Title: Twisted Lattice Supersymmetry

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Abstract:

Twisted Lattice Supersymmetry

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Introduction

- Supersymmetry and discrete spacetime appear incompatible (no p_μ, Leibniz etc)

- Special class of theories exist where discretization preserves part of supersymmetry – super-QCD with $\geq 2^D$ supercharges.

- Twisted formulation. $Q^2 = 0$

- Allow us to give rigorous definition - investigate non-perturbative structure – eg. quantum vacua, bound states,...

- Dual to gravitational systems ?

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Talk

- 2D Twist
- Kähler-Dirac interpretation
- Lattice $\mathcal{N} = 2$ SYM in D=2
- Simulations
- 4D Twist, Lattice $\mathcal{N} = 4$ SYM
- Conclusions/To do ...

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Twisting in 2D

Simplest theory contains 2 fermions λ^i_α
Action invariant under global symmetry:

$$SO(2)_E \times SO(2)_R$$

Construct twisted rotation group

$$SO(2)' = \text{diagonal subgroup}(SO(2)_E \times SO(2)_R)$$

Consider fermions as matrix

$$\lambda^i_\alpha \rightarrow \Psi_{\alpha\beta}$$

Natural to expand:

$$\Psi = \frac{\eta}{2} I + \psi_\mu \gamma^\mu + \chi_{12} \gamma^1 \gamma^2$$

Fermions: set of p-forms – (twisted) components!

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Dirac equation

Original Dirac action

$$S_D = \sum_{i=1}^{2} \overline{\lambda}^i \gamma.\partial\lambda^i$$

becomes

$$S_D = \mathrm{Tr}\overline{\Psi}\gamma.\partial\Psi$$

which may be written as

$$S_D = \psi_\mu\partial_\mu\eta/2 + \frac{1}{2}\chi_{\mu\nu}\partial_{[\mu}\psi_{\nu]}$$

Fermionic action recast in geometrical form! Set of fields $\Psi = (\eta/2, \psi_\mu, \chi_{12})$ called Kähler-Dirac field.

$$0 = (\gamma.\partial)\,\lambda^i \equiv (d - d^\dagger)\Psi$$

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Scalar supersymmetry

Twisted supercharges also

$$q = QI + Q_\mu \gamma^\mu + Q_{12}\gamma^1\gamma^2$$

Original SUSY algebra:

$$\{q^I_\alpha, \overline{q}^J_\beta\} = 2\delta^{IJ}\gamma^\mu_{\alpha\beta}P_\mu$$

Replaced by twisted algebra

$$\{q, q\} = 4\gamma^\mu P_\mu$$

In components

$$\begin{aligned}
\{Q, Q\} &= \{Q_{12}, Q_{12}\} = 0 \\
\{Q, Q_{12}\} &= \{Q_\mu, Q_\nu\} = 0 \\
\{Q, Q_\mu\} &= P_\mu \\
\{Q_{12}, Q_\mu\} &= -\epsilon_{\mu\nu}P_\nu
\end{aligned}$$

Furthermore, generically

$$S = Q\Lambda(X)$$

Thus if I can ensure $Q^2_{\text{lattice}} = 0$ can easily build lattice actions invariant under Q.

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Continuum twisted 2D SYM

Superpartners:

$$(\eta, \psi_\mu \chi_{12}) \xrightarrow{Q} (\overline{\phi}, A_\mu, B_{12})$$

Action:

$$S = \beta Q \int \mathrm{Tr}\left(\frac{1}{4}\eta[\phi,\overline{\phi}] + \psi_\mu D_\mu \overline{\phi} + \chi_{12}B_{12} + \chi_{12}F_{12}\right)$$

Q-symmetry:

$$\begin{aligned} QA_\mu &= \psi_\mu \\ Q\psi_\mu &= -D_\mu \phi \\ Q\phi &= 0 \\ Q\chi_{12} &= B_{12} \\ QB_{12} &= [\phi, \chi_{12}] \\ Q\overline{\phi} &= \eta \\ Q\eta &= [\phi, \overline{\phi}] \end{aligned}$$

Field ϕ parametrizes gauge transformations:

$$f(x) \to G(x) f(x) G(x)^\dagger \text{ where } G = e^{\phi(x)}$$

$$Q^2 = \delta^\phi_G$$

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Other twisted supersymmetries

Continuum fermion action $S_F = \mathrm{Tr}\Psi^T\gamma.\partial\Psi$ invariant under $\Psi \to \Psi^{(a)} = \Psi\Gamma^a$
where $\Gamma^a = \{I, \gamma^\mu, \gamma^1\gamma^2\}$
Transforms twisted fermions.
Eg for $\Gamma^4 = \gamma^1\gamma^2$

$$(Q, Q^\mu, Q^{12}) \to (Q^{12}, \epsilon_{\mu\nu}Q_\nu, Q)$$

Thus

$$\begin{aligned} Q^{12}A_\mu &= \epsilon_{\mu\nu}\psi_\nu \\ Q^{12}\psi_\mu &= -\epsilon_{\mu\nu}D_\nu\phi \\ Q^{12}\eta &= B_{12} \\ Q^{12}B_{12} &= [\phi, \eta] \\ Q^{12}\overline{\phi} &= \chi_{12} \\ Q^{12}\chi_{12} &= [\phi, \overline{\phi}] \end{aligned}$$

Lattice prescription

Natural to map:

Field	Lattice object	Gauge transformation
scalar	points	$G(x)f(x)G^{\dagger}(x)$
vector	links	$G(x)f_{\mu}(x)G^{\dagger}(x+\mu)$
tensor	squares	$G(x)f_{\mu\nu}(x)G^{\dagger}(x+\mu+\nu)$

Wilson prescription: replace $A_{\mu} \rightarrow U_{\mu} = e^{A_{\mu}}$

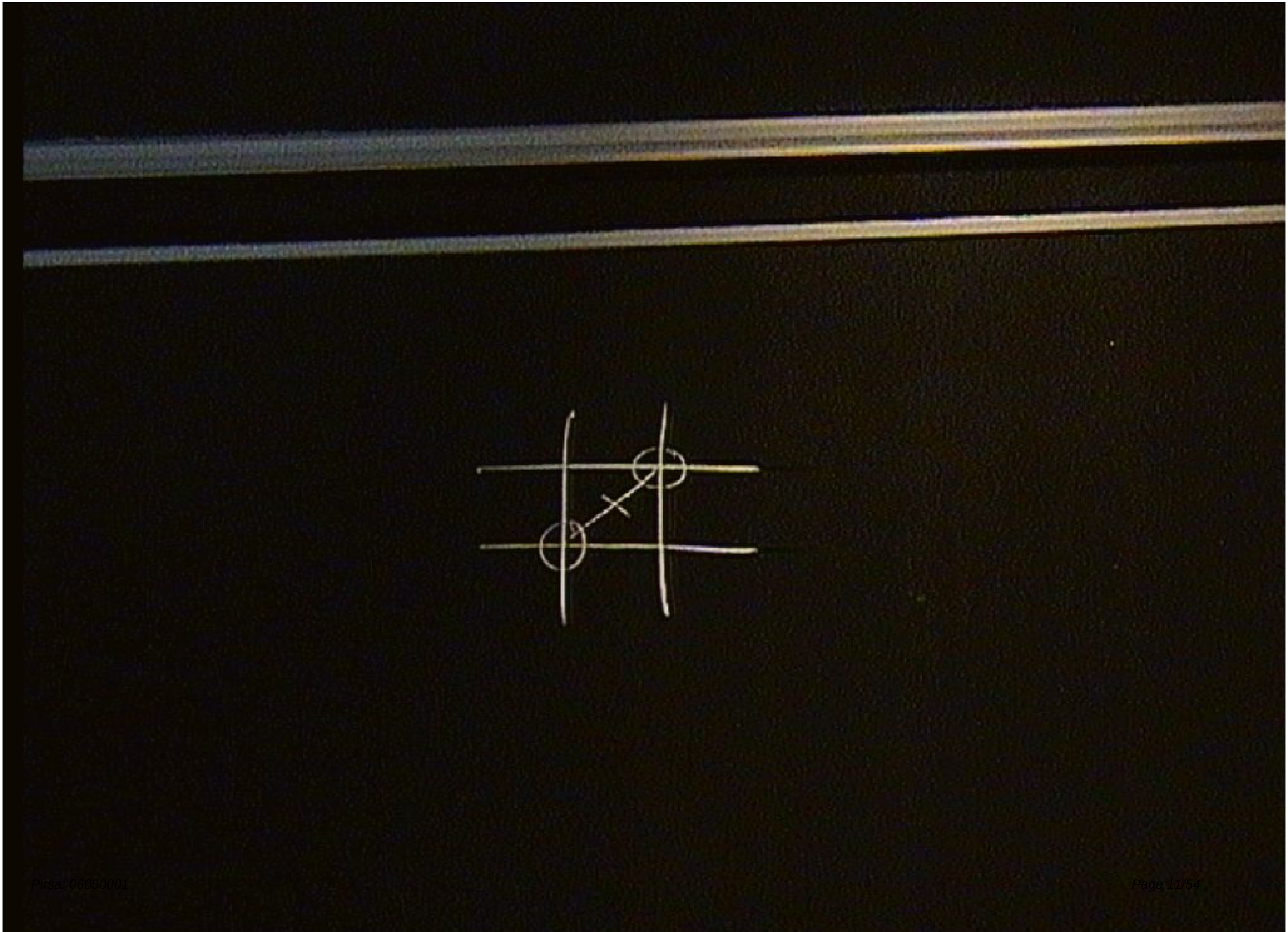
Subtlety: 2 orientations – Use complex fields f and $\overline{f}$. Group $U(N) \rightarrow GL(N,C)$

Q-symmetry same:

$$\begin{aligned} QU_{\mu} &= \psi_{\mu} \\ Q\psi_{\mu} &= -D^{+}_{\mu}\phi \\ QB_{12} &= [\phi, \chi_{12}]' \\ &\cdots \end{aligned}$$

where

$$[\phi, \chi_{12}]' = \phi(x)\chi_{12}(x) - \chi_{12}(x)\phi(x+1+2)$$



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Derivatives

To complete discretization need to replace derivatives by covariant difference operators

$$D_\mu f \;\to\; D_\mu^+ f = U_\mu(x) f(x+\mu) - f(x) U_\mu(x)$$
$$D_\mu f_\nu \;\to\; D_\mu^+ f_\nu = U_\mu(x) f_\nu(x+\mu) - f_\nu(x) U_\mu(x+\nu)$$

and

$$D_\mu^- f_\mu \;=\; f_\mu(x) U_\mu^\dagger(x) - U_\mu^\dagger(x-\mu) f_\mu(x-\mu)$$
$$D_\mu^- f_{\mu\nu} \;=\; f_{\mu\nu}(x) U_\mu^\dagger(x+\nu) - U_\mu^\dagger(x-\mu) f_{\mu\nu}(x-\mu)$$

