Title: More on adiabatic quantum computation - Part 2

Date: Feb 11, 2006 09:50 AM

URL: http://pirsa.org/06020023

Abstract:

Grover $f(z) = \begin{cases} \\ -1 & z = w \end{cases}$

$0 \leq z \leq N-1$

Find w

Classically N calls

Q.M. $N^{1/2}$ calls $\leftarrow$ Best possible

$$H_p = H_w = E\left(1 - |w\rangle\langle w|\right)$$

Can apply H_p but do not know w

$$H(t) = H_0(t) + H_w$$

$\uparrow$ You design this

Want $|\psi(T)\rangle = |w\rangle$

$$T \geq \frac{\sqrt{N}}{2E}$$

E.F.
S. Gutmann

- <u>Proof</u> Use $H_w = E|w\rangle\langle w|$

$$H = H_D(t) + H_w$$

$t=0$ $\quad |\text{initial}\rangle$

$t=T$ $\quad |w\rangle$ want

$$i\frac{d}{dt}|\psi_w, t\rangle = [H_D(t) + H_w]\,|\psi_w, t\rangle$$

$$|\psi_w, 0\rangle = |\text{initial}\rangle \,;\; |\psi_w, T\rangle = |w\rangle$$

- Reference state $|\psi, t\rangle$ evolves with just $H_D(t)$

$$i\frac{d}{dt}|\psi, t\rangle = H_D(t)\,|\psi, t\rangle$$

$$|\psi, 0\rangle = |\text{initial}\rangle$$

$$\sum_w \big|\, |\psi_w, T\rangle - |\psi, T\rangle \big|^2 = \sum_w \big|\, |w\rangle - |\psi, T\rangle \big|^2$$

$$= 2N - \sum_w \left[\langle w|\psi, T\rangle + \langle \psi, T|w\rangle\right]$$

$$\geq N$$

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

$$|\psi(T)\rangle$$

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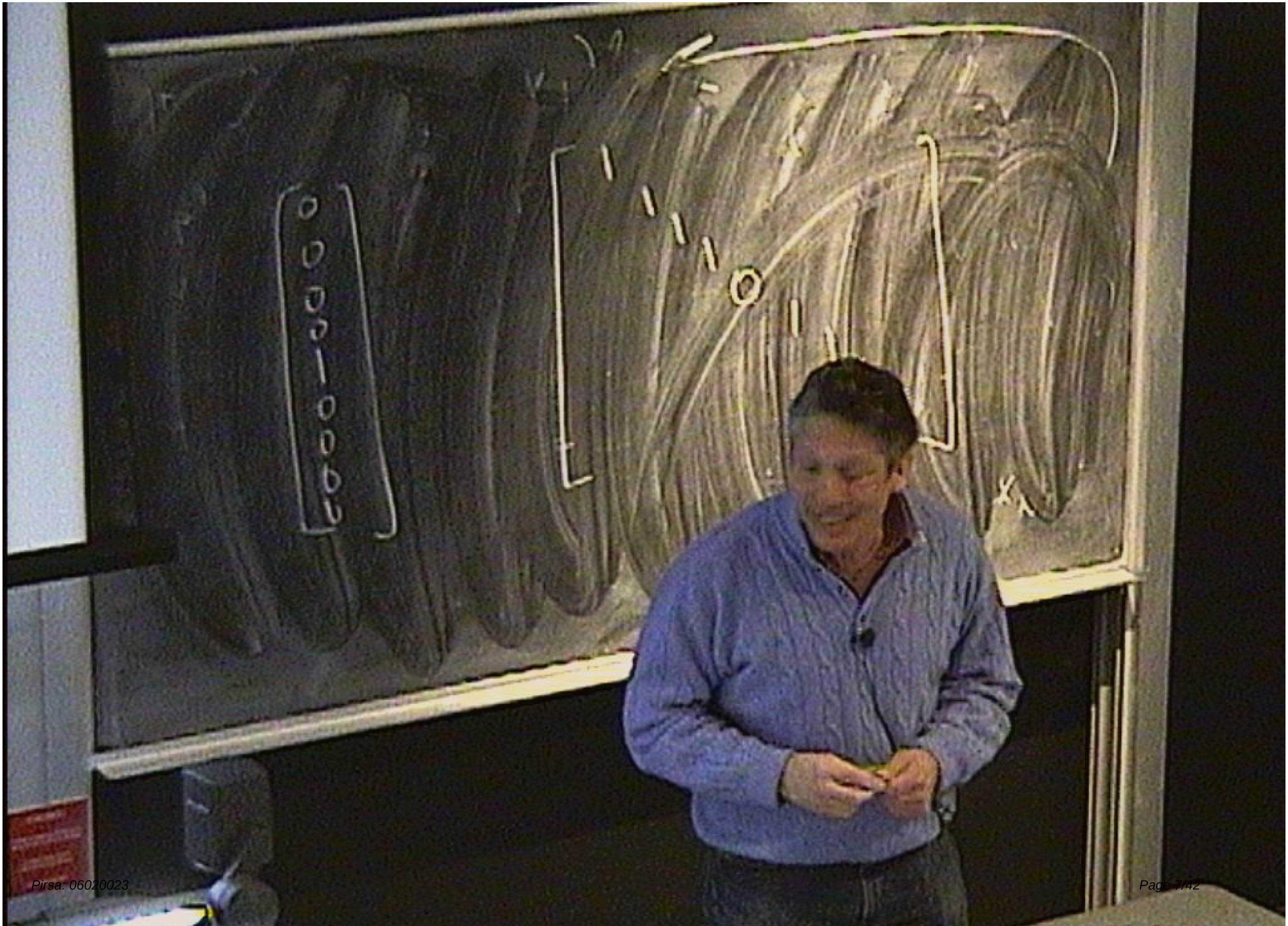

1.

Proof Use $H_\omega = E|\omega\rangle\langle\omega|$

$$H = H_D(t) + H_\omega$$

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$t = T$ $\quad |\omega\rangle$ want

