Title: Making sense of non-Hermitian Hamiltonians

Date: Apr 13, 2006 11:00 AM

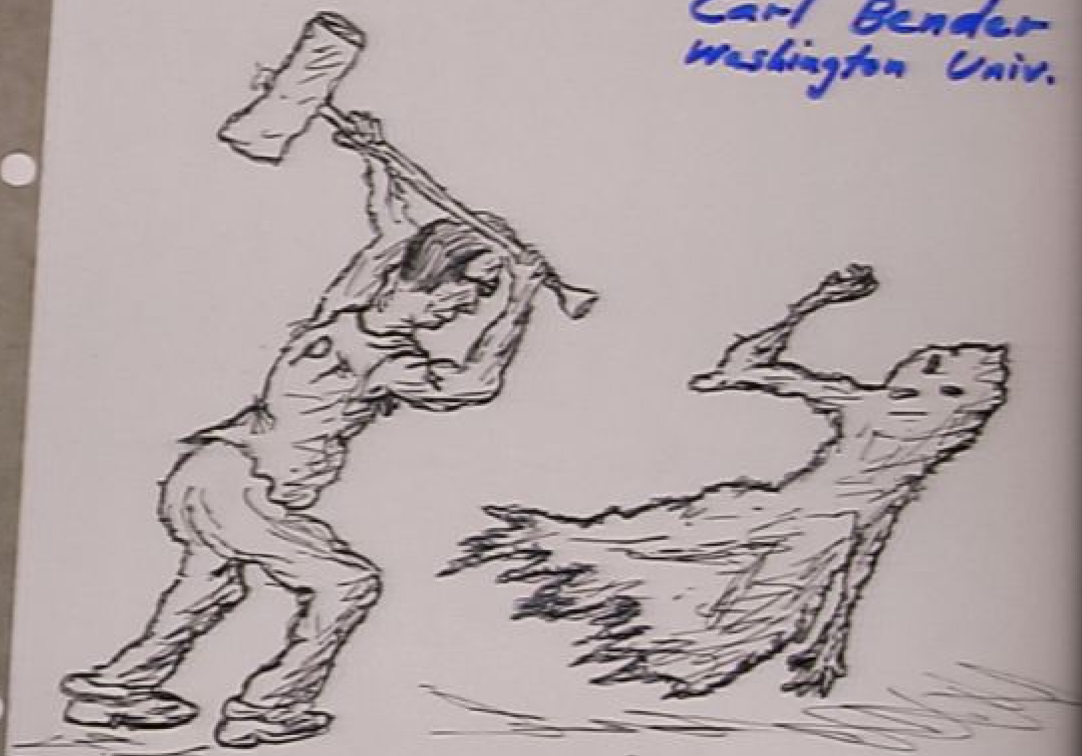
URL: http://pirsa.org/06040016

Abstract: It is a standard axiom of quantum mechanics that the Hamiltonian H must be Hermitian because Hermiticity guarantees that the energy spectrum is real and that time evolution is unitary. In this talk we examine an alternative formulation of quantum mechanics in which the conventional requirement of Hermiticity is replaced by the more general and physical condition of space- time reflection (PT) symmetry. We show that if the PT symmetry of H is unbroken, Then the spectrum of H is real. Examples of PT-symmetric non-Hermitian Hamiltonians are $H=p^2+ix^3$ and $H=p^2-x^4$. Amazingly, the energy levels of these Hamiltonians are all real and positive despite the ``wrong" sign in the x^4 potential! We show that such PT-symmetric Hamiltonians specify physically acceptable quantum-mechanical theories in which the norms of states are positive and time evolution is unitary. To do so we demonstrate that a Hamiltonian that has an unbroken PT symmetry also possesses a new physical symmetry that we call C. Using C, we construct an inner product whose associated norm is positive definite. The result is a new class of consistent complex quantum theories. In effect, we have extended and generalized quantum mechanics into the complex domain. We then discuss PT-symmetric quantum field theories. PT-symmetric scalar field-theoretic Hamiltonians corresponding to the above quantum-mechanical Hamiltonains have interaction terms $ig\phi^3$ and $-g\phi^4$. The latter Theory is interesting because (1) it is asymptotically free and (2) the expectation value of ϕ is nonzero. (Thus, such a theory might be useful in describing the Higgs sector.) PT symmetry resolves the long-standing problem of ghosts in the Lee model. When the renormalized coupling constant in this model increases past a critical value, the Hamiltonian ceases to be Hermitian and a negative-norm ghost state appears. At this transition the Hamiltonian becomes PT-symmetric, and the ghost is a physical particle. PT-symmetric QED and the PT-symmetric massive Thirring model will also be discussed. Finally, we mention recent papers which suggest that PT-symmetry may provide insight into cosmological problems.

GHOST BUSTING —

Making sense of non-Hermitian Hamiltonians

Carl Bender
Washington Univ.



Examples:

$$H = p^2 + ix^3$$

$$H = p^2 - x^4$$

$$H = \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}\pi^2 + \frac{\mu^2}{2}\varphi^2 + ig\varphi^3$$

$$H = \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}\pi^2 + \frac{\mu^2}{2}\varphi^2 - g\varphi^4$$

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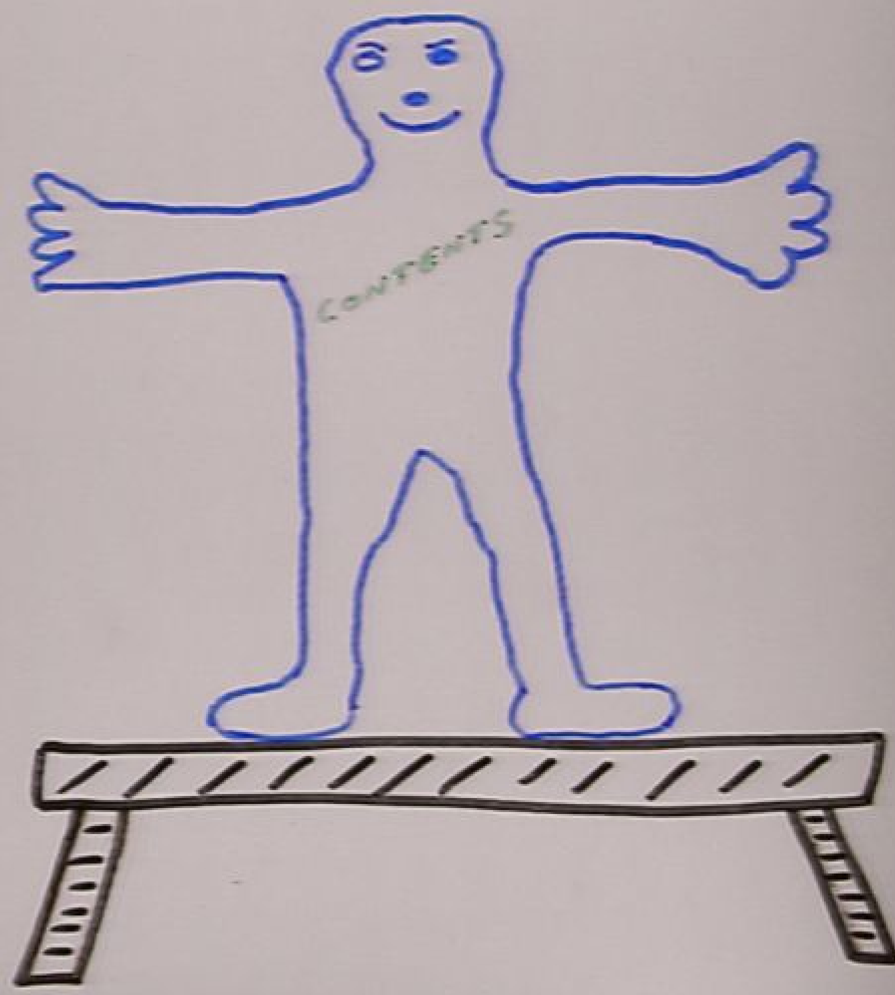
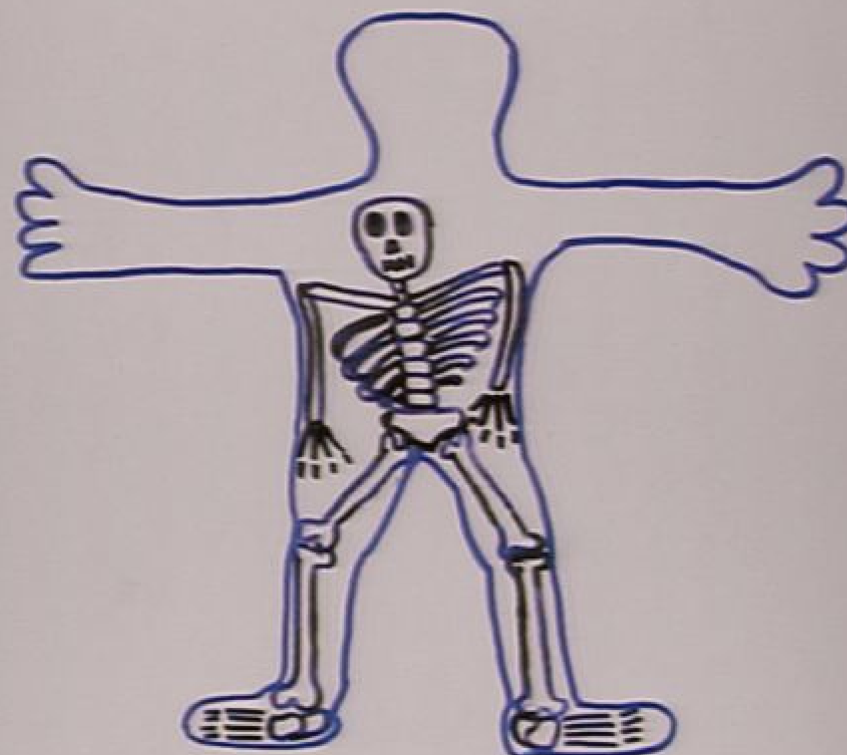


Contents



Contents may have settled during shipping.

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Rogues' Gallery
JONES
BERRY
MEISINGER
MILTON
BOETTCHER
The PT BUNCH
DUNNE
SAVAGE
WANG

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Dunning
Tateo
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Pham
Delabaere
Shin
Kleefeld
Cerveró
Levai
Mielnik
Mostafazadeh
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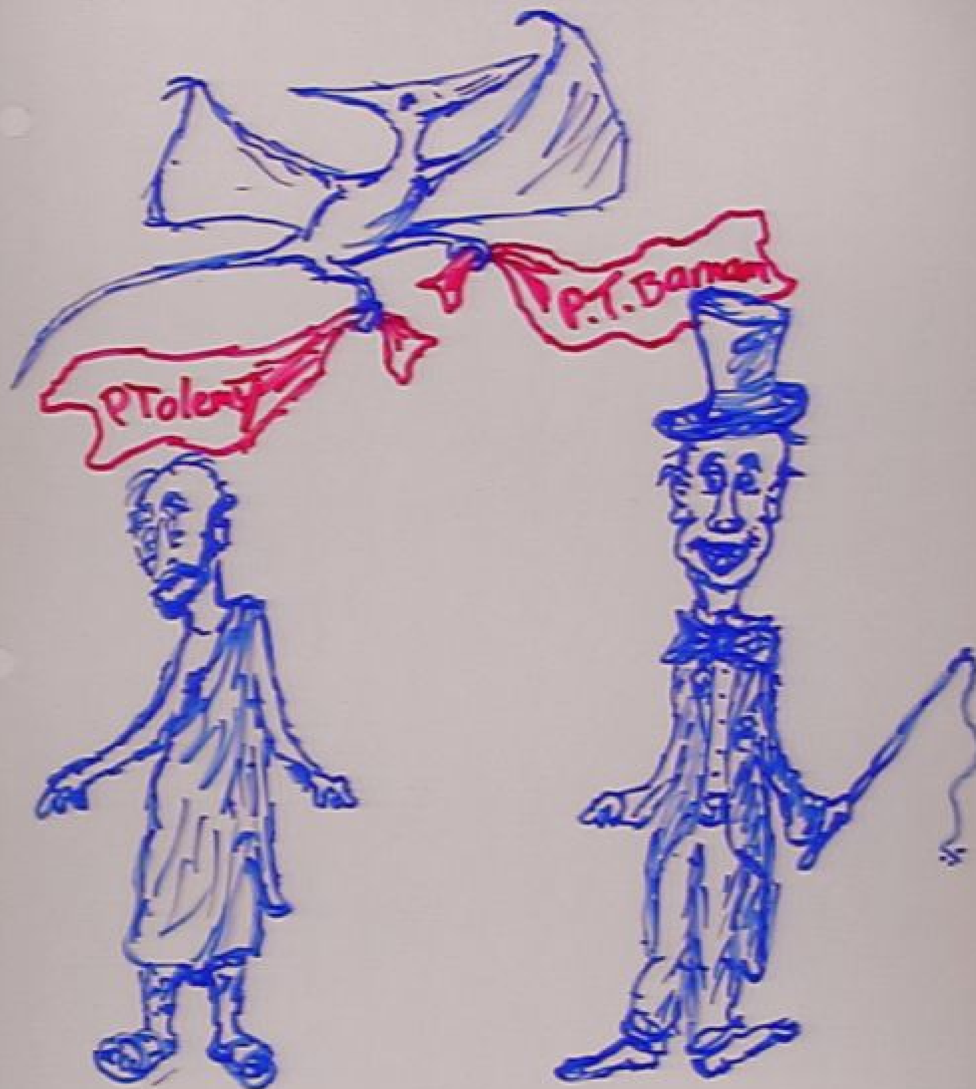
Sinha
Roy
Bagchi
Weigert
Tretter
Scolarici
Navratil
Langer
Jakubsky
Geyer
Bila
Bohacik
Caliceti
Guenther
Heiss
Handy
Graffi
Sjöstrand
⋮

et al! — with apologies



THE ORIGINAL DISCOVERERS OF PT SYMMETRY

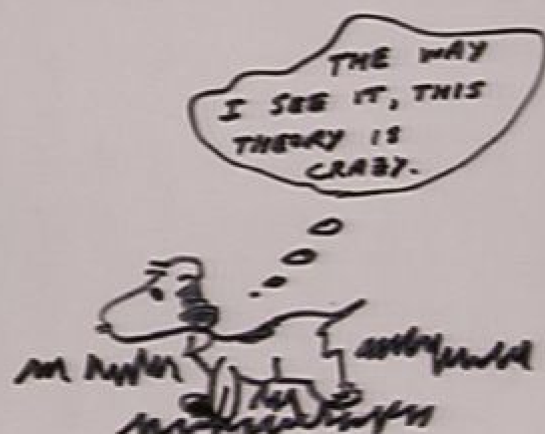
(died from PTomaine poisoning)

HISTORY OF IDEA

$$H = p^2 + ix^3$$

Conjecture by D. Bessis and J. Zinn-Justin: spectrum real

$$H = p^2 + ix^3 \quad (?!)$$



WAIT A MINUTE...

$H = p^2 + ix^3$

EXHIBITS

PT

SYMMETRY

Symmetric under

$PT: \; x \to -x$

$i \to -i$

$p \to p$

parity (space reflection)

time reversal

Idea:

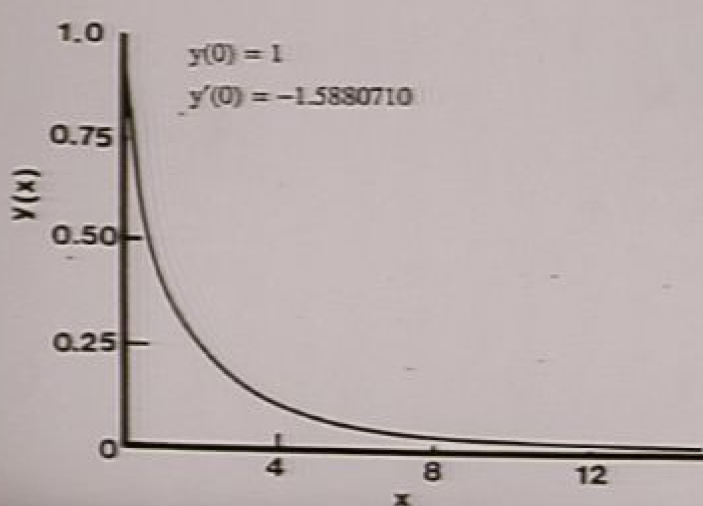
Try $H = p^2 + x^2 (ix)^\epsilon$

because $\mathcal{PT}$-symmetric

for all ϵ real

THOMAS-FERMI EQUATION

$$y'' = \frac{y^{3/2}}{\sqrt{x}}, \quad y(0)=1, \quad y(\infty)=0$$



THOMAS-FERMI EQUATION

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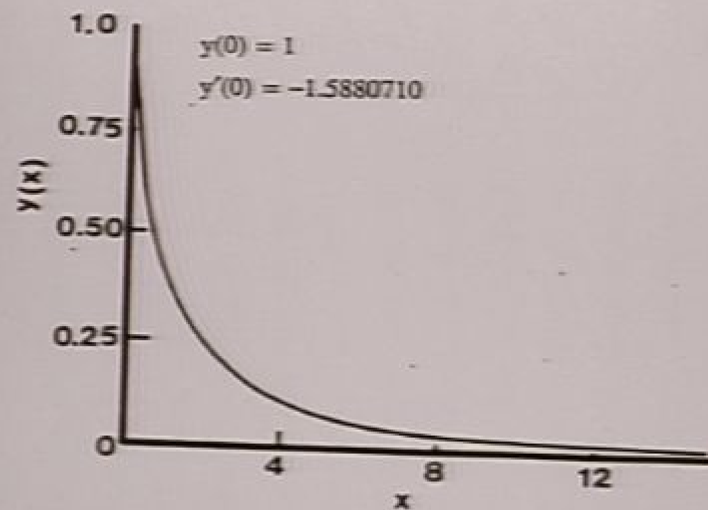


Expand in powers of nonlinearity:

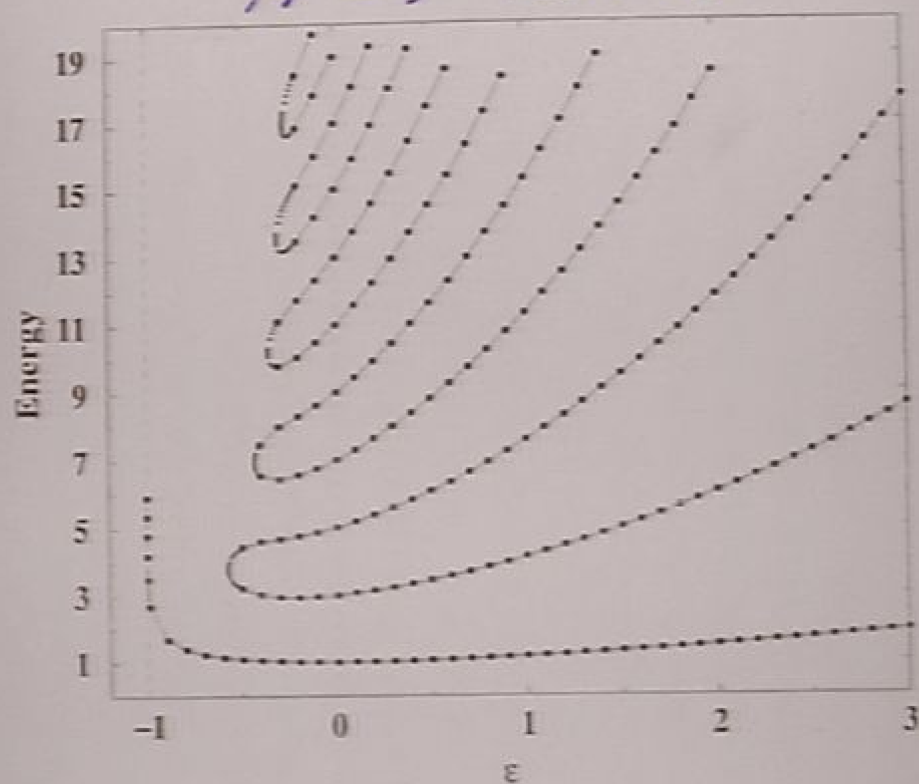
$$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$$

$$y = \sum_{n=0}^{\infty} y_n \epsilon^n$$

$$y_0 = e^{-x}$$

Set $\epsilon = \frac{1}{2}$ at end of calculation

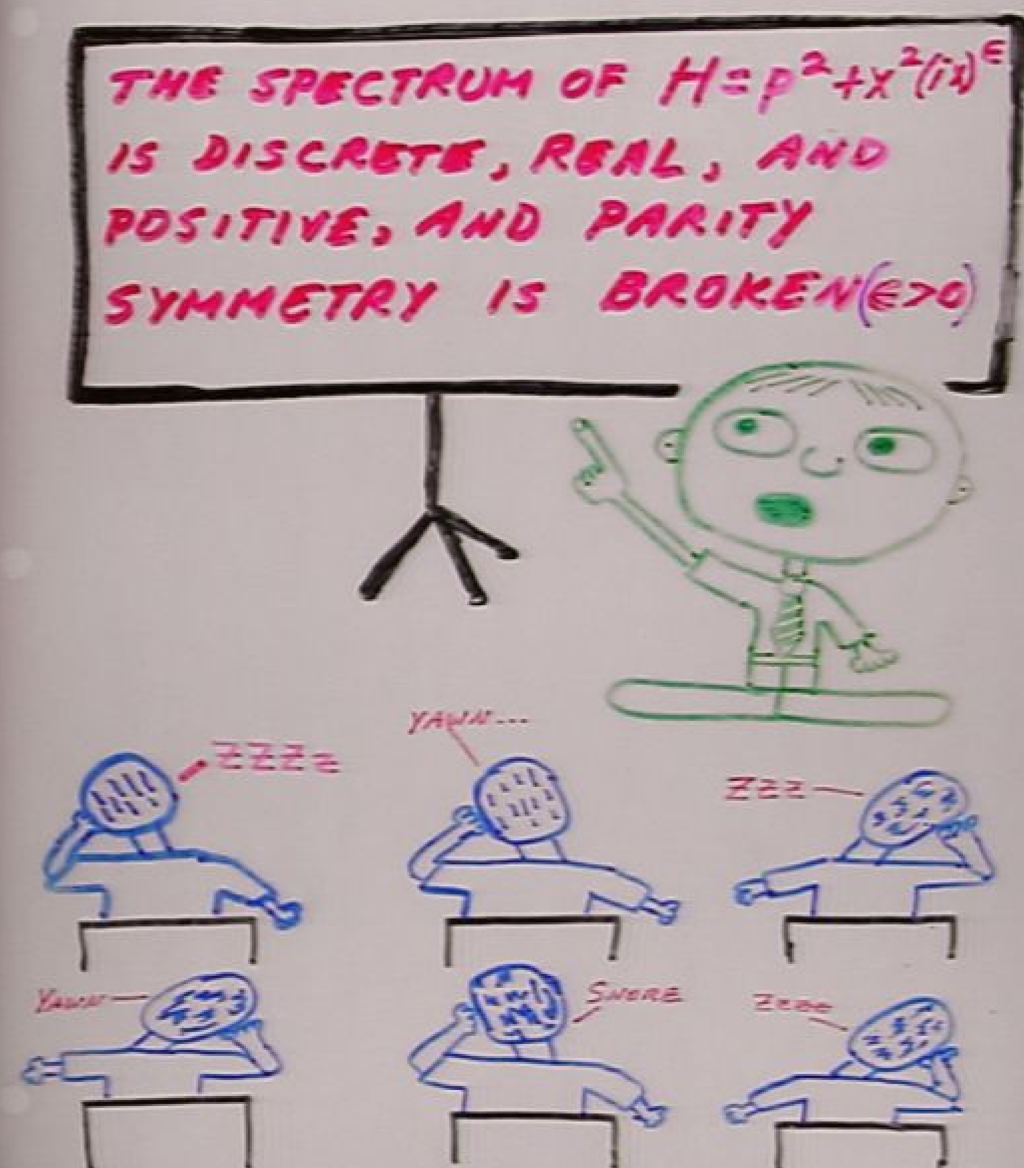
$$H = p^2 + x^2(ix)^\epsilon$$

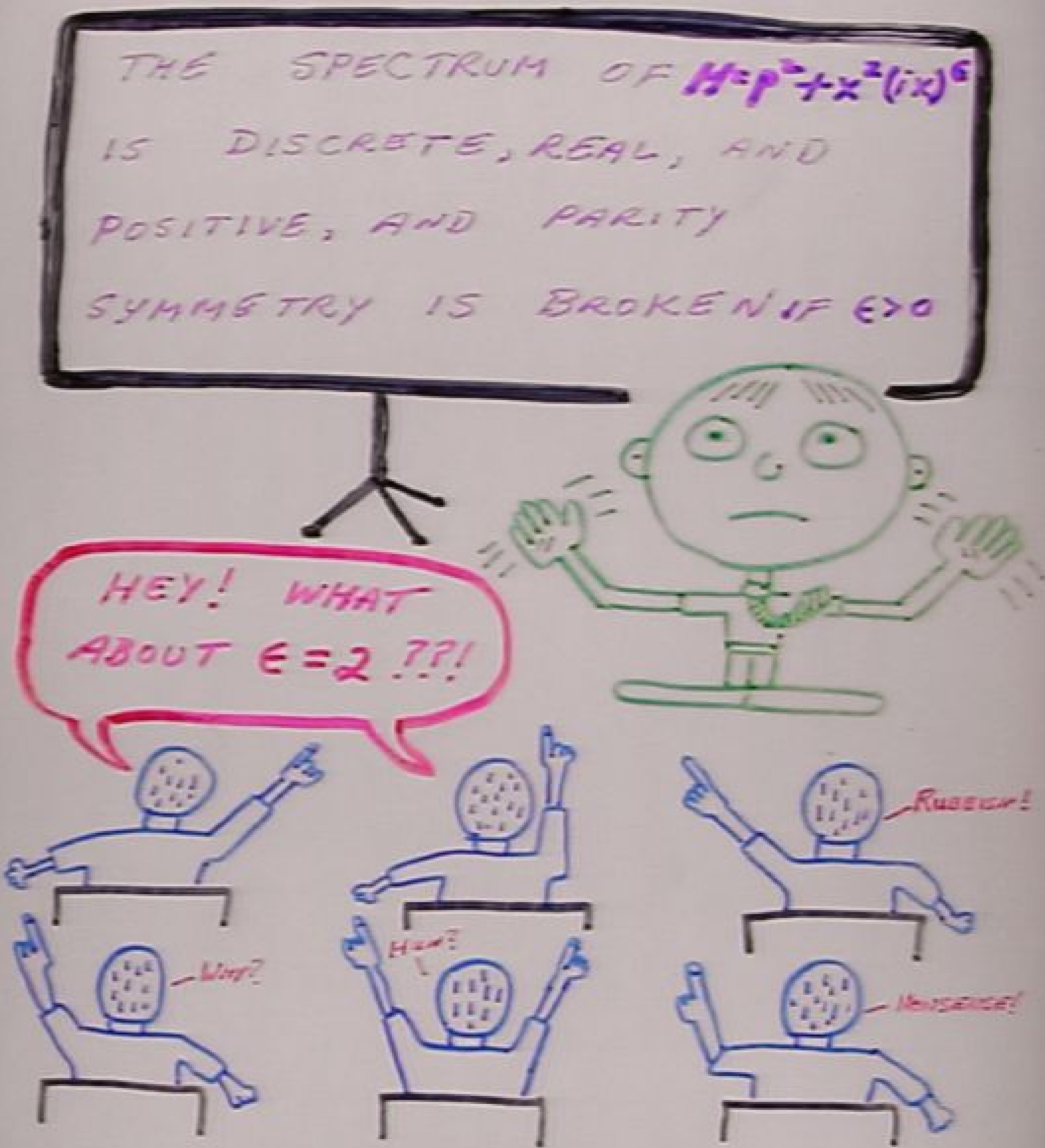


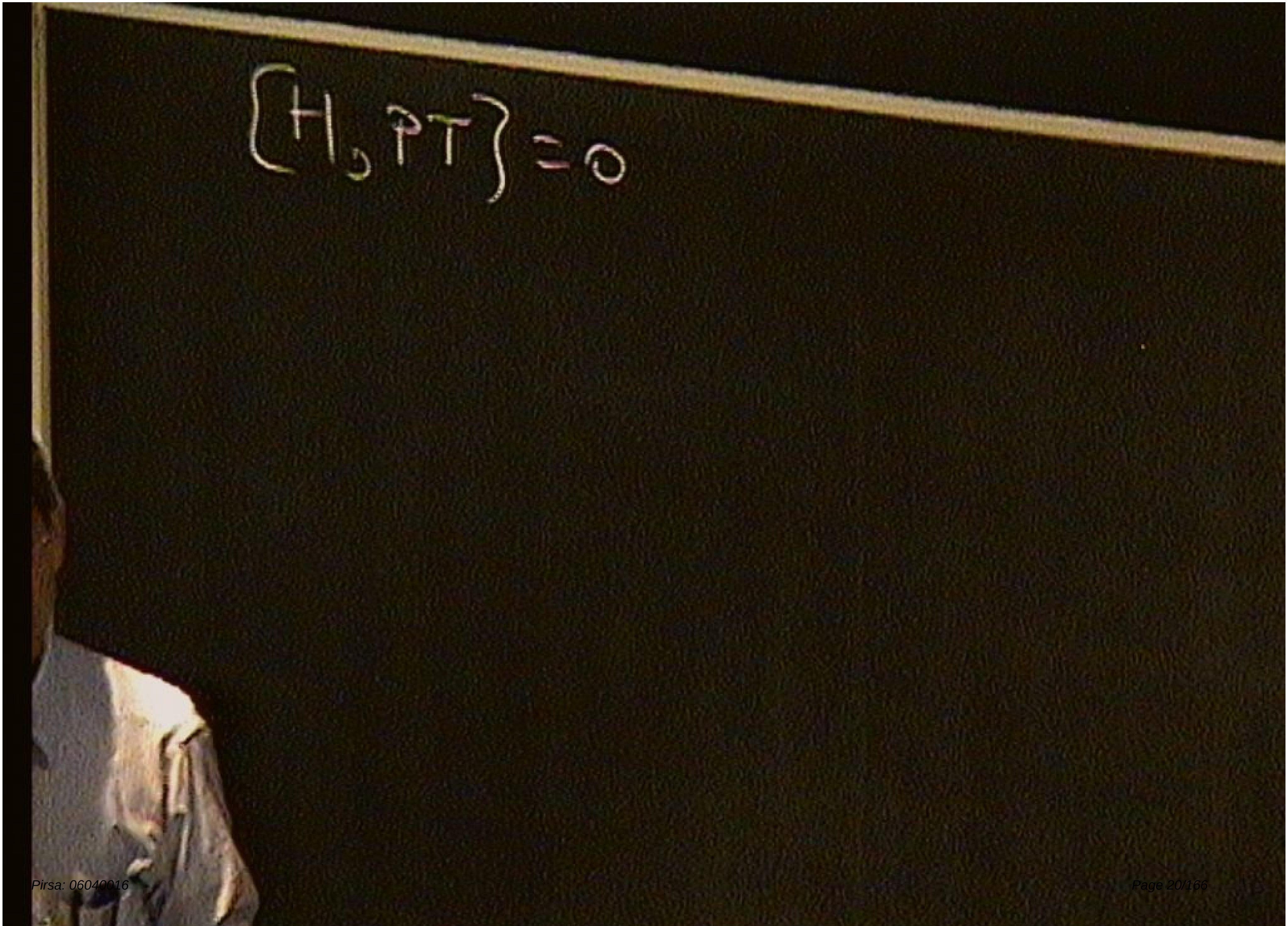
at $\epsilon = 0$

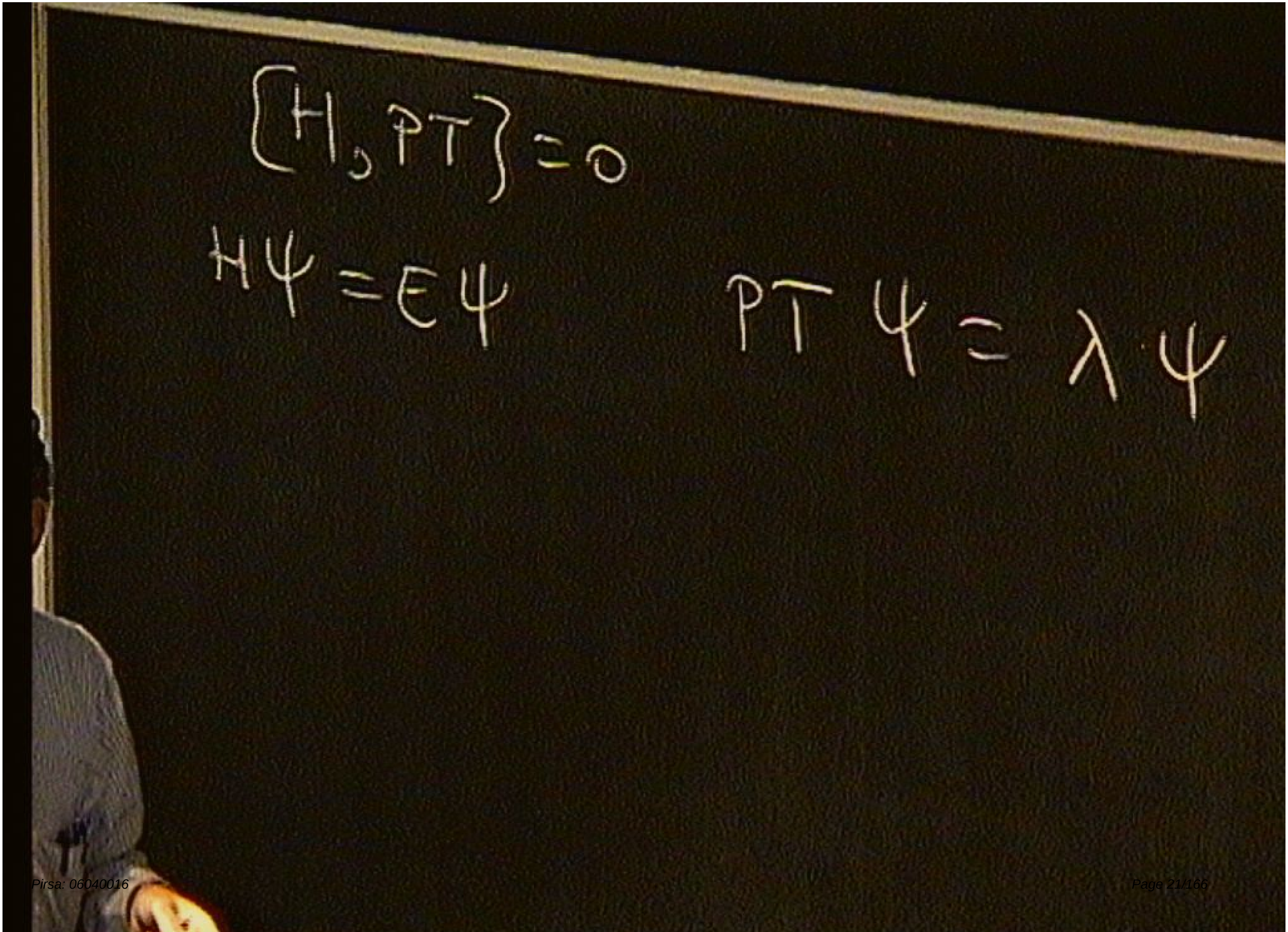
$H = p^2 + x^2$

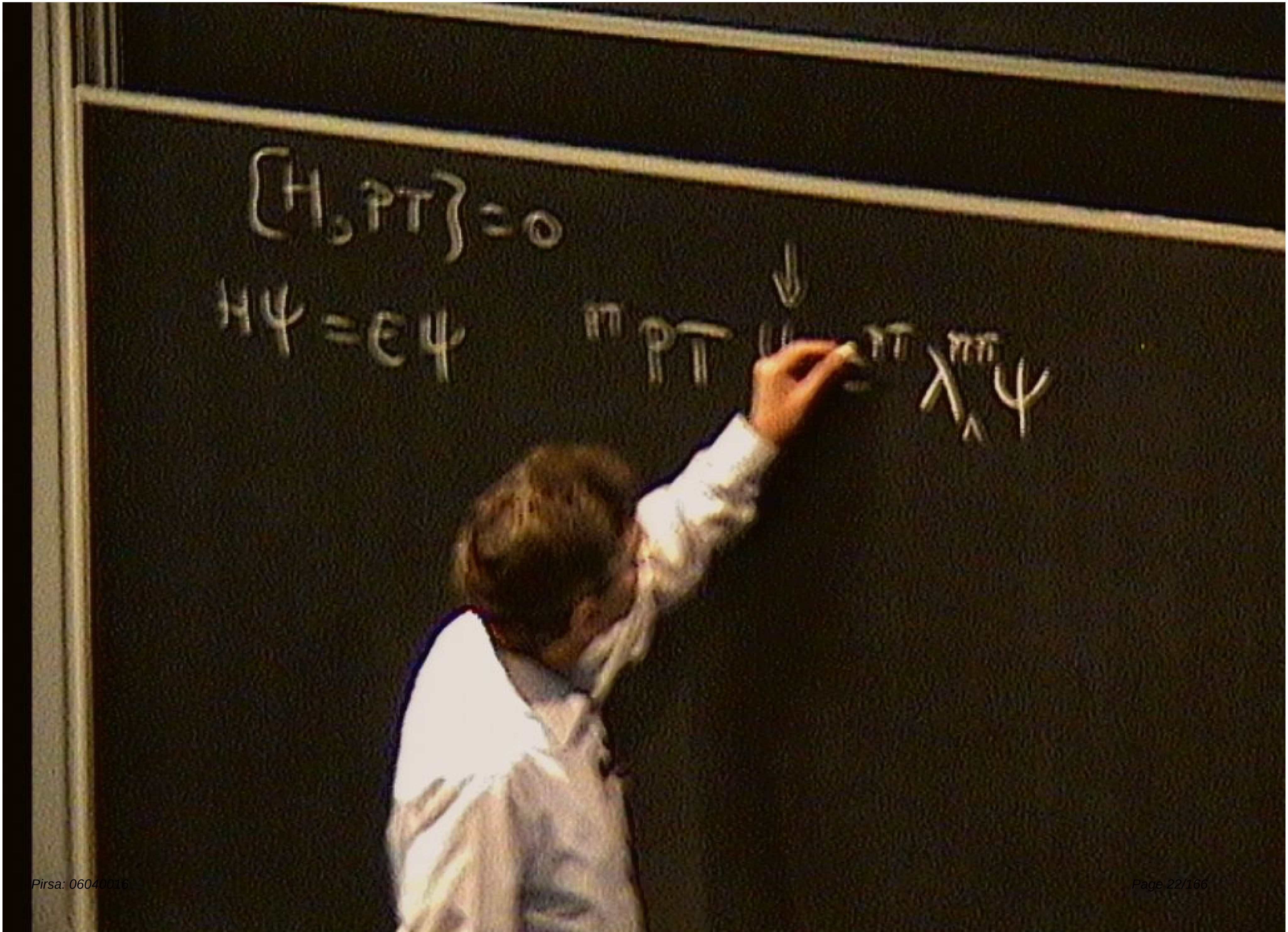
$E_n = 2n+1$

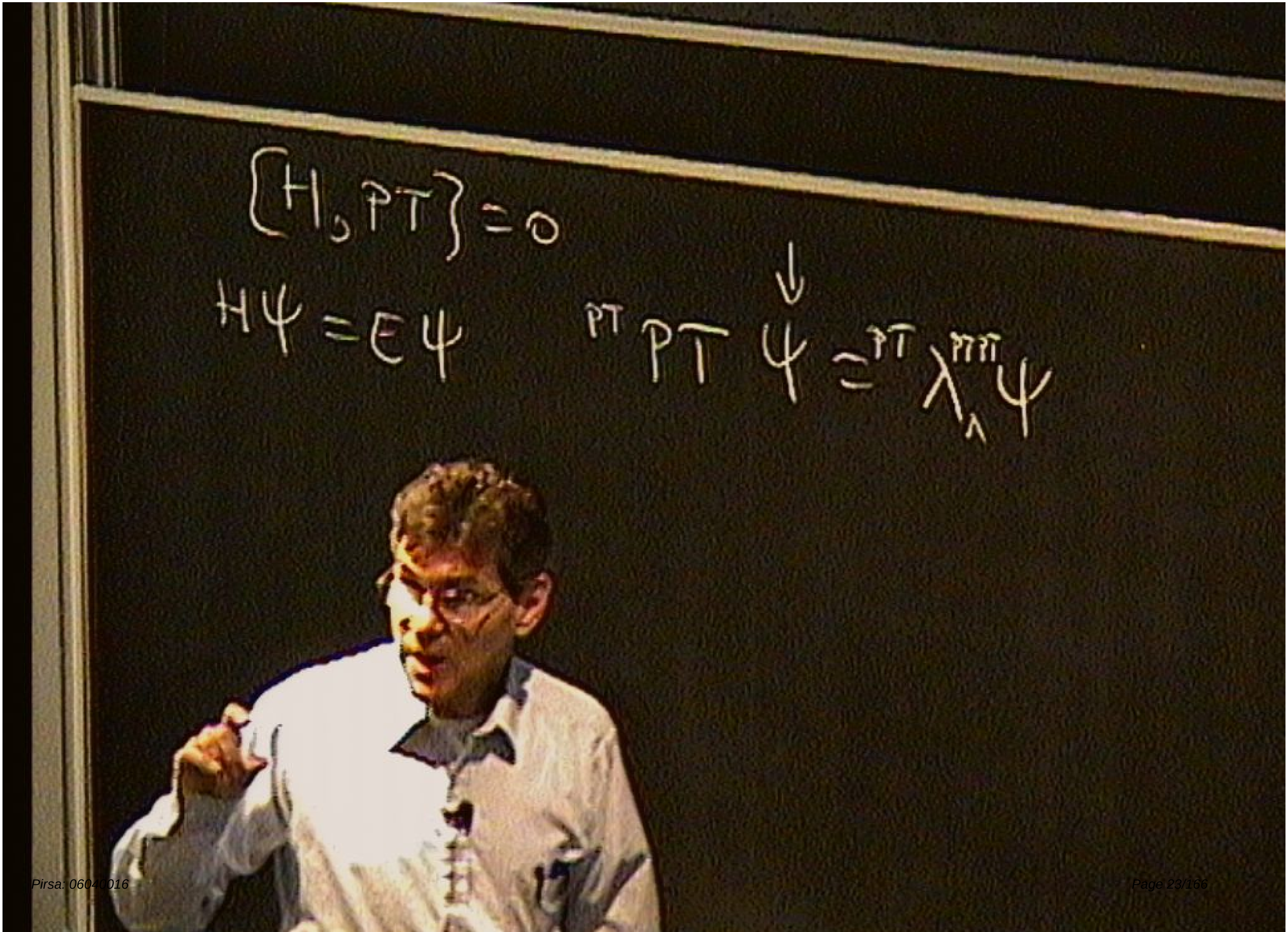

THE SPECTRUM OF $H=p^2+x^2(ix)^\epsilon$
IS DISCRETE, REAL, AND
POSITIVE, AND PARITY
SYMMETRY IS BROKEN ($\epsilon>0$)
ZZZZ
YAWN...
ZZZ
YAWN
SNORE

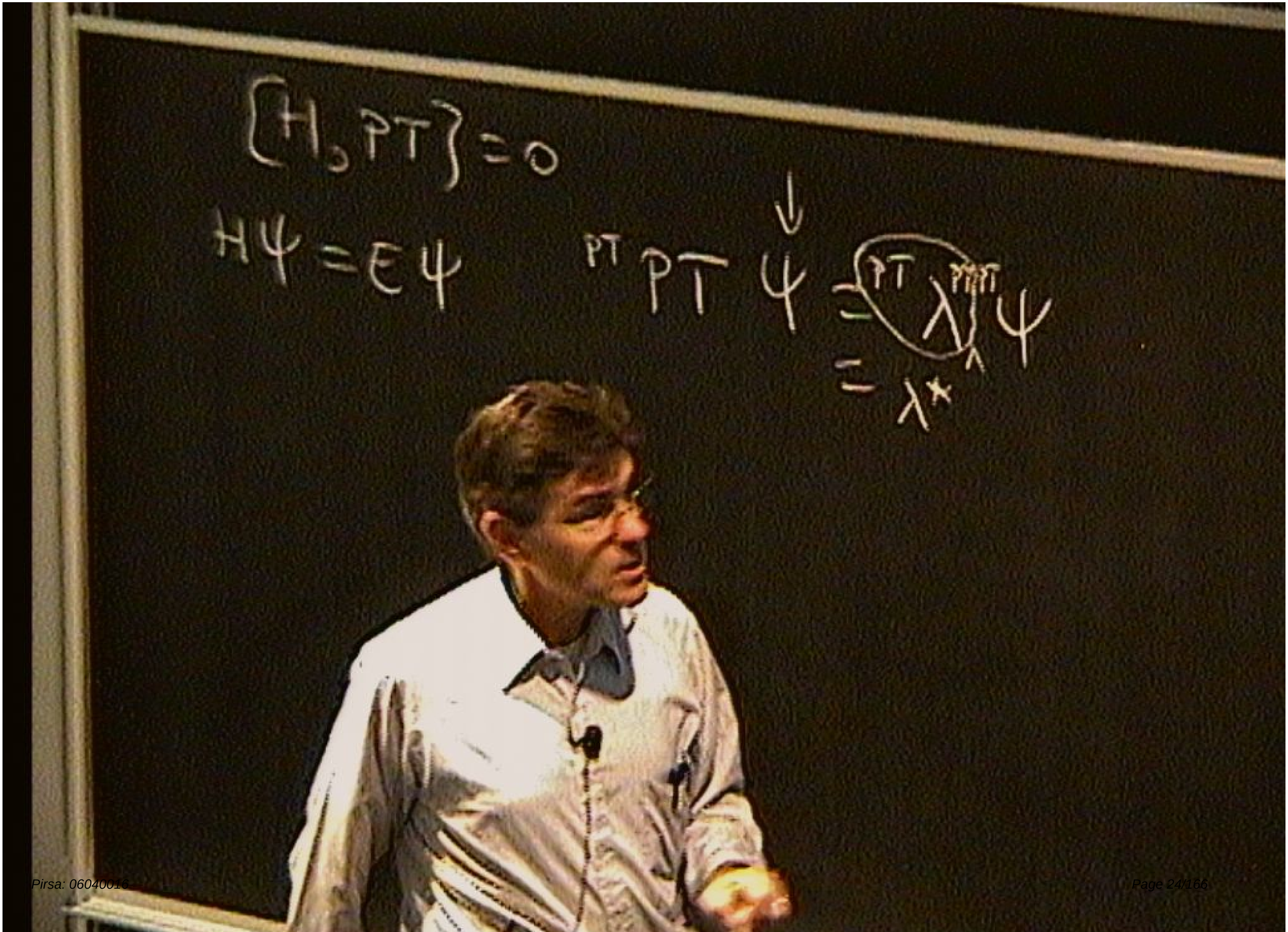

THE SPECTRUM OF $H=p^2+x^2(ix)^\epsilon$
IS DISCRETE, REAL, AND
POSITIVE, AND PARITY
SYMMETRY IS BROKEN IF $\epsilon>0$
HEY! WHAT ABOUT $\epsilon=2$???!
RUBBISH!
WHY?
HOW?
NONSENSE!

