

Title: Introduction to quantum gravity - Part 20

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Abstract: This is an introduction to background independent quantum theories of gravity, with a focus on loop quantum gravity and related approaches.

Basic texts:

-Quantum Gravity, by Carlo Rovelli, Cambridge University Press 2005 -Quantum gravity with a positive cosmological constant, Lee Smolin, hep-th/0209079

-Invitation to loop quantum gravity, Lee Smolin, hep-th/0408048 -Gauge fields, knots and gravity, JC Baez, JP Muniain

Prerequisites:

- undergraduate quantum mechanics
- basics of classical gauge field theories
- basic general relativity
- hamiltonian and lagrangian mechanics
- basics of lie algebras

• 4d (Barrett-Prange)

$$S_{BF} = \int k(e^I \wedge F(\omega)) \longrightarrow$$

$$S_{BF} = \sum_c \int k(E_c \Theta_c) q_c^I(x) \longrightarrow E_c^I = \int e^I$$

$$\omega^J \longrightarrow \omega^I = \int \omega^I(x)$$

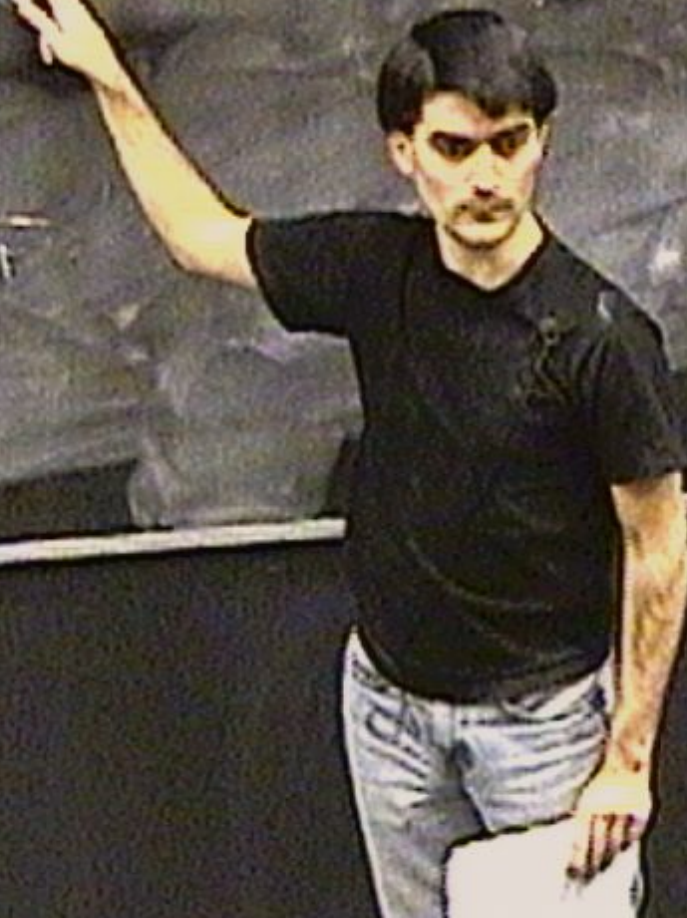
$$g_e = e^{i\omega}$$

$$G_c = \prod_e g_e \leftarrow F$$



$\Sigma \times [0,1]$ F. Dowker, 92-96/0508109 ©

$$Z = \prod_p \int dE_p$$



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$$Z = \prod_p \int E_p^3$$

$$Z = \left(\prod_e \right) \prod_w e^{i \sum_{\text{BP}} S_e(e, w)}$$

• 4d (Barrett-Prane)

$$S_{BF} = \int t(e^I \wedge F^I(\omega)) \rightarrow$$

$$S_{BF} = \sum_c \int t_2(E_c \Theta_c) q_c^I(x) \rightarrow E_c^I = \int e^I$$

$$g_c = e^{i\omega}$$

$$G_c = \prod e^{i\omega} \leftarrow F$$



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$$Z = \prod_p E_p$$

$$Z = \{D_e\} D_w e^{i \sum_{8p} (e, w)}$$

• 4d (Barrett-Prane)

$$S_{\text{BF}} = \int \ell(\mathbf{e}^I \wedge F(\omega)) \rightarrow$$

$$S_{\text{BF}} = \sum_{\mathbf{e}^I} \int \ell_2(E_2 \Theta_2) \mathcal{G}_2^I(x) \rightarrow E_2^I = \int \omega^I(x)$$

$$g_e = e^{i\omega}$$

$$\mathcal{G}_2 = \prod_e g_e \leftarrow F$$



• 4d (Barnett-Prane)

$$S_{BF} = \int t(E \wedge F(\omega)) \rightarrow$$

$$S_{BF} = \sum_c \int t_2(E \wedge \omega_c) \rightarrow E_c = \int e^I$$

$$g_c = e^{i\omega}$$

$$G_c = \prod g_c \leftarrow F$$



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \int_{E_e} \prod_{g \in \text{su}(2)} \int_{g_e} e^{i \sum_e \text{tr}(E_e B_e)}$$

$$= (\text{Det}) \text{Det} e^{i \sum_{\text{BA}} (e, \omega)}$$

$\mathbb{C} \times \mathbb{C}^n$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \left(\prod_{E_e} \prod_{\substack{e' \\ \text{su}(e)}} \right) e^{i \sum_e k_2(E_e, e)}$$

$$Z = \left(\prod_e \right) D_{\omega} e^{i \sum_{\omega} k_2(e, \omega)}$$

$\mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-94/05 08109

$$Z = \prod_p \left(\prod_{E_p} \prod_{\substack{g \in \text{Gal}(E_p/\mathbb{Q})}} e^{i \sum_{\alpha \in E_p} \log(\alpha)} \right) =$$

$$Z = \prod_p e^{i \sum_{\alpha \in E_p} \log(\alpha)}$$

$\mathbb{C} \times \mathbb{C}^*$ F. Vowker, 92-96/0508109

$$Z = \prod_p \left(\int_{E_p} \prod_{g \in \text{Gal}(E_p)} \prod_{\sigma \in \text{Gal}(E_p)} e^{i \cdot \text{tr}(E_p(E_p))} \right) =$$

$$= \left(\prod_p \right) \prod_p e^{i \cdot \sum_{\sigma \in \text{Gal}(E_p)} \text{tr}(E_p(E_p))}$$

• 4d (Barrett-Prange)

$$S_{BF} = \int t_2 (E^I \wedge F^I(\omega)) \rightarrow$$

$$S_{BF} = \sum_c \int t_2 (E_c^I \wedge F_c^I) \rightarrow E_c^I = \int_{\omega^I} \omega^I$$

$$g_c = e^{i \int \omega}$$

$$G_c = \prod_c g_c \leftarrow F$$



$\mathbb{C} \times \mathbb{C}^*$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \left(\int_{E_e} \prod_{g \in \pi_e} \prod_{h \in \pi_e} e^{i \int_{g,h} \omega} \right) =$$

$$Z = \left(\prod_e \int_{D_e} e^{i \int_{D_e} \omega} \right)$$

$\mathbb{C} \times [0,1]$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \int dE_e \prod_{g \in \text{SU}(2)} \int dg_g \prod_e e^{i \int_e k_2(E_e, g_e)} =$$

$$Z = \left(\prod_e \int d\omega_e \right) \prod_e e^{i \int_{\partial A} S_A(e, \omega)} = \prod_e \int d\delta_e \prod_e \delta(\dots)$$

$\leq \times (0,1)$ F. Vowker, 92-94/0508109

$$Z = \prod_e \int_{E_e} \prod_{g \in \text{SU}(2)} \prod_{\ell} e^{i \int_{\ell} \text{tr}(E_e g_e)} =$$

$$\prod_e \int_{\text{SU}(2)} d g_e \prod_{\ell} \delta(g_e)$$

$\leq \times (0,1)$ F. Vowker, 92-94/05 08109

$$\begin{aligned}
 Z &= \prod_c \int dE_c \prod_{\substack{e \\ \text{sur}(c)}} \int dg_e \prod_{e'} \delta(E_{e'} E_e) \\
 &= \prod_c \int dg_c \prod_{\substack{e \\ \text{sur}(c)}} \delta(E_e) \\
 &= \prod_c \int dg_c \prod_{\substack{e \\ \text{sur}(c)}} \delta\left(\prod_{e' \in e} g_{e'}\right)
 \end{aligned}$$

$\mathbb{C} \times (0,1)$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \int_{E_e} \prod_{g \in \text{su}(2)} \int dg_r \prod_l e^{i \int_{\mathbb{C}} \text{tr}(E_e \mathcal{Q}_e)}$$

$$\delta(\mathcal{Q} - \mathbf{I})$$

$$= \prod_e \int_{\text{su}(2)} dg_r \prod_l \delta(\mathcal{Q}_e)$$

$$= \prod_e \int dg_r \prod_{f \in e} \delta\left(\prod_{\mathcal{Q}_{ef} \mathcal{Q}_f}\right)$$

$\in \times (0,1)$

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$$Z = \prod_p \left(\int_{E_p} \prod_{g \in \text{SU}(2)} \prod_{e \in E_p} e^{i \sum_e k_2(E_e, g_e)} \right) =$$

$$\delta(G) \equiv \begin{cases} 0 & \forall e \in I \\ \infty & \end{cases} \quad \prod_p \left(\int_{g \in \text{SU}(2)} \prod_{e \in E_p} \delta(g_e) \right) = \prod_p \left(\int_{g \in \text{SU}(2)} \prod_{e \in E_p} \delta\left(\prod_{e \in E_p} g_e\right) \right)$$

