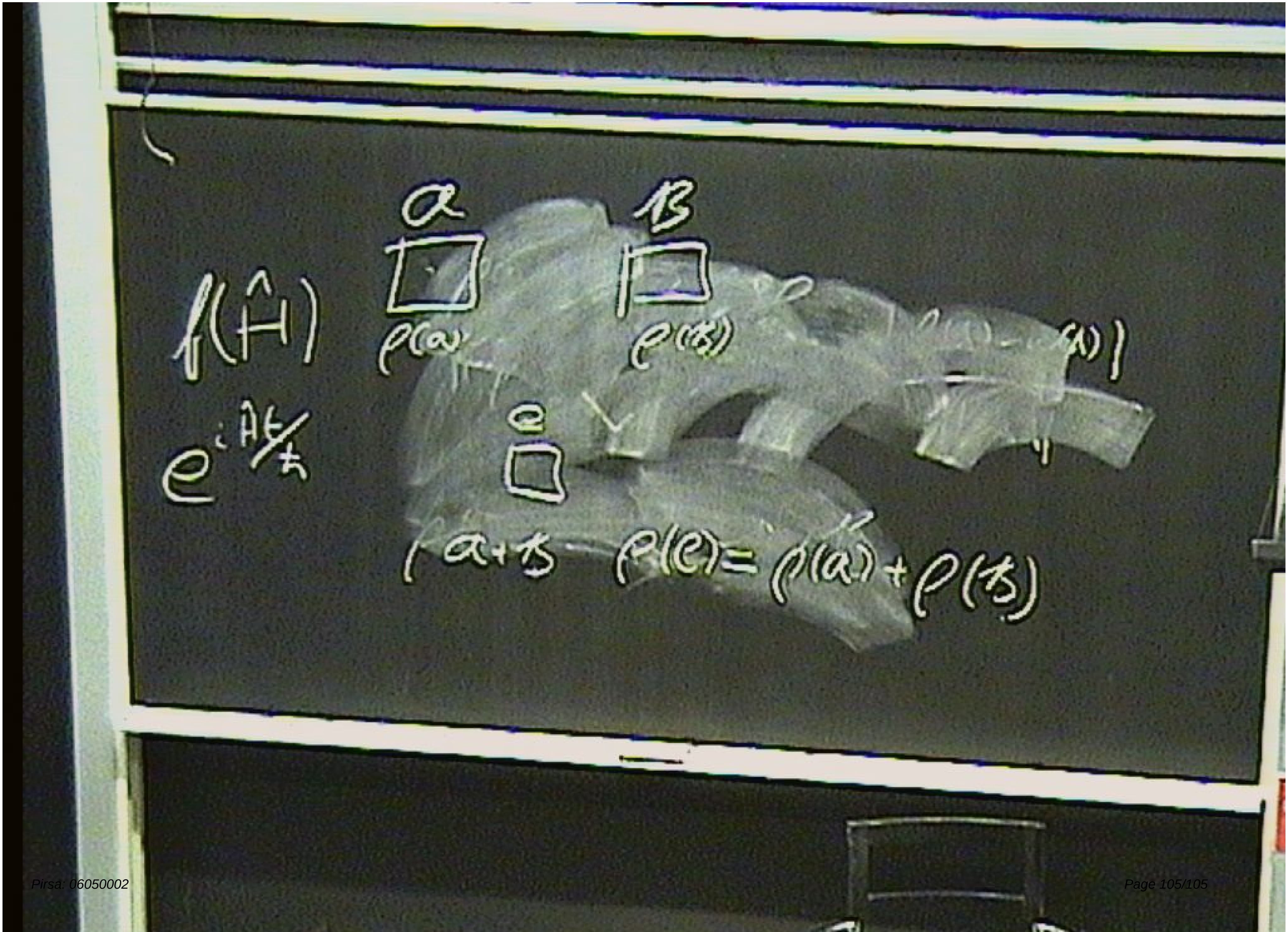
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