Title: Relation of adiabatic QC to other models

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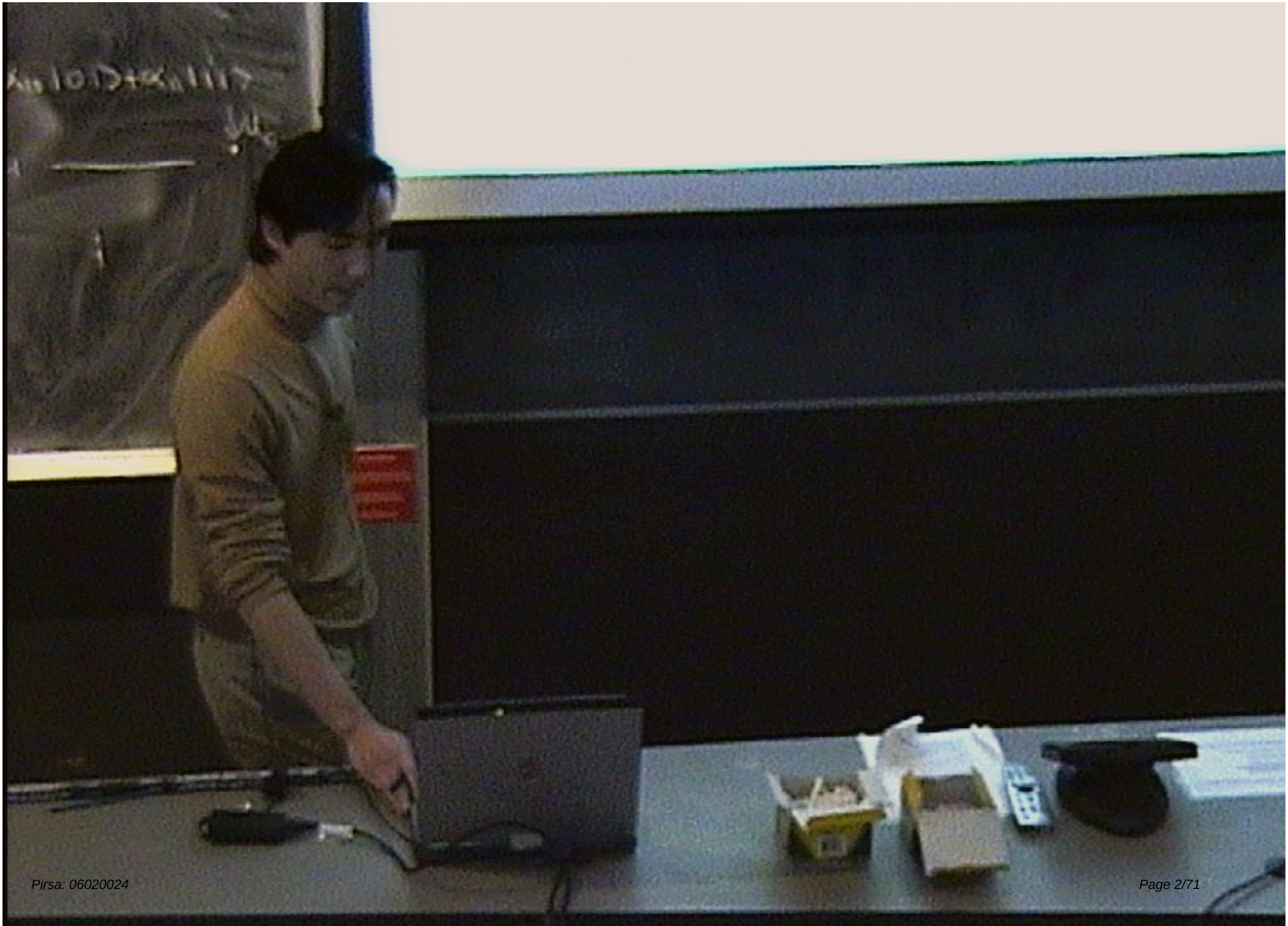
Abstract:



Workshop on Mathematical Aspects of the Quantum Adiabatic Approximation

Relation of Adiabatic Quantum Computing to Other Models

M. Stewart Siu, Stanford University

Reference:

1. Aharonov, Kempe, Regev, Van Dam, Landau & Lloyd, "Adiabatic Quantum Computation is Equivalent to Standard Quantum Computation", quant-ph/0405098

2. Kempe, Kitaev & Regev, "The Complexity of the Local Hamiltonian Problem", quant-ph/0406180

3. Oliveira & Terhal, "The Complexity of Quantum Spin Systems on a Two-Dimensional Square Lattice", quant-ph/0504050

4. Siu, "From Quantum Circuits to Adiabatic Algorithms", Phys. Rev. A 71, 062314 (2005), quant-ph/0409024

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Theme: Encoding a quantum circuit into a (smooth, time-dependent) Hamiltonian

OUTLINE

1. The Issue of Locality

2. Kitaev's History Approach (analog of Cook-Levin's proof of 3-SAT NP-completeness) and Its Adaption to AQC

3. Eigenvalue Gap and Stochastic Matrices

4. Connection to Holonomic Quantum Computing

5. Perturbation Theory Gadgets and Square Lattice Constructions

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1. The Issue of Locality

<u>A straightforward way to map the circuit</u>

Suppose $H|n\rangle = E_n|n\rangle \qquad n = 0, 1, \ldots$

$(H + ?)\, U|n\rangle = E_n' U|n\rangle$

Note that this implies

$?\, U|n\rangle = [U, H]|n\rangle + (E_n' - E_n)\, U|n\rangle$

which completely specifies "?"

Easy Case: $E_n' = E_n \Rightarrow H + ? = UHU^{+}$

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We can use time-dependent similarity transform

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We can use time-dependent Similarity transform

Given Quantum Gate U,

Let $K = -i \log U$ and $\tilde{U}(t) = e^{iKt}$

$\tilde{U}(t) H \tilde{U}(t)^\dagger$ takes $|0\rangle$ to $U|0\rangle$ as $t \to 1$

Problem: If H is m-local and U is 2-local, $[U, H]$ can be $m+1$-local.

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Problem: If H is m-local and U is 2-local, $[U, H]$ can be $m+1$-local.

Can we avoid the problem if we are willing to vary the gap?