Compatible with gauge transformations.
Field strength $F_{\mu\nu} = D_\mu^+ U_\nu$
Furthermore, if

$$D_\mu \;\to\; D_\mu^+ \quad \text{if acts like } d$$
$$D_\mu \;\to\; D_\mu^- \quad \text{if acts like } d^\dagger$$

Avoids fermion doubling problem

Dirac equation

Original Dirac action

$$S_D = \sum_{i=1}^{2} \bar{\lambda}^i \gamma.\partial\lambda^i$$

becomes

$$S_D = \mathrm{Tr}\bar{\Psi}\gamma.\partial\Psi$$

which may be written as

$$S_D = \psi_\mu\partial_\mu\eta/2 + \frac{1}{2}\chi_{\mu\nu}\partial_{[\mu}\psi_{\nu]}$$

Fermionic acti… cast in geometrical form!
Set of fields … $/2, \psi_\mu, \chi_{12})$ called Kähler-Dirac field.

$$(\gamma.\partial)\,\lambda^i \equiv (d - d^\dagger)\Psi$$

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Lattice action

$$S = \frac{\beta}{2} Q \sum \mathrm{Tr}\left(\frac{1}{4}\overline{\eta}[\phi,\overline{\phi}] + \overline{\psi}_\mu D_\mu^+\overline{\phi} + \overline{\chi}_{12}B_{12} + \overline{\chi}_{12}F_{12} + \mathrm{h.c}\right)$$

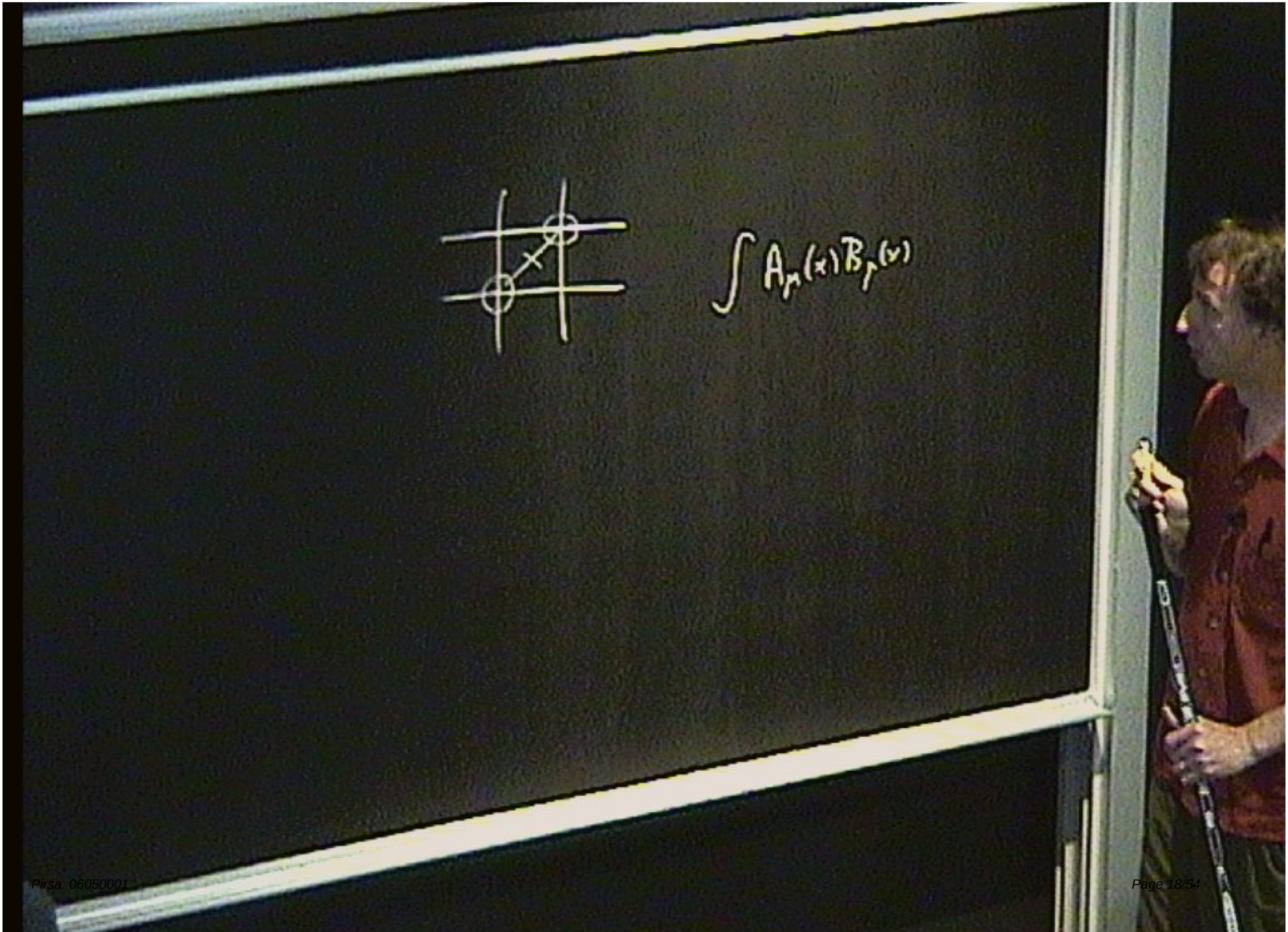
Carrying out Q-variation and int. B

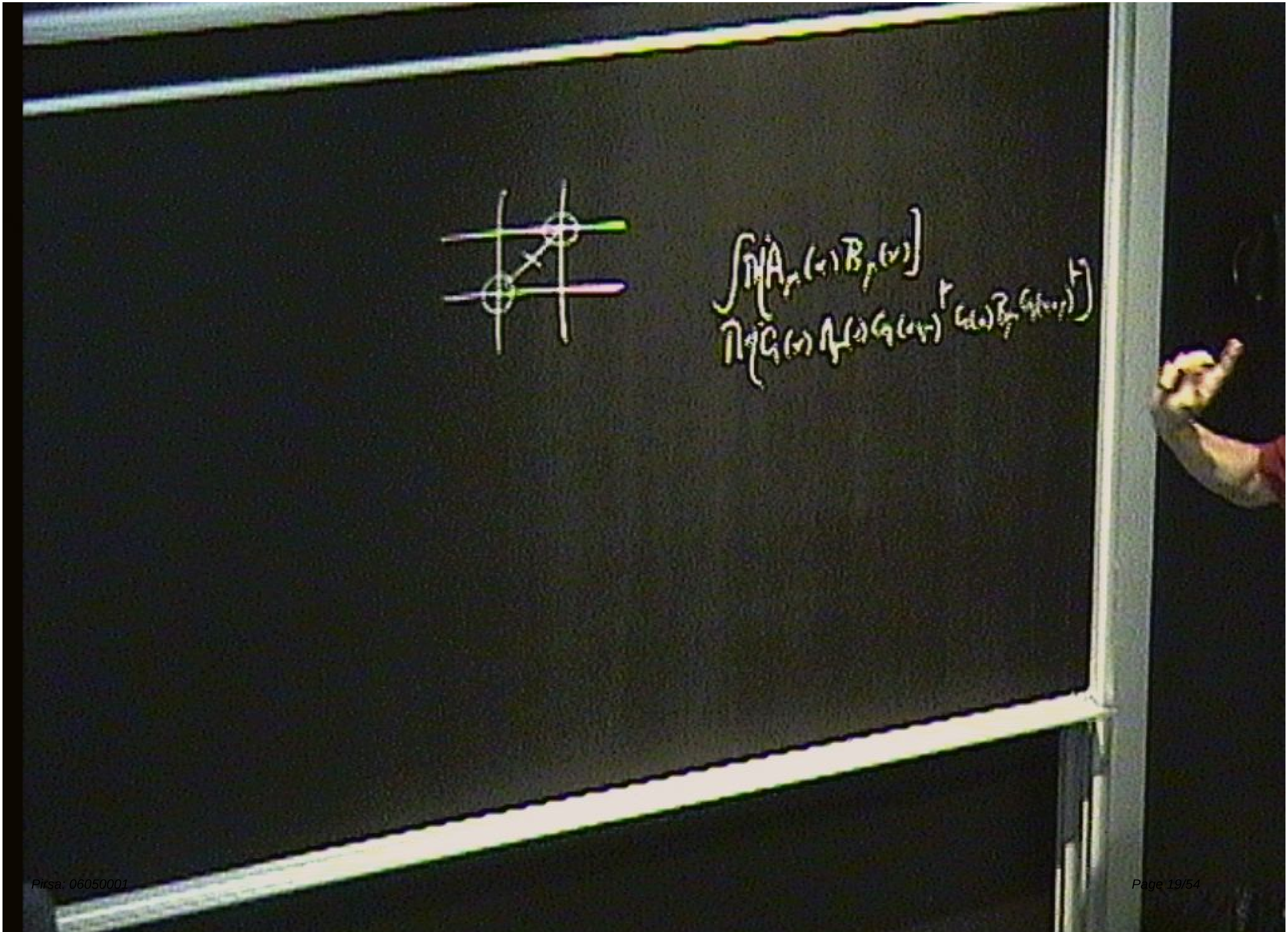
$$\begin{aligned} S_L = \frac{\beta}{2}\mathrm{Tr}\sum_x \Big(&\frac{1}{4}[\phi,\overline{\phi}]^2 + F_{12}^\dagger F_{12} \\ &- \frac{1}{4}\eta^\dagger[\phi,\eta] - \chi_{12}^\dagger[\phi,\chi_{12}]^{(12)} + \psi_\mu^\dagger[\overline{\phi},\psi_\mu]^{(\mu)} \\ &+ (D_\mu^+\phi)^\dagger D_\mu^+\overline{\phi} - 2\chi_{12}^\dagger\left(D_1^+\psi_2 - D_2^+\psi_1\right) \\ &- 2\psi_\mu^\dagger D_\mu^+\frac{\eta}{2} + \mathrm{h.c}\Big)\end{aligned}$$