$\mathbb{Z} \times \mathbb{Z}(1)$

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$$Z = \prod_e \int_{E_e} \prod_{e' \in \text{Sur}(e)} \int g_{e'} \prod_{e'} e^{i \int_{e'} \tau_2(E_{e'} G_{e'})} =$$

$$\prod_{e'} \int dg_{e'} \prod_{e'} \delta(G_{e'})$$

$$= \prod_{e'} \int dg_{e'} \prod_{e'} \delta\left(\prod_{e' \in \text{Sur}(e)} g_{e'}\right)$$

$\mathbb{C} \times [0,1]$ F. Vowker, 92-94/05 08109

$$Z = \prod_e \int_{E_e} \prod_{g \in \text{su}(2)} \int dg_g \prod_e e^{i \int_{\mathbb{C}} \text{tr}(E_e g_e)} =$$

$$\prod_e \int_{\text{su}(2)} dg_e \prod_e \delta(g_e)$$

$$= \prod_e \int dg_e \prod_e \delta\left(\prod_{g \in \text{su}(2)} g_e\right)$$

$\in \times (0,1)$

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$$Z = \prod_c \int dE_c \prod_{g \in S(c)} \int dg_c \prod_e e^{i \int_{\mathcal{C}_e} F_2(E_c, g_c)} =$$

$$\delta(x) = \int dP e^{iPx}$$

$$\prod_c \int dg_c \prod_e \delta(\mathcal{C}_e)$$

$$= \prod_c \int dg_c \prod_e \delta\left(\prod_{\mathcal{C}_e} g_c\right)$$

$\mathbb{Z} \times [0,1]$ F. Dowker, 92-94/0508109

$$Z = \prod_e \int_{E_e} \prod_{g \in \text{SU}(2)} \prod_e e^{i \int g \omega_e} =$$

$$\delta(x) = \int_{\mathbb{R}} dp e^{ipx}$$

$x \in \mathbb{R}$

$$= \prod_e \int_{\text{SU}(2)} dg_e \prod_e \delta(g_e)$$

$$= \prod_e \int dg_e \prod_e \delta\left(\prod_{(e,f)} g_e\right)$$

$\mathbb{Z} \times [0,1]$ F. Dowker, 92-96/0508109

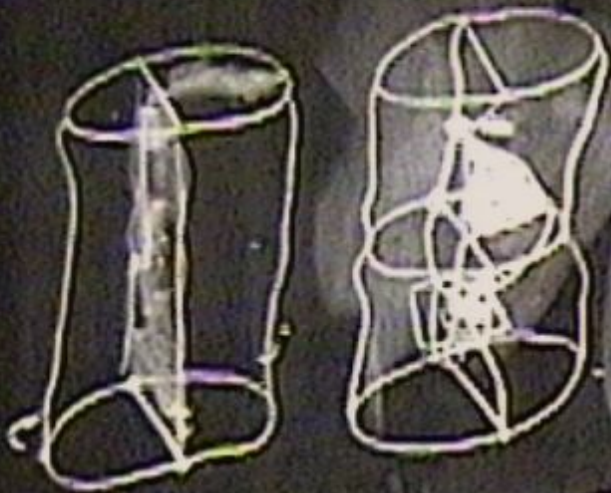
$$Z = \prod_e \int_{E_e} \prod_{g \in \text{su}(1)} \int dg_e \prod_e e^{i \sum_e k_2(E_e G_e)} =$$

$$\delta(x) = \int dp e^{ipx} \quad \prod_e \int dg_e \prod_e \delta(G_e)$$

$$x \in \mathbb{R} \sim T^1 = \prod_e \int dg_e \prod_e \delta\left(\prod_{e \in \text{su}(1)} g_e\right)$$

$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$\delta(\mathcal{C}) = \sum_j$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\frac{t}{\tau}) \dots$$

$$\delta(\mathcal{C}) = \sum_j (2j+1) \chi_j(\mathcal{C})$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{C}) = \sum_j (2j+1) \chi^j(\mathcal{C})$$

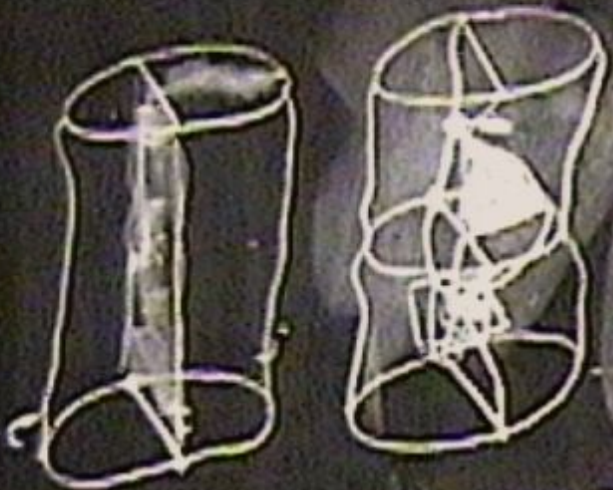
$$j^i |j m\rangle$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{C}) = \sum_j (2j+1) \chi_j(\mathcal{C})$$

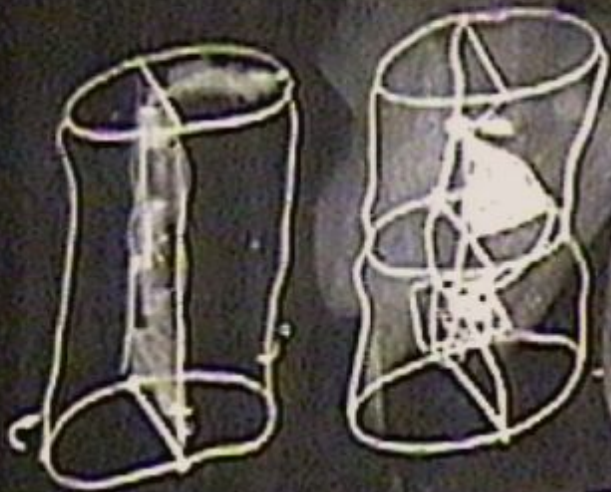
$$\langle j m' | \mathcal{C}^j | j m \rangle$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\frac{\tau}{\hbar}) \dots$$

$$\delta(e) = \sum_j (2j+1) \chi^j(e)$$

$$\langle j m' | \mathcal{O}^j | j m \rangle = D_{m m'}^j(e)$$

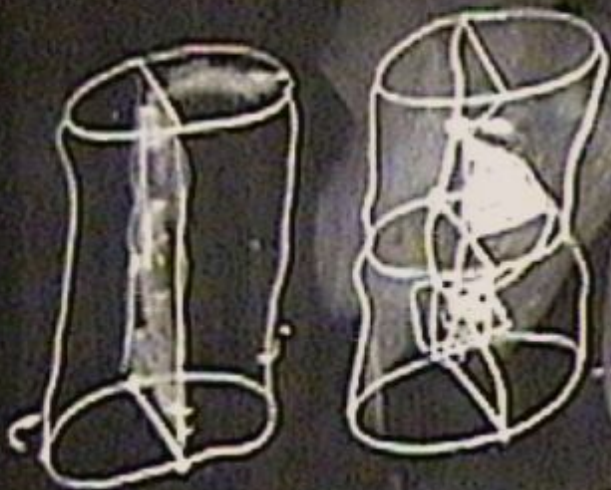


$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{C}) = \sum_j (2j+1) \chi^j(\mathcal{C})$$

$$\langle j m' | \mathcal{C} | j m \rangle = D_{mm'}^j(\mathcal{C})$$

$$\chi^j(\mathcal{A}) = \sum_m D_{mm}^j(\mathcal{A})$$



$\mathbb{C} \times \mathbb{C}^n$

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$$Z = \prod_e \int_{E_e} \prod_{g \in \text{Sur}(e)} \prod_c e^{i \sum_c k_2(E_e, E_c)} =$$

$$\delta(x) = \int dP e^{iPx} \quad \prod_e \int_{\text{Sur}(e)} dg_e \prod_c \delta(e_c)$$

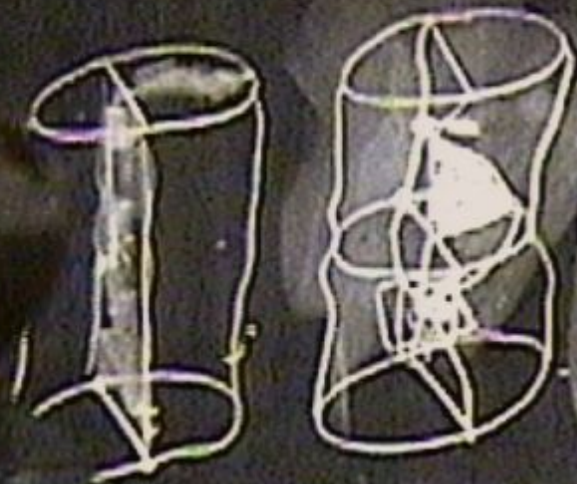
$$x \in \mathbb{R} \sim T^1 = \prod_e \int dg_e \prod_c \delta\left(\prod_{c \in e} g_c\right)$$