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Theorem (Haselgrove, Nielsen & Osborne):

$\forall$ density matrix ρ w/ eigenvalues $\rho_1 \leq \rho_2 \leq \ldots$ and $\mathrm{tr}(\rho H) = \langle\psi|H|\psi\rangle$

$$\sum_{j=1}^{d-1} (E_j - E_0)\rho_{j+1} \leq \left(1 - |\langle\psi|E_0\rangle|^2\right) E_{tot}$$

Suppose a n-qubit circuit generates $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|000\ldots\rangle + |111\ldots\rangle\right)$ at some stage & we want $|\psi\rangle$ to be the ground state of k-local Hamiltonian H (where $k<n$),

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$$\sum_{j=1} (E_j - E_0) p_{j+1} = \ldots$$

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$tr(\rho H) = \langle\psi|H|\psi\rangle$ since $k<n$

Inequality $\Rightarrow$ $E_1 - E_0 \leq (1 - |\langle\psi|E_0\rangle|^2) E_{tot}$

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What can we do?

1) Track approximately
2) Add ancillary qubits
3) Allow degeneracy

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2. The History Approach

Technique not developed for Adiabatic QC, but for QMA-Completeness.

Def. of QMA: $L \in QMA$ if $\exists$ Verifier circuit U $\ni$.

$$\forall x \in L \ \exists |y\rangle \ni. \ P(U_x|y\rangle, |1\rangle) \geq \frac{2}{3}$$

$$\forall x \notin L \ \forall |y\rangle \ni. \ P(U_x|y\rangle, |1\rangle) \leq \frac{1}{3}$$

Def. of Local Hamiltonian Problem:

$H = \sum$ k-local hermitian operators on n-particles

Is the smallest eigenvalue $> b$ or $< a$, where $b - a = poly(n)^{-1}$

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Like 3-SAT, the Local Hamiltonian is a set of constraints.

To prove any QMA problems can be written as LHP, follow Cook-Levin and constrain the history of verifier:

$$\mathcal{H} = \left(\sum_{\text{all time step}} H_{prop}\right) + H_{out}$$

(For convenience, all terms are semi +ve definite)

H_{prop}: $=0$ iff each gate of verifier U is correctly applied

H_{out}: $=0$ iff final state $= |1\rangle$

Initial state is left unconstrained. ∵ we don't know if $\exists |y\rangle$ ∋. $U|y\rangle \sim |1\rangle \otimes \ldots$

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Use this for Adiabatic QC:

$$\hookrightarrow \quad H = \left(\sum_{\text{all time step}} H_{prop}\right) + \cancel{H_{out}} \; + H_{in}$$

- Verifier circuit is replaced by the circuit we are mapping from.
- Instead of constraining the output, we constrain the input.
- The ground state is always the history of the circuit (Unless we keep H_{out}, in which case we'd have to measure the energy)

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Let $|\gamma_0\rangle \ldots |\gamma_L\rangle$ be the intermediate states of the quantum circuit. Make them orthogonal by adding clock ancillae.

The history state is

$$\frac{1}{\sqrt{L+1}}\left(|\gamma_0\rangle + \ldots + |\gamma_L\rangle\right)$$

What H would have an equal superposition as ground state?

Observe that $\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

we can stack these up:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

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$$\begin{bmatrix} -1 & 2 & & \\ & \ddots & \ddots & \\ & & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\sum_t |\gamma_t\rangle\langle\gamma_t| + |\gamma_{t+1}\rangle\langle\gamma_{t+1}| - |\gamma_t\rangle\langle\gamma_{t+1}| - |\gamma_{t+1}\rangle\langle\gamma_t|$$

etc.

☆ The key to making the Hamiltonian k-local, where k is small, is finding good k-local approx. to the terms $|\gamma_{t+1}\rangle\langle\gamma_t|$ etc.

unary clock: $|t\rangle^{clock} = |\underbrace{111\ldots}_{t}00\ldots\rangle$

3-local example:

$$|\gamma_{t+1}\rangle\langle\gamma_t| \approx U_{t+1} \otimes |1\rangle^{clock}_{t+1}\langle 0|^{clock}_{t+1} + J\,|01\rangle^{clock}_{t,t+1}\langle 01|^{clock}_{t,t+1}$$

for large J

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The terms $|\gamma_{t+1}\rangle\langle\gamma_t|$ etc.

analog clock:

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What is the gap of $H_{in} + \sum_t H_{prop}$?

Both H_{in} and $\sum H_{prop}$ have very simple spectra

if both have gaps $\geq \lambda$, then

$$\text{gap of } H_{in} + \sum H_{prop}$$

$$\leq \lambda \max\{ |\langle v|w\rangle|^2 \mid v \in \text{kernel of } H_{in},\ w \in \text{kernel of } \sum H_{prop} \}$$

$$\sim \frac{1}{\text{poly}(n)}$$

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$$\begin{bmatrix} & -1 & 2 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} \\ 1 \end{bmatrix}$$

So $\sum H_{prop}$ in the basis $|\gamma_0\rangle, \ldots |\gamma_L\rangle$ looks like

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

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$$\sim \frac{1}{\text{poly}(L)}$$

combined effect
$\simeq \sum_{t \neq 0} |\gamma_t\rangle\langle\gamma_t|$

Adiabatic Algorithm:

$$H_{initial} = H_{in} + |1\rangle\langle 1|^{clock}_{t=0} \otimes |1\rangle\langle 1|^{clock}_{t=0} + H_{clock}$$

$\uparrow$ make sure clock states are legal

$$\downarrow$$

$$H_{final} = H_{in} + \sum H_{prop} + H_{clock}$$

In the basis $\{|\gamma_0\rangle \ldots |\gamma_L\rangle\}$, this looks like

$$H_{initial} = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

$$\downarrow$$

$$H_{final} = \frac{1}{2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

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3. Eigenvalue Gap and Random Walk

(Cf. Spitzer's talk)

What is the gap of $H(s) = (1-s) H_{initial} + s H_{final}$?