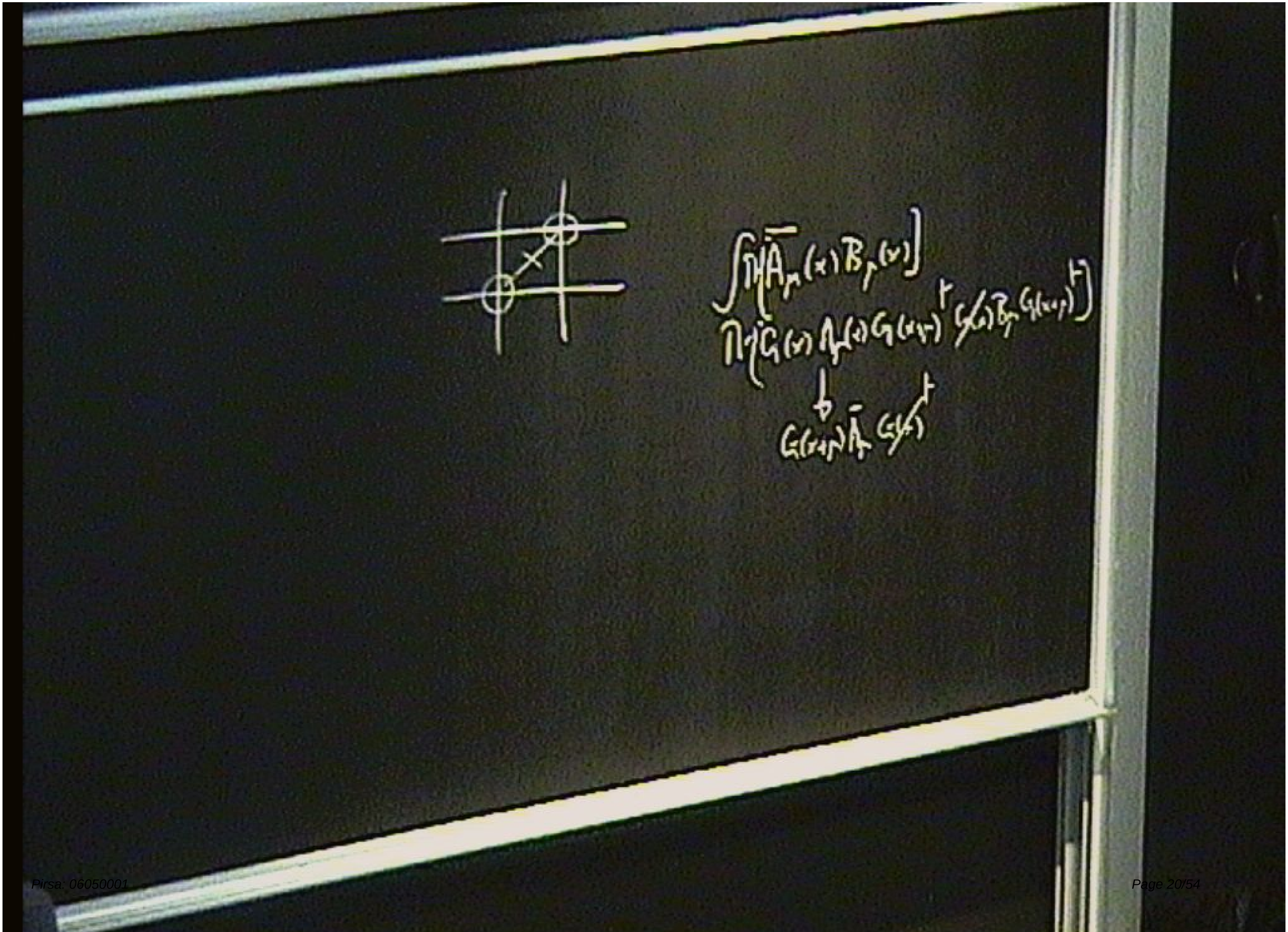
Invariant under Q, finite gauge transformations and $U(1)$

$$\psi_\mu \to e^{i\alpha}\psi_\mu,\quad \eta,\chi_{12} \to e^{-i\alpha}\eta,\chi_{12}$$

$$\overline{\phi} \to e^{-2i\alpha}\overline{\phi},\quad \phi \to e^{2i\alpha}\phi$$


$\int A_\mu(x) B_\nu(x)$





Lattice action

$$S = \frac{\beta}{2} Q \sum \mathrm{Tr} \left(\frac{1}{4}\overline{\eta}[\phi, \overline{\phi}] + \overline{\psi}_\mu D^+_\mu \overline{\phi} + \overline{\chi}_{12} B_{12} + \overline{\chi}_{12} F_{12} + \mathrm{h.c} \right)$$

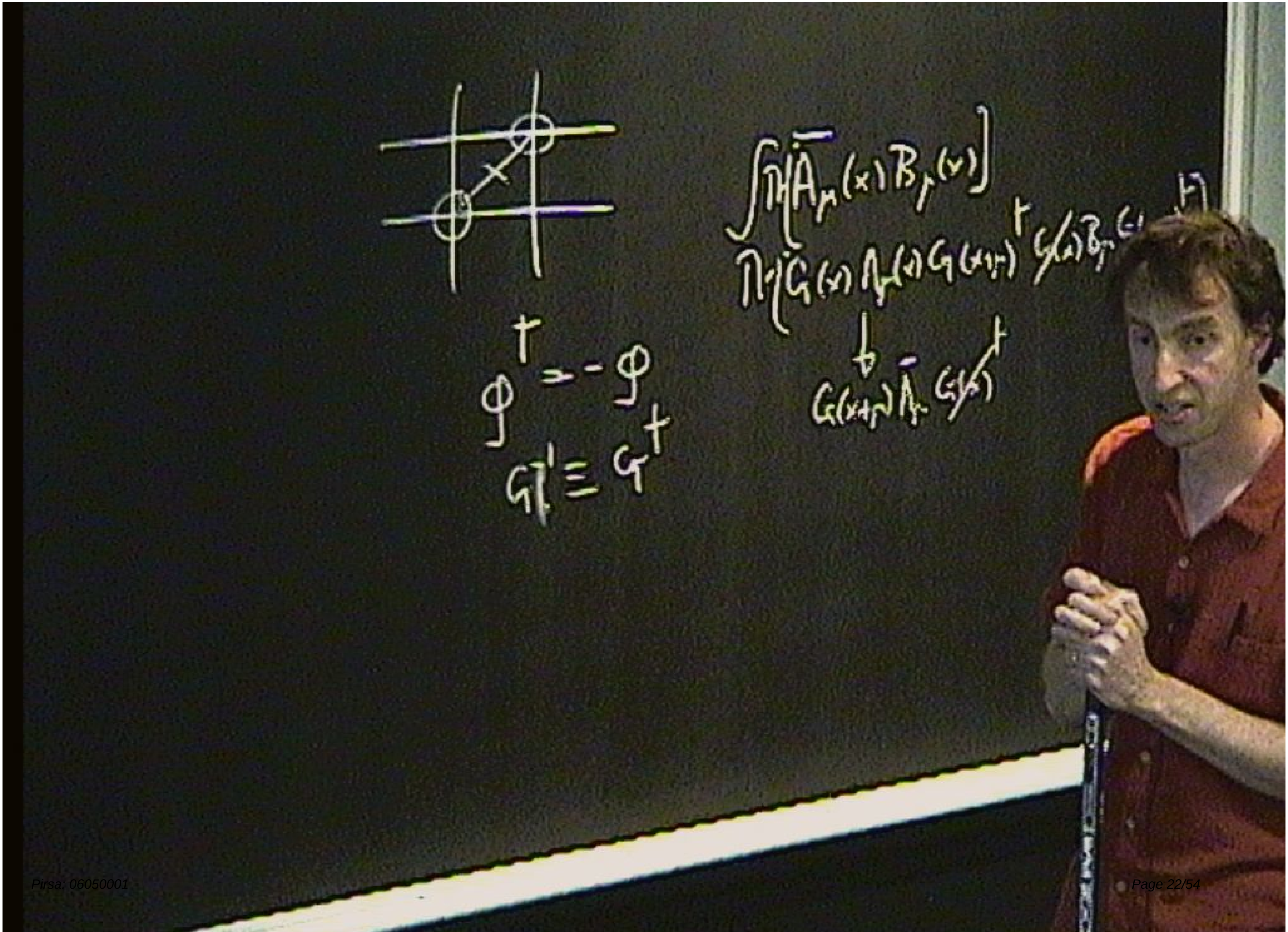
Carrying out Q-variation and int. B

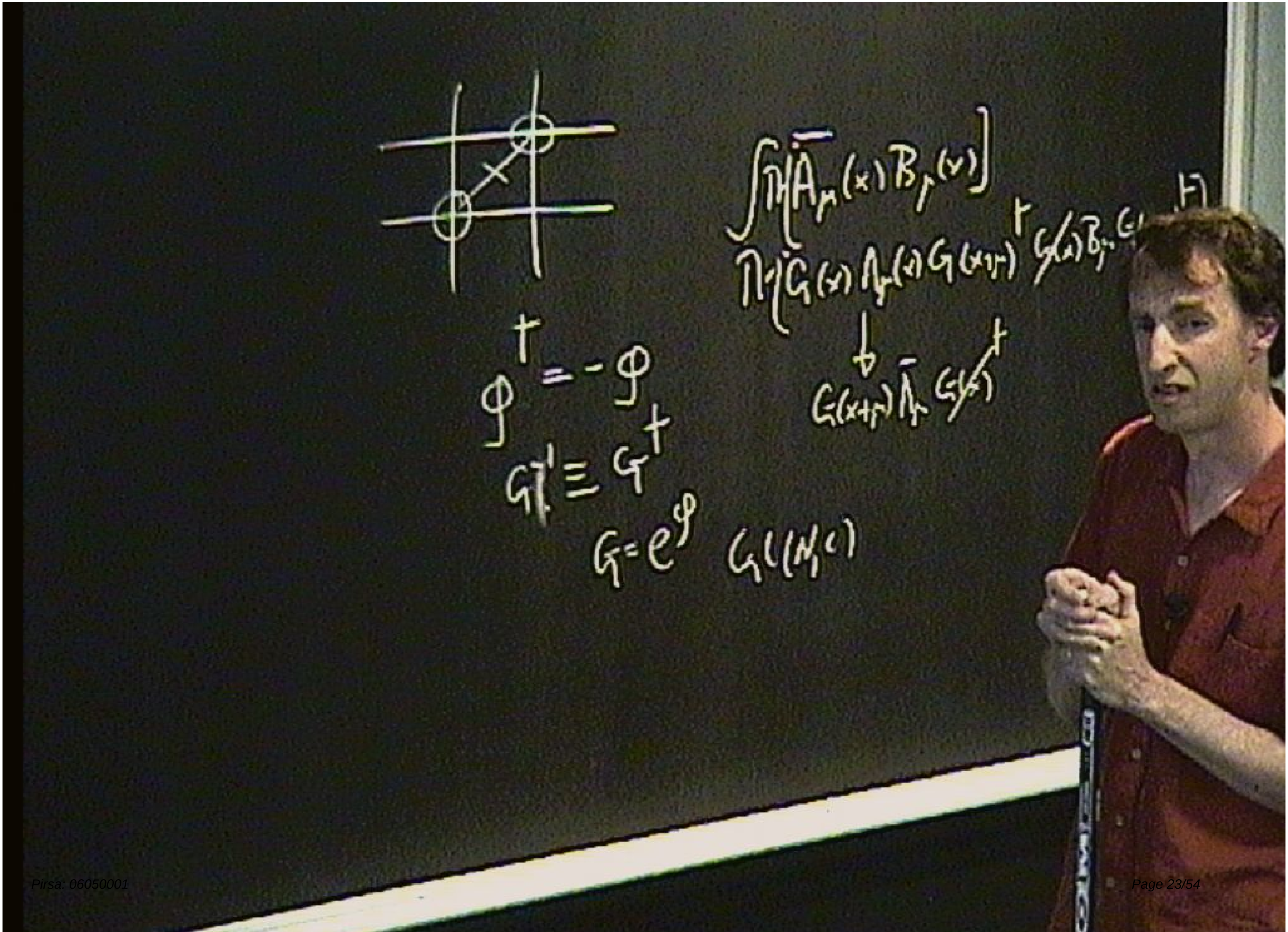
$$\begin{aligned} S_L &= \frac{\beta}{2} \mathrm{Tr} \sum_x \Big(\frac{1}{4}[\phi, \overline{\phi}]^2 + F^\dagger_{12} F_{12} \\ &- \frac{1}{4}\eta^\dagger [\phi, \eta] - \chi^\dagger_{12}[\phi, \chi_{12}]^{(12)} + \psi^\dagger_\mu [\overline{\phi}, \psi_\mu]^{(\mu)} \\ &+ (D^+_\mu \phi)^\dagger D^+_\mu \overline{\phi} - 2\chi^\dagger_{12} \left(D^+_1 \psi_2 - D^+_2 \psi_1 \right) \\ &- 2\psi^\dagger_\mu D^+_\mu \frac{\eta}{2} + \mathrm{h.c} \Big) \end{aligned}$$

