Title: Gap estimates in adiabatic transport

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Abstract:

Gap estimates in adiabatic transport

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Perimeter Institute, Waterloo
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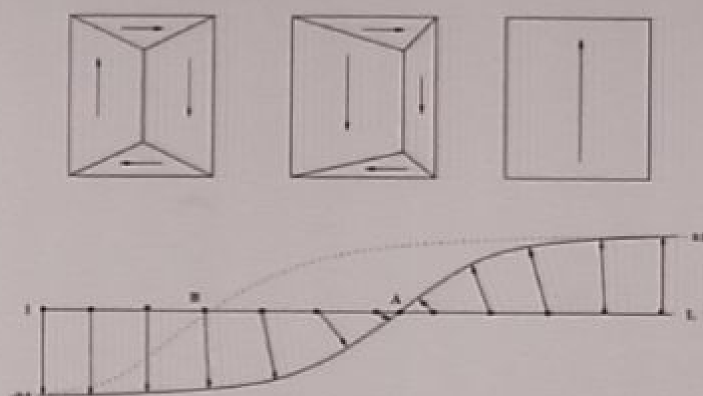
Motivation comes from the motion of magnetic domain walls (DW) under the influence of an exterior magnetic field.



- How can we describe this in a microscopic model?
- Understand the motion of a DW in terms of microscopic interaction, applied magnetic field, temperature, etc.

Starting with Alcaraz-Salinas-Wreszinski in 1995, it was realized that a ferromagnetic Heisenberg Hamiltonian, H, has DW as ground states (GS).

Now, start with such a GS, ψ_A, centered around lattice site A. Then add a transverse magnetic field, $V(t)$, which at $t = 0$ is localized near A, and at time T localized near B. State at time t is

$$\psi_A(t) = \mathbb{T}\{\exp[-\mathrm{i}\int_0^t (H + V(s))\,\mathrm{d}s]\}\psi_A .$$

- If we do this slowly (adiabatically), $\psi_A(T)$ will be close to ψ_B, which is DW centered at lattice site B.
- We want to apply the Adiabatic Theorem. In real applications, a DW may stretch over 100 atoms. So we need gap estimates uniform in the size of the whole system. Martingale method.
- There is no hope to compute $\psi_A(t)$ or adiabatic constants explicitly. In order to get a quantitative numerical picture we have implemented a time dependent DMRG algorithm (Vidal).

Background to Martingale Method (MM):

- In the context of "frustration free" quantum spin systems (QSS), MM was invented by Nachtergaele in 1995. He proved excellent gap estimates for the Heisenberg chain.
- In 2002, jointly with Starr we improved MM and obtained sharp gap estimates for Heisenberg model and good gap estimate for AKLT (*anti*-ferromagnetic spin-1 Heisenberg) model.
- Recently, we have used MM for Heisenberg in a magnetic field (non-translation invariant).
- MM is inductive. We have to know GS. Suppose, we know the gap for QSS on chain $[0, L-1]$ and how much the GS change(s) if we add another site. Then we get an estimate about the gap of QSS on chain $[0, L]$.

We consider a QSS on the finite chain, $[0, L-1]$. We cover this interval with connected intervals (subsystems), C_i, $i = 0, \ldots, N$ s.t.

- $\bigcup_{i=0}^{N} C_i = [0, L-1]$,
- two such intervals share at most 1 lattice point.

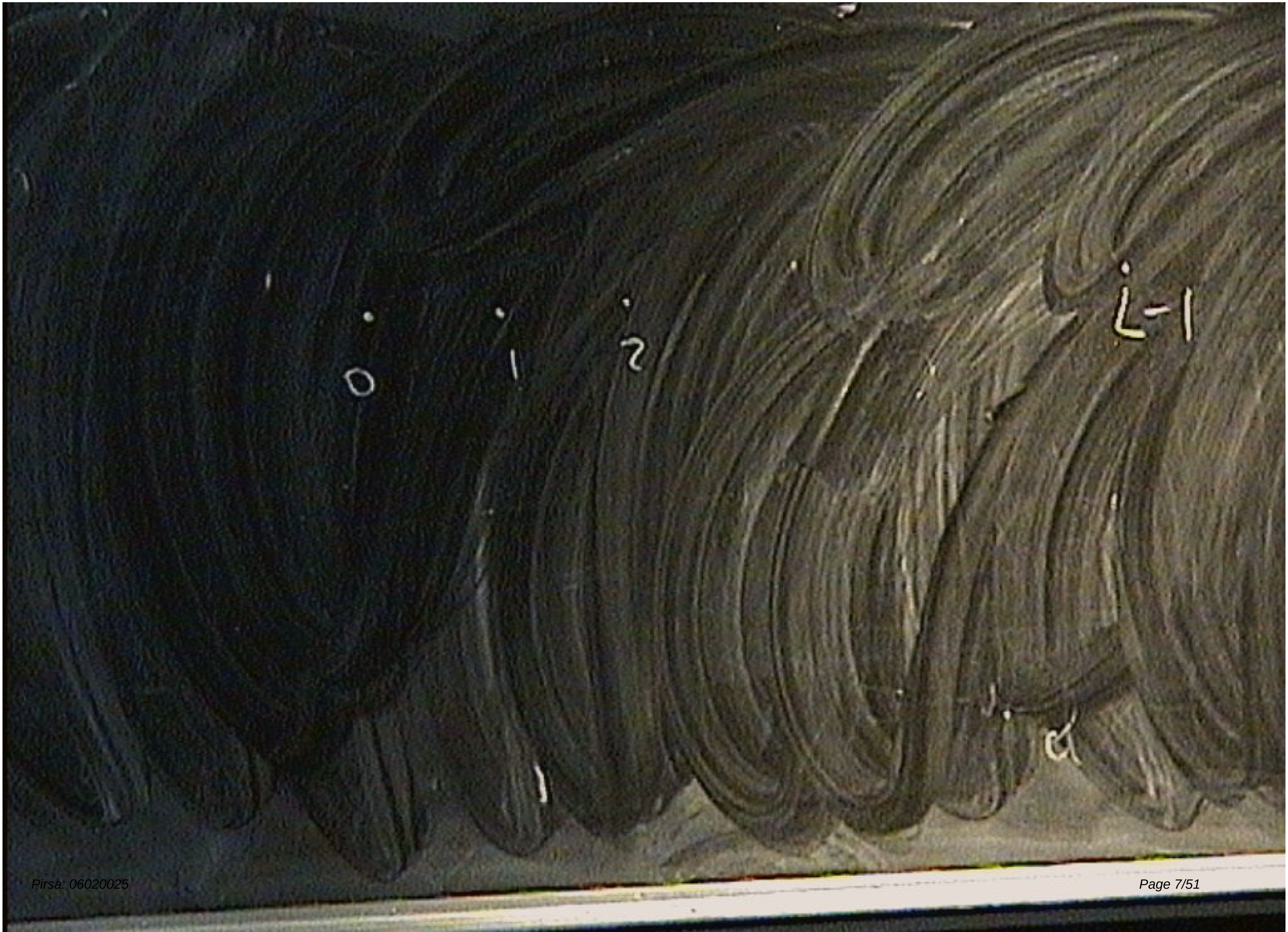
E.g.,

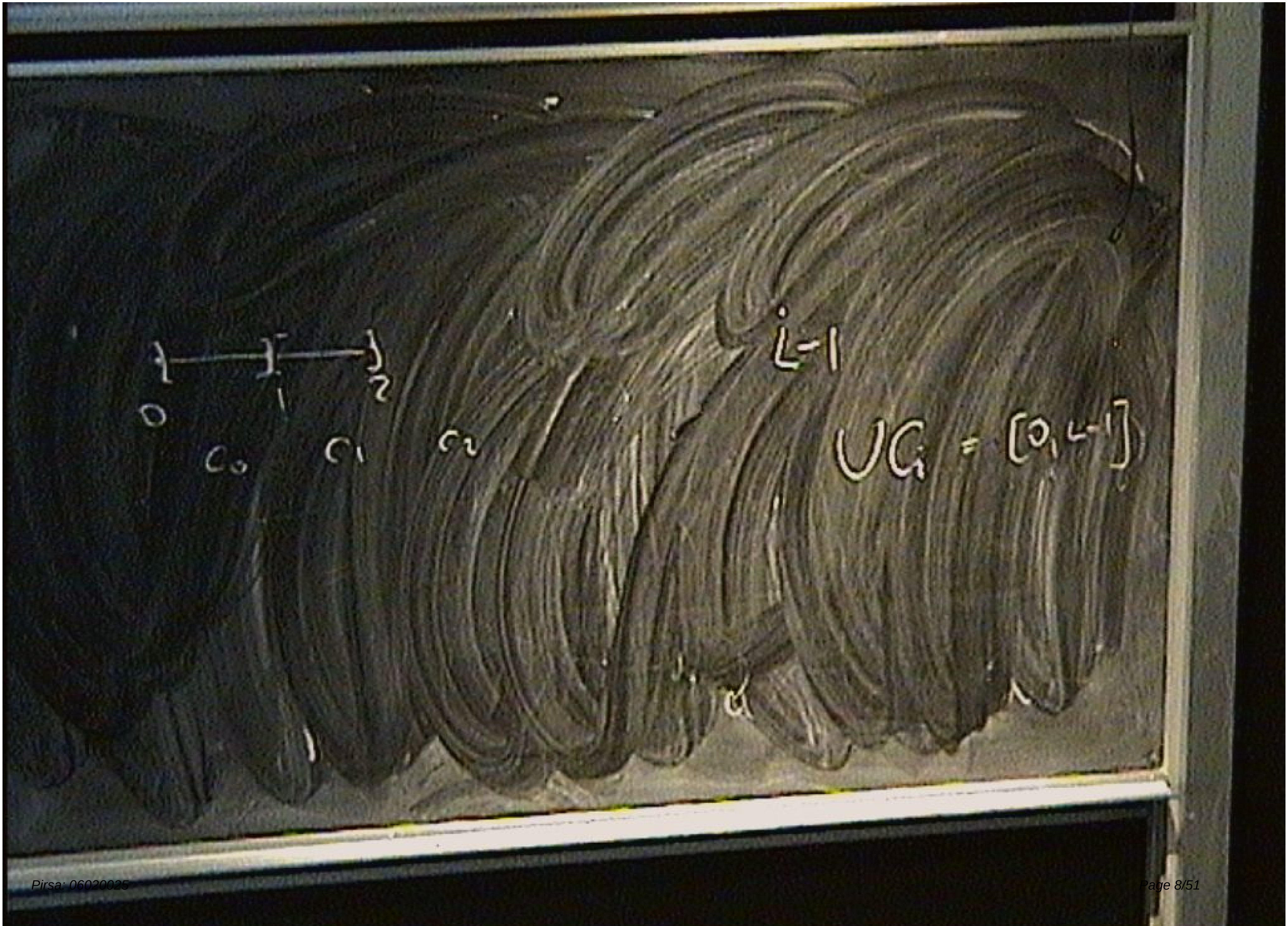
- $C_i = [i, i+1]$ for $i = 0, \ldots, L-2$.
- For $0 < x_0 < y_0 < L-1$ start with $C_0 = [x_0, y_0]$. Then define $C_1 = [x_0 - 1, x_0]$, $C_2 = [y_0, y_0 + 1]$, $C_3 = [x_0 - 2, x_0 - 1]$, etc.

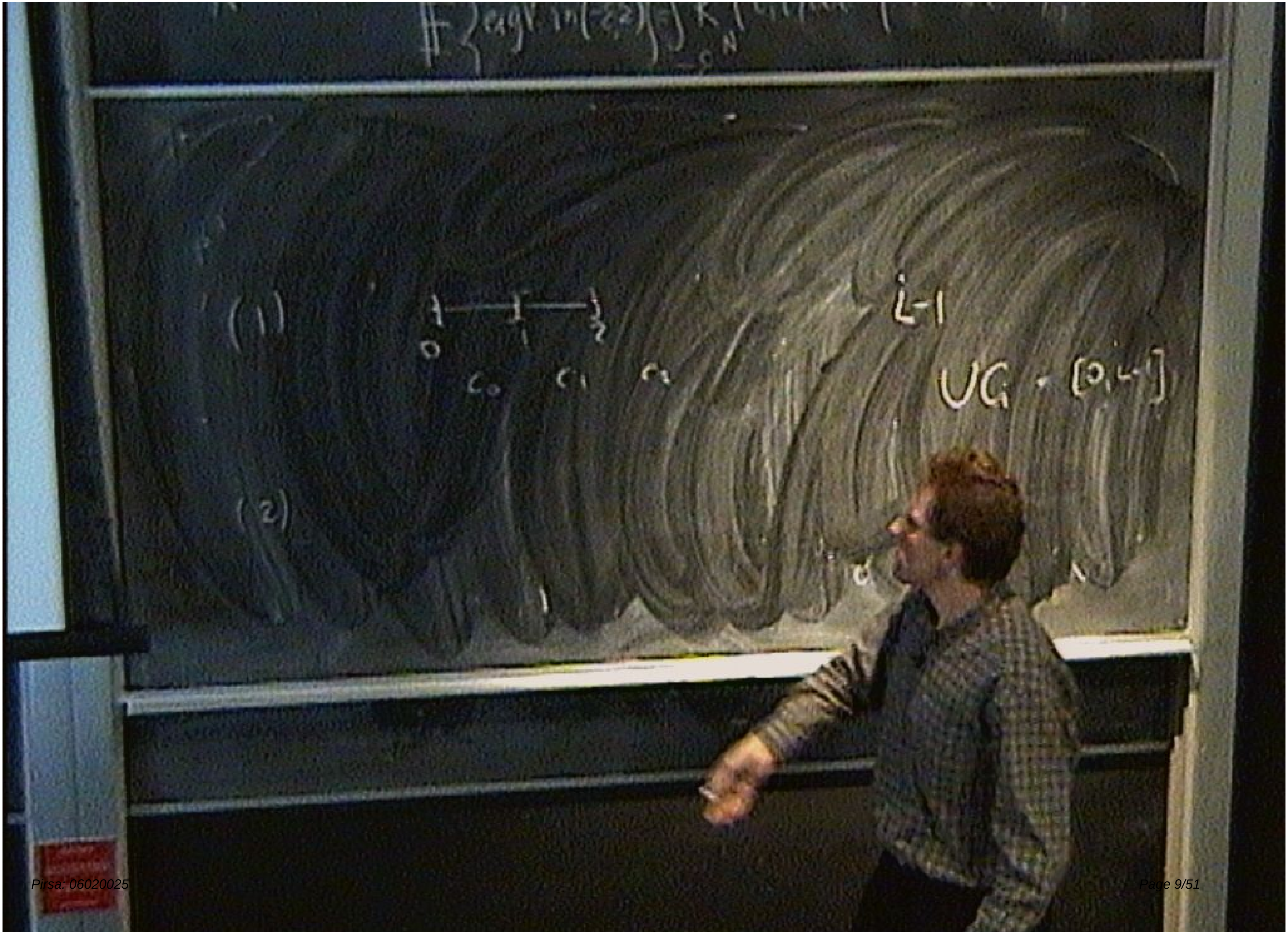
For every subsystem C_i we have a (local) Hamiltonian, h_{C_i}.

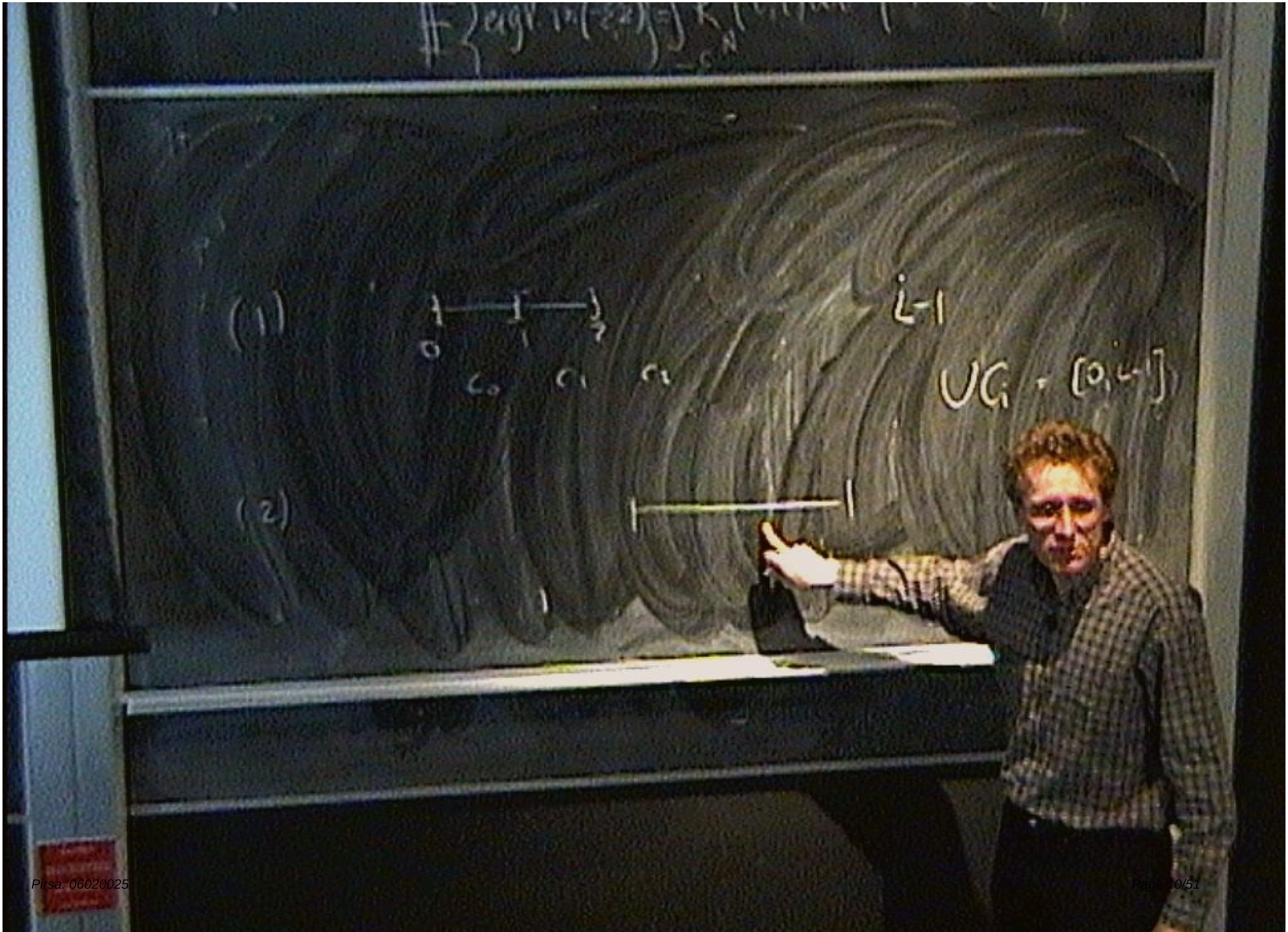
Total Hamiltonian of QSS is

$$H_L = \sum_{i=0}^{N} h_{C_i}.$$


0
1
2
i-1


0
1
2
i-1
c0
c1
c2
∪Ci = [0, L-1]