Rough Idea :

Case I, $s \leq \frac{1}{3}$

Circle theorem :

$$\begin{bmatrix} A \cdots \\ \cdots B \cdots \end{bmatrix} \quad H_{ii} - \sum_{i \neq j} |H_{ij}| \leq \overset{\text{eigenvalue}}{E_i} \leq H_{ii} + \sum_{i \neq j} |H_{ij}|$$

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Since $H(s)$ close to $H_{int} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & \ddots \end{bmatrix}$

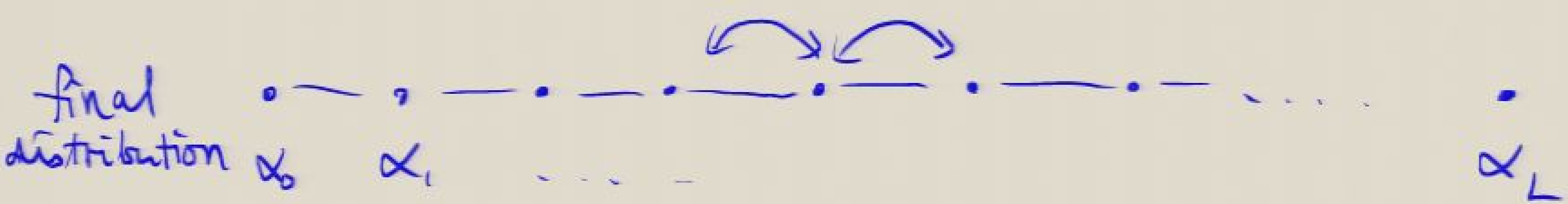
$$A + \cdots\cdots \leq \frac{1}{3} \qquad \Rightarrow \text{gap} \geq \frac{1}{3}$$
$$B - \cdots\cdots \geq \frac{2}{3}$$

Case II, $s > \frac{1}{3}$ (sketch only)

We want to use the property that the gap of stochastic matrix (positive entries, sum of row/column = 1) can be related to its conductance (expansion parameter).

Define $G = 4I - H(s)$ and $P_{ij} = \frac{\alpha_j}{\mu \alpha_i} G_{ij}$, then P is stochastic.

Random walk on the line $\gamma_0 \ldots \gamma_L$



Limiting distribution is monotonic : $\alpha_0 \geq \alpha_1 \ldots \geq \alpha_L$

$$\text{gap} \sim \text{conductance} = \min_B \left(\sum_{\substack{i \in B \\ j \notin B}} \pi_i P_{ij} \right) \Big/ \left(\sum_{i \in B} \pi_i \right) \leq \frac{1}{2}$$

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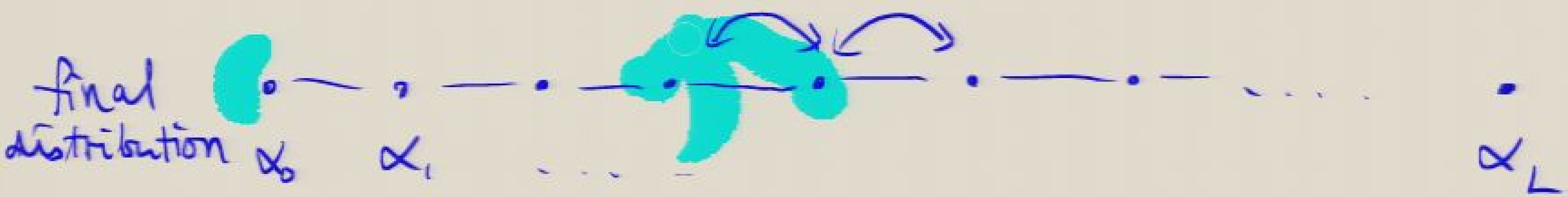

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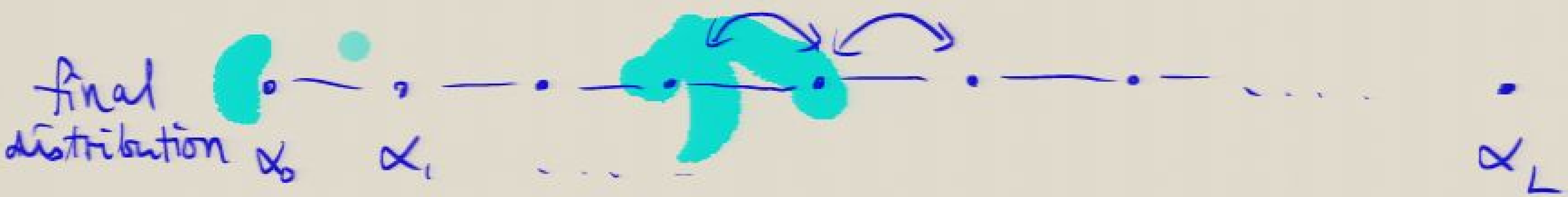

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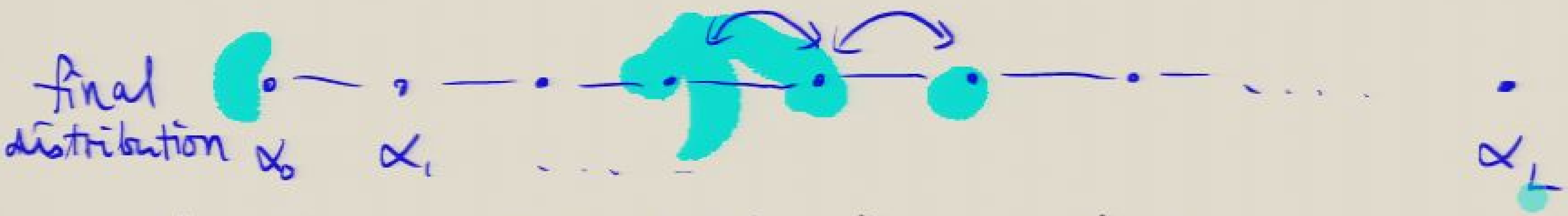