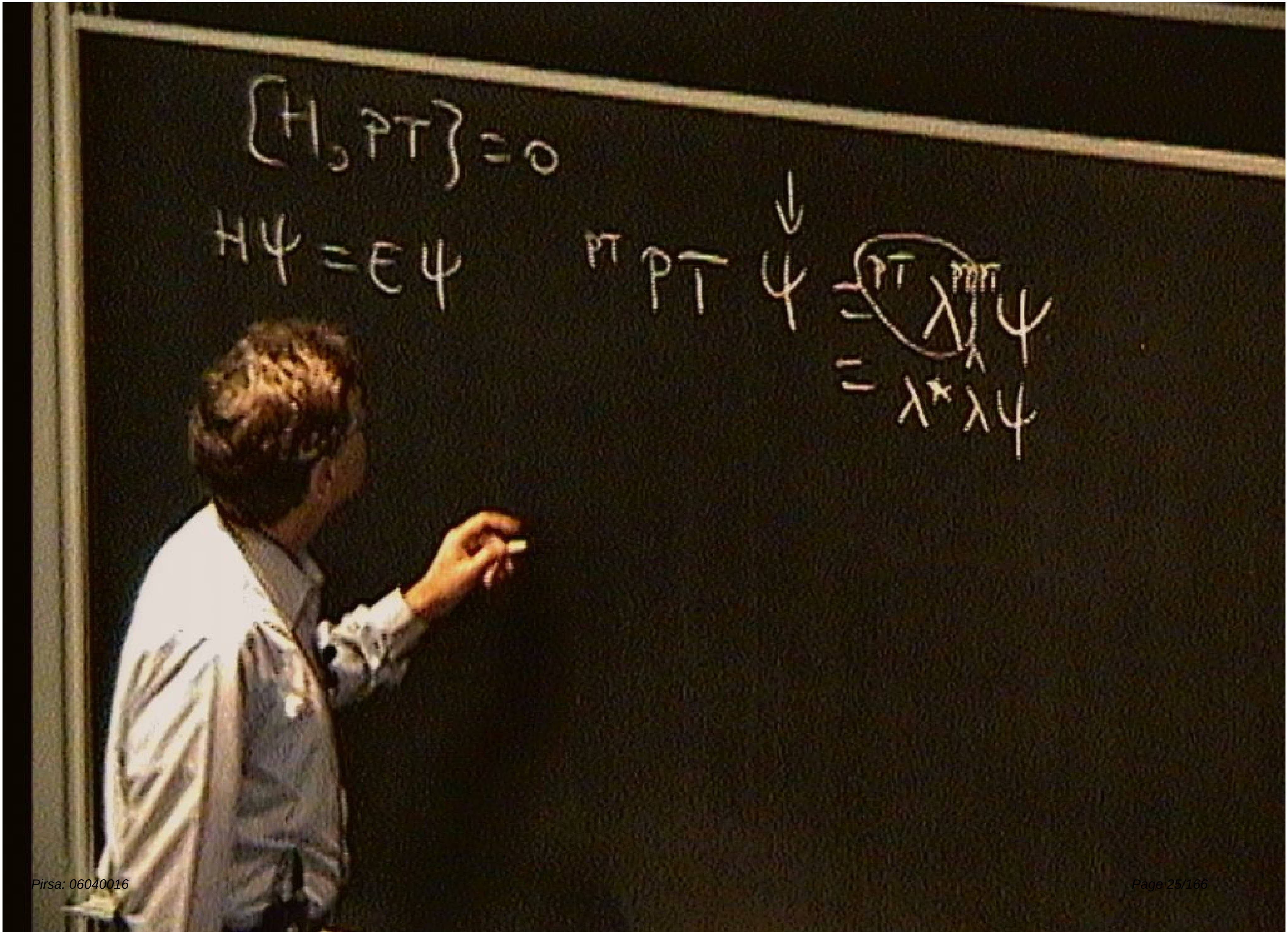

{H, PT} = 0

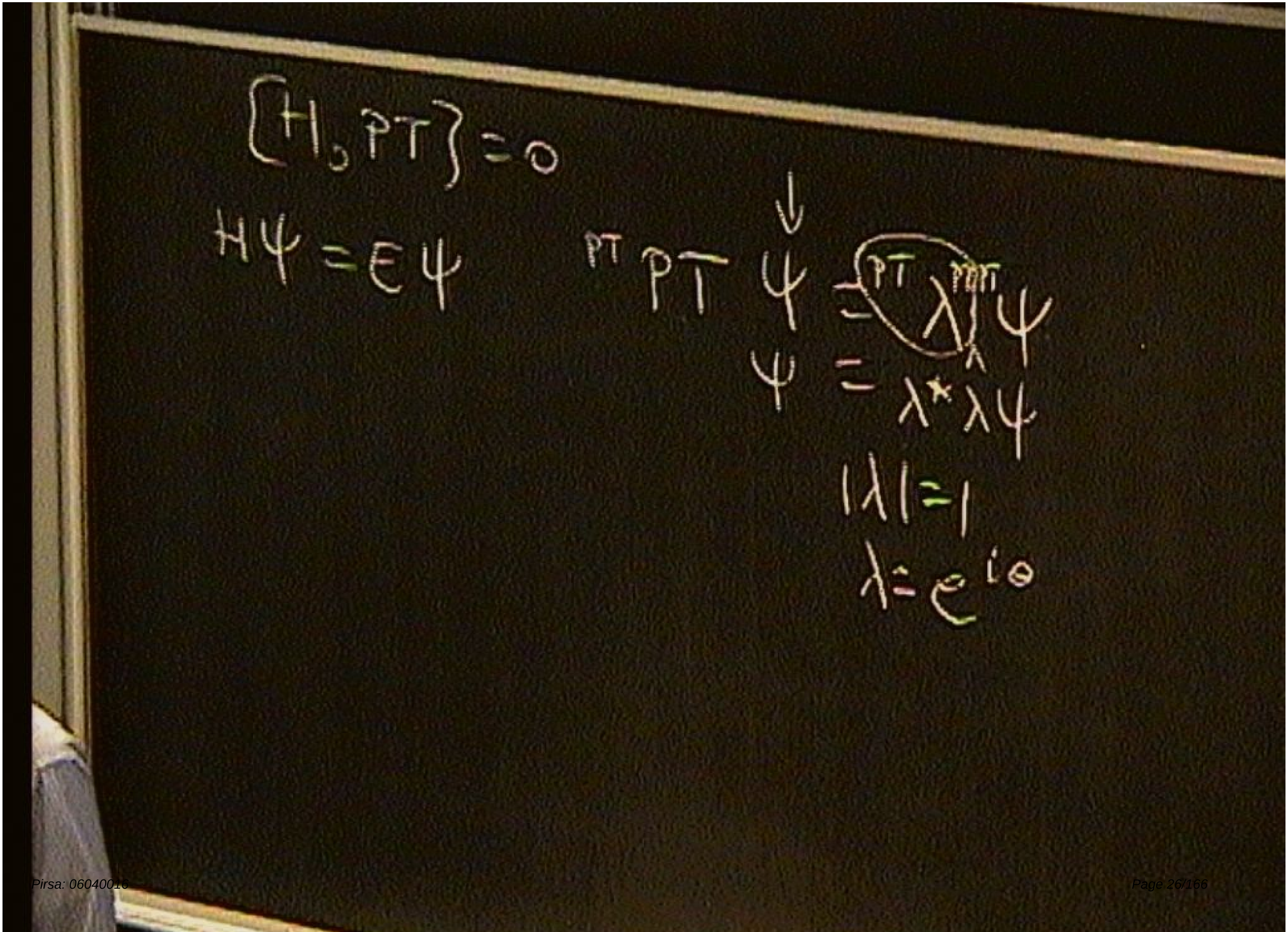

[H, PT] = 0
Hψ = Eψ
PT ψ = λ ψ


[H, PT] = 0
Hψ = Eψ


[H, PT] = 0
Hψ = Eψ


[H, PT] = 0
Hψ = Eψ
PT PT ψ = PT λ ψ
= λ*


[H, PT] = 0
Hψ = Eψ
PT PT ψ = PT λ ψ
= λ* λ ψ



$[H, PT] = 0$
$H\psi = E\psi$
$PT\,PT\,\psi = PT\,\lambda\,\psi$
$\psi = \lambda^* \lambda \psi$
$|\lambda| = 1$
$\lambda = e^{i\theta}$

$[H, PT] = 0$

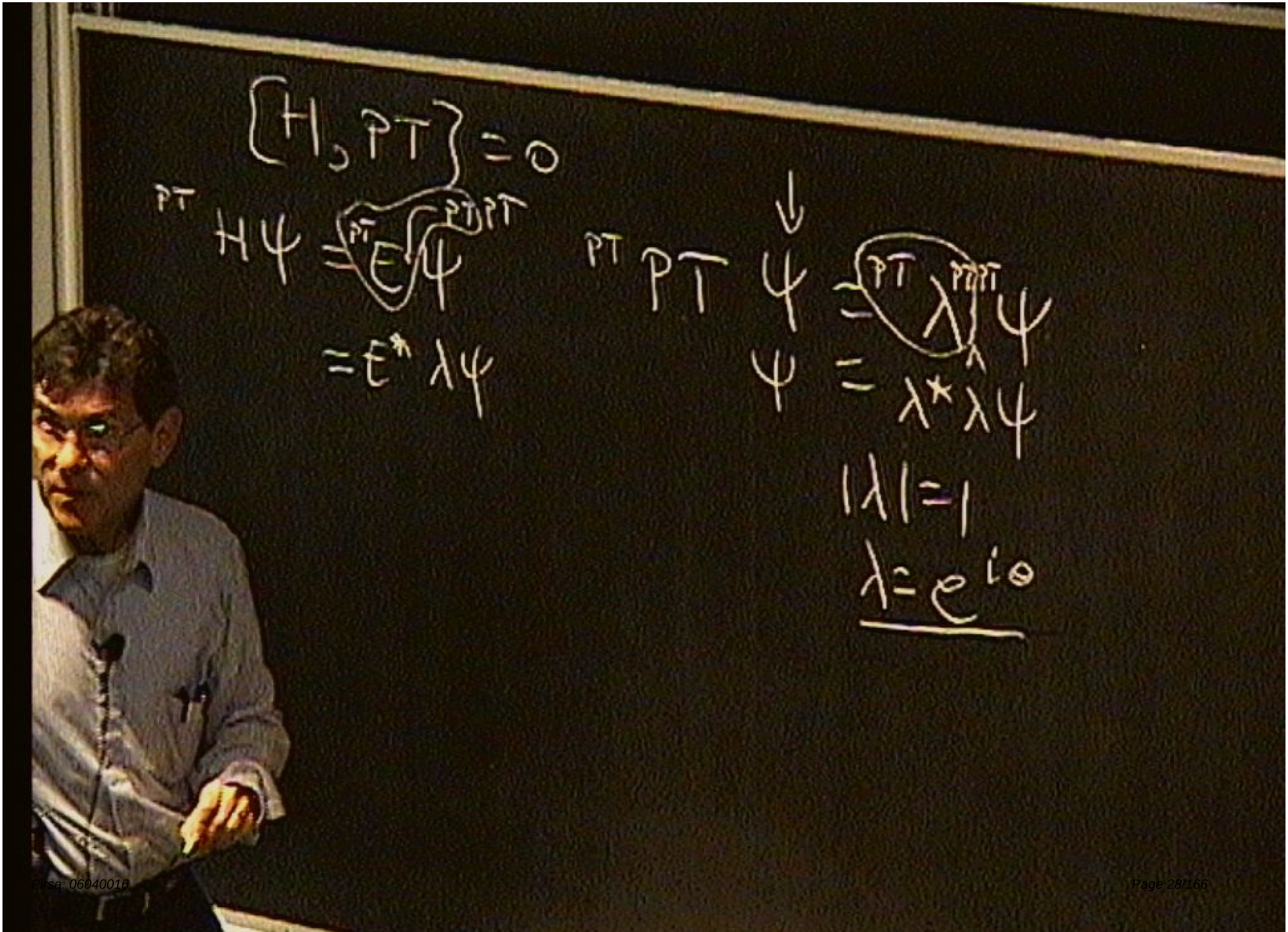
$H\psi = E\psi$

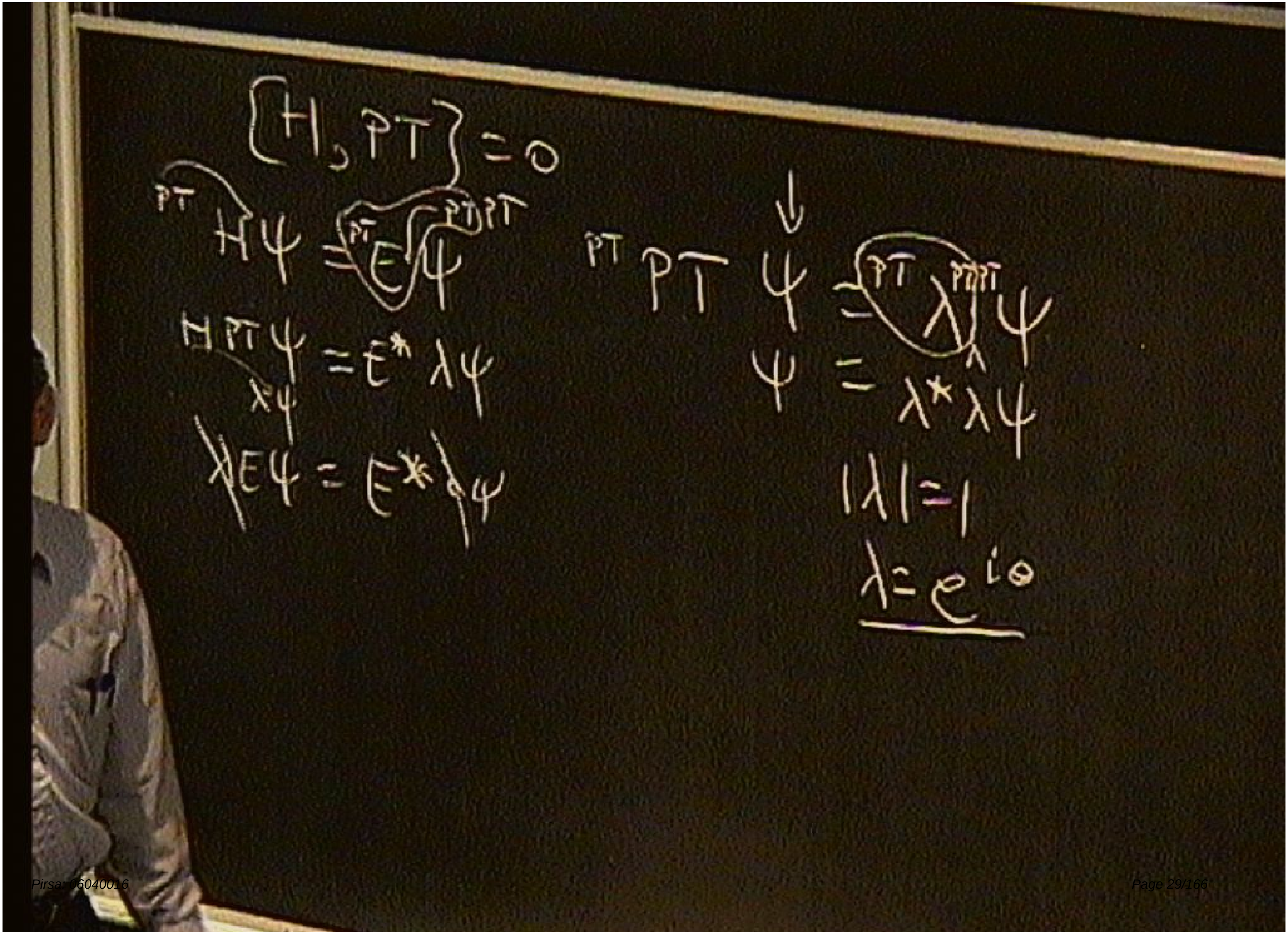
$PT\,PT\,\psi = PT\,\lambda\,\psi$

$\psi = \lambda^* \lambda \psi$

$|\lambda| = 1$

$\lambda = e^{i\theta}$


[H, PT] = 0
Hψ = Eψ
= E* λψ
PT ψ = λψ
ψ = λ* λψ
|λ| = 1
λ = e^{iθ}


[H, PT] = 0
Hψ = Eψ
H PT ψ = E* λψ
λEψ = E* λψ
PT ψ = λψ
ψ = λ*λψ
|λ| = 1
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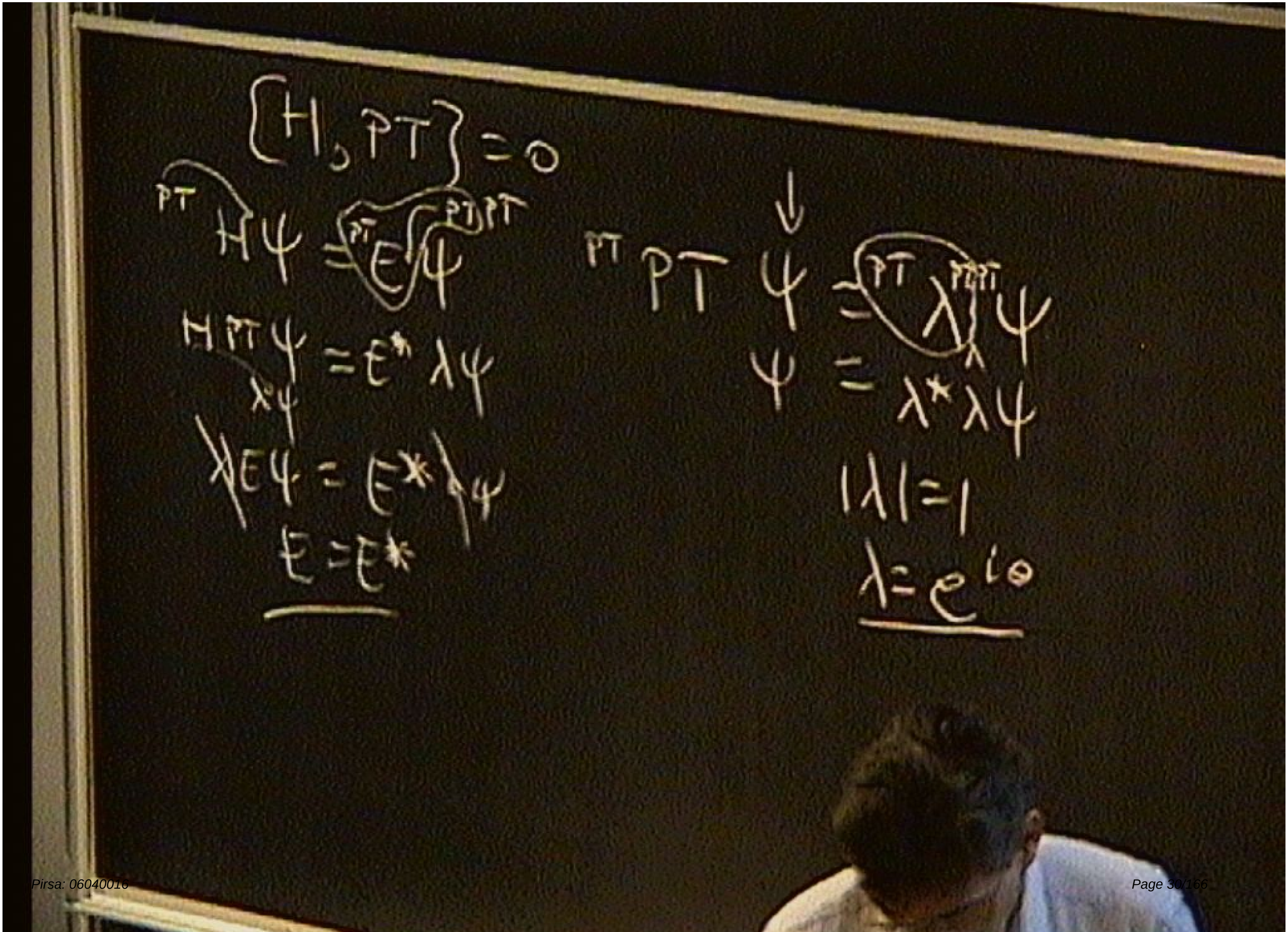
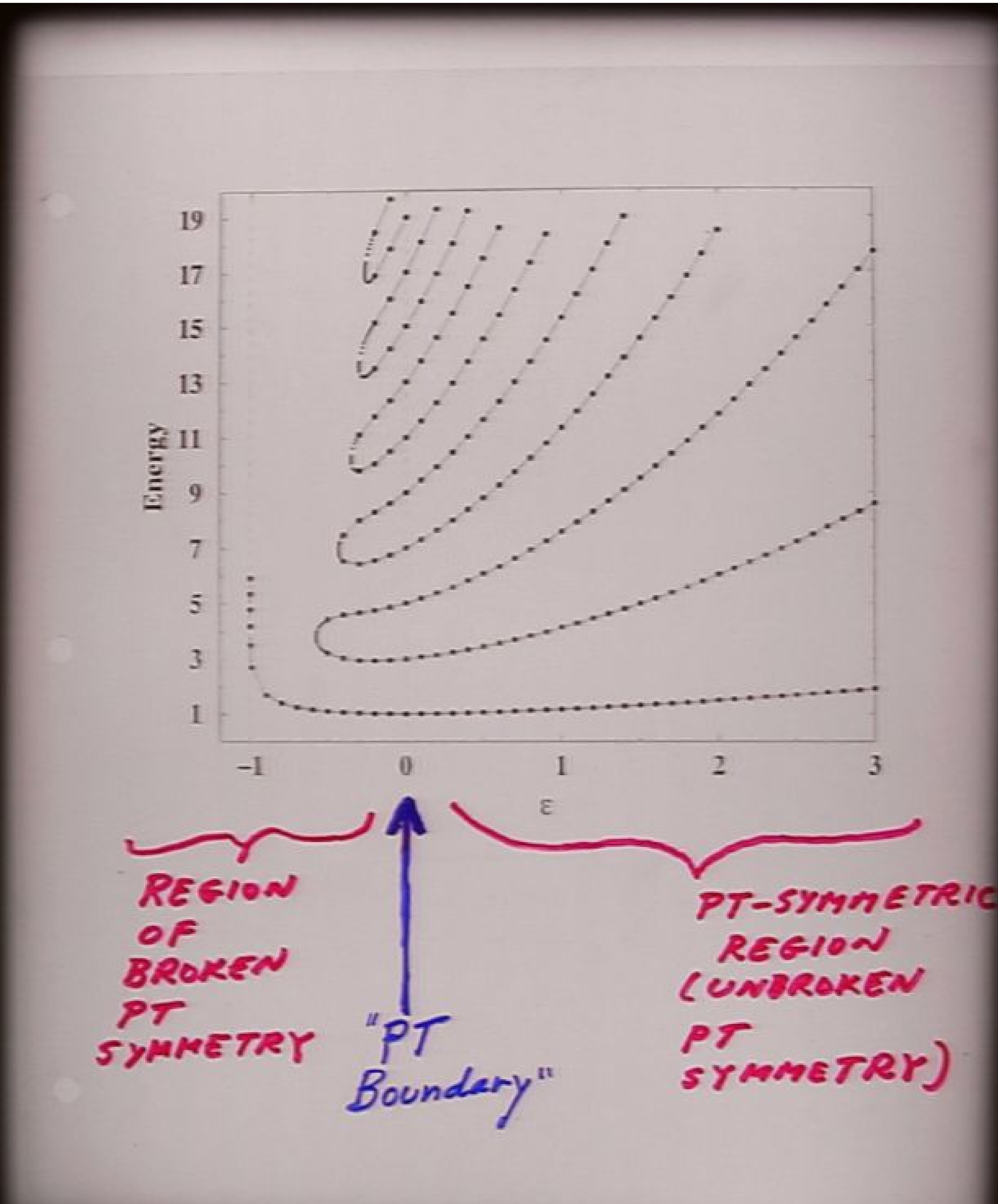
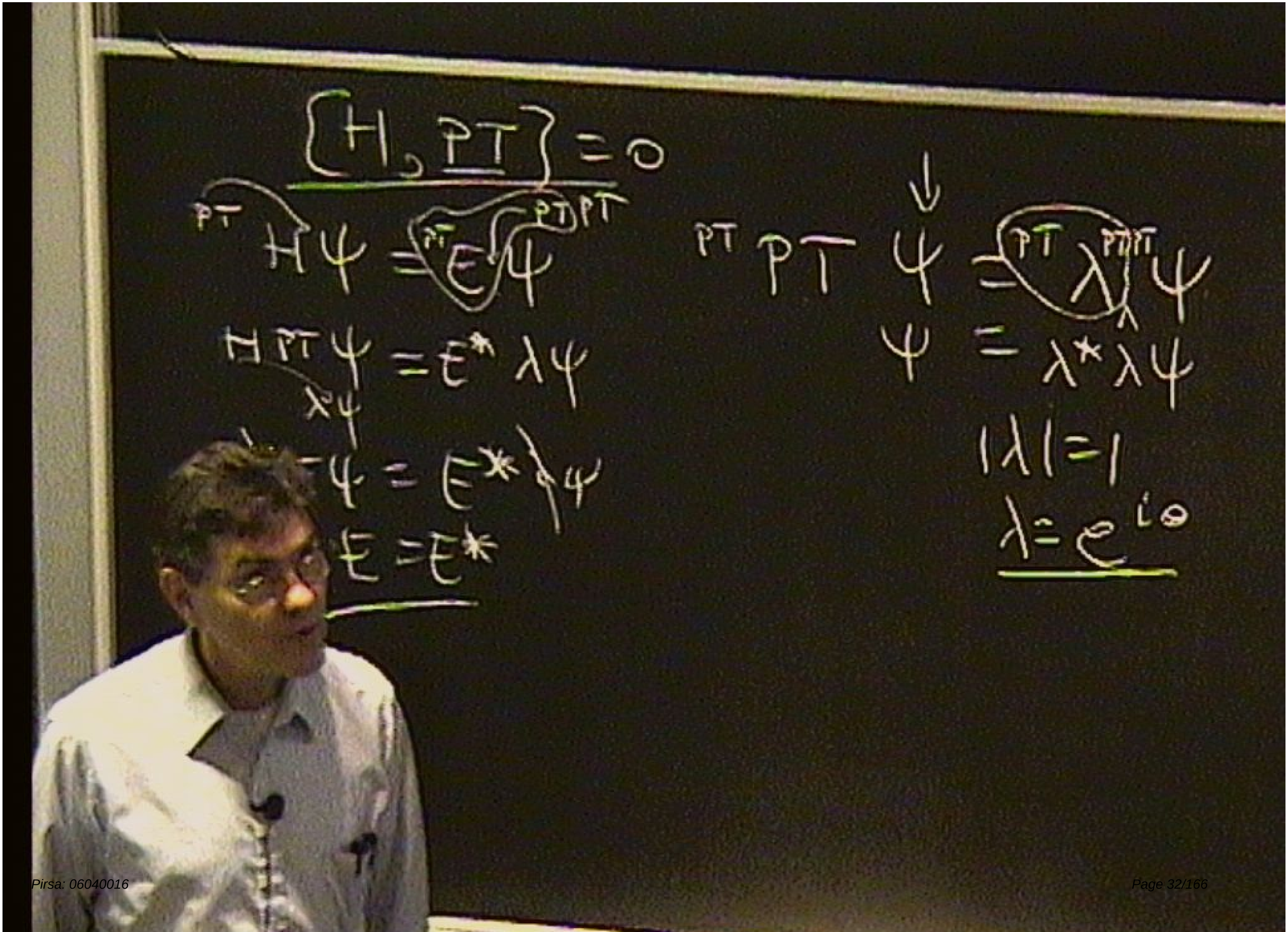
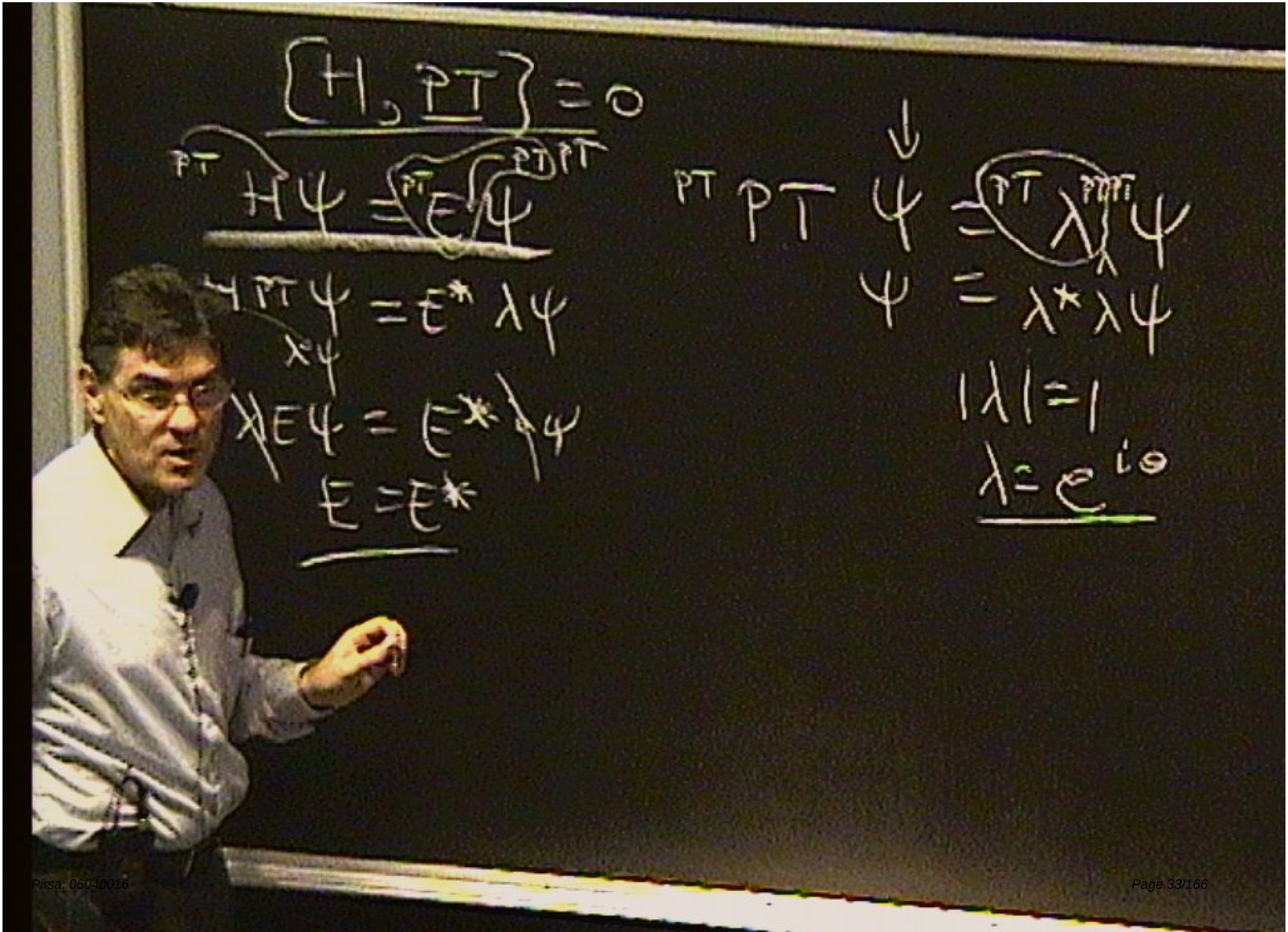
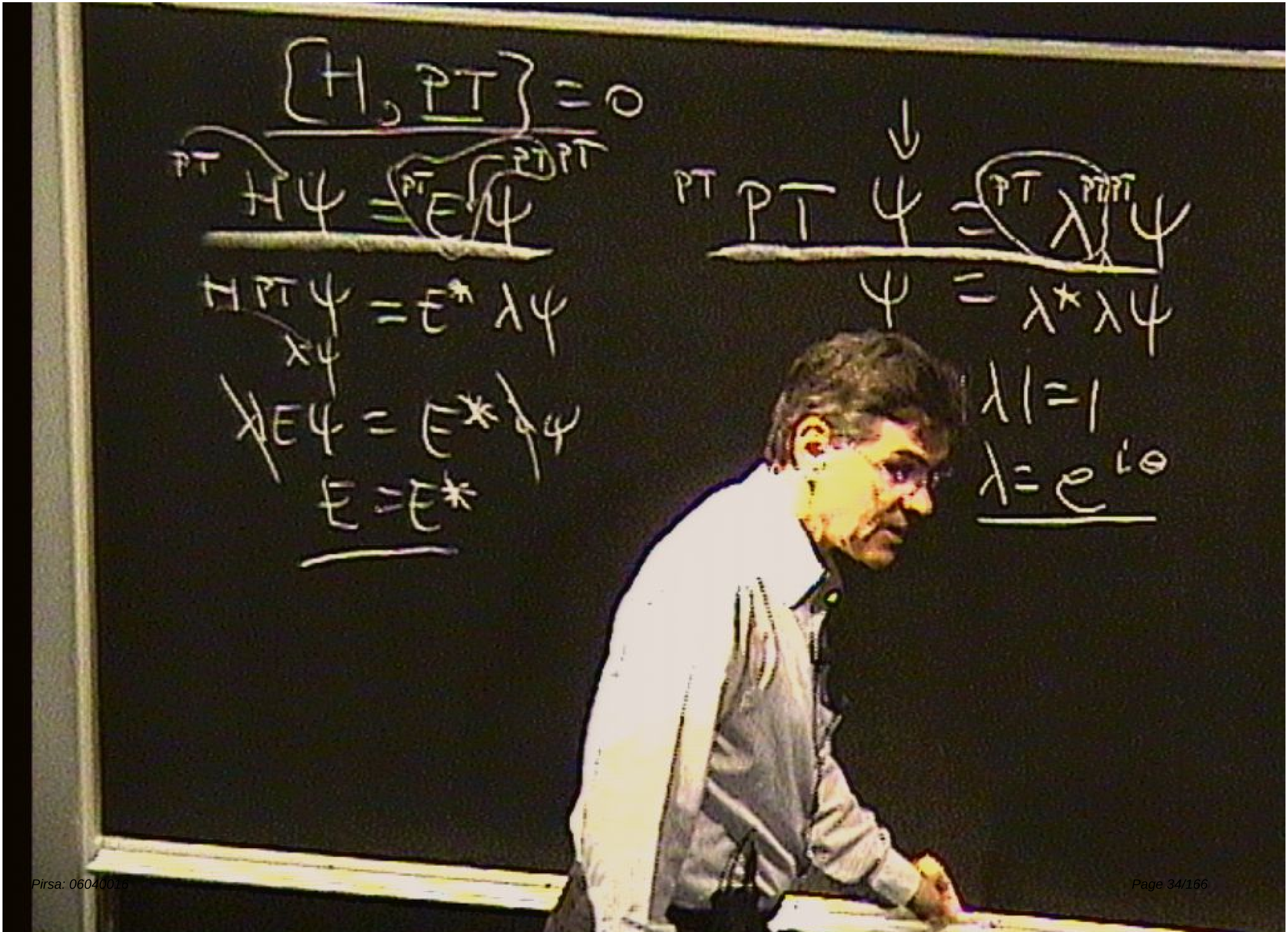

[H, PT] = 0
H ψ = E ψ
H PT ψ = E* λ ψ
λ*ψ
λ E ψ = E* λ ψ
E = E*
PT PT ψ = λ ψ
ψ = λ* λ ψ
|λ| = 1
λ = e^{iθ}

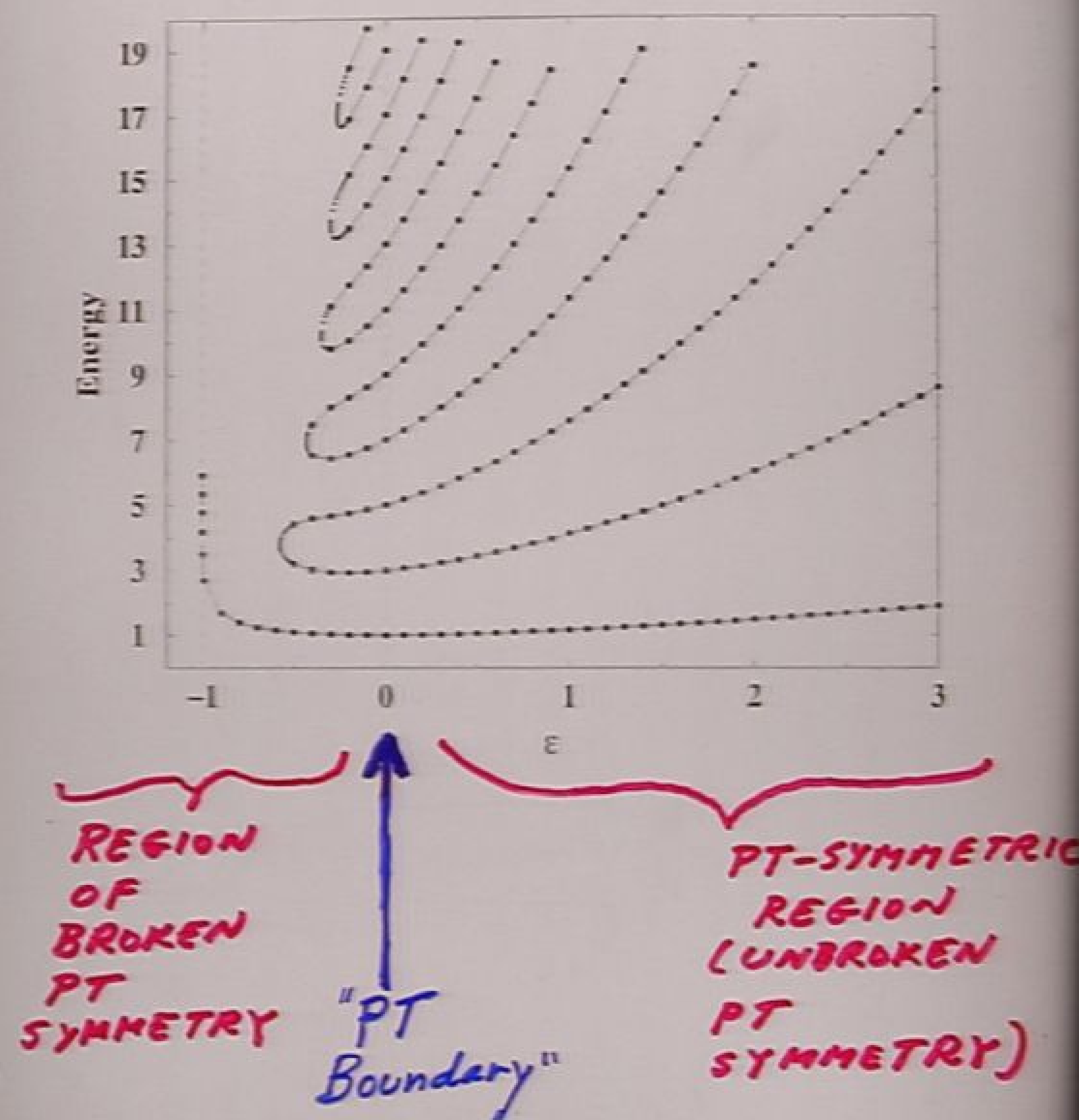


Energy
19
17
15
13
11
9
7
5
3
1
−1
0
1
2
3
ε
REGION OF BROKEN PT SYMMETRY
"PT Boundary"
PT-SYMMETRIC REGION (UNBROKEN PT SYMMETRY)


[H, PT] = 0
HΨ = EΨ
H PT Ψ = E* λΨ
E = E*
PT PT Ψ = λΨ
Ψ = λ*λΨ
|λ| = 1
λ = e^{iθ}


[H, PT] = 0
PT HΨ = EΨ
H PT Ψ = E* λΨ
λEΨ = E* λΨ
E = E*
PT PT Ψ = λΨ
Ψ = λ*λΨ
|λ| = 1
λ = e^{iθ}


$[H, PT] = 0$
$H\psi = E\psi$
$PT\,\psi = \lambda\psi$
$H\,PT\,\psi = E^* \lambda\psi$
$\lambda E\psi = E^* \lambda\psi$
$E = E^*$
$\psi = \lambda^* \lambda\psi$
$|\lambda| = 1$
$\lambda = e^{i\theta}$


Energy
19
17
15
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11
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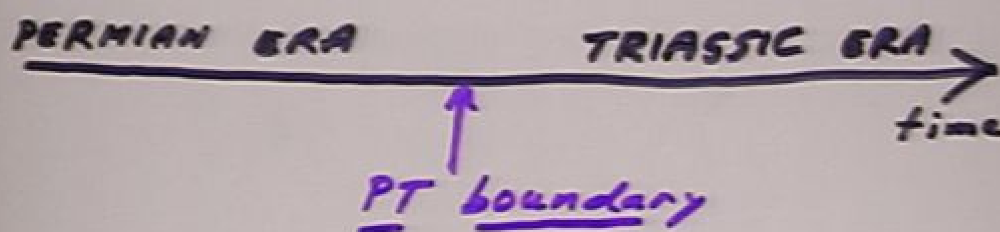
7
5
3
1
−1
0
1
2
3
ε
REGION OF BROKEN PT SYMMETRY
"PT Boundary"
PT-SYMMETRIC REGION (UNBROKEN PT SYMMETRY)

PT Boundary .....

The greatest murder mystery of all time!



extinction of over 90% of species!

Rigorous mathematical contributions

RIGOROUS proof that spectrum is real by

Dorey, Dunning, Tateo

Shin

Mostafazadeh

Caliceti, Graffi, Sjöstrand

Pham, Delabaere, Trinh

Weigert

and many more...!
(apologies!)


PT


PT
PT

To define a theory you give the

HAMILTONIAN

H

→ specifies the dynamics of a theory

→ embodies symmetries

Usual
symmetries:

CONTINUOUS

Lorentz invariance

DISCRETE

Hermiticity $H^\dagger = H$

$\dagger$ = (transpose)(complex conj.)

Guarantees that

E is real

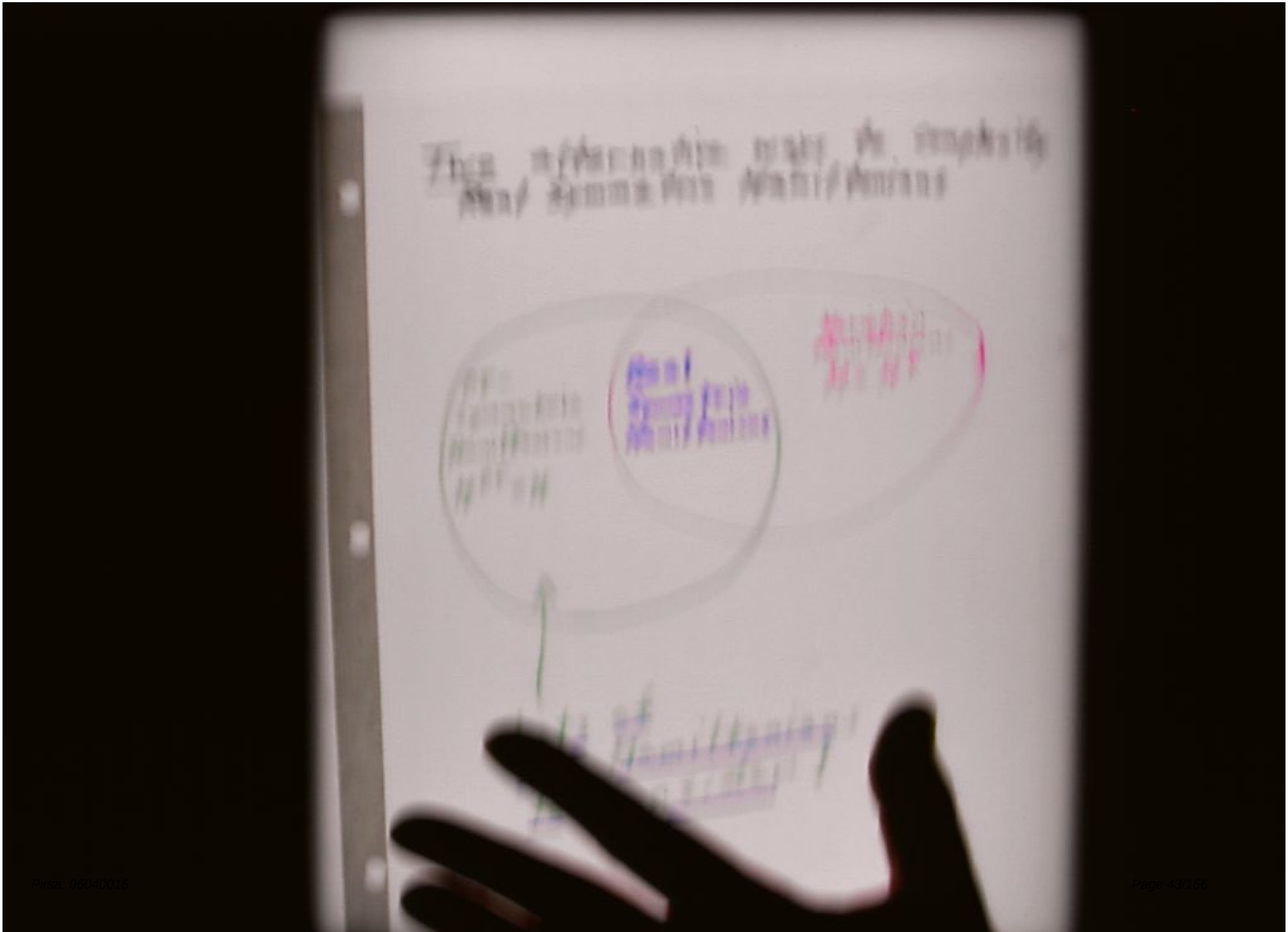
MAIN IDEA OF TALK:

$H = H^{\dagger}$ is mathematically convenient but physically obscure and remote
($\dagger$ = transpose + cx conjugate)

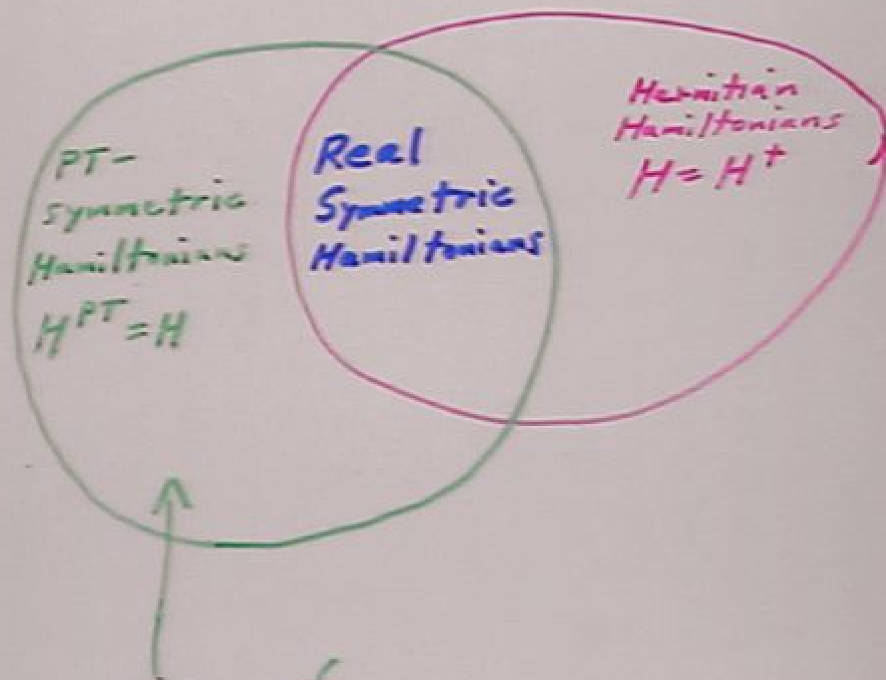
Propose a more physical discrete symmetry condition:

SPACE-TIME REFLECTION (PT) SYMMETRY:

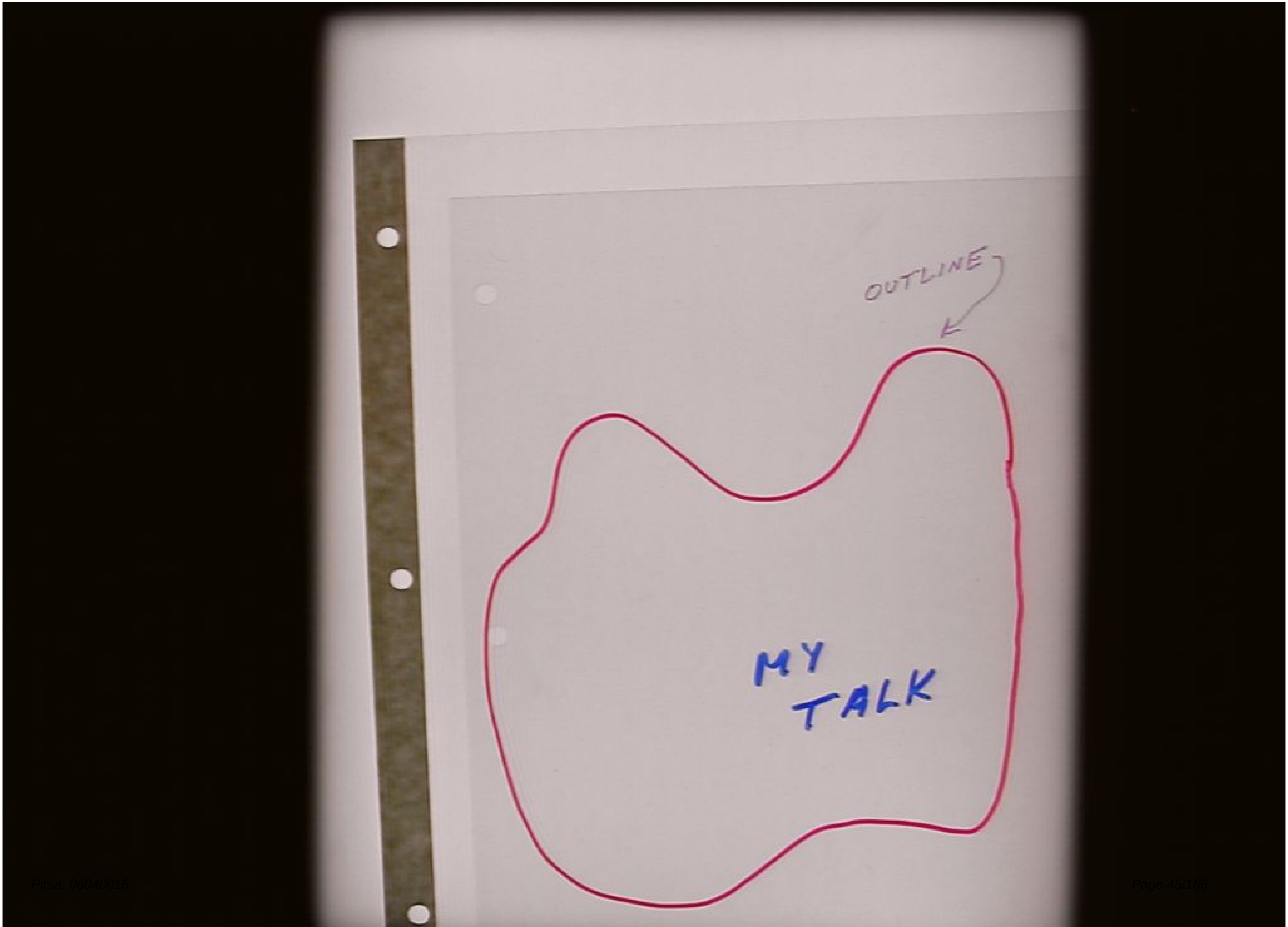
$$H^{PT} = H$$



Two alternative ways to complexify Real Symmetric Hamiltonians



Lots of new Hamiltonians to consider!


OUTLINE
MY TALK

OUTLINE OF TALK:

1. Beginning

2. Middle

3. End

(4. applause!)

$H = p^2 + x^2 (ix)^\epsilon$

Has a transition at $\epsilon = 0$ in both the classical and quantum versions of the theory

The spectrum is
real and positive

but what about
unitarity ?

i.e. Is this QM or just a
catSturm-Liouville problem ?

CMB., D. Brody, H. Jones
PRL 2002

PT-symmetric quantum mechanics in 9 easy steps:

#1 Given $H = p^2 + x^2(ix)^\epsilon$ $(\epsilon \geq 0)$

solve $H\phi_n = E_n \phi_n$:

$$-\phi_n''(x) + x^2(ix)^\epsilon \phi_n(x) = E_n \phi_n(x)$$

Verify:

(a) E_n real, positive

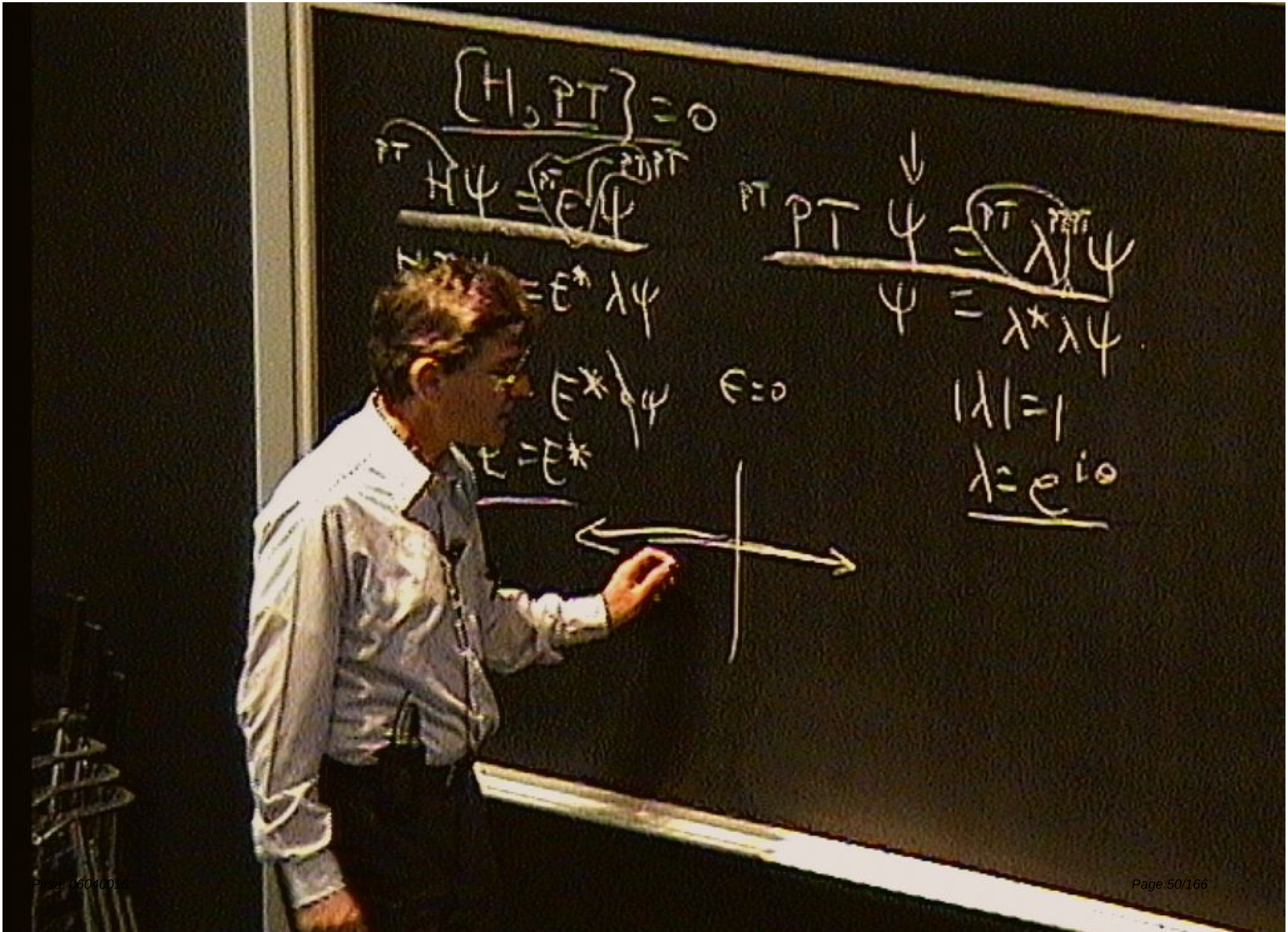
(b) $\phi_n(x)$ is also an eigenfunction of PT for all n

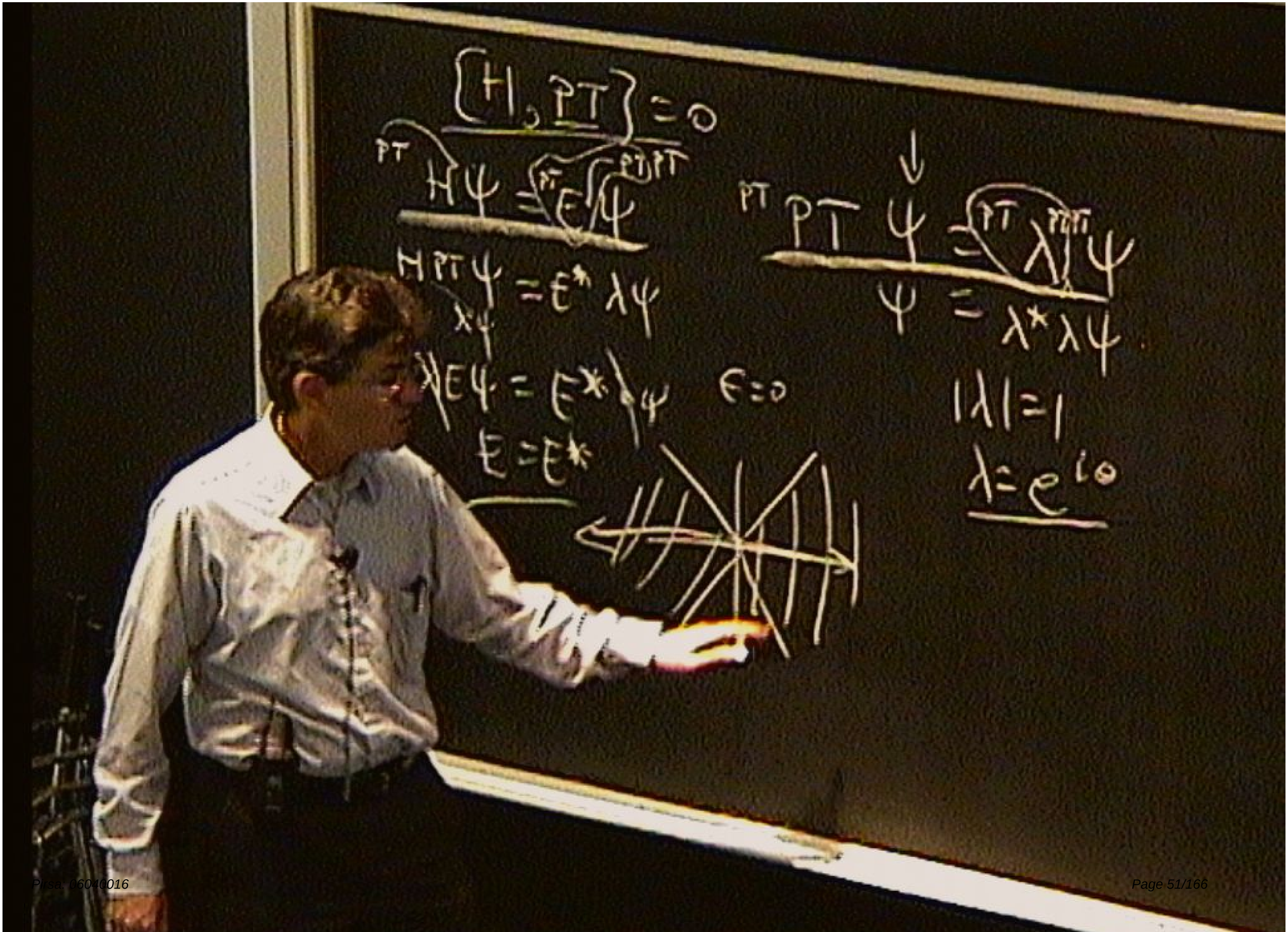
$$PT\ \phi_n = \lambda_n \phi_n, \quad |\lambda_n| = 1$$