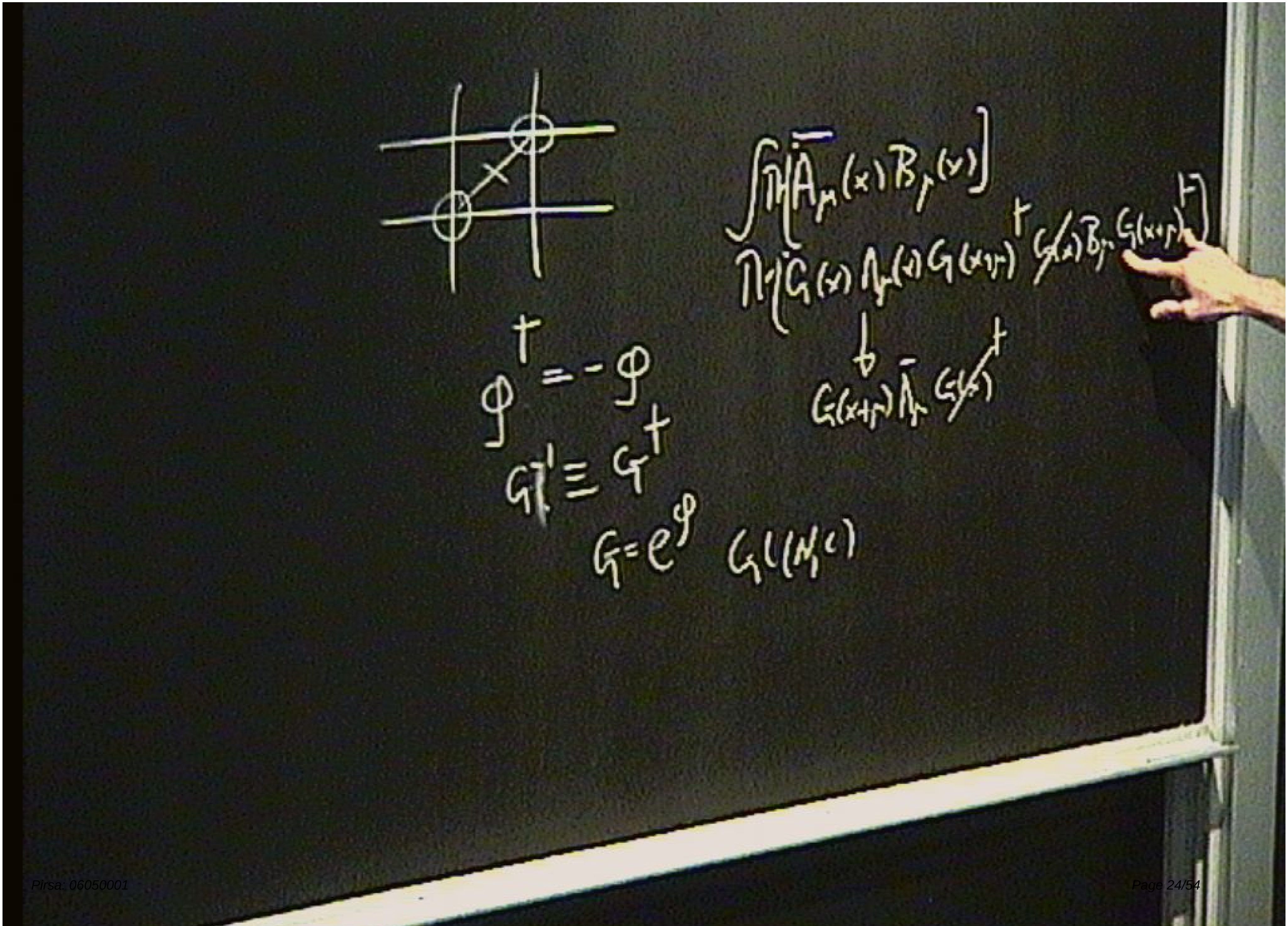
Invariant under Q, finite gauge transformations and $U(1)$

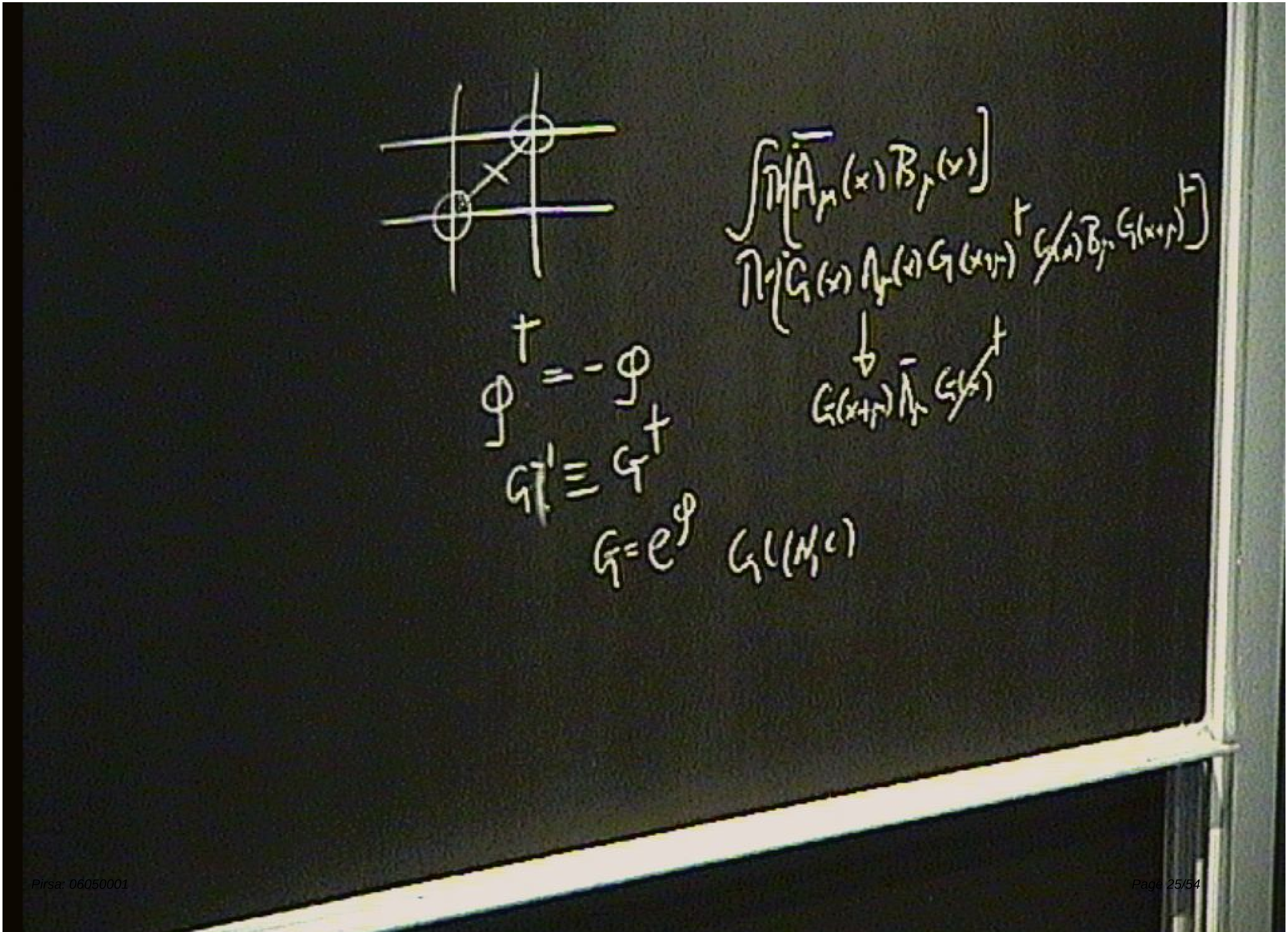
$$\psi_\mu \to e^{i\alpha}\psi_\mu, \;\; \eta, \chi_{12} \to e^{-i\alpha}\eta, \chi_{12}$$

$$\overline{\phi} \to e^{-2i\alpha}\overline{\phi}, \;\; \phi \to e^{2i\alpha}\phi$$


φ† = - φ
G' ≡ G†




G=e^φ
GL(N,C)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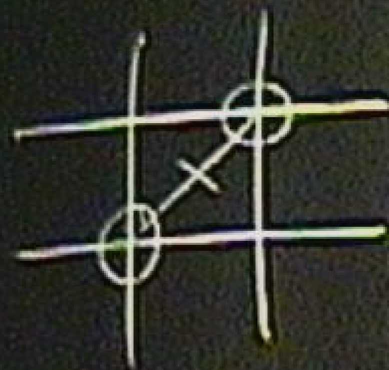

$$\int \Pi d\bar{A}_\mu(x) B_\mu(y)]$$

$$\Pi d[G(x) A_\mu(x) G(x+\mu)^\dagger G(x) B_\mu G(x+\mu)^\dagger]$$

$$\downarrow$$

$$G(x+\mu)\bar{A}_\mu G(x)^\dagger$$

$$\varphi^\dagger = -\varphi$$

$$G' \equiv G^\dagger$$

$$G = e^\varphi \quad GL(N,C)$$

Lattice action

$$S = \frac{\beta}{2} Q \sum \mathrm{Tr}\left(\frac{1}{4}\overline{\eta}[\phi,\overline{\phi}] + \overline{\psi}_\mu D^+_\mu \overline{\phi} + \overline{\chi}_{12} B_{12} + \overline{\chi}_{12} F_{12} + \mathrm{h.c}\right)$$