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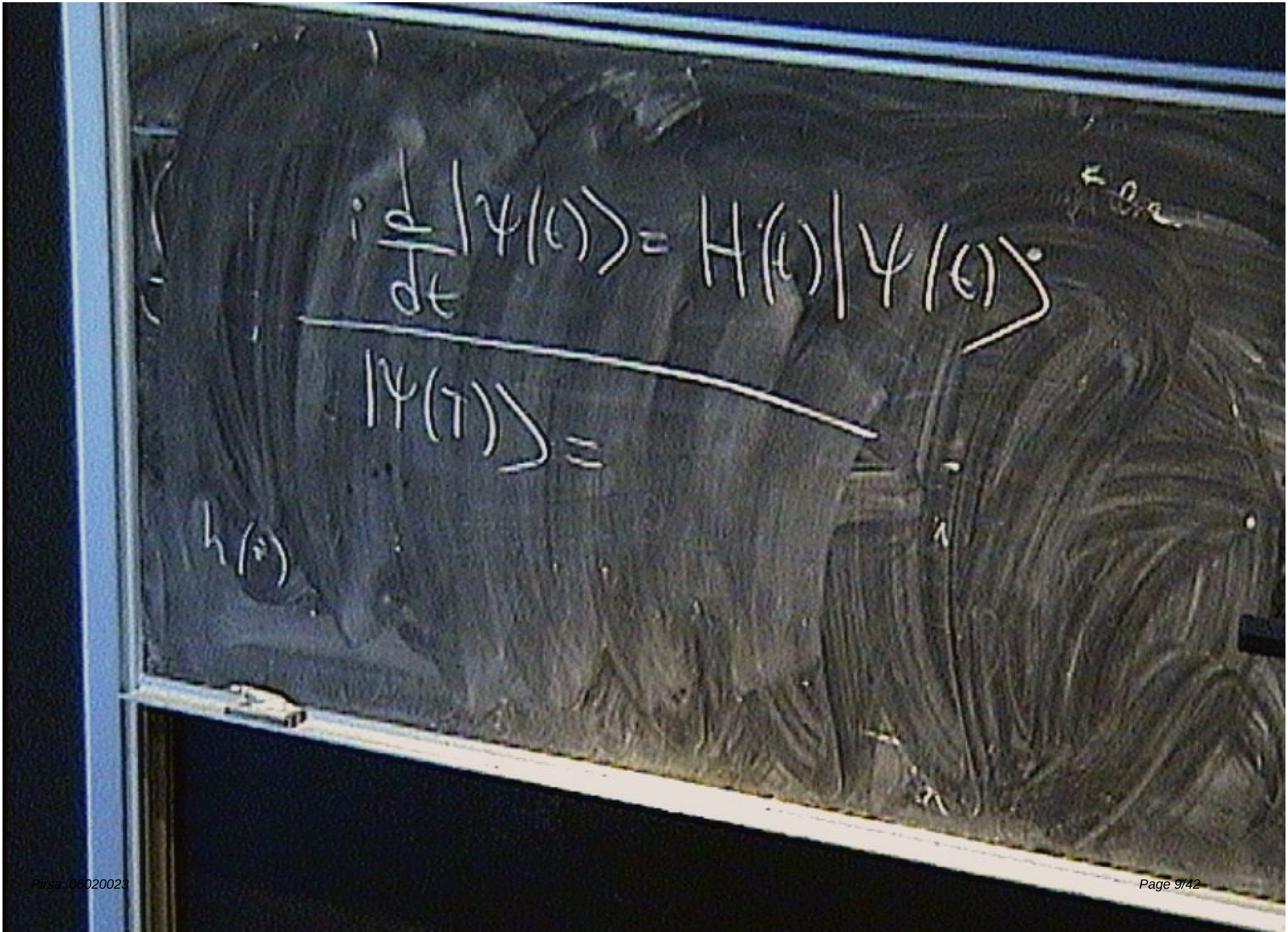
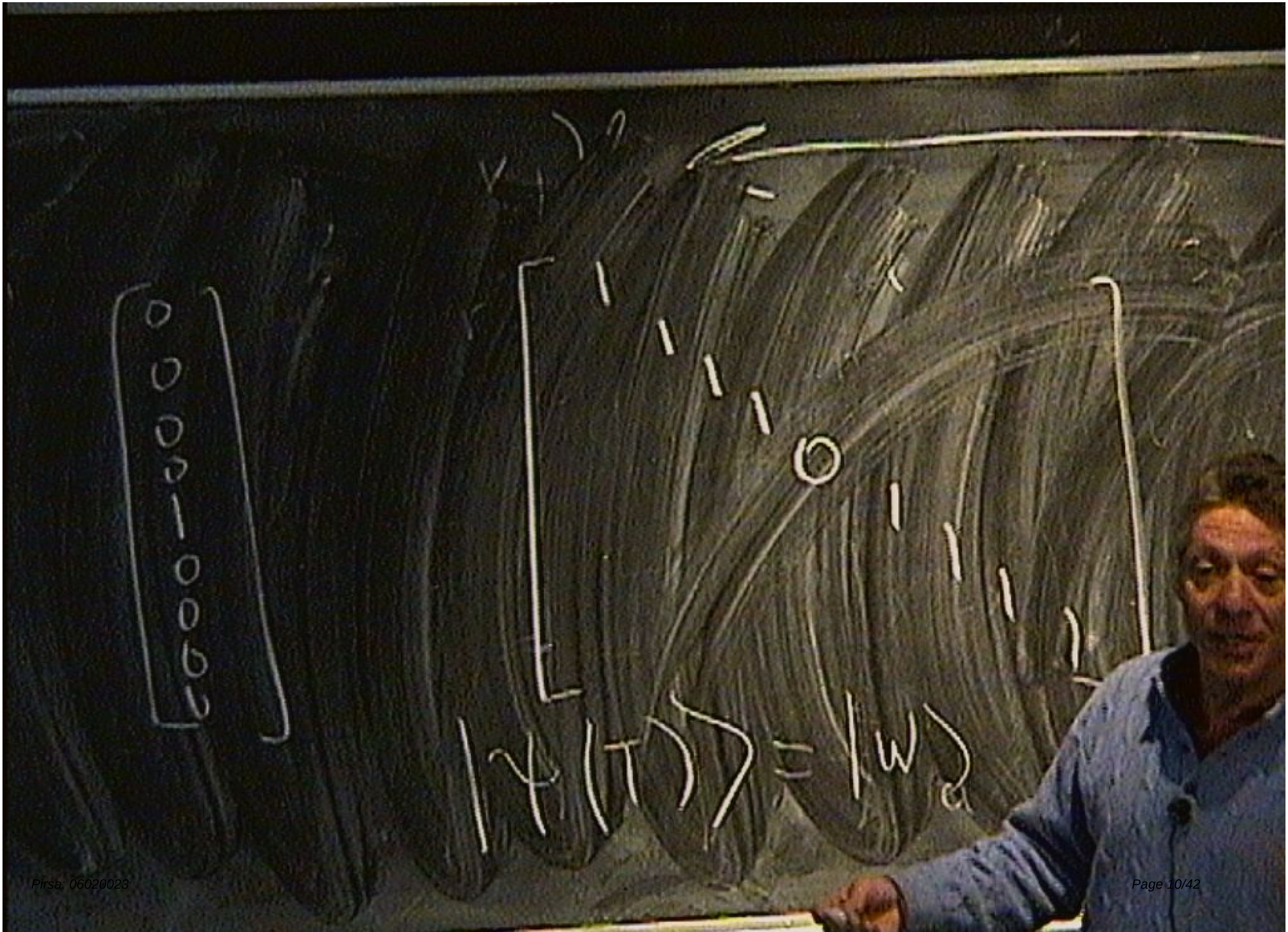


$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$
$|\psi(T)\rangle =$
$h(t)$


|ψ(t)⟩ = |w⟩

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$$\geq N$$

$$\frac{d}{dt}\left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 = -2\,\mathrm{Re}\,\frac{d}{dt}\langle\psi_\omega,t|\psi,t\rangle$$

$$= 2\,\mathrm{Im}\,\langle\psi_\omega,t|H_\omega|\psi,t\rangle$$

$$\leq 2\left|\,H_\omega|\psi,t\rangle\right|$$

Now $\sum_\omega$: $\sum_\omega E|\omega\rangle\langle\omega| = E$

$$\frac{d}{dt}\sum_\omega\left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 \leq 2EN^{1/2}$$

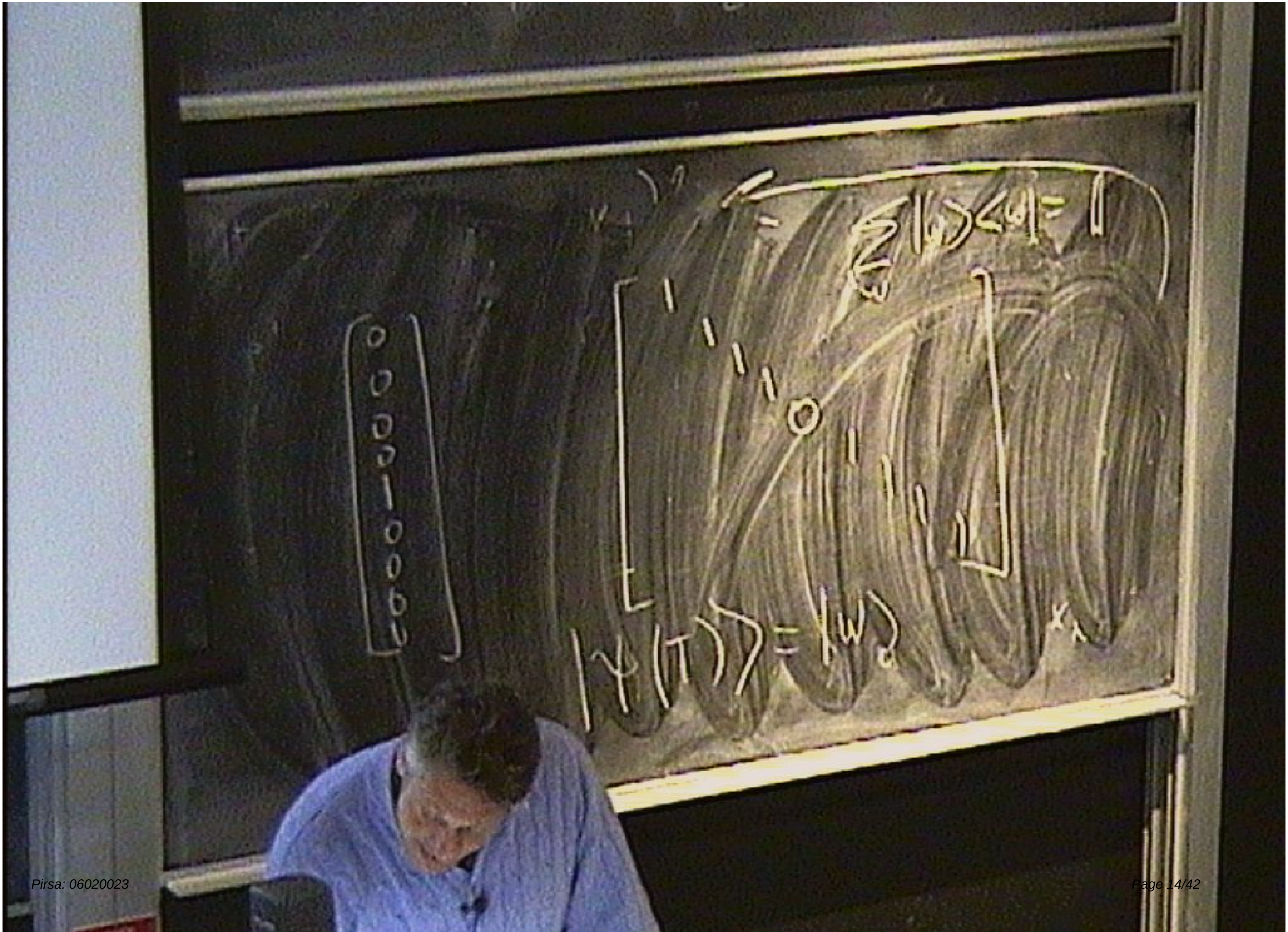
Now $|\psi_\omega,0\rangle = |\psi,0\rangle$