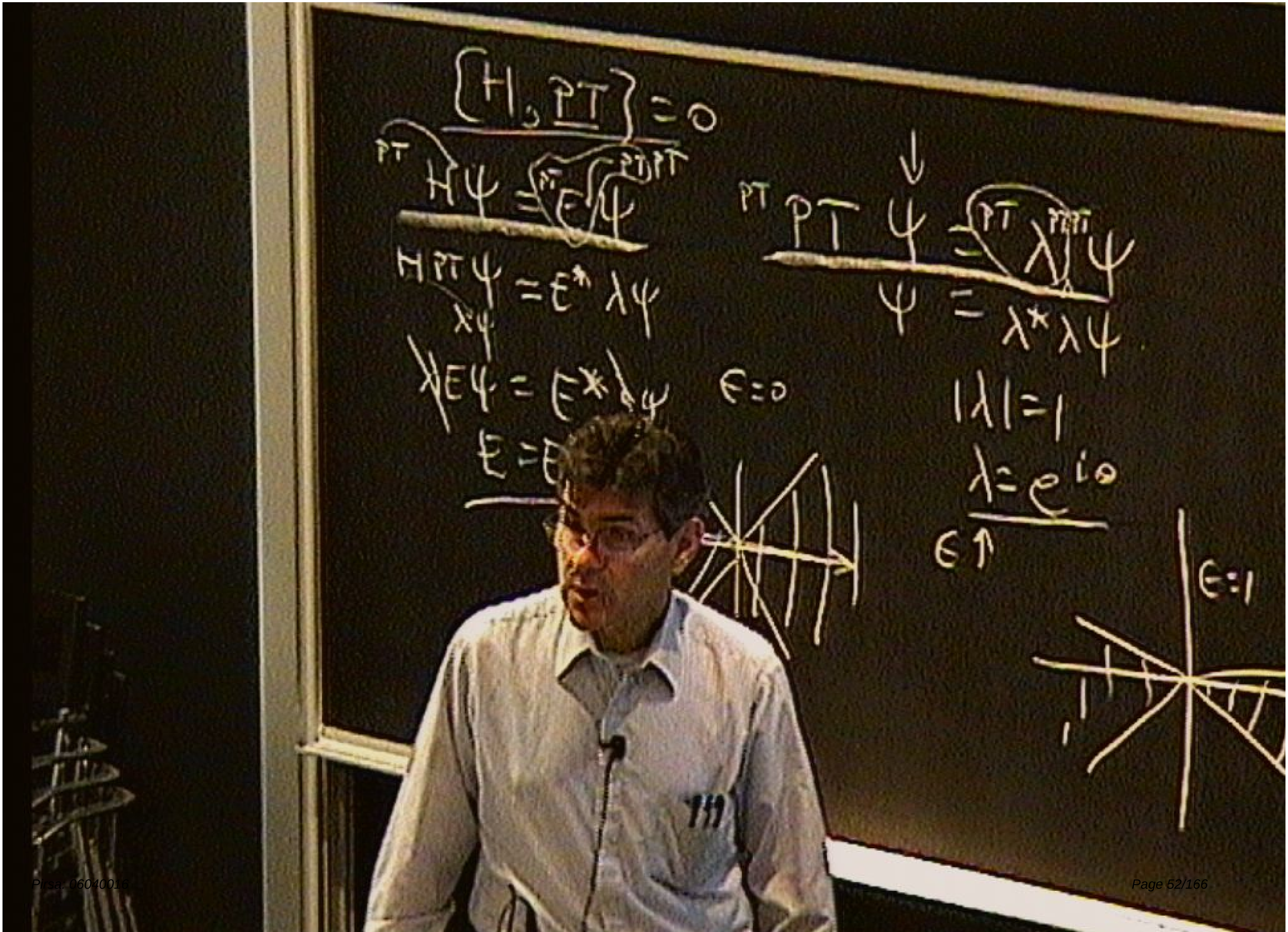
(c) Convention: normalize ϕ_n so that

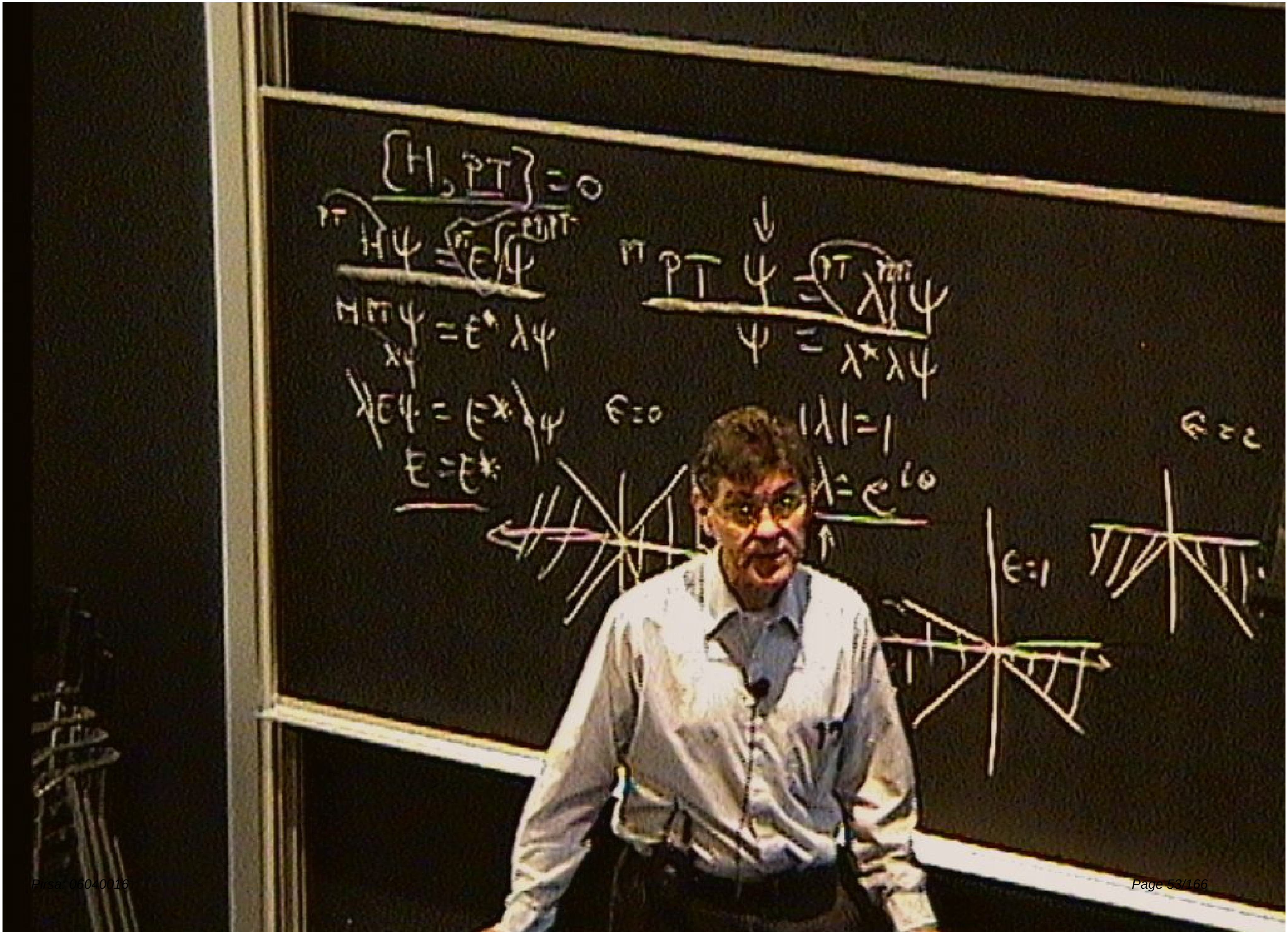
$$PT\ \phi_n(x) = \phi_n(x)$$

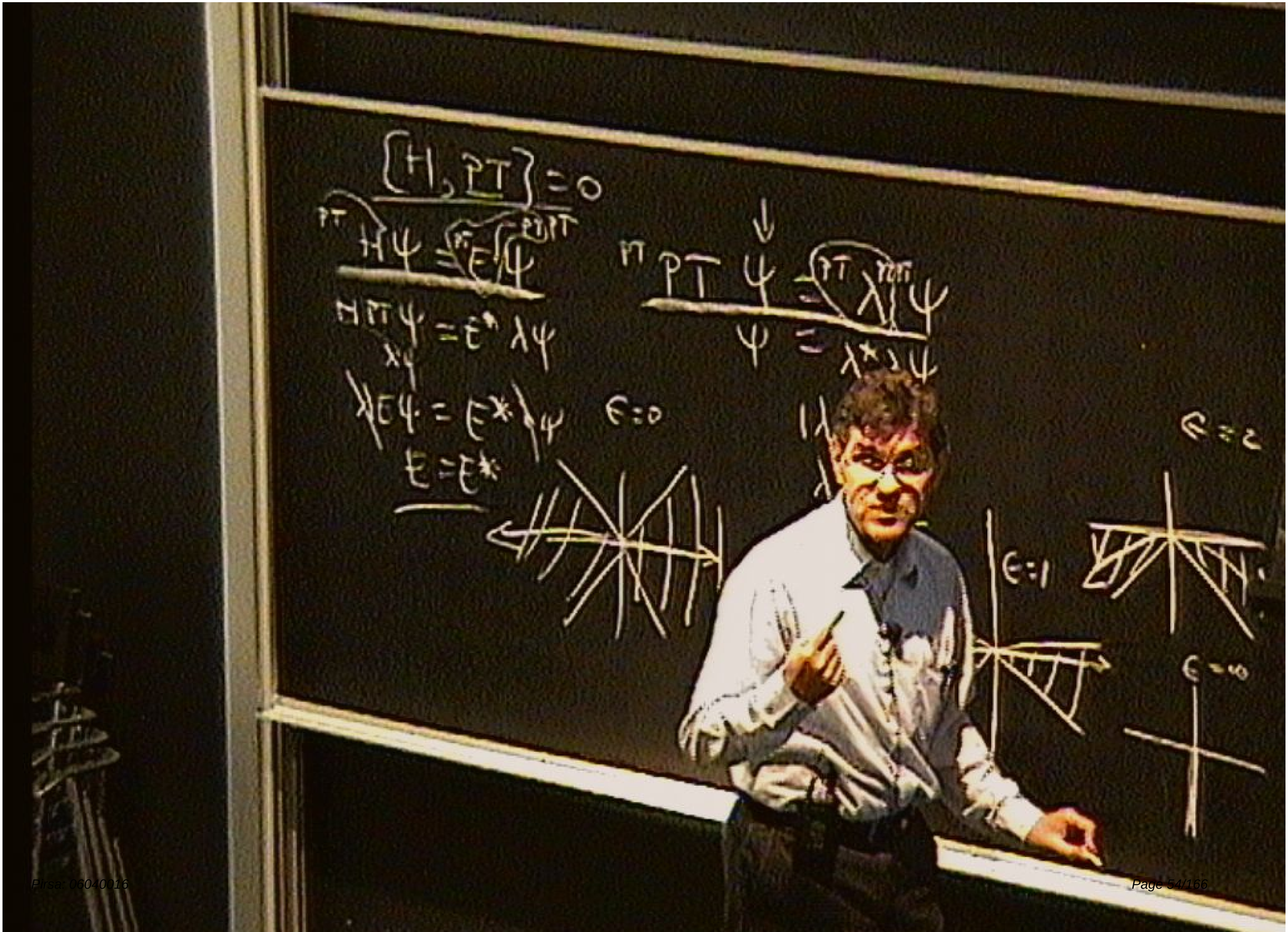
by appropriate choice of phase

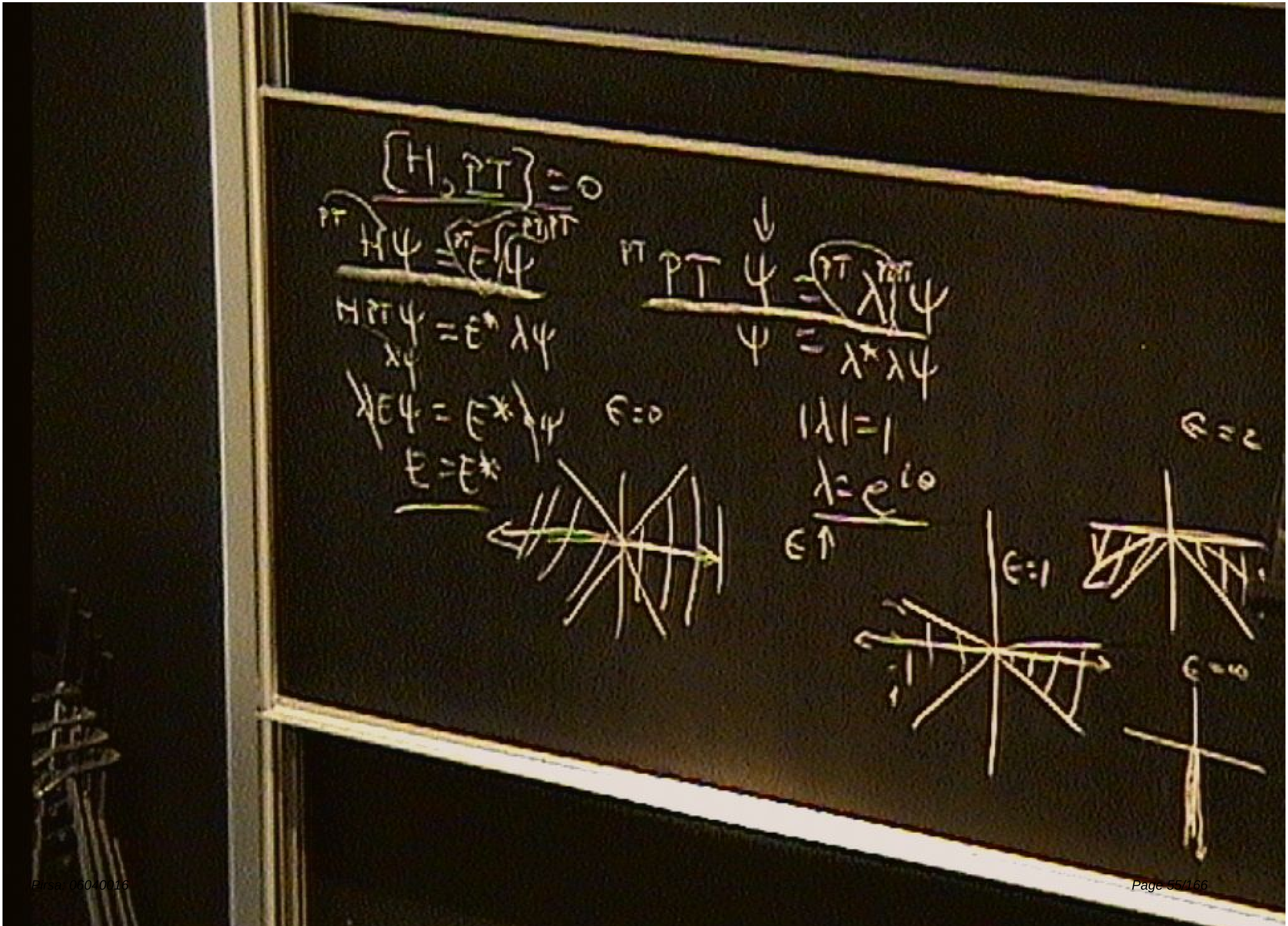

[H, PT] = 0
E = E*
ε = 0
|λ| = 1
λ = e^{iθ}


[H, PT] = 0
PT Hψ = PT Eψ
H PT ψ = E* λψ
E = E*
PT PT ψ = λψ
ψ = λ* λψ
|λ| = 1
λ = e^{iθ}


[H, PT] = 0
ε=0
|λ|=1
λ=e^{iθ}
ε=1


[H, PT] = 0
ε = 0
|λ| = 1
ε = ε*
ψ = λ*λψ
ε = 1


$[H, PT] = 0$
$E = E^*$
$\epsilon = 0$
$\epsilon = 1$
$\epsilon = 2$
$\epsilon = \infty$


[H, PT] = 0
E = E*
|λ| = 1
λ = e^{iθ}
ε = 0
ε = 1

PT - symmetric quantum mechanics in 9 easy steps:

#1 Given $H = p^2 + x^2(ix)^\epsilon$ $(\epsilon \geq 0)$

solve $H\phi_n = E_n\phi_n$:

$$-\phi_n''(x) + x^2(ix)^\epsilon \phi_n(x) = E_n\phi_n(x)$$

Verify:

(a) E_n real, positive

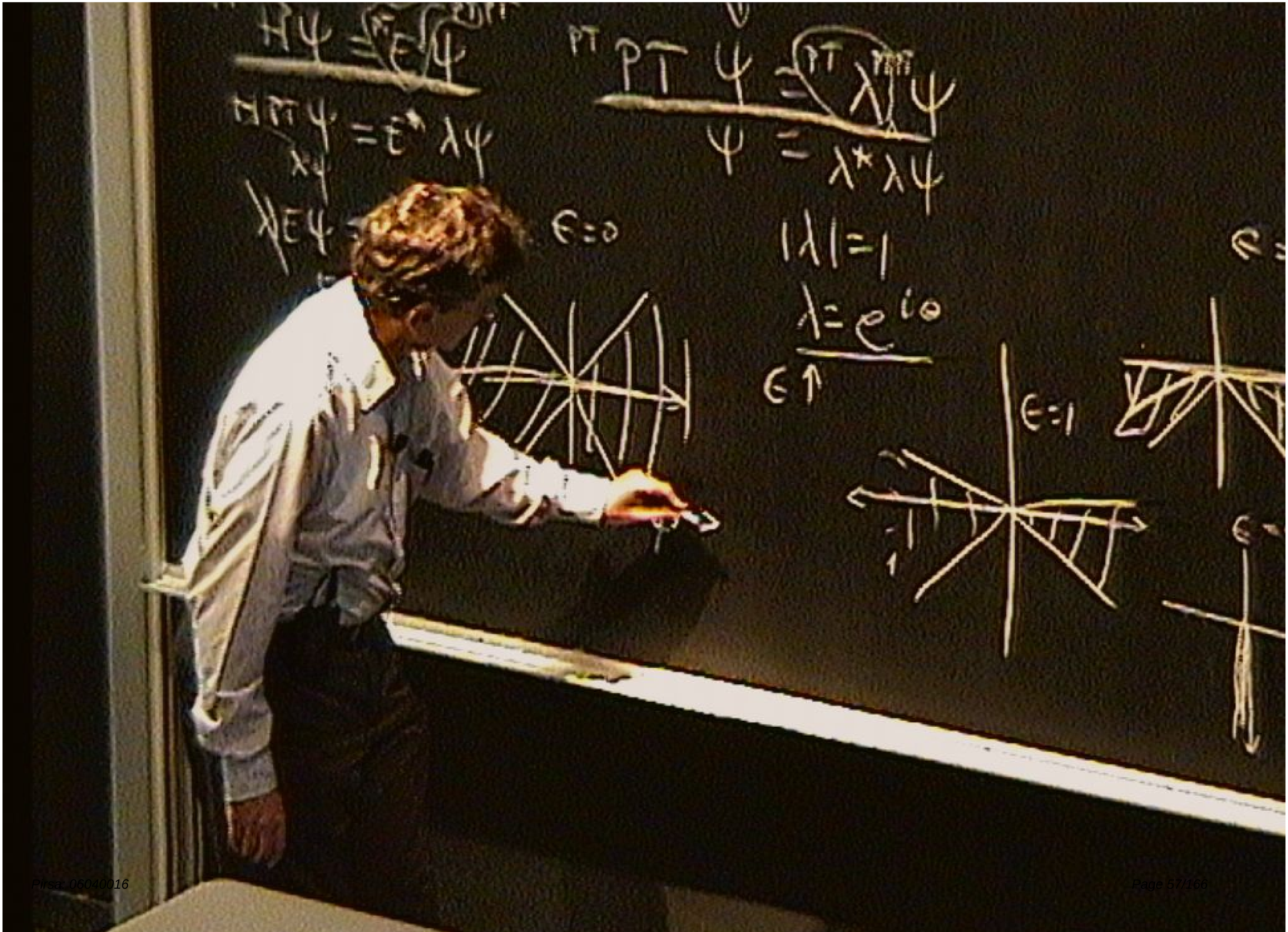
(b) $\phi_n(x)$ is also an eigenfunction of PT for all n

$$PT\,\phi_n = \lambda_n\phi_n, \quad |\lambda_n| = 1$$

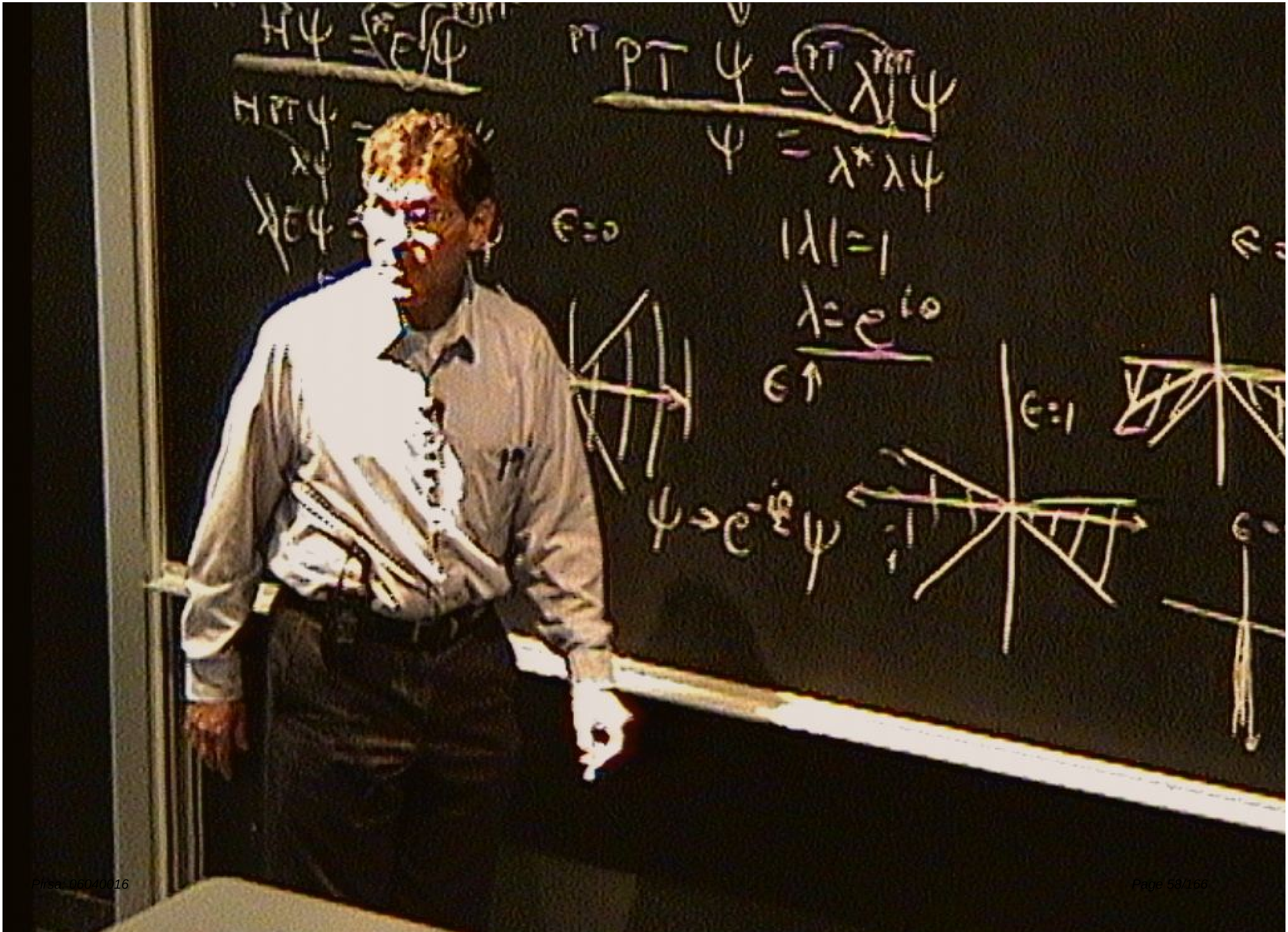
(c) convention: normalize ϕ_n so that

$$PT\,\phi_n(x) = \phi_n(x)$$

by appropriate choice of phase



Hψ = Eψ
PT ψ = λ ψ
ψ = λ* λψ
ε=0
|λ|=1
λ=e^{iθ}
ε↑
ε=1



PT - symmetric quantum mechanics in 9 easy steps:

#1 Given $H = p^2 + x^2(ix)^\epsilon$ $(\epsilon \geq 0)$

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Verify:

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$$PT\ \phi_n = \lambda_n \phi_n, \quad |\lambda_n| = 1$$

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$$PT\ \phi_n(x) = \phi_n(x)$$

by appropriate choice of phase

#2 Orthogonality

Guess that inner product is

$$\langle a | b \rangle = a^{PT} b$$

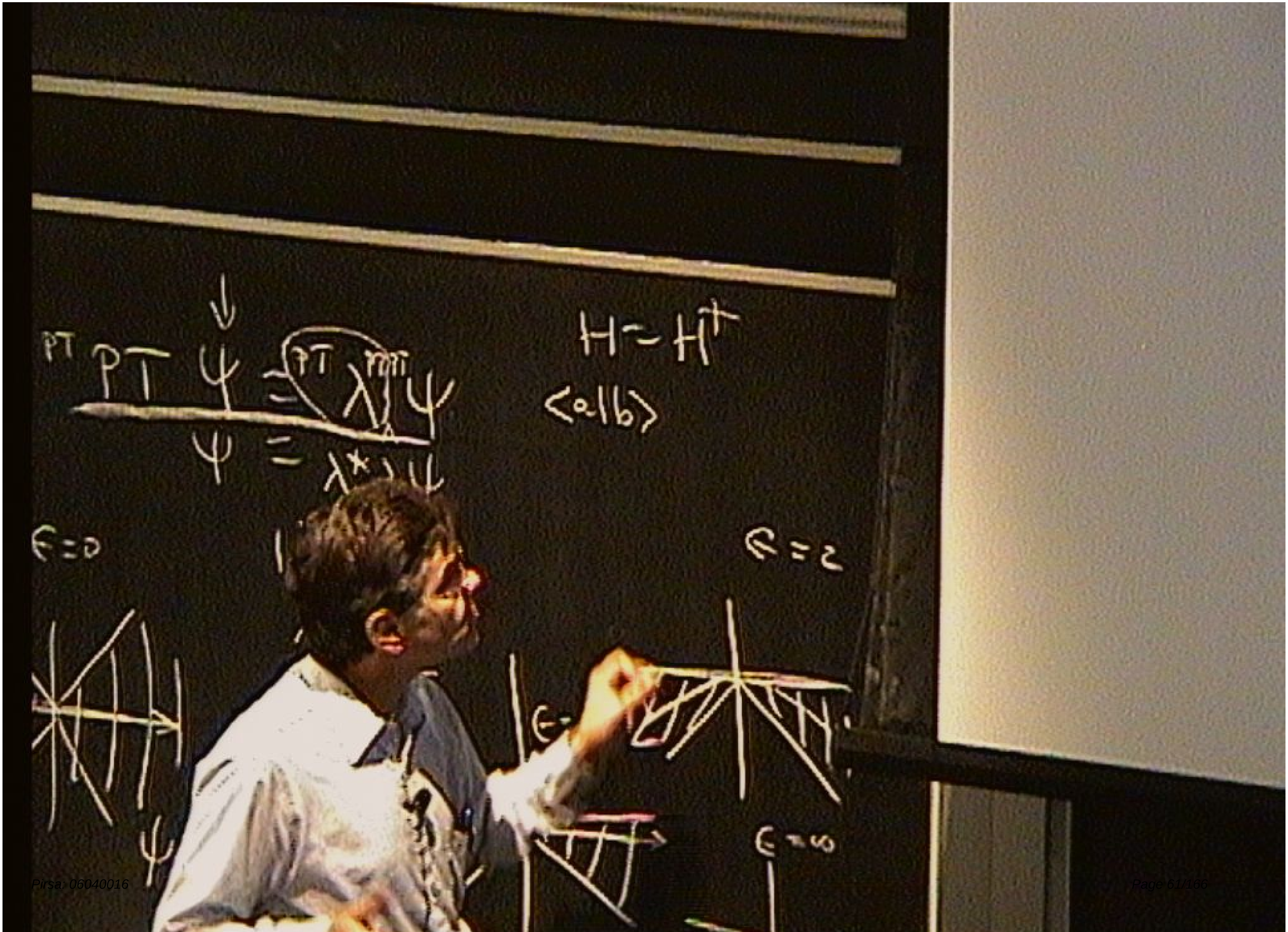
verify: $\langle \phi_n | \phi_m \rangle = 0 \quad (n \neq m)$

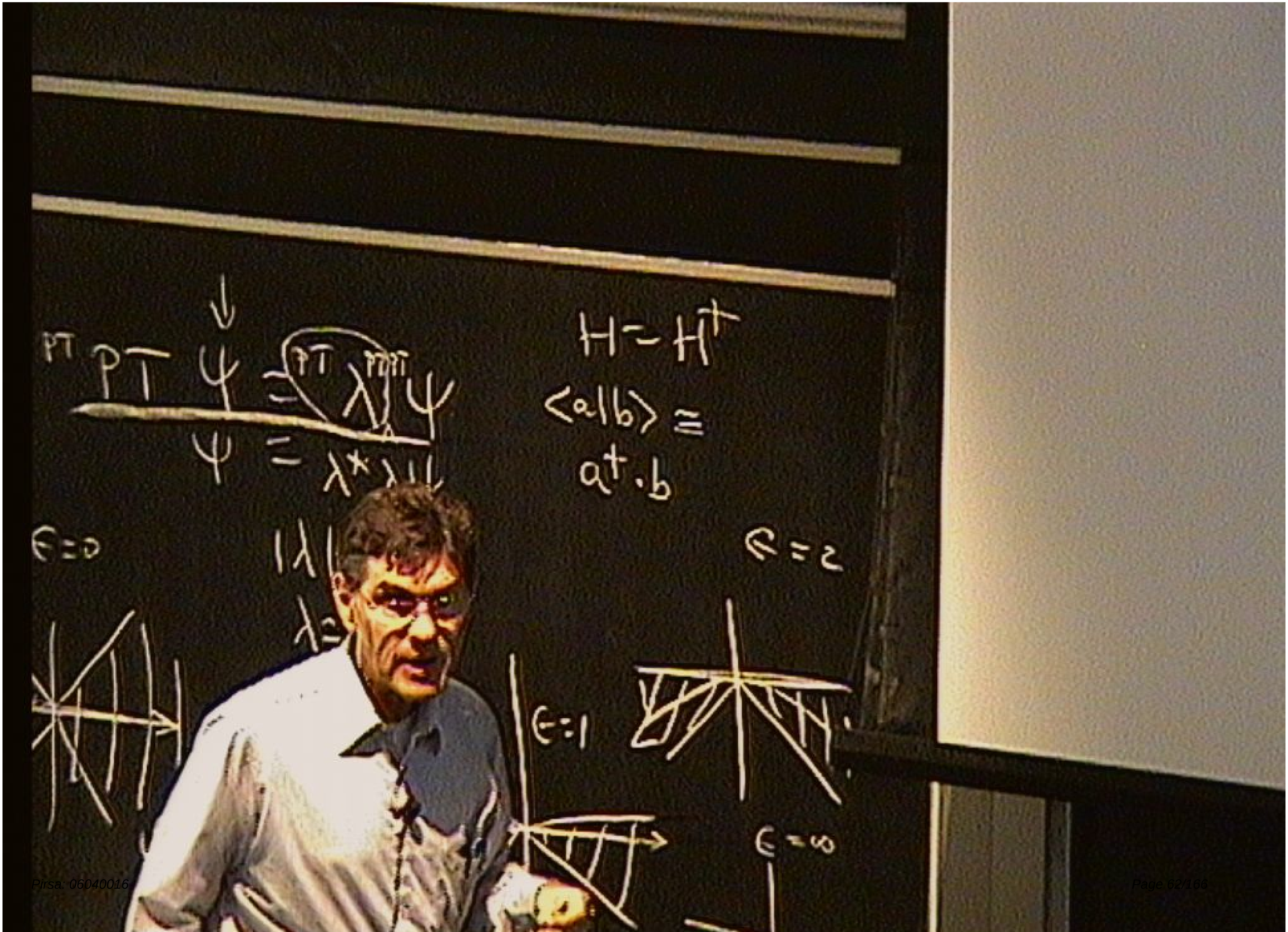
i.e. $\langle \phi_n | \phi_m \rangle = \int_C \phi_n^{PT}(x)\, \phi_m(x)\, dx$

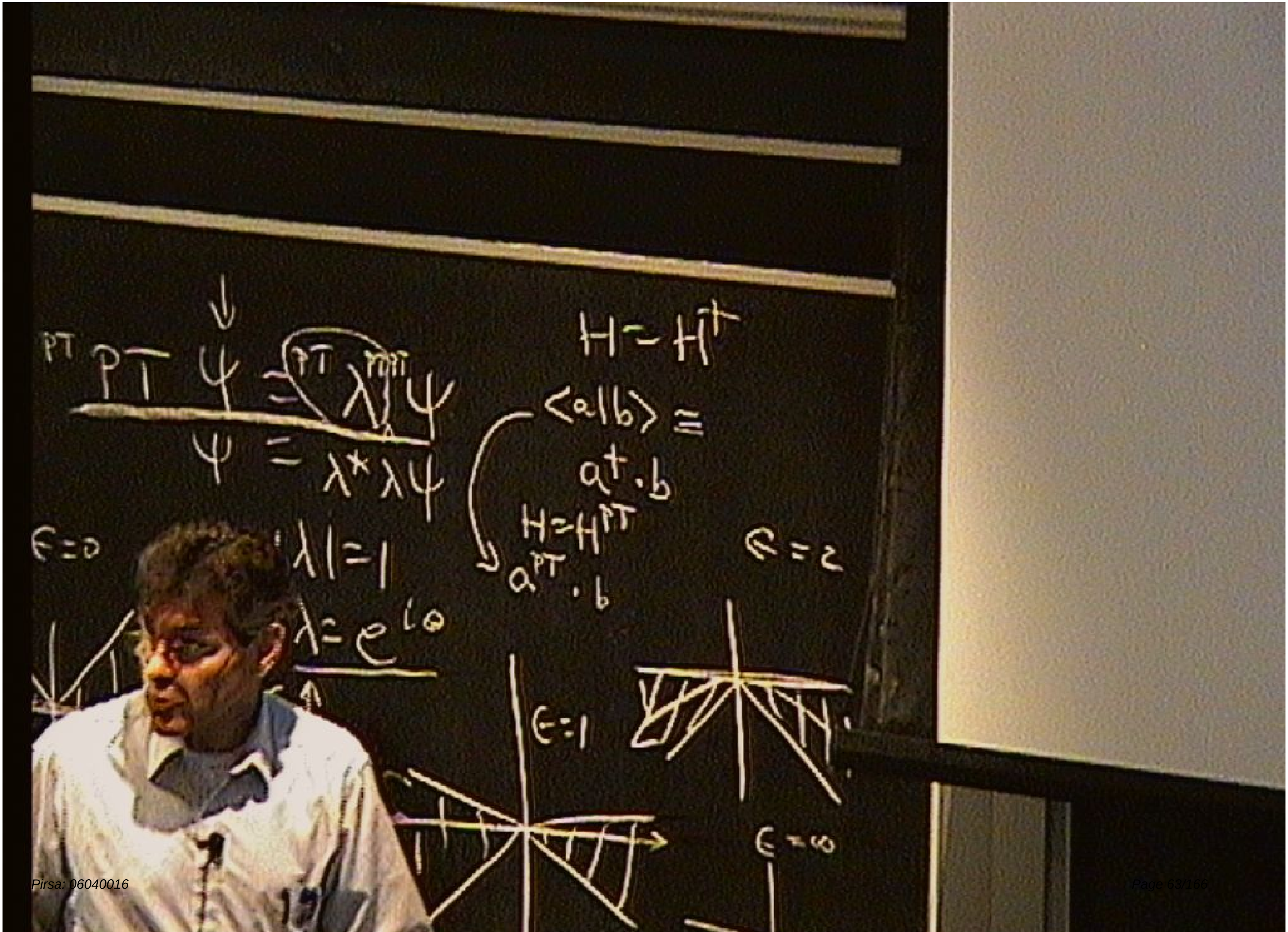
$$= \int_C \phi_n(x)\, \phi_m(x)\, dx$$

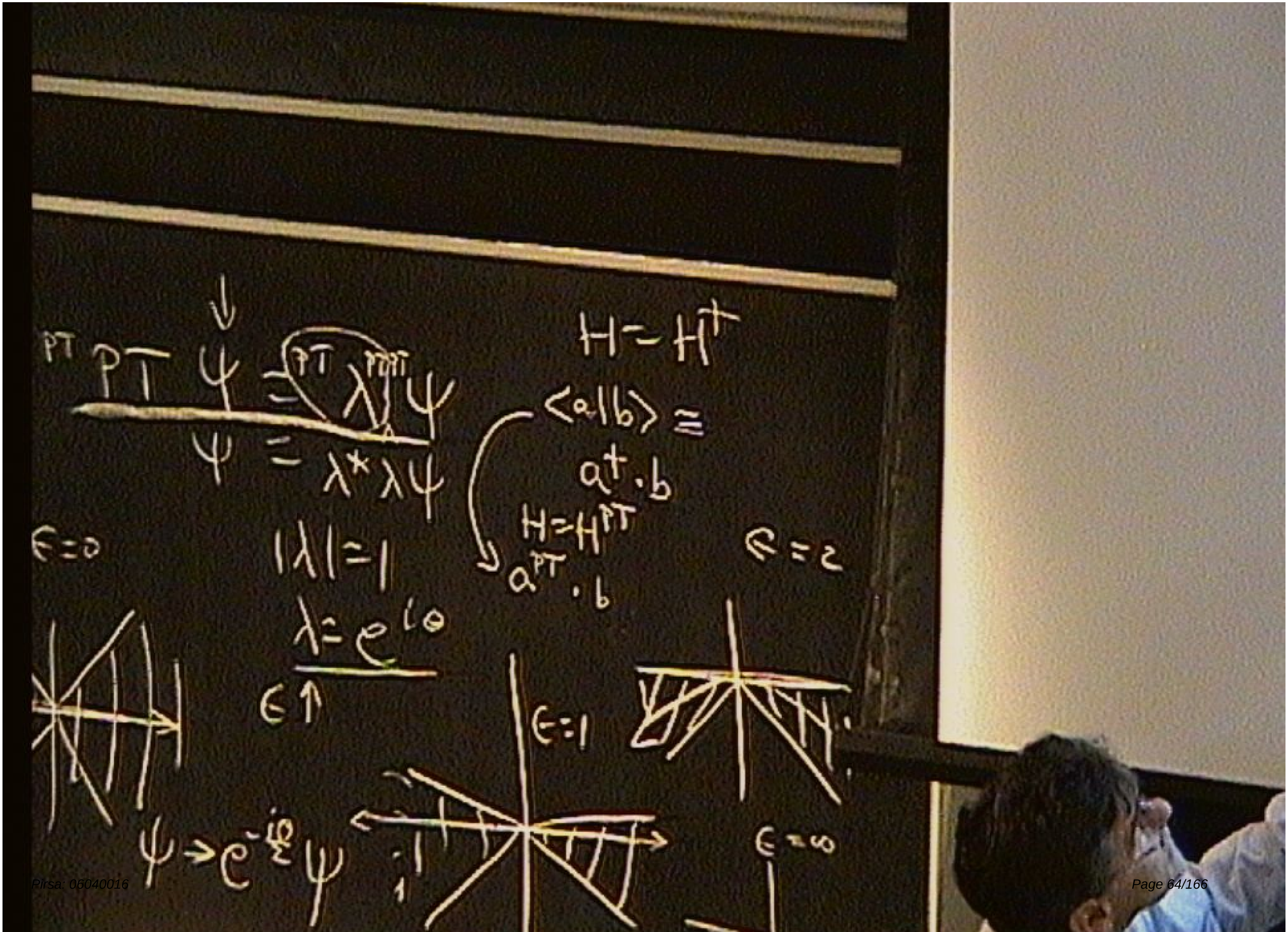
$$= 0$$

So far, so good, but....


H ~ H†
<a|b>
ε=0
ε=2
ε=∞


H = H†
<a|b> ≡
a†·b
ε=0
ε=2
ε=1
ε=∞


H = H†
⟨a|b⟩ ≡
a†·b
H = H^PT
a^PT·b
|λ|=1
λ = e^iθ
ε=0
ε=1
ε=2
ε=∞


H = H†
<a|b> ≡
a†·b
H=H^PT
a^PT·b
ε=0
|λ|=1
λ=e^iθ
ε=2
ε=1
ε=∞

#2 Orthogonality

Guess that inner product is

$$\langle a|b\rangle = a^{PT} b$$

verify: $\langle \phi_n | \phi_m \rangle = 0 \quad (n \neq m)$

i.e. $$\langle \phi_n | \phi_m \rangle = \int_C \phi_n^{PT}(x)\, \phi_m(x)\, dx$$

$$= \int_C \phi_n(x)\, \phi_m(x)\, dx$$

$$= 0$$

So far, so good, but....

#3 The problem:

$$\int_C dx\, \phi_n^2(x) = (-1)^n$$

Thus, $\overline{\underline{PT}}$ norm not positive

Not a disaster .....
just continue on .....

#4 Completeness

$$\mathbb{1} = \sum_{n=0}^{\infty} (-1)^n \phi_n(x)\, \phi_n(y) = \delta(x-y)$$

#5 Reconstruct operators

(a) $P(x,y) = \delta(x+y) = \sum_{n=0}^{\infty} (-1)^n \phi_n(x) \phi_n^*(-y)$

note: $P^2 = 1$

i.e. $\int dz \; P(x,z) P(z,y) = \delta(x-y)$

(b) $H(x,y) = \sum_{n=0}^{\infty} (-1)^n \phi_n(x) \phi_n(y) E_n$

(c) $G(x,y) = \sum_{n=0}^{\infty} (-1)^n \phi_n(x) \phi_n(y) \frac{1}{E_n}$

note: $HG = \delta(x-y)$

Can calculate spectral zeta function exactly

$$\sum_{n=0}^{\infty} \frac{1}{E_n} \qquad (\epsilon > 0)$$

Can calculate spectral zeta function exactly

$$\sum_{n=0}^{\infty} \frac{1}{E_n} \qquad (\epsilon > 0)$$

$$H = p^2 - (ix)^N \quad (N > 2)$$

$$Z_N = \frac{4 \sin^2\left(\frac{\pi}{N+2}\right) \Gamma\left(\frac{1}{N+2}\right) \Gamma\left(\frac{2}{N+2}\right) \Gamma\left(\frac{N-2}{N+2}\right)}{(N+2)^{\frac{2N}{N+2}} \Gamma\left(\frac{N-1}{N+2}\right) \Gamma\left(\frac{N}{N+2}\right)}$$

Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results. The physical interpretation of relativistic quantum mechanics that one gets by a natural development of the non-relativistic theory involves these things and is thus in contradiction with experiment. We therefore have to consider ways of modifying or supplementing this interpretation.

— P.A.M. Dirac

Bakerian Lecture

June 19, 1941

Proc. Royal Society

#6 Construct new symmetry operator C

$$C(x,y) \equiv \sum_{n=0}^{\infty} \phi_n(x)\,\phi_n(y)$$

verify: (a) $C^2 = \mathbb{1}$

(b) $C \neq P$ (2 square-roots of 1)

(c) $[C,H]=0$

(d) $C\phi_n = (-1)^n \phi_n$

(e) $[C, PT]=0$ (but $[C,P]\neq 0$)

(f) as $\epsilon \to 0$, $C \longrightarrow P$

We have 3 commuting operators: H, PT, and C. H and C are observables.

#7 Construct Hilbert Space

Space is all complex linear combinations of $\phi_n(x)$

Inner product:

$$\langle a|b\rangle = a^{CPT} b$$

Verify: norm is positive

i.e. $\langle a|a\rangle \geq 0$

(and $=0$ iff $|a\rangle = 0$)

Novelty: Dynamically determined norm — i.e. Hilbert space determined by H

#8 Time evolution

Time evolution operator:

e^{iHt}

inner product preserved ⇒ unitarity

#9 The rest....

(a) probability density

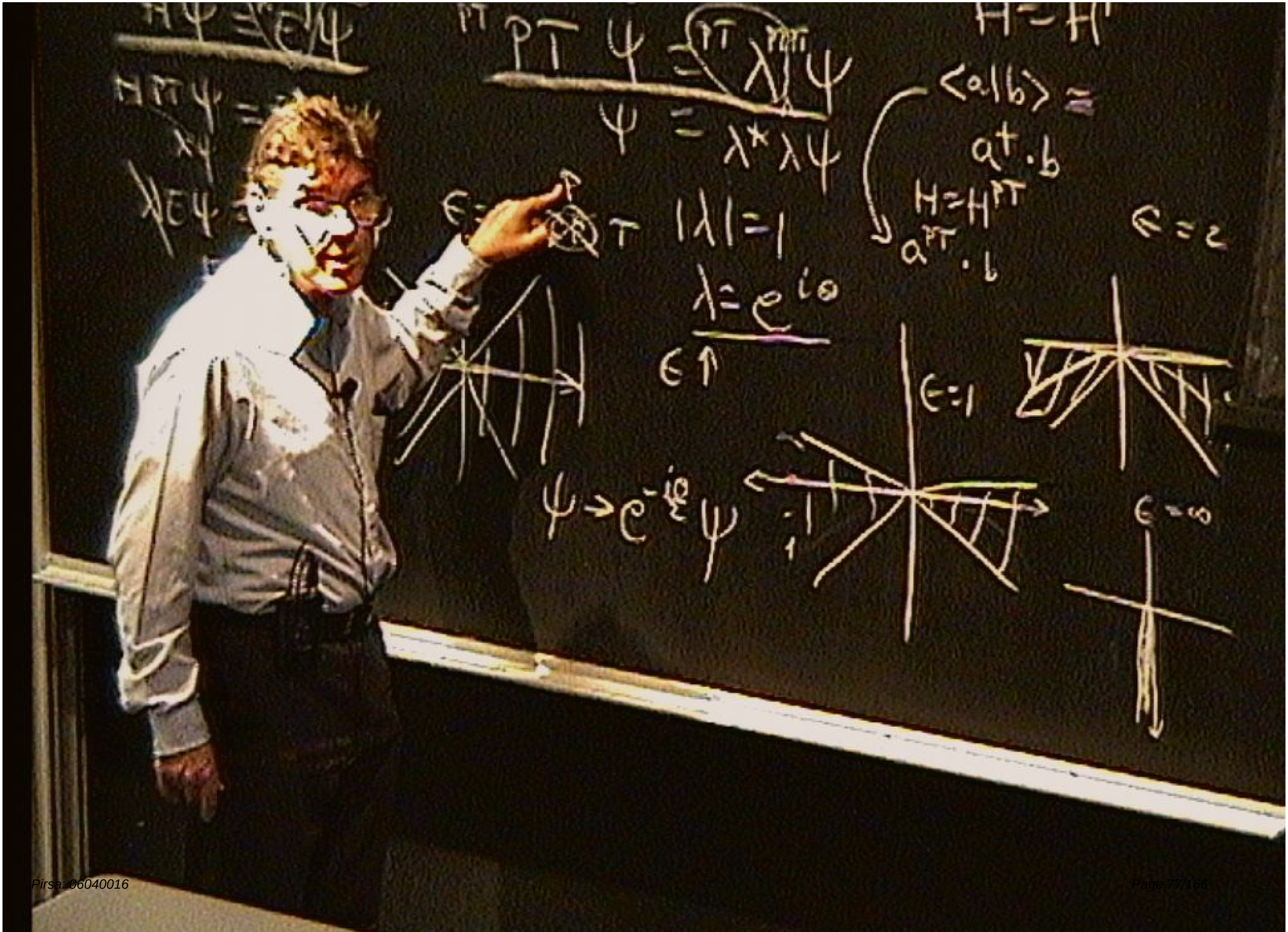
(b) probability currents

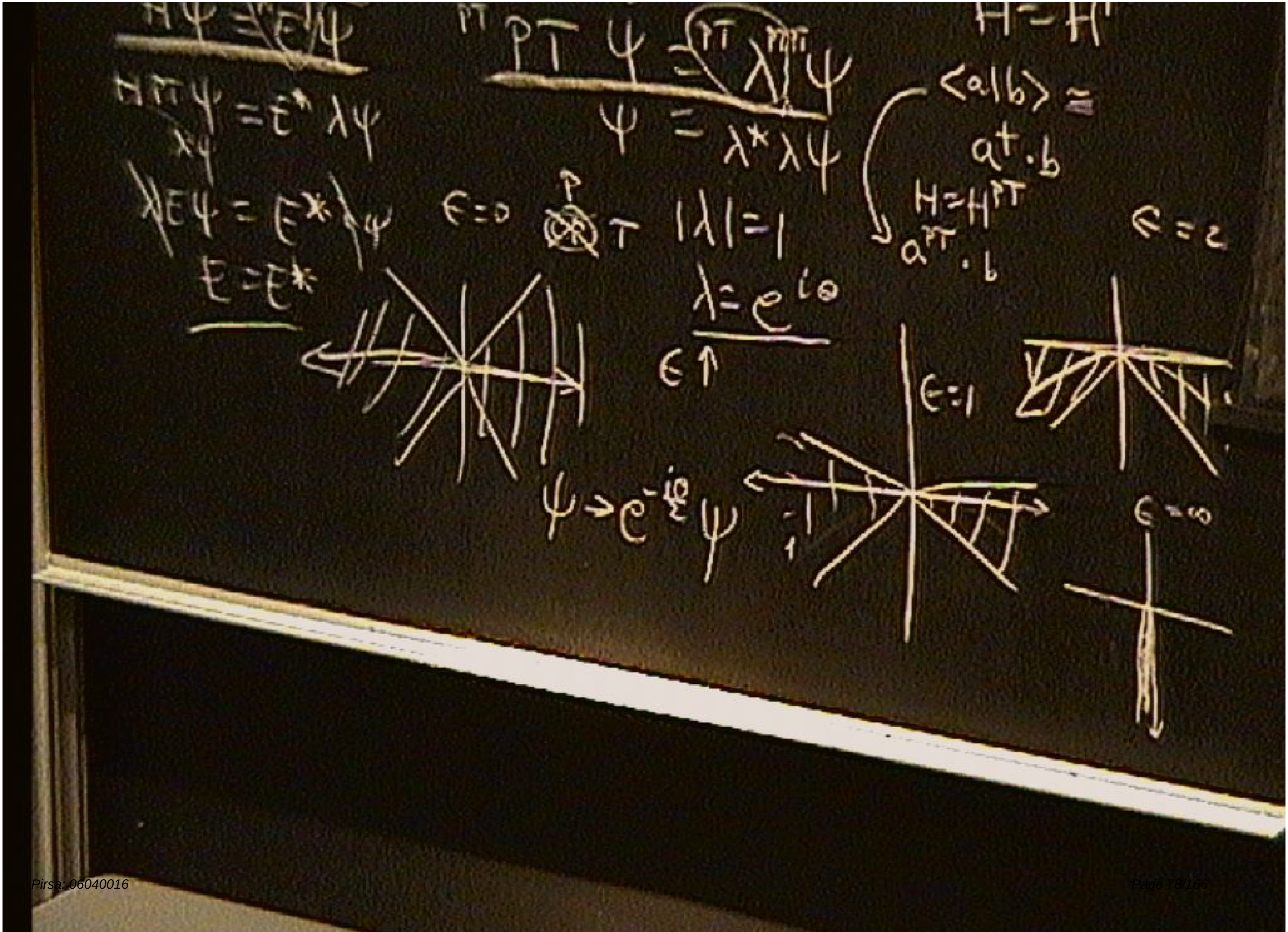
(c) Ehrenfest theorem

(d) definition of observables

⋮

Result: Conventional quantum mechanics based on non-Hermitian operator H


⟨a|b⟩ =
a†·b
H = H^PT
a^PT·b
|λ| = 1
λ = e^iθ
ε = 1
ε = 2
ε = ∞


ψ = λ*λψ
⟨a|b⟩ =
a†·b
H=H^PT
a^PT·b
ε=0
|λ|=1
E=E*
λ=e^iθ
ε=1
ε=2
ε=∞
ψ→e^-iθ ψ

Novelties

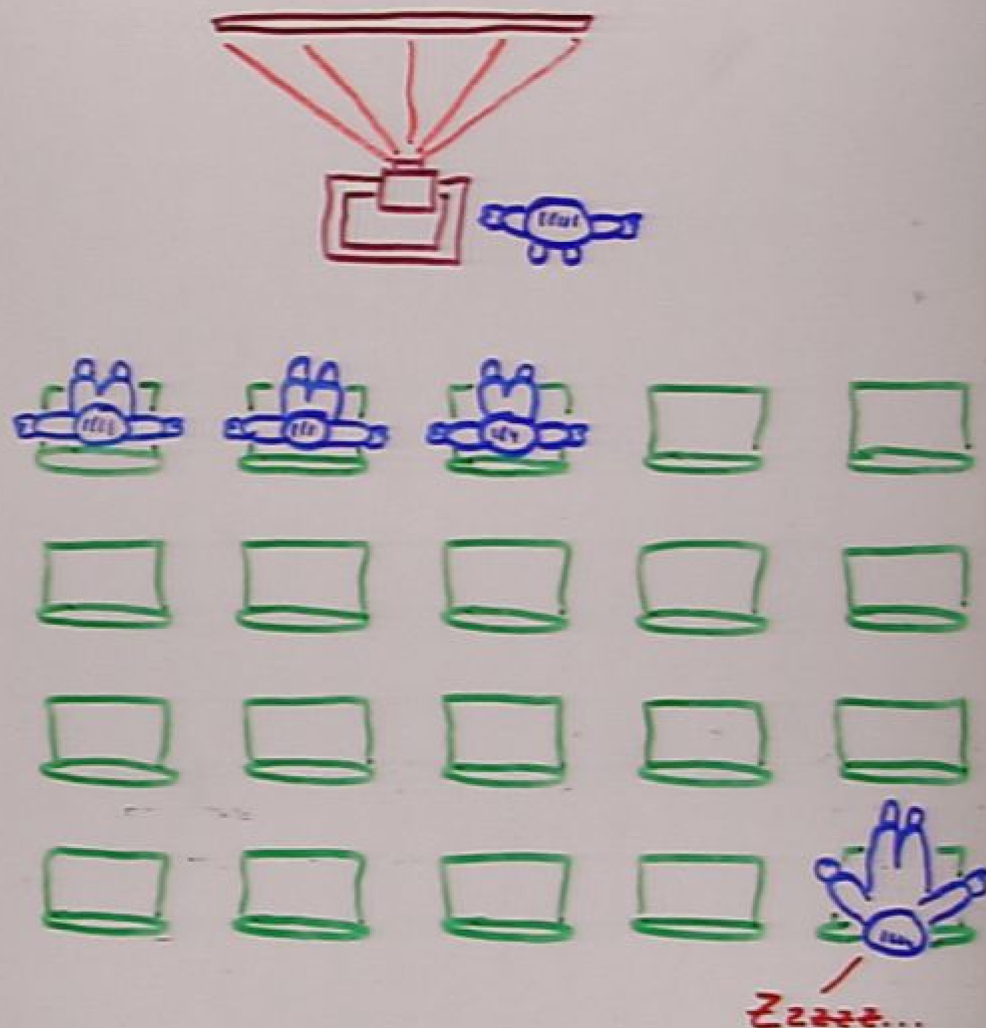
(1) Complex extension of conventional QM
(Remember - no C operator in ordinary Hermitian QM!)