$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{C}) = \sum_j (2j+1) \chi^j(\mathcal{C})$$

$$\langle j m' | \mathcal{C}^j | j m \rangle = D_{mm'}^j(\mathcal{C})$$

$$\chi^j(\mathcal{A}) = \sum_m D_{mm}^j(\mathcal{A})$$

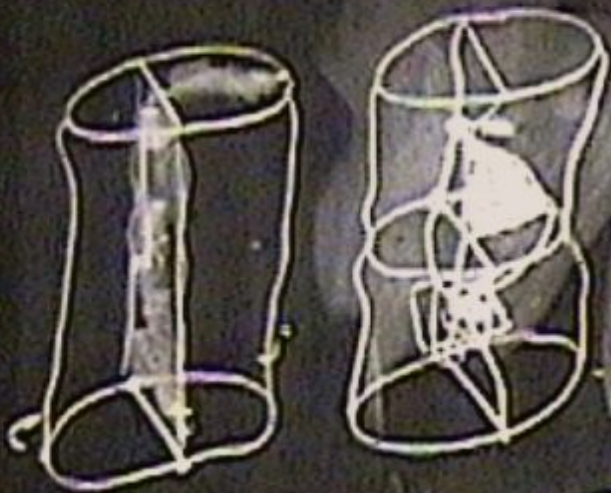


$$U_{N,0} \rightarrow U(T) \rightarrow U = \int d\tau U(\frac{t}{T}) \dots$$

$$\delta(\mathcal{C}_f) = \sum_j (2j+1) \chi^j(\mathcal{C})$$

$$\langle j m' | \mathcal{C}^j | j m \rangle = D_{m m'}^j(\mathcal{C})$$

$$\chi^j(\mathcal{C}) = \sum_m D_{m m}^j(\mathcal{C})$$



$\times \in \mathbb{R}^n$

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$$Z = \prod_e \int dE_e \prod_{g \in \text{SU}(2)} \int dg_e \prod_e e^{i \int dt \mathcal{L}_2(E_e, g_e)} =$$

$$\delta(x) = \int dp e^{ipx}$$

$$x \in \mathbb{R} \sim T^1$$

$$\prod_e \int dg_e \prod_e \delta(g_e)$$

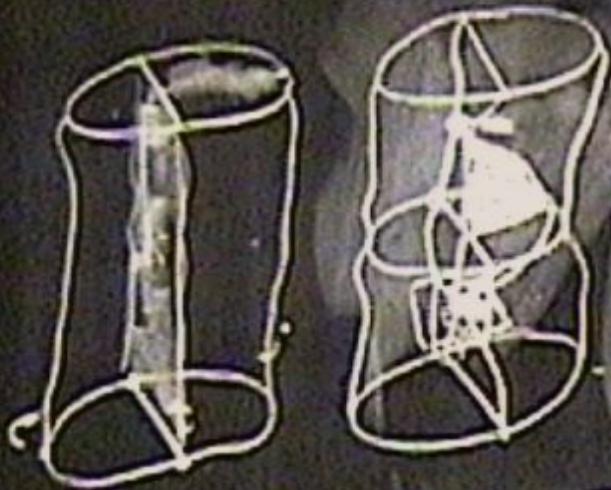
$$= \prod_e \int dg_e \prod_f \delta\left(\prod_{e \in f} g_e\right)$$

$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{G}_i) = \sum_j (z_{j+1}) \chi_j(\mathcal{G})$$

$$\langle j m | \mathcal{G} | j m \rangle = D_{m m}^j(\mathcal{G})$$

$$\chi_j(\mathcal{G}) = \sum_m D_{m m}^j(\mathcal{G})$$



• 4d (Barrett-Crane)

$$\sum_c Z(\Delta)$$

• 4d (Barrett-Crane)

$$Z(\Delta) = \left[\prod_F \sum_{\sigma_F} (2j_F + 1) \right] \left[\prod_{F \in \text{int}(\Delta)} \int dj_F \right] \prod_F \chi^{j_F}(\prod_i g_{iF})$$

$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(\tau) \dots$$

$$\delta(\mathcal{G}_r) = \sum_j (2j+1) \chi^j(\mathcal{G}_r)$$

$$\langle j m' | \mathcal{G}^j | j m \rangle = D_{m m'}^j(\mathcal{G})$$

$$\chi^j(\mathcal{G}) = \sum_m D_{m m}^j(\mathcal{G})$$



$\mathbb{C} \times \mathbb{C}^n$ F. Vowker, 92-94/0508109

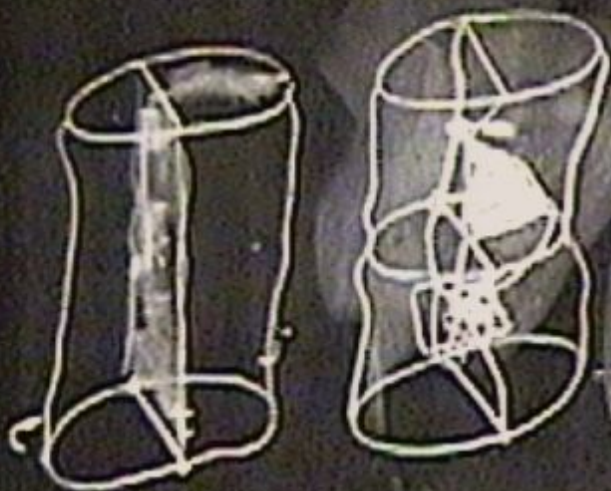
$$Z = \prod_e \int_{E_e} \prod_{g \in \text{SU}(2)} \prod_{\ell} e^{i \int_{\ell} \text{Tr}(E_e g_e)} =$$

$$\delta(x) = \int dp e^{ipx} \quad \prod_e \int_{\text{SU}(2)} dg_e \prod_{\ell} \delta(g_e)$$

$$x \in \mathbb{R} \sim T^1 = \prod_e \int dg_e \prod_{\ell} \delta\left(\prod_{e \in \ell} g_e\right)$$

$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \left(d\tau U(\frac{\tau}{\hbar}) \right) \dots$$

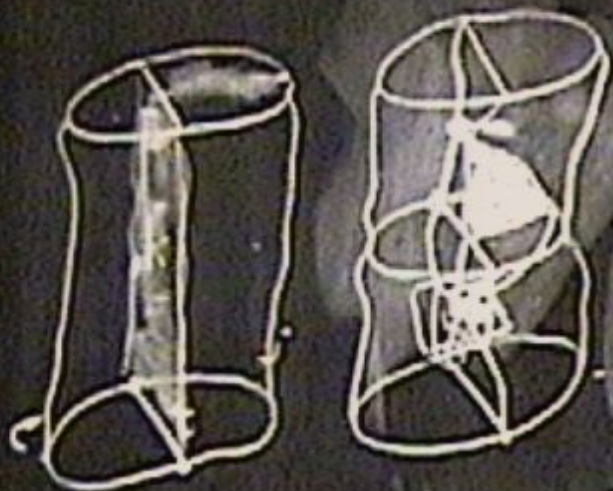
$$\chi(\prod g_i) = >$$



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int d\tau U(t) \dots$$

$$\chi(\prod g_i) = \sum_{mm} (\prod \chi_i) =$$

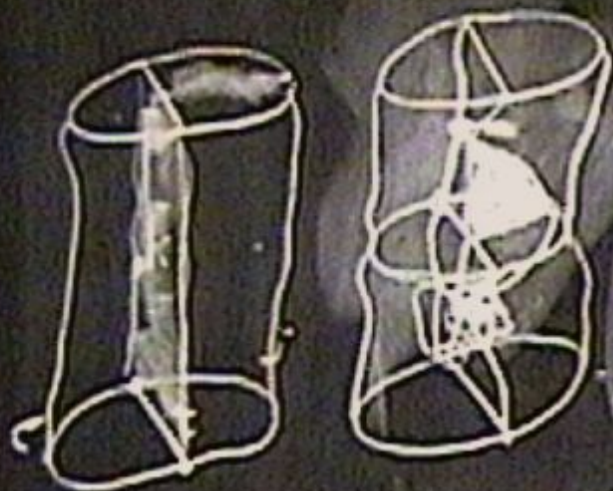
$$= D$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \left(d\tau U(t) \right) \dots$$

$$\chi(\prod_{i \in I} g_i) = \chi_{\text{mm}}(\prod_{i \in I} g_i) =$$

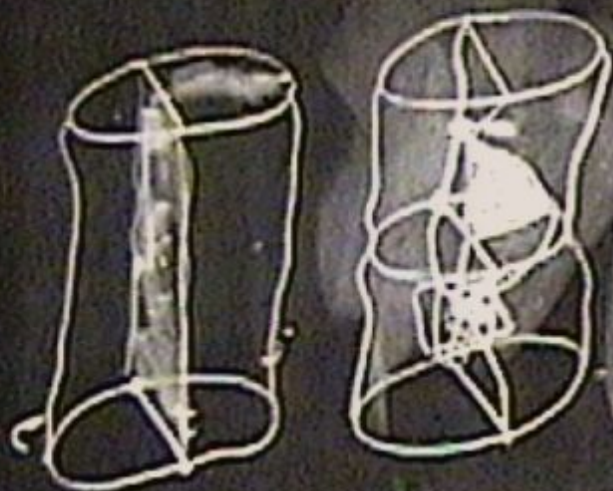
$$= D_{\text{mx}}^{i \in I}(g_i) D_{\text{ke}}^{i \in I}(g_i) \dots D^{i \in I}$$