$$\sum_\omega\left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 \leq 2EN^{1/2}t$$

for $t = T$

$$N \leq 2EN^{1/2}T$$

$$\boxed{\frac{N^{1/2}}{2E} \leq T}$$


$\sum_{\omega} |\omega\rangle\langle\omega| = I$
$|\psi(T)\rangle = |\omega\rangle$

- $$\frac{d}{dt}\left|\, |\psi_\omega, t\rangle - |\psi_1, t\rangle \right|^2 = -2\,\mathrm{Re}\,\frac{d}{dt}\langle \psi_\omega, t | \psi_1, t\rangle$$

$$= 2\,\mathrm{Im}\,\langle \psi_\omega, t | H_\omega | \psi_1, t\rangle$$

$$\leq 2\left|\, H_\omega |\psi_1, t\rangle \right|$$

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5.

$$\frac{d}{dt}\left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 = -2\,\mathrm{Re}\frac{d}{dt}\langle\psi_\omega,t|\psi,t\rangle$$

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$$\frac{d}{dt}\sum_\omega \left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 \leq 2E\,N^{1/2}$$

Now $|\psi_\omega,0\rangle = |\psi,0\rangle$

$$\sum_\omega \left|\,|\psi_\omega,t\rangle - |\psi,t\rangle\right|^2 \leq 2E\,N^{1/2}\,t$$

for $t = T$

$$N \leq 2E\,N^{1/2}\,T$$

$$\boxed{\frac{N^{1/2}}{2E} \leq T}$$

$$\frac{d}{dt}\left|\,|\psi_\omega,t\rangle - |\psi_1 t\rangle\right|^2 = -2\,\mathrm{Re}\frac{d}{dt}\langle\psi_\omega,t|\psi_1 t\rangle$$

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General Search Starting with a One-Dimensional Projector:

$$H_P = \sum_z h(z)\,|z\rangle$$

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle$$

$$H_B = E\left(1 - |s\rangle\langle s|\right)$$

$$|\psi(0)\rangle = |s\rangle$$

Theorem: Let H_P be diagonal in the z-basis with a ground state subspace of dimension K. Let

$$H(t) = \left(1 - \frac{t}{T}\right) E\left(1 - |s\rangle\langle s|\right) + \frac{t}{T} H_P$$

Let P be the projector onto the ground state subspace of H_P and let $b>0$ be the success probability

$$b = \langle\psi(T)|P|\psi(T)\rangle$$

Then

$$T \geq \frac{b}{E}\sqrt{\frac{N}{K}} - \frac{2\sqrt{b}}{E}$$

Simple example

Decoupled problem

For each bit j $\quad h_j(z_j) = \begin{cases} 0 & z_j = 0 \\ 1 & z_j = 1 \end{cases}$

$$h(z) = z_1 + z_2 \dots + z_n$$

Good: $H_B = \sum_j \frac{1}{2}\left(1 - \sigma_x^{(j)}\right)$

Ground state of H_B is $|s\rangle$

Qubits rotate to winner. $\quad T \sim \sqrt{n}$

Bad $\quad H_B = E\left(1 - |s\rangle\langle s|\right)$

T must be greater than $2^{n/2}$

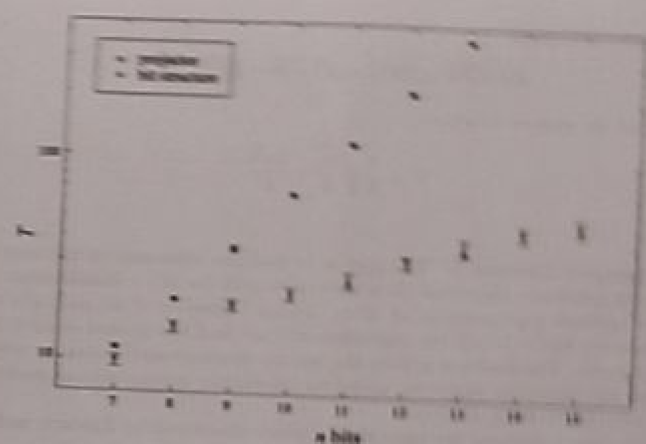


FIG. 1: Median required run time T versus bit number. At each bit number there are 50 random instances of Exact Cover with a single satisfying assignment. We choose the required run time to be the value of T for which quantum adiabatic algorithm has success probability between 0.2 and 0.21. For the projector beginning Hamiltonian we use (8) with $E = n/2$. The plot is log-linear. The error bars show the 95% confidence interval for the true medians.

The total cost function is then

$$h(z) = \sum_c h_c(z).$$

To get H_B to reflect the bit and clause structure we may pick

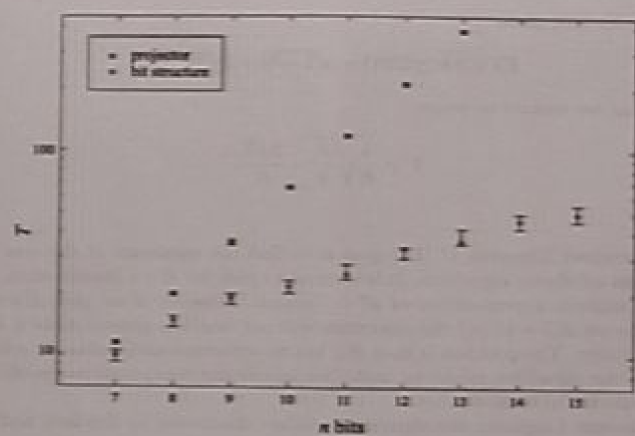
$$H_{B,c} = \frac{1}{2}\left[(1-\sigma_x^{(i_c)}) + (1-\sigma_x^{(j_c)}) + (1-\sigma_x^{(k_c)})\right]$$

with

$$H_B = \sum_c H_{B,c}. \tag{14}$$

In this case the ground state of H_B is again $|s\rangle$. With this setup, Theorem 1 does not apply.