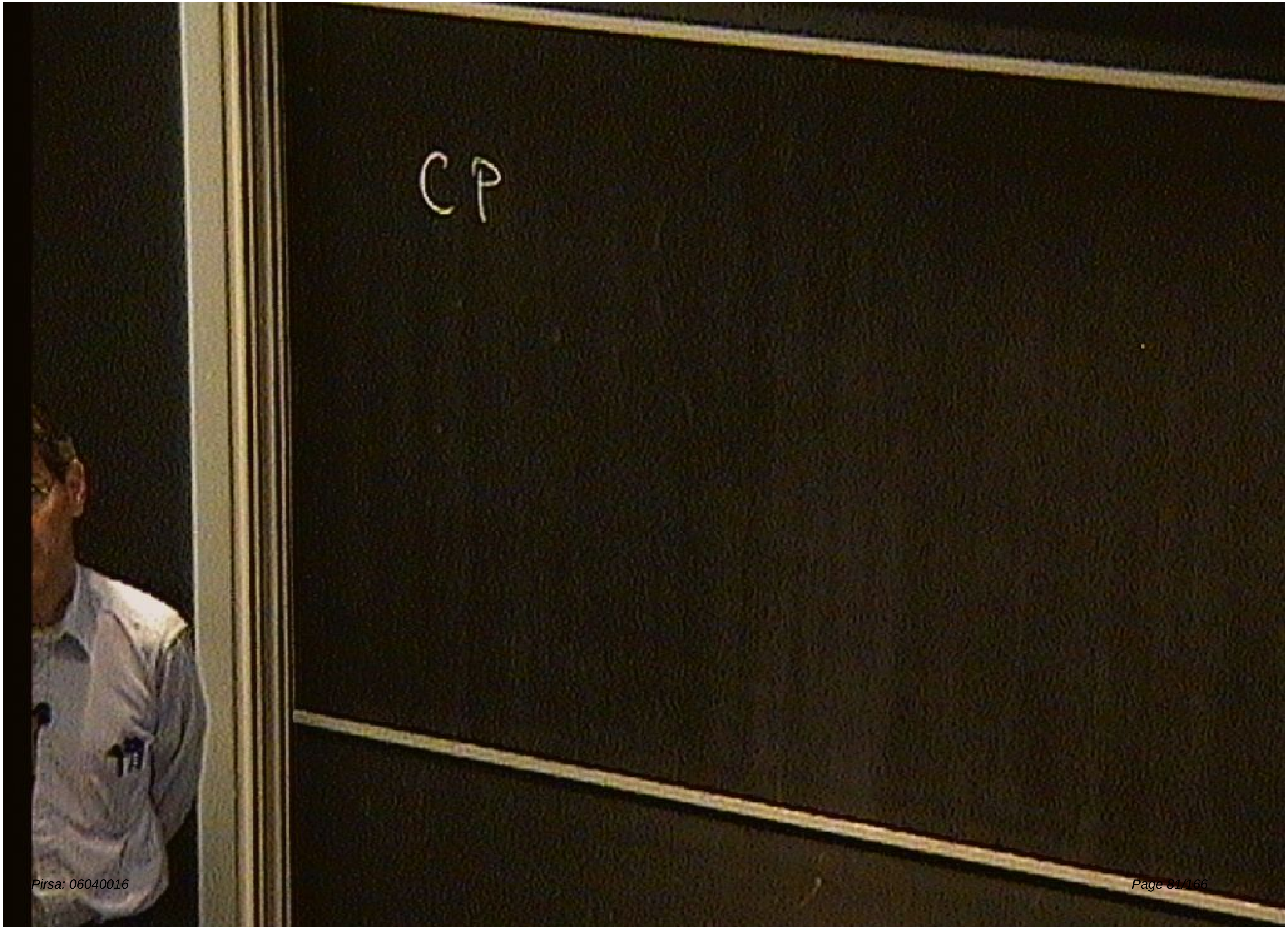
(2) dynamically determined Hilbert space and inner product —"bootstrap" theory

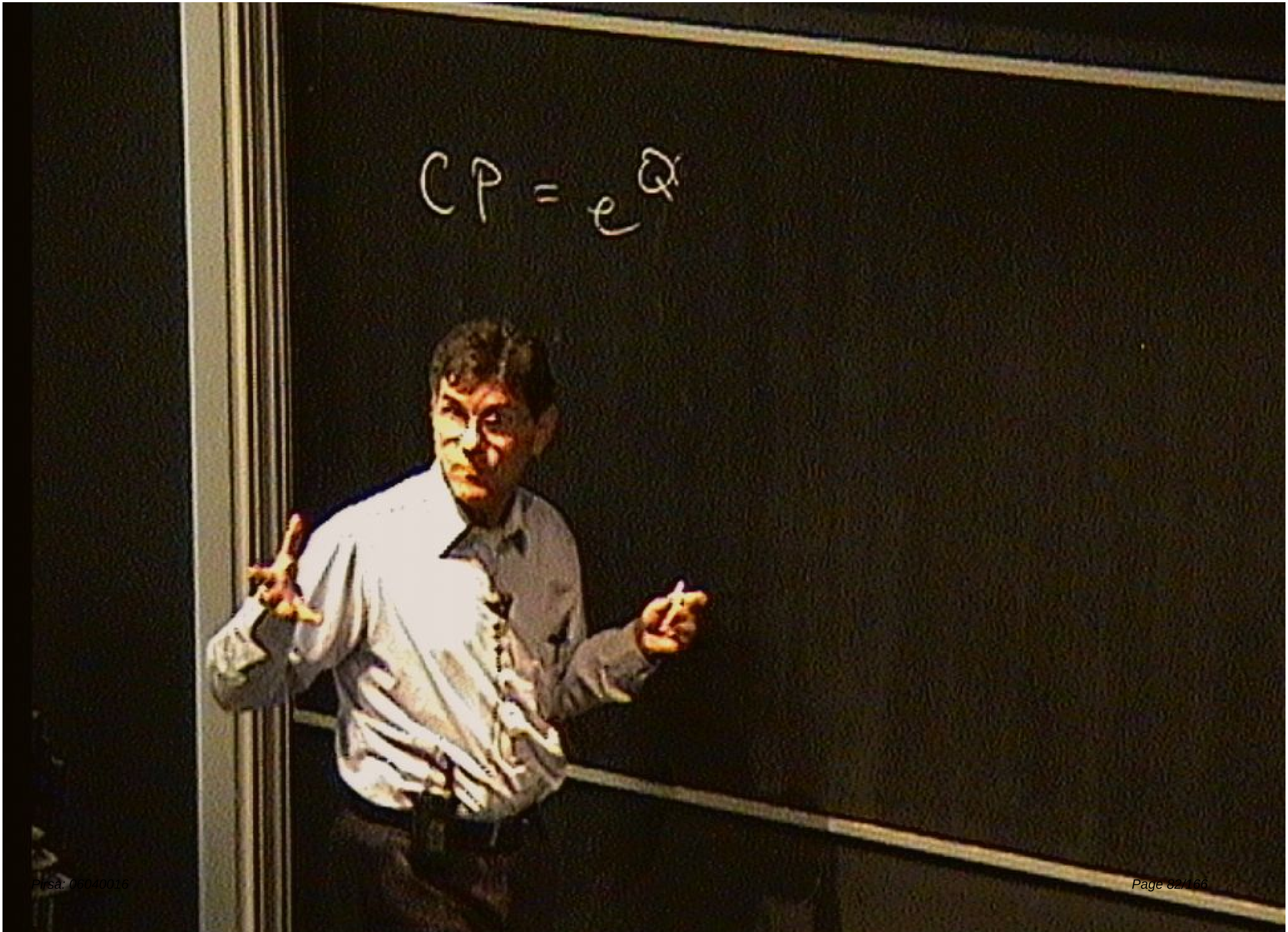
(3) $[C, P] \neq 0$ so "particles" and "antiparticle" states need not have same mass

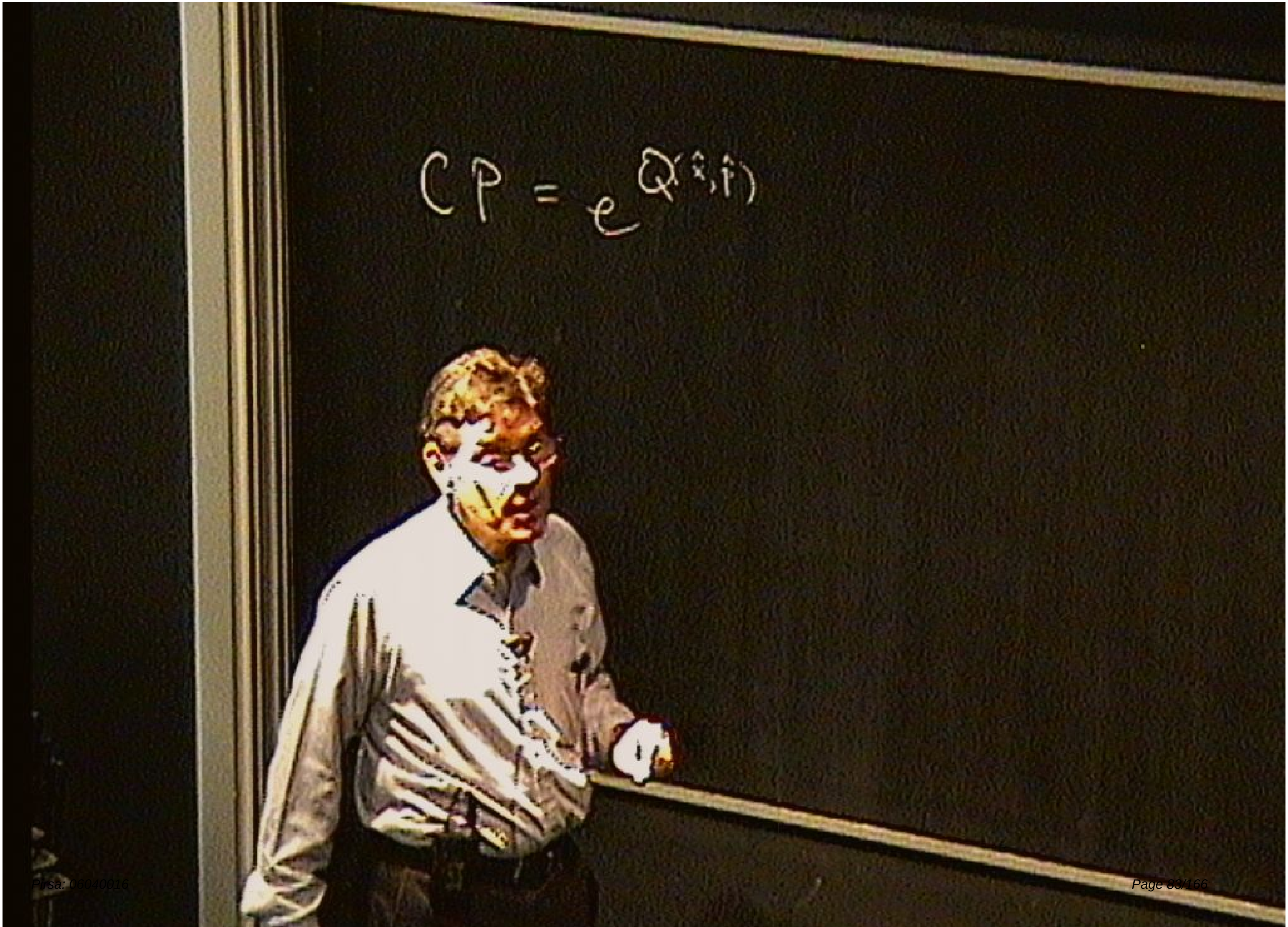
(4) Lots of new models — $i\varphi^3$, $-\varphi^4$, ...

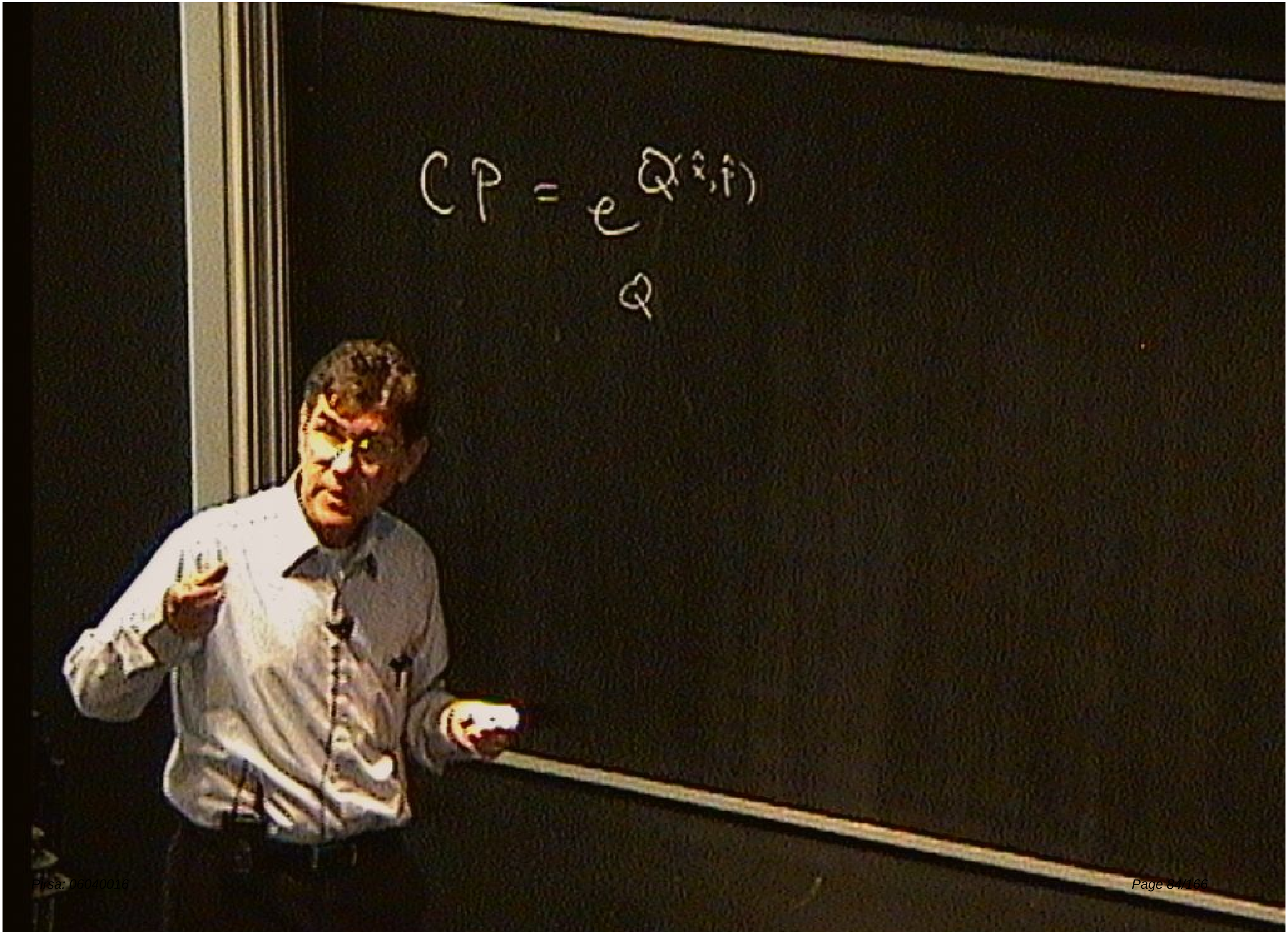
Overview of Talk

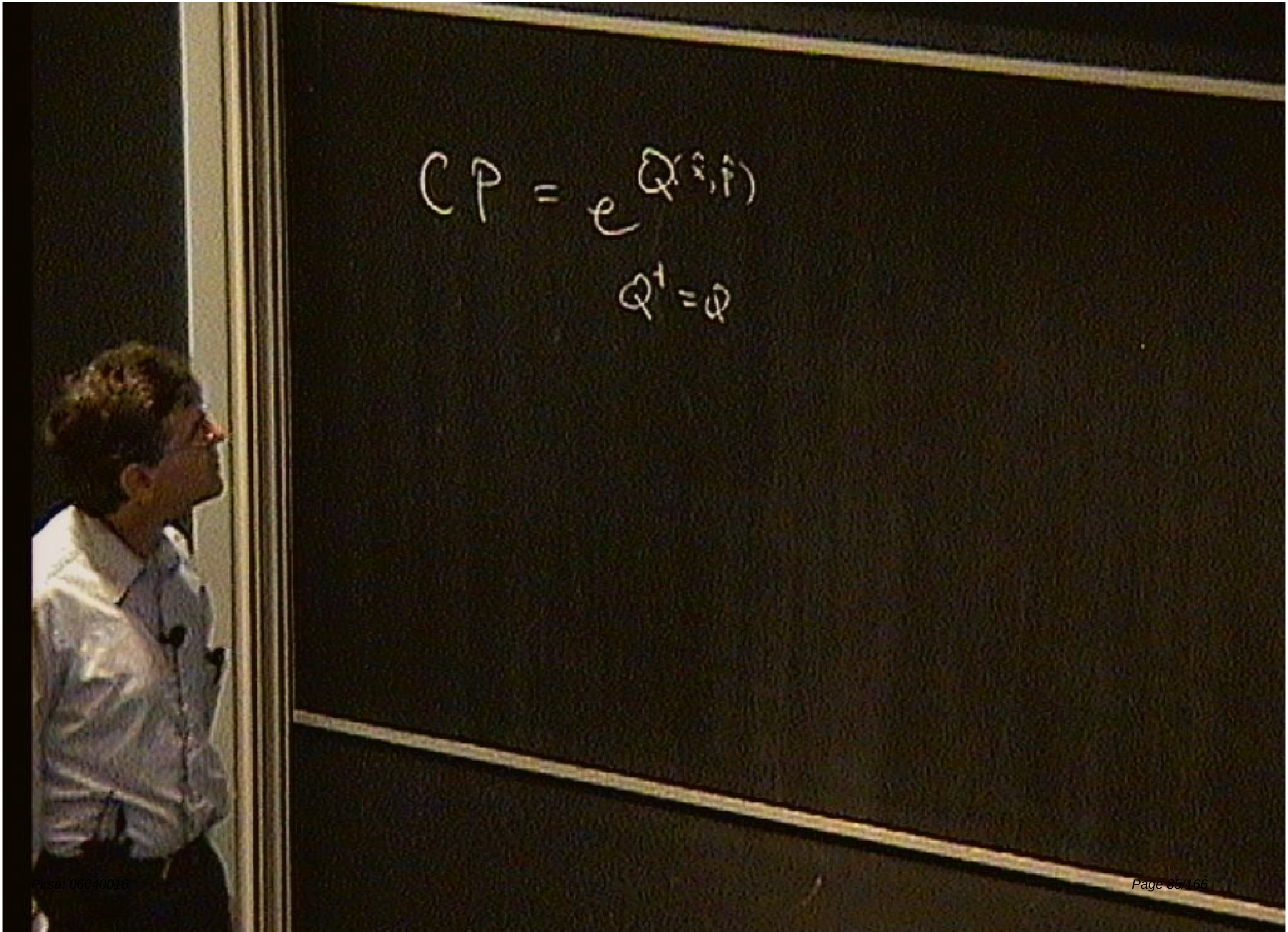


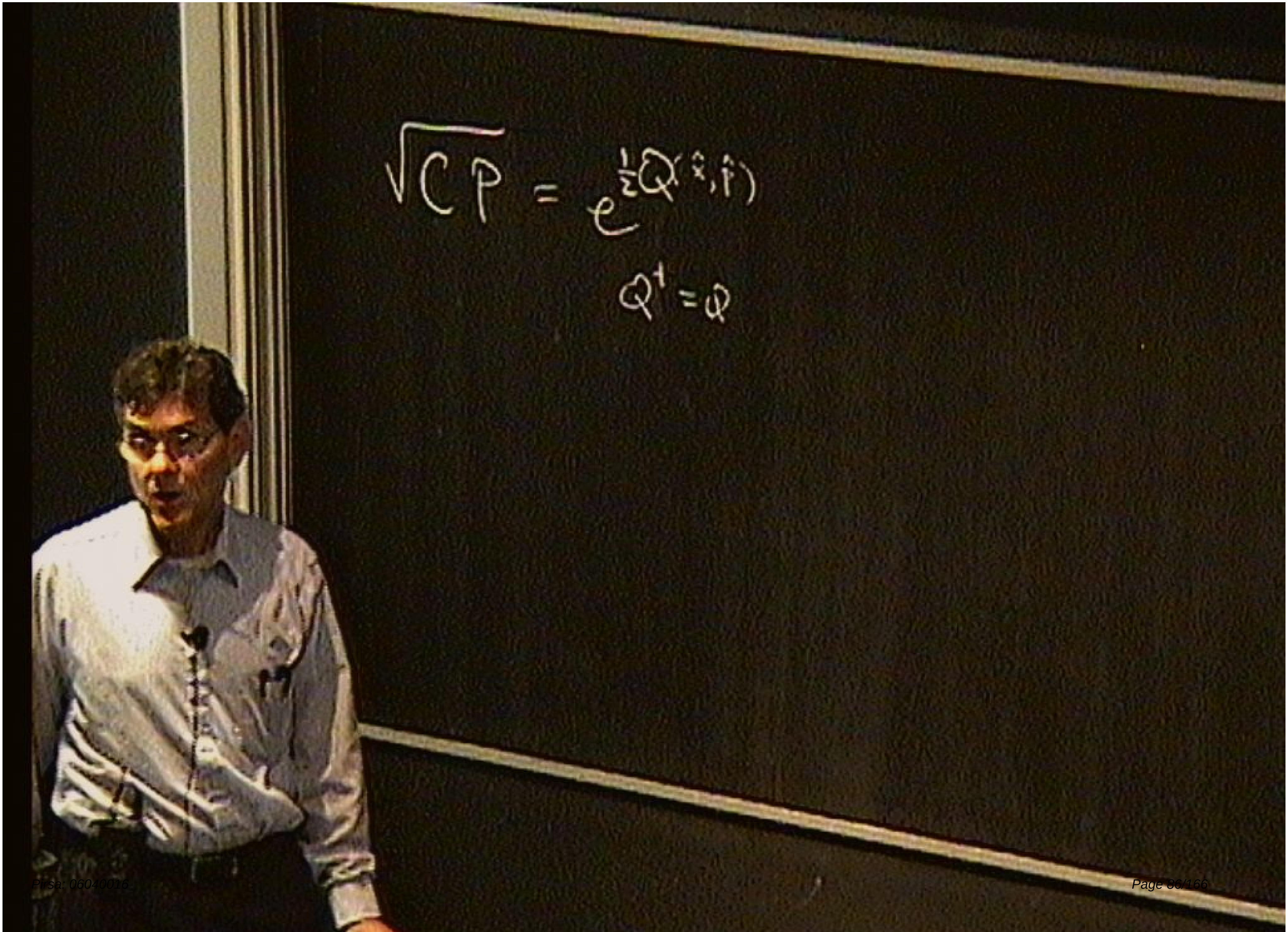

CP

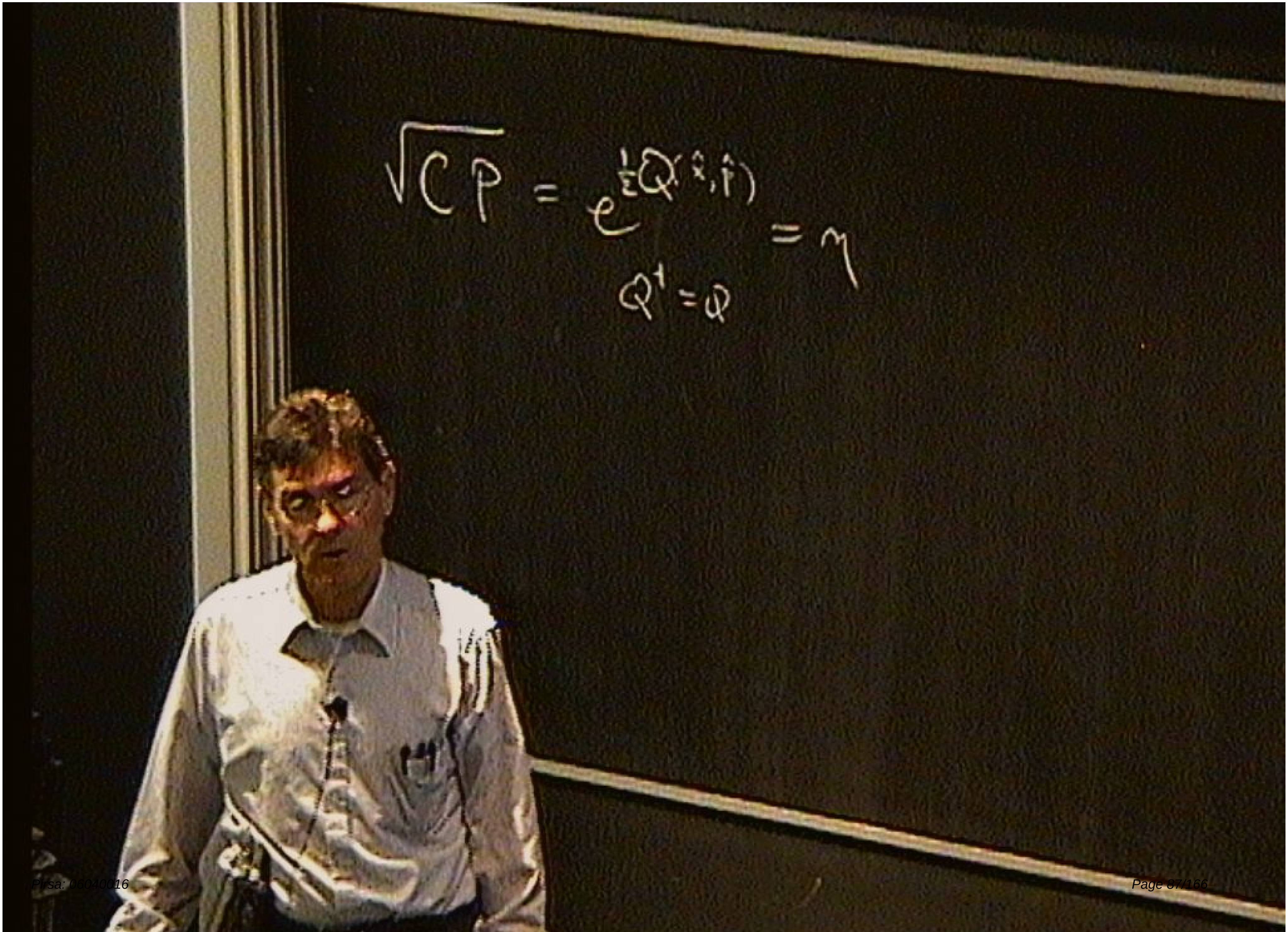

$CP = e^{Q}$

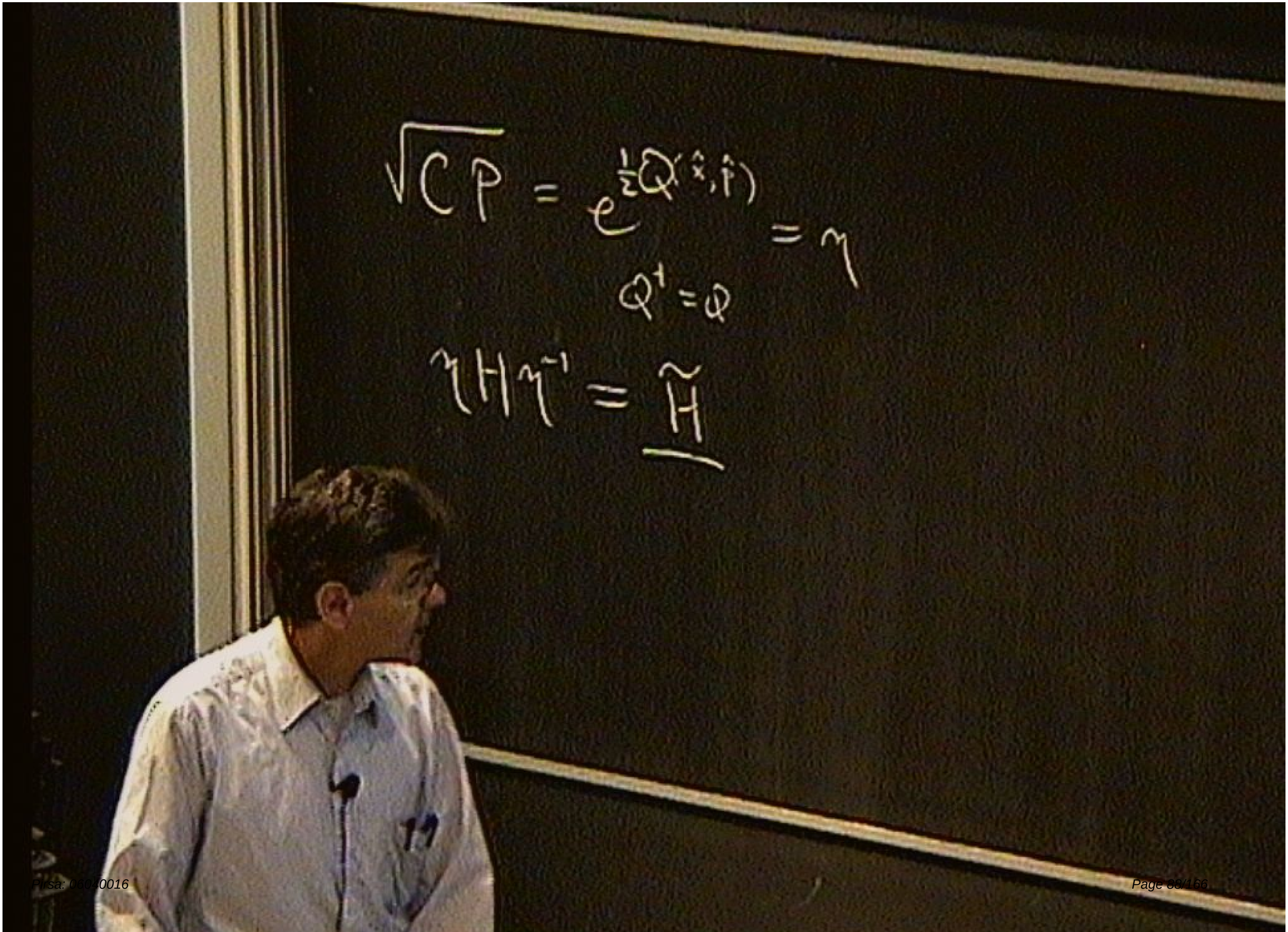

$CP = e^{Q(\hat{x},\hat{p})}$

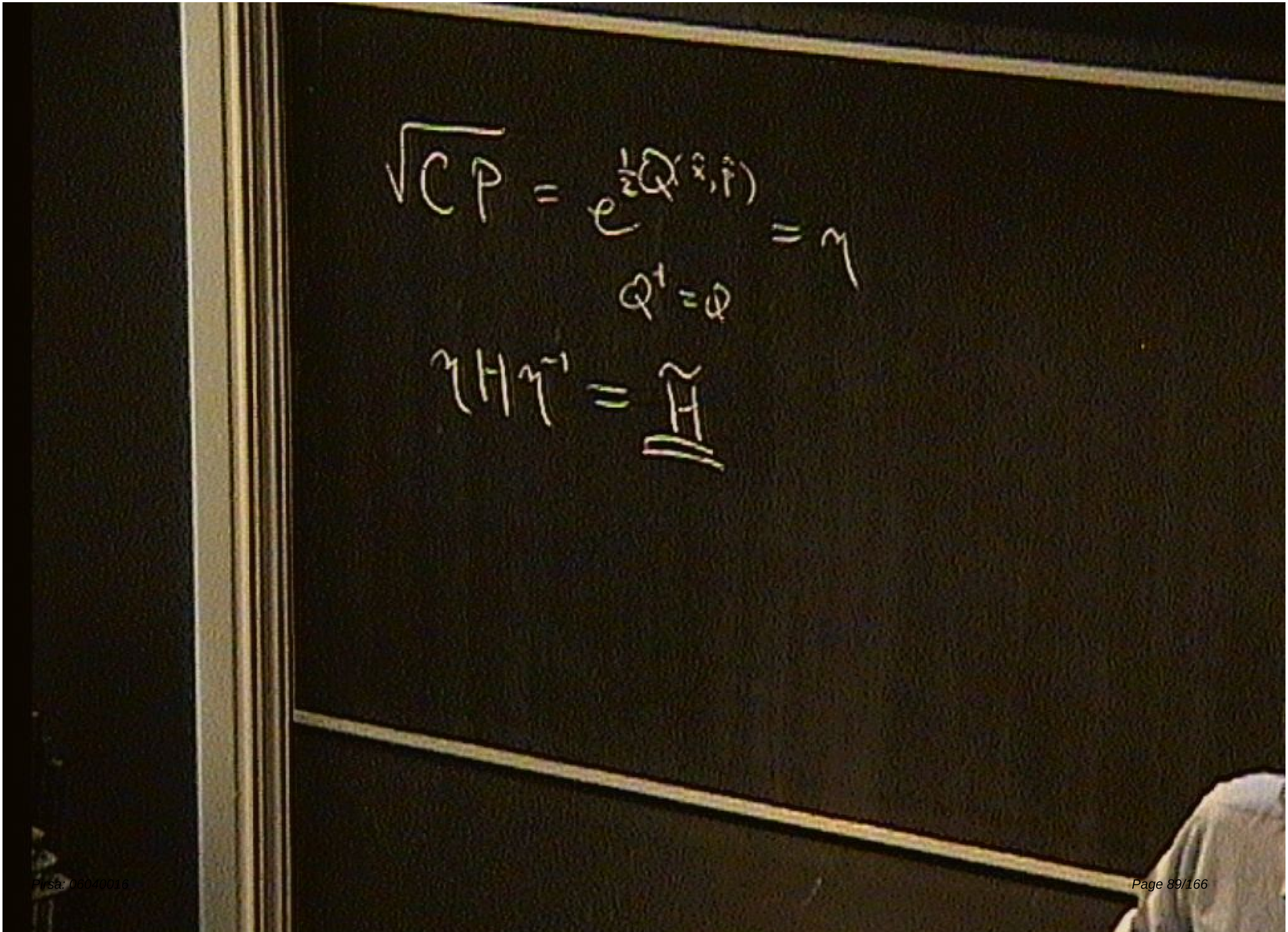

$CP = e^{Q(\hat{x},\hat{p})}$
Q


$CP = e^{Q(\hat{x},\hat{p})}$
$Q^{\dagger} = Q$


$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})}$
$Q^\dagger = Q$


$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})} = \eta$
$Q^\dagger = Q$


$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})} = \eta$
$Q^\dagger = Q$
$\eta H \eta^{-1} = \tilde{H}$



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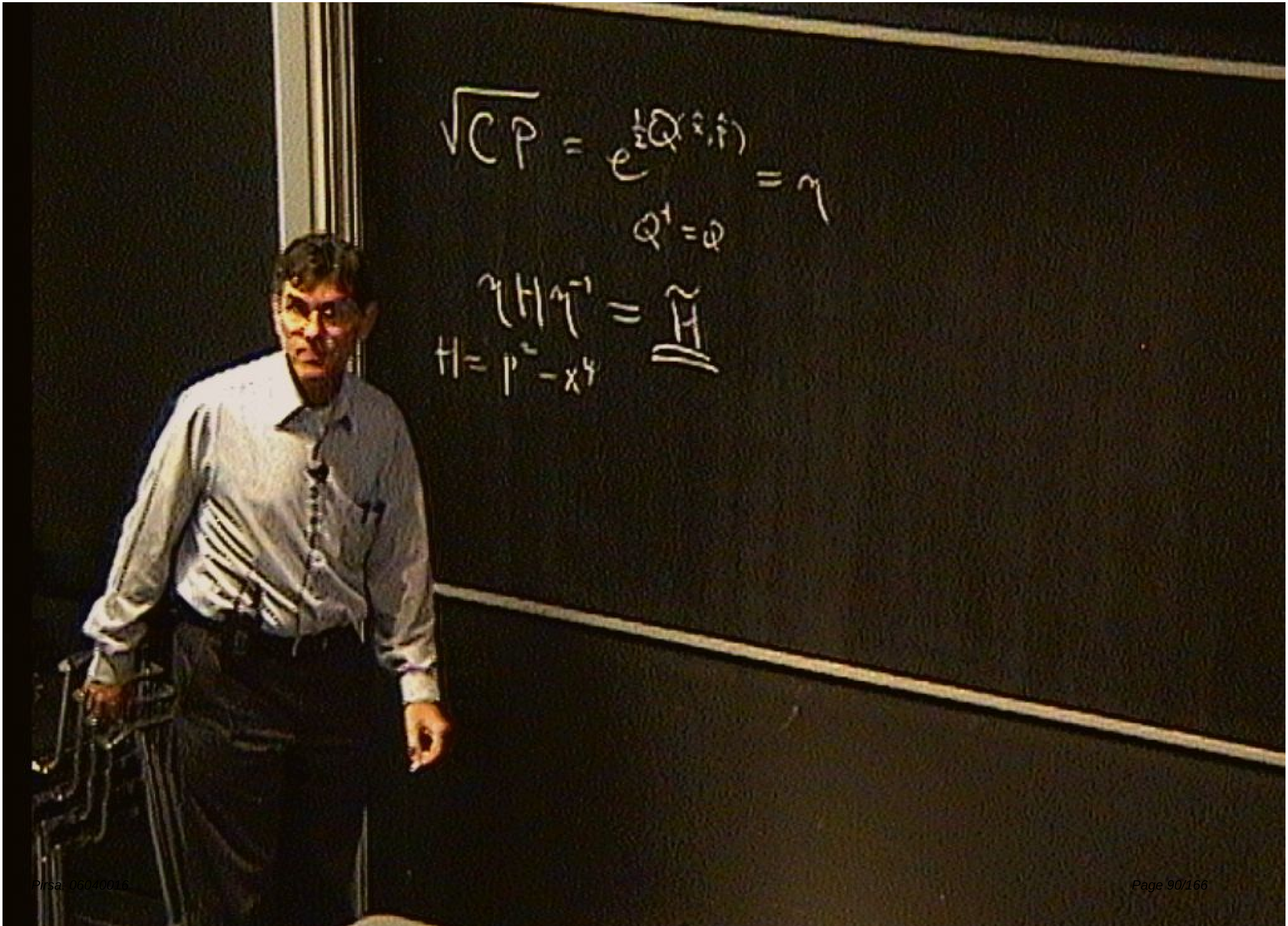
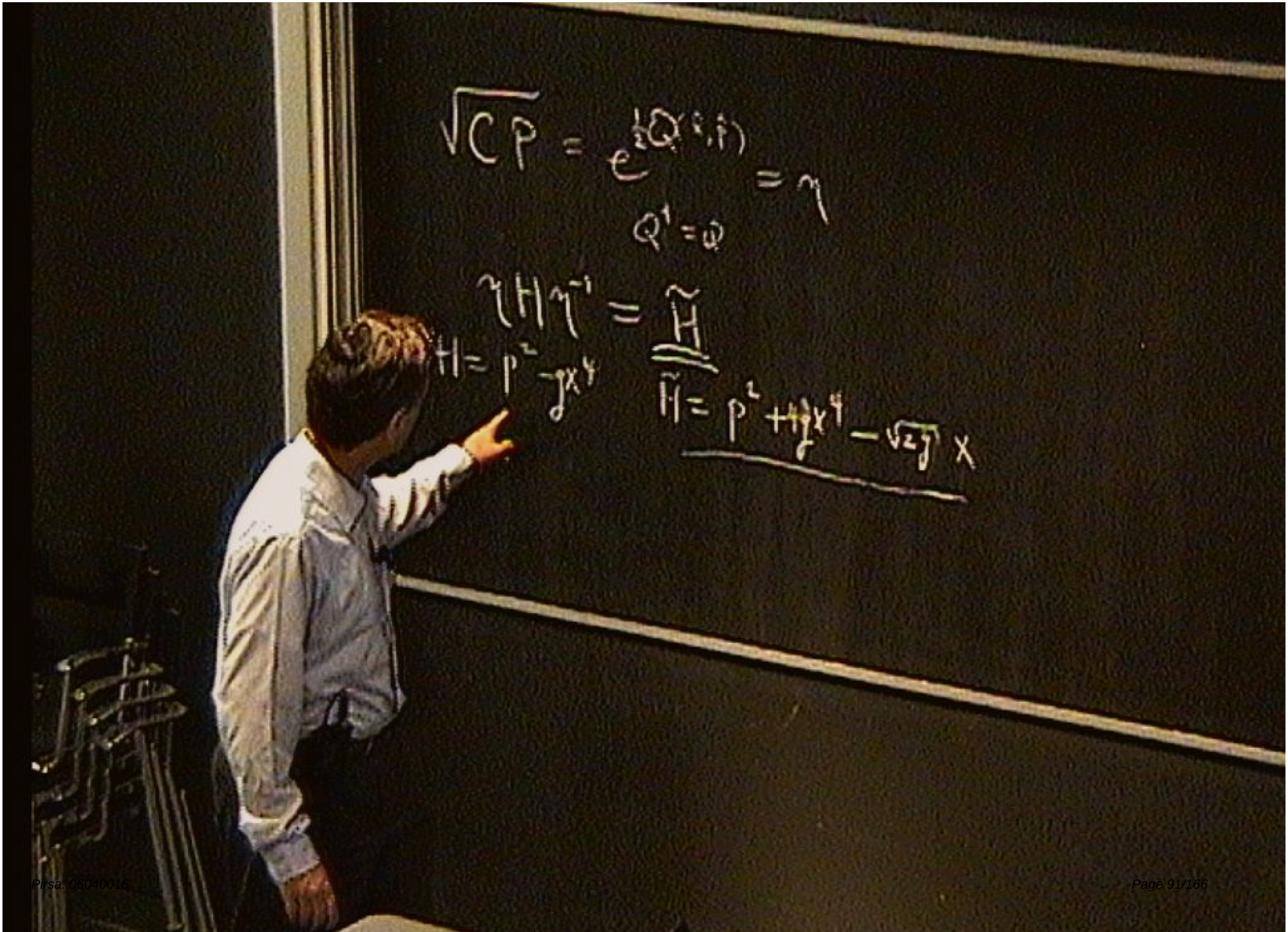
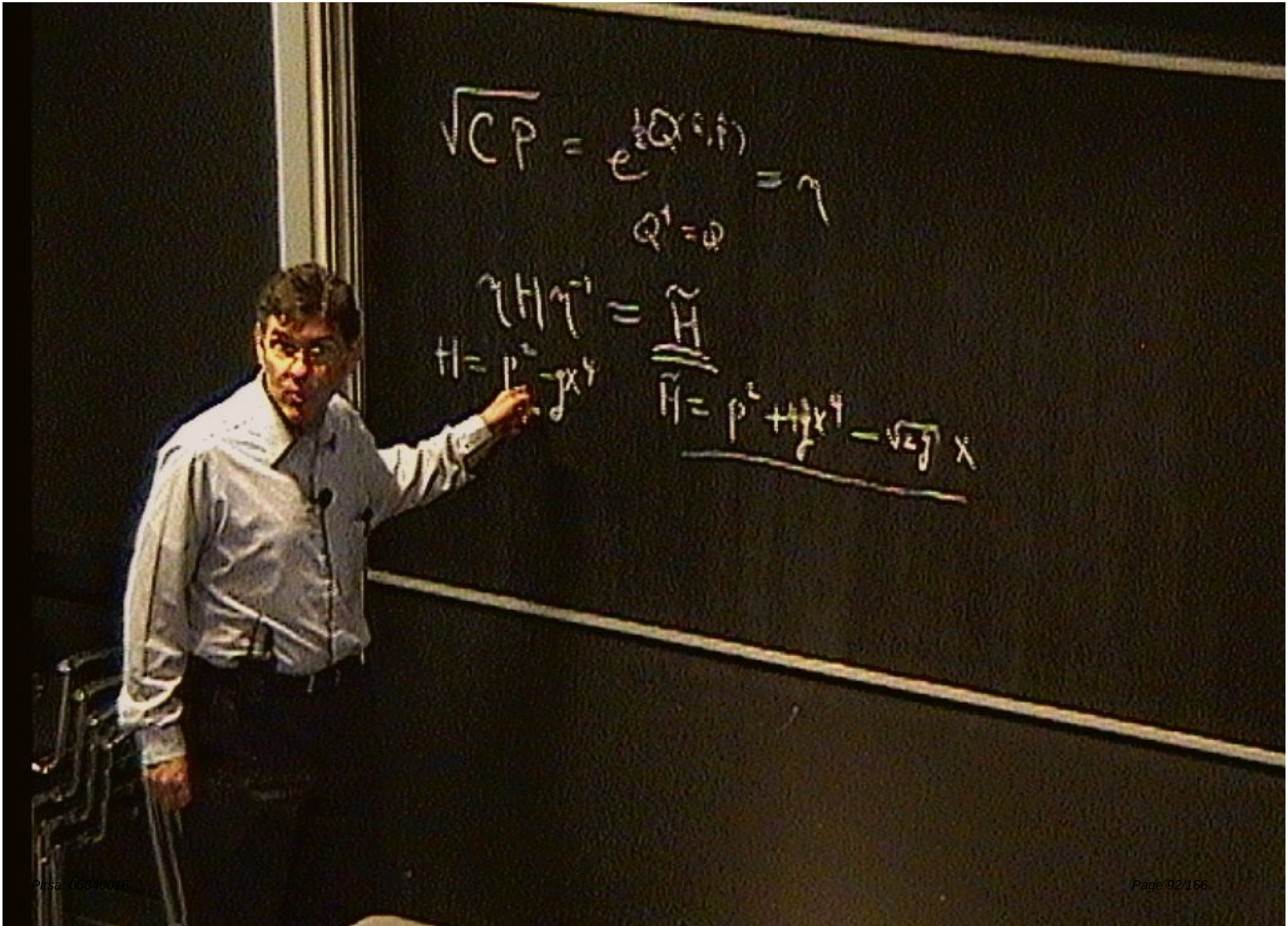
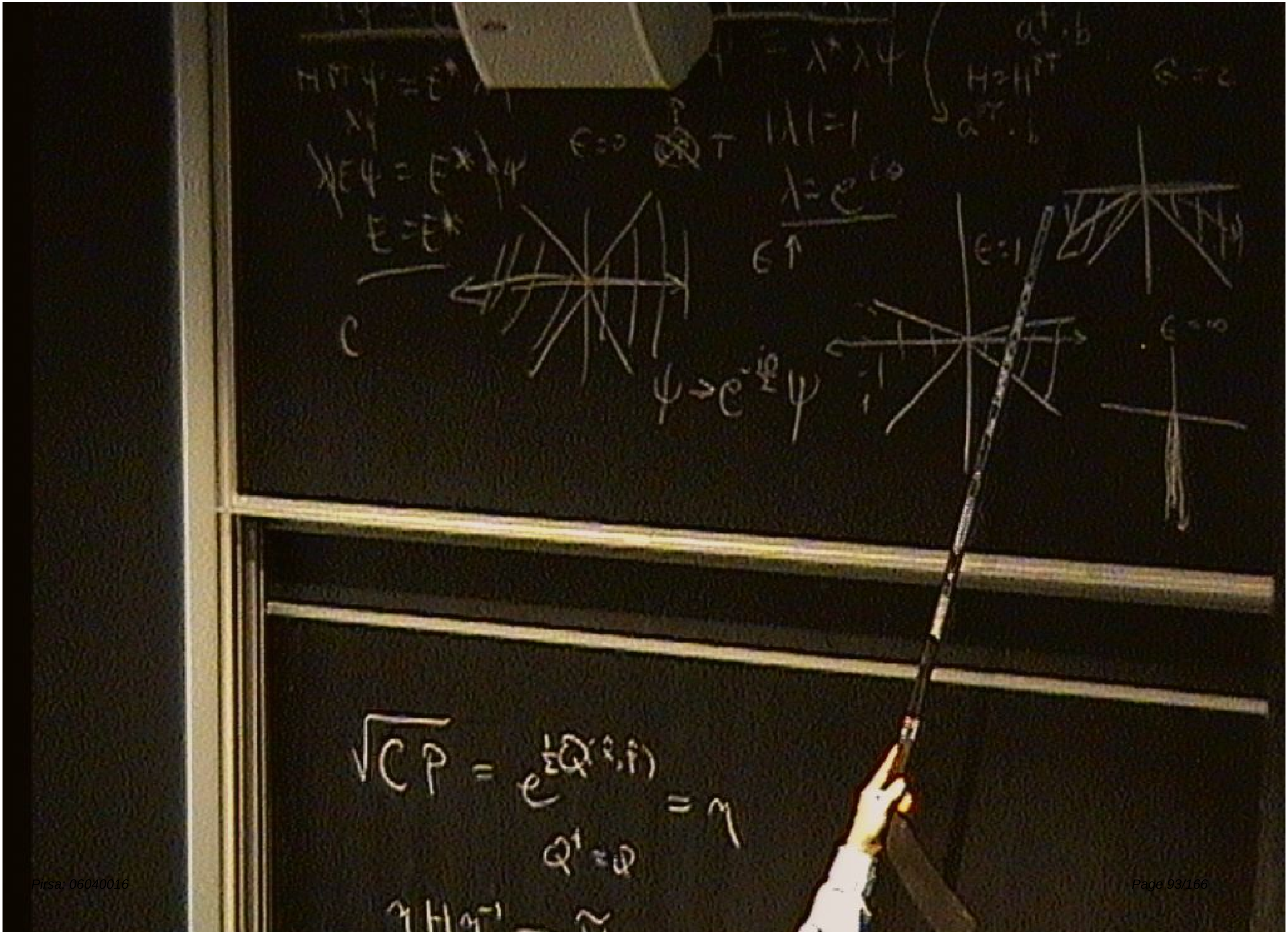
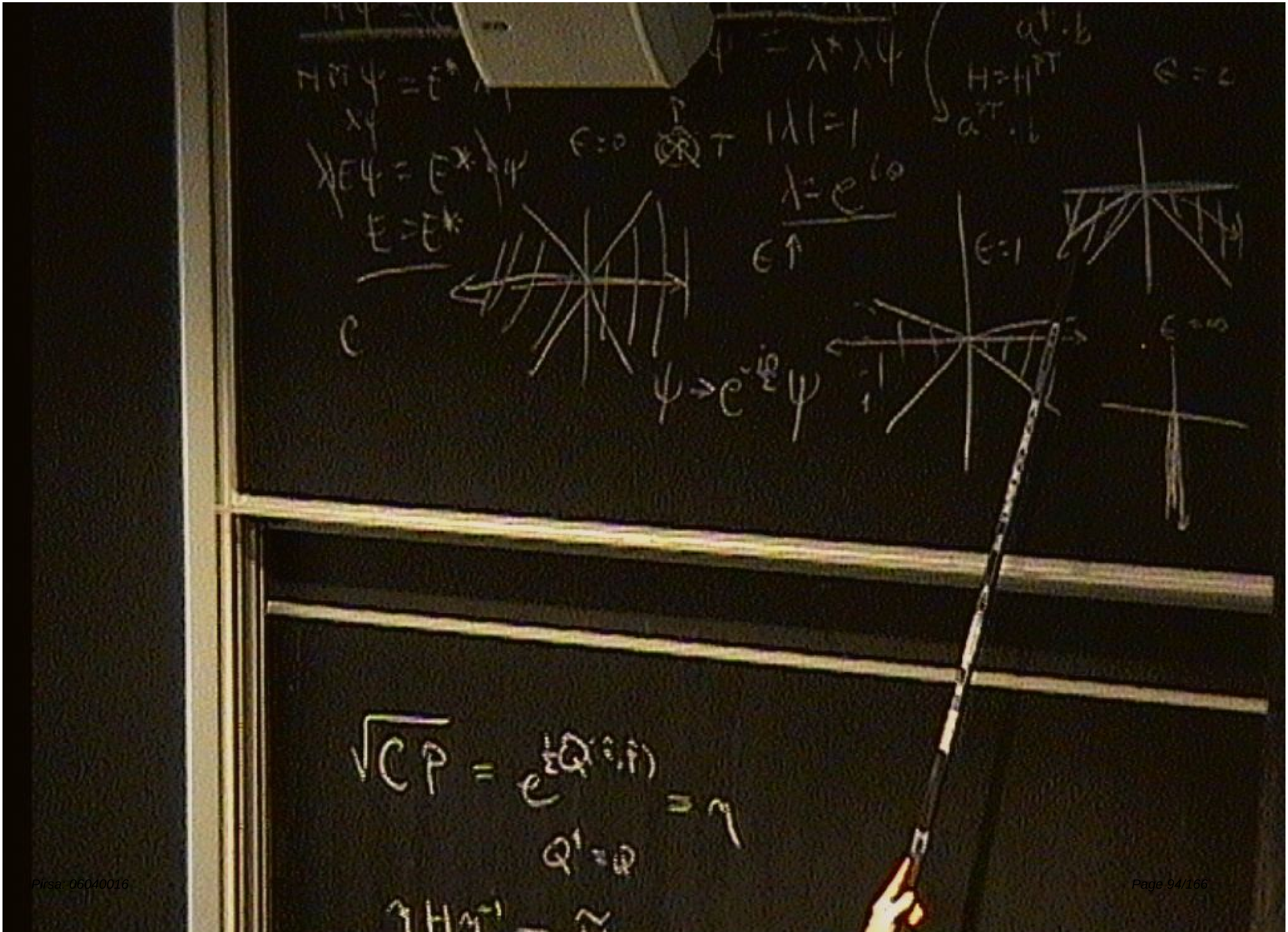
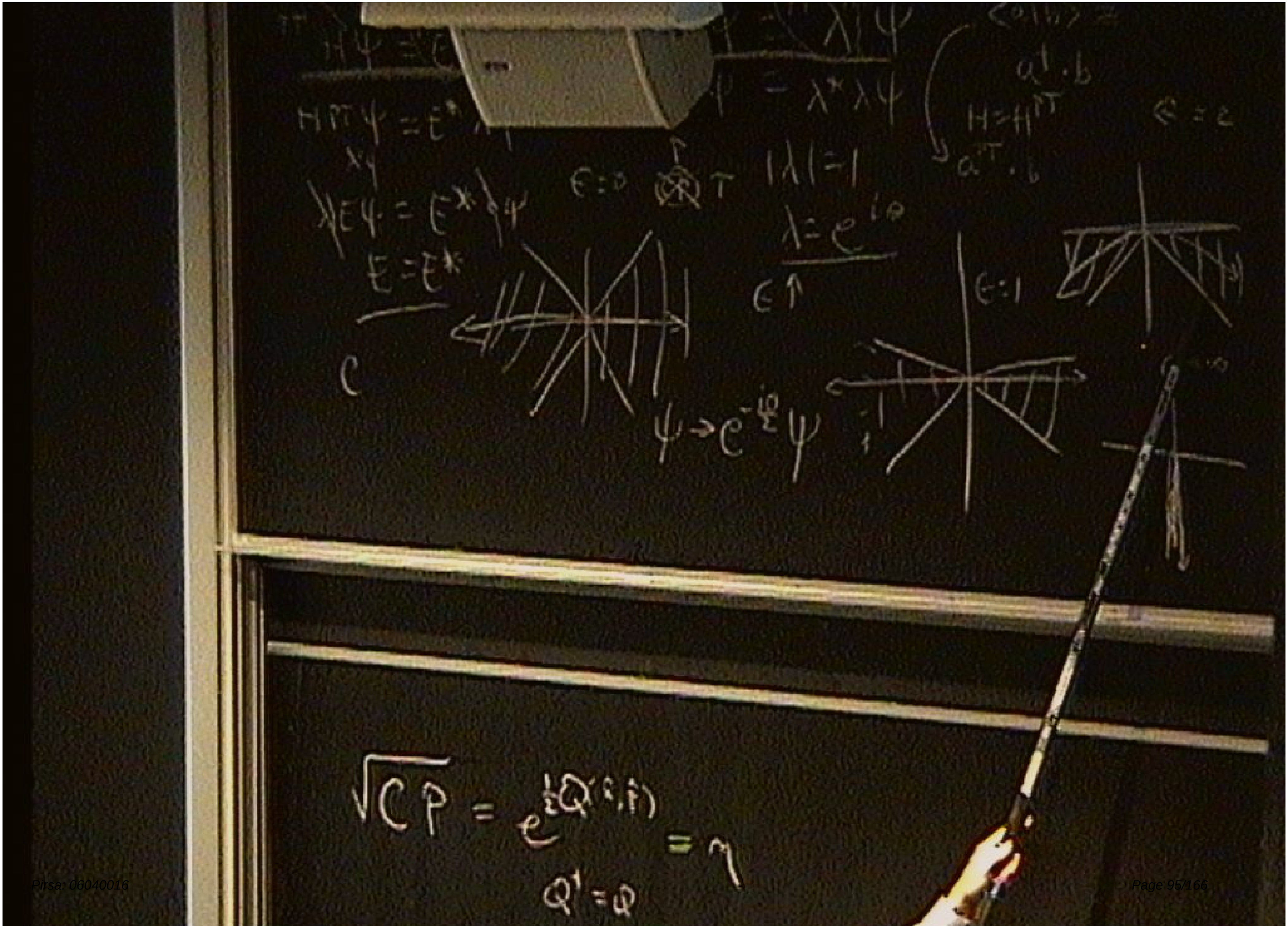

$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})} = \eta$
$Q^{\dagger} = Q$
$\eta H \eta^{-1} = \tilde{H}$
$H = p^2 - x^4$