(1)
(2)


(1)
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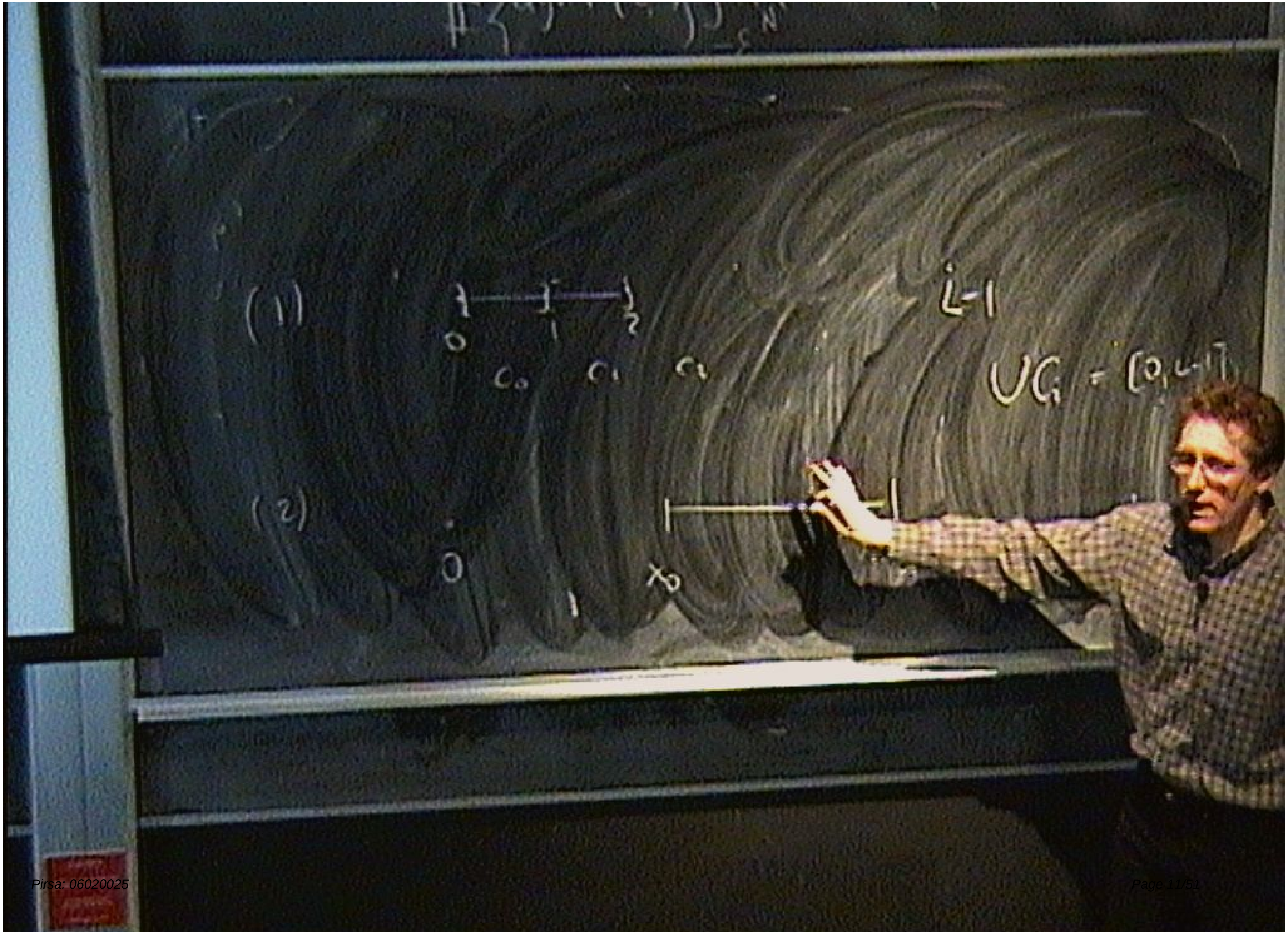
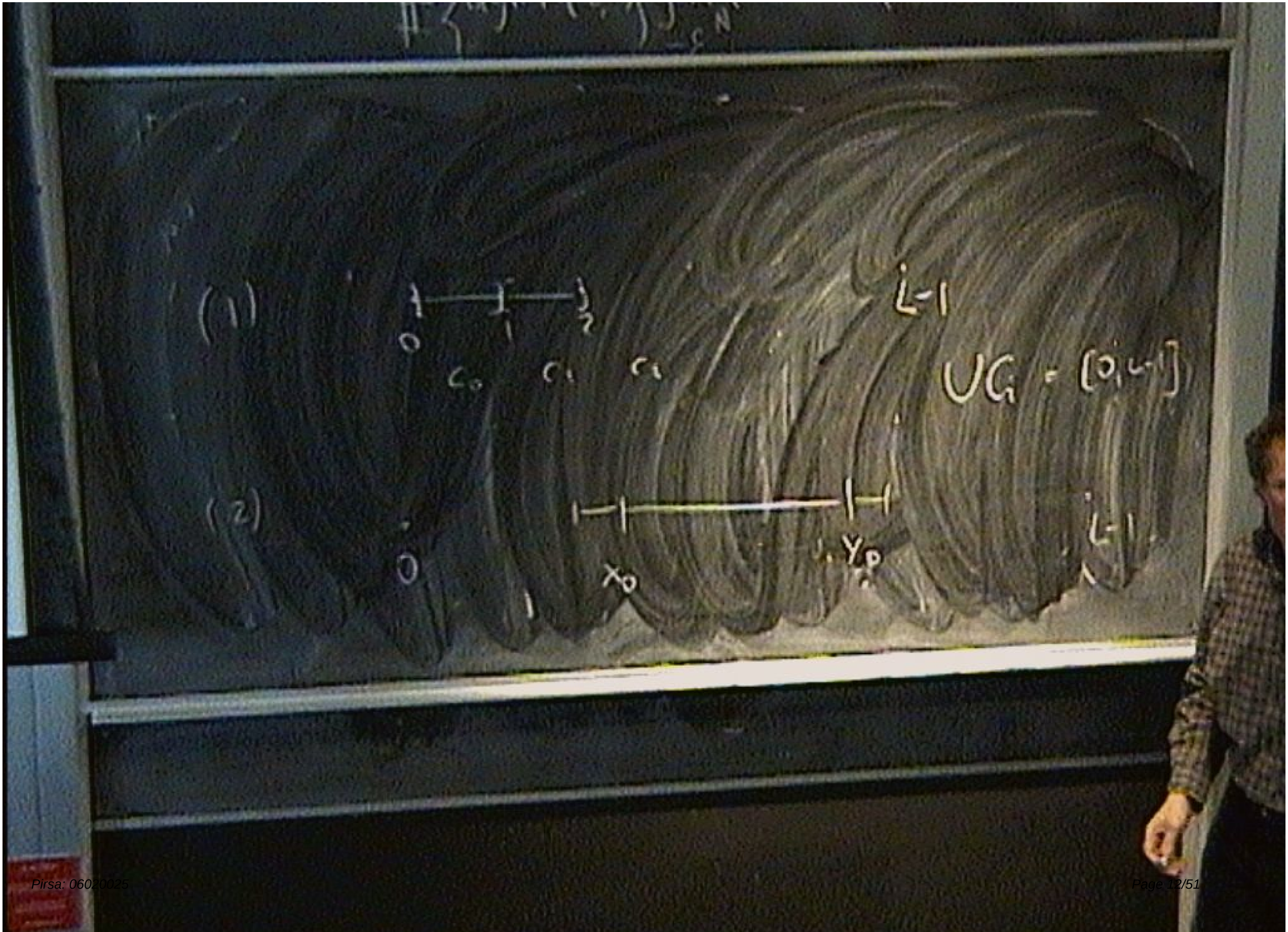
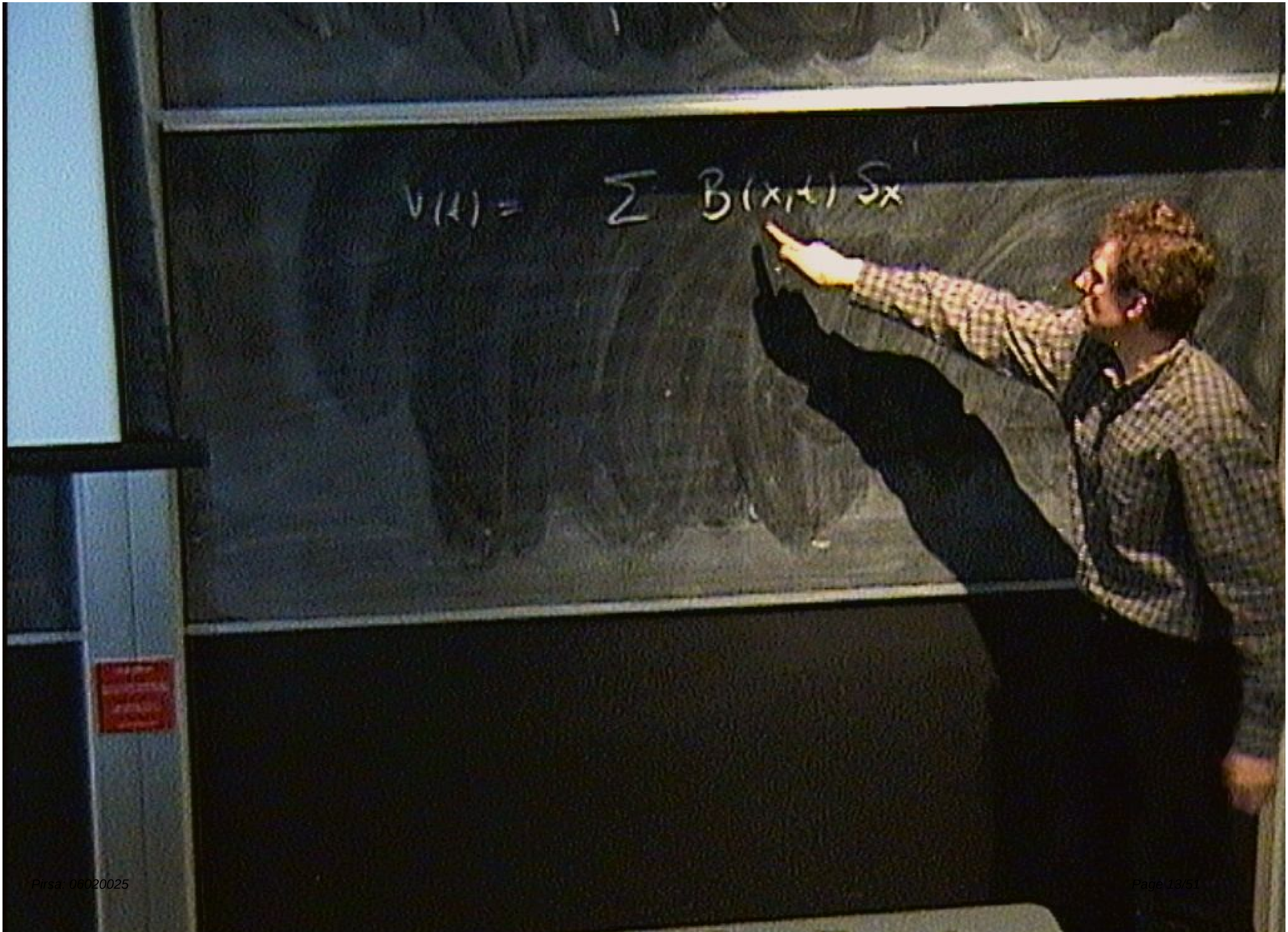
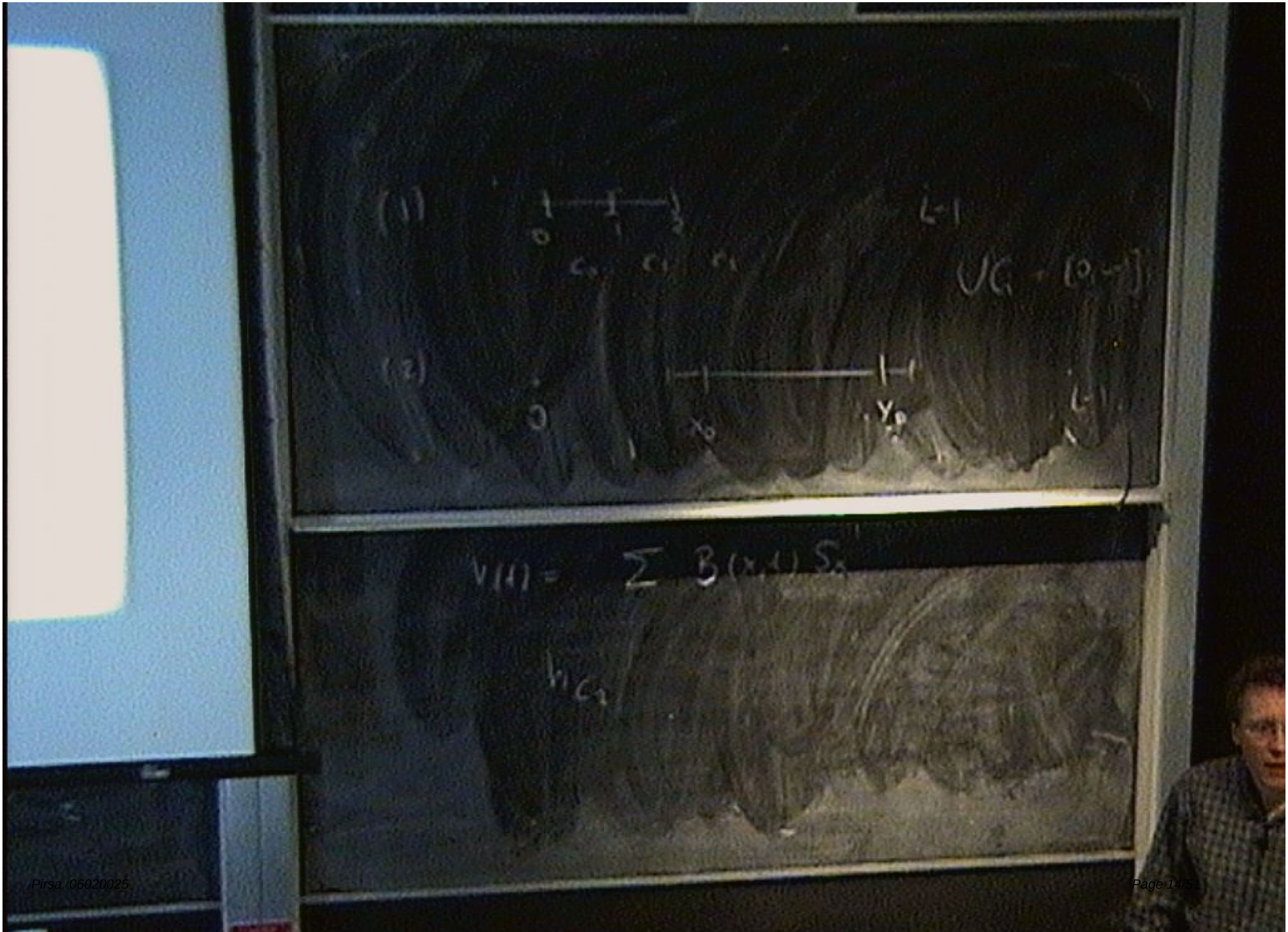
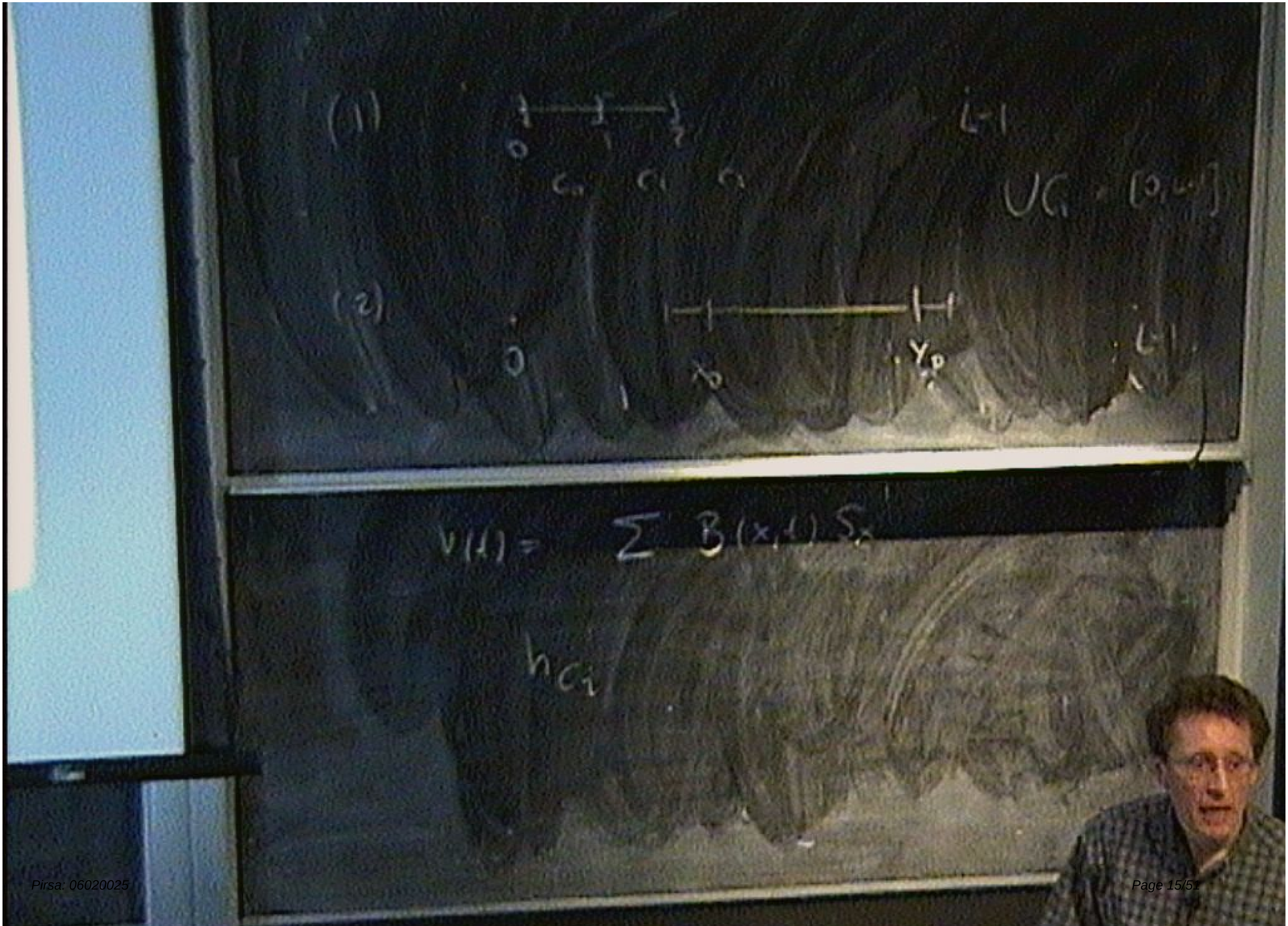
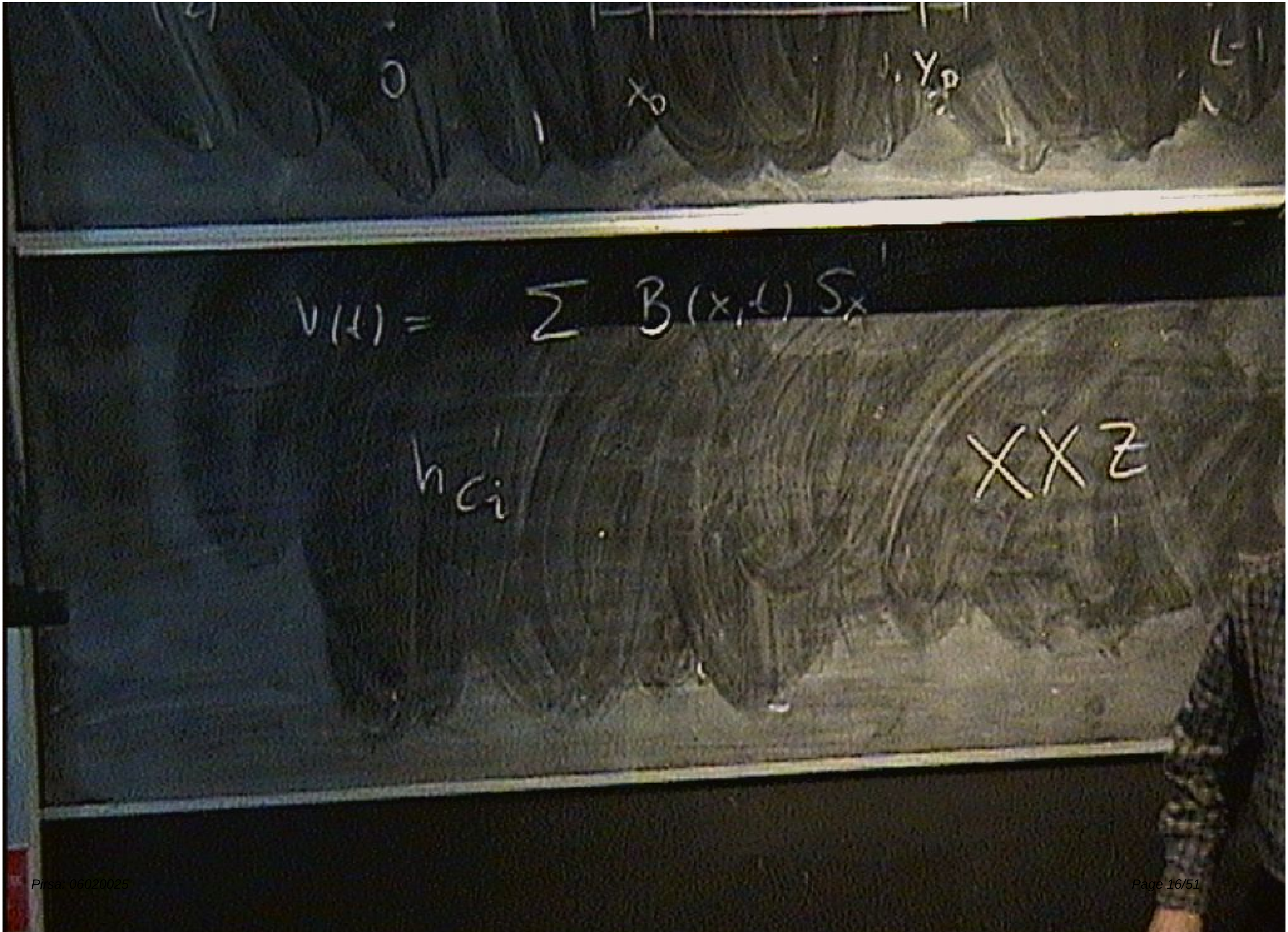