Carrying out Q-variation and int. B

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Invariant under Q, finite gauge transformations and $U(1)$

$$\psi_\mu \to e^{i\alpha}\psi_\mu,\ \ \eta,\chi_{12} \to e^{-i\alpha}\eta,\chi_{12}$$

$$\overline{\phi} \to e^{-2i\alpha}\overline{\phi},\ \ \phi \to e^{2i\alpha}\phi$$

Gauge action

$$\beta \mathrm{Tr} \sum_x F_{12}^\dagger(x) F_{12}(x)$$

$$\beta \mathrm{Tr} \sum_x \left(2I - U_P - U_P^\dagger\right) + \beta \mathrm{Tr} \sum_x (M_{12} + M_{21} - 2I)$$

where

$$U_P = U_1(x) U_2(x+1) U_1^\dagger(x+2) U_2^\dagger(x)$$

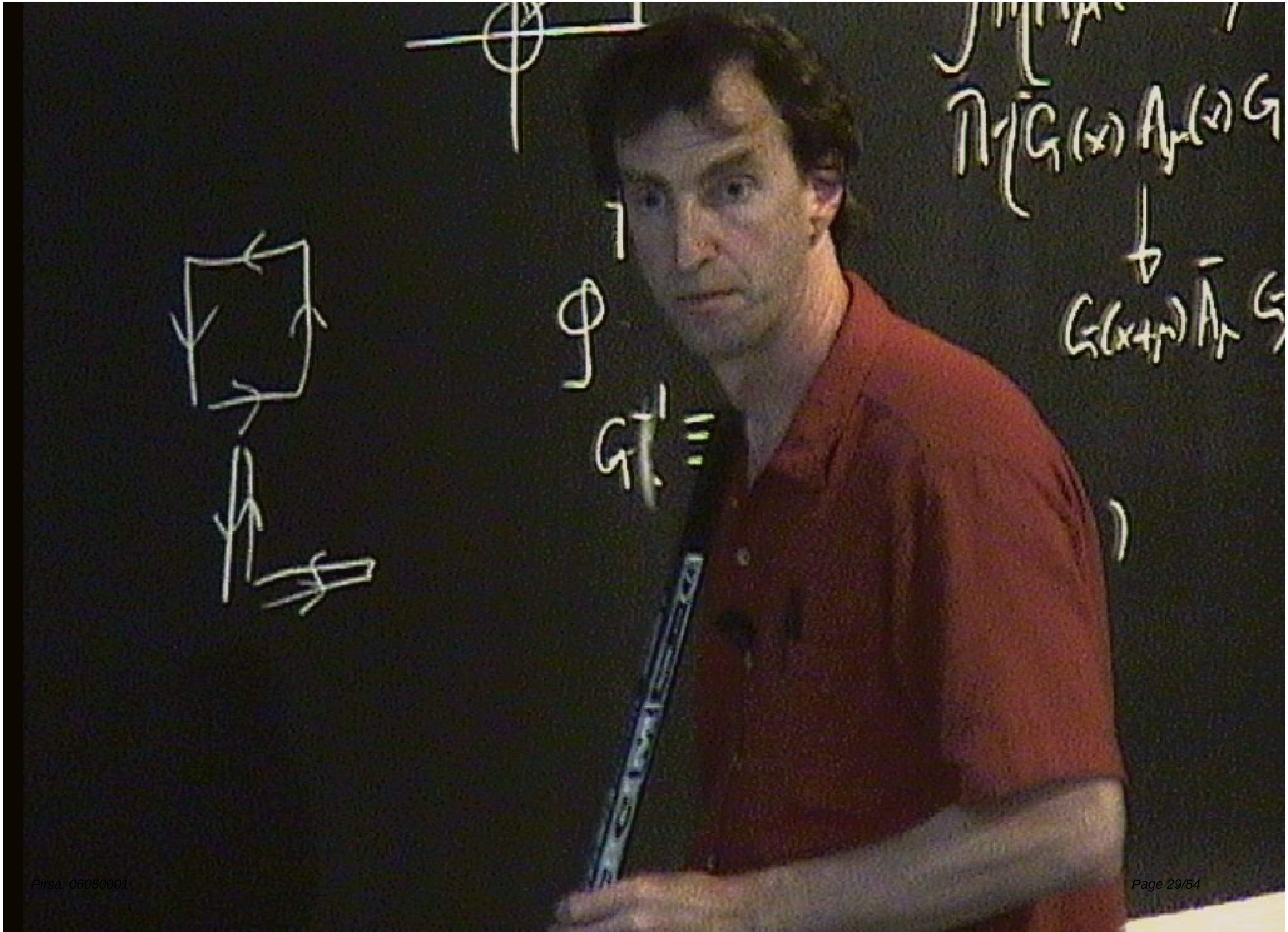
and

$$M_{12} = U_1(x) U_1^\dagger(x) U_2^\dagger(x+1) U_2(x+1)$$

Notice:

2nd term is zero if $U_\mu^\dagger(x) U_\mu(x) = I$.

Gauge action collapses to usual Wilson action!



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Notice:

2nd term is zero if $U_\mu^\dagger(x) U_\mu(x) = I$.

Gauge action collapses to usual Wilson action!

Twisted Fermions

$$\Psi = \begin{pmatrix} \eta/2 \\ \chi_{12} \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

Action $\Psi^\dagger M \Psi$

$$M = \begin{pmatrix} -[\phi,]^{(p)} & K \\ -K^\dagger & [\overline{\phi},]^{(p)} \end{pmatrix}$$

$$K = \begin{pmatrix} D_2^+ & -D_1^+ \\ -D_1^- & -D_2^- \end{pmatrix}$$

After integration – Pf(M).

In free limit Pf(M) = det(K) = det($D_\mu^+ D_\mu^-$)–
no doubles!

Simulations I.

Lattice formulation given in terms of complex fields.

Target continuum theory corresponds to setting

$$\begin{aligned} \mathrm{Im}X^a_\mu &= 0 \text{ all fields } X \text{ bar scalars} \\ \overline{\phi} &= -\phi^\dagger \end{aligned}$$

Question: are Ward identities corresponding to Q still satisfied?

Conjecture: yes for $\beta \to \infty$

Why?

Q-exact S – compute W.I exactly for $\beta \to \infty$

Setting $U_\mu(x) = R_\mu(x)u_\mu(x)$, $u_\mu(x)$ in $SU(N)$ find $R_\mu(x) = 1$!

Fermion operator $M(R, u) = M(u) = -M^T$ Fluctuations of real and imaginary parts of fields decouple

W.I of truncated theory at large β are reproduced !

Simulations II

Integrate out grassman fields – non-local determinant

Employ RHMC algorithm. Efficient, exact

Trick:

$$\det^{\frac{1}{4}}(M^\dagger M) = \int DFDF^\dagger e^{-F^\dagger (M^\dagger M)^{-\frac{1}{4}} F}$$

Fractional power approximated by:

$$x^{-\frac{1}{4}} \sim \sum_{i=1}^{n} \frac{\alpha_i}{x+\beta_i}$$

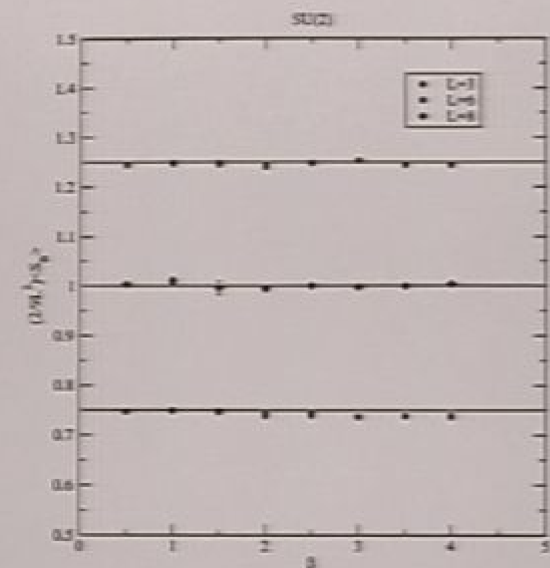
Optimal minimax set $\{\alpha_i, \beta_i\}$ in some interval $\epsilon \leq x \leq 1$ found by remez alg.

Linear systems handled using multimass CG-solver.

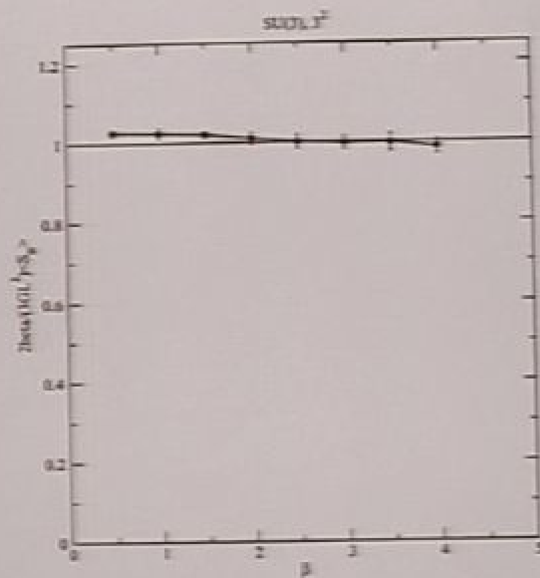
$SU(2)$, $SU(3)$ results for $L \times L$ lattice with $L = 2-8$, $\beta = 0.5-4.0$

Check supersymmetry – eg.