We did a numerical study of a particular satisfiability problem, Exact Cover. For this problem if clause c involves bits i_c, j_c and k_c, the cost function is



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$$h_c(z) = \begin{cases} 0 & \text{if } z_{i_c} + z_{j_c} + z_{k_c} = 1, \end{cases}$$

Simple example

Decoupled problem

For each bit j $h_j(z_j) = \begin{cases} 0 & z_j = 0 \\ 1 & z_j = 1 \end{cases}$

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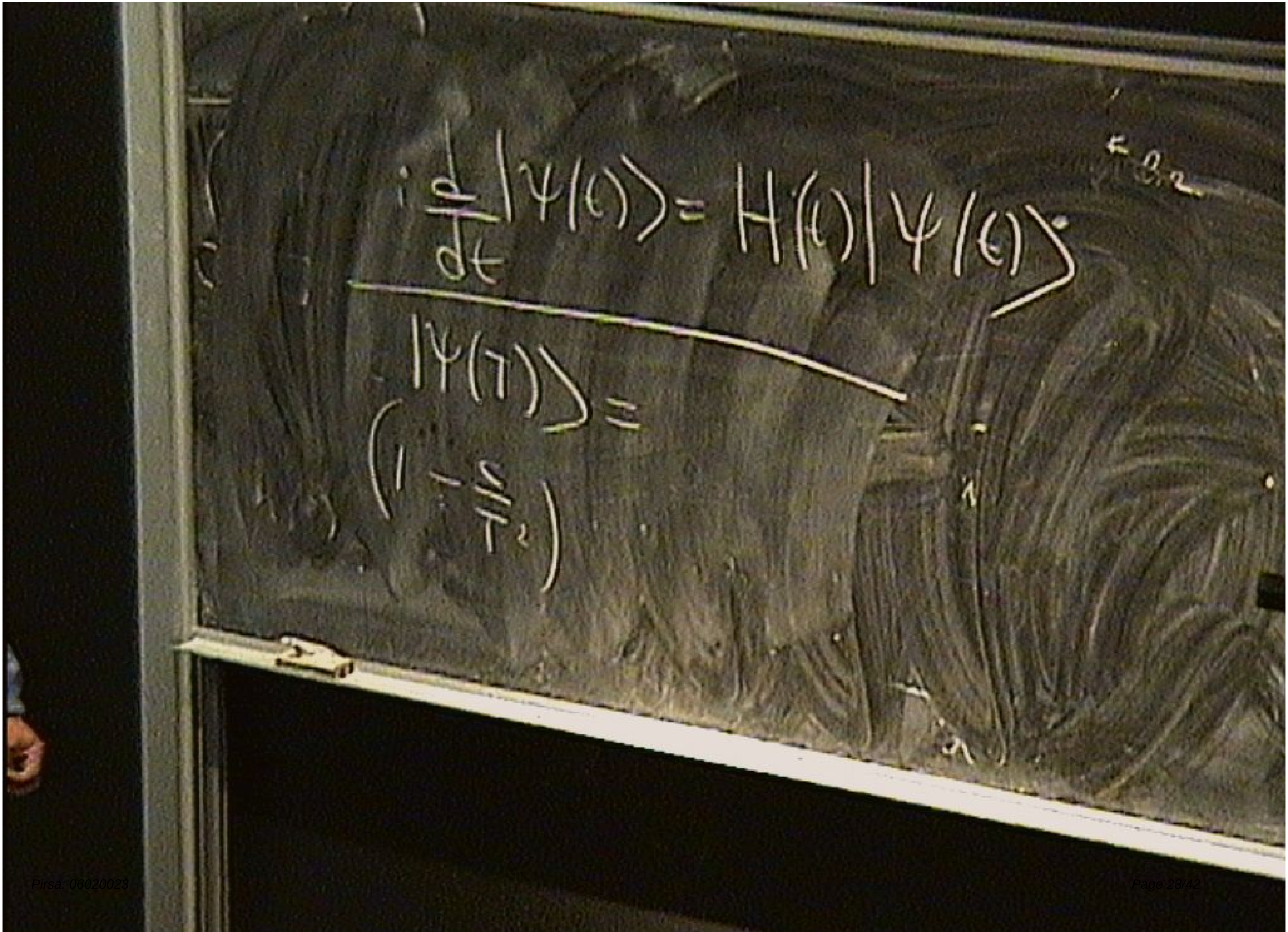
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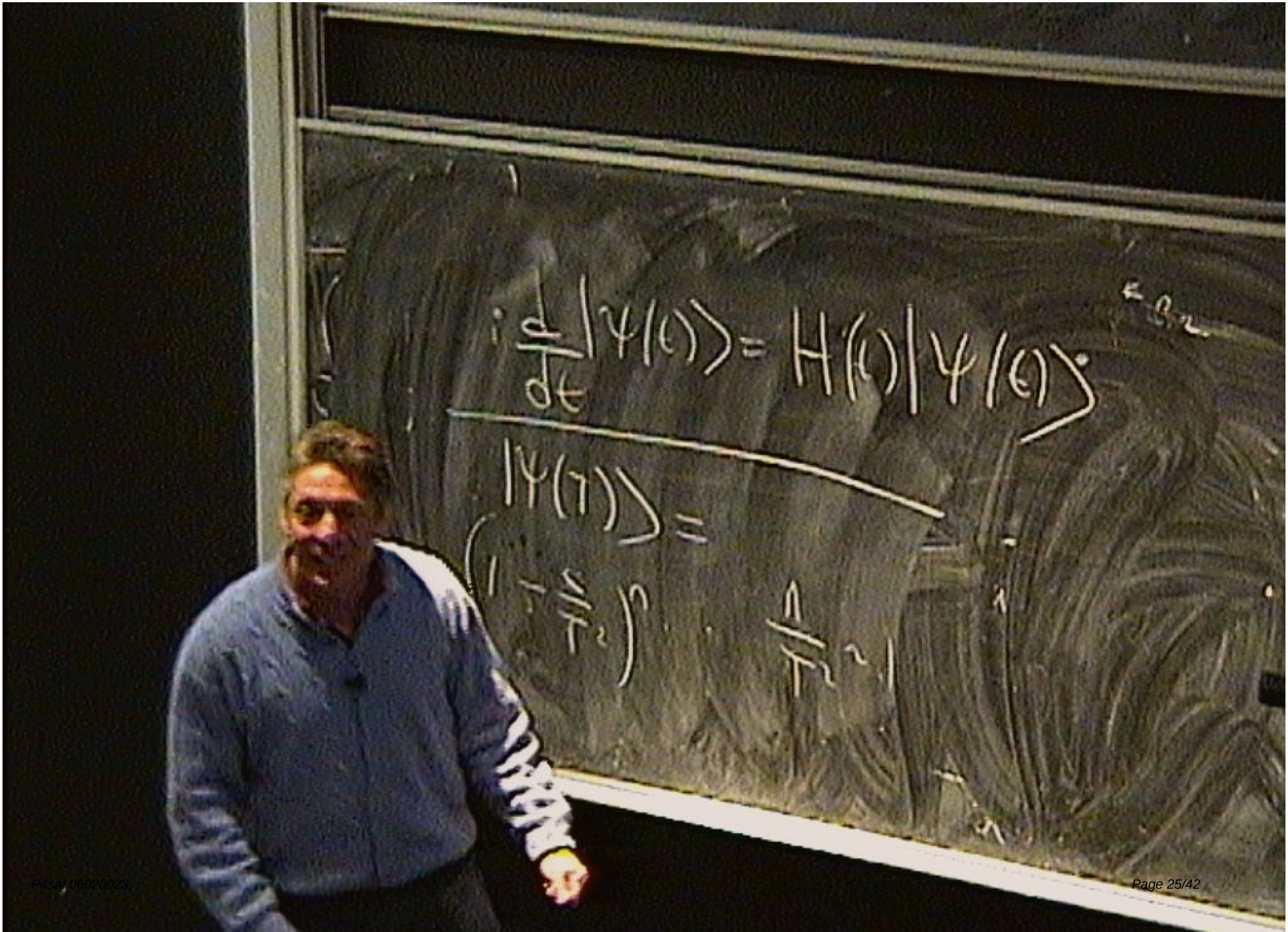

i d/dt |ψ(t)⟩ = H(t)|ψ(t)⟩
|ψ(T)⟩ =

8.