(1)
0
1
2
C_0 C_1 C_2
L-1
$\cup C_i = [0, L-1]$
(2)
0
x_0

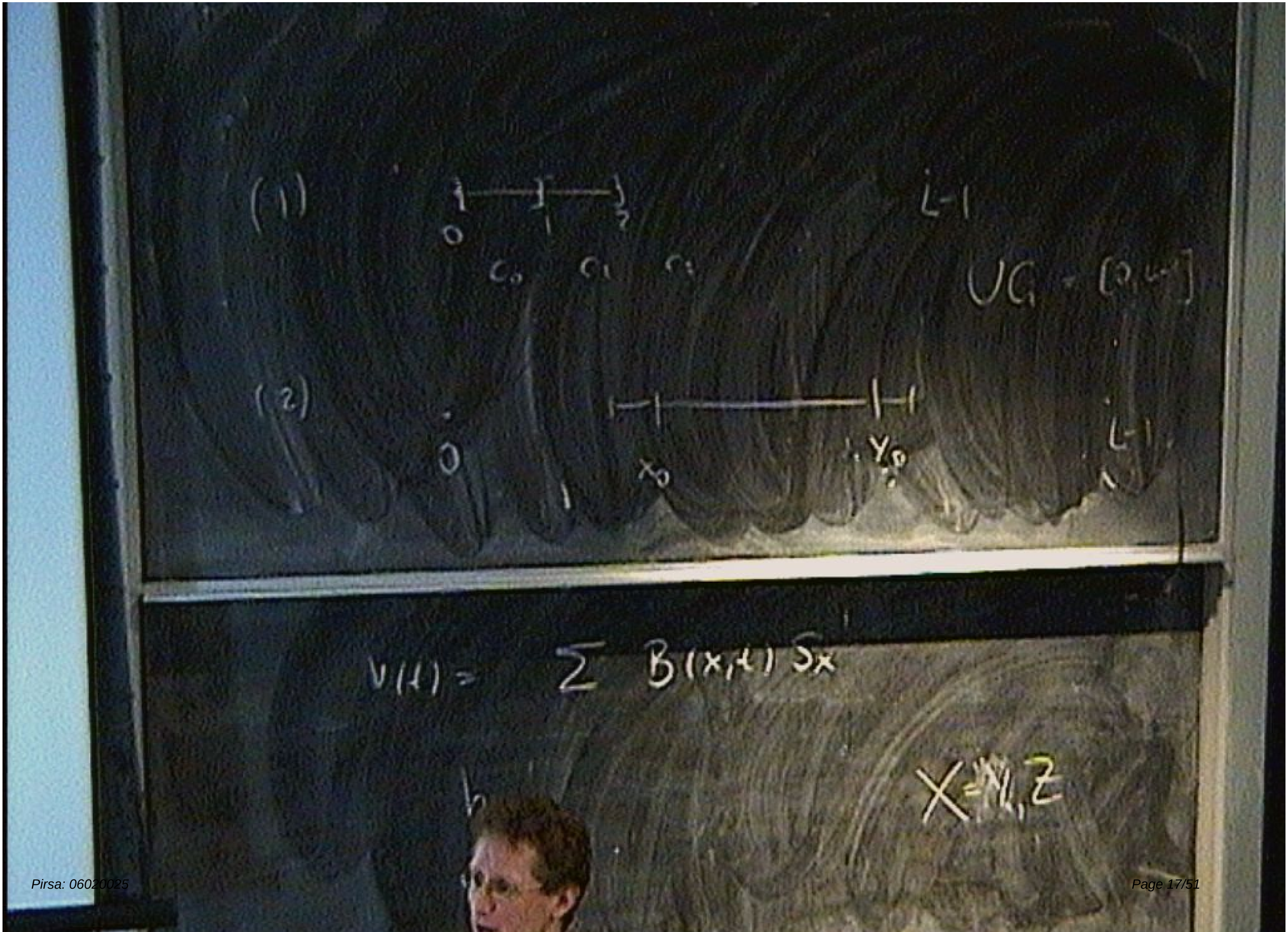

(1)
0 1 2
L-1
UG = [0, L-1]
(2)
0
L-1

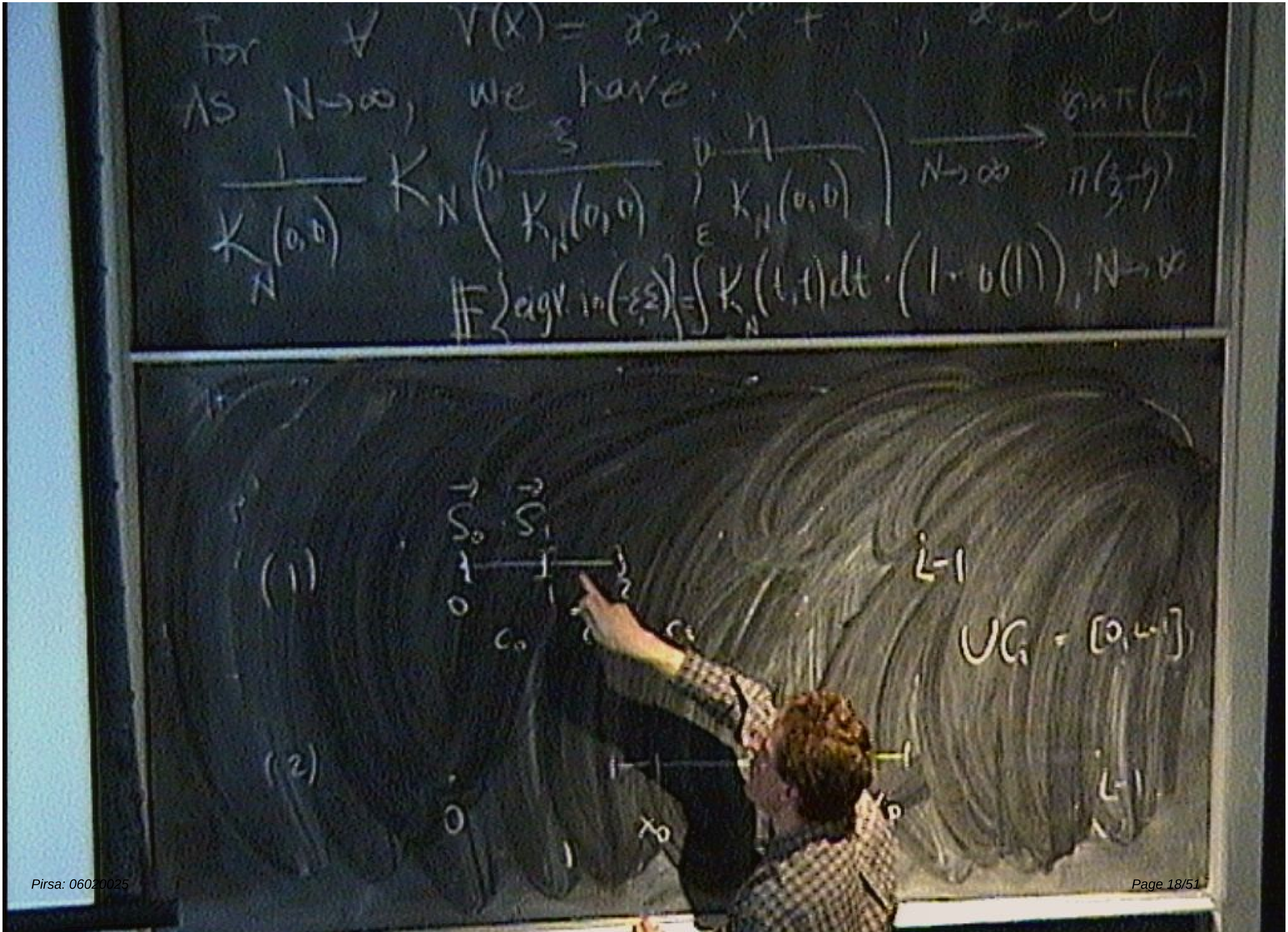

$V(t) = \sum B(x,t) S_x$

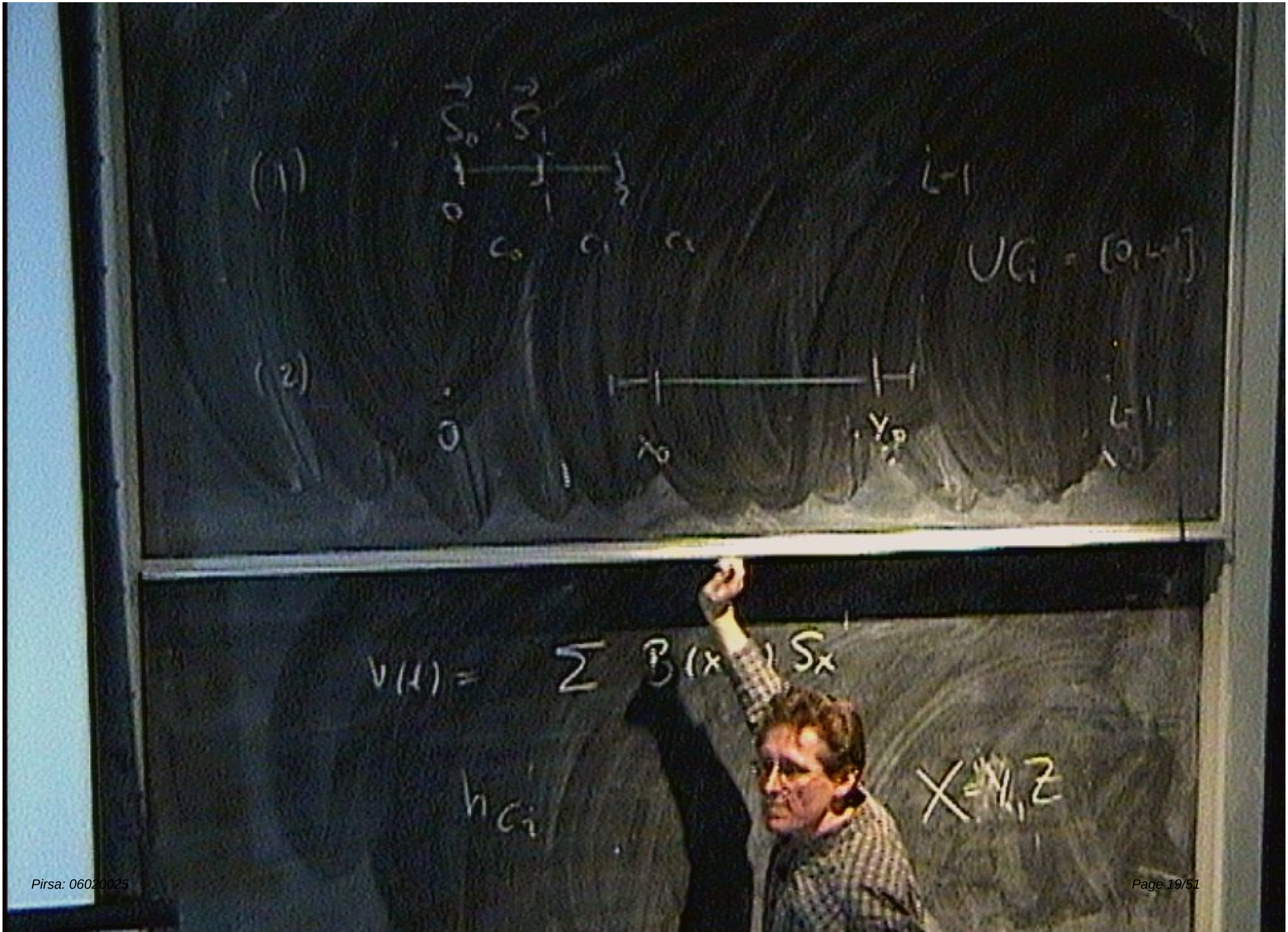

(1)
(2)
L-1

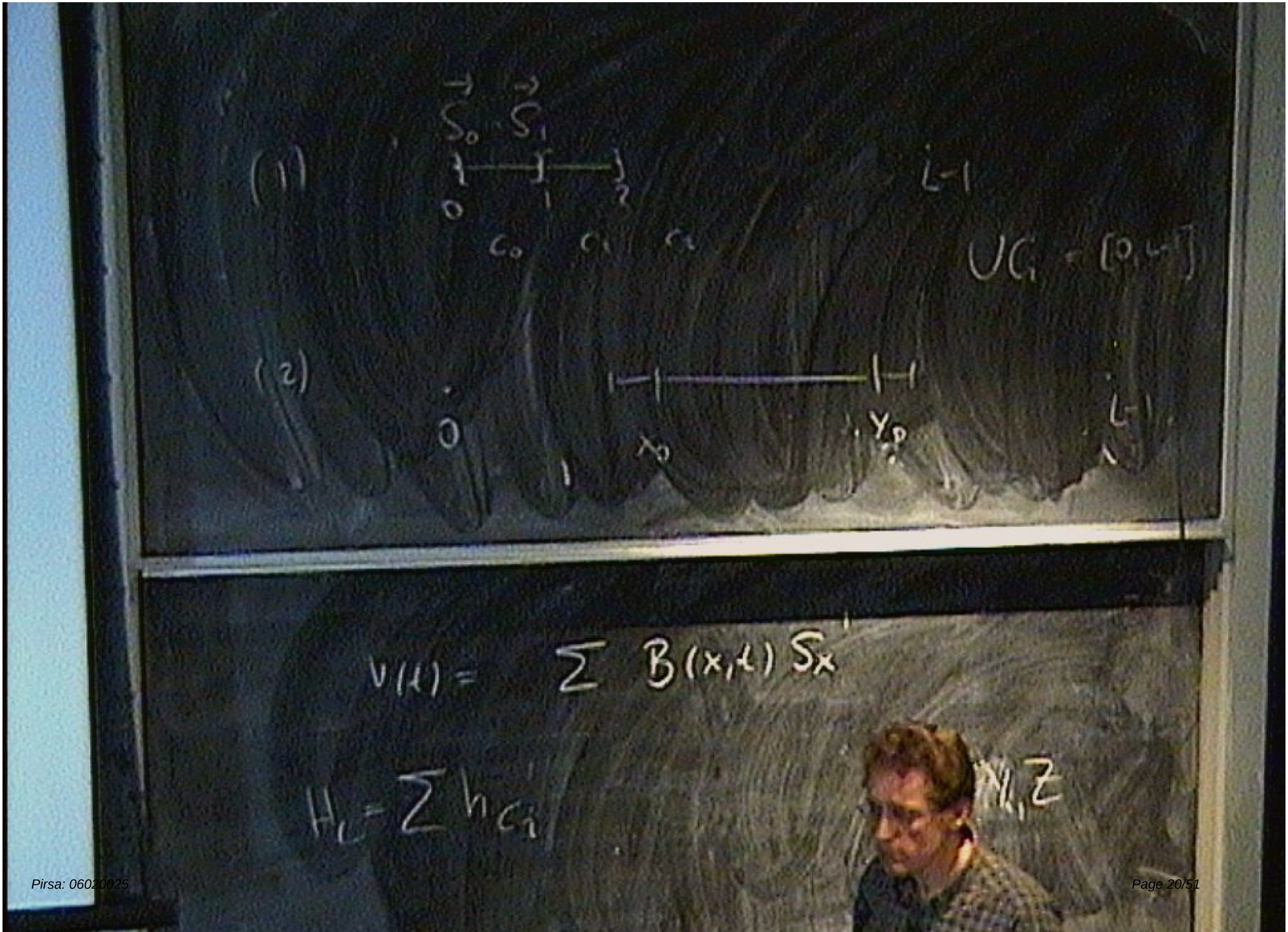

(1)
L-1
(2)


V(t) = Σ B(x,t) S_x
h_{c_i}
XXZ


(1)
0 1 2
L-1
UG
(2)
0
L-1
V(t) = Σ B(x,t) Sx


For ∀ V(x) =
As N→∞, we have
K_N(0,0)
N→∞
(1)
(2)
L-1
UC_i = [0, L-1]


(1)
(2)
UG = [0, L-1]
L-1
V(t) = Σ B(x, t) S_x
h_{C_i}
X, Y, Z


(1)
(2)
UG = [0, L-1]
V(t) = Σ B(x,t) S_x
H_L = Σ h_{c_i}

“Frustration free” assumptions on QSS,

(i) $h_{C_i} \geq 0$. Hence, $H_L \geq 0$.