$$< S > = < Q\Lambda > = 0 \rightarrow \frac{2\beta}{3RL^2} < S_B > = 1$$



Mean bosonic action
Ward identities satisfied



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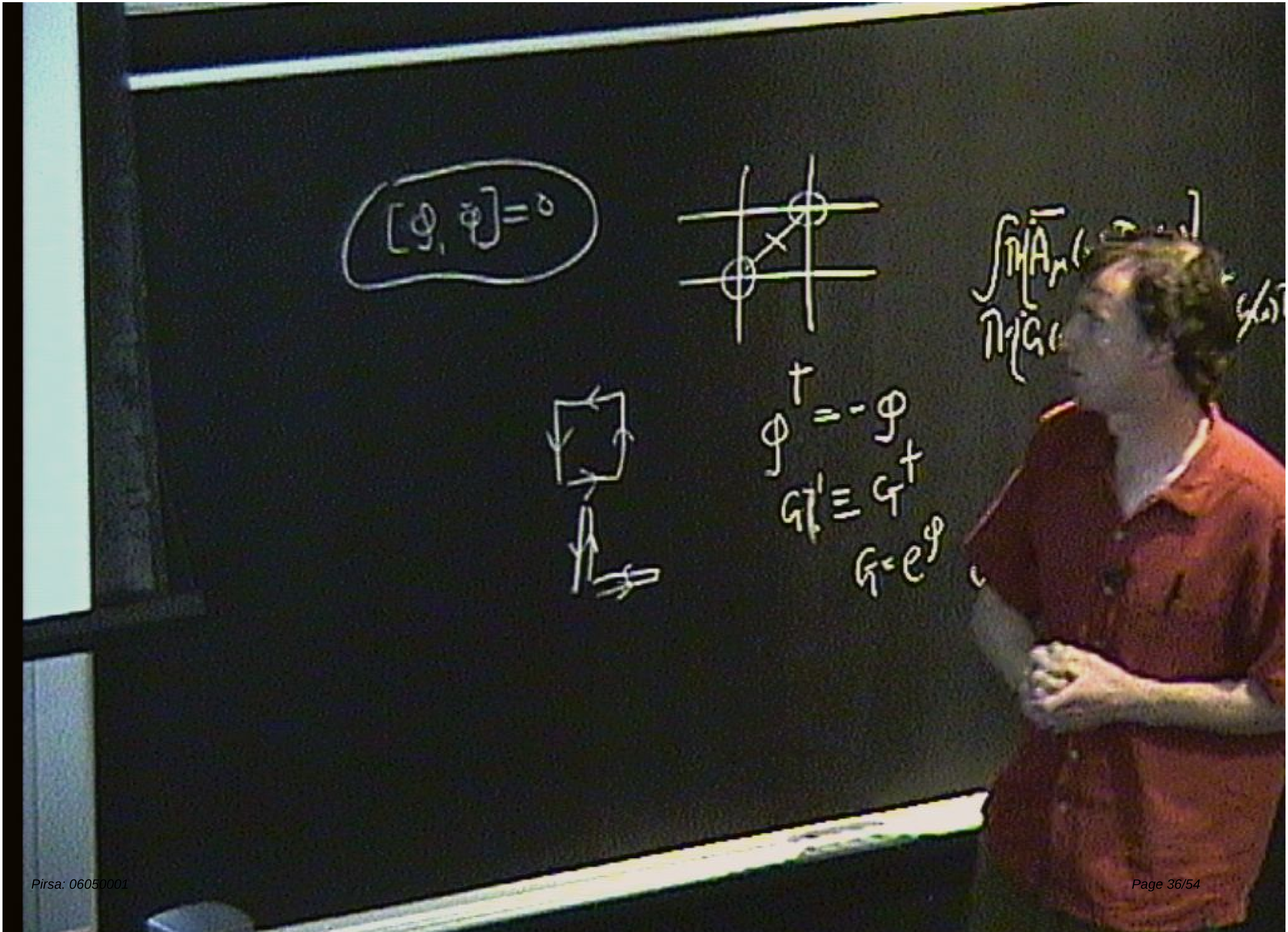
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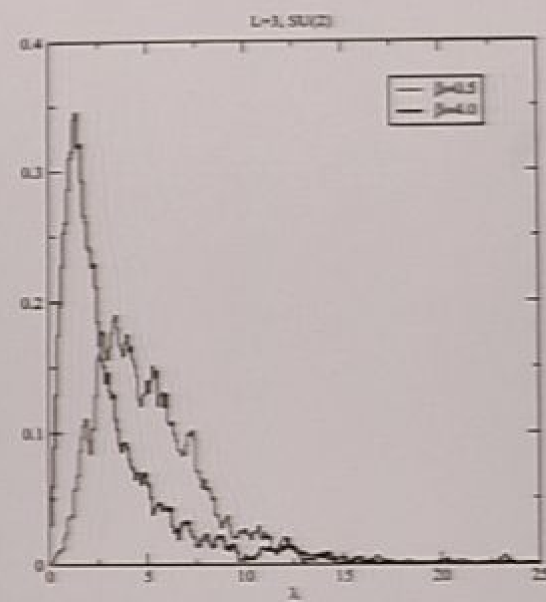
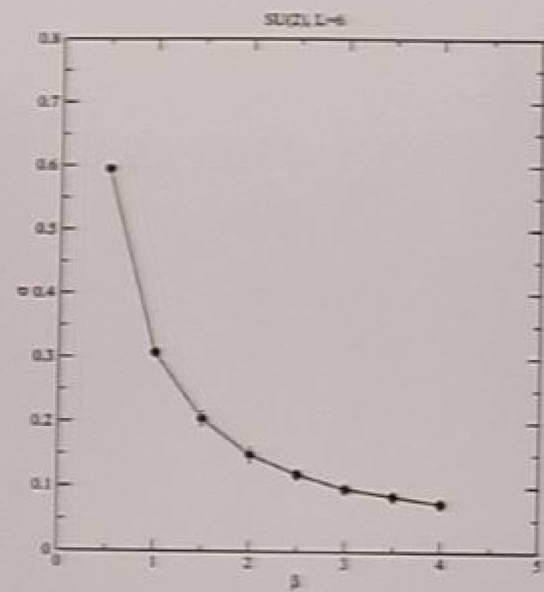


Distribution of eigenvalues of scalars

Flat directions lifted via quantum effects



String tension

Consistent with single phase – Z is just Witten index

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Twisting $\mathcal{N}=4$ SYM in D=4

4 Majorana spinors Ψ^I_α. Kähler-Dirac twist:

$$SO(4)' = \text{diag}(SO(4) \times SO(4)_R)$$

Regard supercharges and fermions as matrices:

$$\begin{aligned}\Psi &= \eta I + \psi_\mu \gamma_\mu + \frac{1}{2!}\chi_{\mu\nu}\gamma_\mu\gamma_\nu + \\ &+ \frac{1}{3!}\theta_{\mu\nu\lambda}\gamma_\mu\gamma_\nu\gamma_\lambda + \frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\end{aligned}$$

16 real components needed for single Kähler-Dirac field!

Twisted fermions paired with superpartners $(\overline{\phi}, A_\mu, B_{\mu\nu}, W_{\mu\nu\lambda}, C_{\mu\nu\lambda\rho})$ through scalar Q supersymmetry

(B and C multipliers)

4D Q transformations

Simple generalization of D=2

$$
\begin{aligned}
Q\bar{\phi} &= \eta & Q\eta &= [\phi, \bar{\phi}] \\
QA_\mu &= \psi_\mu & Q\psi_\mu &= -D_\mu\phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] & Q\chi_{\mu\nu} &= B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} & Q\theta_{\mu\nu\lambda} &= [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] & Q\kappa_{\mu\nu\lambda\rho} &= C_{\mu\nu\lambda\rho} \\
Q\phi &= 0
\end{aligned}
$$

Notice $Q^2 = \delta_G^\phi$

Gauge Fermion of $\mathcal{N} = 4$ SYM

$S = \beta Q \Lambda$ with

$$
\begin{aligned}
\Lambda = & \int d^4x \, \mathrm{Tr} \Big[\chi_{\mu\nu} \Big(F_{\mu\nu} + \frac{1}{2} B_{\mu\nu} - \frac{1}{2} [W_{\mu\lambda\rho}, W_{\nu\lambda\rho}] \\
& + D_\lambda W_{\lambda\mu\nu} \Big) \\
& + \psi_\mu D_\mu \bar{\phi} + \frac{1}{4} \eta [\phi, \bar{\phi}] + \frac{1}{3!} \theta_{\mu\nu\lambda} [W_{\mu\nu\lambda}, \bar{\phi}] \\
& + \frac{1}{4!} \kappa_{\mu\nu\lambda\rho} \Big(\sqrt{2} D_{[\mu} W_{\nu\lambda\rho]} + \frac{1}{2} C_{\mu\nu\lambda\rho} \Big) \Big]
\end{aligned}
$$

Carry out Q-variation and integrate out $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$

Dualize 3 and 4-form fields – connect to Marcus topological twist

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Gauge Fermion of $\mathcal{N}=4$ SYM

$S=\beta Q\Lambda$ with

$$\Lambda = \int d^4x \,\mathrm{Tr}\Big[\chi_{\mu\nu}\Big(F_{\mu\nu}+\frac{1}{2}B_{\mu\nu}-\frac{1}{2}[W_{\mu\lambda\rho},W_{\nu\lambda\rho}]$$
$$+ \; D_\lambda W_{\lambda\mu\nu}\Big)$$
$$+ \; \psi_\mu D_\mu\overline{\phi}+\frac{1}{4}\eta[\phi,\overline{\phi}]+\frac{1}{3!}\theta_{\mu\nu\lambda}[W_{\mu\nu\lambda},\overline{\phi}]$$
$$+ \; \frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\Big(\sqrt{2}D_{[\mu}W_{\nu\lambda\rho]}+\frac{1}{2}C_{\mu\nu\lambda\rho}\Big)\Big]$$

Carry out Q-variation and integrate out $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$

Dualize 3 and 4-form fields – connect to Marcus topological twist

Continuum geometric action

$$S_F = \int d^4x \mathrm{Tr}\left[-\chi_{\mu\nu} D_{[\mu}\psi_{\nu]} - \chi_{\mu\nu} D_\lambda \theta_{\lambda\mu\nu} - \eta D_\mu \psi_\mu - \frac{\sqrt{2}}{4!}\kappa_{\mu\nu\lambda\rho} D_{[\mu}\theta_{\nu\lambda\rho]}\right]$$

$$S_B = \int d^4x \mathrm{Tr}\left[-\frac{1}{2}\left(\left(F_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]\right)^2 + \left(D_\lambda W_{\lambda\mu\nu}\right)^2 + \frac{2}{4!}\left(D_{[\mu} W_{\nu\lambda\rho]}\right)^2\right) - D_\mu \phi D_\mu \overline{\phi} + \frac{1}{4}[\phi, \overline{\phi}]^2 - \frac{1}{3!}[\phi, W_{\mu\nu\lambda}][\overline{\phi}, W_{\mu\nu\lambda}]\right]$$

$$S_Y = \ldots$$

Notice 4 W's plus $\phi, \overline{\phi}$ make up 6 scalars.

Spinor formulation

Making another change of variables

$$\begin{aligned} X^\mu &= V_\mu \;\; \mu = 0 \ldots 3 \\ X^4 &= \phi_1 \\ X^5 &= \phi_2 \end{aligned}$$

$$S_B = -\frac{1}{2} F_{\mu\nu}^2 - \left(D_\mu X^i\right)^2 - \frac{1}{2} \sum_{ij} [X_i, X_j]^2$$

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \;\; \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}$$

$$S_F = \frac{1}{2} \sum_{\alpha=1,2} \lambda_\alpha^\dagger \gamma . D \lambda_\alpha$$

where λ^1 and λ^2 columns of KD matrix.