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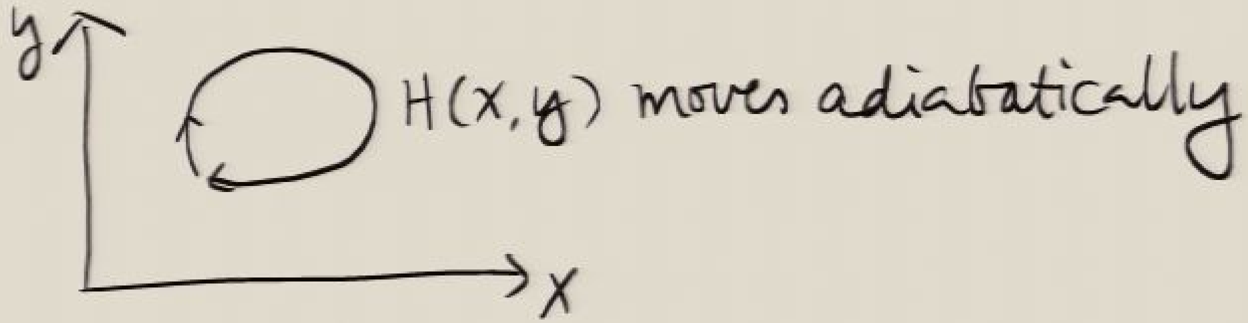
4. Connection to Holonomic QC

HQC :



Non-abelian Berry phase to implement U.

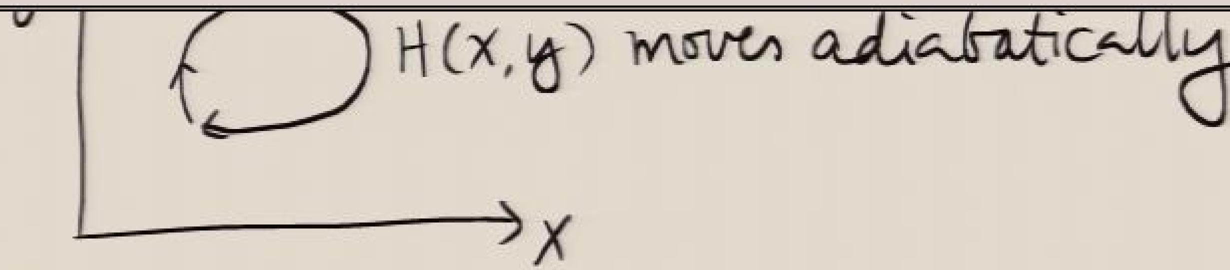
Degenerate eigenspace is like a gauge group.

What does it have to do with the history state?

$H = \cancel{H_{in}} + \sum H_{prop}$

↳ without H_{in} still works since it's always zero on the correct states.

But we could have started with any H_{in}.



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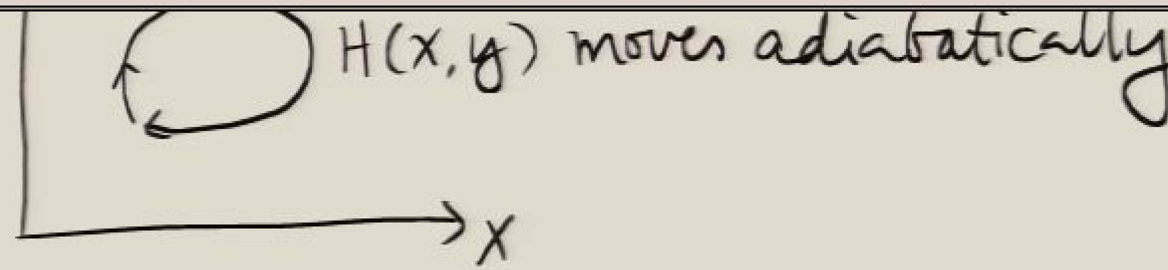

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Consider a one gate circuit mapped using the history approach. The initial and final states are:

$$|\psi_i\rangle = |0\rangle \otimes |0\rangle^{clock}$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle^{clock} + U|0\rangle \otimes |1\rangle^{clock} \right)$$

The corresponding Hamiltonians are

$$H_{initial} = |1\rangle\langle 1|^{clock}$$

$$H_{final} = \frac{1}{2}\left(I \otimes |0\rangle\langle 0|^{clock} + I \otimes |1\rangle\langle 1|^{clock} - U \otimes |1\rangle\langle 0|^{clock} + U^{\dagger} \otimes |0\rangle\langle 1|^{clock} \right)$$

Note that the same Hamiltonians could have taken

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Idea:

$$H(t) = \frac{1}{2}\Big(P(t) \otimes |0\rangle\langle 0|^{clock} + Q(t)|1\rangle\langle 1|^{clock} - R(t)\, U \otimes |1\rangle\langle 0|^{clock} - R(t)\, U^{+} \otimes |0\rangle\langle 1|^{clock} \Big)$$

The ground state $\propto R(t)|0\rangle|0\rangle^{clock} + P(t)\, U|0\rangle|1\rangle^{clock}$

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This works for multiple gates as well.

Let $|t\rangle^c$ denote the clock states at time step t,

i.e. $|\gamma_t\rangle = U_t \dots U_1 |0\rangle \otimes |t\rangle^c$

Suppose $H(s) = (1-s)H_i + sH_f$ takes

$$|\psi_i\rangle = |0\rangle \otimes |0\rangle^c$$

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Up to relabelling of the clock, $|\psi_f'\rangle = |\psi_f\rangle$

$\Rightarrow$ If we implement the evolution

$$H_i \rightarrow H_f \,(H_f') \longrightarrow H_i'$$

We can take $|0\rangle$ to $U_L U_{L-1} \ldots |0\rangle$, exactly the desired output.

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Lesson: Adiabatic QC using history state is in fact "half" of Holonomic QC. This also tells us

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Lesson: Adiabatic QC using history state is in fact "half" of Holonomic QC. This also tells us there is much room for manipulation of gap size.