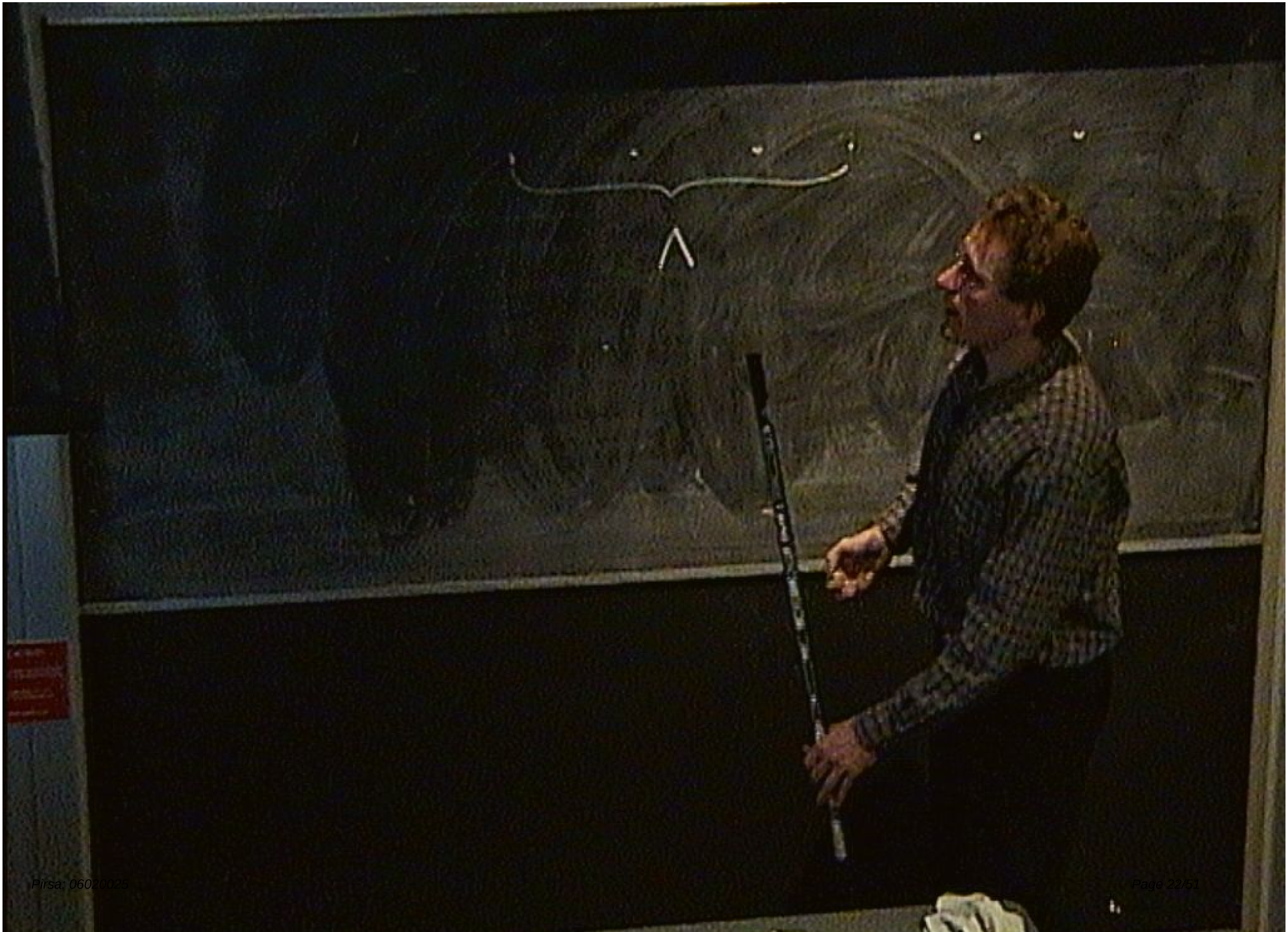
(ii) $\ker H_L \neq \{0\}$.

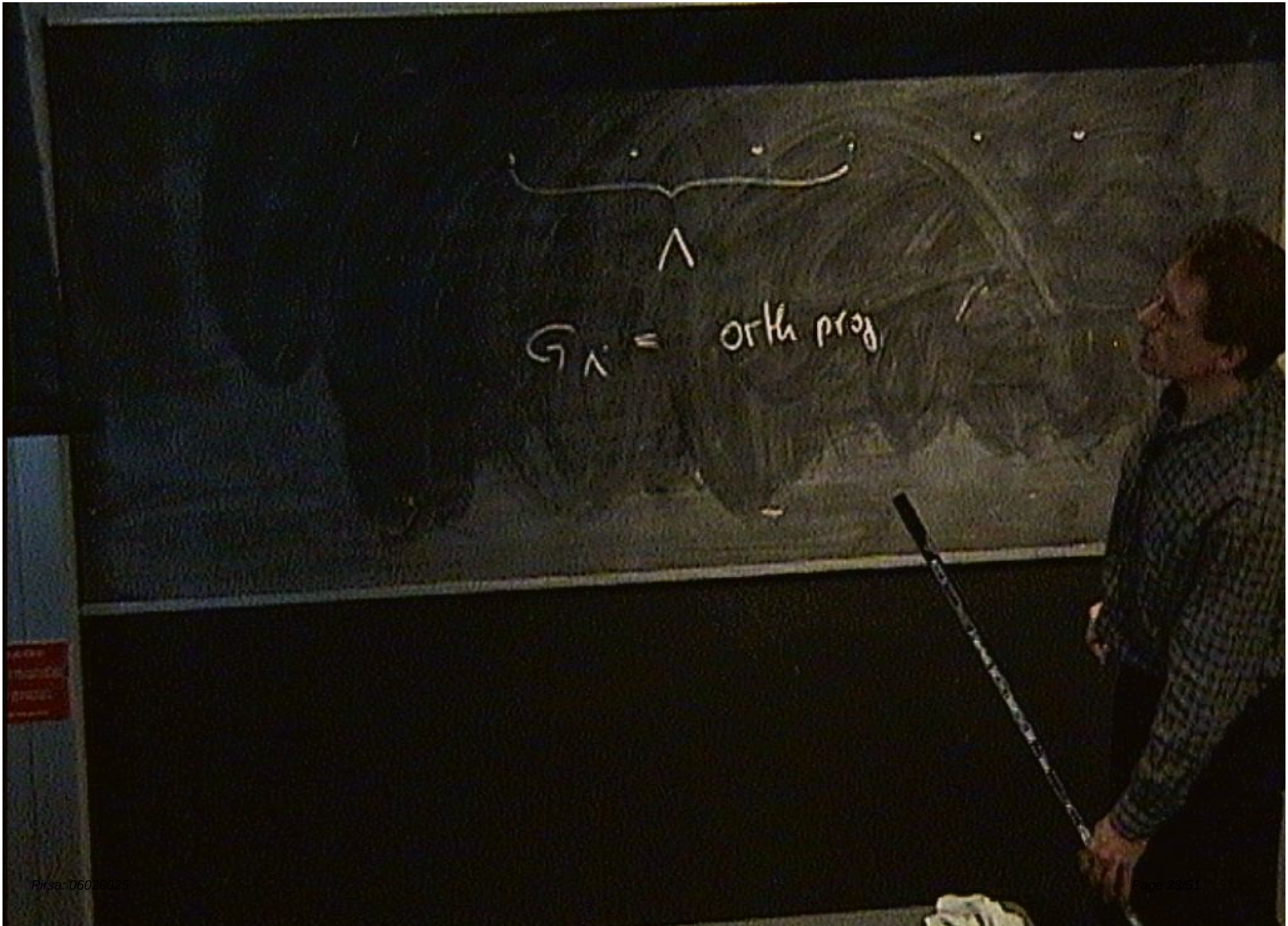
Note: $\ker H_L = \bigcap_{i=0}^{N} \ker(h_{C_i})$. Therefore, if $\psi \in \ker(H_L)$ i.e., is a GS, then its restriction to subsystem on C_i is also a GS.

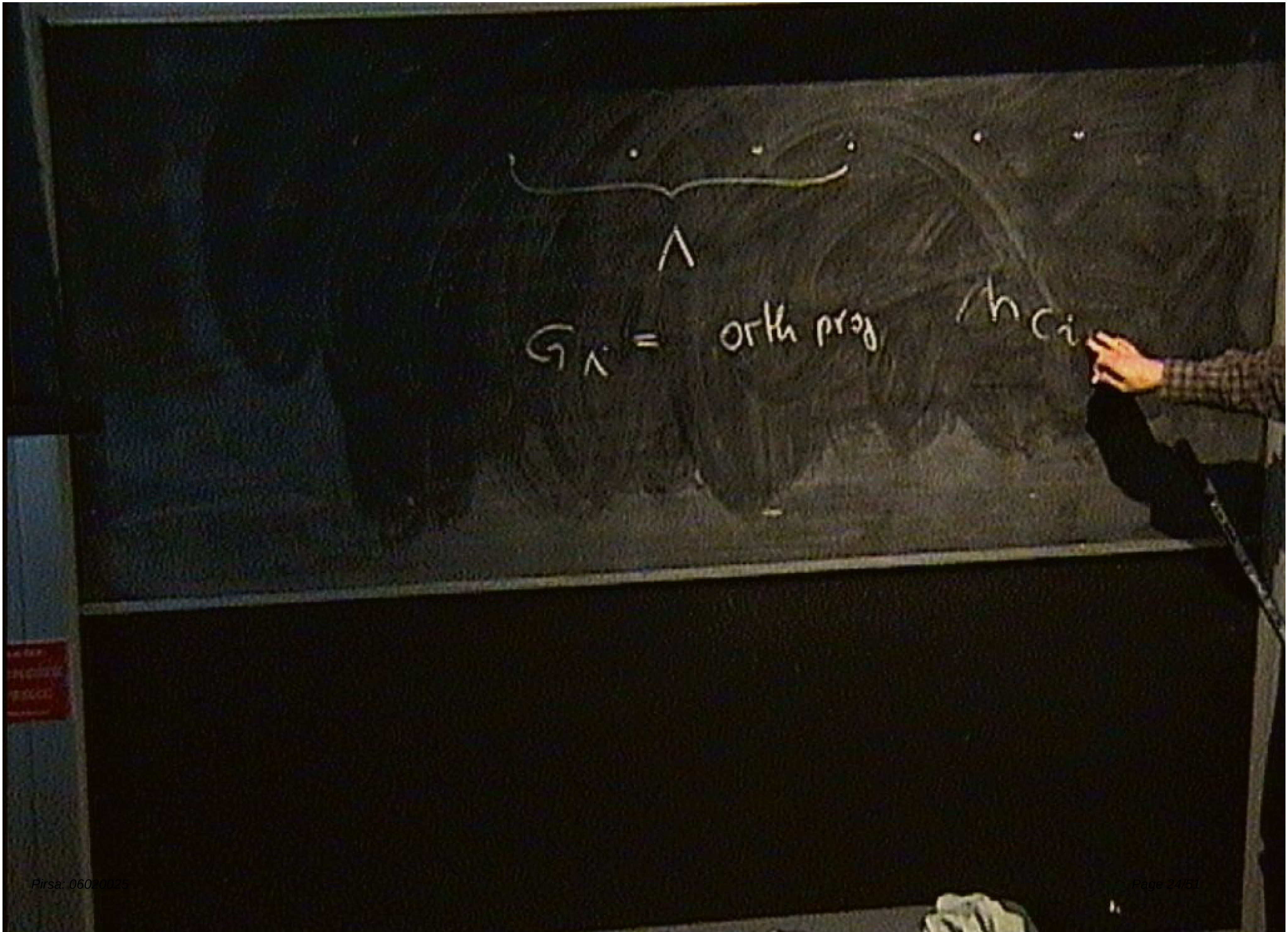
- For $\Lambda \subset [0, L-1]$ define $G_\Lambda = \text{orth proj}(\ker \sum_{i:C_i \subset \Lambda} h_{C_i})$.
- Let $\Lambda_i = \bigcup_{j \leq i} C_j$. Define the projections,

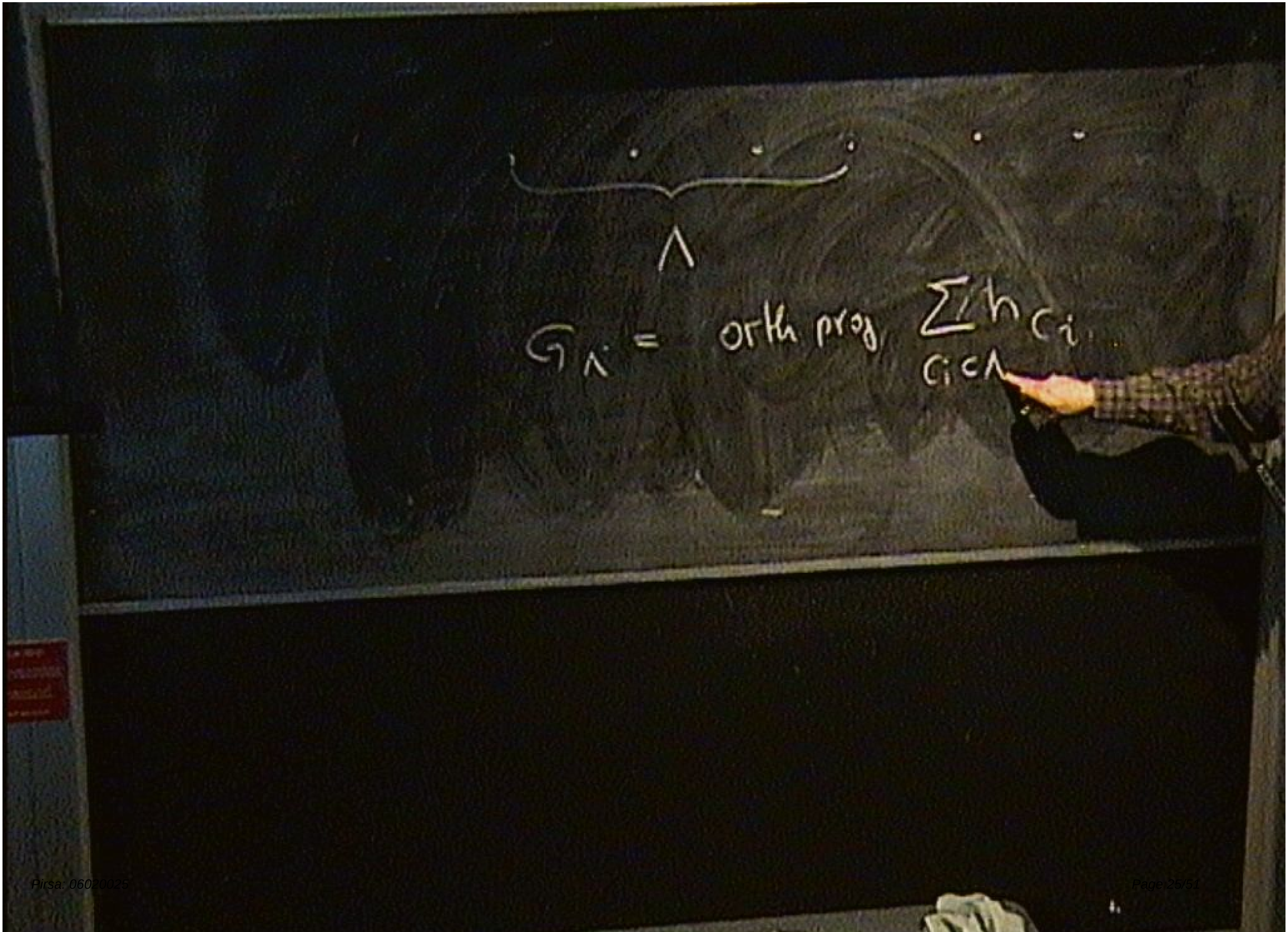
$$E_i = \begin{cases} \mathbf{1} - G_{\Lambda_0} & i = 0 \\ G_{\Lambda_i} - G_{\Lambda_{i+1}} & 1 \leq i \leq N-1 \\ G_{[0,L-1]} & i = N \end{cases} .$$