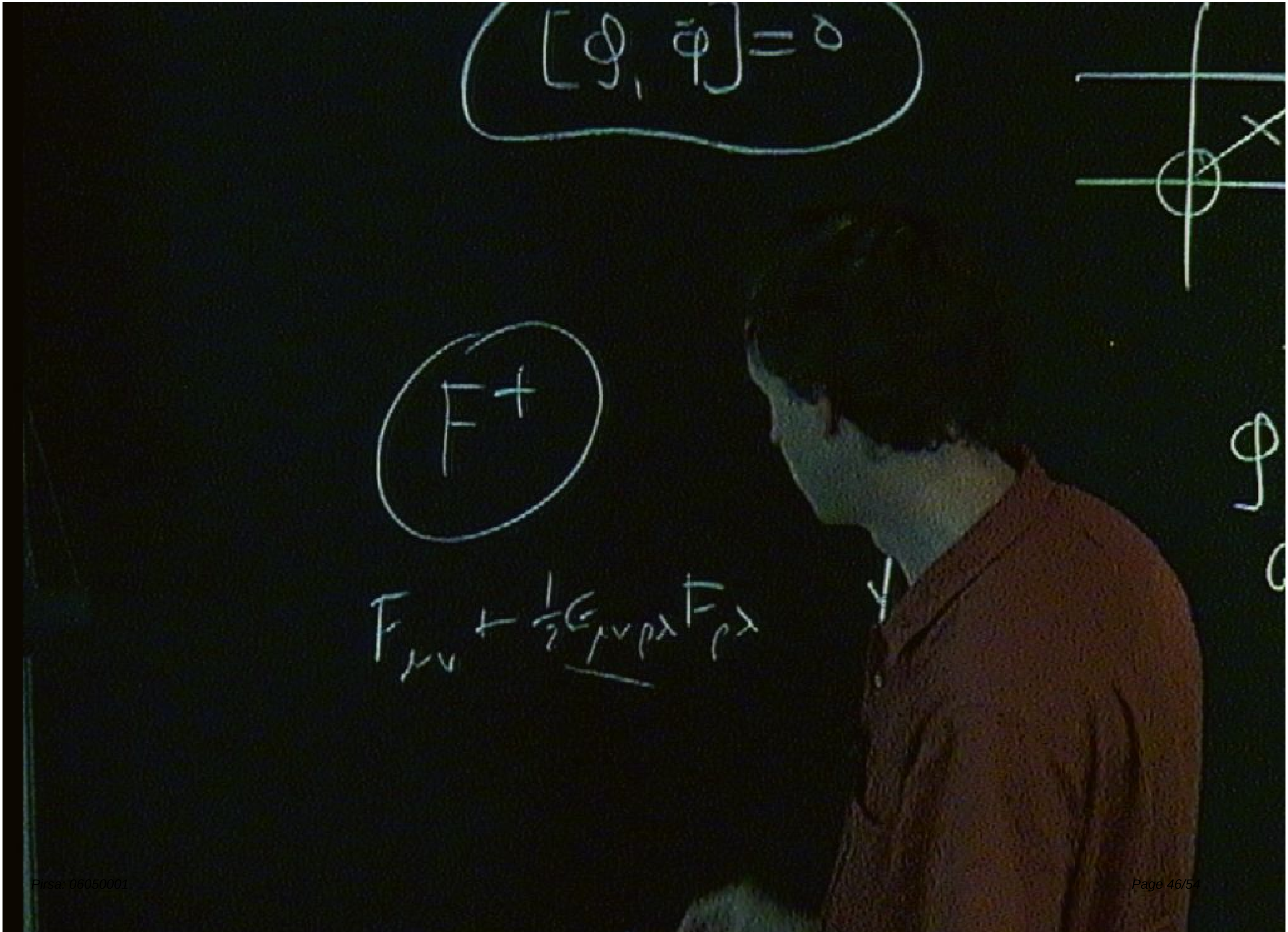
Gauge Fermion of $\mathcal{N} = 4$ SYM

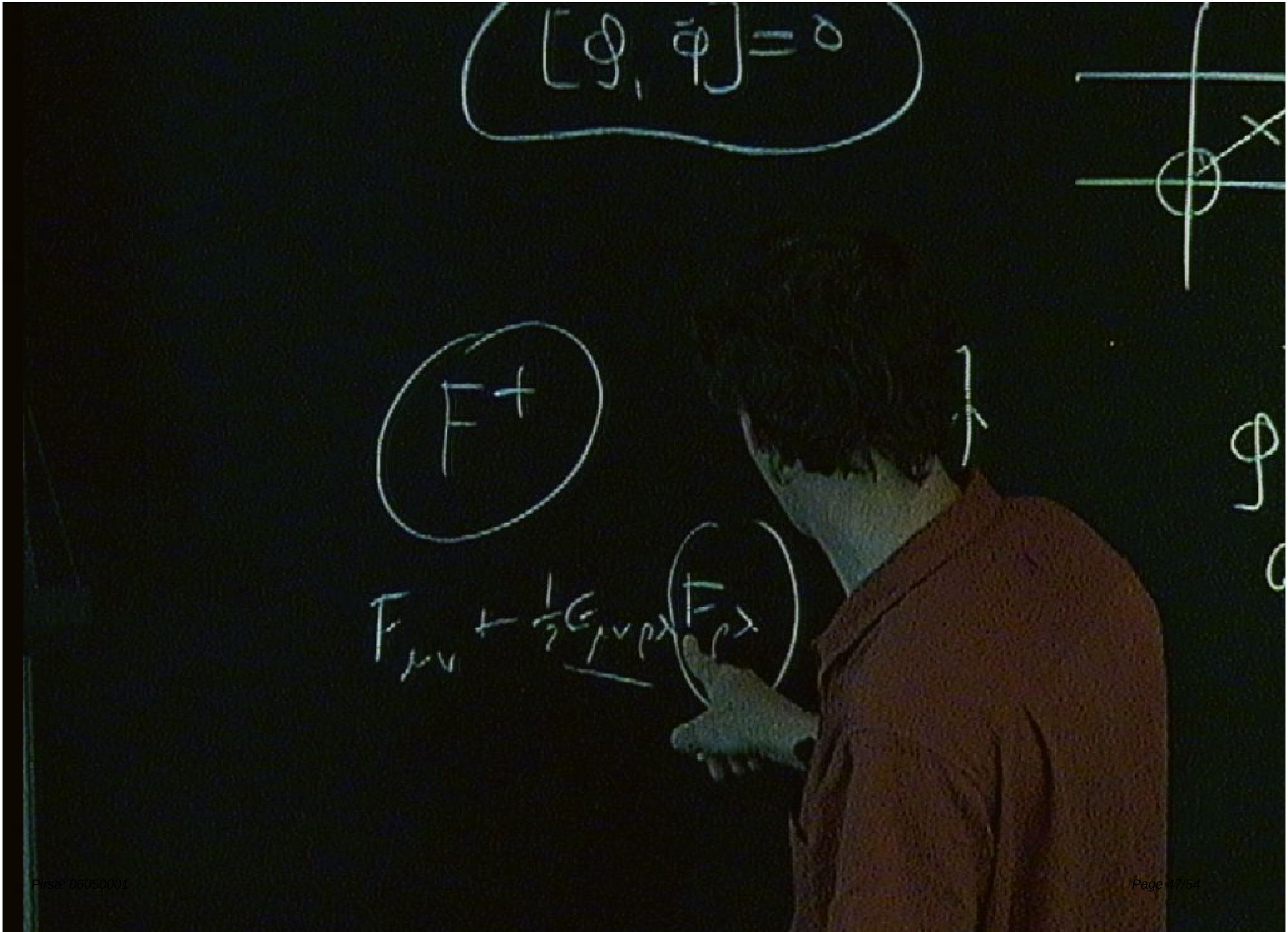
$S = \beta Q \Lambda$ with

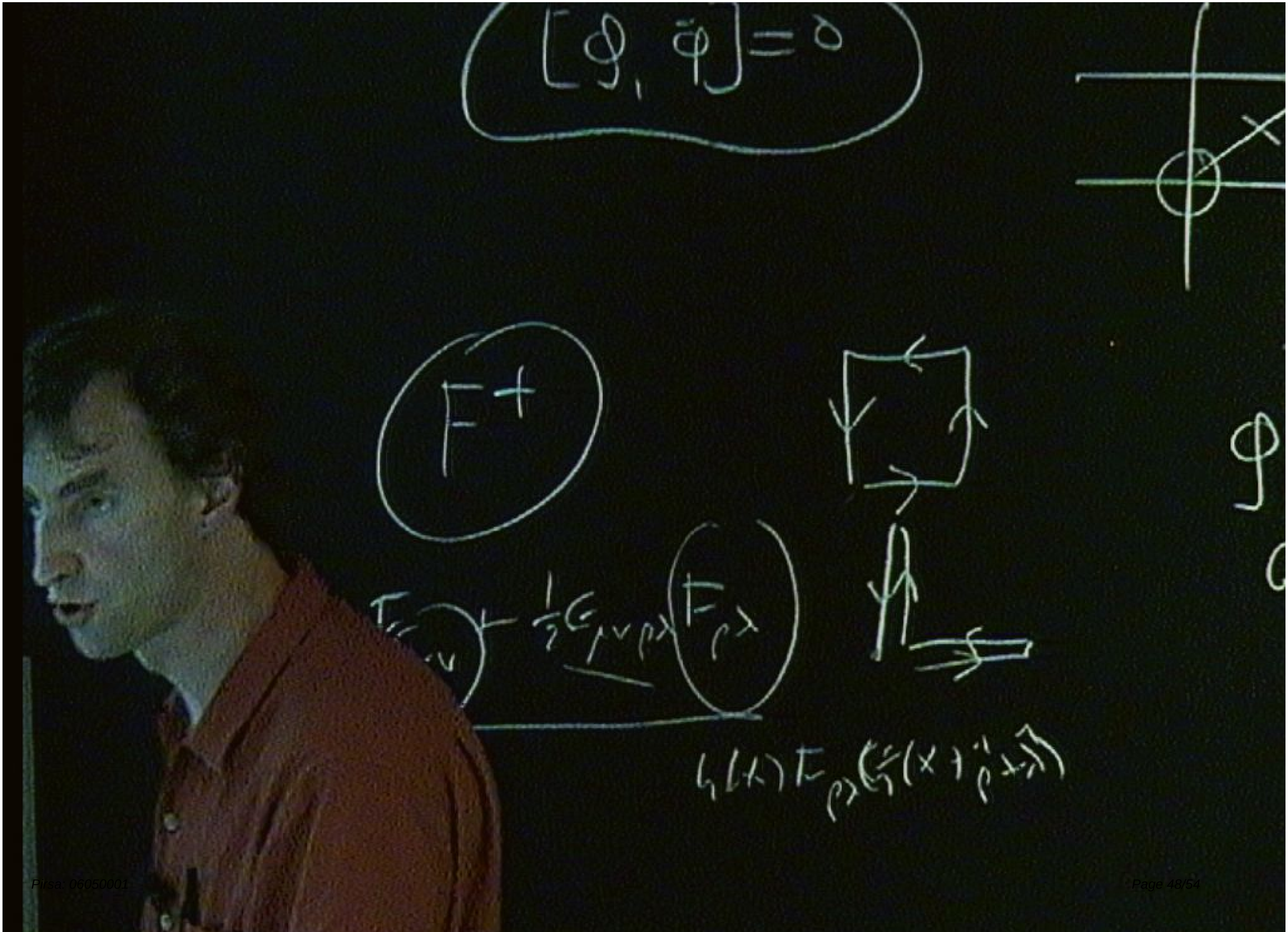
$$\begin{aligned}\Lambda &= \int d^4x \,\mathrm{Tr}\left[\chi_{\mu\nu}\left(F_{\mu\nu} + \frac{1}{2}B_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]\right.\right.\\ &+ \left. D_\lambda W_{\lambda\mu\nu}\right)\\ &+ \psi_\mu D_\mu \overline{\phi} + \frac{1}{4}\eta[\phi, \overline{\phi}] + \frac{1}{3!}\theta_{\mu\nu\lambda}[W_{\mu\nu\lambda}, \overline{\phi}]\\ &+ \left.\frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\left(\sqrt{2}D_{[\mu}W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho}\right)\right]\end{aligned}$$