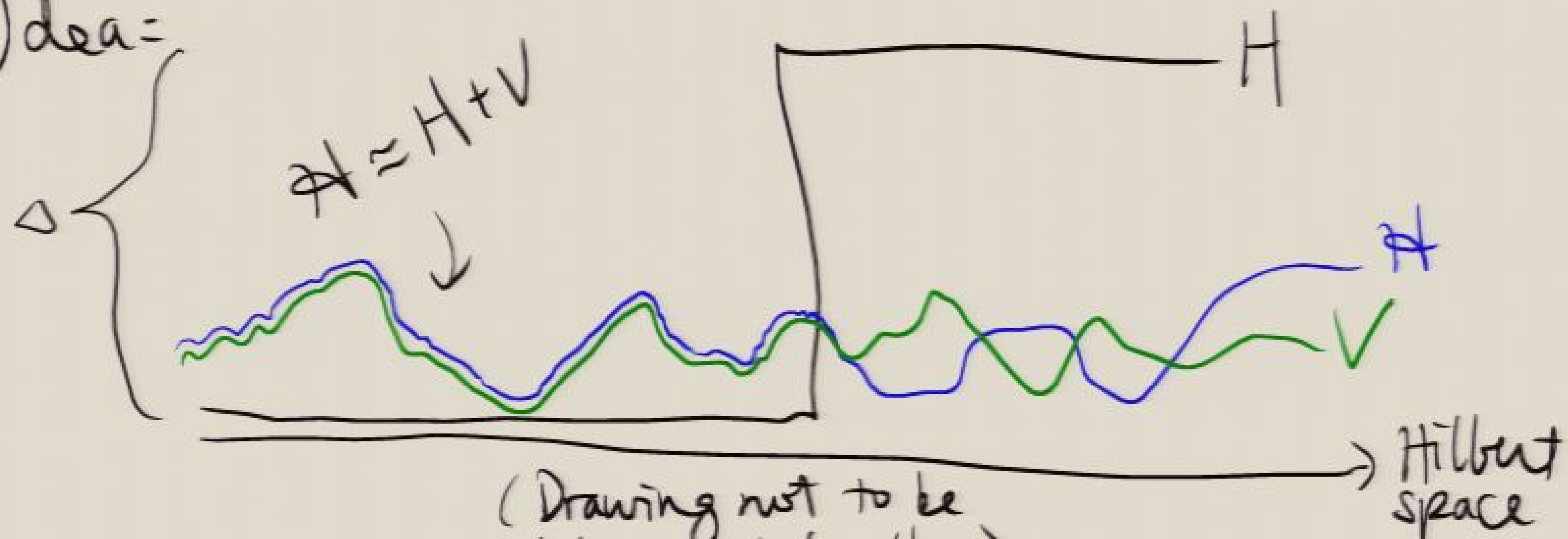
5. Perturbation Theory Gadgets

Interactions in Nature tends to be between nearest neighbors, i.e. they are 2-local and fit in a lattice. Can we implement AQC/HQC using 2-local Hamiltonian?

Yes, ∃ approximations that reduces nonlocality and preserves approximately both eigenvalues & eigenvectors.

Price we pay = Gap reduced by poly(n) factor

Basic Idea:



(Drawing not to be taken too literally)

Using ancillary qubits, we can often replace k-local Hamiltonian $\mathcal{H}$ by less-than-k-local Hamiltonians $H+V$, where H has a large gap Δ and ground state space $\mathcal{L}_-$ (write total Hilbert space $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$) ∋. $V|_{\mathcal{L}_-} \simeq \mathcal{H}$, Theorem (roughly stated):

Let spectrum of $\mathcal{H} \in [a, b]$. $\Delta/2 > a, b$

For $z \in [a, b]$, if $\Sigma_{--}(z)$ (defined below) is close to $\mathcal{H}$, low energy eigenvalues and eigenvectors of $H+V$ are close to $\mathcal{H}$.

$$\Sigma_{--} \equiv H_{--} + V_{--} + V_{-+} G_{++} V_{+-} + V_{-+} G_{++} V_{++} G_{++} V_{+-} + \ldots$$

where "$+-$" denotes projections to $\mathcal{L}_+$, $\mathcal{L}_-$ etc.

$$G(z) = \frac{1}{z - H}$$

Remark: Why Σ_{--} and not just $H_{--} + V_{--}$?

Because $\mathcal{L}_-$, $\mathcal{L}_+$ are eigenspaces of H,

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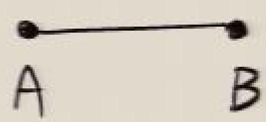
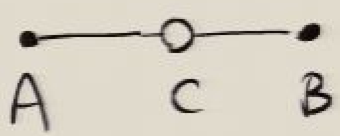
Because $\mathcal{L}_-$, $\mathcal{L}_+$ are eigenspaces of H, not V

Simple example of a gadget



$\mathcal{H}$ acts on AB

$\downarrow$ approximation



$H+V$, H acts on C, V acts on AC and CB

$$\mathcal{H} = -(P_A + P_B)^2/2 \quad \left(\text{meaning } -(P_A \otimes I_B + I_A \otimes P_B)^2/2 \right)$$

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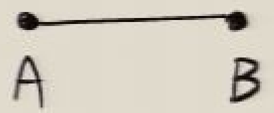

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$$H = \Delta |1\rangle\langle 1|_c \quad (\text{on qubit } c)$$

$$V = \sqrt{\frac{\Delta}{2}}\,(P_A + P_B) \otimes \left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)_c$$

From these we get

$$G_{++}(z) = \frac{|1\rangle\langle 1|_c}{z - \Delta}, \quad H_{--} + V_{--} = 0,$$

$$V_{+-} = \sqrt{\frac{\Delta}{2}}\,(P_A + P_B) \otimes |1\rangle\langle 0|_c$$

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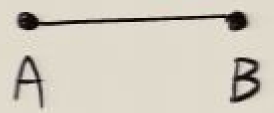

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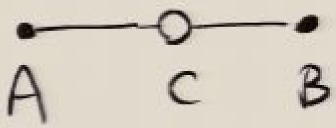

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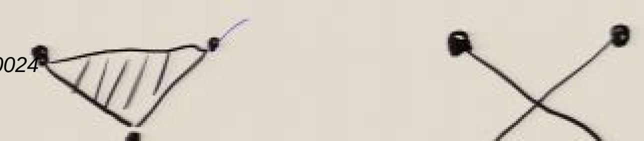
$$\Sigma_{--} = \frac{\Delta}{2(z-\Delta)}(P_A + P_B)^2 \otimes |0\rangle\langle 0|_c + O(\Delta^{-\frac{1}{2}})$$

we can choose Δ such that Σ_{--} is close to $\mathcal{H}$ on A, B.