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(t) \dots$$

$$\chi(\prod g_i) = \chi(\prod_{mm} g_i) =$$

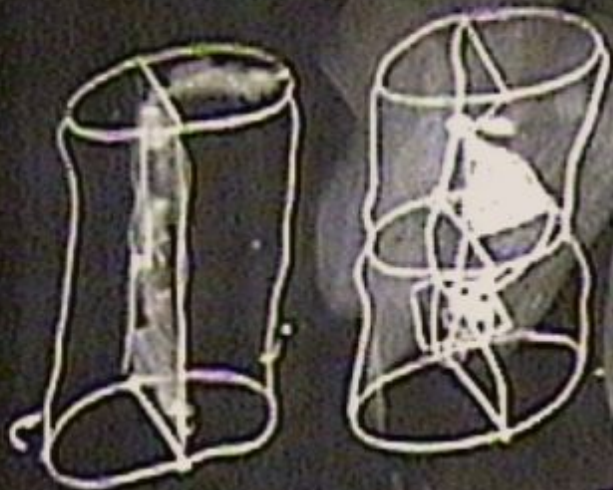
$$= D_{mk}^{j_k}(g_k) D_{ke}^{j_e}(g_e) \dots D_{nm}^{j_n}(g_n)$$



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(t) \dots$$

$$\chi(\prod g_i) = \prod_{mm} (\prod g_i) =$$

$$= D_{mx}^{j_F}(g_1) D_{xe}^{j_F}(g_2) \dots D_{nm}^{j_F}(g_n)$$



48 (Barrett-Crane)

$$Z(\Delta) = \left[\prod_F \sum_{\sigma_F} (z_F^{\sigma_F}) \right] \left[\prod_{F \in \mathcal{S}(1)} dg_F \right] \prod_F \chi^{j_F}(\prod_i g_{iF})$$

4d (Barnett-Prone)

$$Z(\Delta) = \left[\prod_{F'} \sum_{\mathcal{G}_F} (2j_F) \right] \left[\prod_{F' \in S_{U(1)}} \int dg_F \right] \prod_{F'} \chi^{j_F}(\prod_{e \in F} g_e)$$

$$= \left[\prod_{F'} \sum_{\mathcal{G}_F} (2j_F) \right] \prod_{F' \in S_{U(1)}} \int dg_F D_{j_F}^{j_F}(g_F) D_{j_F}^{j_F}(g_F) |$$

4d (Barnett-Prange)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{f'} \sum_{\Delta_f} (z_{f'})^{j_{f'}} \right] \left[\prod_{f' \in S_{\Delta}(\Delta)} dg_{f'} \right] \prod_{f'} \chi^{j_{f'}}(\prod_{e'} g_{e'}) \\
 &= \left[\prod_{f'} (z_{f'})^{j_{f'}} \right] \prod_{f' \in S_{\Delta}(\Delta)} \left(dg_{f'} \prod_{k_1}^{j_{f'}} D_{k_1}^{j_{f'}}(g_{f'}) \dots D_{k_{j_{f'}}}^{j_{f'}}(g_{f'}) \right)
 \end{aligned}$$

4d (Barrett-Crane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{f \in F} \sum_{\sigma_f} (z_{f, \sigma_f}) \right] \left[\prod_{f \in F} \int_{S_{U(1)}} dg_f \right] \prod_f \chi^{j_f}(\prod_c g_c) \\
 &= \left[\prod_{f \in F} \sum_{\sigma_f} (z_{f, \sigma_f}) \right] \prod_{f \in F} \int_{S_{U(1)}} dg_f D_{k_{f1}}^{j_f}(g_f) \dots D_{k_{fs}}^{j_f}(g_f)
 \end{aligned}$$

4d (Barnett-Crane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_F \sum_{\vec{g}_F} (2j_F) \right] \left[\prod_{S_{\text{int}}} dg_{\vec{g}_c} \right] \prod_F \chi^{j_F}(\prod_c g_c) \\
 &= \left[\prod_F \sum_{\vec{g}_F} \right] \prod_{S_{\text{int}}} \left(dg_{\vec{g}_c} D_{\vec{g}_c}^{j_F}(\vec{g}_c) \dots D_{\vec{g}_c}^{j_F}(\vec{g}_c) \right)
 \end{aligned}$$



4d (Barnett-Prone)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{F'} \sum_{\mathcal{G}_F} (z_{F'}^{j_{F'}}) \right] \left[\prod_{F'} \int_{S_{U(F)}} dg_F \right] \prod_{F'} \chi^{j_F}(\prod_i g_{F,i}) \\
 &= \left[\prod_{F'} \sum_{\mathcal{G}_F} (z_{F'}^{j_{F'}}) \right] \prod_{F'} \int_{S_{U(F)}} dg_F D_{k_1}^{j_{F'}}(g_F) \dots D_{k_s}^{j_{F'}}(g_F)
 \end{aligned}$$

• 4d (Barnett-Prone)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{F'} \sum_{\mathfrak{g}_F} (2j_F + 1) \right] \left[\prod_{F' \in S_U(1)} dg_{F'} \right] \prod_{F'} \chi \left(\prod_{F'} g_{F'} \right) \\
 &= \left[\prod_{F'} \sum_{\mathfrak{g}_F} (2j_F + 1) \right] \left[\prod_{F' \in S_U(1)} dg_{F'} D_{k_1}^{j_F}(g_{F'}) \dots D_{k_s}^{j_F}(g_{F'}) \right]
 \end{aligned}$$

$\mathcal{Z} \times \mathcal{L}(0,1)$ F. Vowker, 92-94/05 08109

$$\mathcal{Z} = \left[\prod_{f^*} \sum_{\tilde{g}_{f^*}} (z \tilde{g}_{f^*}) \right] \prod_{e^*}$$

$$\prod_{e^*} \int d\tilde{g}_{e^*} \prod_{f^*} \delta(\tilde{g}_{e^*})$$

$$x \in \mathbb{R} \sim T^{-1}$$

$$= \prod_{e^*} \int d\tilde{g}_{e^*} \prod_{f^*} \tilde{g}_{e^*}$$

$\mathcal{Z} \propto \mathcal{L}(y)$ F. Vowker, 92-94/05 08109

$$\mathcal{Z} = \left[\prod_{f^*} \sum_{\vec{j}_{f^*}} (z_{\vec{j}_{f^*}}) \right] \prod_{e^*} \mathcal{L}_{x, x_2, x_3}^{j_1, j_2, j_3} \mathcal{L}_{u, v, p, q}^{j_1, j_2, j_3}$$

$$\begin{aligned} & \prod_{e^*} \int dg_{e^*} \prod_i \delta(Q_i) \\ & \in \mathbb{R} \sim T^{-1} = \prod_{e^*} \int dg_{e^*} \prod_{f^*} \delta\left(\prod_{(e,f)} g_{e^*}\right) \end{aligned}$$

$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{\ell^*} \mathcal{C}_{k,k,k}^{j_1,j_2,j_3} \mathcal{C}_{\ell,\ell,\ell}^{j_1,j_2,j_3} \mathcal{C}_{k,k,k}^{j_1,j_2,j_3} \in H^1 \otimes H^1 \otimes H^1$$

$$x \in \mathbb{R} \sim T^1 = \prod_{\ell \in \mathbb{R}} \delta(\prod_{\ell \in \mathbb{R}} g_{\ell})$$

$\mathcal{Z} \times \mathcal{L}(0,1)$ F. Vowker, 92-94/05 08109

$$\mathcal{Z} = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{i_1, i_2, i_3}^{j_1, j_2, j_3} C_{i_1, i_2, i_3}^{j_1, j_2, j_3} \in H^1 \otimes H^1 \otimes H^1$$

$\mathcal{Z}$

$\times R \sim T^1$

$$\prod_{e^*} \int dg_{e^*} \prod_{e^*} \delta(Q_{e^*})$$

$$= \prod_{e^*} \int dg_{e^*} \prod_{f^*} \delta(\prod_{e^*} g_{e^*})$$

$\mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-96/0508109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{k,k,k}^{j_1,j_2,j_3} C_{l,e,l}^{j_1,j_2,j_3}$$

$C_{k,k,k}^{j_1,j_2,j_3} \in H^1 \otimes H^1 \otimes H^1$



$$\langle j_1, m_1, j_2, m_2 | j_3, m_3 \rangle \prod_{e^*} \int dg_{e^*} \prod_l \delta(Q_l)$$

$$k \in \mathbb{R} \sim T^1$$

$$= \prod_{e^*} \int dg_{e^*} \prod_{f^*} \delta\left(\prod_{l \in f^*} g_l\right)$$

$\mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-96/0508109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{k_1 k_2 k_3}^{j_1 j_2 j_3} C_{l_1 l_2 l_3}^{j_1 j_2 j_3} C_{k_1 k_2 k_3}^{i_1 i_2 i_3} \in H^0(H^0 H^0)$$


 $\langle j_1 m_1, j_2 m_2 | j_3 m_3 \rangle \prod_{e^*} \int dg_{e^*} \prod_l \delta(\mathcal{Q}_l)$

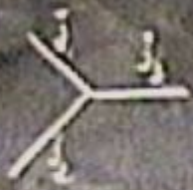
$x \in \mathbb{R} \sim T^1 = \prod_{e^*} \int dg_{e^*} \prod_{f^*} \delta(\prod_{e \in f^*} g_e)$

• 4d (Barrett-Peane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_F \sum_{\sigma_F} (2j_F + 1) \right] \left[\prod_{S \cup \{1\}} \int dg_c \right] \prod_F \chi \left(\prod_c g_c \right) \\
 &= \left[\prod_F \sum_{\sigma_F} (2j_F + 1) \right] \left[\prod_{S \cup \{1\}} \int dg_c \right] \prod_{k \in S} D_{k_1}^{j_k}(g_c) \dots D_{k_S}^{j_k}(g_c)
 \end{aligned}$$