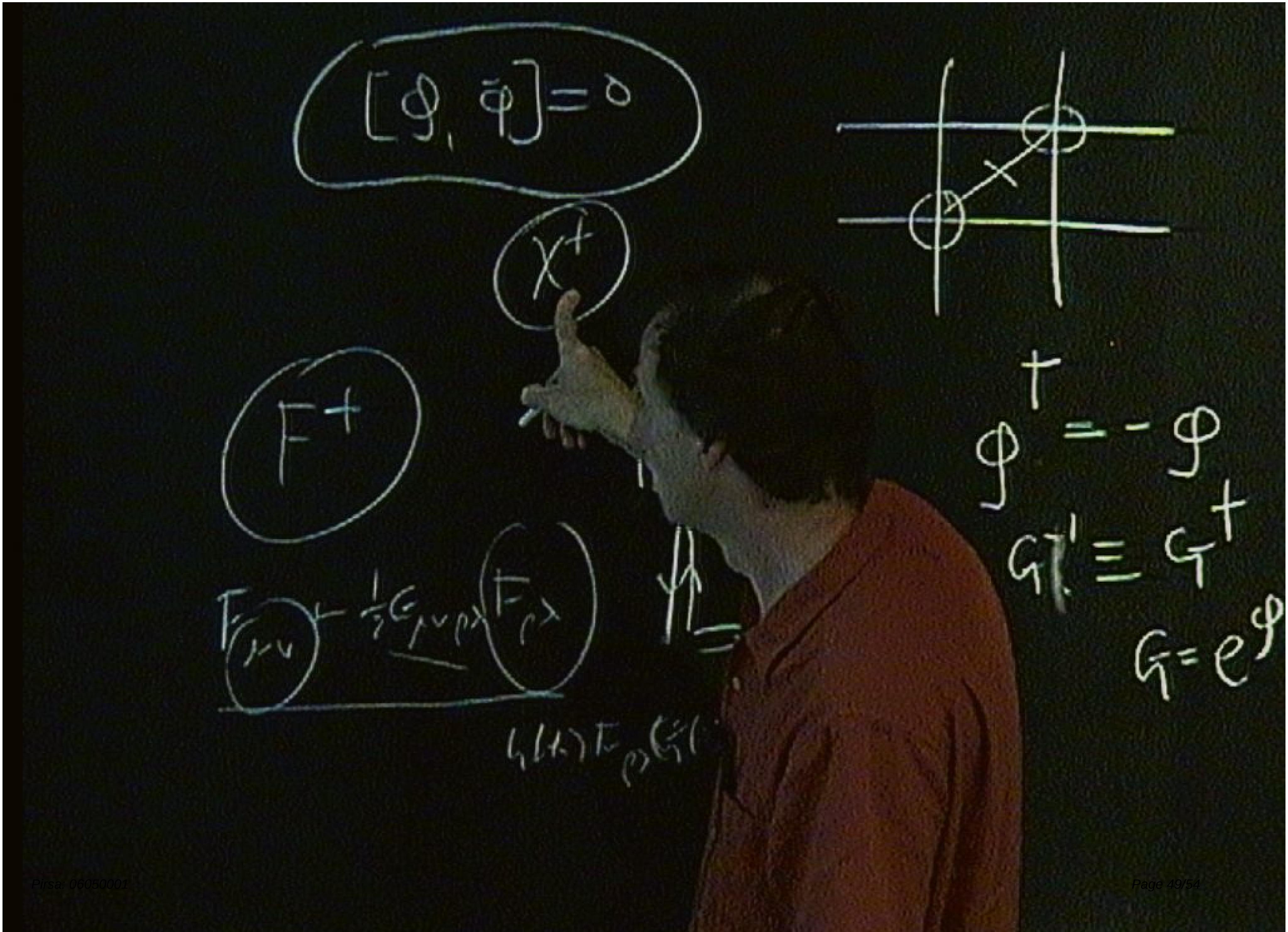
Carry out Q-variation and integrate out $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$

Dualize 3 and 4-form fields – connect to Marcus topological twist


$[\varphi, \bar{\varphi}] = 0$
F^+
$F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda}$

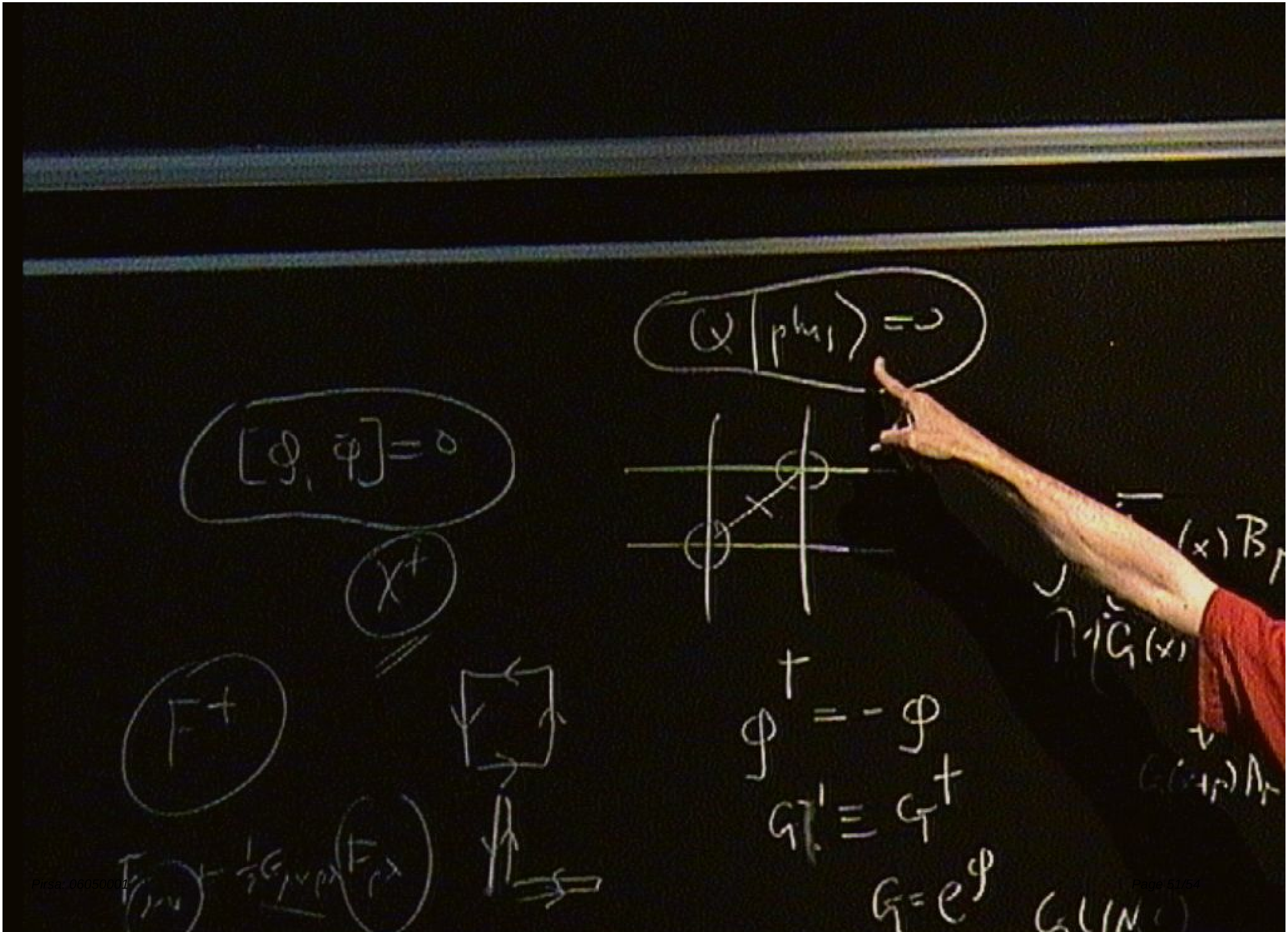

[φ, φ̄] = 0
F+
F_μν ← ½ ε_μνρλ F_ρλ




[φ, φ̄] = 0
χ⁺
F⁺
φ† = −φ
G' ≡ G†
G = e^φ

Conclusions

- Theories with extended SUSY (in Eucl. space) can be discretized while preserving G.I and a (twisted) SUSY
- Lattice theories are local and free of spectrum doubling.
- Discretization proceeds from reformulation in geometrical terms (Dirac-Kähler fields)
- For $D = 2$ evidence (σ-models, WZ models) that all supersymmetries recovered without fine tuning in continuum limit.
- Non-perturbative formulation – numerical simulation possible



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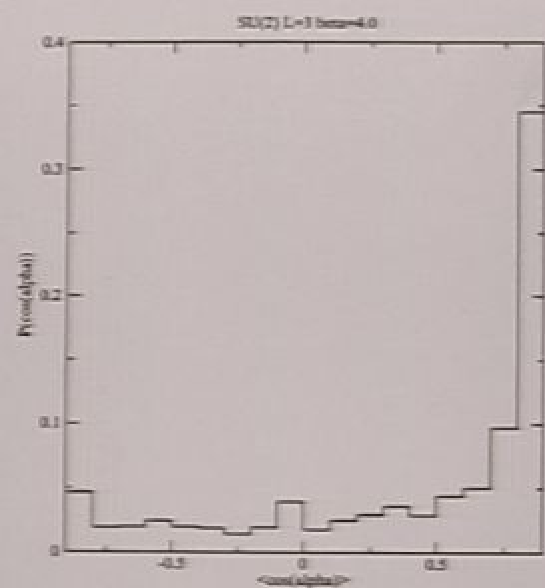
To do ...

- Check all Ward identities for $\beta \to \infty$
- Phase of Pfaffian at non-zero a ?
- Structure of moduli space as function of β and N and number of supercharges
- Numerical simulation of $\mathcal{N} = 4$ SYM in $D = 4$?
- Formalism allows for $\mathcal{N} = 2, 4, 8$ in $D = 2$ and $\mathcal{N} = 4, 8$ in $D = 3$.
- Explore connection to supergravity. Use temporal a.p.b.c for fermions and explore t'Hooft limit $N \to \infty$ with $\lambda = \frac{\beta}{N}$ fixed.



Distribution of $\cos\alpha$
Peaks at $\alpha = 0$