- Search with a Scrambled Problem Hamiltonian:

  Cost function $h(z)$: minimum at $z=0$

Let π be a permutation of $0, 1, \ldots, N-1$ (not of the bits!)

$$h^{[\pi]}(z) = h(\pi^{-1}(z)) \quad : \text{minimum at } \pi(0)$$

- $$H_{P,\pi} = \sum_{z=0}^{N-1} h^{[\pi]}(z)|z\rangle\langle z| = \sum_{z=0}^{N-1} h(z)|\pi(z)\rangle\langle\pi(z)|$$

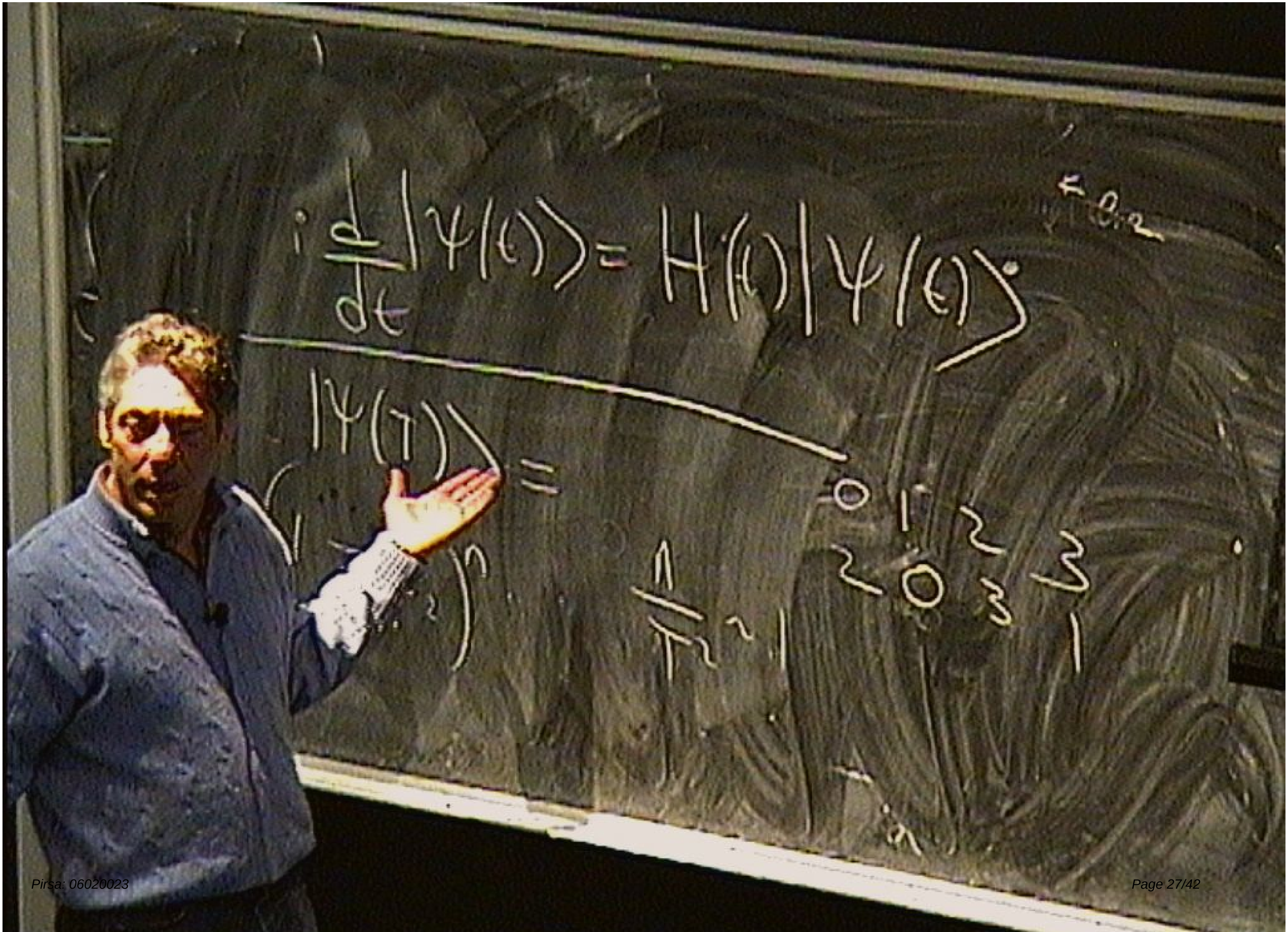
Let $H_\pi(t) = H_0(t) + c(t)\, H_{P,\pi}$

$|c(t)| \leq 1$ for all t

Evolve from a π independent state for time T.

- $$i\frac{d}{dt}|\psi_\pi(t)\rangle = [H_0(t) + c(t) H_{P,\pi}]|\psi_\pi(t)\rangle$$

Want $|\psi_\pi(T)\rangle$ near $|\pi(0)\rangle$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- Theorem: Suppose a continuous time algorithm of the form on the previous page succeeds with a probability b, $|\langle \psi_\pi(T) | \pi(0) \rangle|^2 \geq b$ for a set of $\epsilon N!$ permutations. Then

$$T \geq \frac{\epsilon^2 b}{16 h^*} \sqrt{N-1} - \frac{\epsilon \sqrt{\epsilon/2}}{4 h^*}$$

- with $h^* = \left(\frac{1}{N-1} \sum_z h^2(z) \right)^{1/2}$

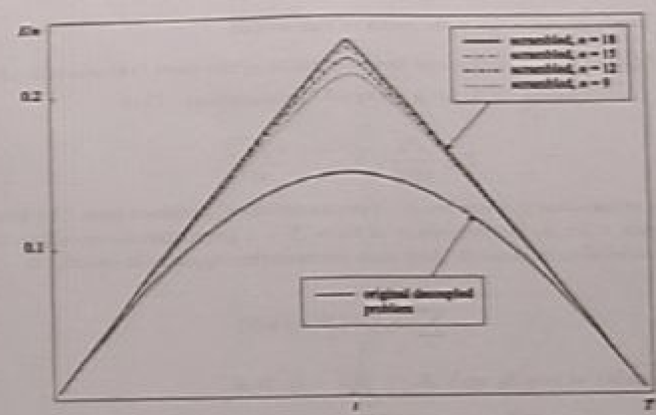


FIG. 2: The scaled ground state energy E/n for a quantum adiabatic algorithm Hamiltonian of a decoupled problem. The lowest curve corresponds to the original decoupled problem. The upper "triangular" curves correspond to single instances of the n-bit decoupled problem, where the problem Hamiltonian was scrambled.