$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{\nu, \kappa, \lambda}^{j_1, j_2, j_3} C_{\mu, \rho, \sigma}^{j_4, j_5, j_6} C_{\kappa, \lambda, \mu}^{j_7, j_8, j_9} \in H \otimes H \otimes H$$



$\langle j, m, j, m \rangle$

$x \in \mathbb{R} \sim T^1$

$$\prod_{e^*} \int dg_{e^*} \prod_{e^*} \delta(\mathcal{Q}_{e^*})$$

$$\prod_{e^*} \int dg_{e^*} \prod_{f^*} \delta(\prod_{e^*} g_{e^*})$$

$\mathbb{Z} \times \mathbb{Z}(0,1)$

F. Vowker, 92-94/0508109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{k_1 k_2 k_3}^{j_1 j_2 j_3} C_{l_1 l_2 l_3}^{j_1 j_2 j_3}$$

$C_{k_1 k_2 k_3}^{j_1 j_2 j_3} \in H^1 \otimes H^1 \otimes H^1$



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_f^*} (z_{j_f^*}^{f^*}) \right] \prod_{\ell} C_{\ell, k, k_3}^{j_1, j_2, j_3} C_{\ell, \ell, \ell}^{j_1, j_2, j_3} C_{k, k_3, k_3}^{j_1, j_2, j_3} \in H^0(H^0 H^0 H^0)$$



$\mathbb{Z} \times \mathbb{Z}_{(1)}$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} c_{k,k,k}^{j_1,j_2,j_3} c_{l,l,l}^{j_1,j_2,j_3}$$

$c_{k,k,k}^{j_1,j_2,j_3} \in H^0 H^0 H^0$



$\mathbb{C} \times (0,1)$

F. Vowker, 92-94/05 08109

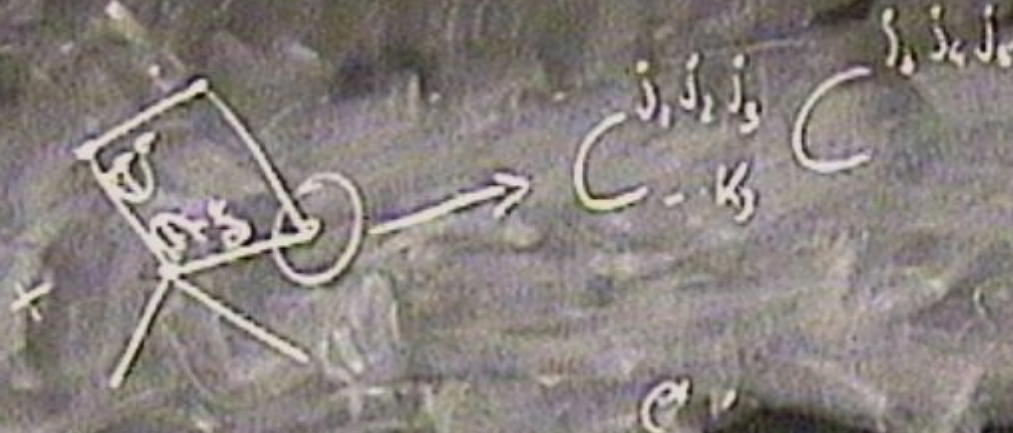
$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{k_1, k_2, k_3}^{j_1, j_2, j_3} C_{l_1, l_2, l_3}^{j_1, j_2, j_3}$$

$C_{k_1, k_2, k_3}^{j_1, j_2, j_3} \in H^0 H^0 H^0$



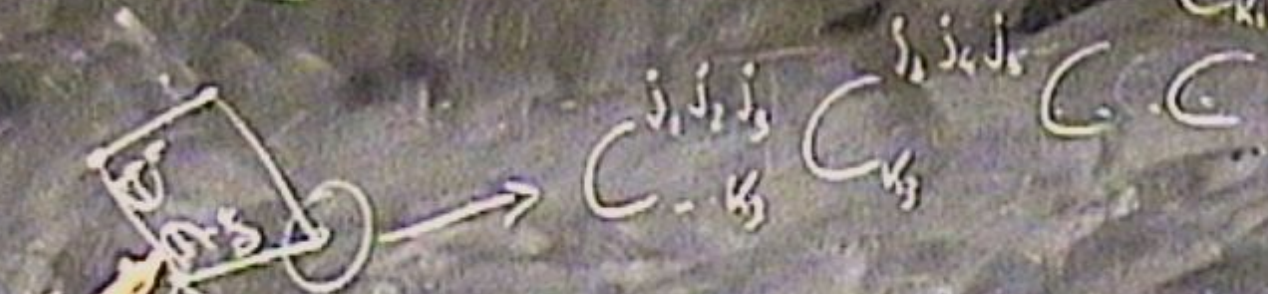
$\in \times (0,1)$ F. Vowker, 92.90/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{k,k,k}^{j_1, j_2, j_3} C_{l,l,l}^{j_1, j_2, j_3} C_{k,k,k}^{j_1, j_2, j_3} \in H \otimes H \otimes H$$



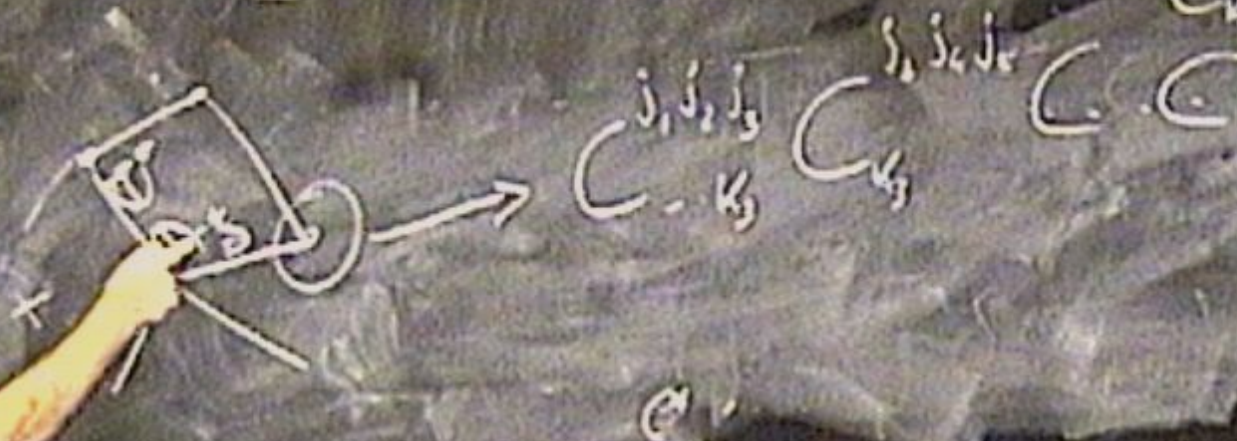
$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ T. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{k_1, k_2, k_3}^{j_1, j_2, j_3} C_{l_1, l_2, l_3}^{j_1, j_2, j_3} C_{k_1, k_2, k_3}^{i_1, i_2, i_3} \in H^0 H^0 H^0$$



$\mathcal{Z} \times [0,1]$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{p^*} C_{k_1 k_2 k_3}^{j_1 j_2 j_3} C_{l_1 l_2 l_3}^{j_1 j_2 j_3} C_{k_1 k_2 k_3}^{i_1 i_2 i_3} \in H^0 H^0 H^0$$



$\mathbb{Z} \times \mathbb{Z}(0,1)$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{\vec{j}_{f^*}} (z_{\vec{j}_{f^*}}) \right] \prod_{\vec{e}} C_{k_1, k_2, k_3}^{j_1, j_2, j_3} C_{l_1, l_2, l_3}^{i_1, i_2, i_3}$$

$C_{k_1, k_2, k_3}^{i_1, j_1, j_2} \in H^0(H^0 H^0)$

$$C_{k_1, k_2, k_3}^{j_1, j_2, j_3} C_{k_3}^{j_3, j_4, j_5} C_{k_2}^{j_2, j_6, j_7} C_{k_1}^{j_1, j_8, j_9}$$



$Z \times [0,1]$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{v, x, y}^{j_1, j_2, j_3} C_{u, e, e_3}^{j_1, j_2, j_3}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{v \downarrow} \text{[Diagram]}$$

The diagram shows a transformation from a graph on the left to a graph on the right. The left graph has a central node with several edges, some of which are labeled with x and y . An arrow points from this graph to the right graph. The right graph is a diamond-shaped graph with a central node labeled j_{f^*} and four nodes at the corners. The edges are labeled with x and y . Below the right graph is a small circle containing the letter C .