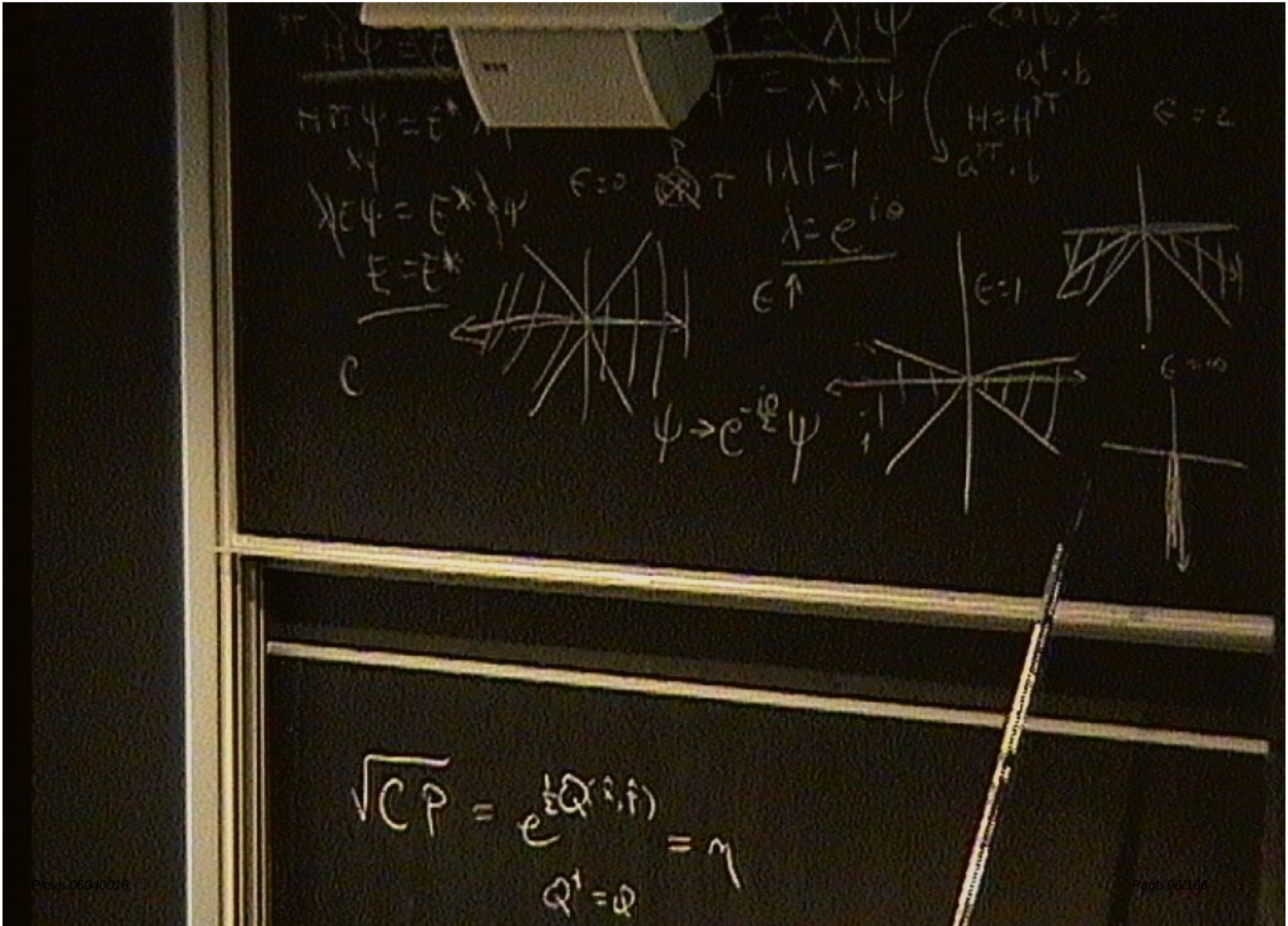

√CP = e^{½Q(x,p)} = η
Q† = Q
ηHη⁻¹ = H̃
H = p² − gx⁴
H̃ = p² + 4gx⁴ − √2g x

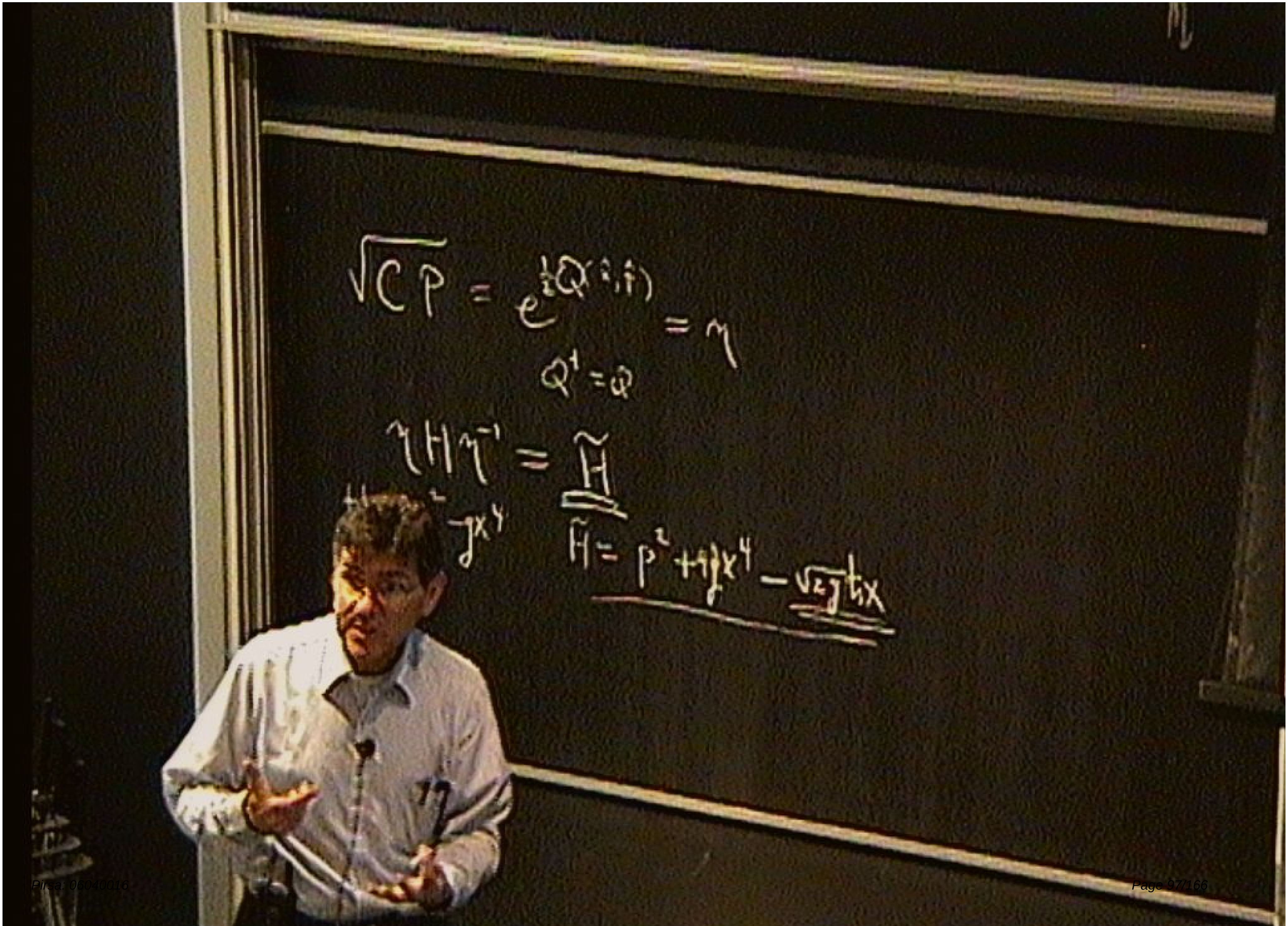

$\sqrt{CP} = e^{Q(q,p)} = \eta$
$Q' = Q$
$\eta H \eta^{-1} = \tilde{H}$
$H = p^2 - gx^4$








ε = ε*
ε↑
ε=1
√CP =
= η
Q† = Q


√CP = e^{Q(x,p)} = η
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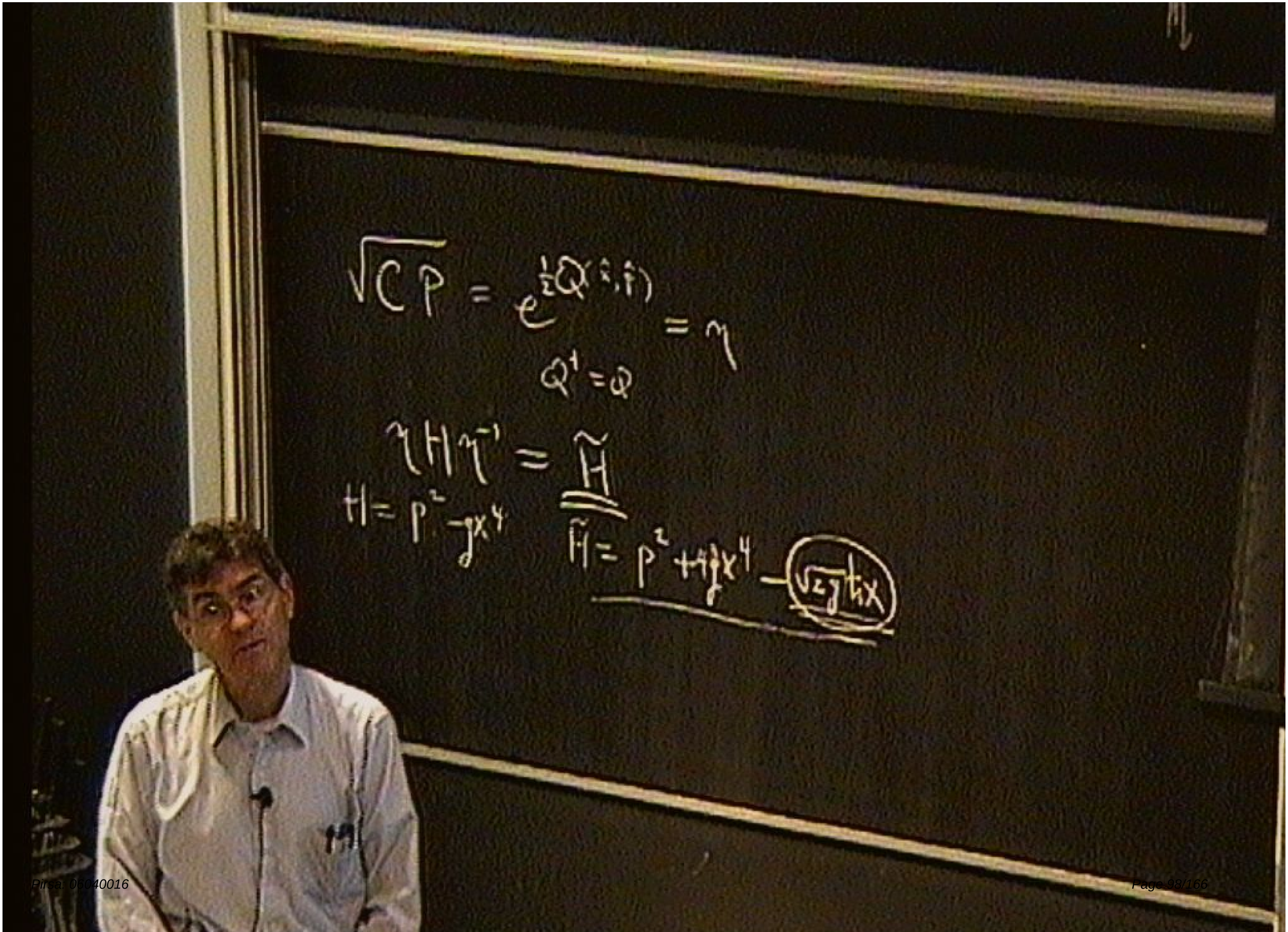
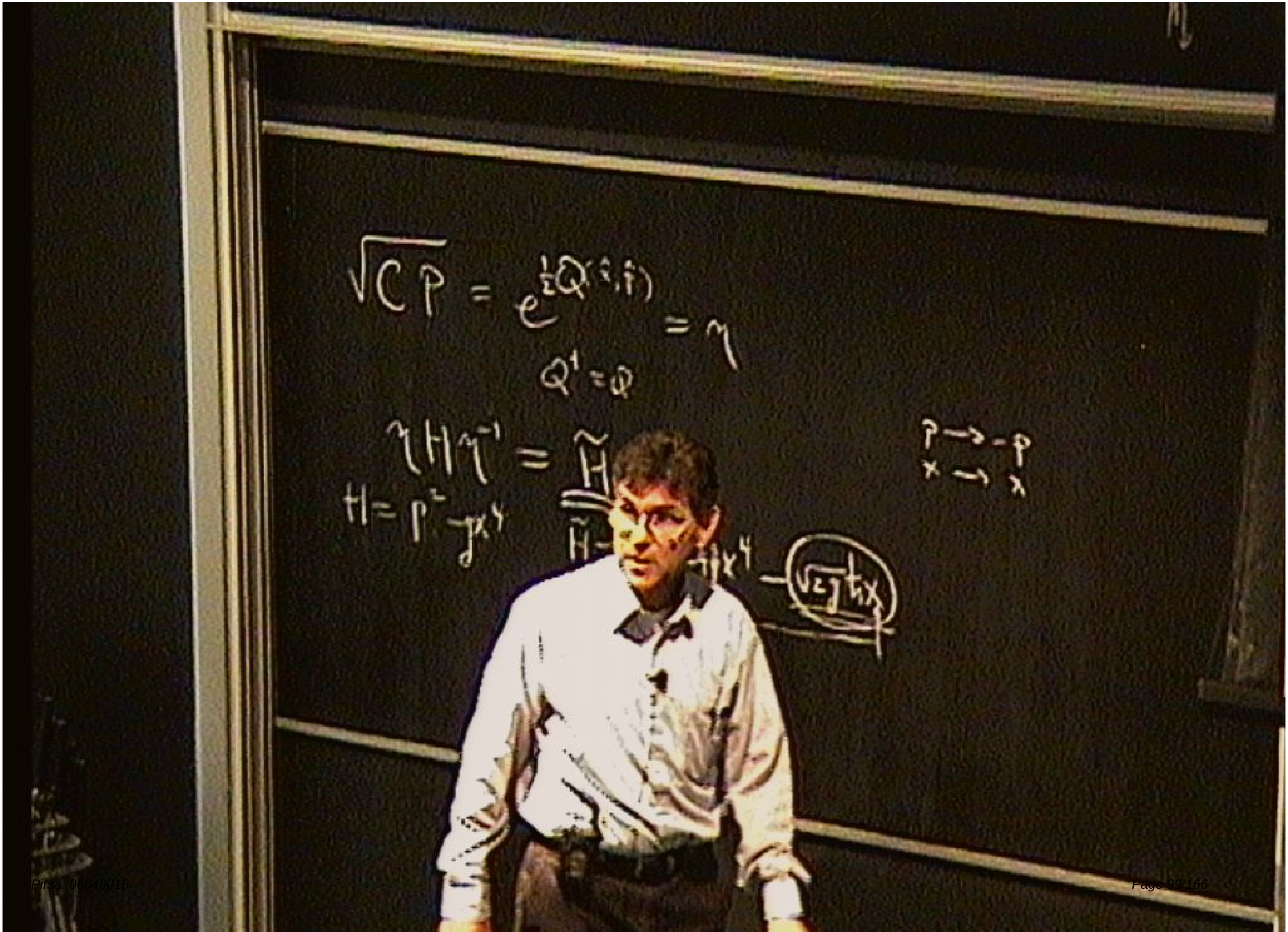
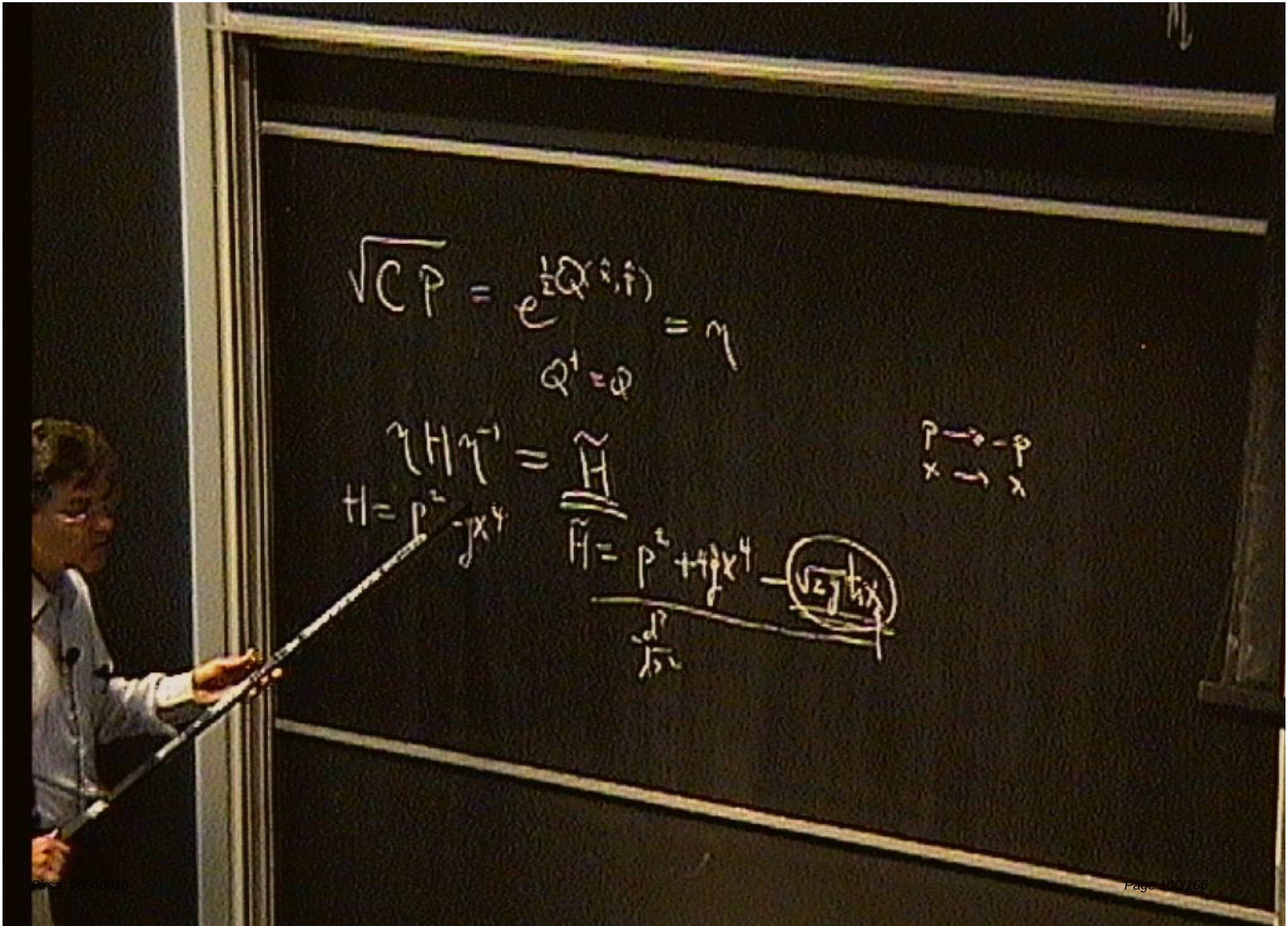
ηHη⁻¹ = H̃
H̃ = p² + 4gx⁴ − √2g ħx


$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})} = \eta$
$Q^\dagger = Q$
$\eta H \eta^{-1} = \tilde{H}$
$H = p^2 - gx^4$
$\tilde{H} = p^2 + 4gx^4 - \sqrt{2g}\,\hbar x$



√CP = e^{½Q(x,p)} = η
Q† = Q
ηHη⁻¹ = H̃
H = p² − gx⁴
p → −p
x → x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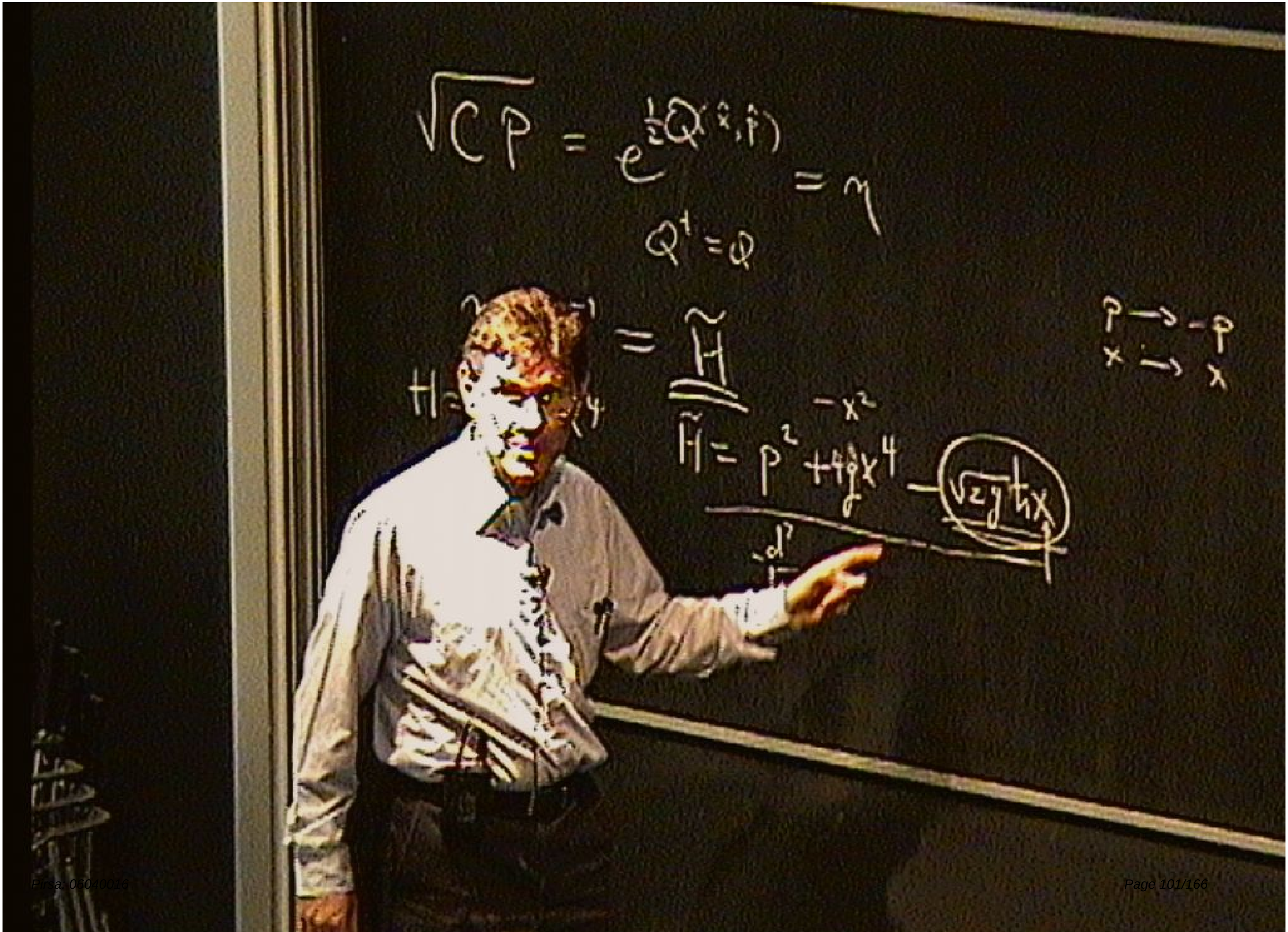
p → −p
x → x



$\sqrt{CP} = e^{\frac{1}{2}Q(\hat{x},\hat{p})} = \eta$
$Q^\dagger = Q$
$p \to -p$
$x \to x$

Applications and properties of theories defined by PT-symmetric Hamiltonians


CHRYSLER
It's PT cool in the open air.
Featuring:
Air Conditioning
See Dealer for Details


Revell
PT-109
PT 109

Welcome to PT's Show Club. You must be 21 or older to enter. Enter birth date bel

(Please enter your birthday in the following format: mm/dd/yyyy, then click the "Card Me" button)

Card Me

1950's T. T. Wu

quantization of interacting hard spheres

1970's H. Abarbanel, J. Bronzan, R. Sugar, A. White, R. Brower, M. Furman, M. Moshe, C.-I Tan, ...

Reggeon field theory $i\varphi^3$

1970's – 80's M. E. Fisher, J. Cardy, G. Mussardo, A. Zamolodchikov, ...

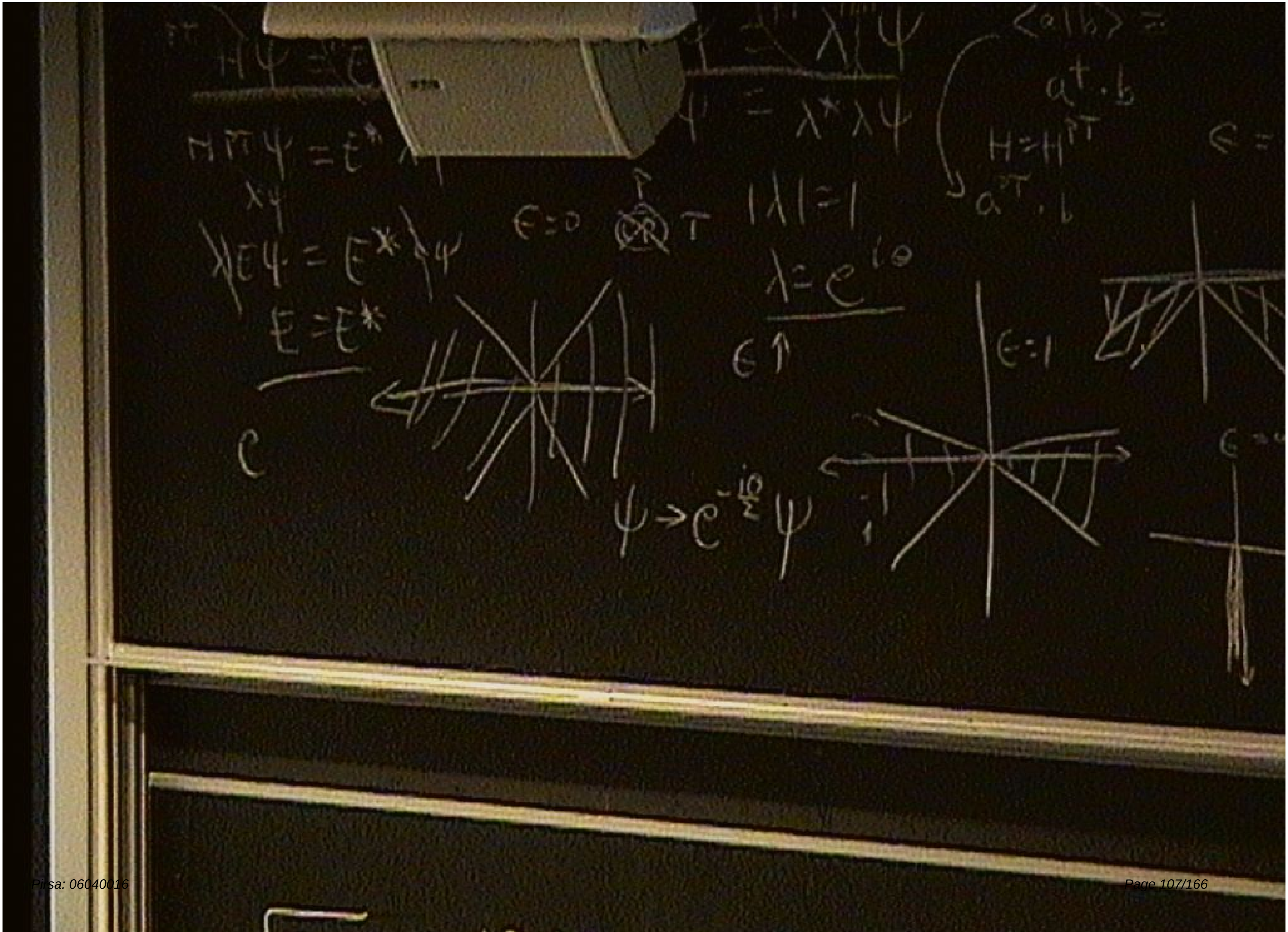
Lee-Yang edge singularity $i\varphi^3$

1980's T. Hollowood, D. Olive

Complex Toda lattices

2000's A. Zeilinger

laser experiments $H = p^2 + i \sin x$, $H = p^2 + e^{ix}$



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2000's A. Zeilinger

laser experiments $H = p^2 + i \sin x$, $H = p^2 + e^{ix}$

Other Interesting models

1) Higgs Sector

$$\mathcal{H}_I = -g\varphi^4$$

- $\langle\varphi\rangle \neq 0$
- dimensionless g
- asymp. free!

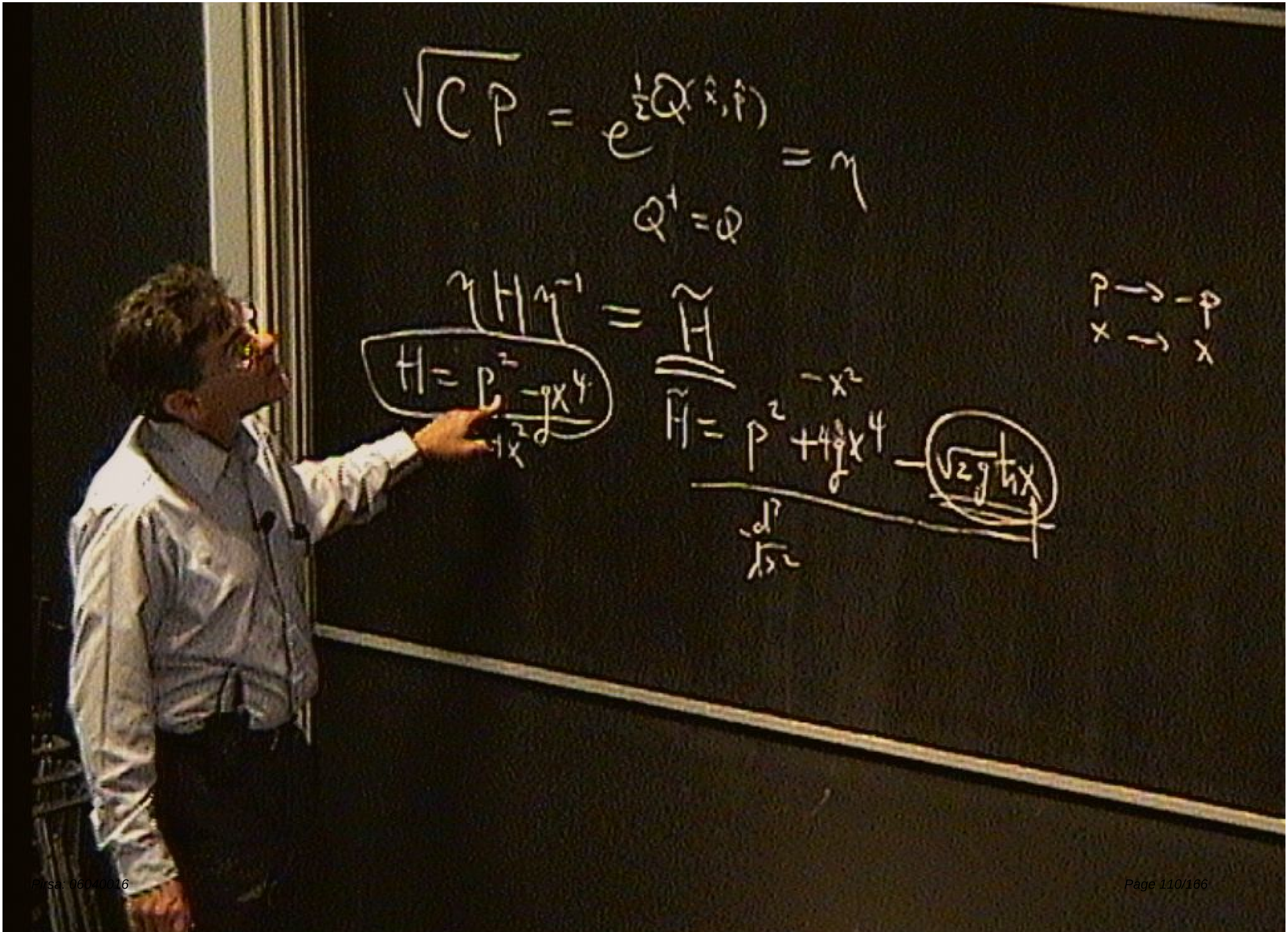
2) PT QED

- asymp. free
- determines own coupling const. (Johnson, Baker, Willey, Adler, Gell-Mann-Low)
- Casimir (Boyer)

3) Bound states of $-g\varphi^4$

4) PT gravity — repulsive force

5) dS-CFT correspondence / AdS-CFT correspondence in cosmological models — Witten, Horowitz, Strominger, etc.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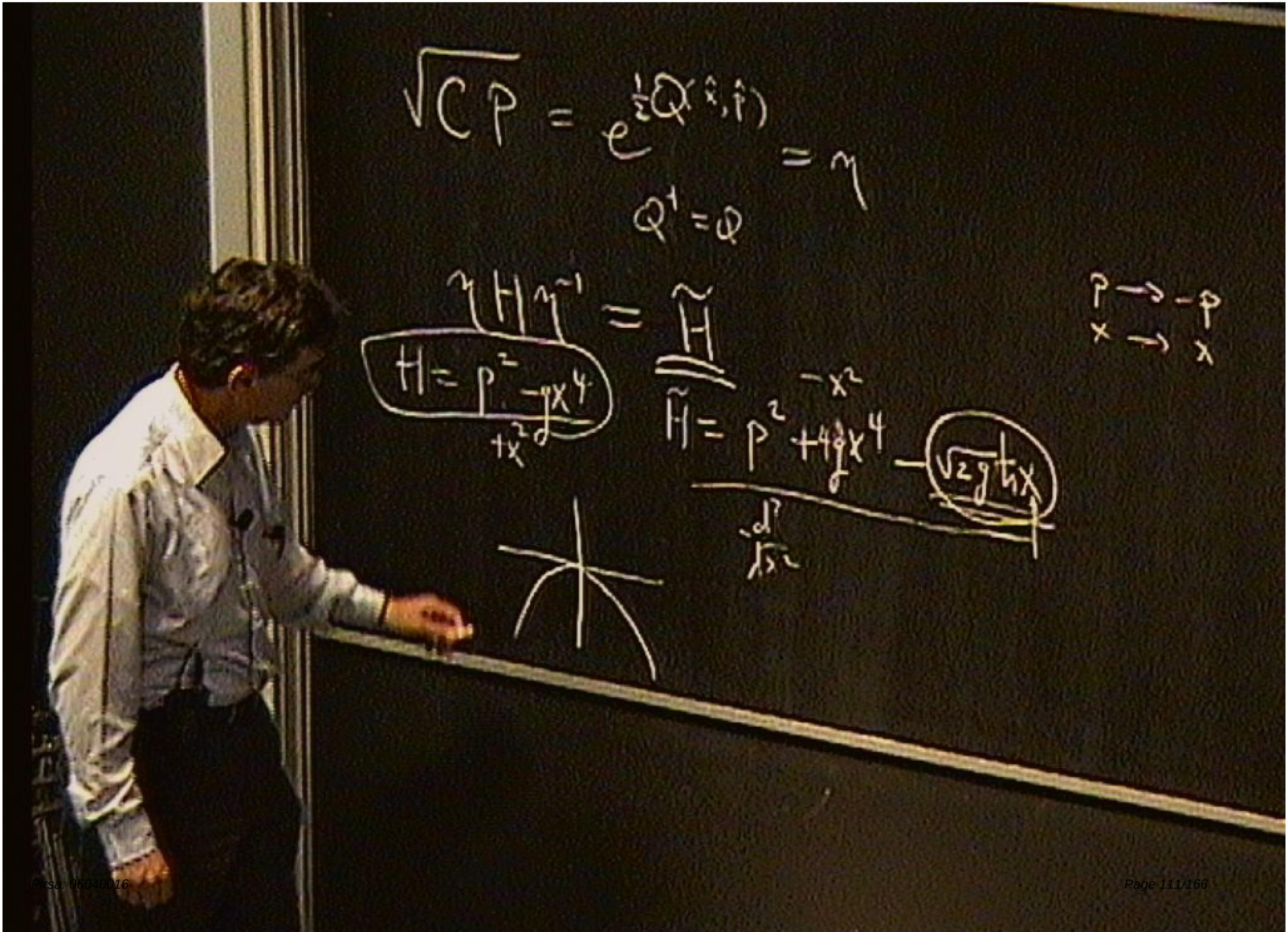
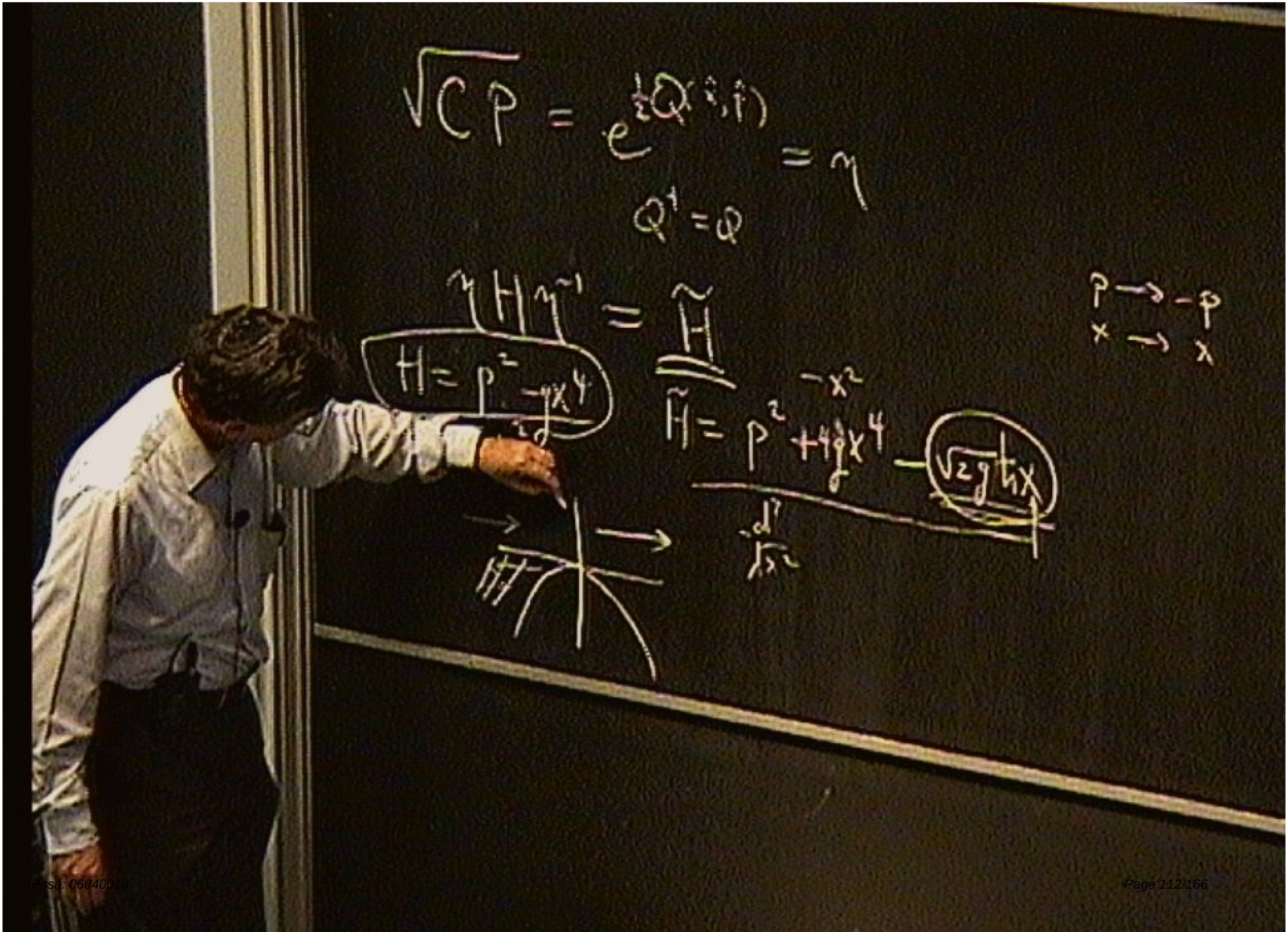
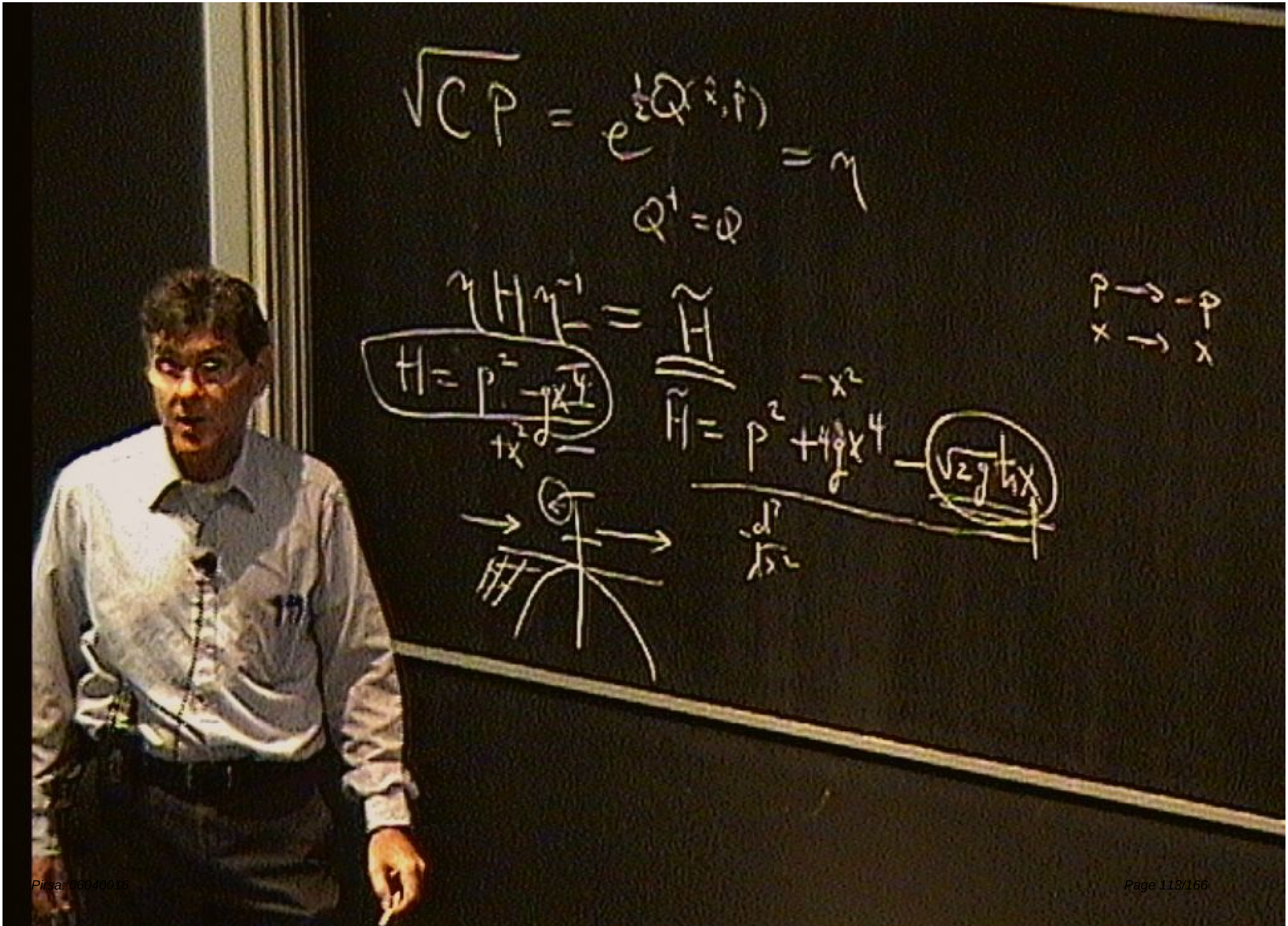
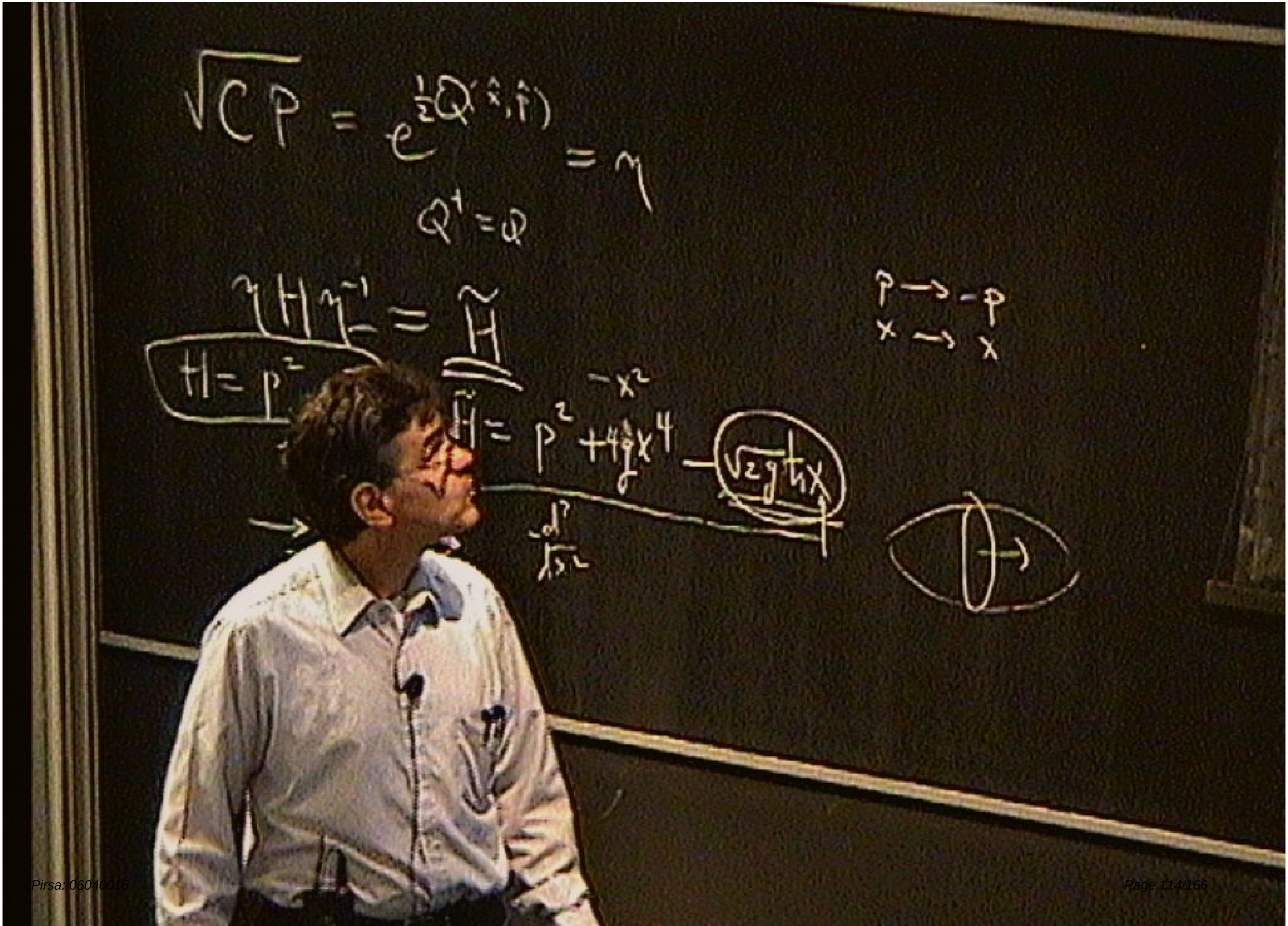
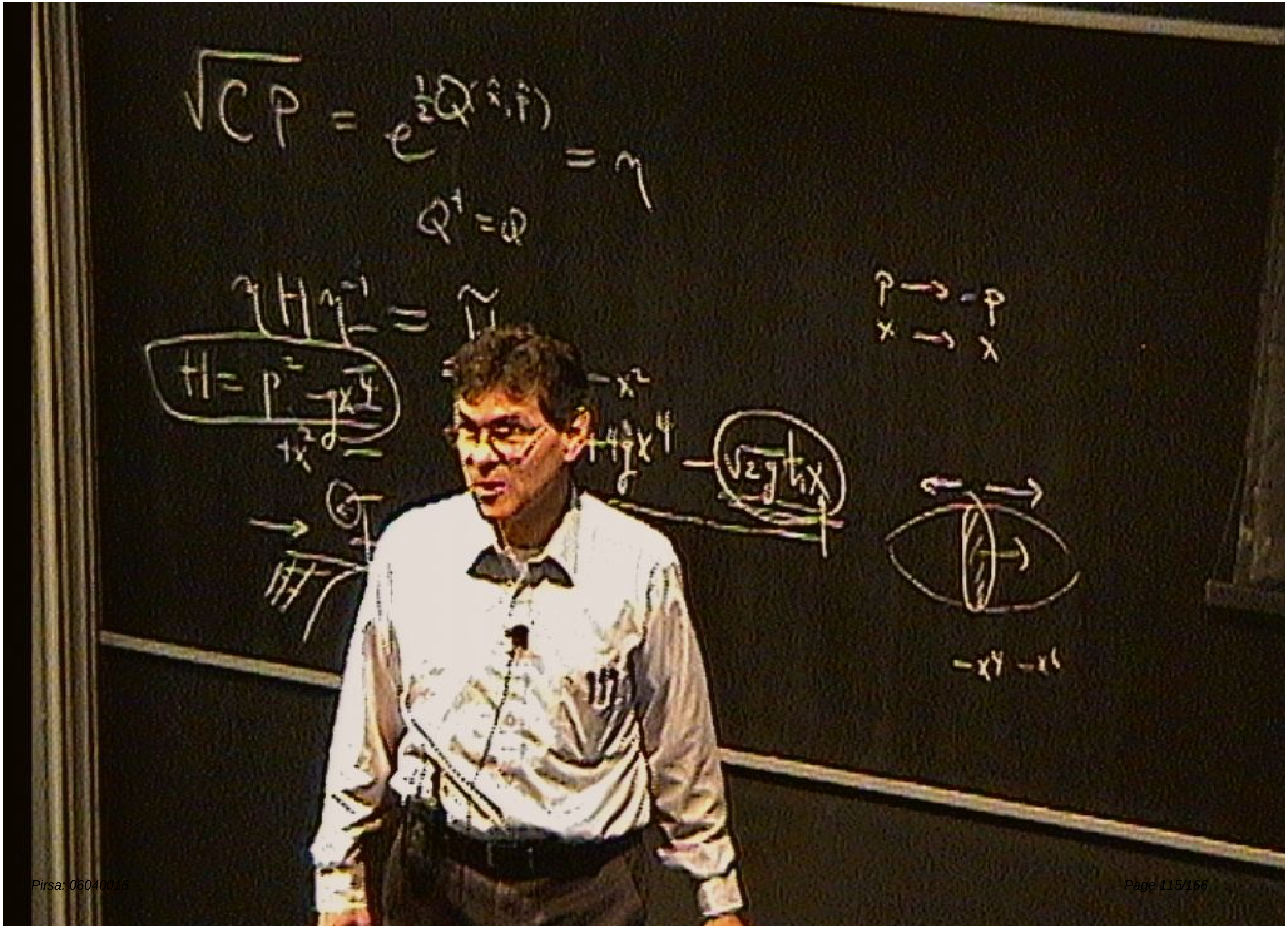
Q† = Q
ηHη⁻¹ = H̃
H = p² − gx⁴
H̃ = p² + 4gx⁴ − √2g ħx
p → −p
x → x


√CP = e^{½Q(x̂,p̂)} = η
Q† = Q
η H η⁻¹ = H̃
H = p² − gx⁴
+x²
H̃ = p² + 4gx⁴ − √2g ħ x
−x²
p → −p
x → x


√CP = e^{½Q(x,p)} = η
Q† = Q
η H η⁻¹ = H̃
H = p² − gx⁴
H̃ = p² + 4gx⁴ − √2g ħx
p → −p
x → x


Q† = Q
H = p² − gx⁴
p → −p
x → x


√CP = e^{½Q(x̂,p̂)} = η
Q† = Q
ηHη⁻¹ = H̃
p → −p
x → x



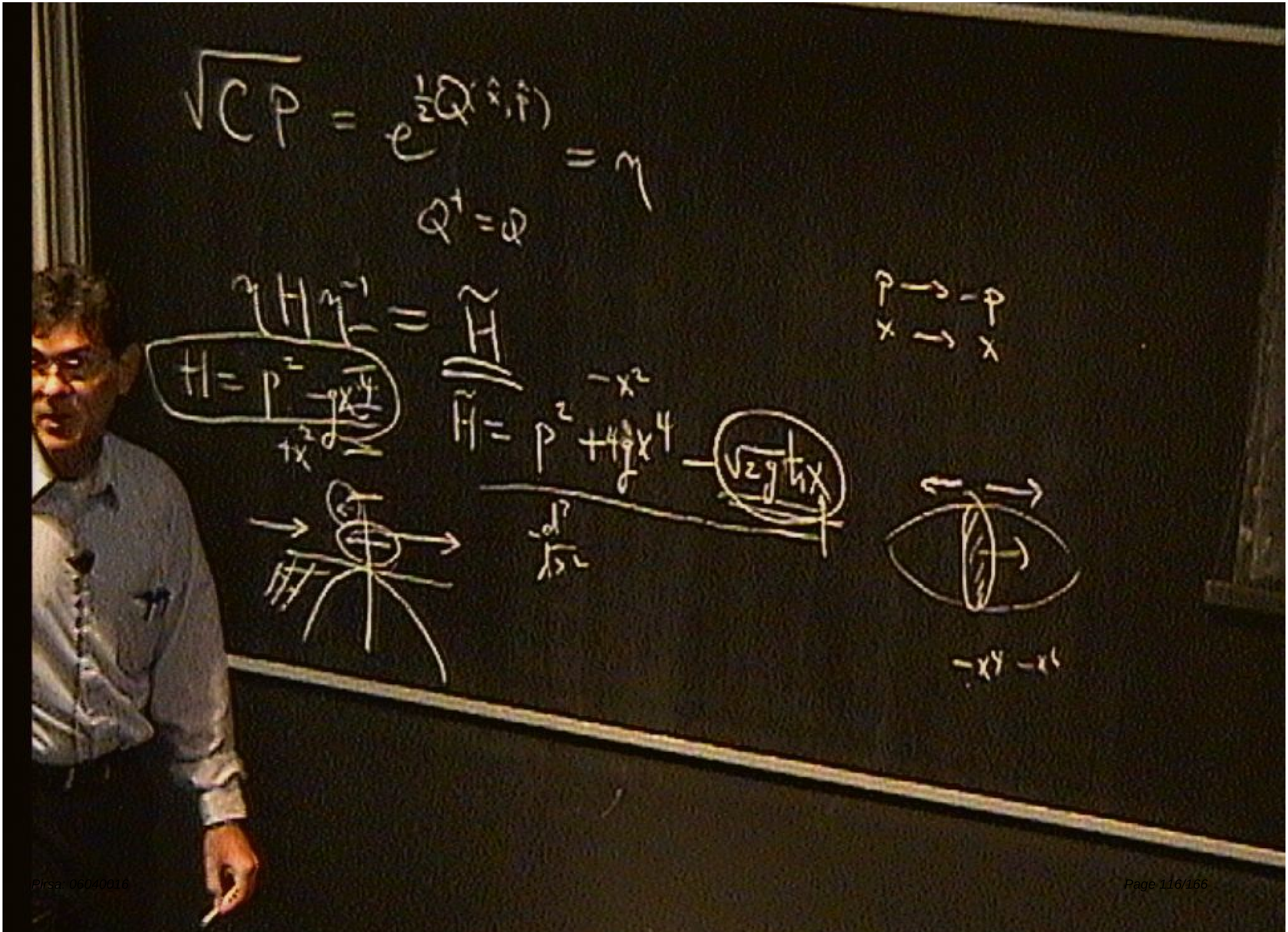
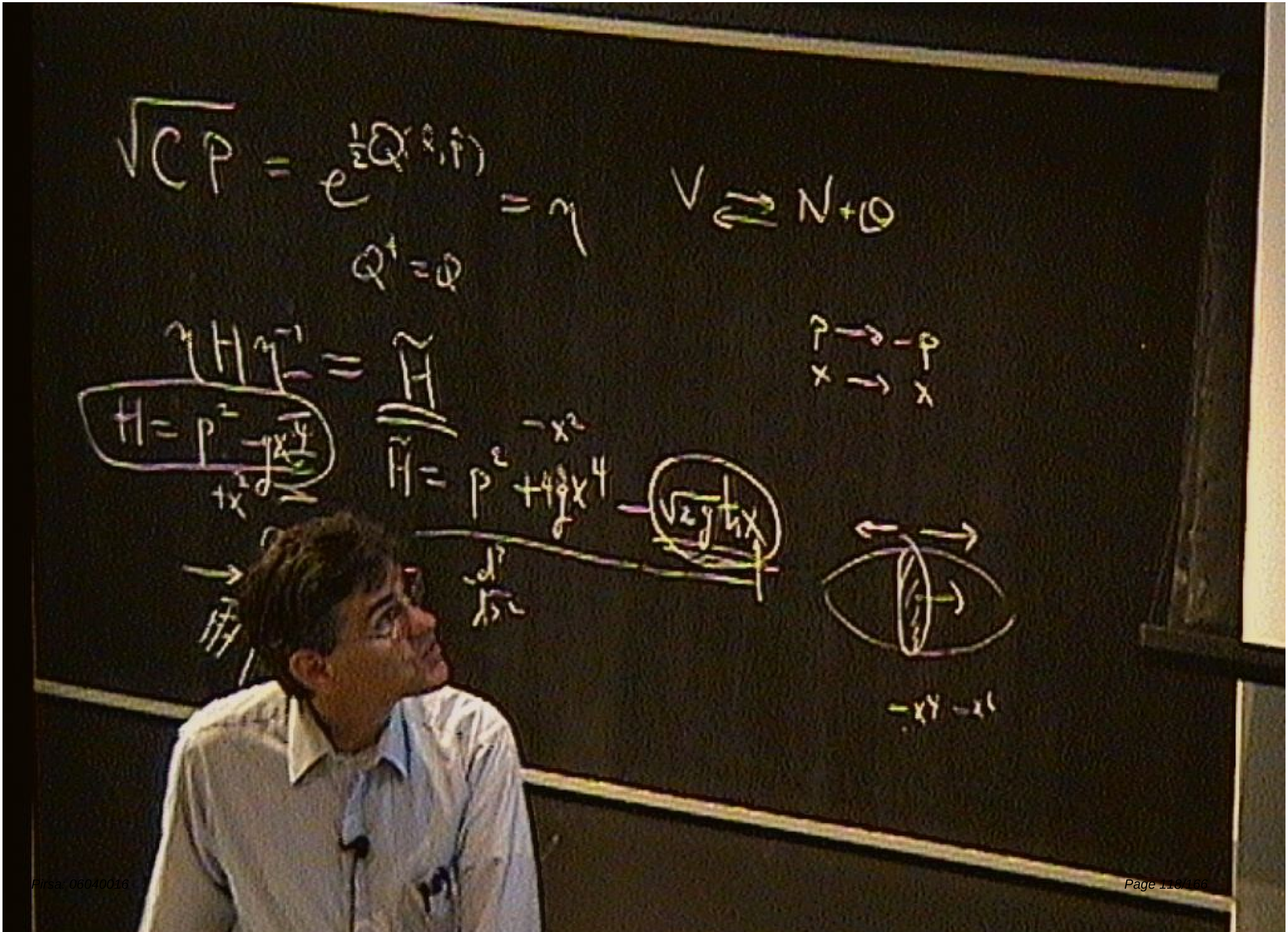

√CP = e^(½Q(x̂,p̂)) = η
Q† = Q
ηHη⁻¹ = H̃
H = p² − gx⁴
H̃ = p² + 4gx⁴
p → −p
x → x

ILLUSTRATION:

Lee Model (1954)

Ghost states found by Källén and Pauli (1955)



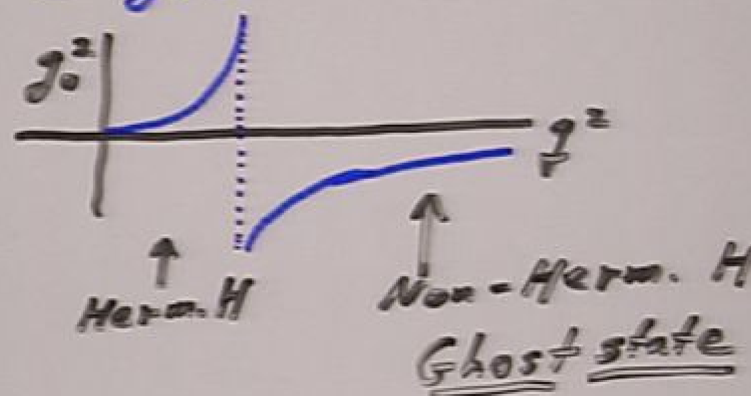
√CP = e^{½Q(x̂,p̂)} = η
Q† = Q
V ⇄ N + Q
ηHη⁻¹ = H̃
H = p² − gx⁴
H̃ = p² + 4gx⁴ − √2 g ħ x
p → −p
x → x

1954 Lee Model

$V \rightleftarrows N + \theta$

renormalizable

$H_I = g_0 (V^+ N\theta + N^+ \theta^+ V)$



Pauli
Wick
Källen
⋮

"A non-Hermitean Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation."