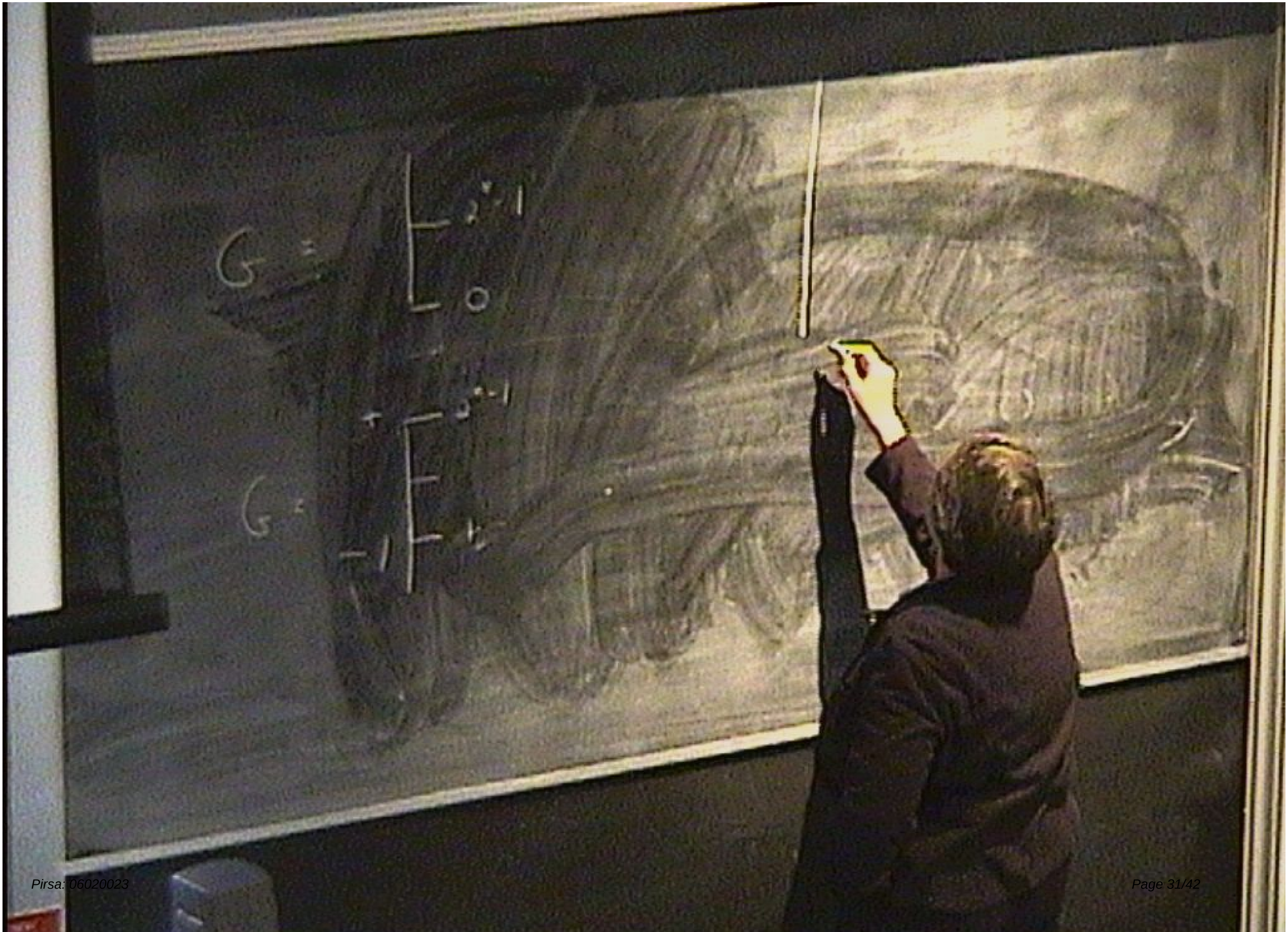
(Since the system is decoupled this is actually the ground state energy of a single qubit.) We then consider the n bit scrambled problem for different values of n. At each n we pick a single random permutation π of $0, \ldots, (2^n - 1)$ and apply it to obtain a cost function $h(\pi^{-1}(z))$ while keeping H_B fixed. The ground state energy divided by n is now plotted for $n = 9, 12, 15$ and 18. From these scrambled problems it is clear that if we let n get large the typical curves will approach a triangle with a discontinuous first derivative at $t = T/2$. For large n, the ground state changes dramatically as t passes through $T/2$. In order to keep the quantum system in the ground state we need to go very slowly near $t = T/2$ and this results in a long required run time.