$Z \times [0,1]$ F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{u,v,w}^{j_1, j_2, j_3} C_{u,v,w}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{\downarrow} \{ \begin{matrix} j_1, j_2, j_3 \\ \downarrow \end{matrix} \}$$



$\mathcal{Z} \times [0,1]$ F. Vowker, 92-94/05 08109

$$\mathcal{Z} = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{e^*} C_{v, x, y}^{j_1, j_2, j_3} C_{u, e, e_3}^{j_1, j_2, j_3}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_v \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$\mathcal{Z} \times \mathcal{L}(0,1)$ F. Vowker, 92-94/05 08109

$$\mathcal{Z} = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{u,v,w}^{j_1, j_2, j_3} C_{u,v,w}^{j_1, j_2, j_3}$$

$$= \left[\prod_{j_{f^*}} \sum (2j_{f^*} + 1) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^+} \sum_{j_{f^+}} (2j_{f^+} + 1) \right] \prod_{e^+} C_{i_1, i_2, i_3}^{j_1, j_2, j_3} C_{i_4, i_5, i_6}^{j_4, j_5, j_6}$$

$$= \left[\prod_{f^+} \sum_{j_{f^+}} (2j_{f^+} + 1) \right] \prod_{v^+} \left\{ \begin{matrix} j_1, j_2, j_3 \\ j_4, j_5, j_6 \end{matrix} \right\}$$



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{F^*} \sum_{j_{F^*}} (2j_{F^*} + 1) \right] \prod_{C^*} C_{k_1, k_2, k_3}^{j_1, j_2, j_3} C_{l_1, l_2, l_3}^{j_1, j_2, j_3}$$



$$= \left[\prod_{F^*} \sum_{j_{F^*}} (2j_{F^*} + 1) \right] \prod_{V^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



• 4d (Barrett-Peterson)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{\mathbf{f}} \sum_{\mathbf{g}_{\mathbf{f}}} (2j_{\mathbf{f}} + 1) \right] \left[\prod_{\mathbf{f}} \int d\mathbf{g}_{\mathbf{f}} \right] \prod_{\mathbf{f}} \chi(\prod_{\mathbf{f}} \mathbf{g}_{\mathbf{f}}) \\
 &= \left[\prod_{\mathbf{f}} \sum_{\mathbf{g}_{\mathbf{f}}} (2j_{\mathbf{f}} + 1) \right] \prod_{\mathbf{f}} \left(\int d\mathbf{g}_{\mathbf{f}} D_{\mathbf{f}}^{\mathbf{j}_{\mathbf{f}}}(\mathbf{g}_{\mathbf{f}}) \dots D_{\mathbf{f}}^{\mathbf{j}_{\mathbf{f}}}(\mathbf{g}_{\mathbf{f}}) \right)
 \end{aligned}$$

$z \in \{0,1\}$

F. Vowker, 92-94/05 08109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}+1}) \right] \prod_{\ell^*} C_{v, k, k_3}^{j_1, j_2, j_3} C_{u, \ell, \ell_3}^{j_1, j_2, j_3}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}+1}) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



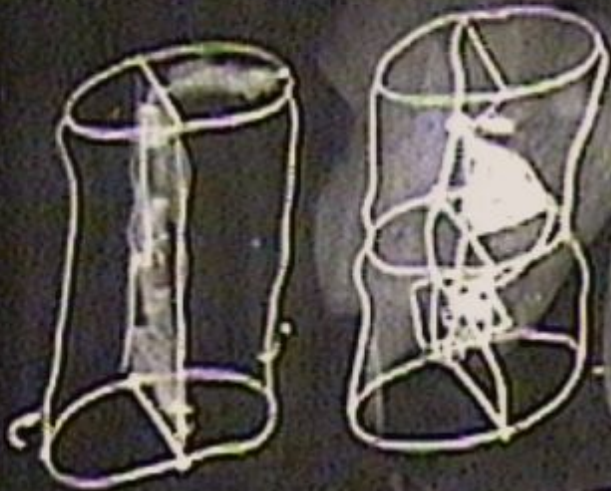
$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$Z = \int \mathcal{D}\phi$$



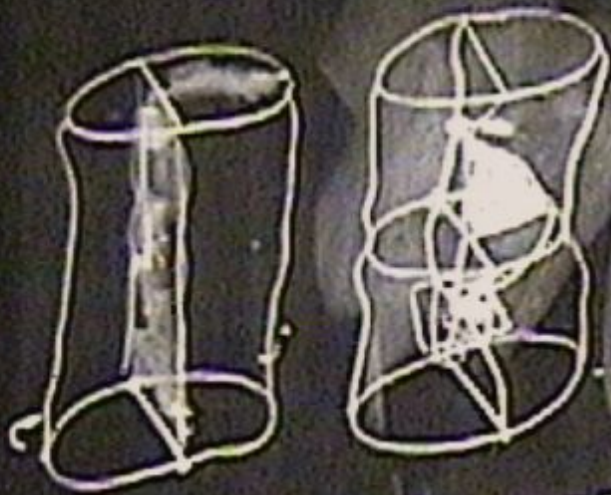
$$U_{N,0} \rightarrow U(T) \rightarrow U = \int d\tau U(t) \dots$$

$$Z_G = \sum_{\{F_0\}} \prod_f A(f_f) \prod_c A_c \prod_v A_v$$



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$Z_G = \sum_{j \in \mathcal{F}_0} \prod_f A(j_f) \prod_e A_e \prod_v A_v(j_{ev})$$



$$U_{N,0} \rightarrow U(\tau) \rightarrow U = \int d\tau U(t) \dots$$

$$Z = \sum_{\{f_0\}} \prod_f A(f_f) \prod_l A_l \prod_v A_v(j_{ev})$$



$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

F. Vowker, 92-94/0508109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{j_1, j_2, j_3}^{j_4, j_5, j_6} C_{j_1, j_2, j_3}^{j_4, j_5, j_6}$$

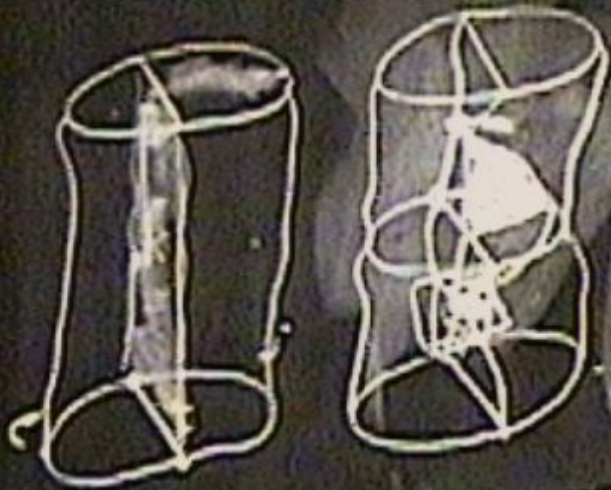


$$= \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$Z_G = \sum_{\mathcal{S}_{Fe}} \prod_f A(\mathcal{S}_f) \prod_l A_l \prod_v A_v(\mathcal{J}_{Fev})$$



$Z \times \mathbb{C}^n$ F. Vowker, 92-94/0508109

$$Z = \left[\prod_{f \in \mathcal{F}} \sum_{j_f \in \mathcal{J}_f} (2j_f + 1) \right] \prod_{e \in \mathcal{E}} \begin{matrix} j_1, j_2, j_3 \\ e, k_1, k_2 \end{matrix} \begin{matrix} j_1, j_2, j_3 \\ e, l_1, l_2 \end{matrix}$$

$$= \left[\prod_{f \in \mathcal{F}} \sum_{j_f \in \mathcal{J}_f} (2j_f + 1) \right] \prod_{e \in \mathcal{E}} \left\{ \begin{matrix} j_1, j_2, j_3 \\ j_4, j_5, j_6 \end{matrix} \right\}$$



$z \in [0,1]$

F. Vowker, 92-94/0508109

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{\ell} C_{u,v,w}^{j_1,j_2,j_3} C_{u,v,w}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (z_{j_{f^*}}) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$\mathbb{Z} \times \mathbb{Z}(0,1)$ F. Vowker, 92-94/0508109

$$Z = \left[\prod_{f^x} \sum_{j_{f^x}} (2j_{f^x} + 1) \right] \prod_{e^x} C_{v, k_1, k_2}^{j_1, j_2, j_3} C_{u, l_1, l_2}^{j_1, j_2, j_3}$$

$$= \left[\prod_{f^x} \sum_{j_{f^x}} (2j_{f^x} + 1) \right] \prod_{v^x} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



M h_1 h_2 $I \rightarrow F$ 1×2 $i S_H(g)$ $i S_R(\Delta)$ $\Sigma \times [0,1]$ $F. Dowker, 92-94/0508109$

$$Z = \int_{\mathcal{M}/h_1, h_2} \left(\int_{\Sigma} \frac{1}{\sqrt{g}} e^{i S_H(g)} \right) \left(\int_{\Sigma} \frac{1}{\sqrt{g}} e^{i S_R(\Delta)} \right) d\mu$$

$\Sigma \times [0,1]$ F. Dowker, 92/94/0508109 $C^{r(\Delta)}$

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{k_1, k_2, k_3}^{h_1, h_2, h_3} C_{l_1, l_2, l_3}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



• 4d (Barrett-Crane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{F'} \sum_{S_F} (2j_F + 1) \right] \left[\prod_{F'} \int dg_F \right] \prod_{F'} \chi^{j_F} \left(\prod_{e \in F} g_e \right) \\
 &= \left[\prod_{F'} \sum_{S_F} (2j_F + 1) \right] \prod_{F'} \left(\int dg_F D_{k_1}^{j_F}(g_F) \dots D_{k_3}^{j_F}(g_F) \right)
 \end{aligned}$$

$\Sigma \times [0,1]$ F. Dowker, 92-94/0508109 $C^{\infty}(D)$

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{j_1, j_2, j_3} C_{j_4, j_5, j_6}$$

$$= \left[\prod_{f^*} (2j_{f^*} + 1) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$$j \rightarrow j+1 \quad \hookrightarrow e = j(j+1)$$

$\Sigma \times [0,1]$ F. Dowker, 92-94/0508109 $\mathbb{C}^{R(\Delta)}$

$$Z = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} \mathbb{C}_{j_1, j_2, j_3}^{j_4, j_5, j_6}$$

$$\left[\prod_{j \in \tau} \sum (2j + 1) \right] \prod_{\nu^*} \left\{ \begin{matrix} j_1, j_2, j_3 \\ j_4, j_5, j_6 \end{matrix} \right\}$$