–*Introduction to Advanced Field Theory* by G. Barton (1963)

Problem Solved in 2005!

CMB, S. F. BRANDT, J.-H. CHEN, Q. WANG
Phys. Rev. D 71, 025014 (2005)

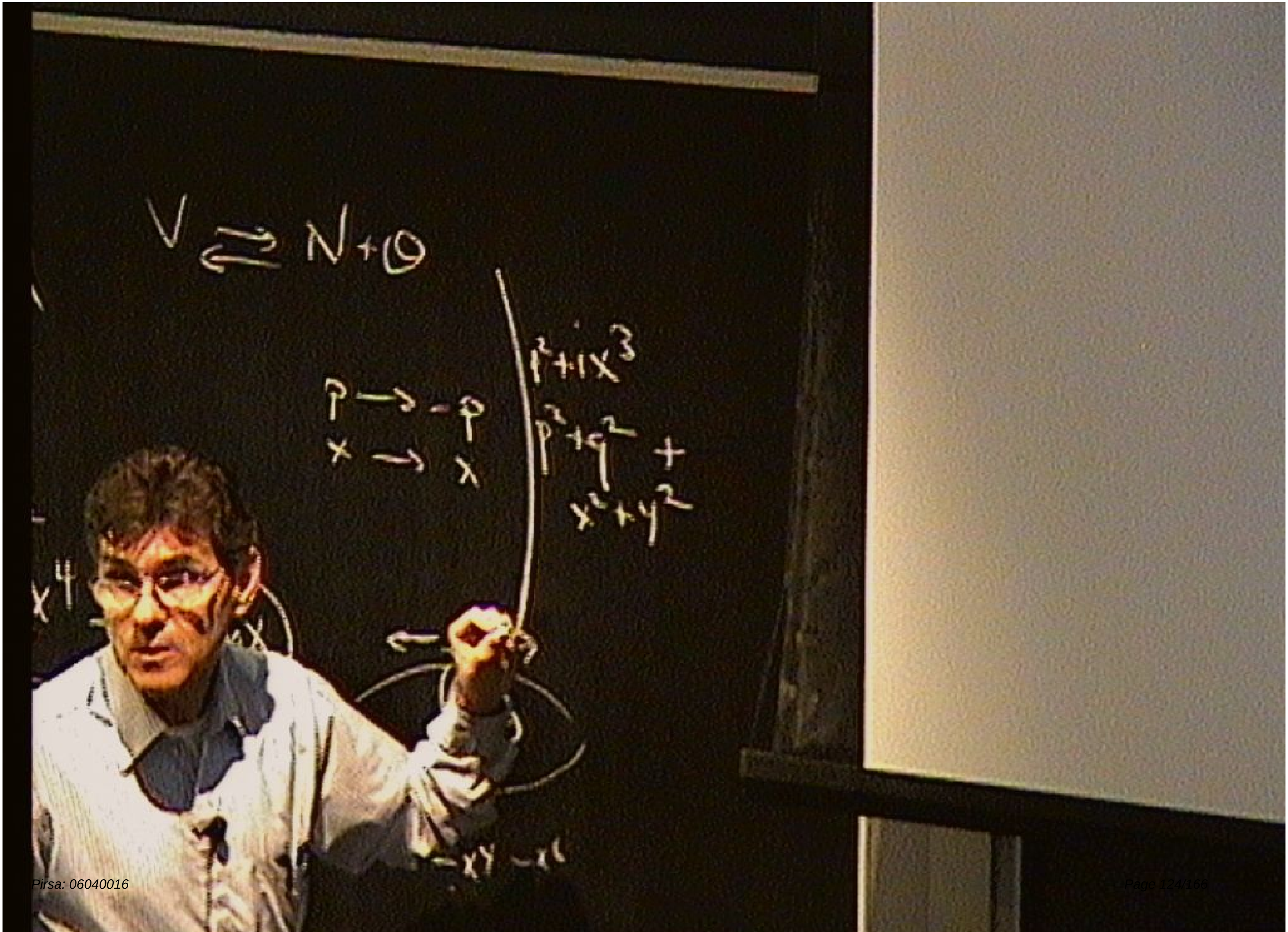
Solution ....
Lee model Hamiltonian
becomes PT symmetric!

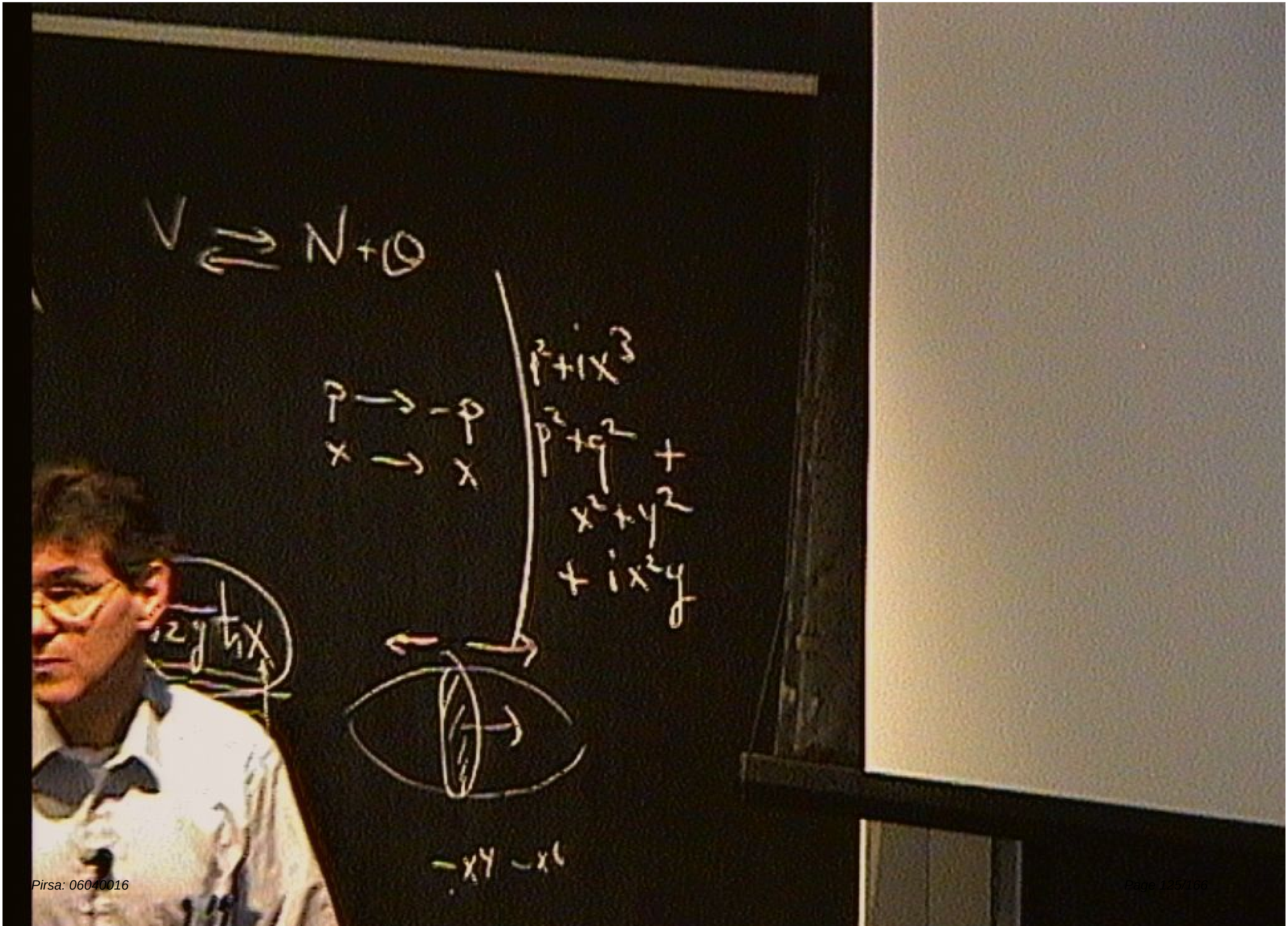
$$
\begin{aligned}
c = \Big[& 1 - n_V - n_N + n_V n_N + \frac{\mu_0 n_N (1 - n_V)}{\sqrt{\mu_0^2 + 4 g_0^2 n_\theta}} \\
& + \frac{\mu_0 n_V (1 - n_N)}{\sqrt{\mu_0^2 + 4 g_0^2 (n_\theta + 1)}} + V^\dagger N a \frac{2 g_0 \sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4 g_0^2 n_\theta}} \\
& - \frac{2 g_0 \sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4 g_0^2 n_\theta}} a^\dagger N^\dagger V \Big] (1 - 2 n_V)(1 - 2 n_N) e^{i \pi n_\theta} .
\end{aligned}
$$

Ghost busted

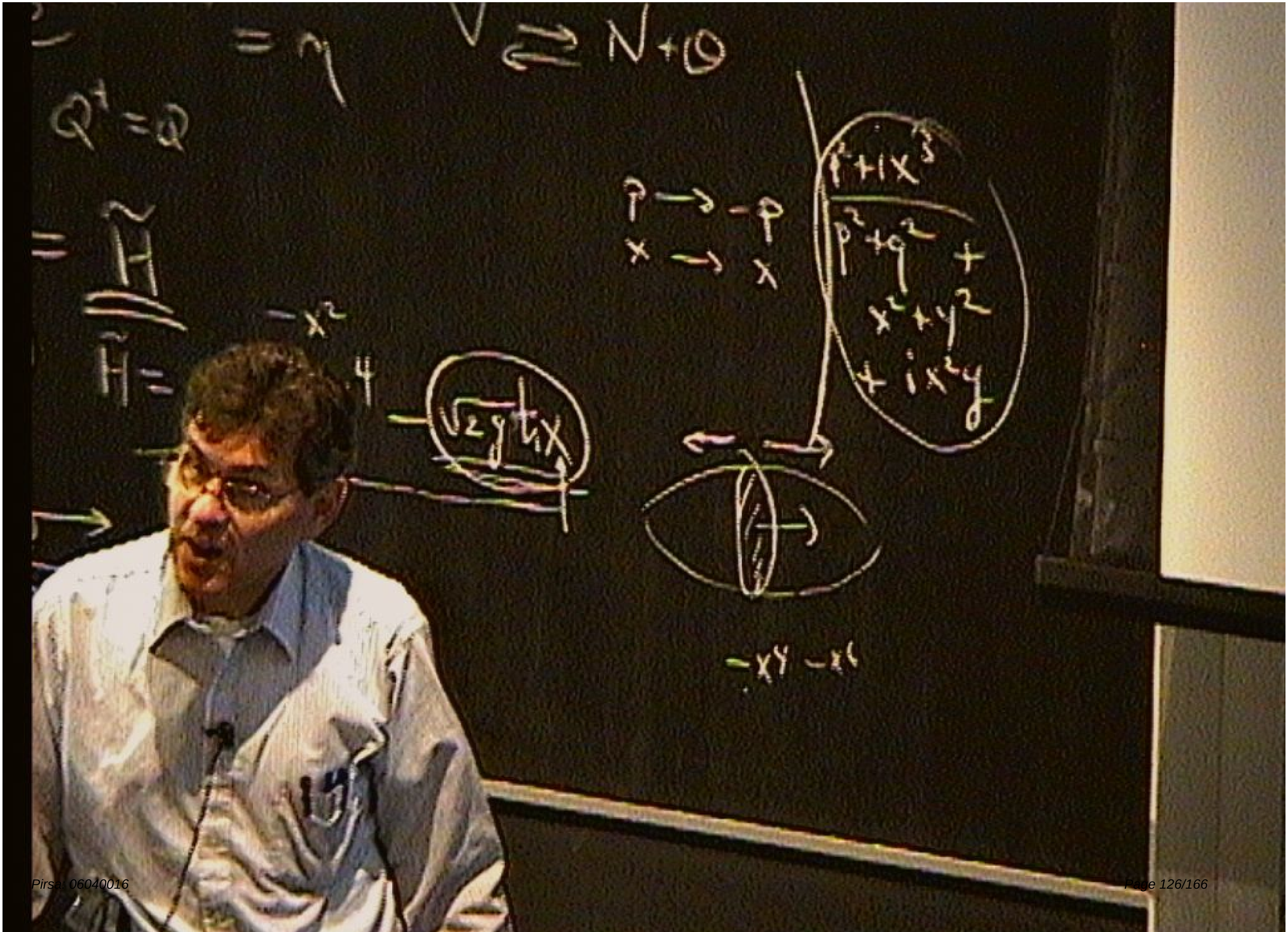
$(|G\rangle)^{CPT} \cdot |G\rangle > 0$

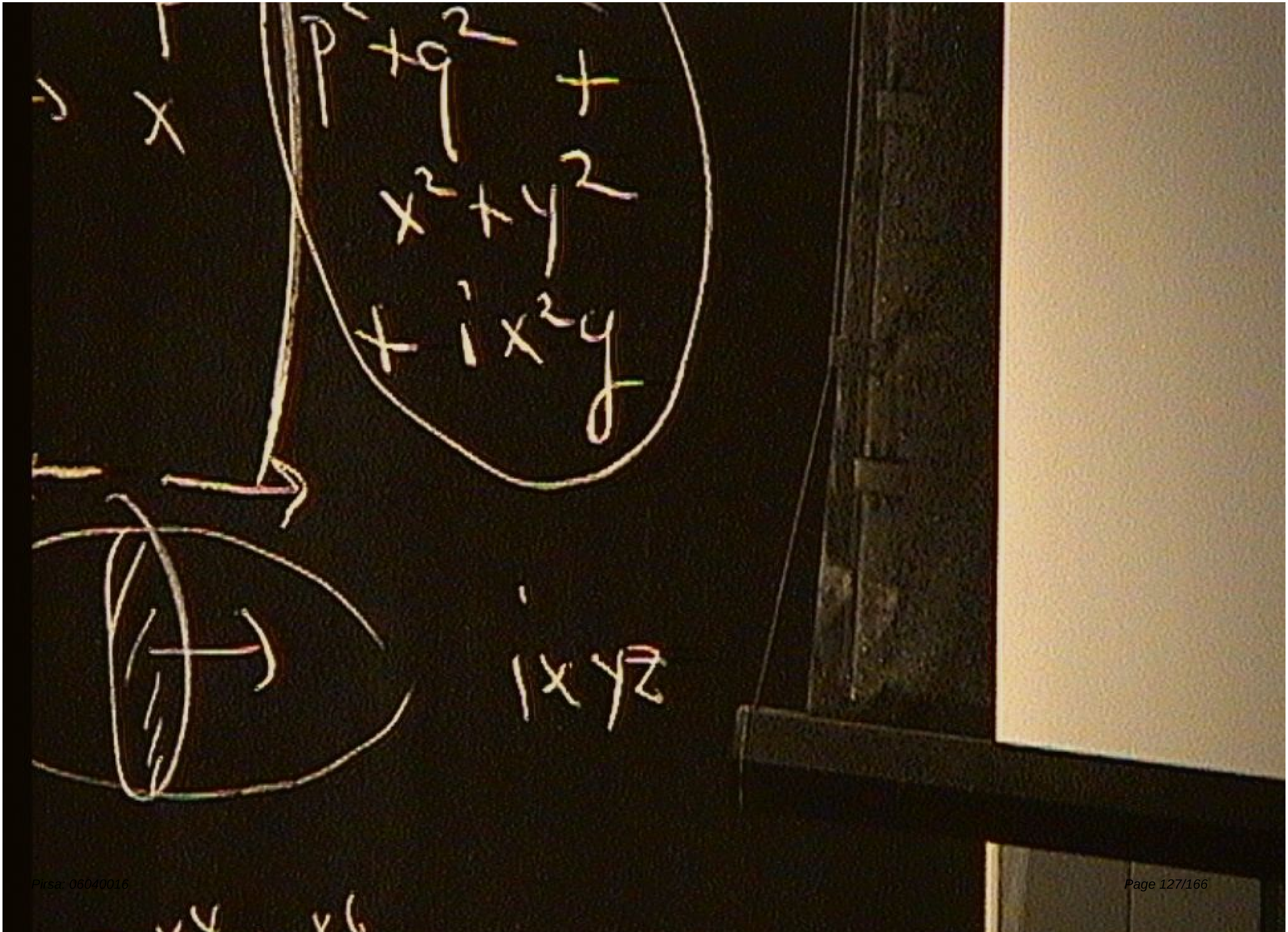
I'm very
excited about
PT symmetry!

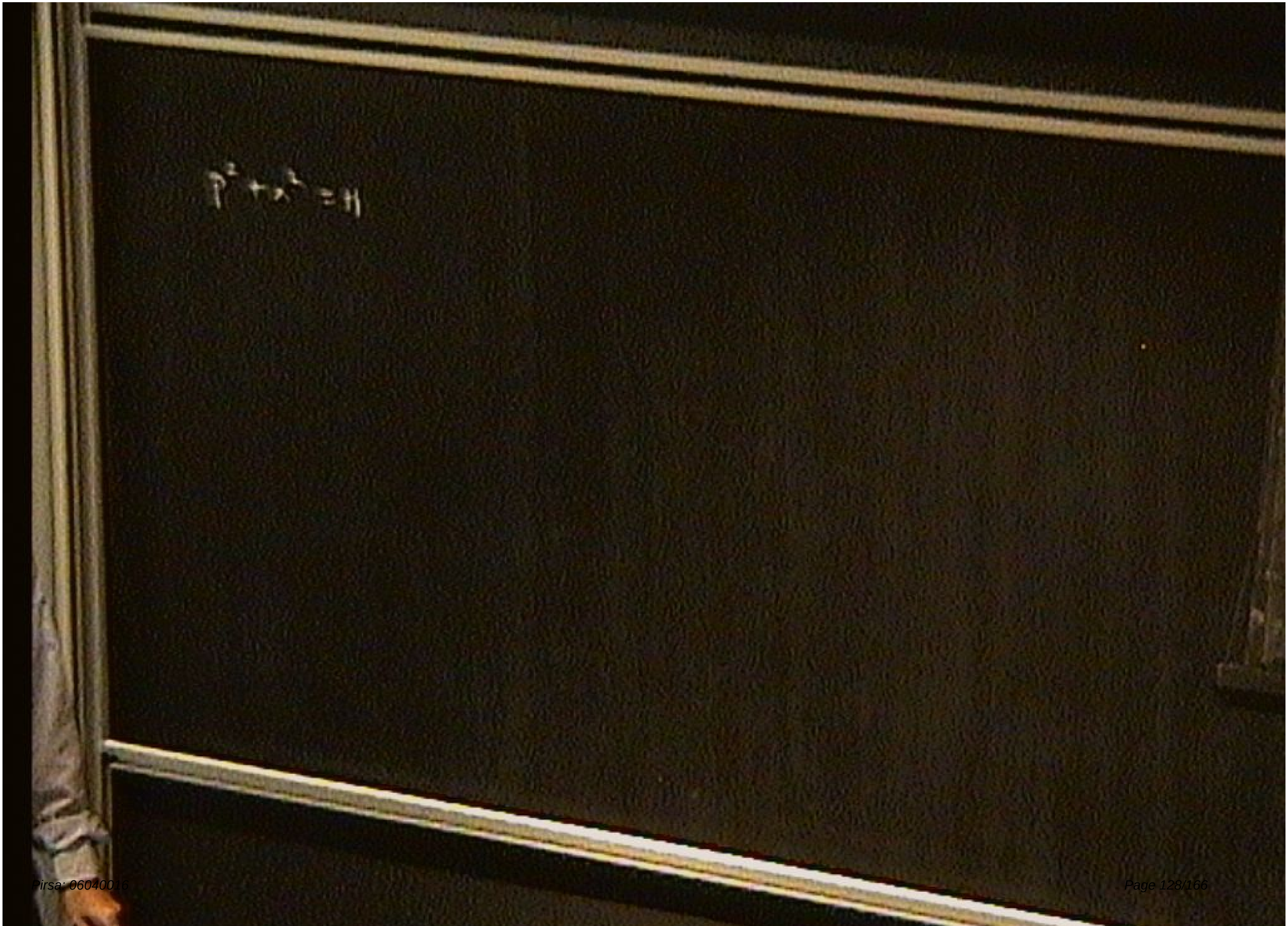

V ⇄ N + Θ
p → -p
x → x
p² + ix³
p² + q² +
x² + y²

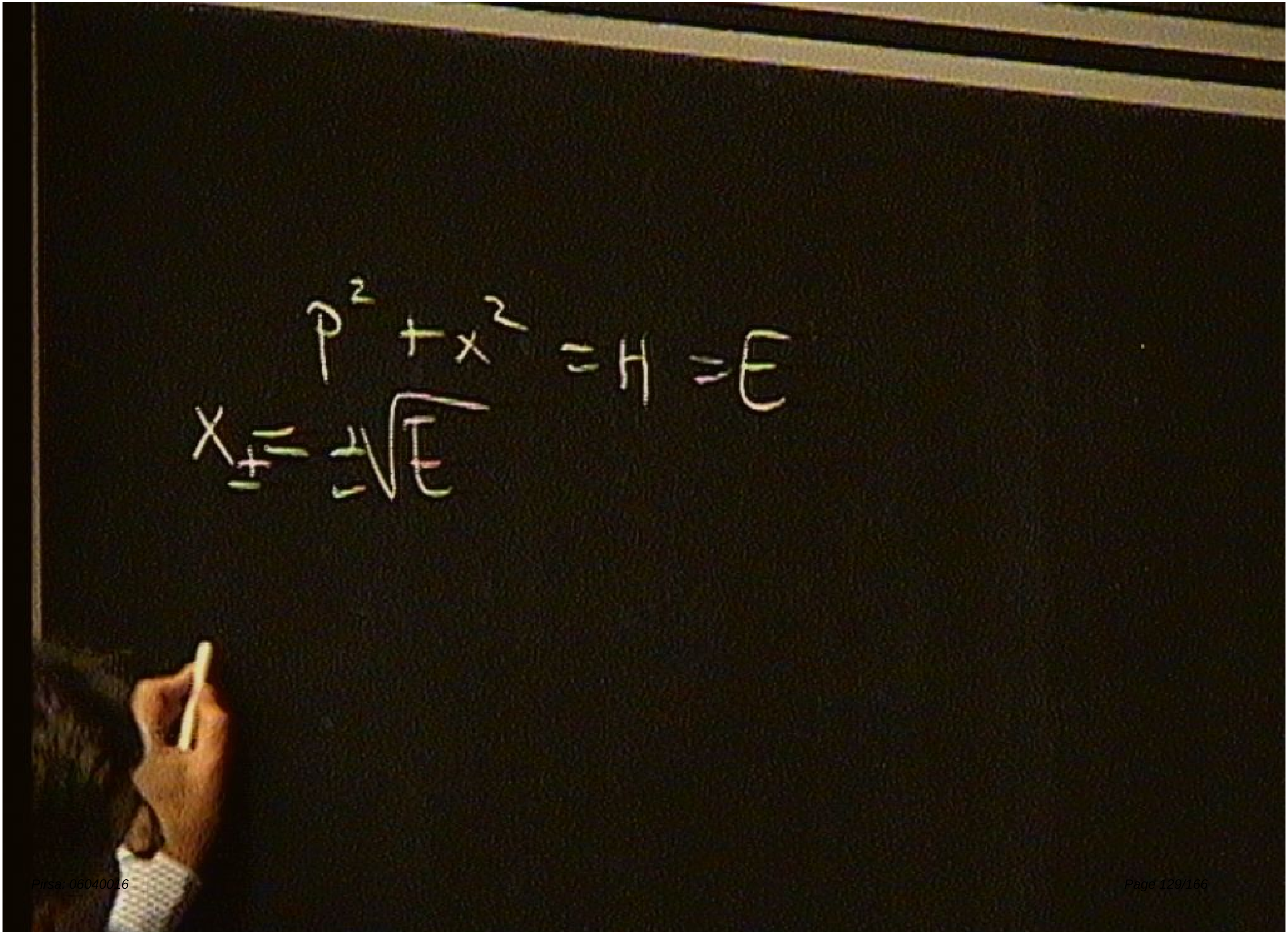


V ⇄ N + Θ
p → -p
x → x
p² + ix³
p² + q² +
x² + y²
+ ix²y




p^2+q^2 +
x^2+y^2
+ ix^2y
$ixyz$

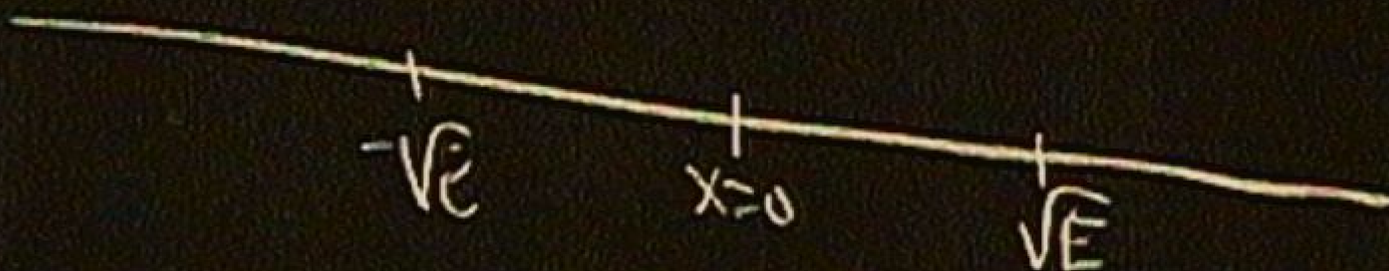

p² + x² = H

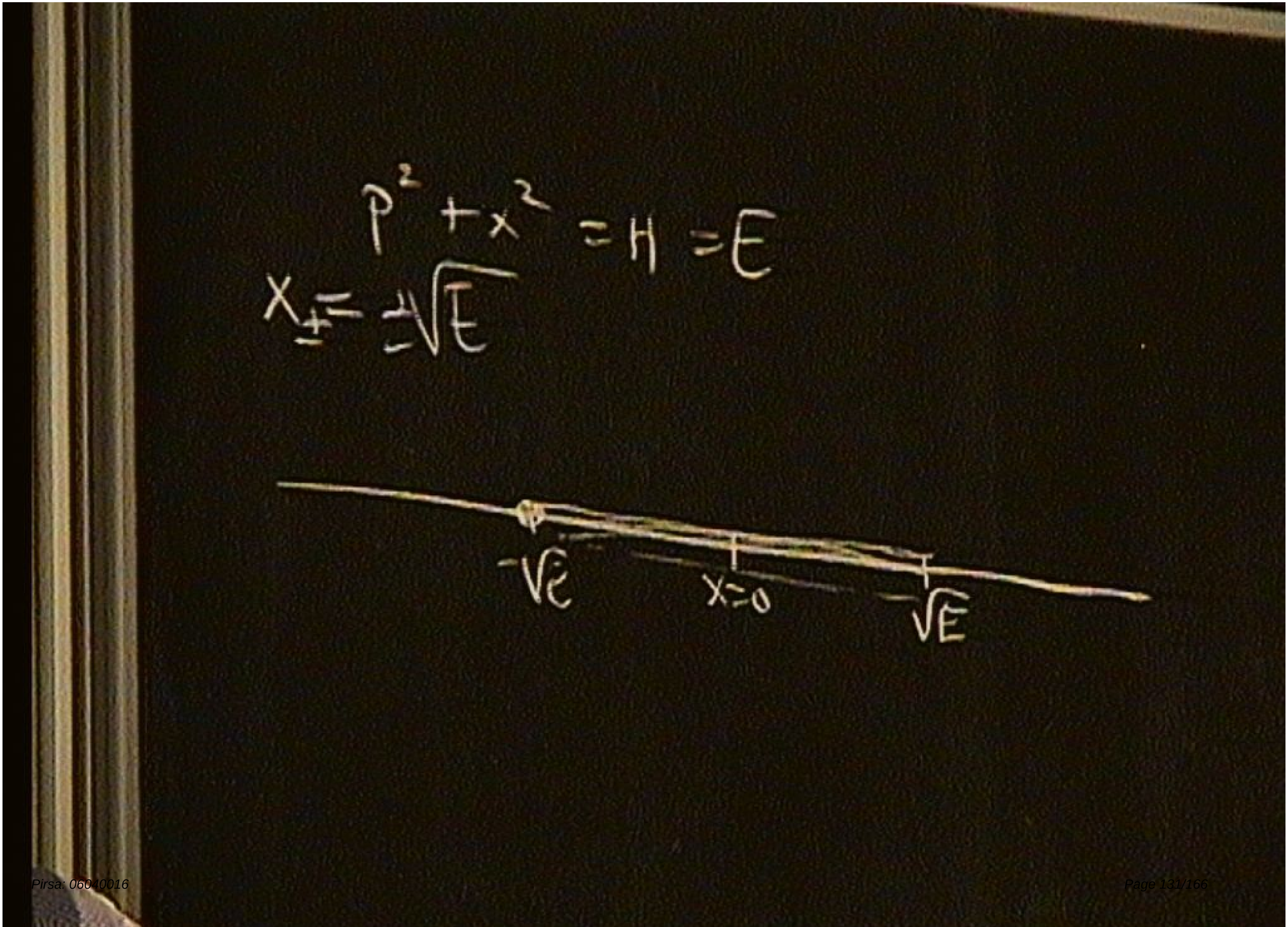


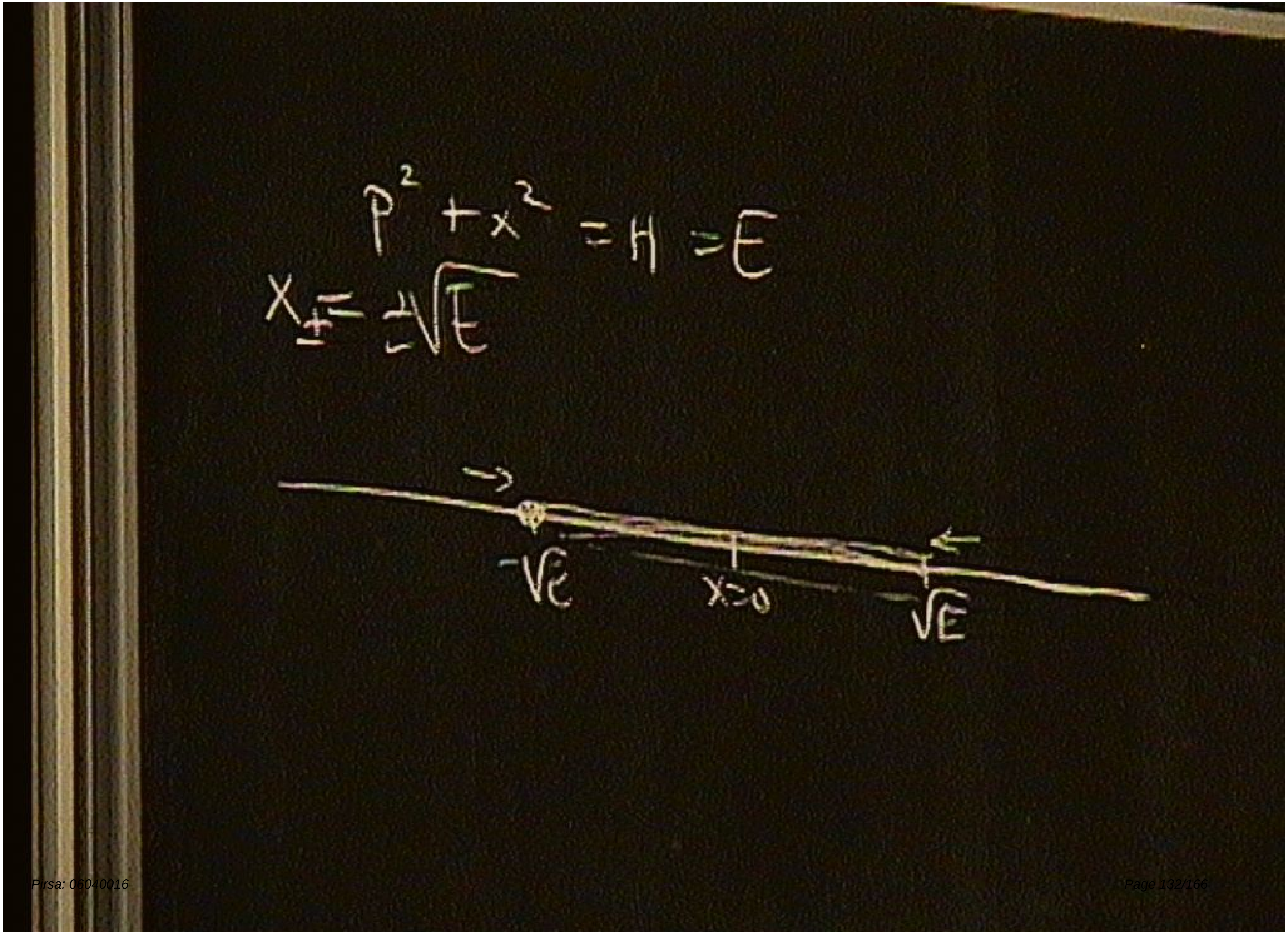
$p^2 + x^2 = H = E$
$x_\pm = \pm\sqrt{E}$