Examples of other gadgets (changing locality/topology)



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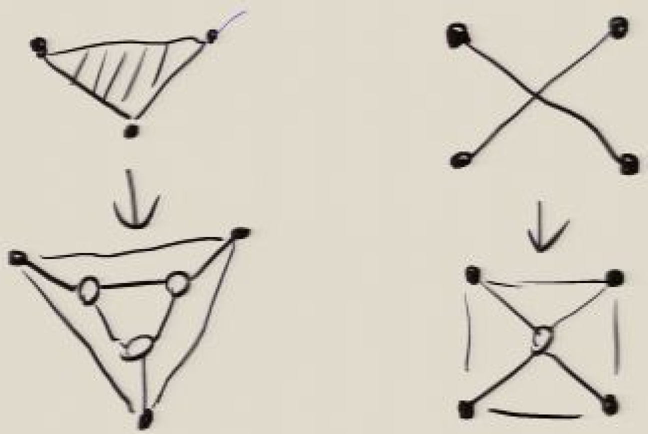

Gadgets can be used on the AQC/HQC

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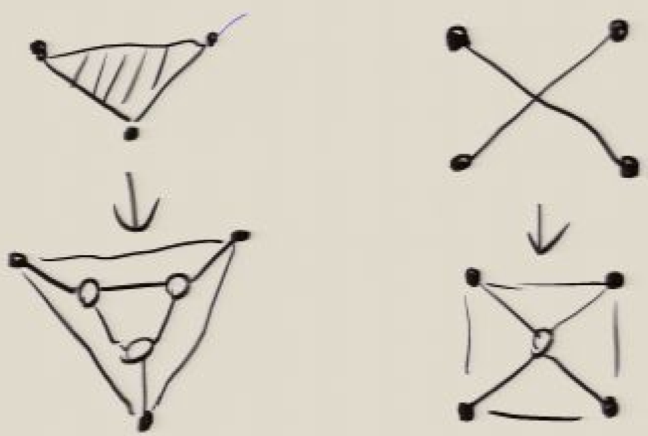

Gadgets can be used on the AQC/HQC Hamiltonians so that they become 2-local

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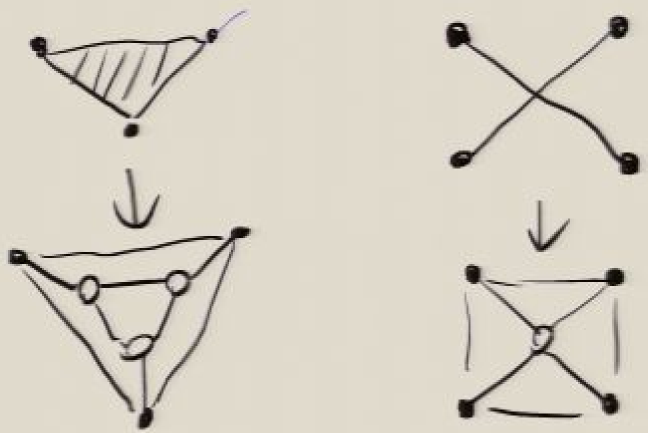
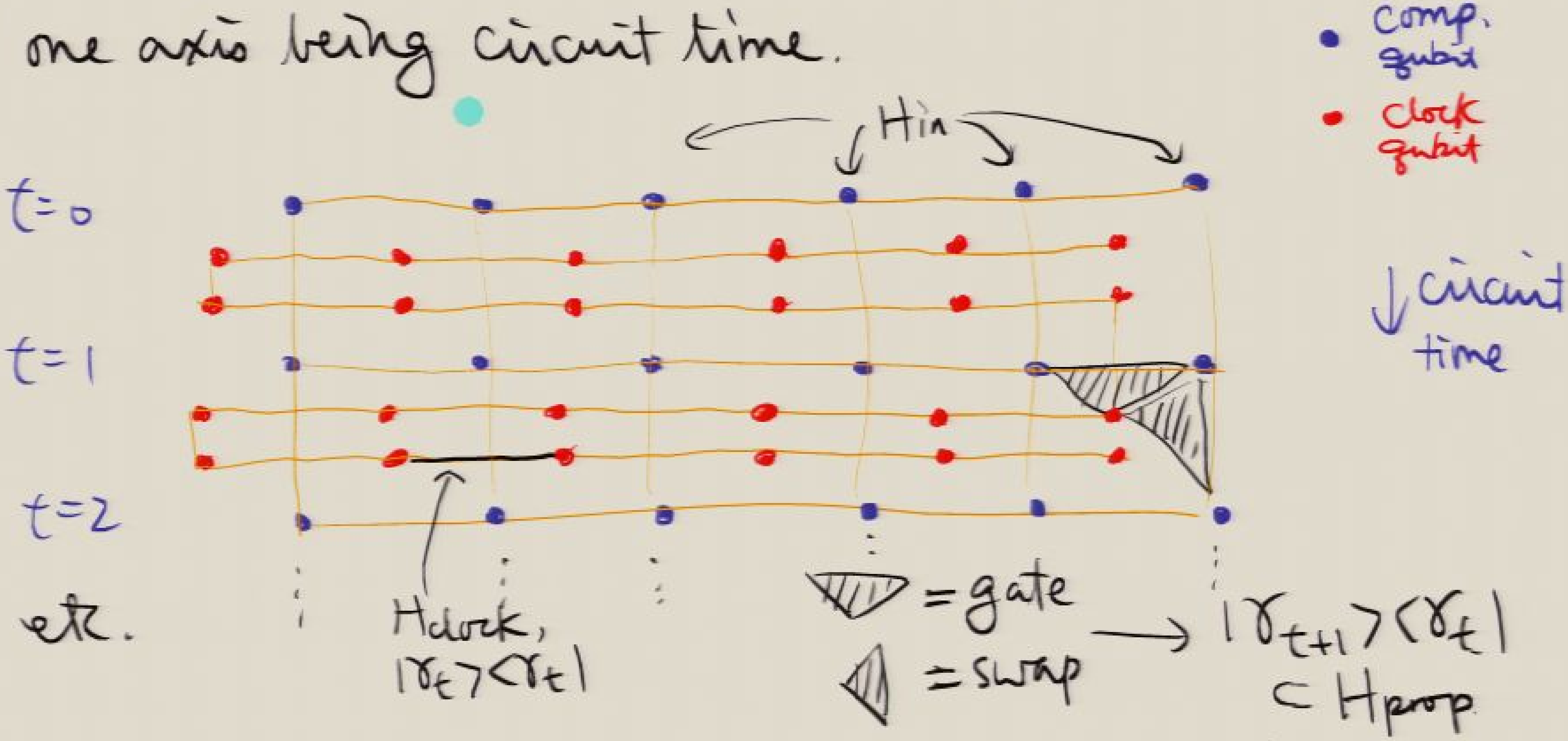


Gadgets can be used on the AQC / HQC Hamiltonians so that they become 2-local

Square lattice construction (Oliveira & Terhal)

$H_f = H_{in} + \sum H_{prop}$ etc. can be implemented 3-locally on a square lattice of spins (qubits), w/ one axis being circuit time.