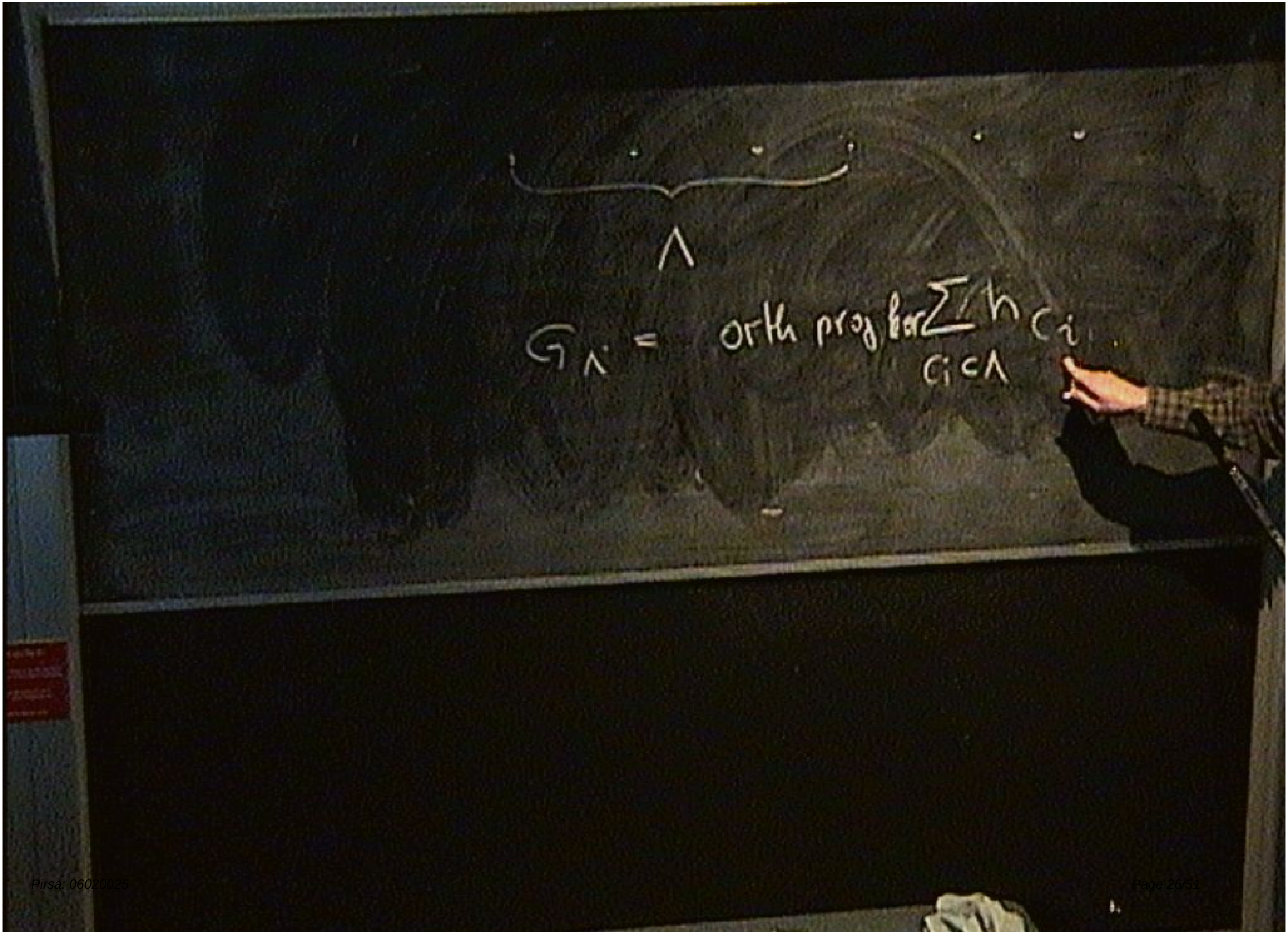
- Let γ_i be the gap of h_{C_i}, i.e., $h_{C_i} \geq \gamma_i(\mathbf{1} - G_{C_i})$.