$$p^2 + x^2 = H = E$$

$$x_\pm = \pm\sqrt{E}$$

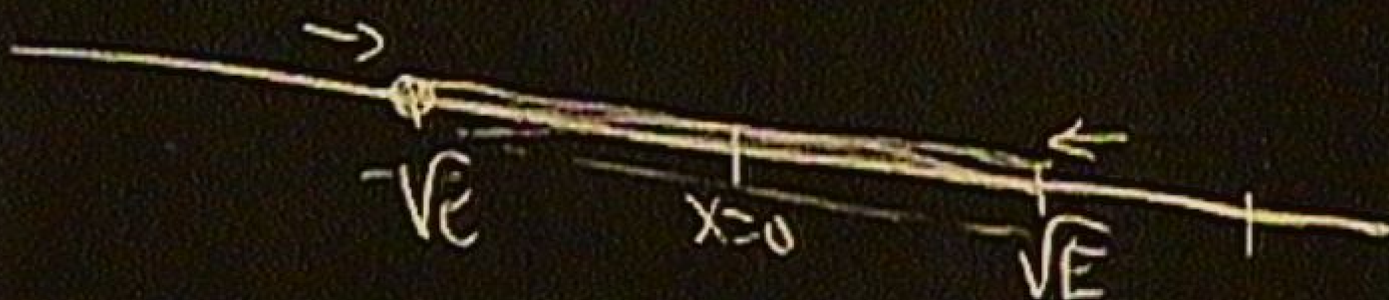


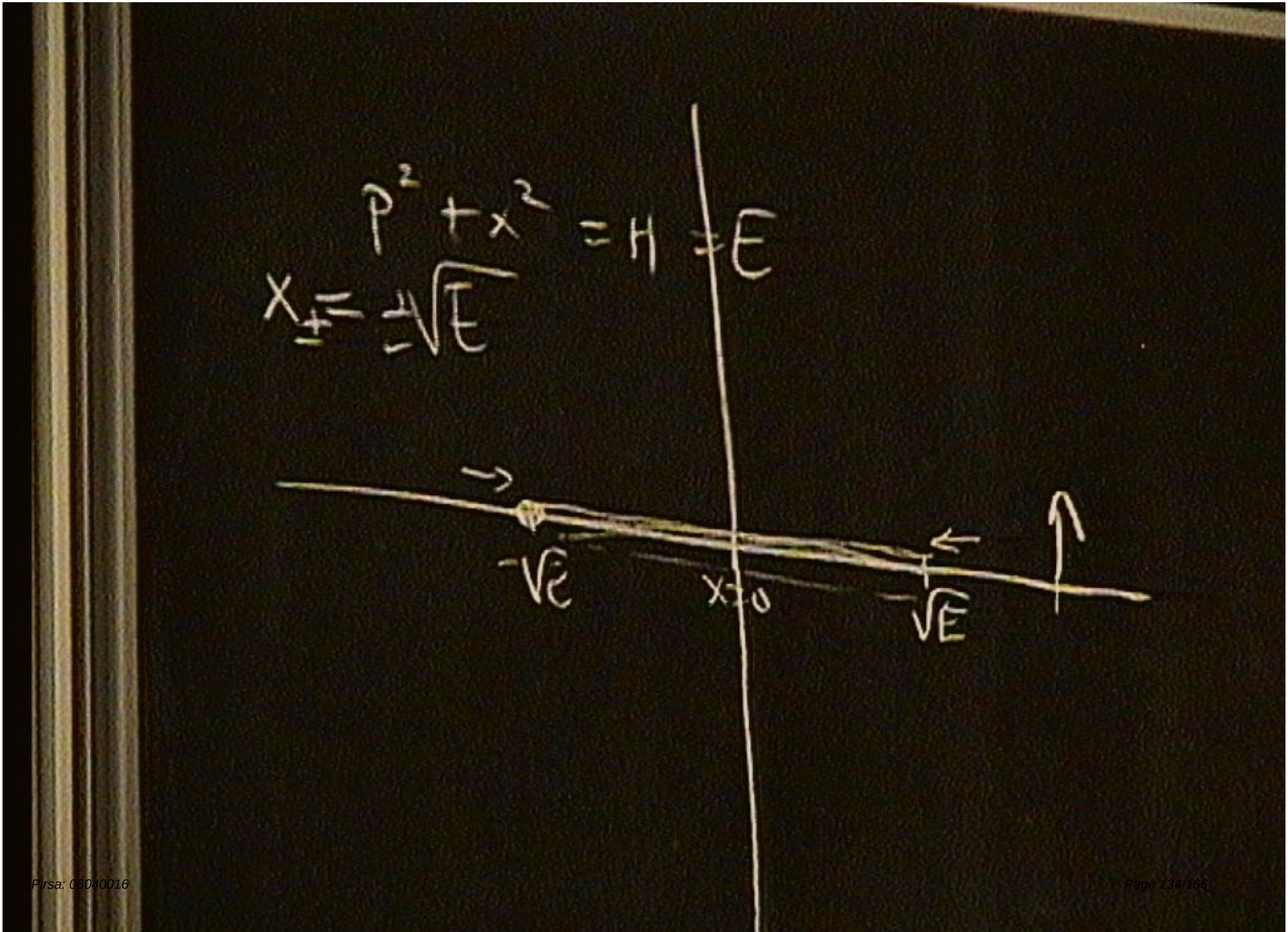

$p^2 + x^2 = H = E$
$x_{\pm} = \pm\sqrt{E}$
$-\sqrt{E}$
$x=0$
$\sqrt{E}$

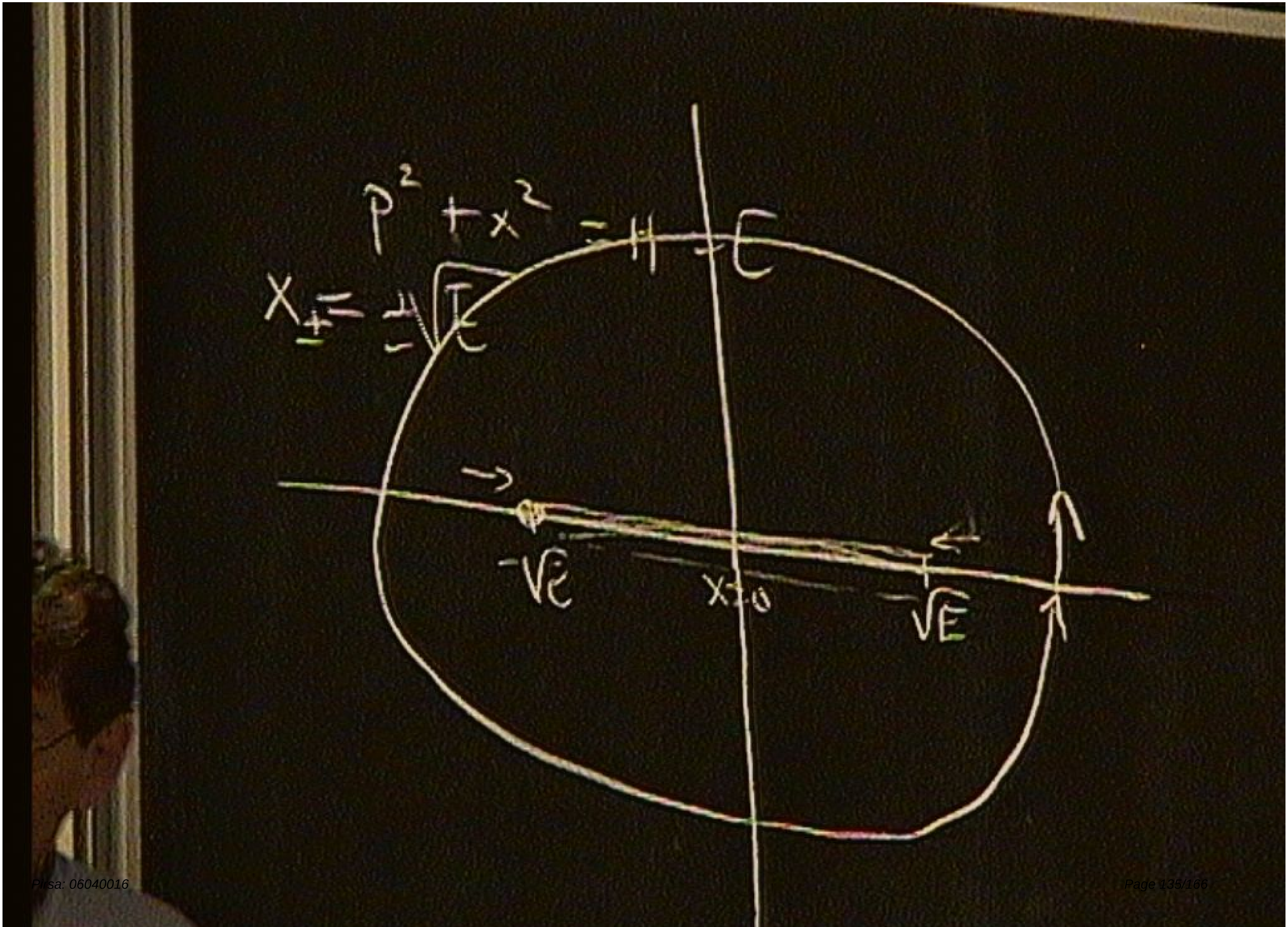

p² + x² = H = E
x± = ±√E
→
←
-√E
x=0
√E

$$p^2 + x^2 = H = E$$

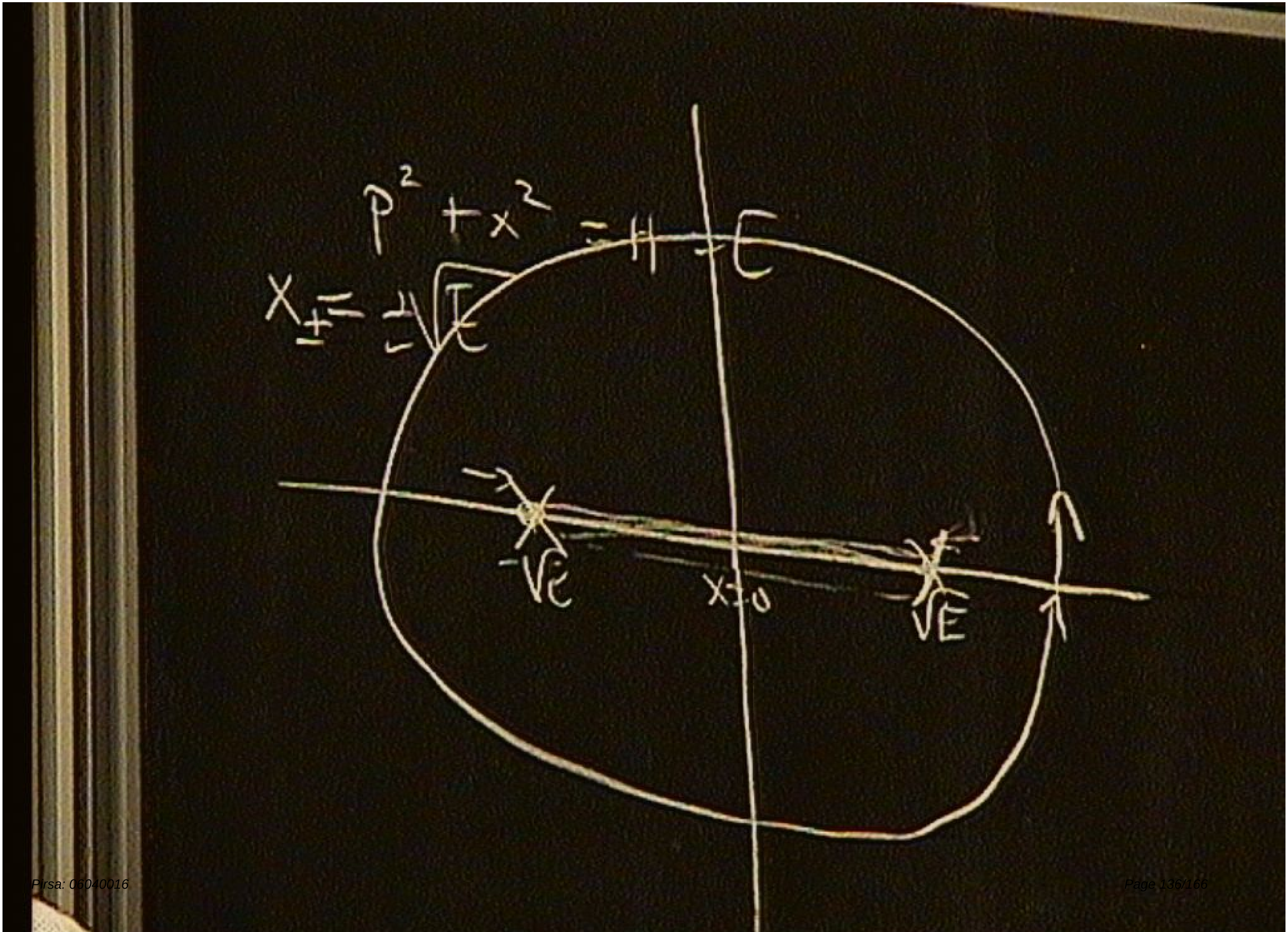
$$x_{\pm} = \pm\sqrt{E}$$

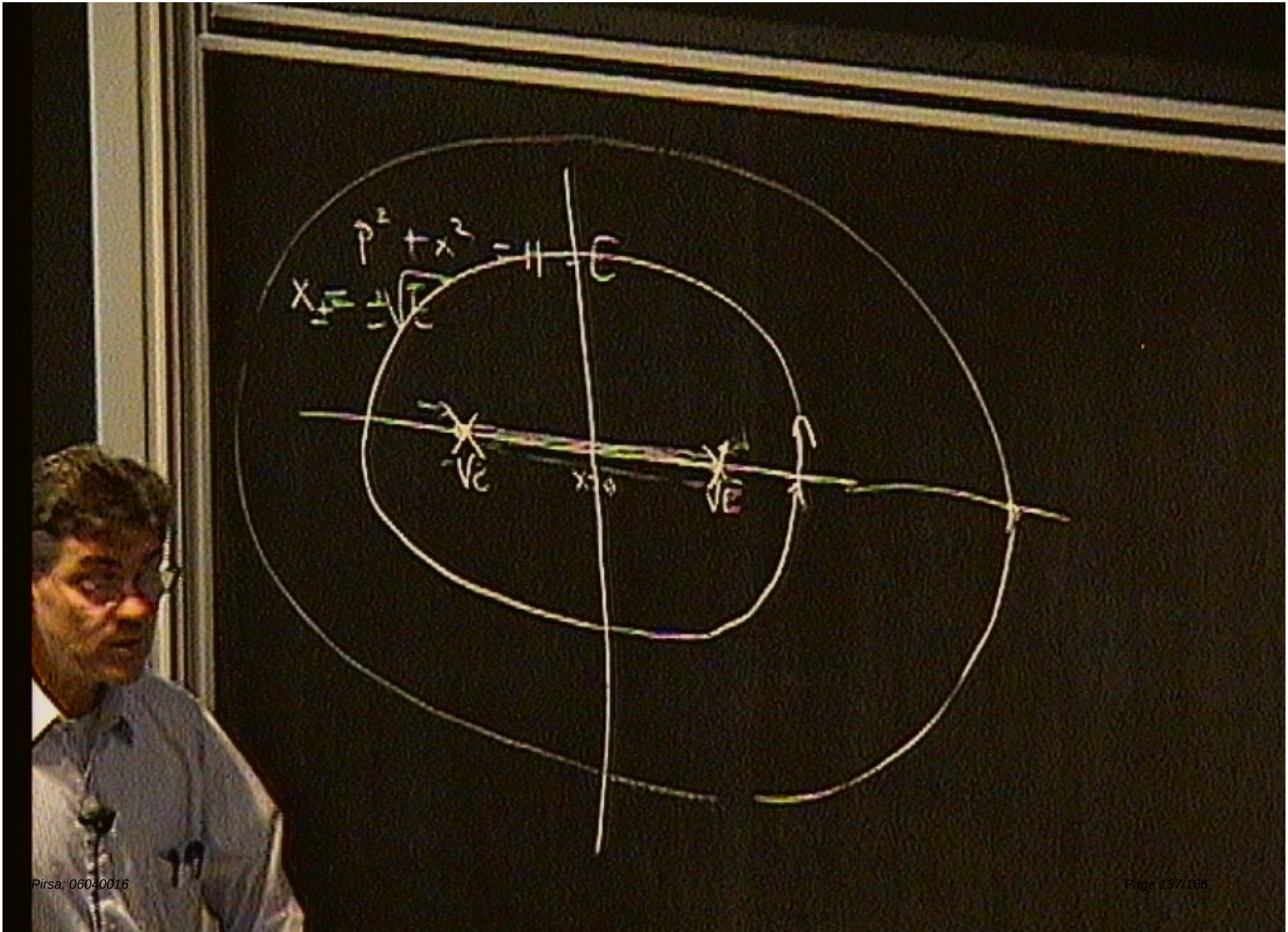



P² + x² = H = E
x± = ±√E
-√E
x=0
√E

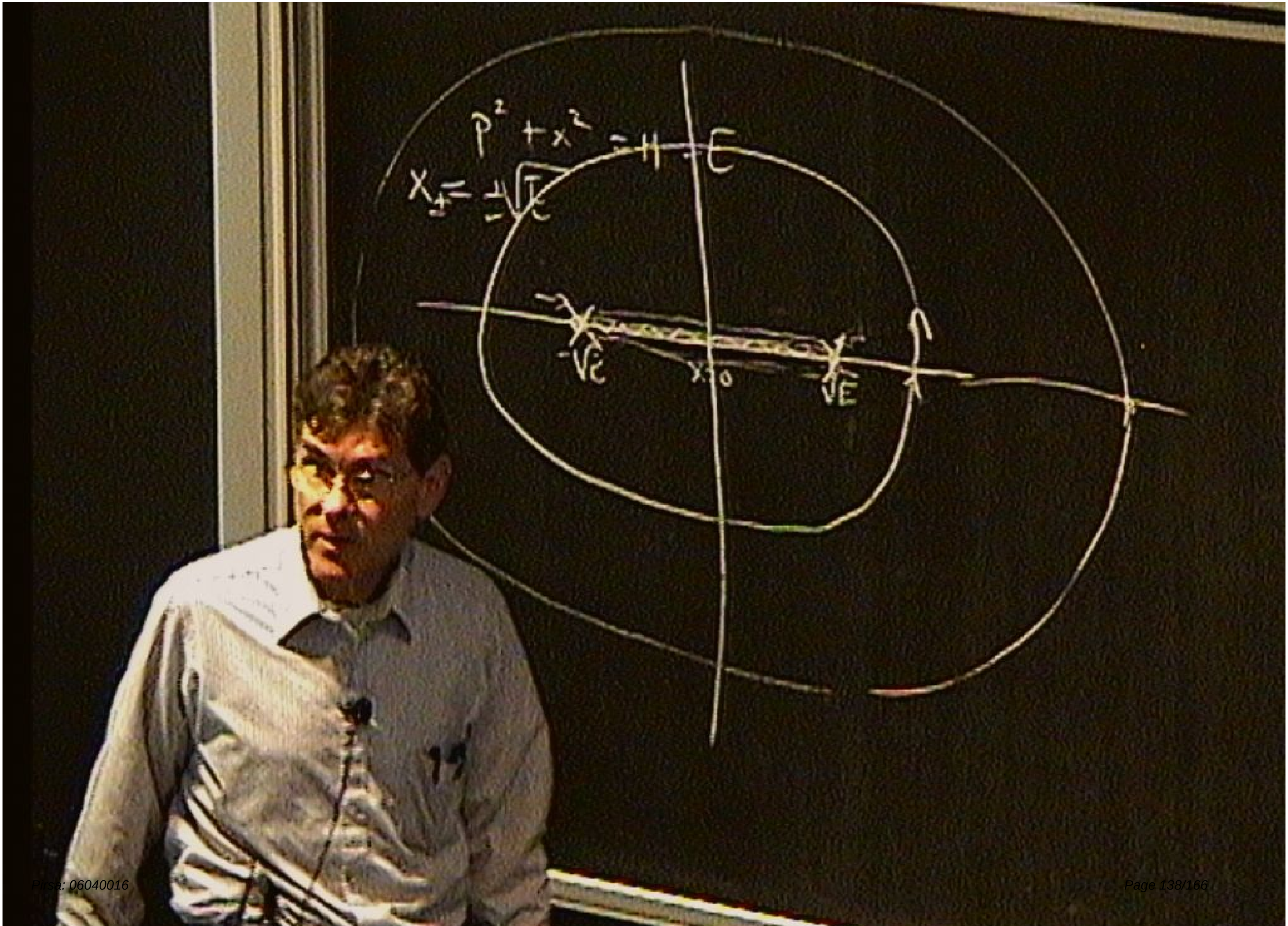


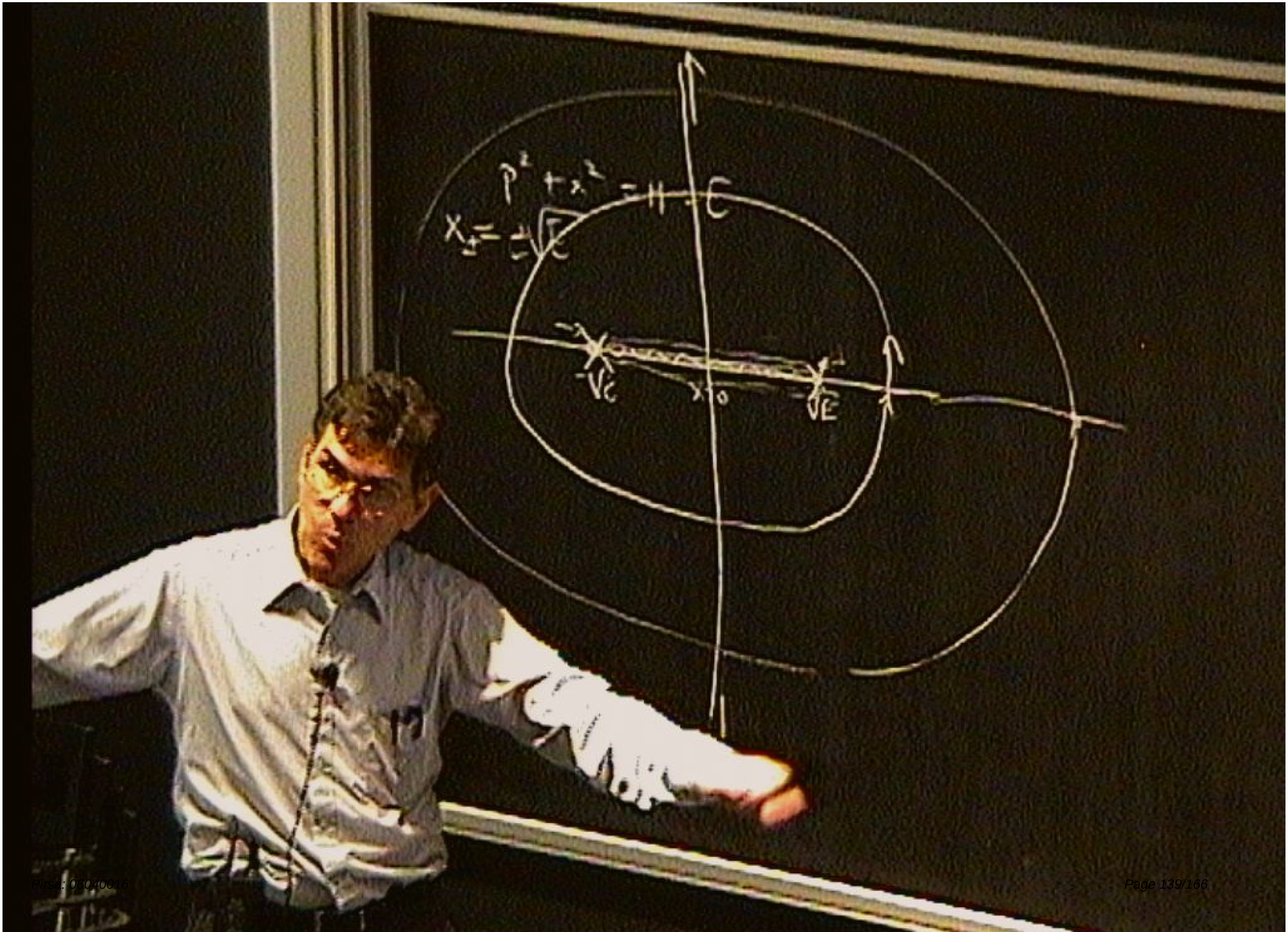
p² + x² = H = E
X± = ±√E
-√E
x=0
√E

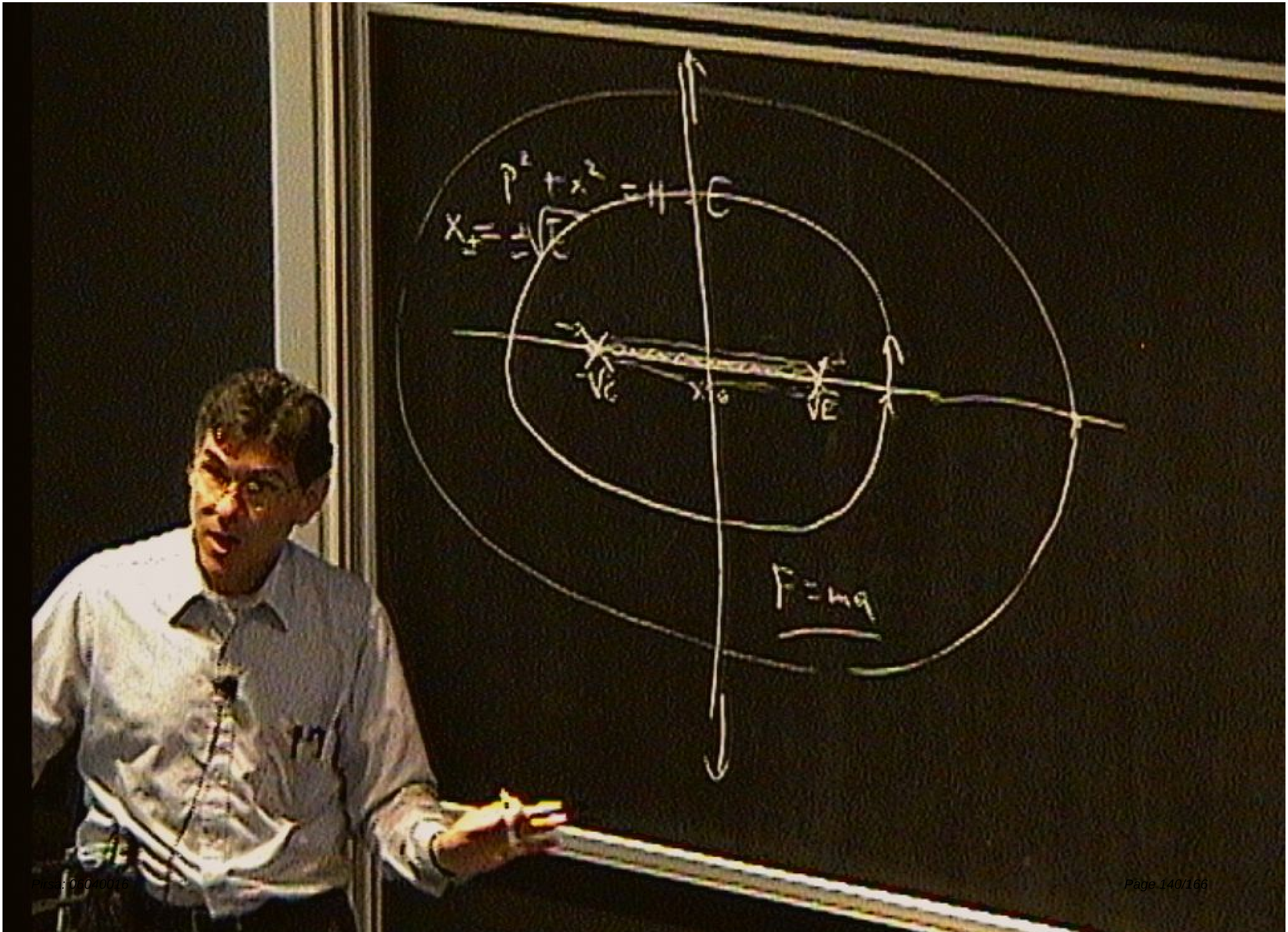

$p^2 + x^2 = H = E$
$x_\pm = \pm\sqrt{E}$
$-\sqrt{E}$
$x = 0$
$\sqrt{E}$

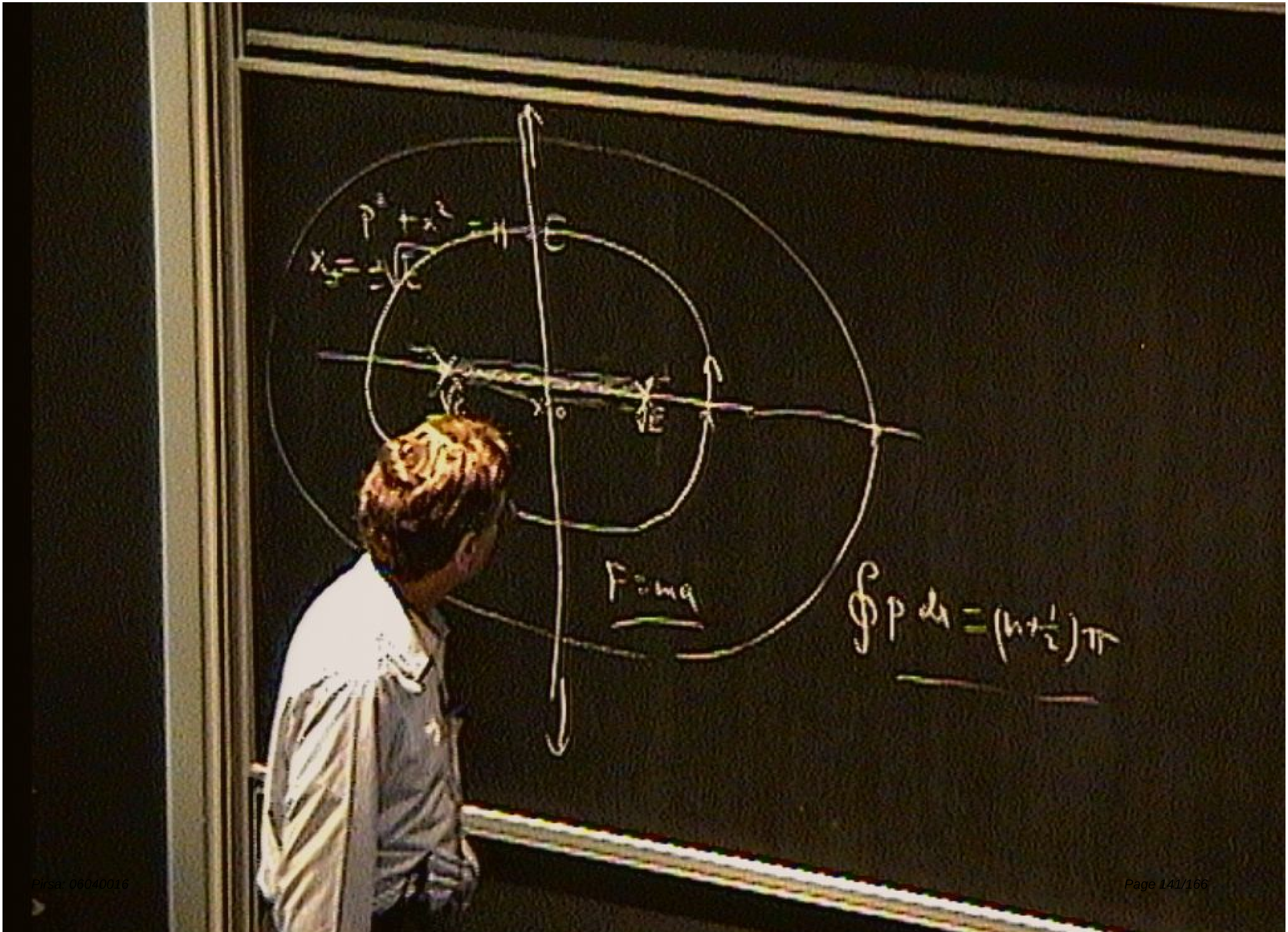


$p^2 + x^2 = \hbar C$
$x_\pm = \pm\sqrt{C}$
$-\sqrt{C}$
$x=0$
$\sqrt{C}$

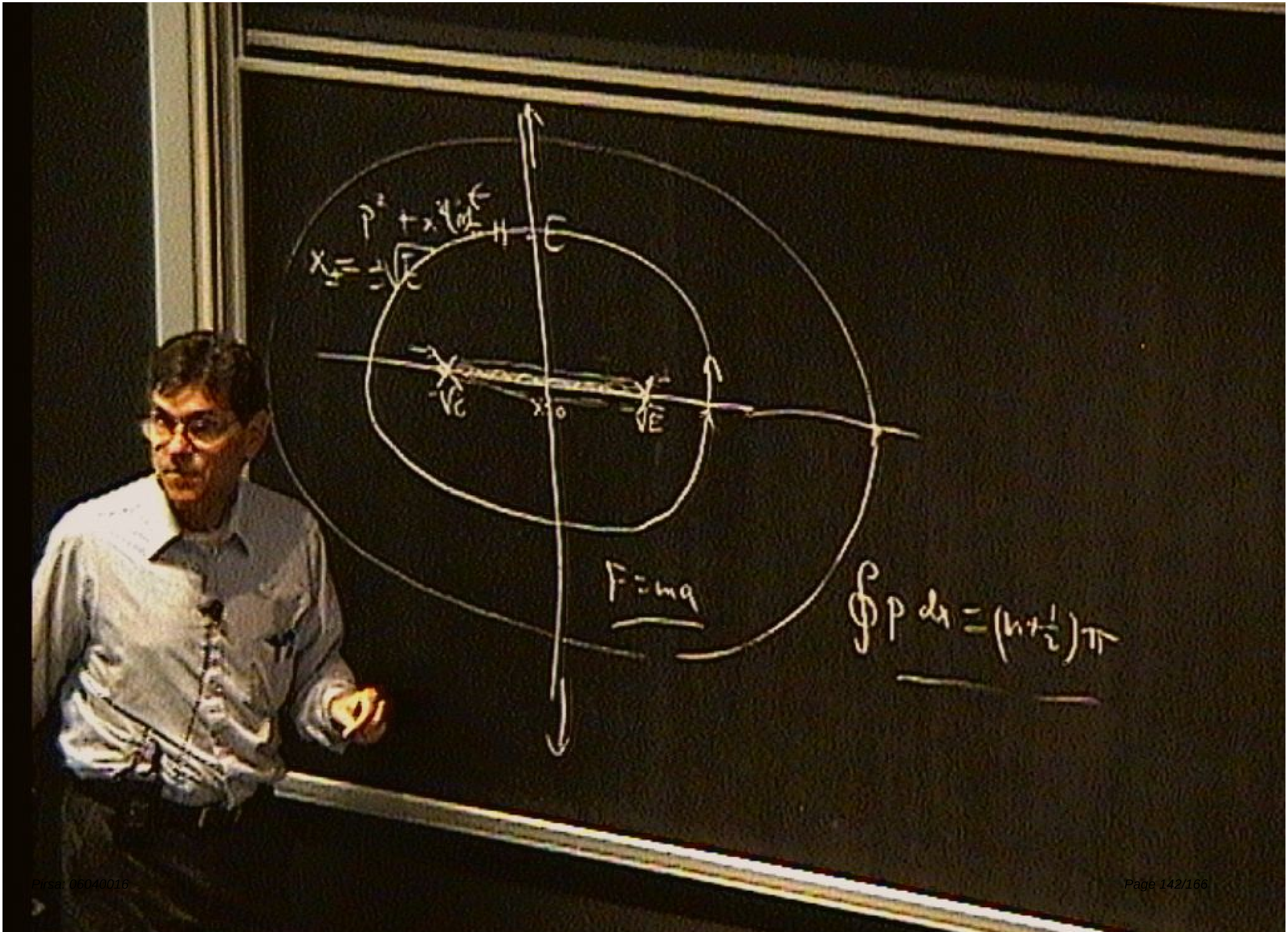

p² + x² = H = E
x± = ±√E
-√E
x=0
√E

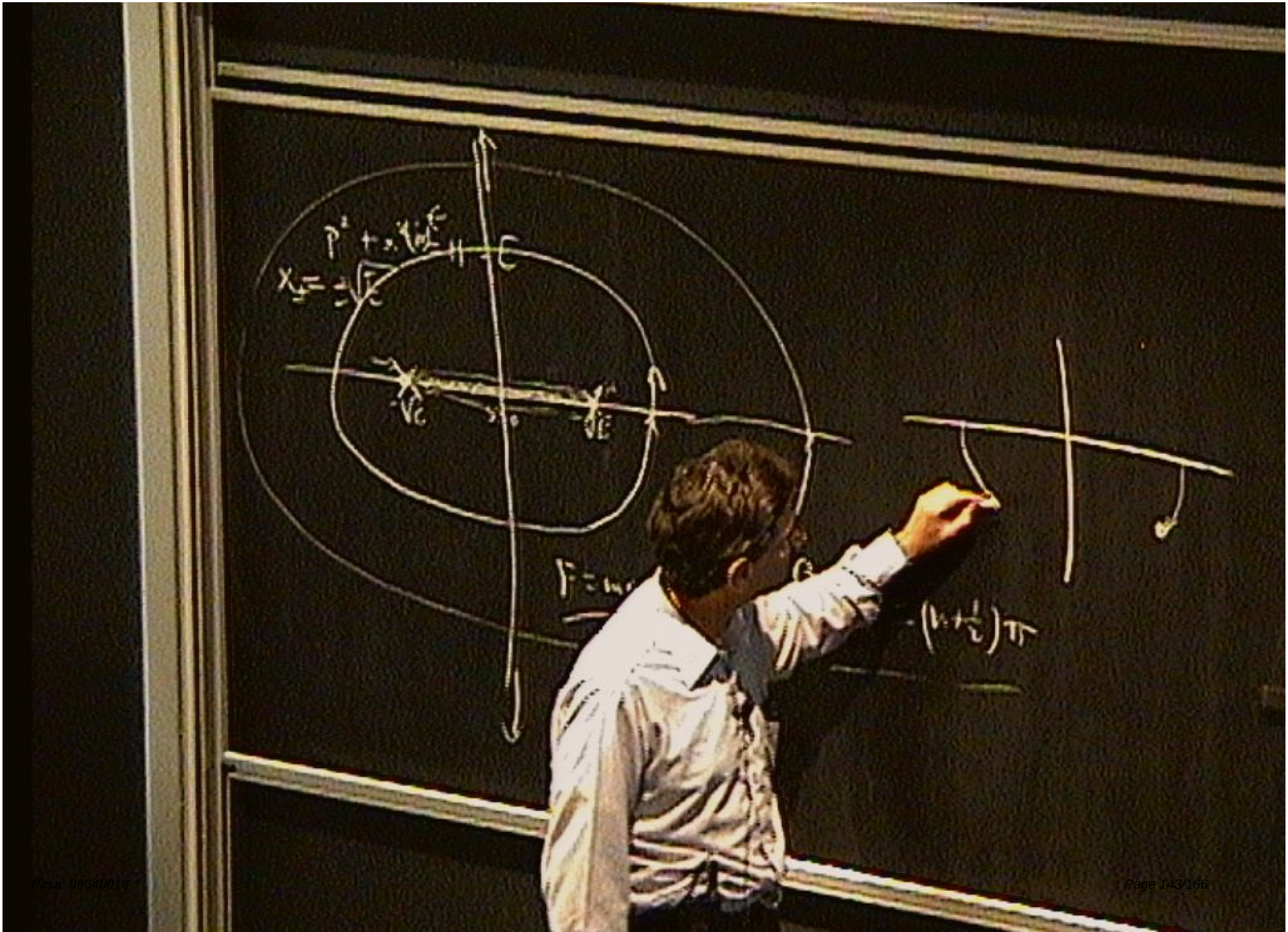

p² + x² = H = E
x± = ±√E
-√E
x₀
√E

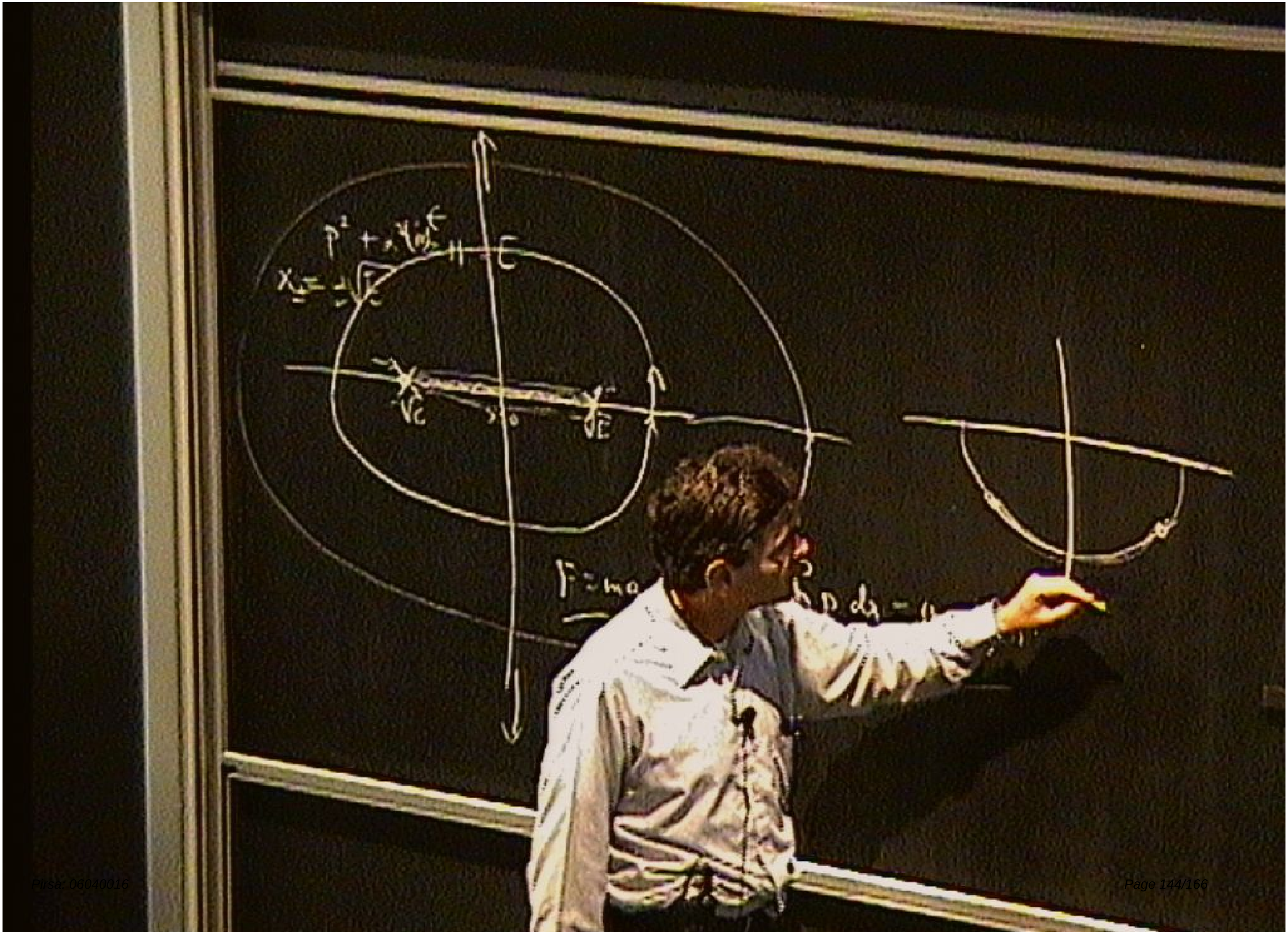

p² + x² = H = E
F = ma

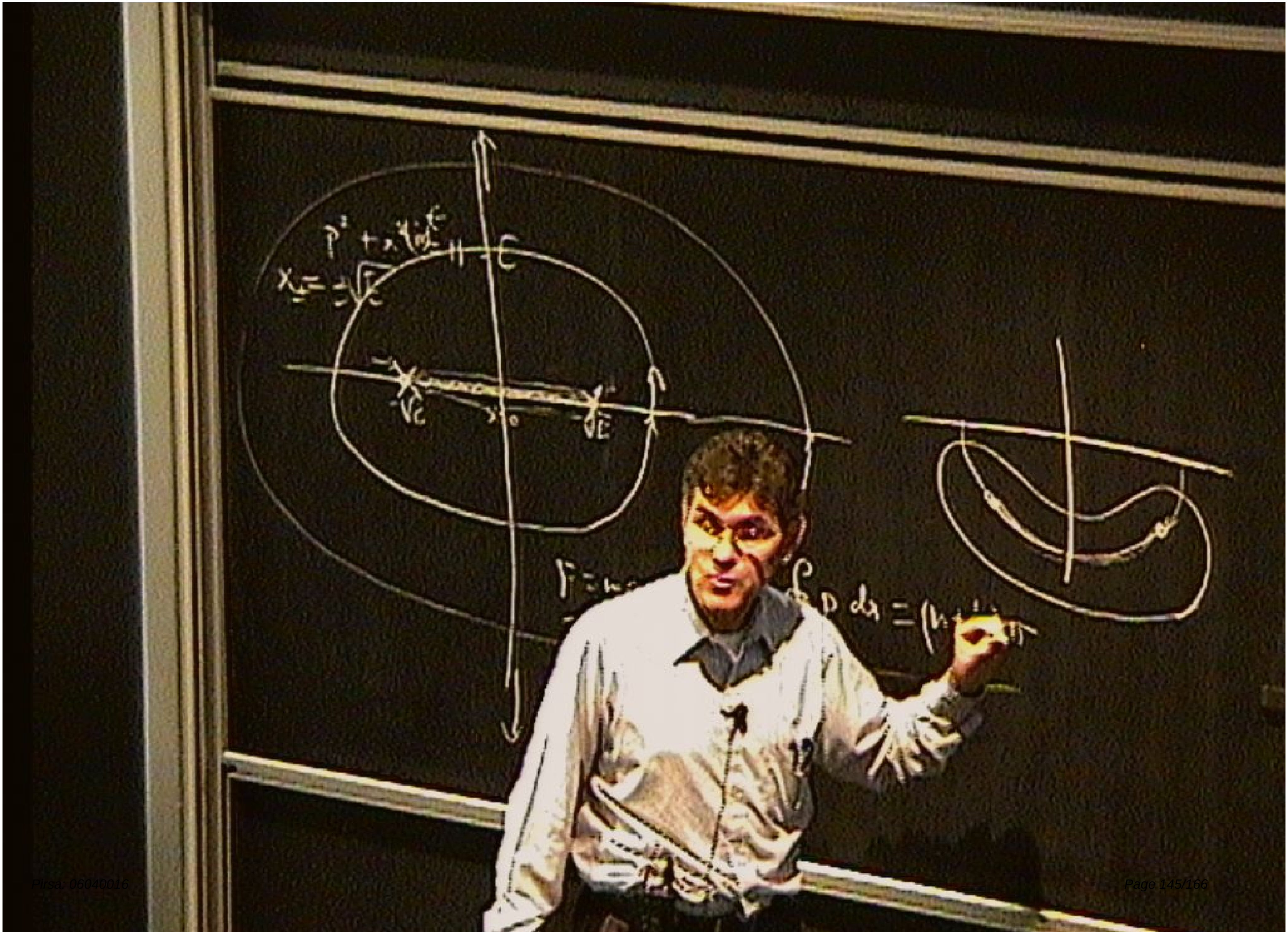


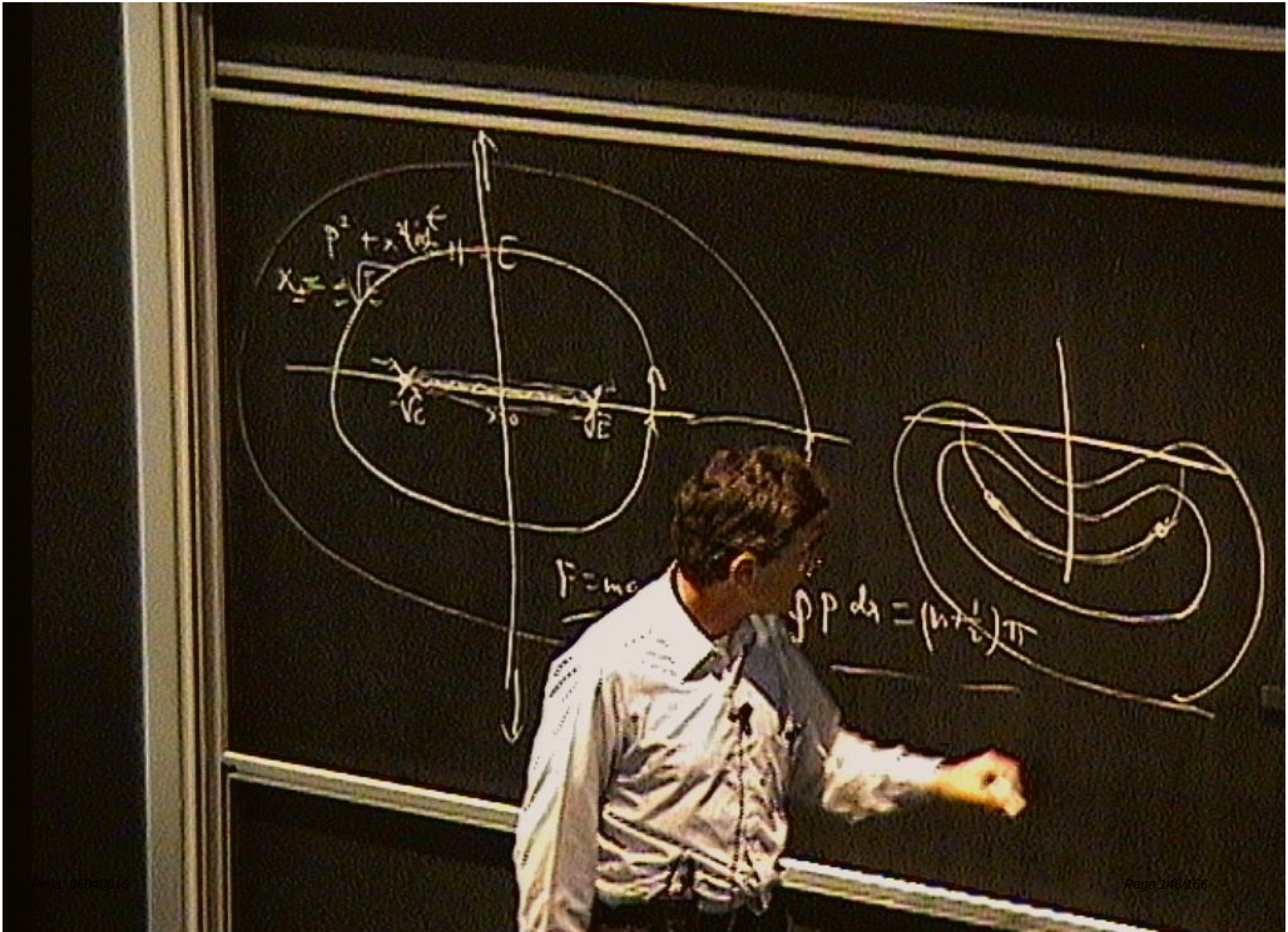
F=ma
∮ p dx = (n+½)π

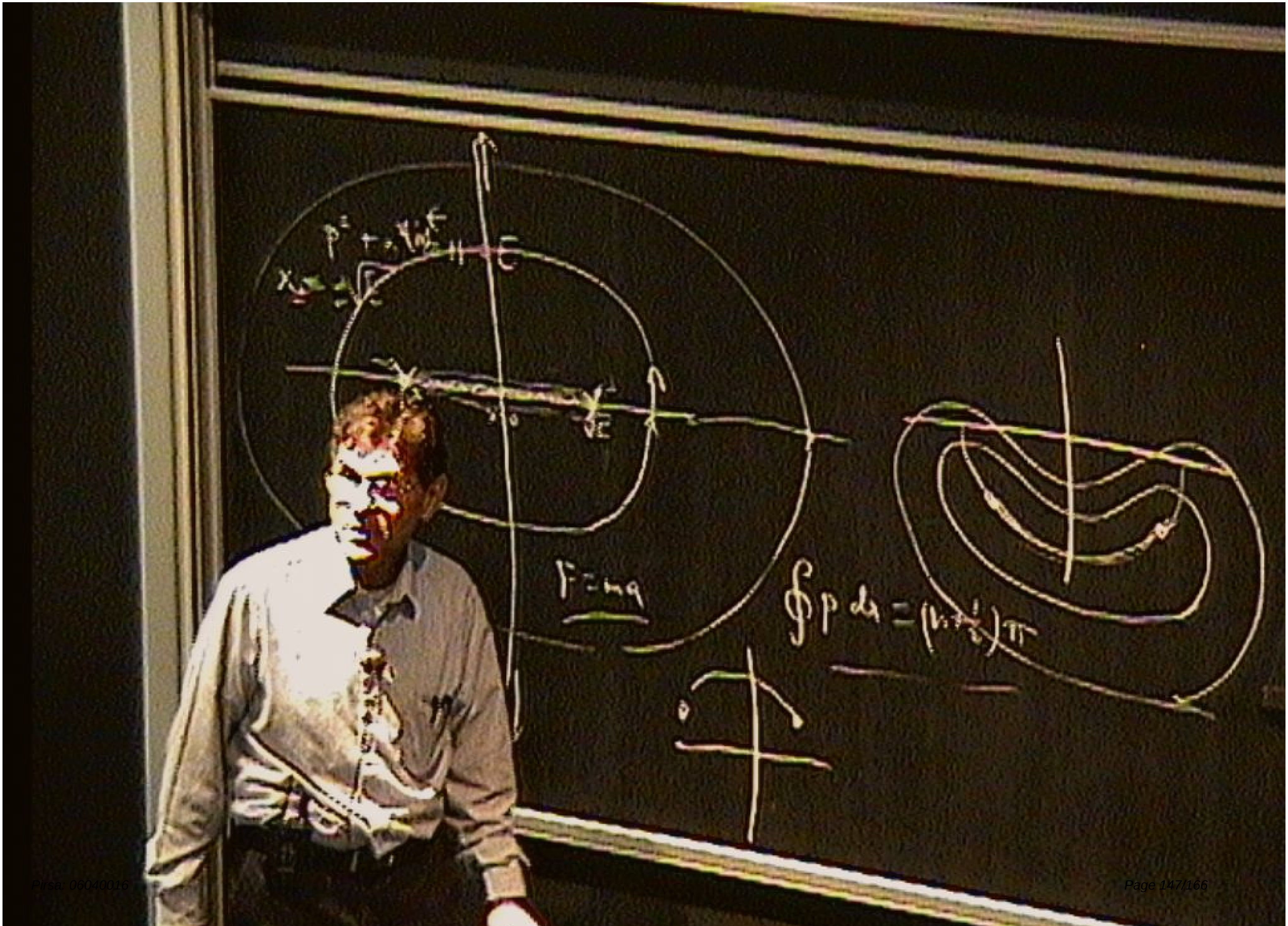