Gadgets used on this can reduce to 2-local neighboring interaction

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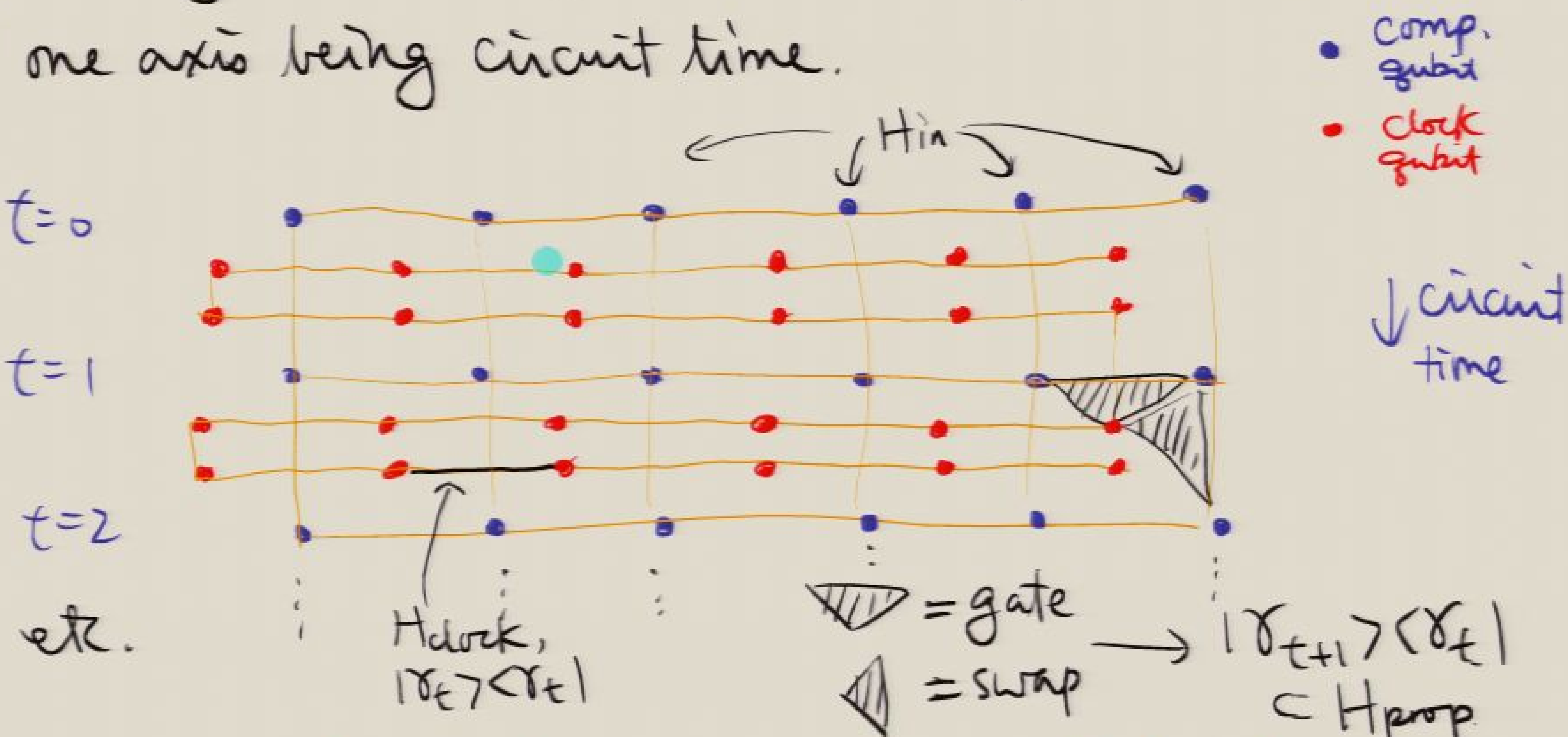

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Adiabatic Algorithm:

$$H_{initial} = H_{in} + \underbrace{1\!\!1 \otimes |\,\rangle\langle\,|^{clock}_{t=0} \ldots}_{t\neq 0} + H_{clock}$$

make sure clock states are legal

$$\downarrow$$

$$H_{final} = H_{in} + \sum H_{prop} + H_{clock}$$

In the basis $\{|\gamma_0\rangle \ldots |\gamma_L\rangle\}$, this looks like

$$H_{initial} = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

$$\downarrow$$

$$H_{final} = \frac{1}{2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

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★ The key to making the Hamiltonian k-local, where k is small, is finding good k-local approx. to the terms $|\gamma_{t+1}\rangle\langle\gamma_t|$ etc.

unary clock:
$|t\rangle^{clock} = |\underbrace{111\ldots}_{t}000\ldots\rangle$

3-local example:

$$|\gamma_{t+1}\rangle\langle\gamma_t| \approx U_{t+1} \otimes |1\rangle^{clock}_{t+1}\langle 0|^{clock}_{t+1} + J\,|01\rangle^{clock}_{t,t+1}\langle 01|^{clock}_{t,t+1}$$

for large J

What is the gap of $H_{in} + \sum H_{prop}$?