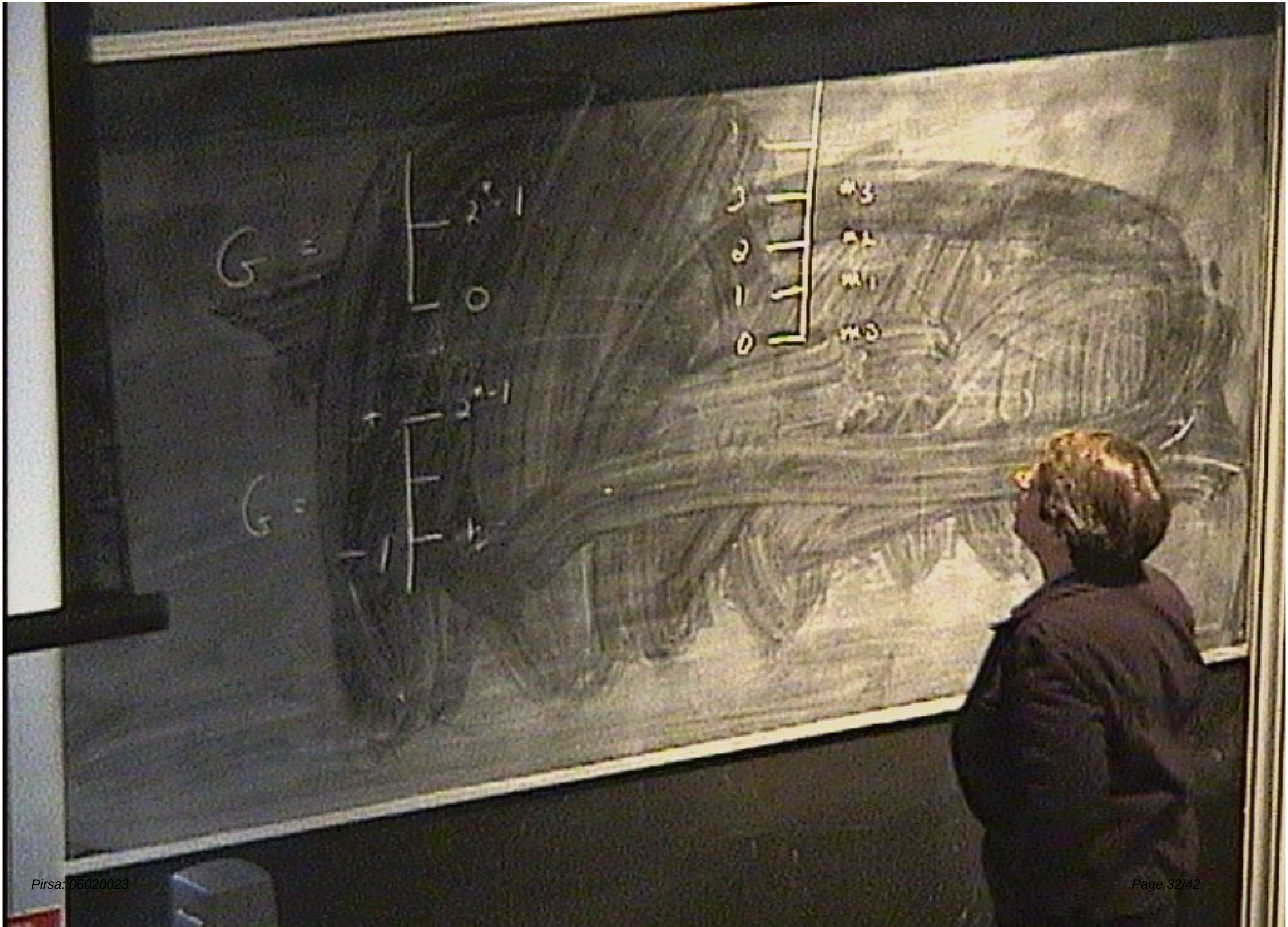
IV. CONCLUSIONS

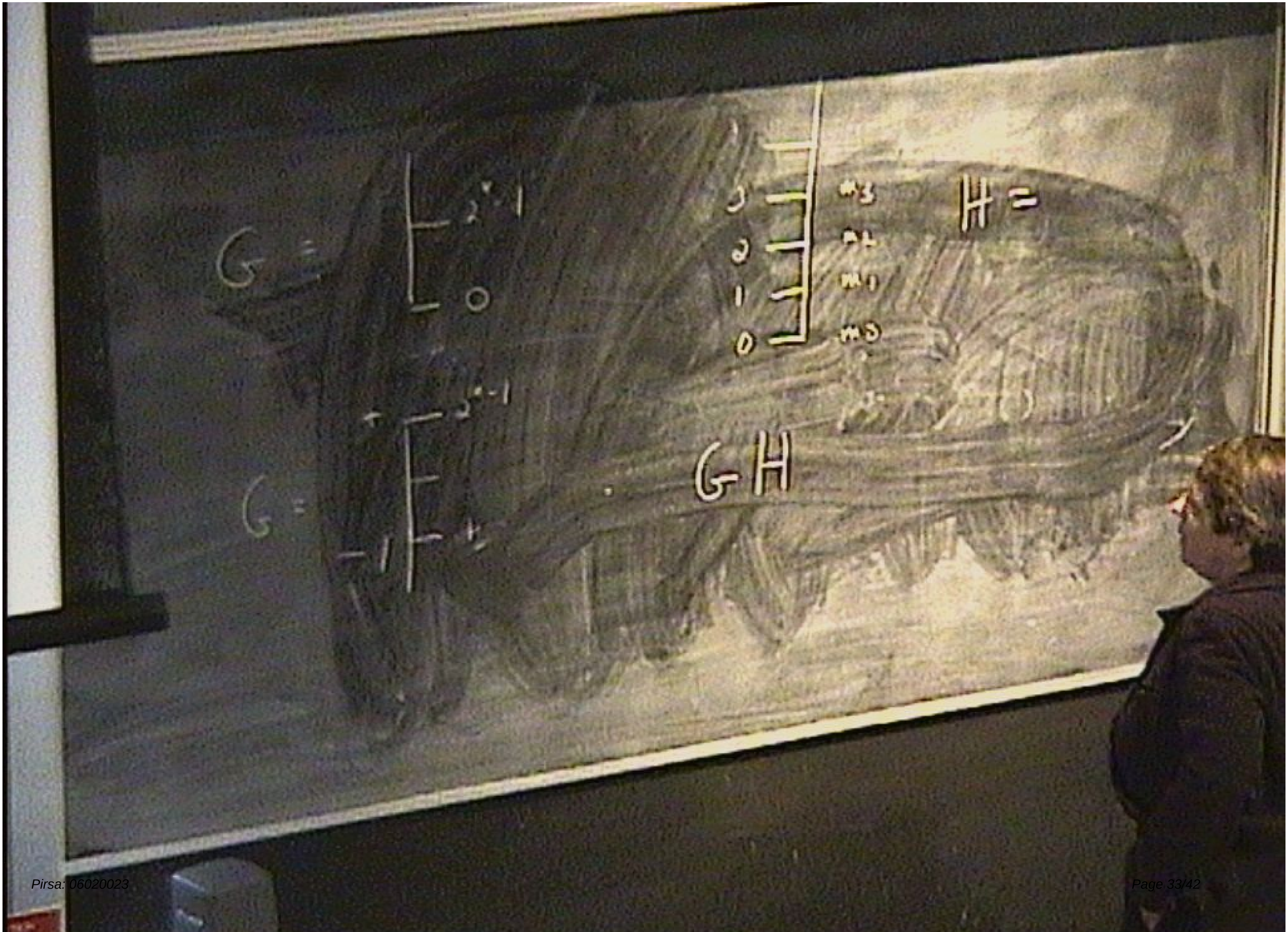
In this paper we have two main results about the performance of the quantum adiabatic algorithm when used to find the minimum of a classical cost function $h(z)$ with $z = 0, \ldots, N-1$. Theorem 1 says that for any cost function $h(z)$, if the beginning Hamiltonian is a one dimensional projector onto the uniform superposition of all the $|z\rangle$ basis states, the algorithm will not find the minimum of h if T is less then of order $\sqrt{N}$. This is true regardless of how simple it is to classically find the minimum of $h(z)$.

In Theorem 2 we start with any beginning Hamiltonian and classical cost function h. Replacing $h(z)$ by a scrambled version, i.e. $h^{[\pi]}(z) = h(\pi(z))$ with π a permutation of 0 to $N-1$, will make it impossible for the algorithm to find the minimum of $h^{[\pi]}$ in time less than order $\sqrt{N}$ for a typical permutation π. For example suppose we have a cost function $h(z)$ and have chosen H_B so that the quantum algorithm finds the minimum in time of order $\log N$. Still scrambling the cost function results in algorithmic failure.

These results do not imply anything about the more interesting case where H_B and H_P are