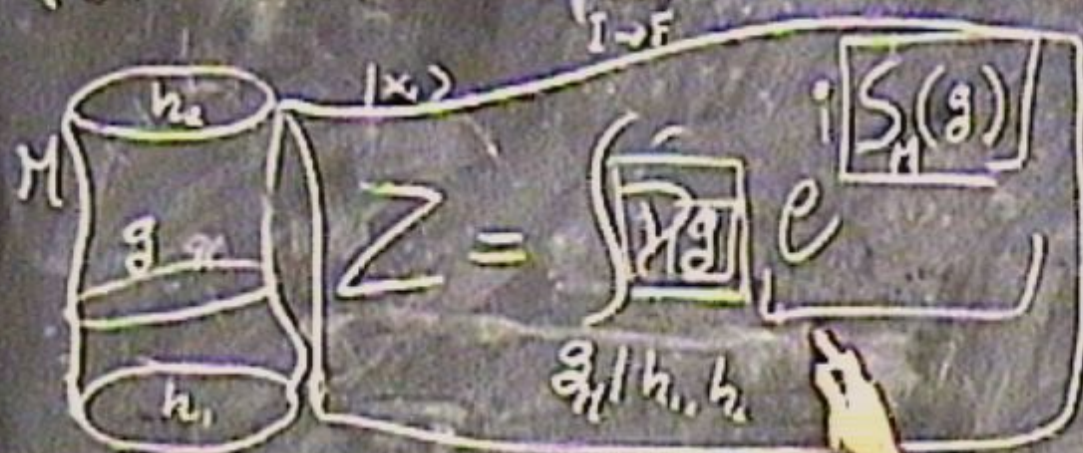
$$L_e = j(j+1)$$

$$\langle \psi | e^{iHt} | \psi \rangle = \sum_{\text{paths } I \rightarrow F} A(\text{path})$$

COMPLEX

$$S = \sum_{\text{paths}} V(\psi) \Delta t$$

$i S_R(\psi)$ शा-0



$$Z = \int_{\mathcal{M}_H} e^{i S_H(g)}$$

$g_H/h_1, h_2$

$$\int_{\mathcal{M}_H} e^{i S_H(g)}$$

$$\sum_C e^{i S_R(\Delta)}$$

$\Sigma \times [0, 1]$

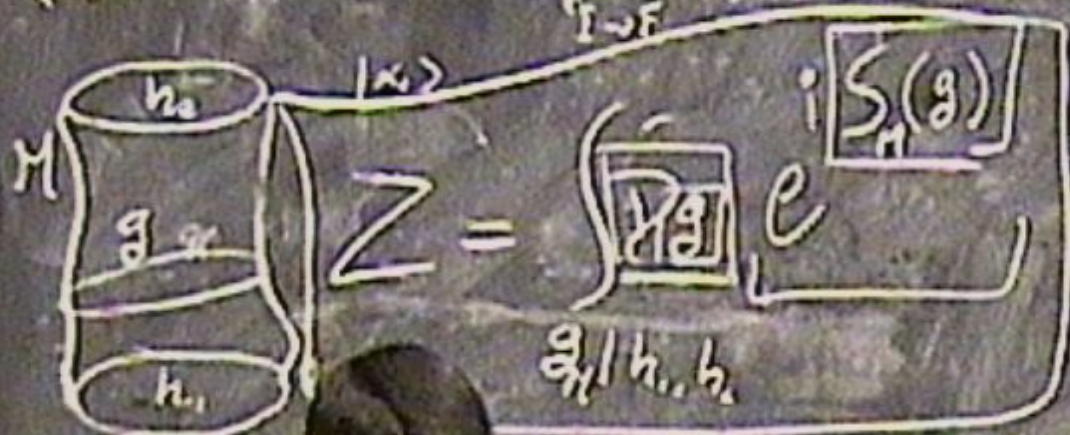
F. Dowker, 94/0508109

$$\langle \psi | e^{iHt} | \psi \rangle = \sum_{\text{paths } I \rightarrow F} A(\text{path})$$

D-DIMENSIONAL
COMPLEX

$$S = \sum_{\Delta} V(\Delta) \epsilon(\Delta)$$

$i S_R(\Delta)$ सा-0



$$Z = \int_{\mathbb{R}/h_1, h_2} \left[\int_{\mathbb{R}/h_1, h_2} e^{i S(g)} \right]$$

$$\int_{\mathbb{R}} d\phi e^{i S_R(\Delta)}$$

$$\sum e^{i S_R(\Delta)}$$

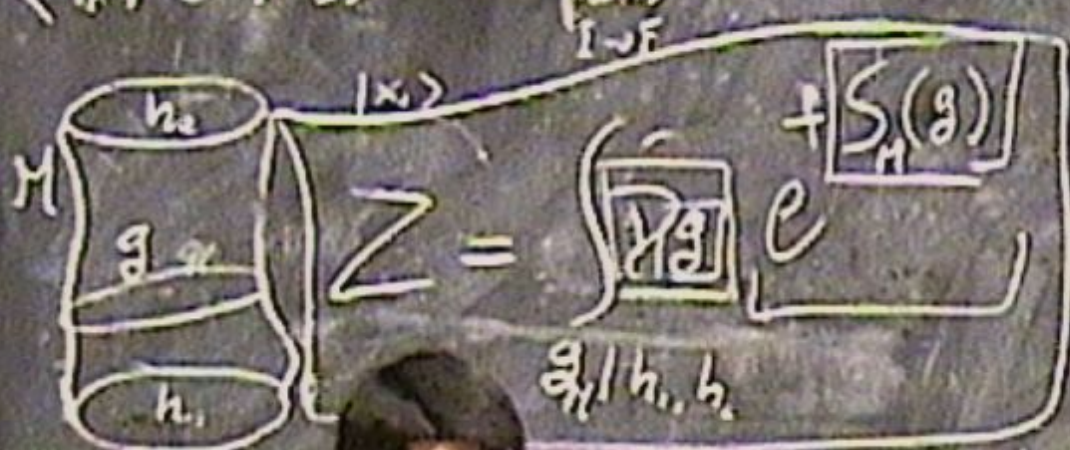
$\Sigma \times [0, 1]$ F. Dowker, 92-94/0508109

$$\langle \psi | e^{iHt} | \psi \rangle = \sum_{\text{paths } I \rightarrow F} A(I, F)$$

D-DIMENSIONAL
COMPLEX

$$S = \sum_{\Delta} V(\Delta) E(\Delta)$$

$i S_R(\Delta)$ सा-0



$$\int_{\mathbb{R}} e^{i S_R(\Delta)}$$

$$\sum e^{i S_R(\Delta)}$$

$$\Sigma \times [0, 1]$$

F. Dowker, 92-94/0508109

$$\langle \psi | e^{iHt} | \psi \rangle = \sum_{\text{paths } I \rightarrow F} A(\text{path})$$

D-DIMENSIONAL
COMPLEX

$$S = \sum_{\Delta} V(\Delta) E(\Delta)$$

$i S_R(\Delta)$ सा- ϕ



$$\int_{\Sigma} e^{i S_R(\Delta)}$$

$$\sum e^{i S_R(\Delta)}$$

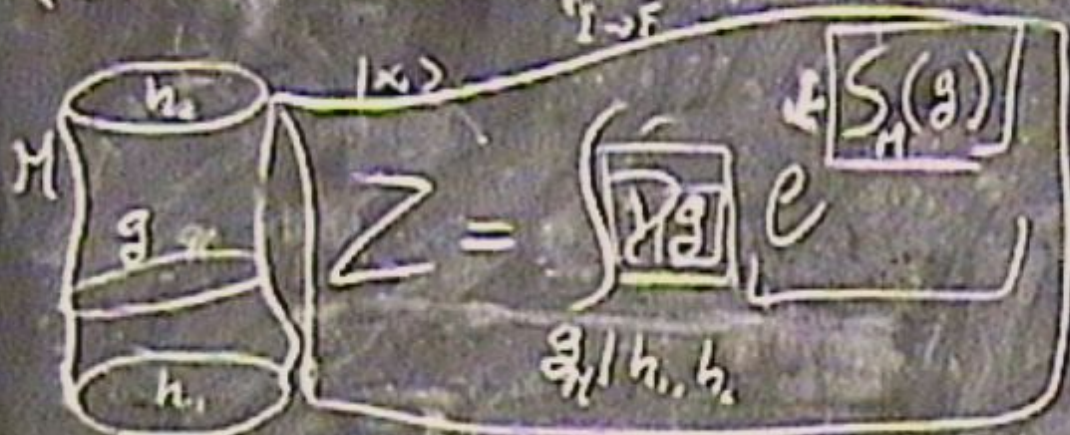
F. Bowker, 92-94/0508109

$$\langle \psi | e^{iHt} | \psi \rangle = \sum_{\text{paths } 1 \rightarrow F} A(\text{path})$$