Both H_{in} and $\sum H_{prop}$ have very simple spectra

if both have gaps $\geq \lambda$, then

gap of $H_{in} + \sum H_{prop}$

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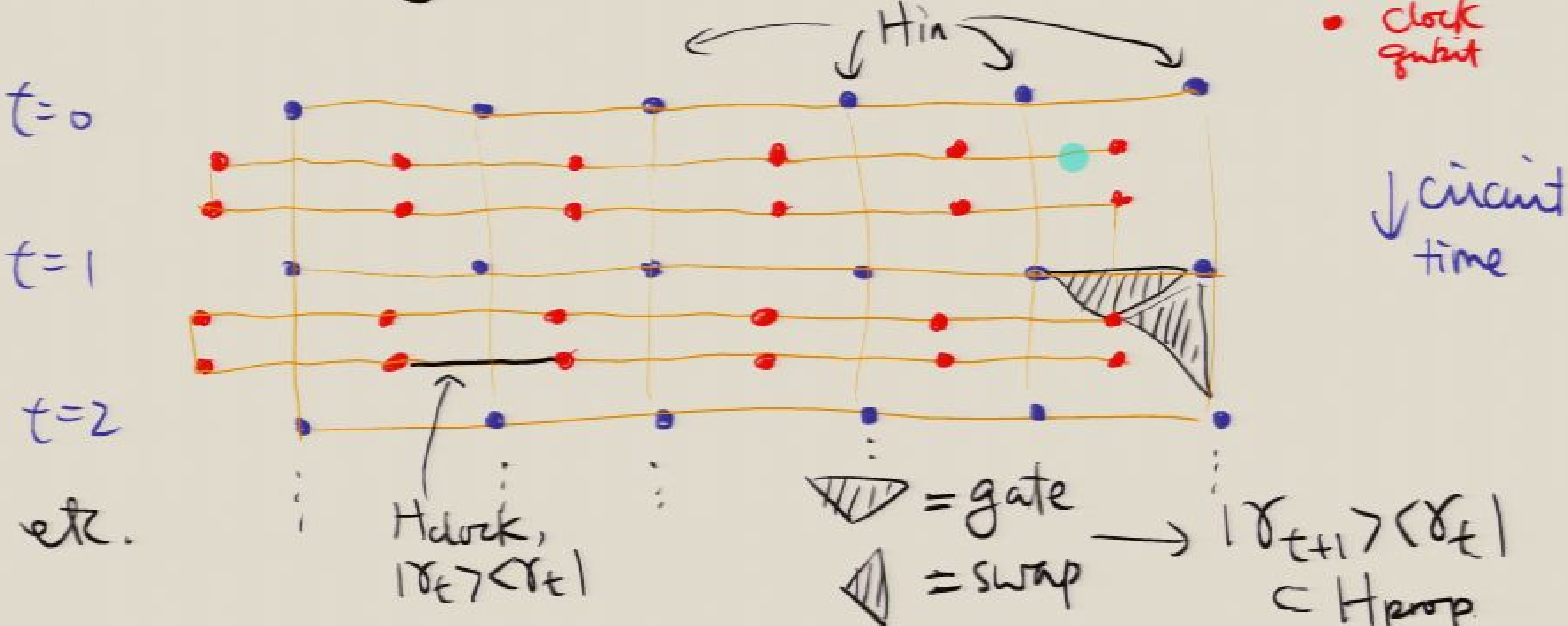

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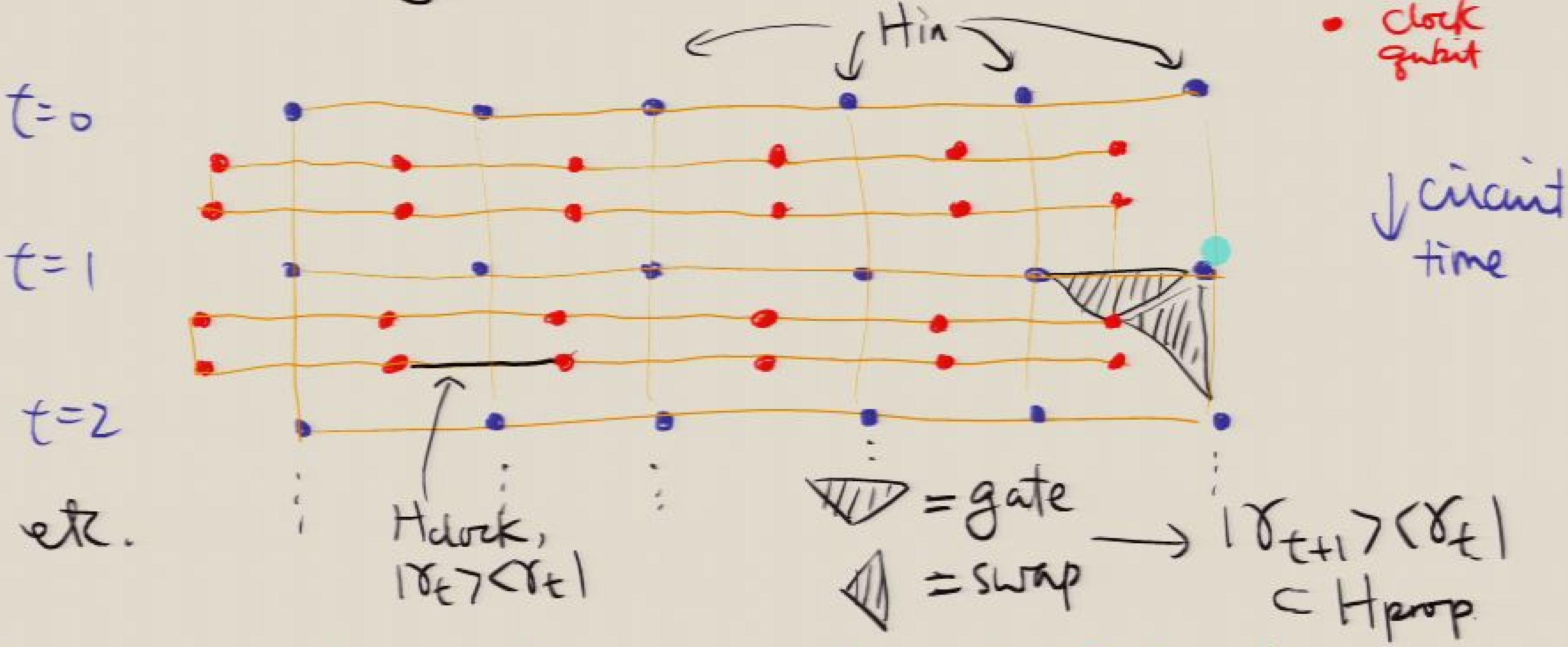

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