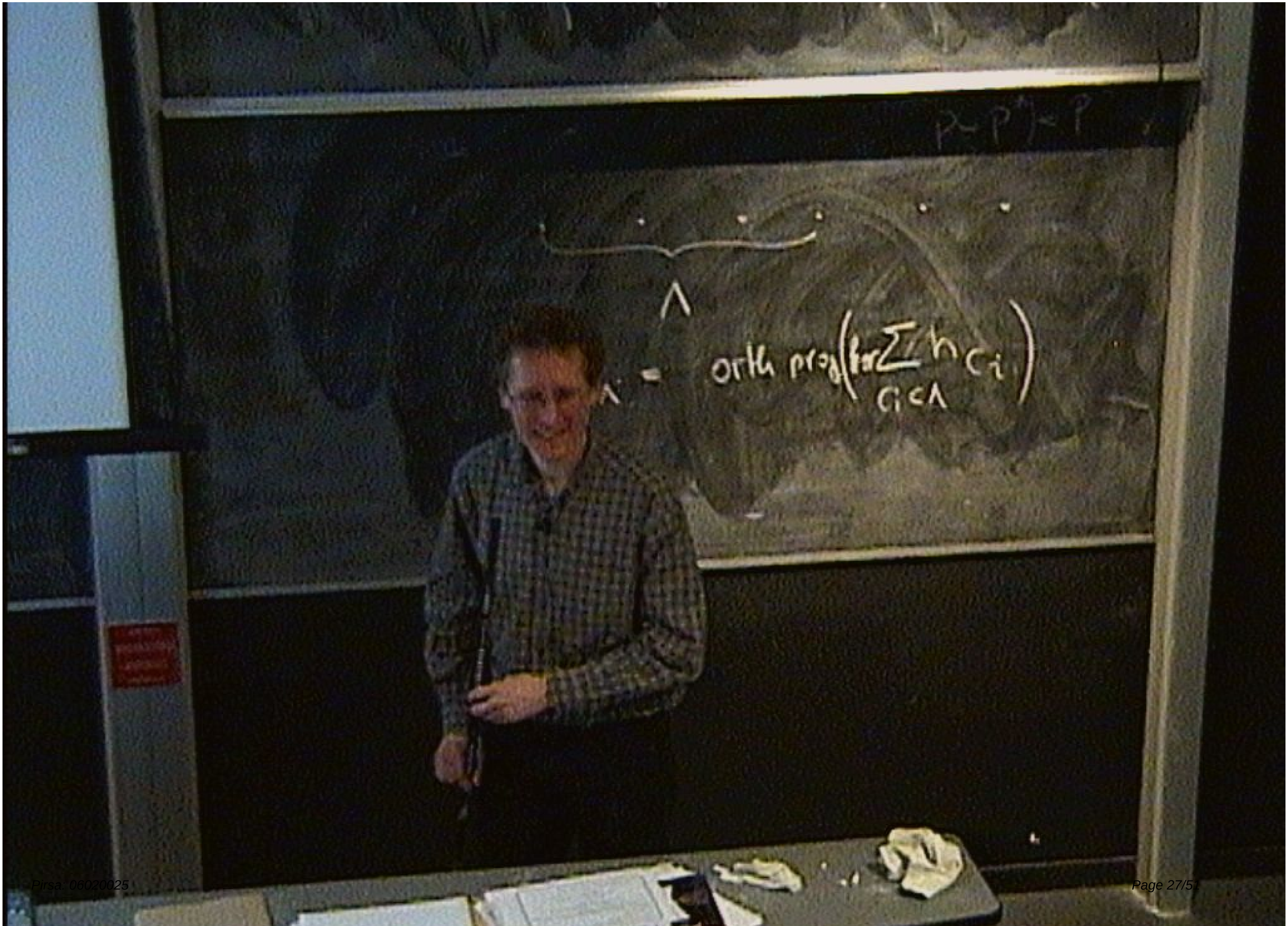


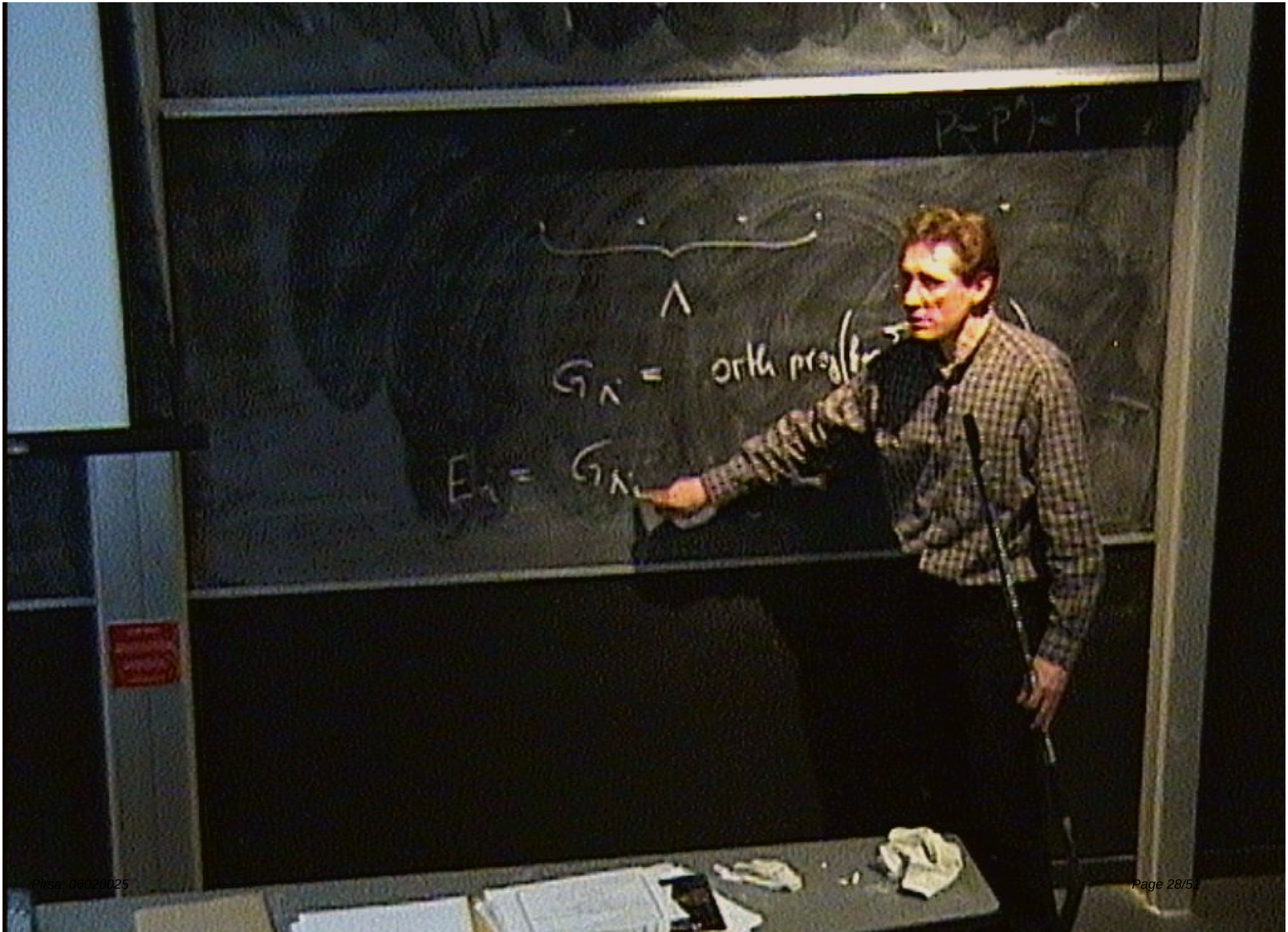

Λ
G_Λ = orth proj

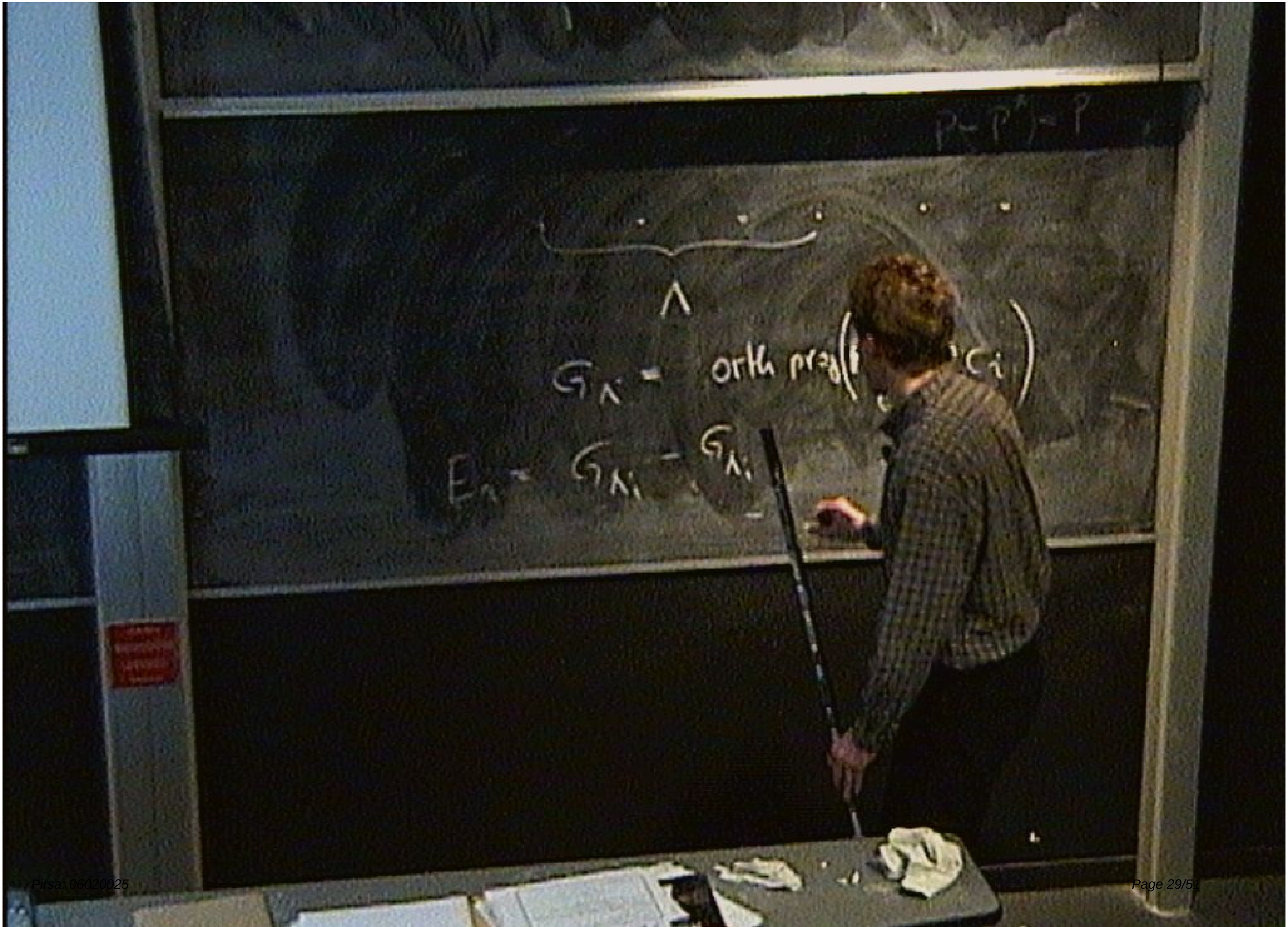

Λ
G_Λ = orth proj

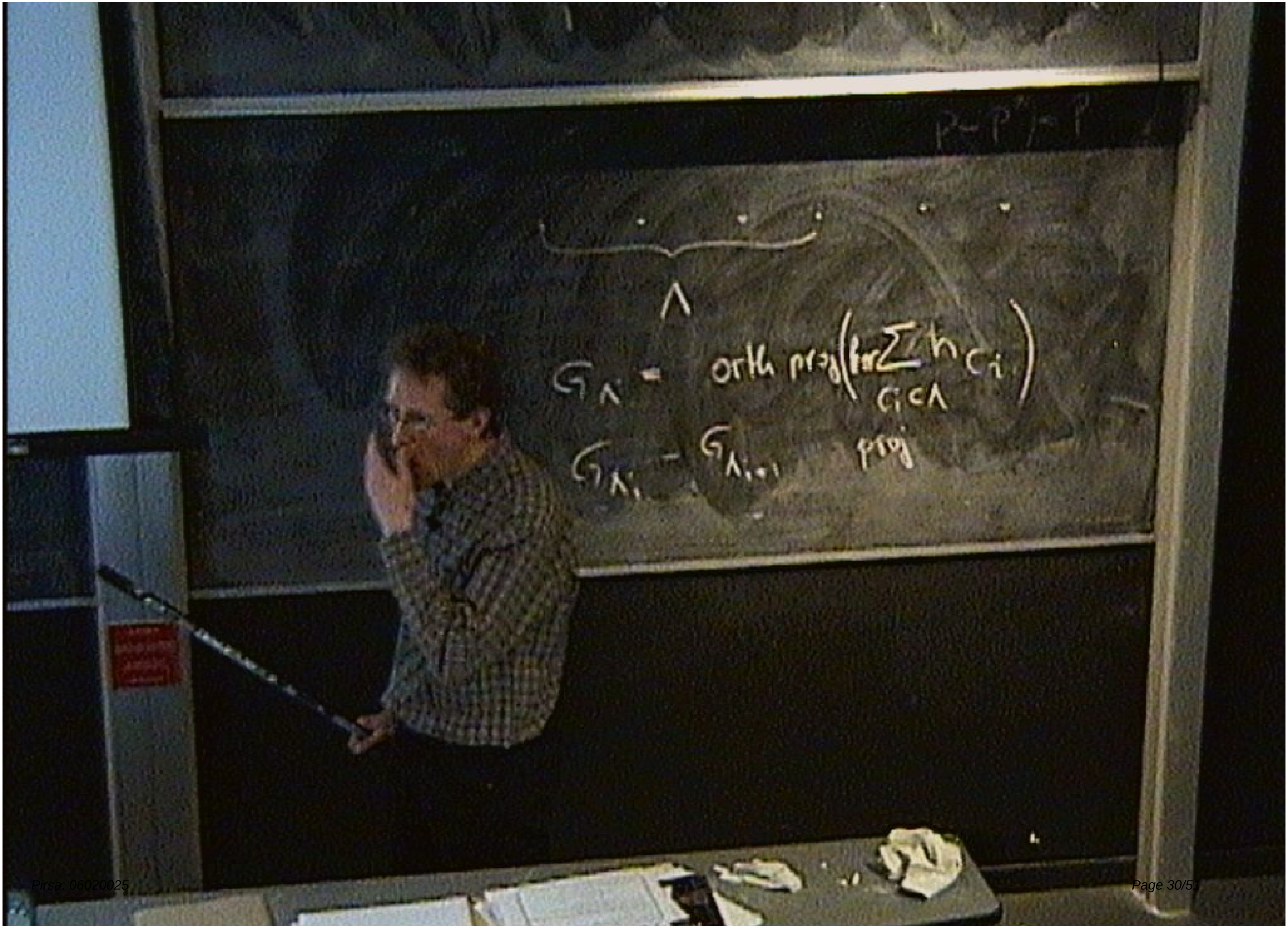

Λ
G_Λ = orth proj Σ h_{c_i}
c_i ⊂ Λ

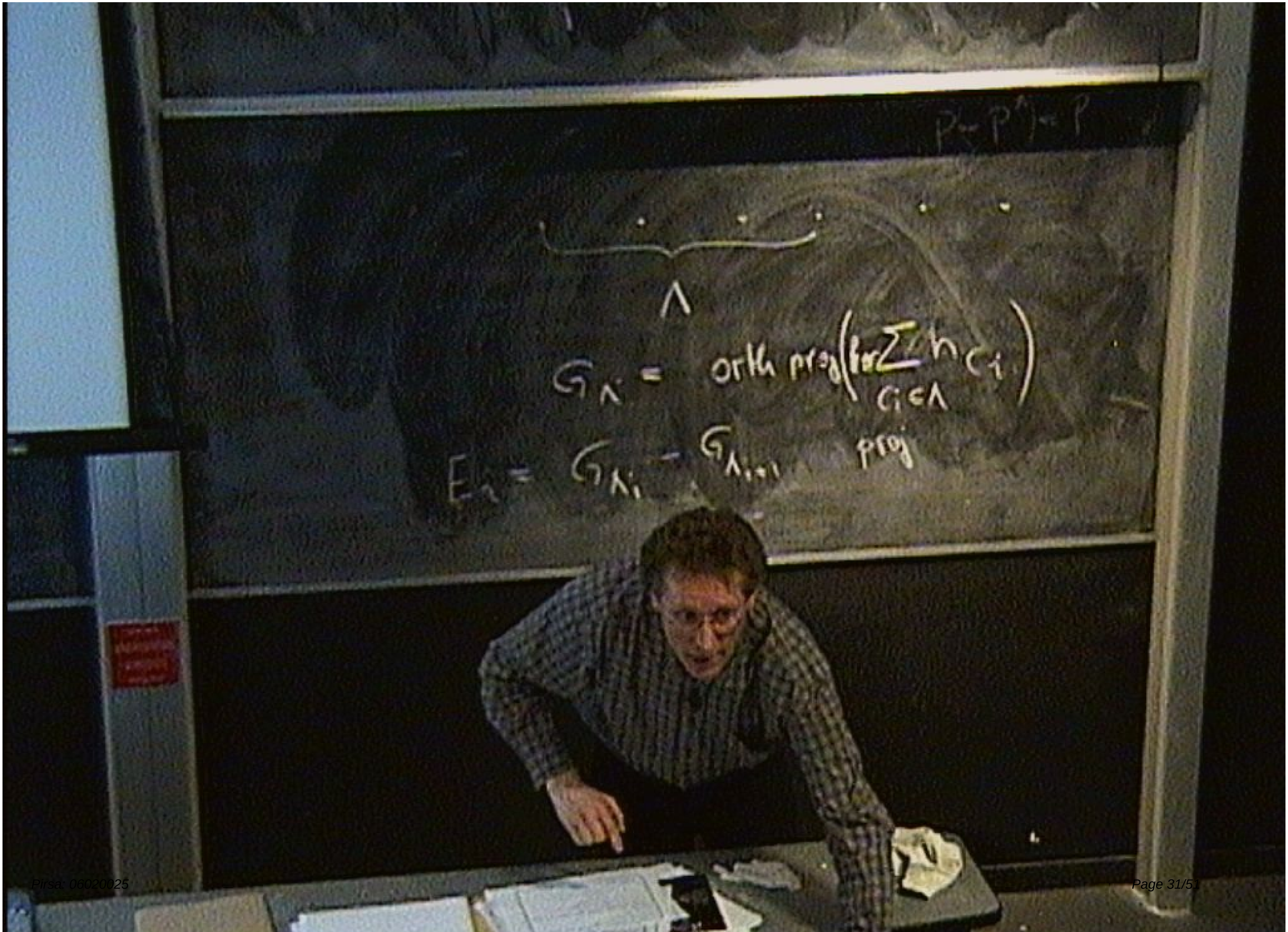

Λ
G_Λ = orth proj for Σ h_{c_i}
c_i ⊂ Λ




Λ
G_Λ = orth proj
E_Λ = G_Λ


orth proj

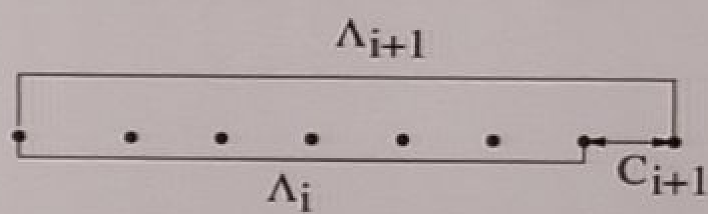



orth proj
proj

THEOREM 1. *Let $\gamma = \min \gamma_i$. Suppose, that $\|G_{C_{i+1}} E_i\| \leq \varepsilon < 1/\sqrt{2}$ for $0 \leq i \leq N-1$. Then, if ψ is orthogonal to the GS of H_L,*

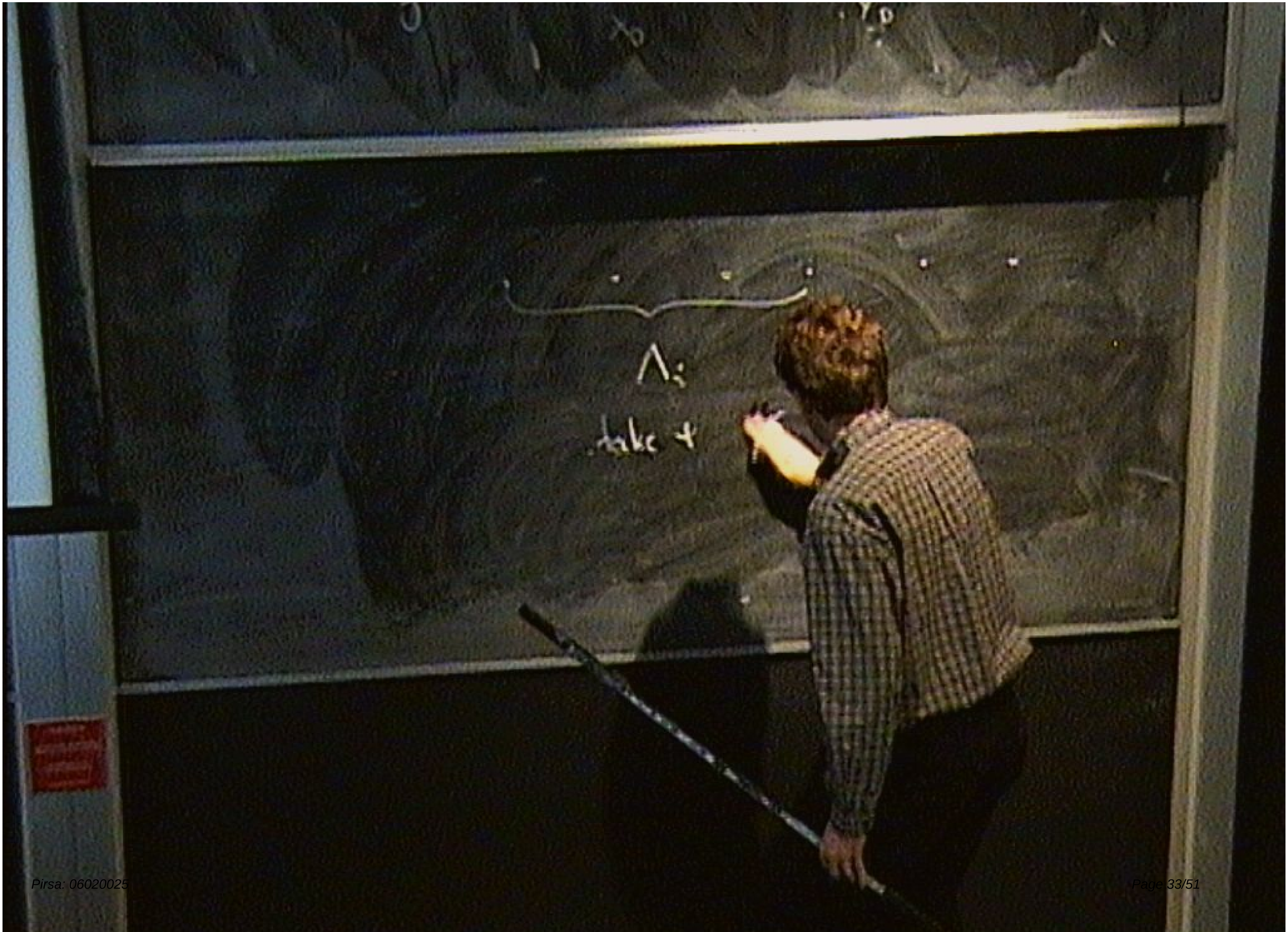
$$\langle \psi, H_L \psi \rangle \geq \gamma (1 - 2\varepsilon\sqrt{1-\varepsilon^2})\, \|\psi\|^2 .$$

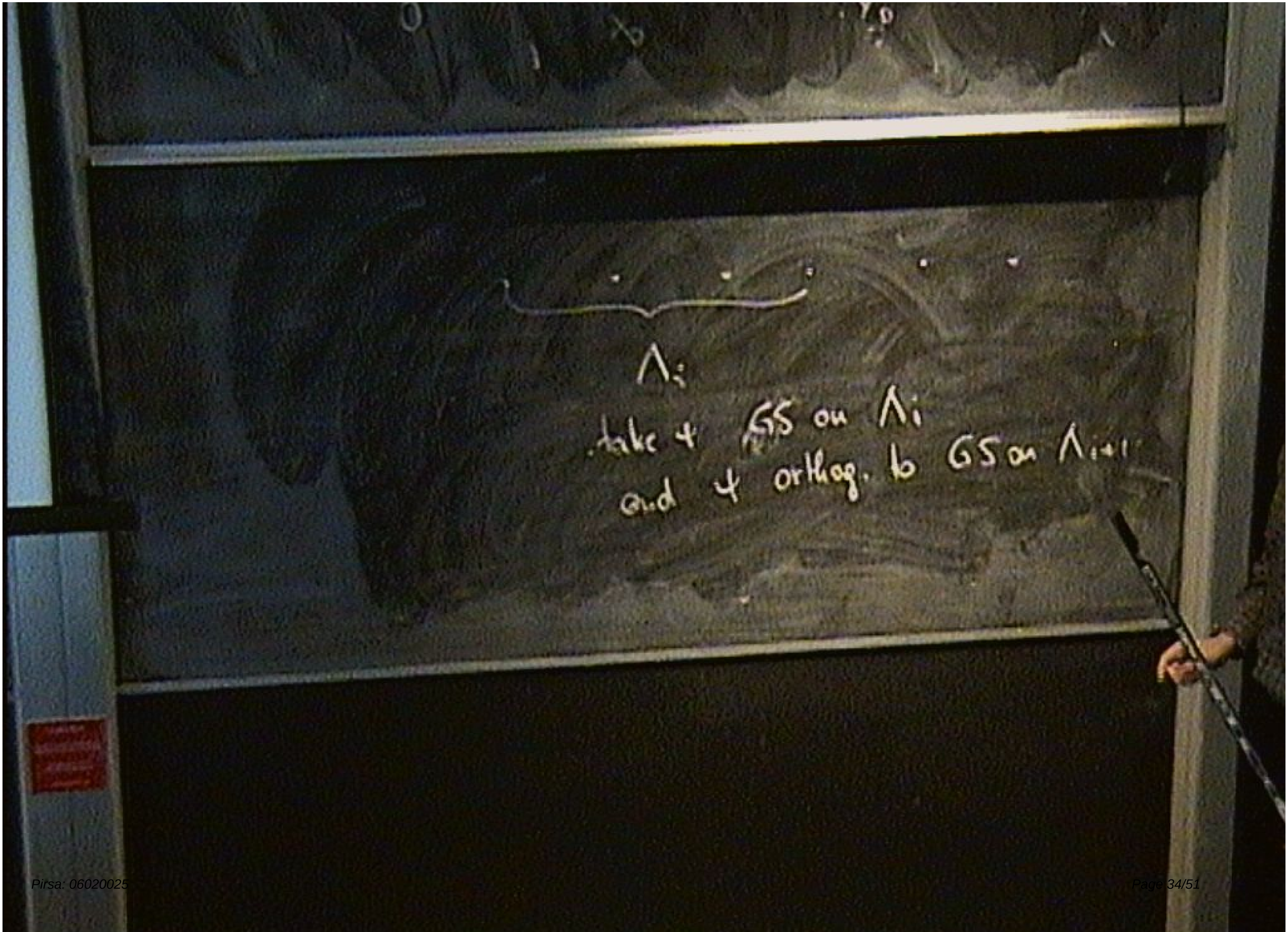
In order to calculate $\|G_{C_{i+1}} E_i\|$, take ψ in the range of $E_i = G_{\Lambda_i} - G_{\Lambda_{i+1}}$. I.e., ψ is a GS on the chain Λ_i and perpendicular to all GS of Λ_{i+1}. Then compute the norm of the incremental projection, $G_{C_{i+1}} \psi$.

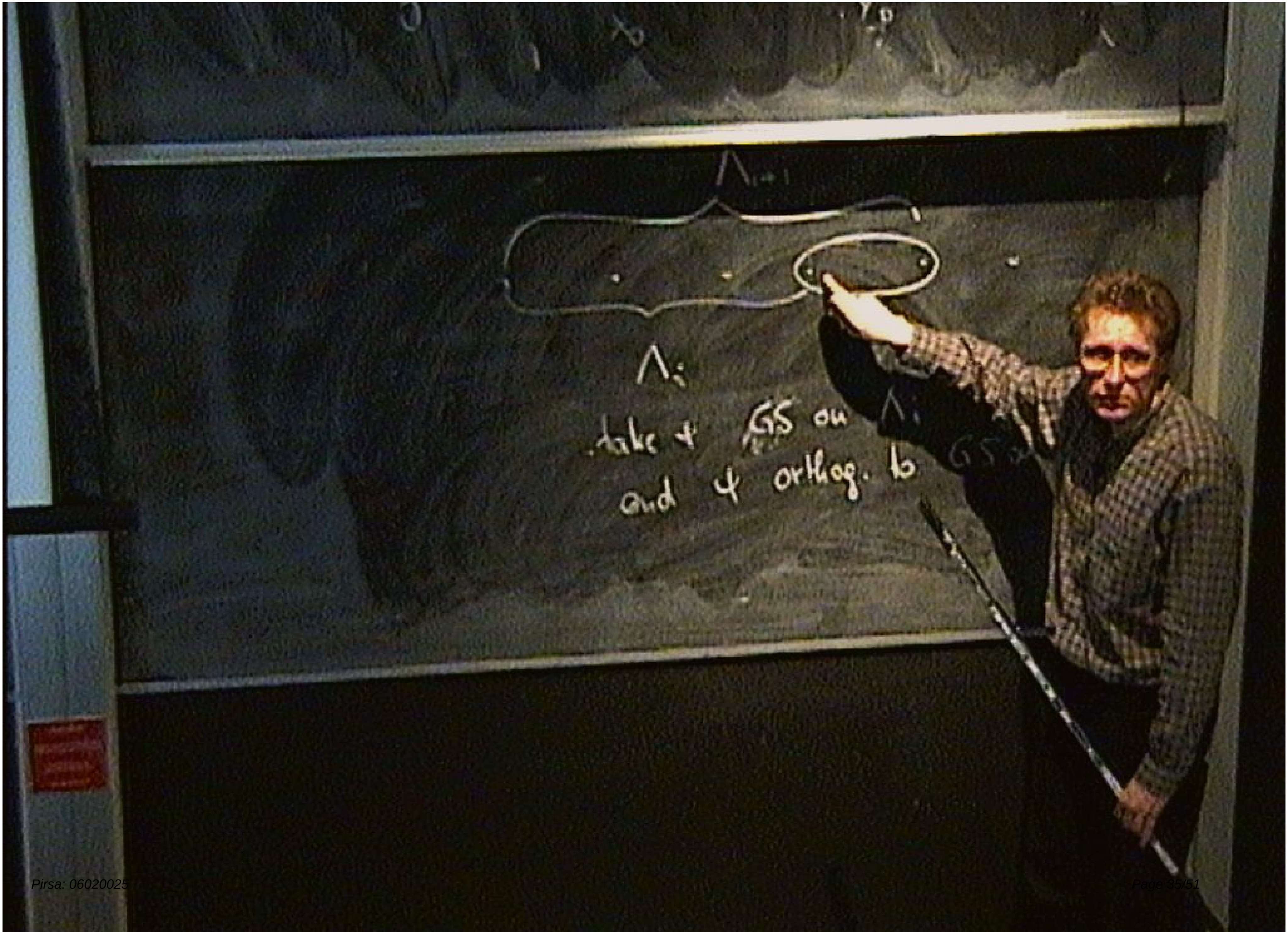


Very useful to prove gap for infinite systems, which is a subtle business.