F = ma
∮ p dx = (n + ½) π

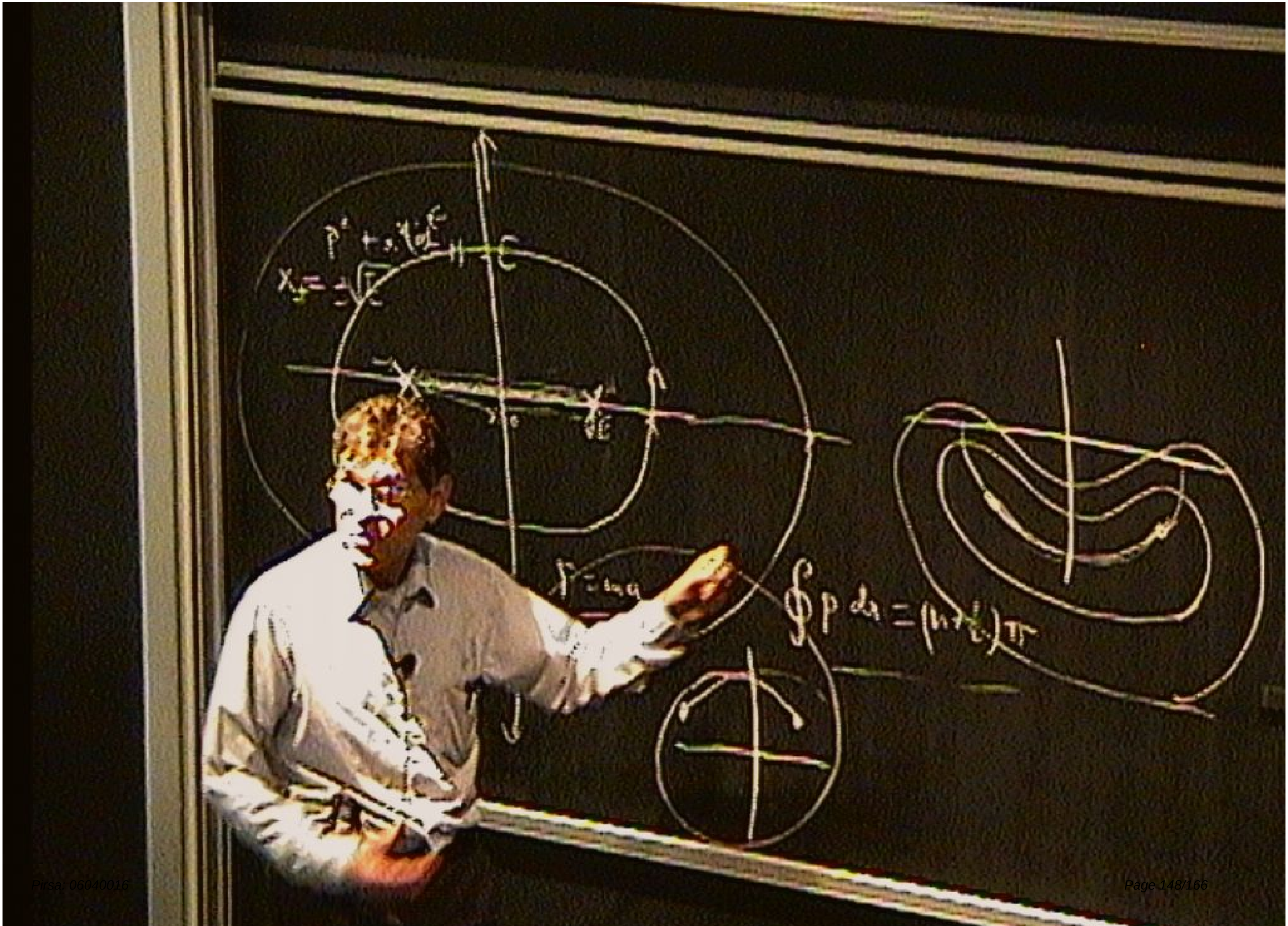


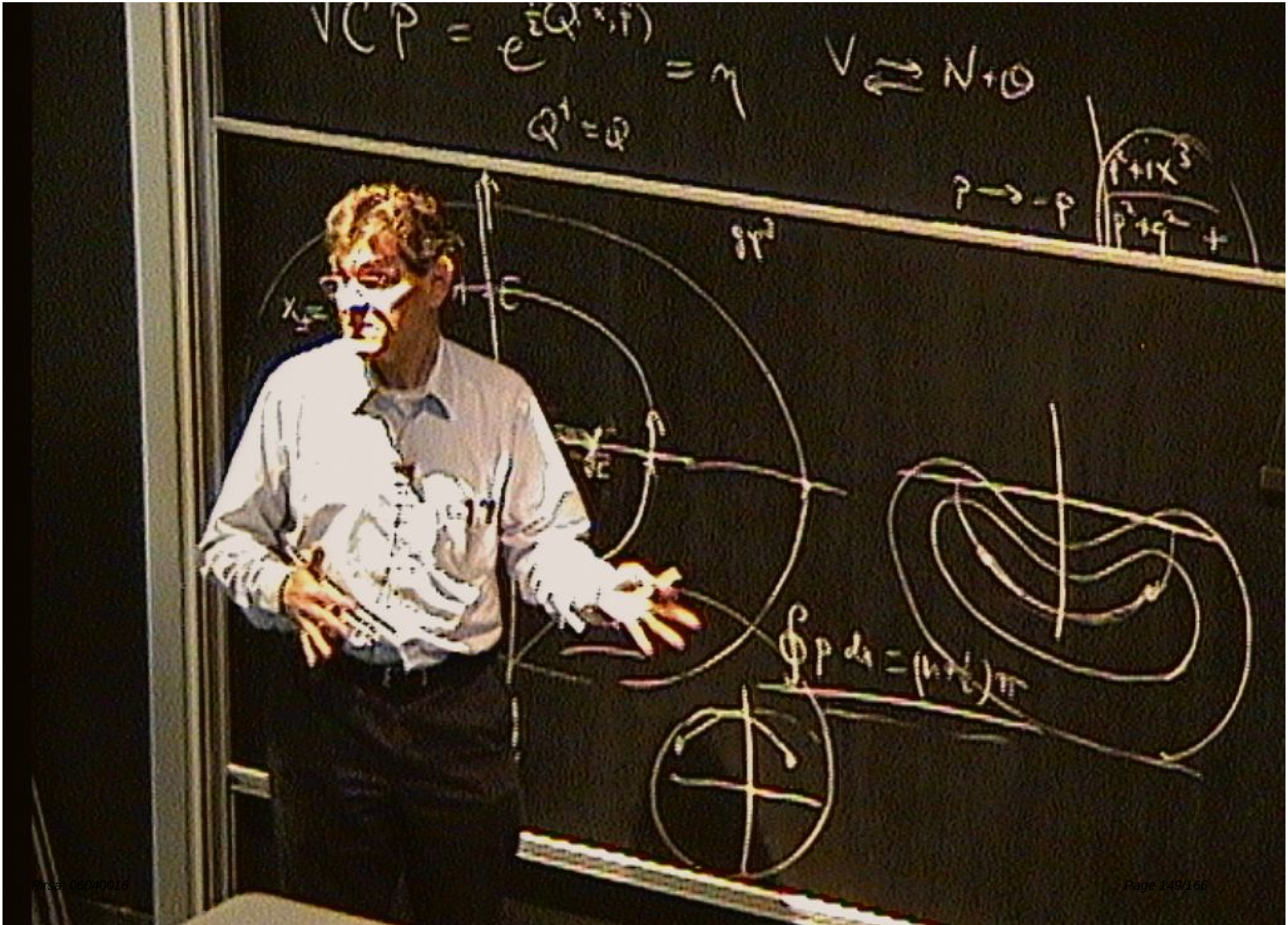


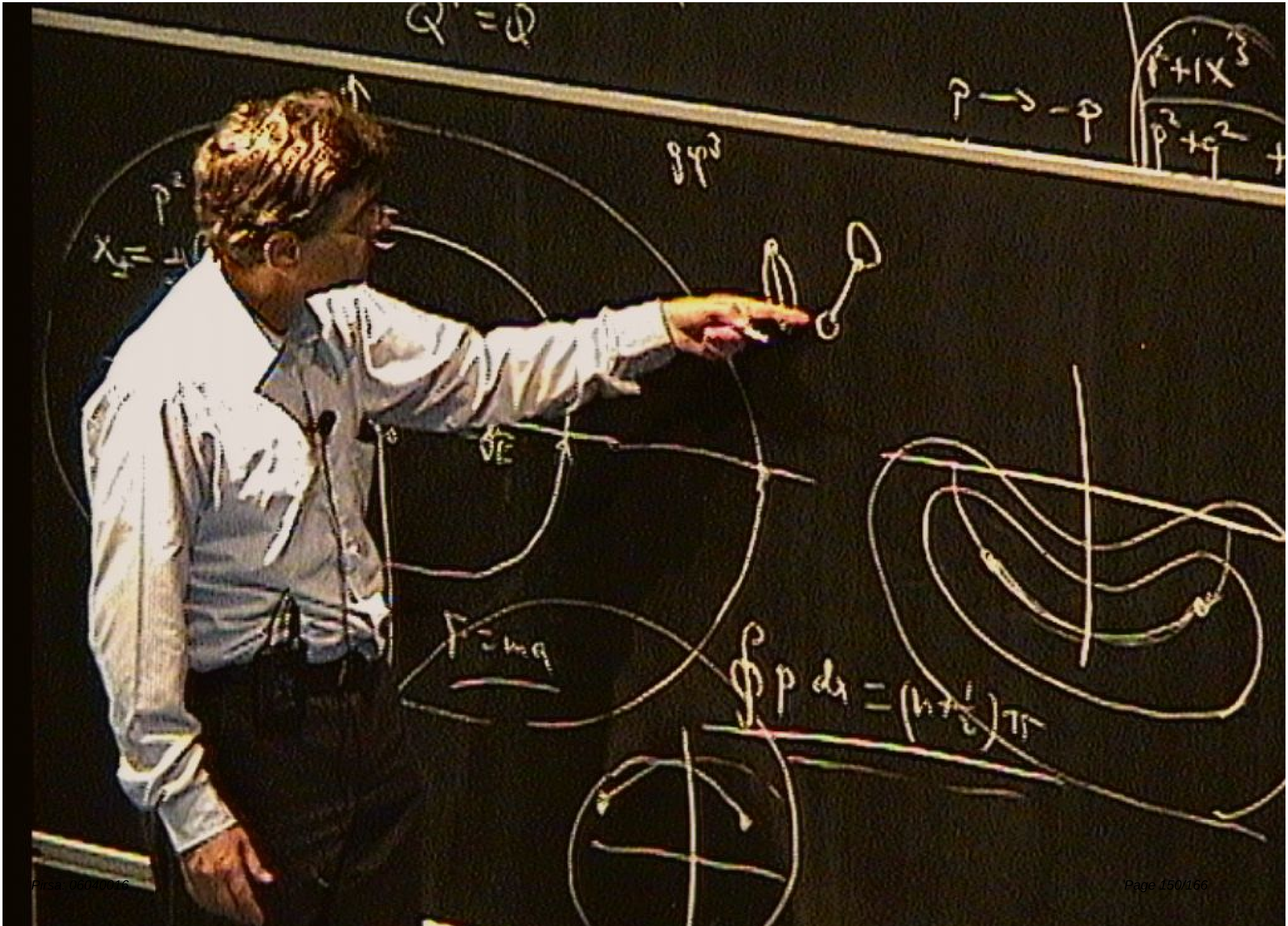


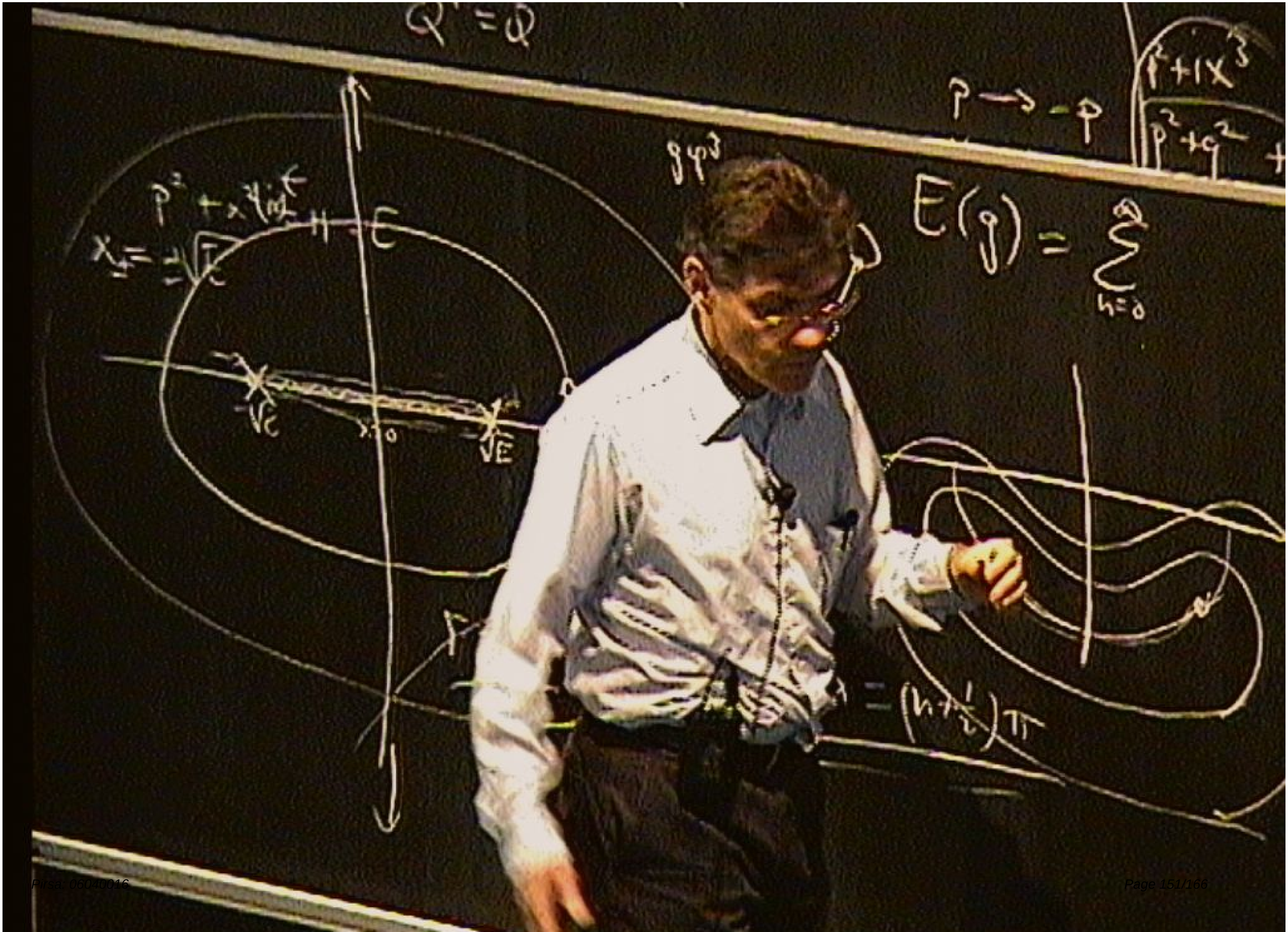


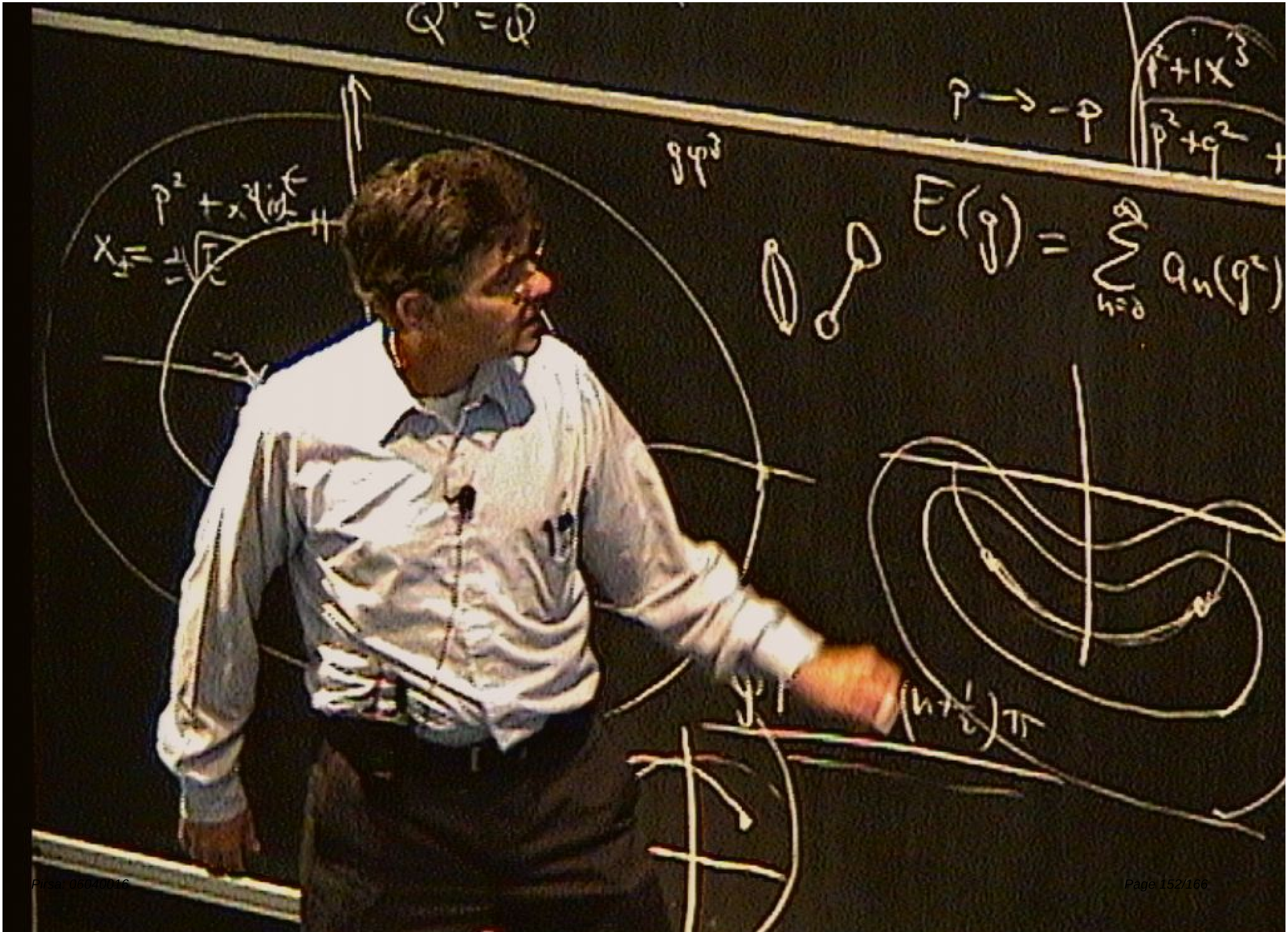

F=ma

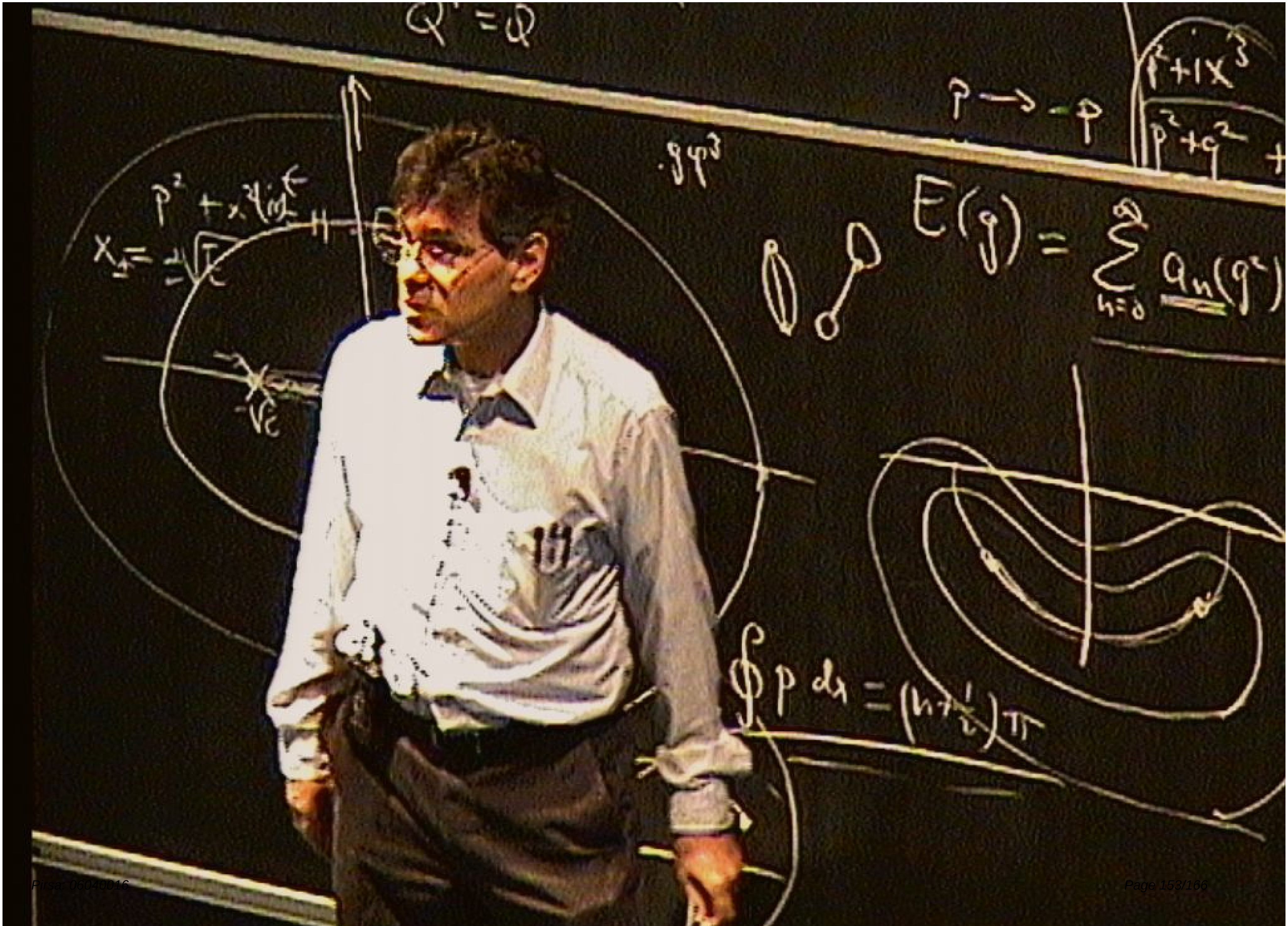


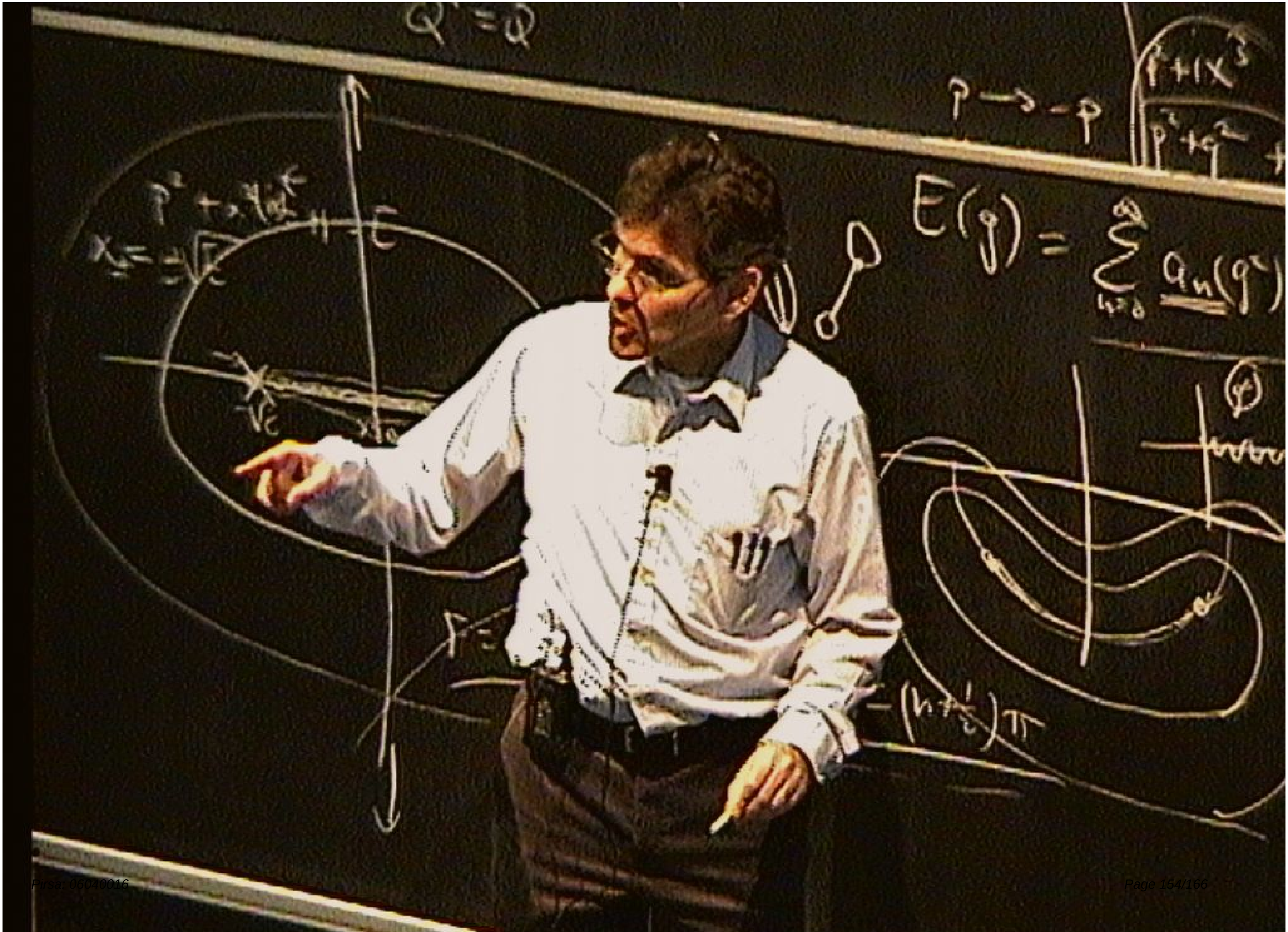


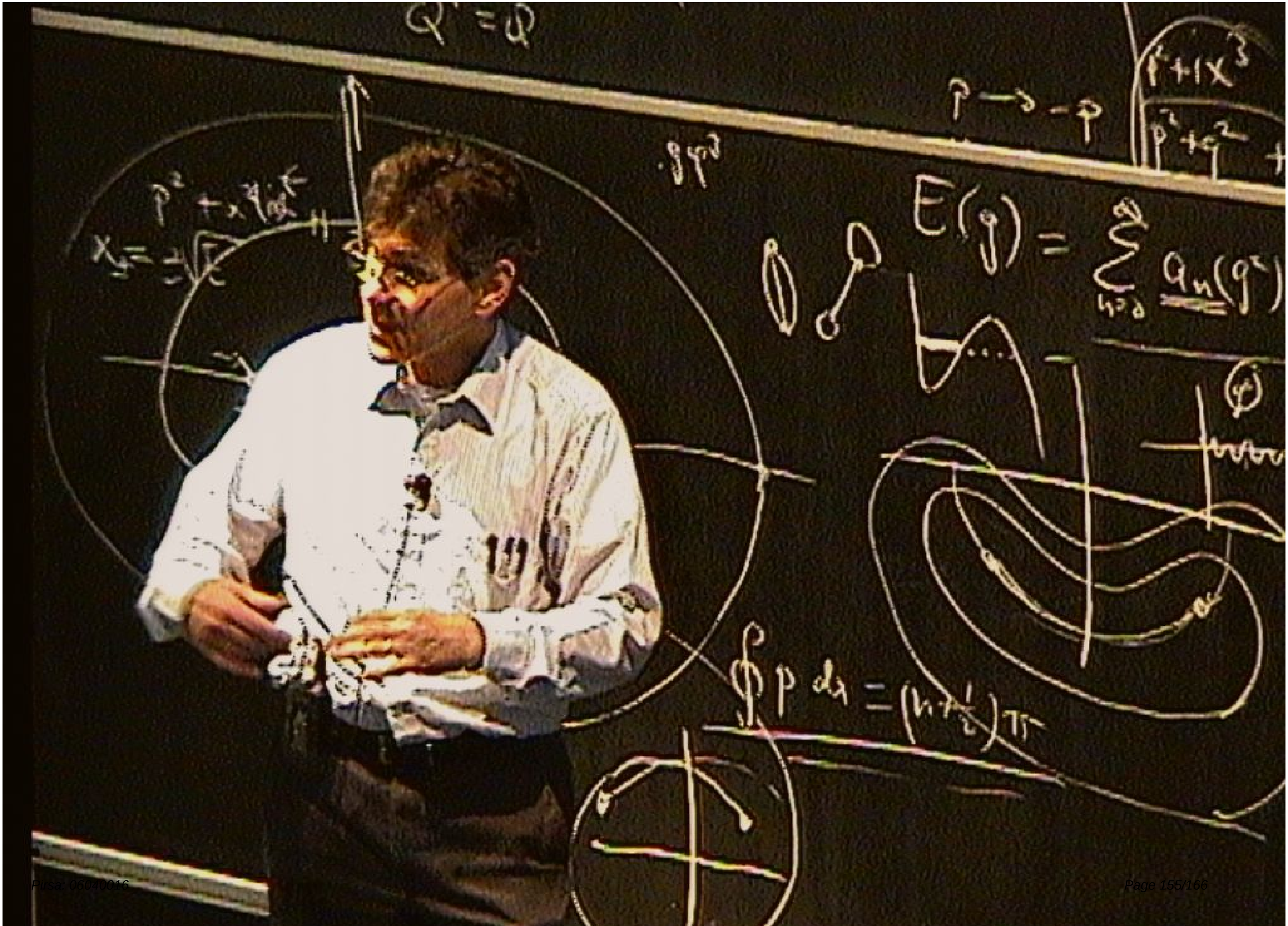

p → -p

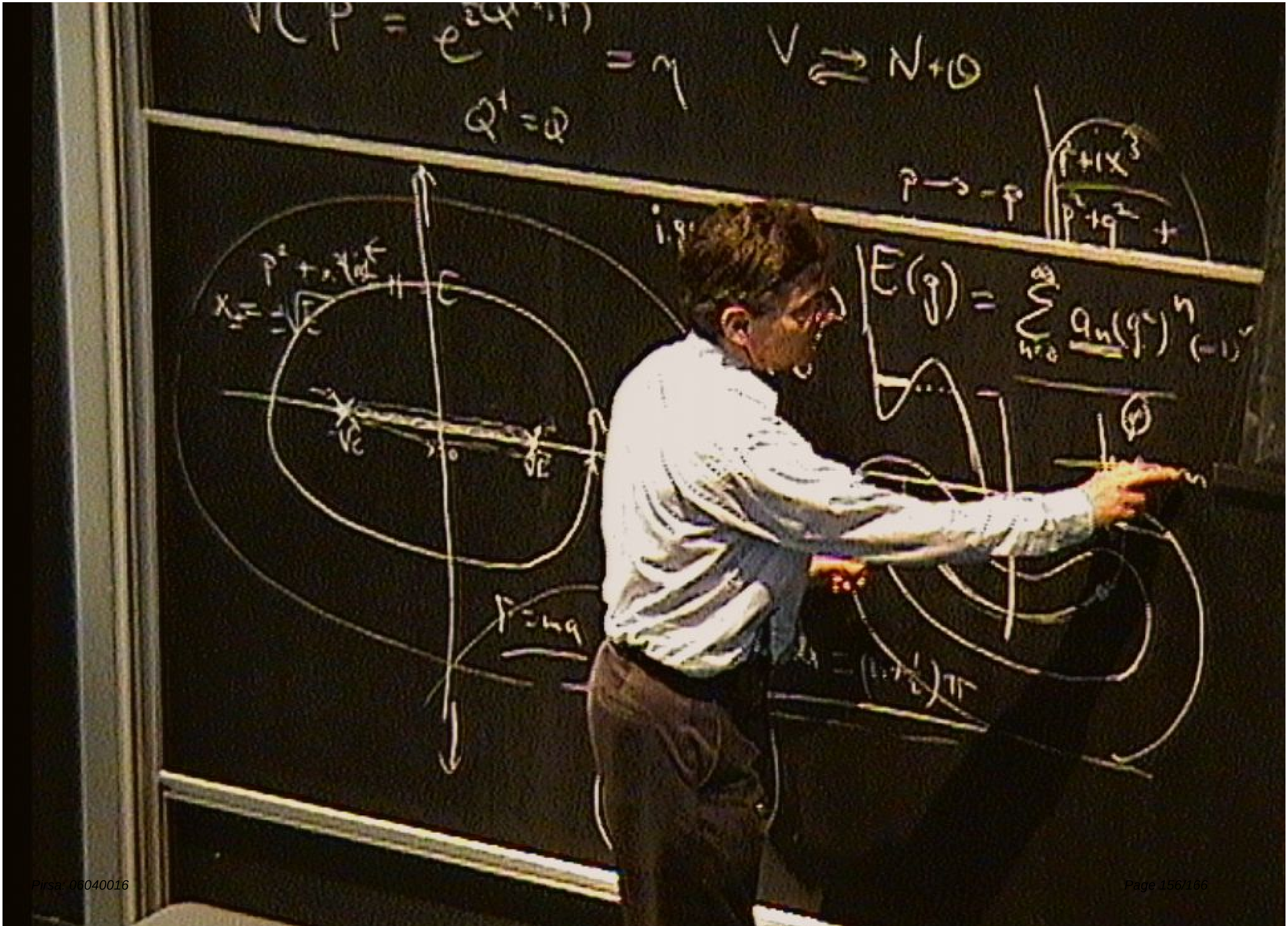

gφ³
p → -p
E(g) =

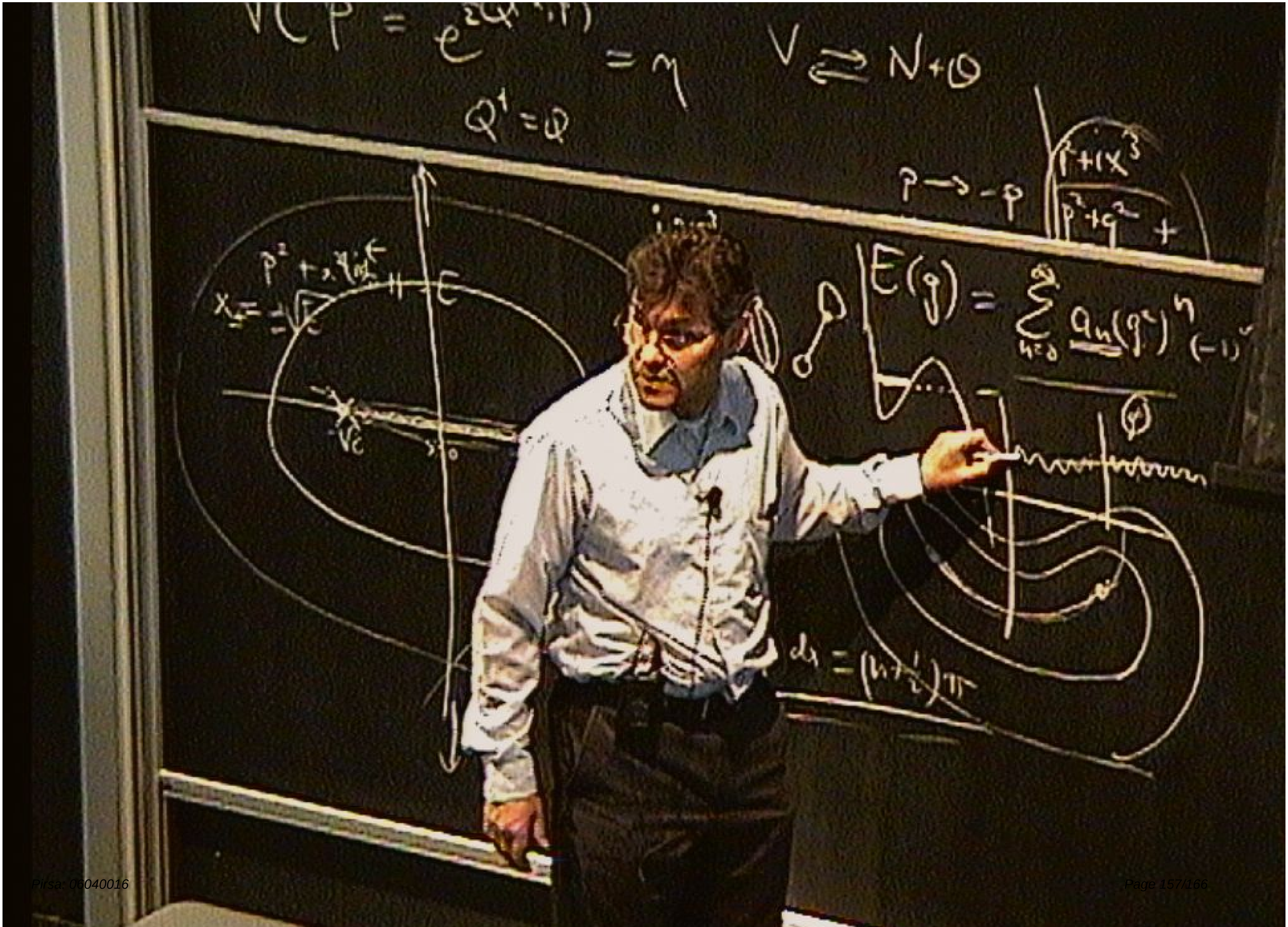

Q' = Q
p → -p
gφ³
E(g) = ∑ aₙ(g²)
n=0
(n+½)π

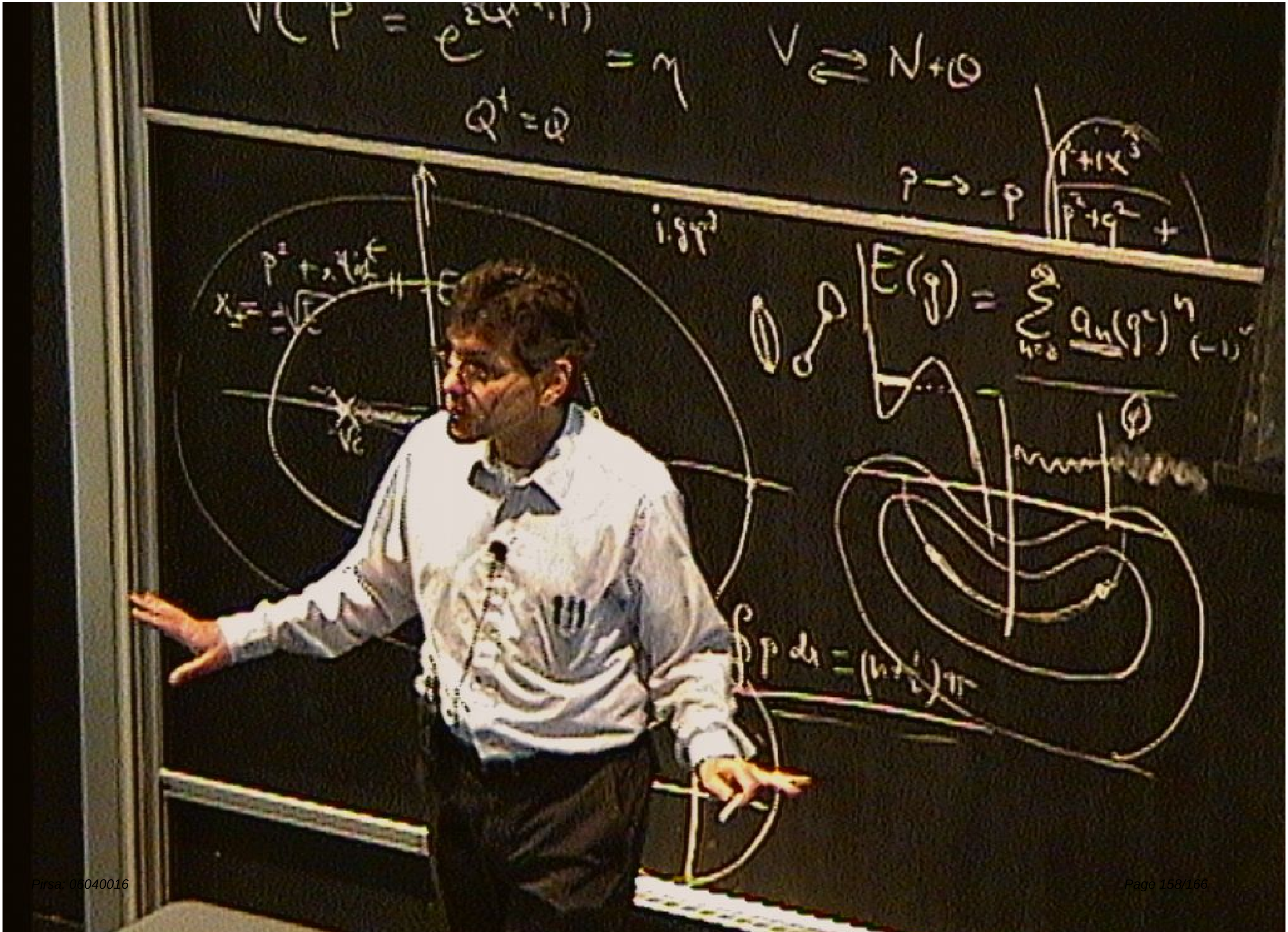

p → -p
E(g) = Σ a_n(g²)
n=0
∮ p dx = (n+½)π

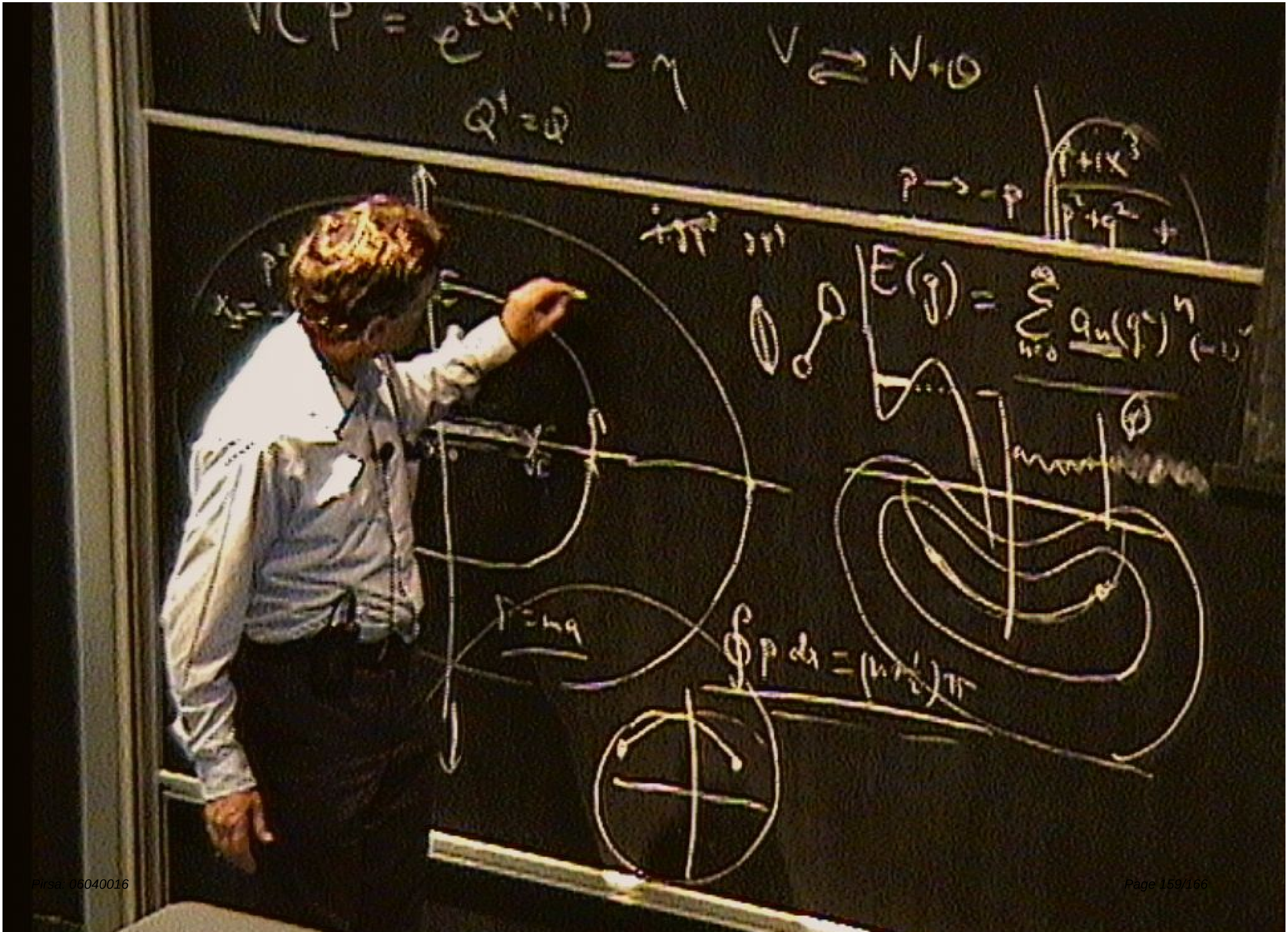

p → -p
E(g) =

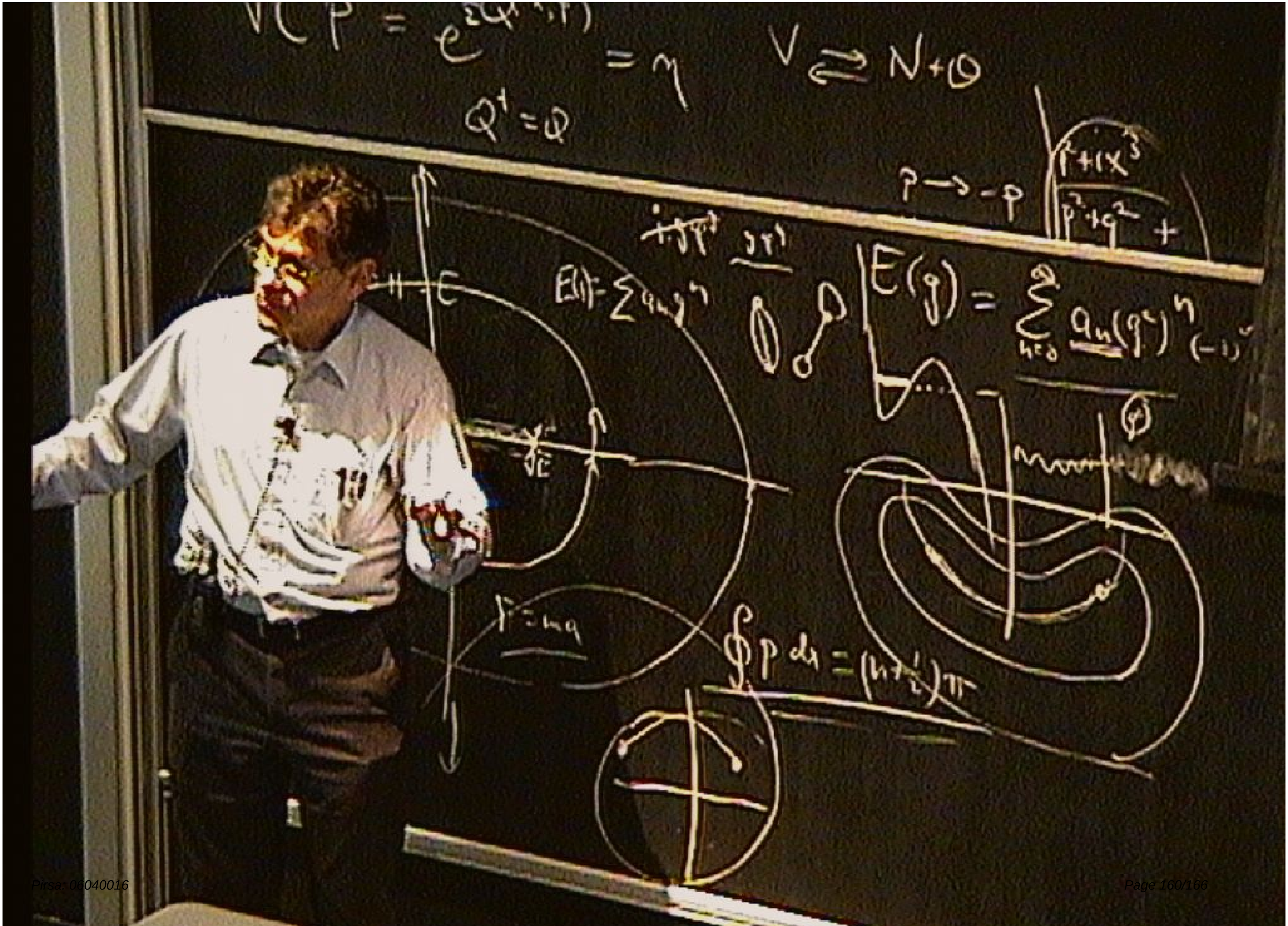

p → -p
∮ p dx = (n+½)π

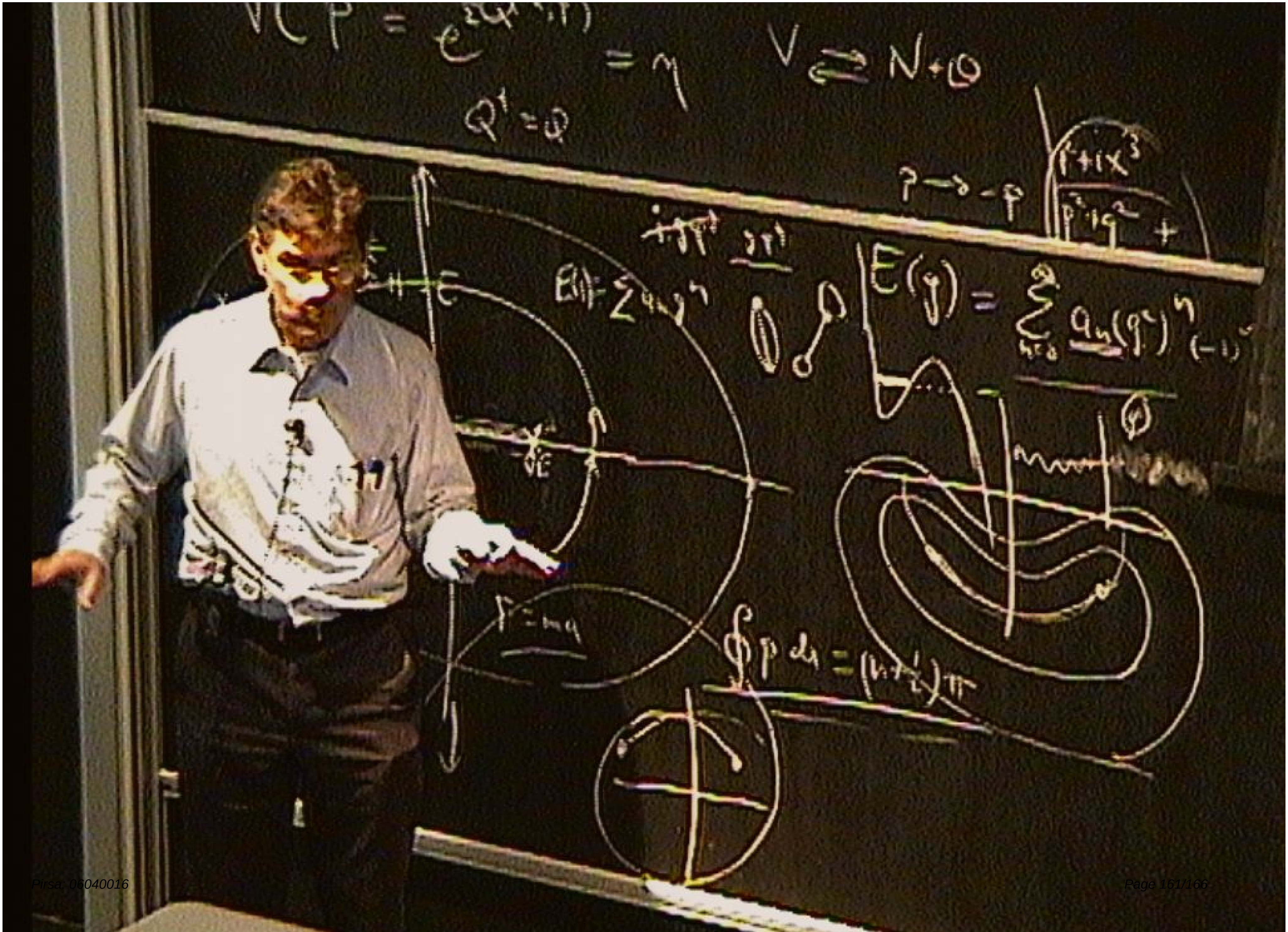

V ⇄ N+Φ
Q† = Q
p → −p
E(g) =
p = ħq

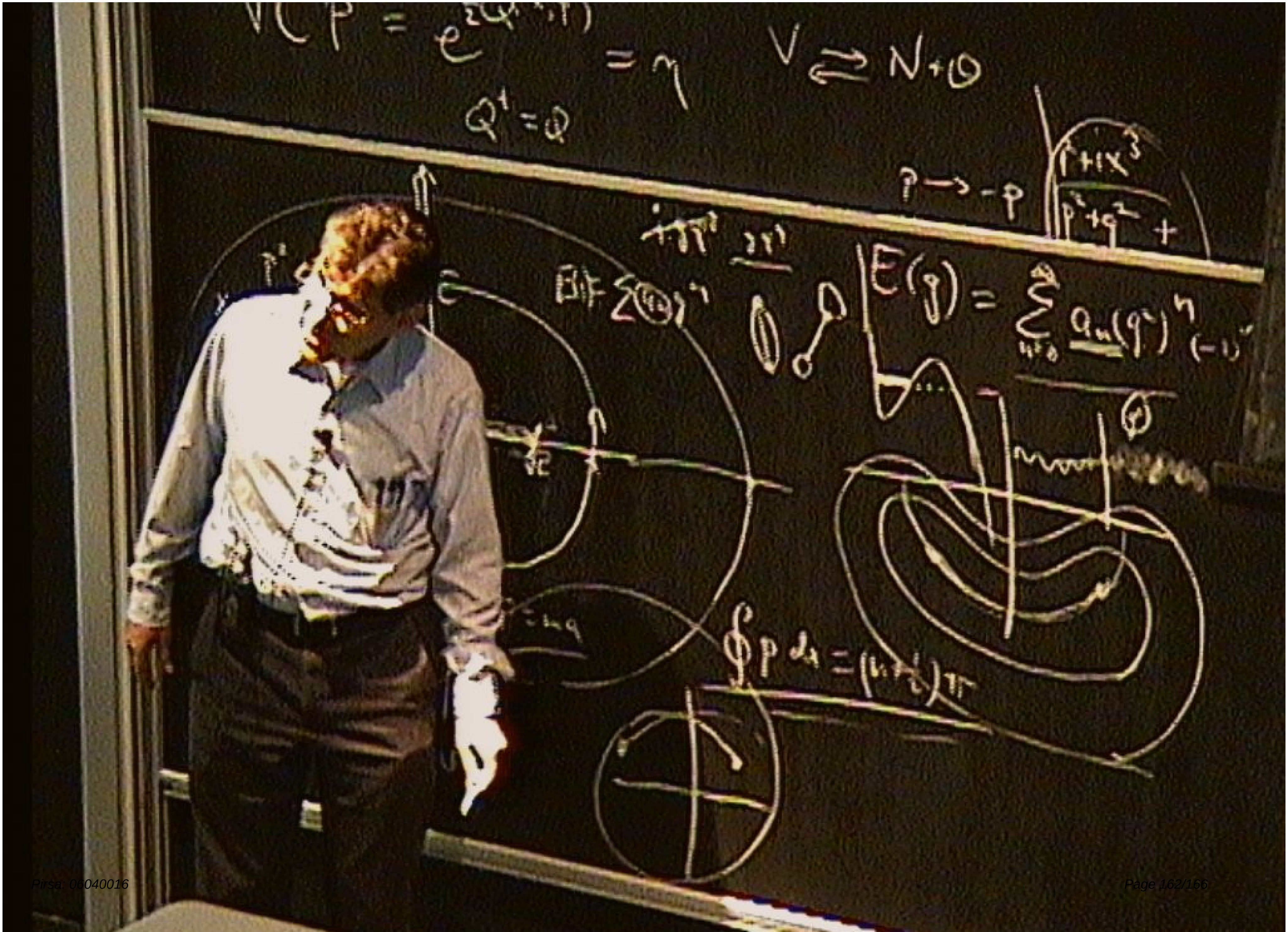

V ⇄ N+Θ
Q† = Q
p → −p

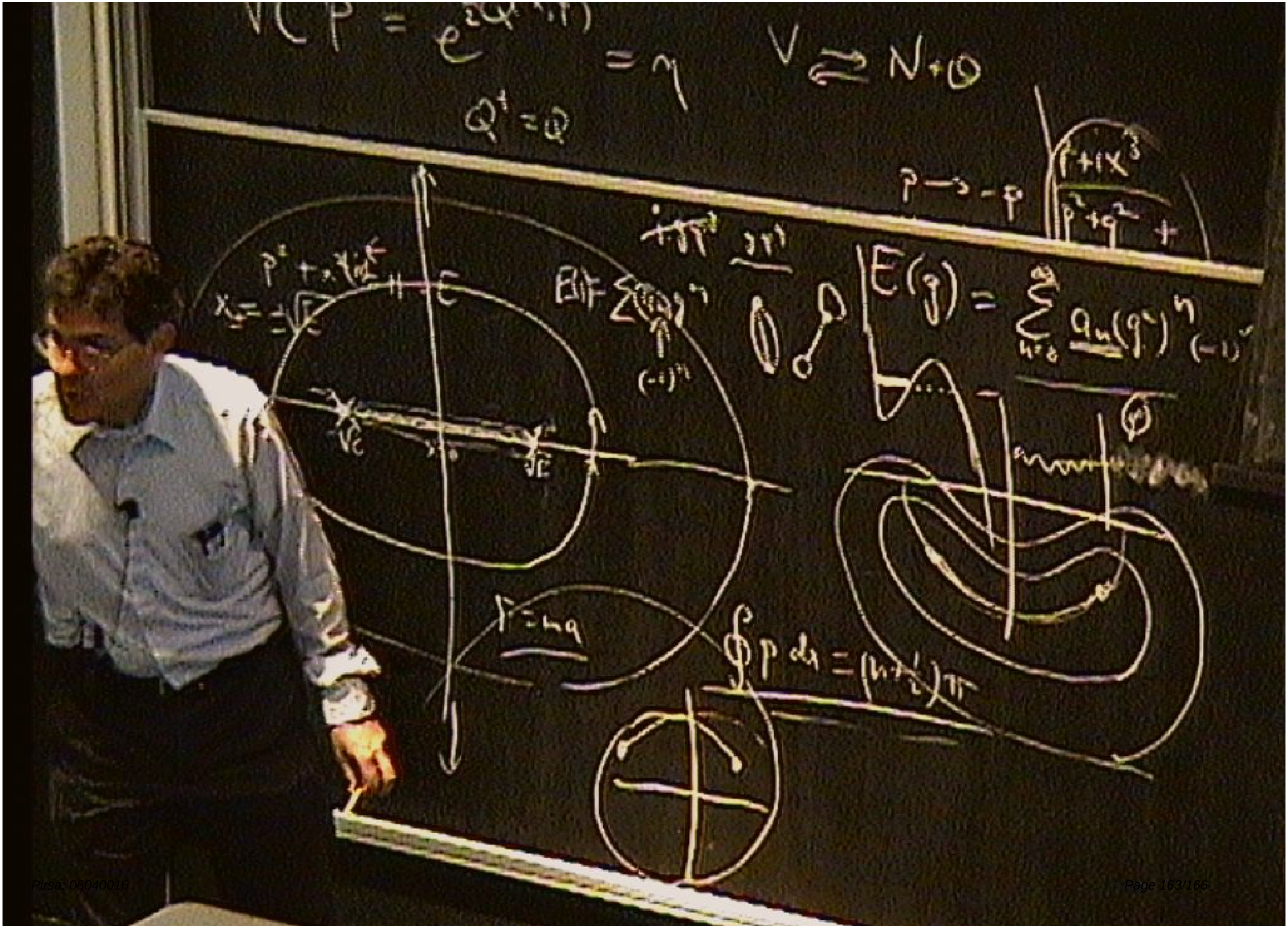


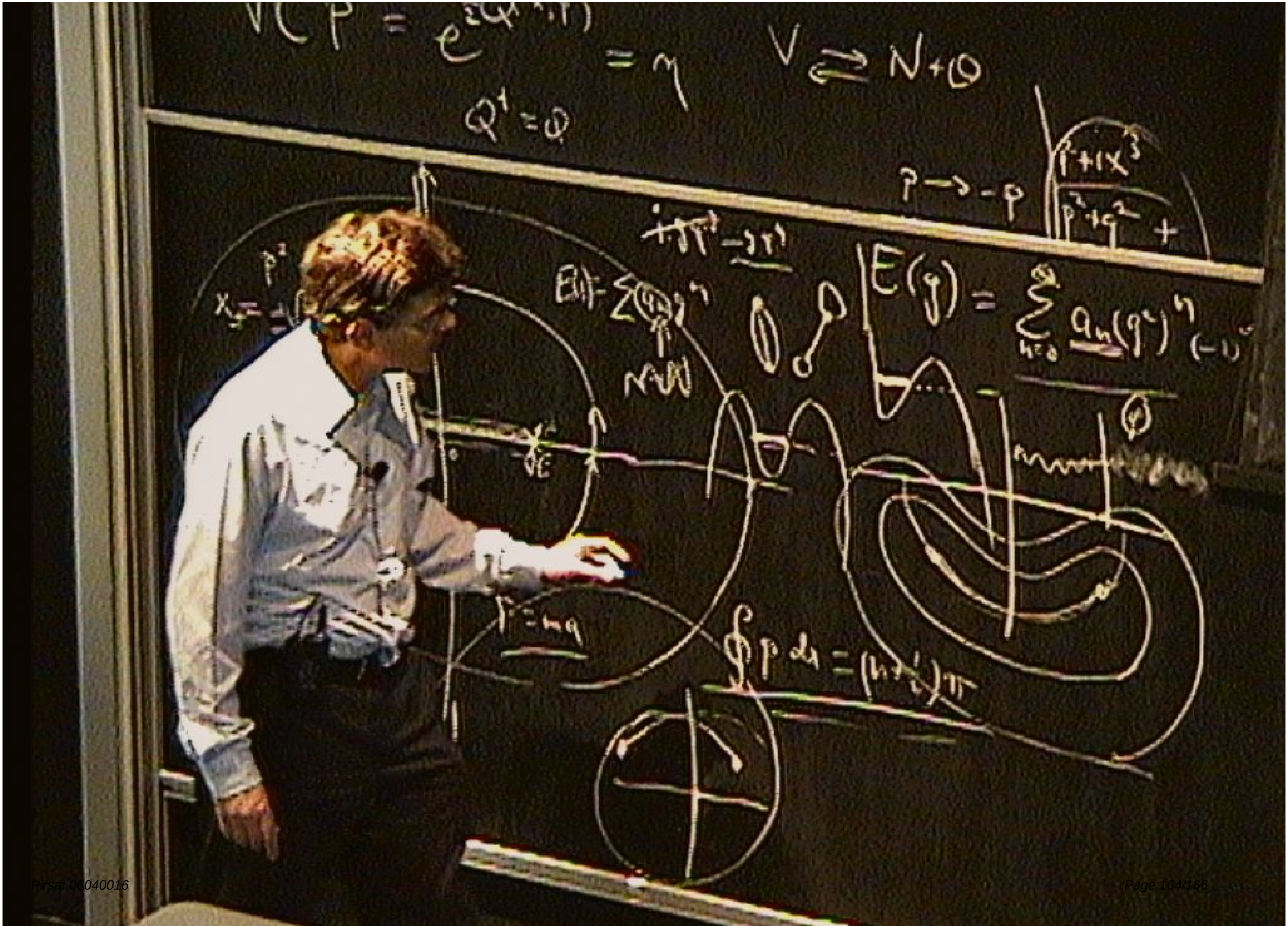




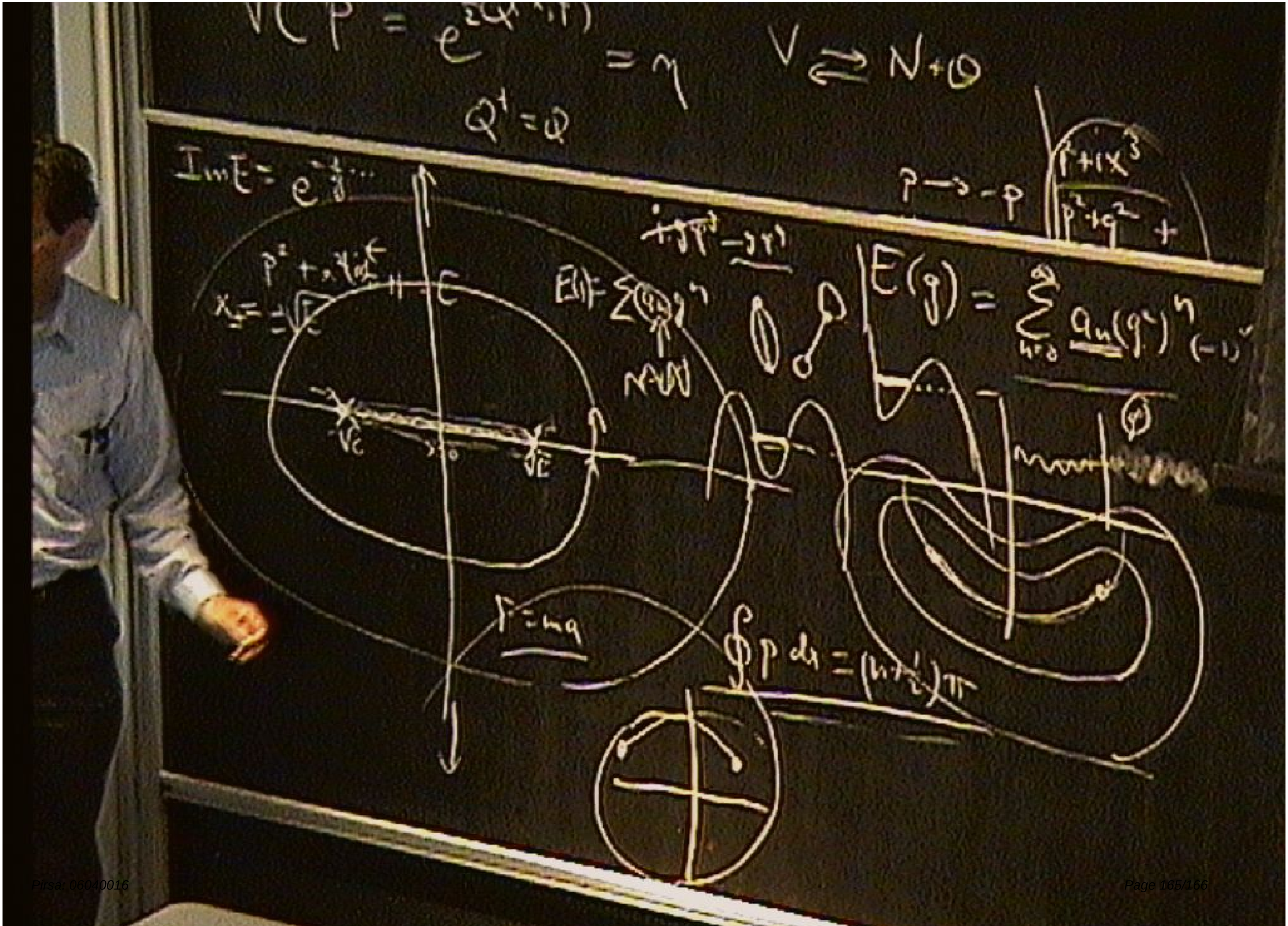


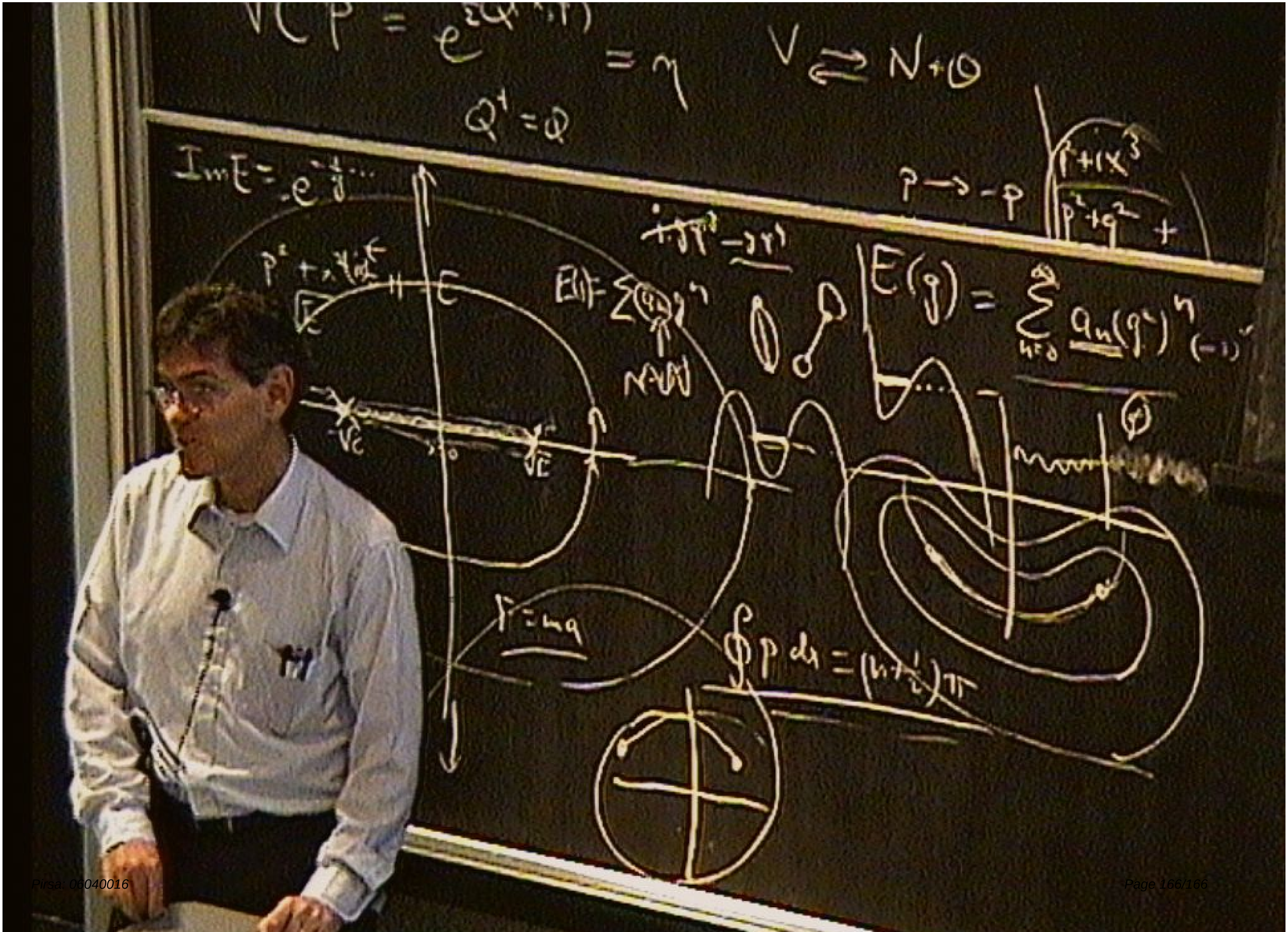

V ⇄ N+Θ
Q† = Q
p → −p





$Q^\dagger = Q$
$p \to -p$


Q† = Q
V ⇄ N+Q
p → −p
E(g) =


Q† = Q
V ⇄ N+Q
p → −p
E(g) =
∮ p dx = (n+½)π