D-DIMENSIONAL
COMPLEX

$$S = \sum_{\Delta} V(\Delta) E(\Delta)$$

$i S_R(\Delta)$ सा-०



$$\int e^{i S(g)}$$

$$\sum e^{i S_R(\Delta)}$$

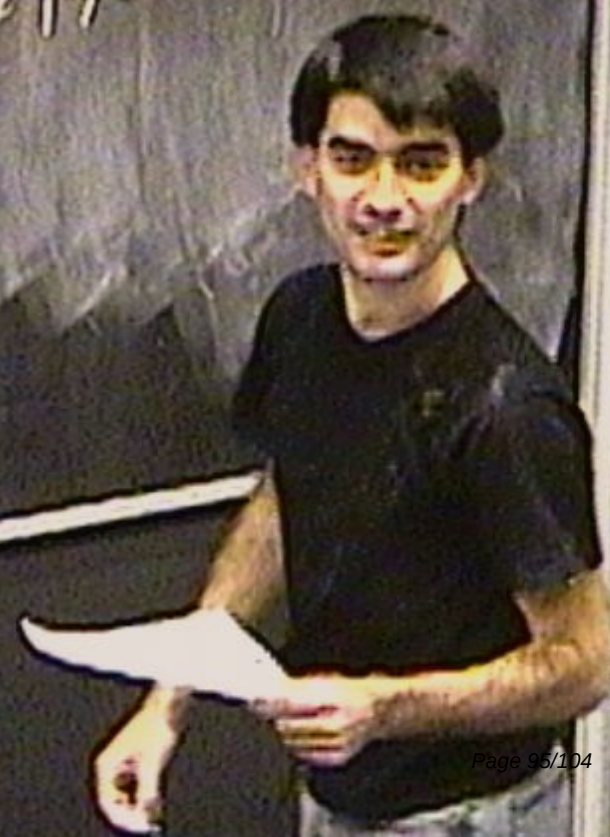
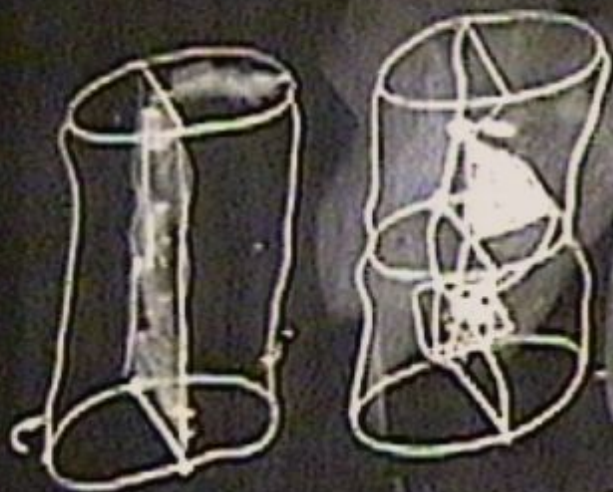
$\Sigma \times [0, 1]$

F. Dowker, १२-१८/०५०८१०९

$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$Z_G = \sum_{j \in \mathcal{F}_G} \prod_f A(j_f) \prod_e A_e \prod_v A_v(j_{fv})$$

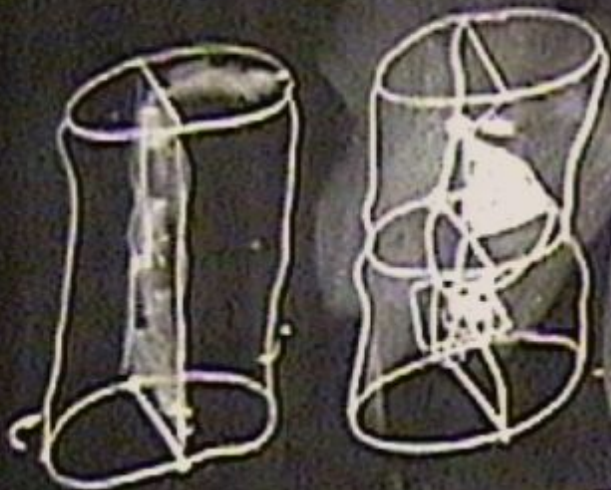
D. ORITI, gr-qc/0311066



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

$$Z = \sum_{\{F_0\}} \prod_F A(F) \prod_C A_C \prod_V A_V(\{F_{CV}\})$$

D. ORITI, 92-96/0311066
A. PEREZ, 92-96/0301113



$$U_{N,0} \rightarrow U(T) \rightarrow U = \int dT U(T) \dots$$

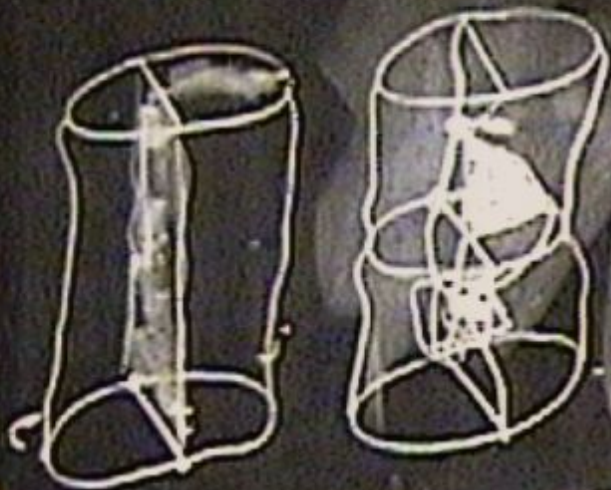
$$Z_G = \sum_{\{F_e\}} \prod_f A(f) \prod_e A_e \prod_v A_v(\{f_{ev}\})$$

D. ORITI, 92-9c/0311066

A. PEREZ, 92-9c/0301113

D. ORITI, 91-9c/0512103

...



• 4d (Barrett-Crane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{f \in F} \sum_{j_f} (2j_f + 1) \right] \left[\prod_{f \in F} \int dg_f \right] \prod_{f \in F} \chi_{j_f}(\prod_l g_{lf}) \\
 &= \left[\prod_{f \in F} \sum_{j_f} (2j_f + 1) \right] \prod_{f \in F} \int dg_f D_{j_f}^{j_f}(g_f) \prod_{l \in L} D_{j_l}^{j_l}(g_l) \left| \Delta \right|_{L_e}
 \end{aligned}$$

$\Sigma \times [0,1]$ F. Dowker, 92-94/0508109 $C^{1,2}_R(\Delta)$

$$Z = \left[\prod_{f \in F^*} \sum_{j_f} (2j_f + 1) \right] \prod_{e \in E} C_{j_e, j_e, j_e}$$

$$= \left[\prod_{f \in F^*} \sum_{j_f} (2j_f + 1) \right] \prod_{v \in V^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$\Sigma \times [0,1]$ $F. Dowker, 92-94/0508109$ $C^i \Sigma_R(\Delta)$

$$\begin{aligned}
 \angle &= \left[\begin{array}{c} \parallel \\ F^* \end{array} \right] \angle_{F^*}^{(2j_F+1)} \parallel \begin{array}{c} C_{K_1, K_2, K_3} \\ C_{L, P, Q} \end{array} \\
 &= \left[\prod_{F^*} \sum (2j_i+1) \right] \prod_{V^*} \left\{ \begin{array}{c} j_1, j_2, j_3 \\ j_4, j_5, j_6 \end{array} \right\}
 \end{aligned}$$



$$\ell_i = i(i+1)$$

$\Sigma \times [0,1]$ $\mathcal{A}/h, h_2$ $\sum c^i s_R(\Delta)$
 F. Dowker, 92-94/0508109

$$\angle = \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{e^*} C_{k_1, k_2, k_3} C_{l_1, l_2, l_3}$$

$$= \left[\prod_{f^*} \sum_{j_{f^*}} (2j_{f^*} + 1) \right] \prod_{v^*} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$



$$L_e = j(j+1)$$

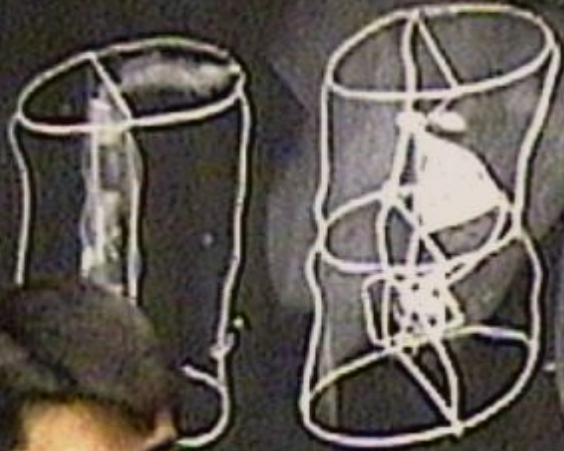
$$Z_0 = \sum_{j \in J_0} \prod_i A(i) \prod_{\ell} A_{\ell} \prod_{\nu} A_{\nu}(j_{\nu})$$

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A. PEREZ, 92-9c/0301113

D. ORITI, 91-9c/0512103

...



• 4d (Barrett-Peane)

$$\begin{aligned}
 Z(\Delta) &= \left[\prod_{F'} \sum_{\mathcal{G}_F} (2j_F + 1) \right] \left[\prod_{F' \in S_U(2)} \int dg_{F'} \right] \prod_{F'} \chi^{j_F}(\prod_{e \in F} g_e) \\
 &= \left[\prod_{F'} \sum_{\mathcal{G}_F} (2j_F + 1) \right] \left[\prod_{F' \in S_U(2)} \int dg_{F'} D_{k_1}^{j_F}(g_{F'}) \dots D_{k_n}^{j_F}(g_{F'}) \right] \triangle_{L_2}^{L_1}
 \end{aligned}$$

$\Sigma \times [0,1]$ F. Dowker, 92-94/0508109 $C^1 \gamma_R(\Delta)$

$$Z = \left[\prod_{f \in F} \sum_{j_f} (2j_f + 1) \right] \prod_{e \in E} C_{j_e, j_e, j_e}$$

$$= \left[\prod_{f \in F} \sum_{j_f} (2j_f + 1) \right] \prod_{v \in V} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}$$