As an application we consider the spin-1/2 ferromagnetic Heisenberg model in a transverse magnetic field on the chain $[0, L-1]$.




Λi
take ψ GS on Λi
and ψ orthog. to GS on Λi+1



Λi+1
Λi
take ψ GS on
and ψ orthog. to

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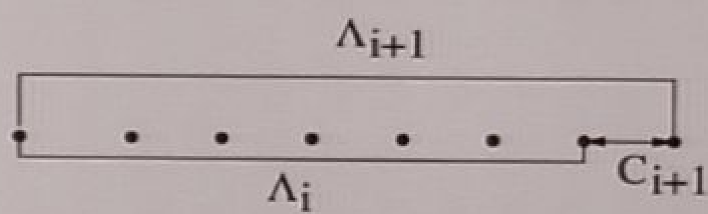

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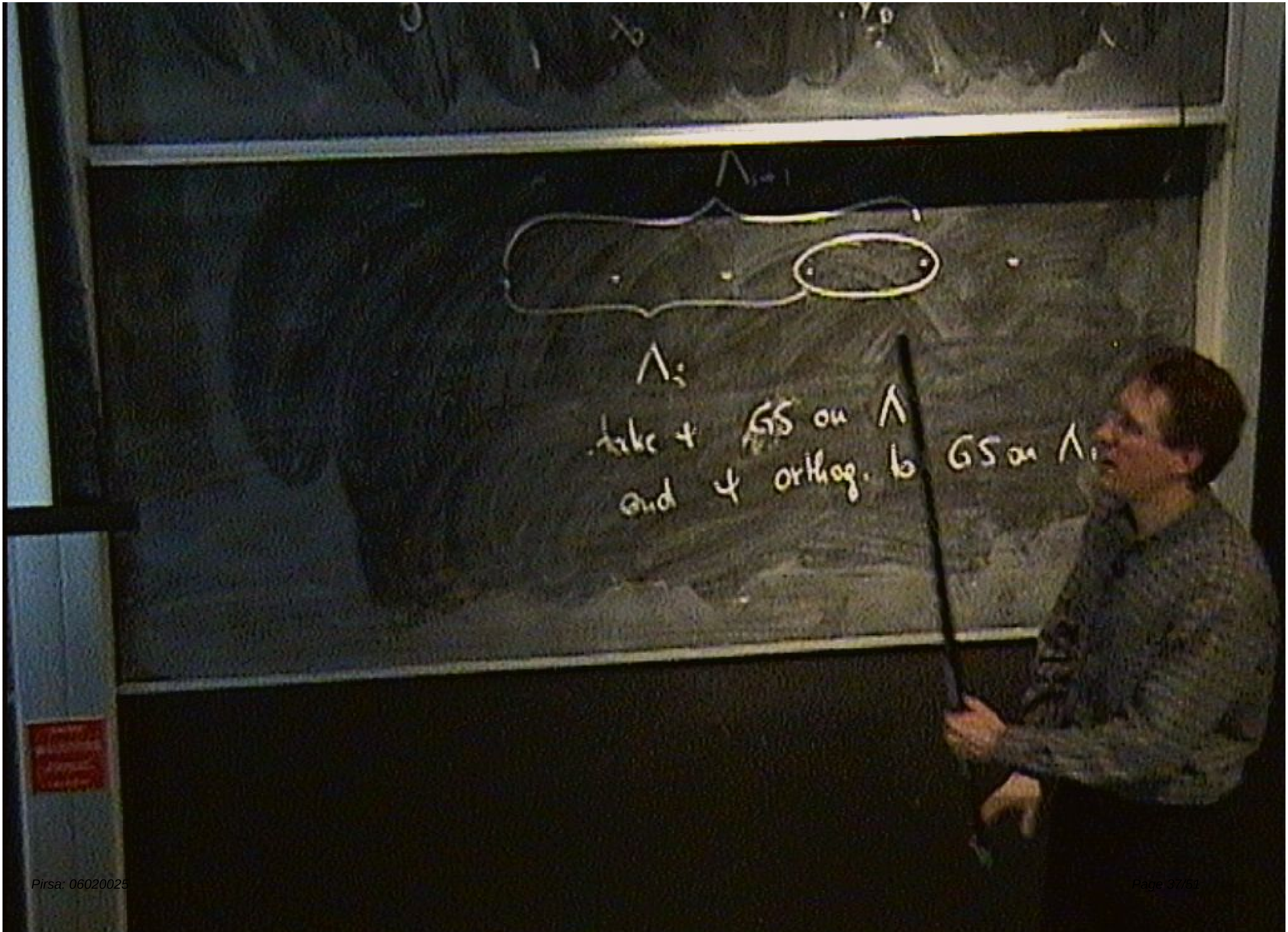
As an application we consider the spin-1/2 ferromagnetic Heisenberg model in a transverse magnetic field on the chain $[0, L-1]$.


Λi+1
Λi
take ψ GS on Λ
and ψ orthog. to GS on Λi

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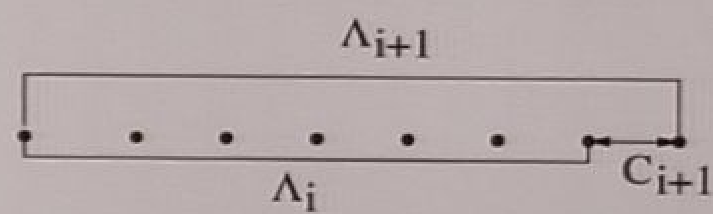

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As an application we consider the spin-$1/2$ ferromagnetic Heisenberg model in a transverse magnetic field on the chain $[0, L-1]$.

7

Let $\Delta > 1$ be the anisotropy parameter. Then for two nearest neighbor sites $x, x+1$ we have the local ferromagnetic Heisenberg interaction,

$$h_{xx+1} = -\frac{1}{\Delta}(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2) - S_x^3 S_{x+1}^3 + \frac{1}{2}\sqrt{1-\Delta^{-1}}(S_x^3 - S_{x+1}^3) + \frac{1}{4}\mathbb{1}\,.$$

The matrices S_x^1, S_x^2, S_x^3 are the usual spin-1/2 matrices,

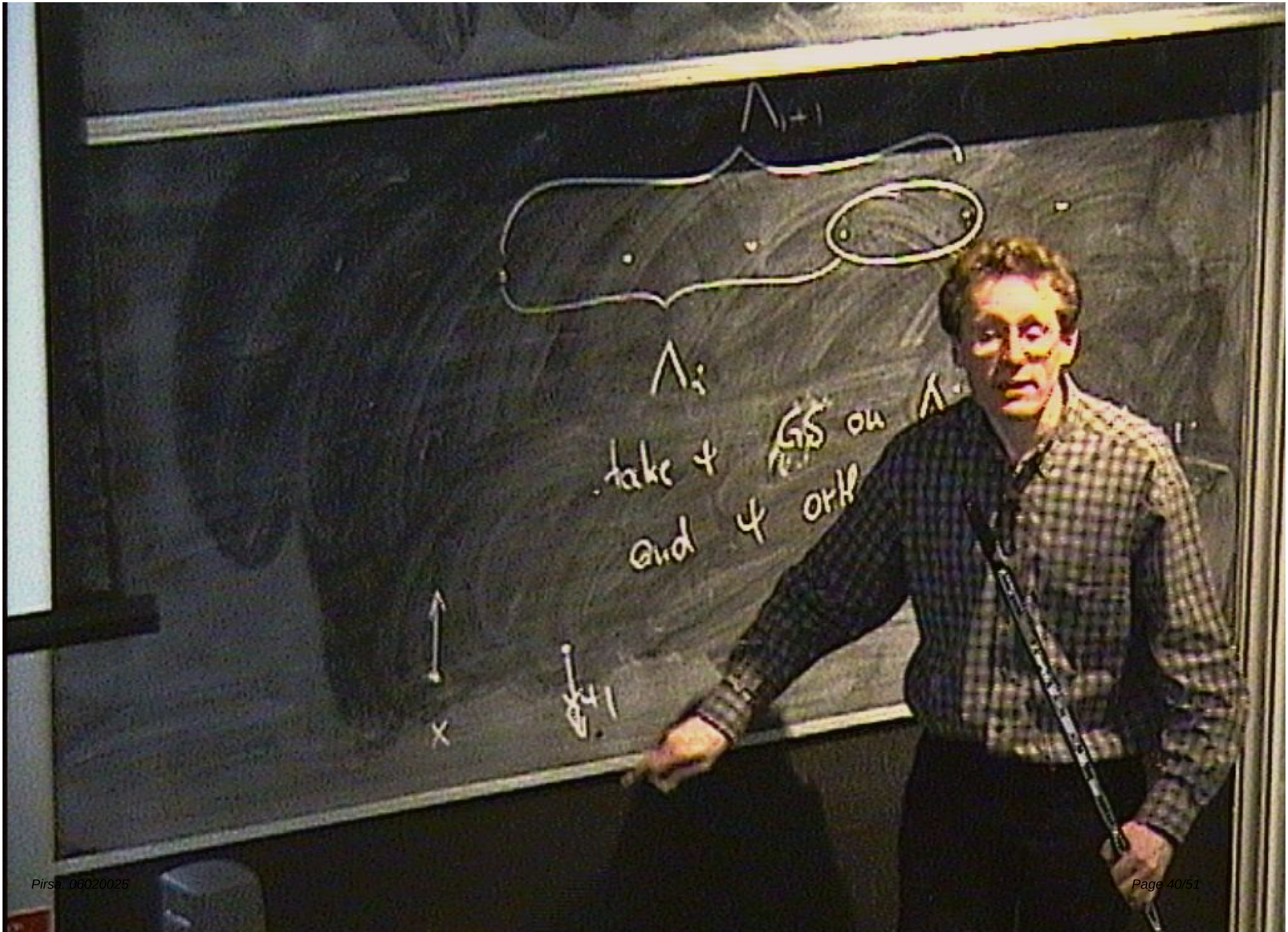
$$S_x^1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_x^2 = \frac{1}{2}\begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad S_x^3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The ferromagnetic Heisenberg model on the chain $[0, L-1]$ is then

$$H_L = \sum_{x=0}^{L-2} h_{xx+1}\,.$$

Facts:

- H_L has $L+1$ GS with energy 0 that describe DW.
- Gap above GS is $1 - \frac{1}{\Delta}\cos(\pi/L) \geq 1 - \frac{1}{\Delta}$. Koma-Nachtergaele, 1995.


Λi+1
Λi
take ψ GS on
and ψ

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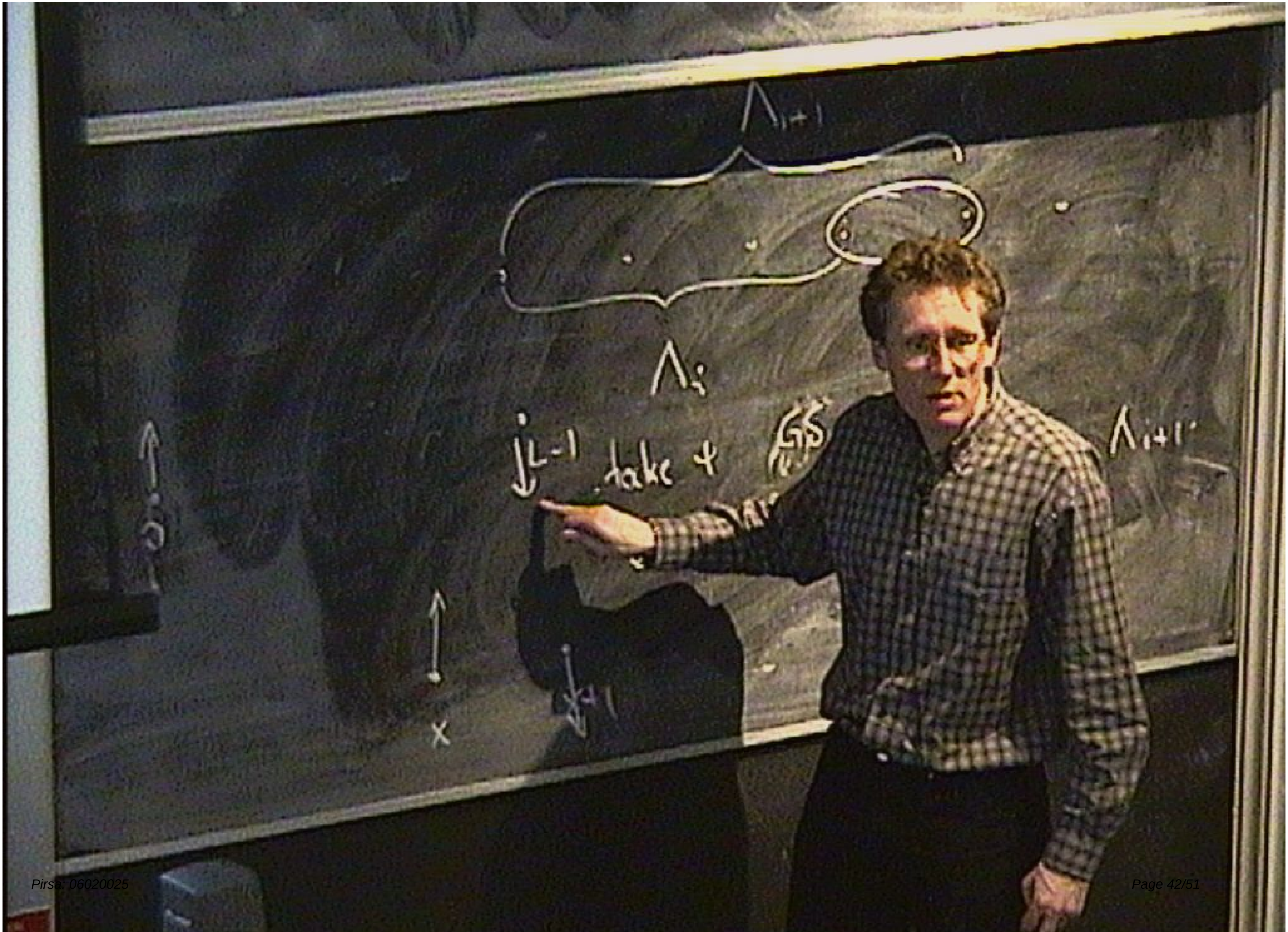
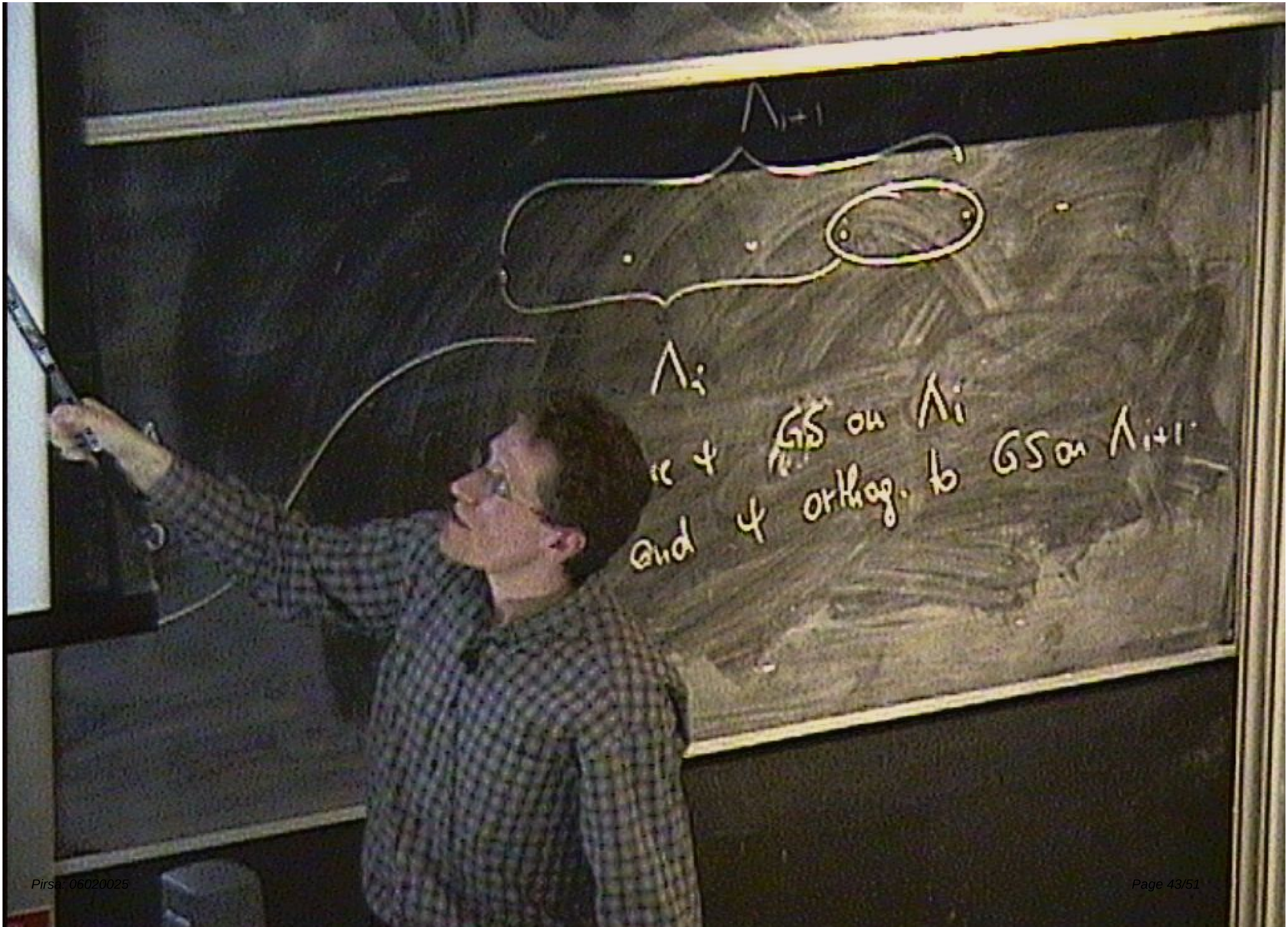

take


Λi+1
Λi
GS on Λi
GS on Λi+1
and ψ orthog. to GS on Λi+1

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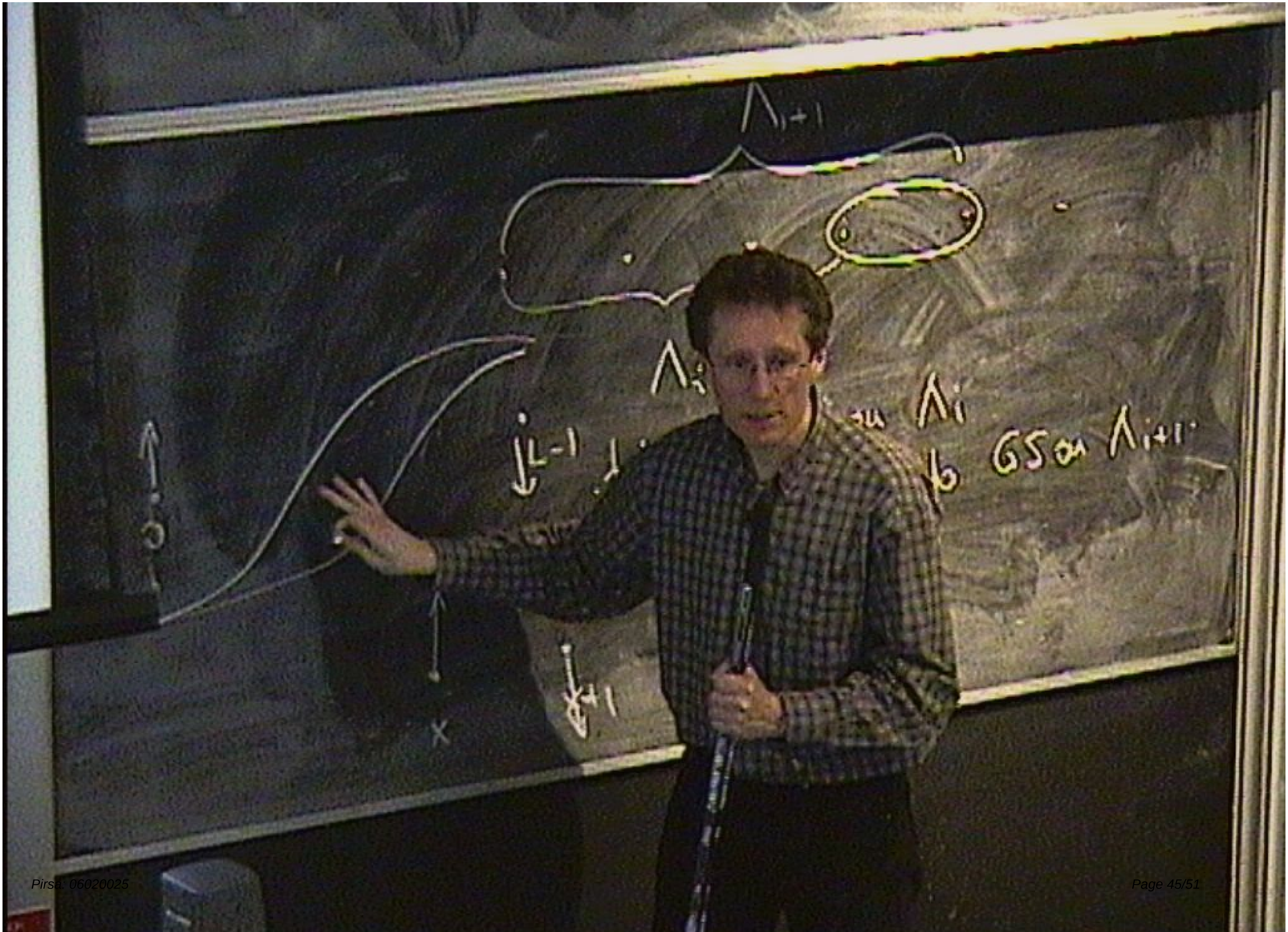

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Transverse magnetic field (for simplicity), $V = \sum B(x) S_x^1$. We assume

(i) support of $B(x)$ is finite uniformly in L,

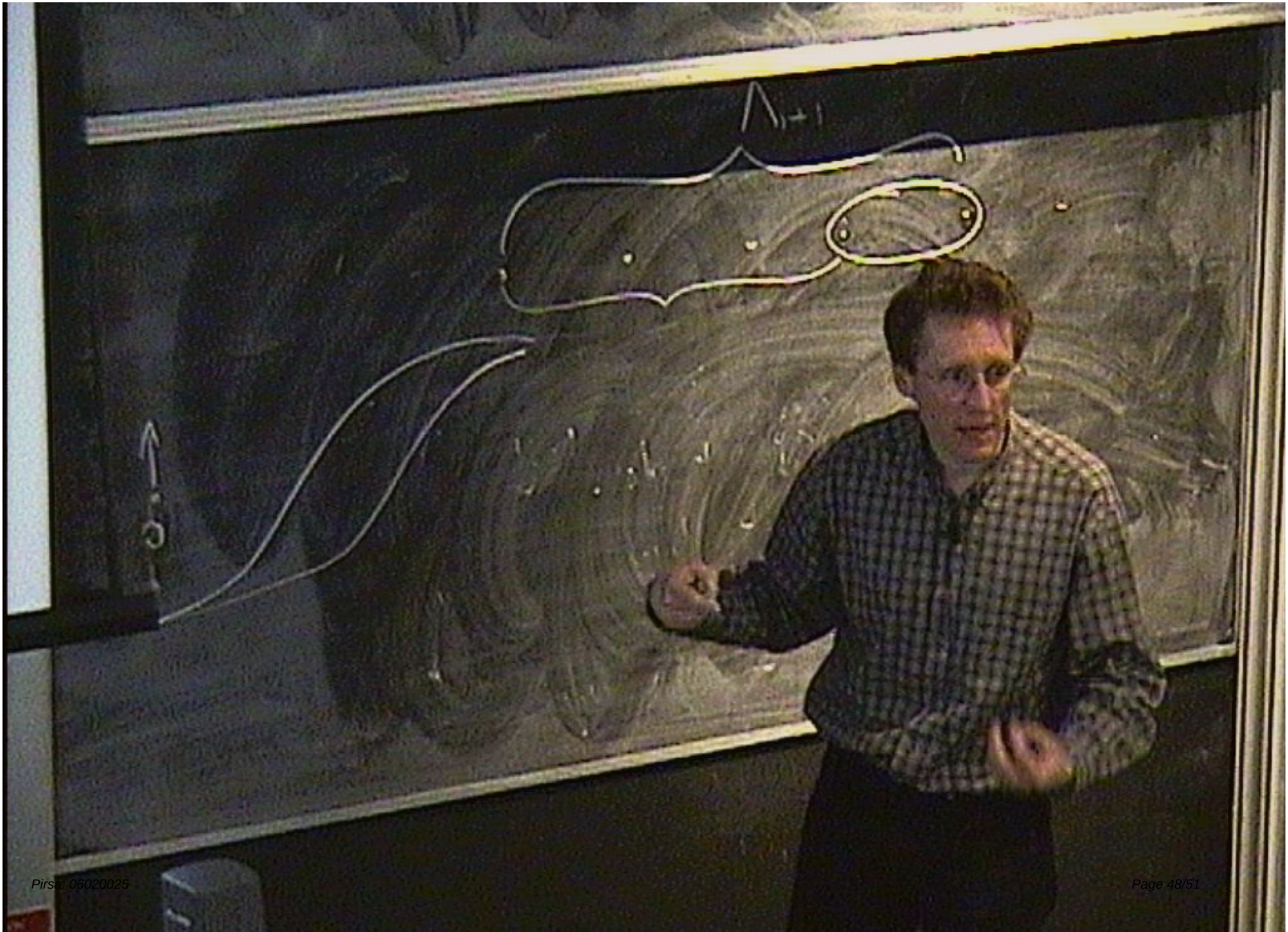
(ii) on support, $B(x) > 0$.

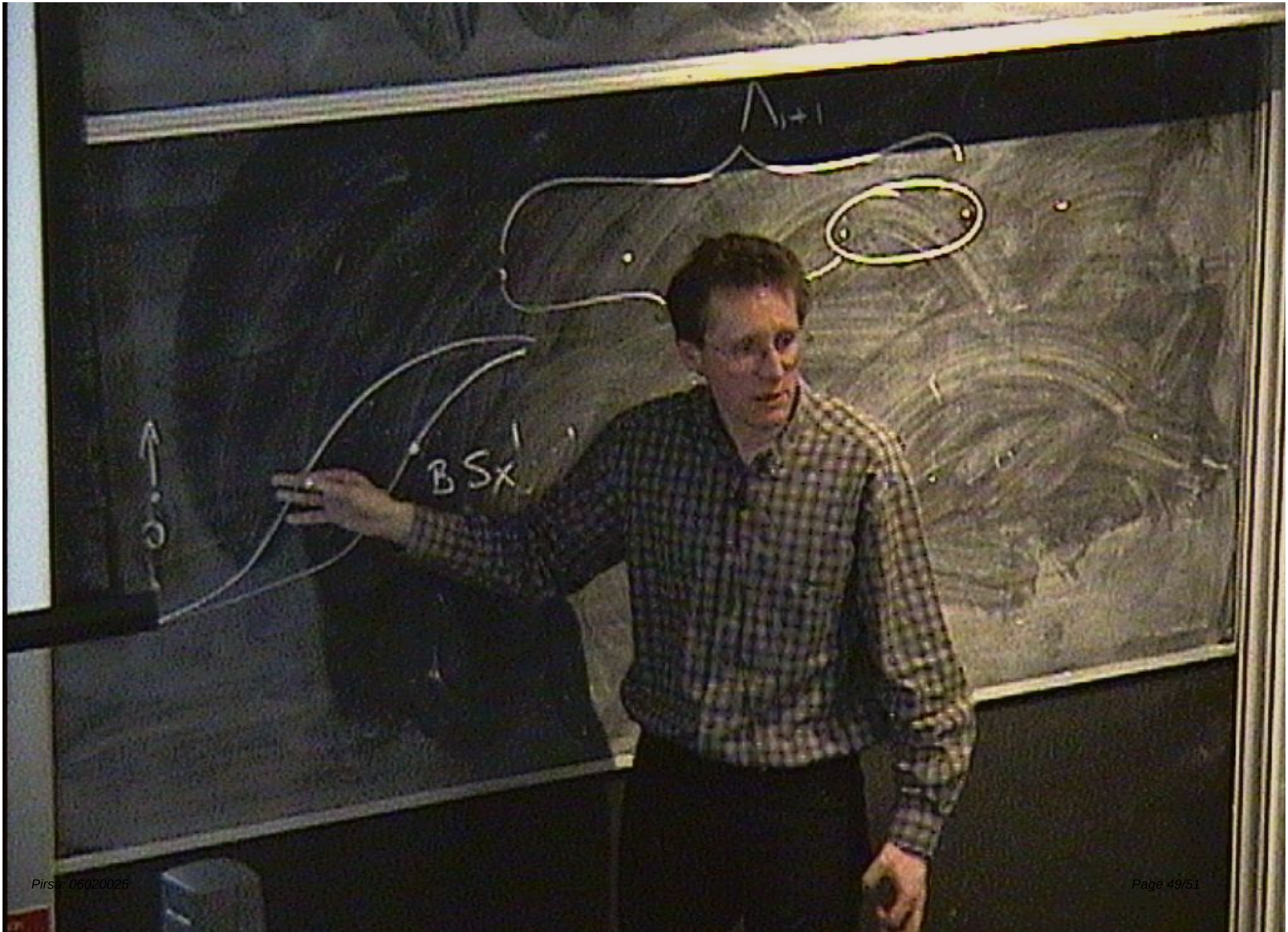
On the chain $[0, L-1]$ we define the Hamiltonian,

$$H_L(B) = H_L + \sum B(x) S_x^1 .$$

Example: Let $B(x) = B\delta(x,y)$. Then, $H_L(B)$ has a unique GS with DW centered at y. This explicitly known GS is gapped uniformly in L.

THEOREM 2. *Under the above assumptions on the magnetic field, $H_L(B)$ has a unique GS with a strictly positive gap, uniformly in L.*


Λi+1


BSx

Transverse magnetic field (for simplicity), $V = \sum B(x) S^1_x$. We assume

(i) support of $B(x)$ is finite uniformly in L,

(ii) on support, $B(x) > 0$.

On the chain $[0, L-1]$ we define the Hamiltonian,

$$H_L(B) = H_L + \sum B(x) S^1_x .$$

Example: Let $B(x) = B\delta(x,y)$. Then, $H_L(B)$ has a unique GS with DW centered at y. This explicitly known GS is gapped uniformly in L.

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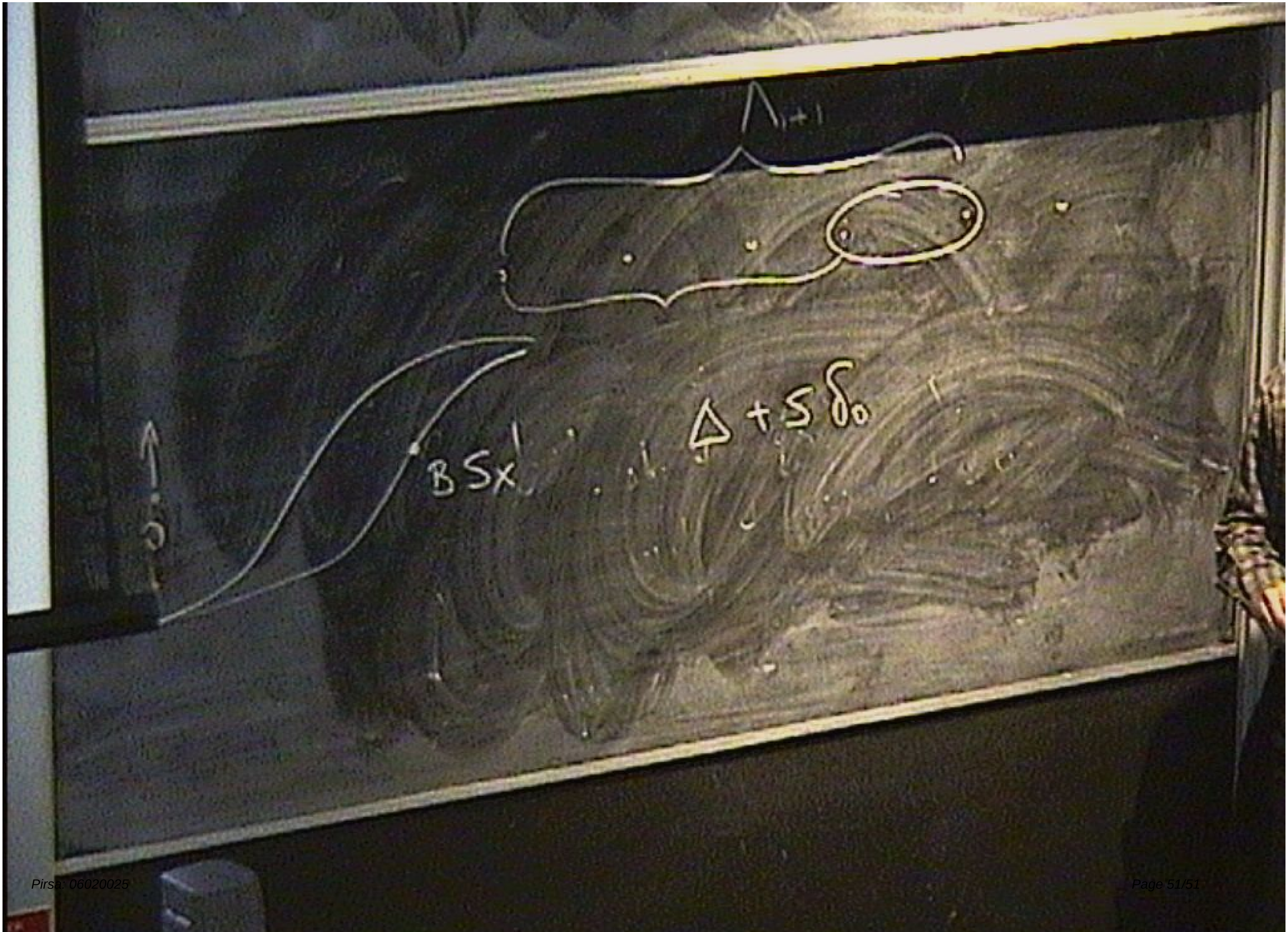

Λ_{i+1}
BSx
$\Delta + S\delta_0$