G =
0
G =
3
2
1
0


G =
H =
GH

$H_0(1-s) + s H_F$

$G = \; 2^n-1, \; 0$

$3, \; 2, \; 1, \; 0$

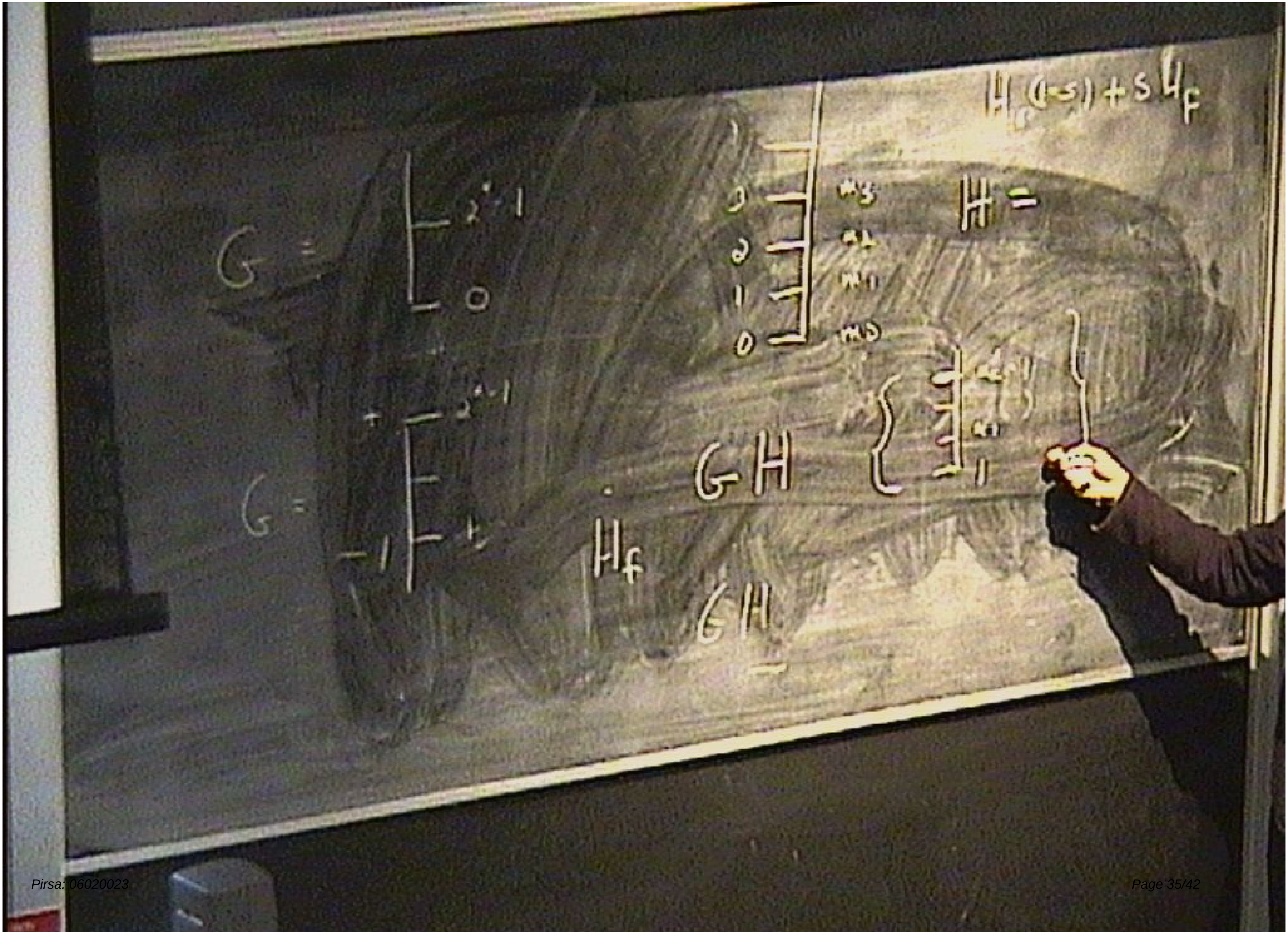
$H =$

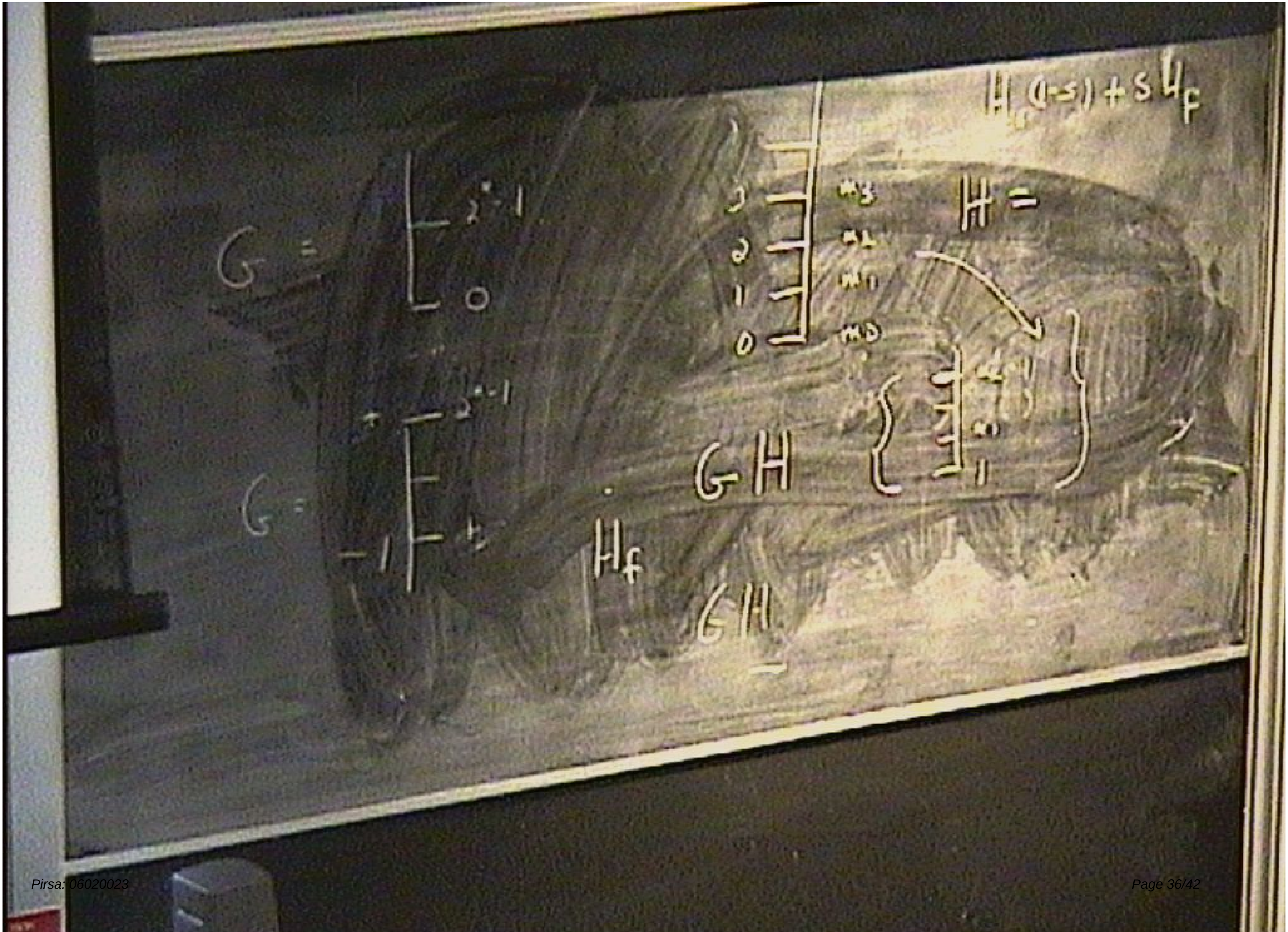
$G = \; 2^n-1, \; 1$

H_f

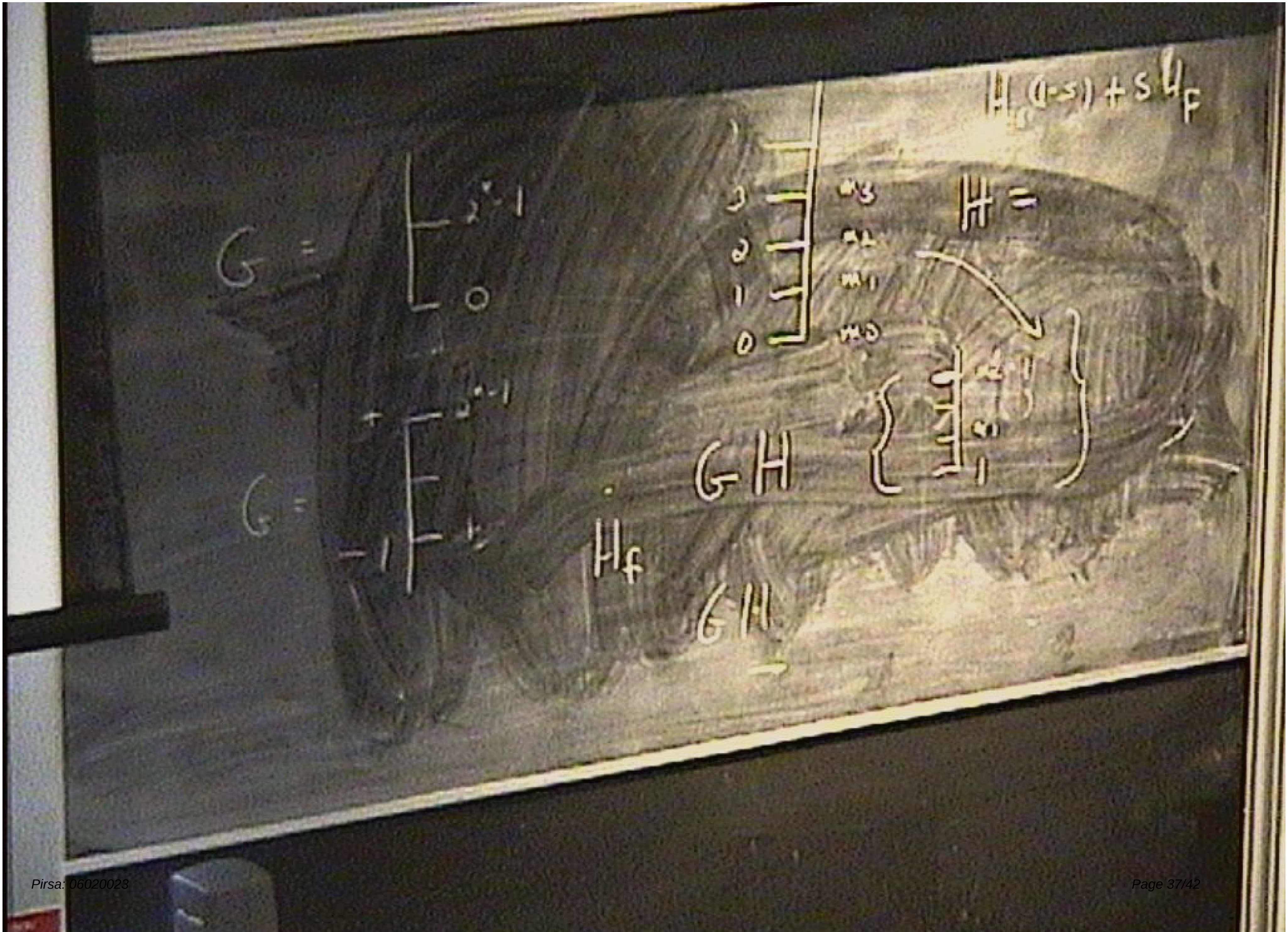
GH

GH

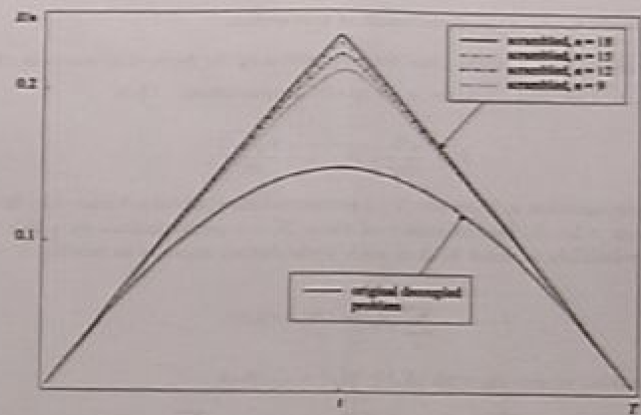

H(1-s) + s H_F
G =
H =
G =
H_f
GH
GH



$H_i(1-s) + s H_f$
H =
G =
G =
GH
H_f
GH



H0(1-s) + s HF
H =
G =
G =
GH
Hf
GH



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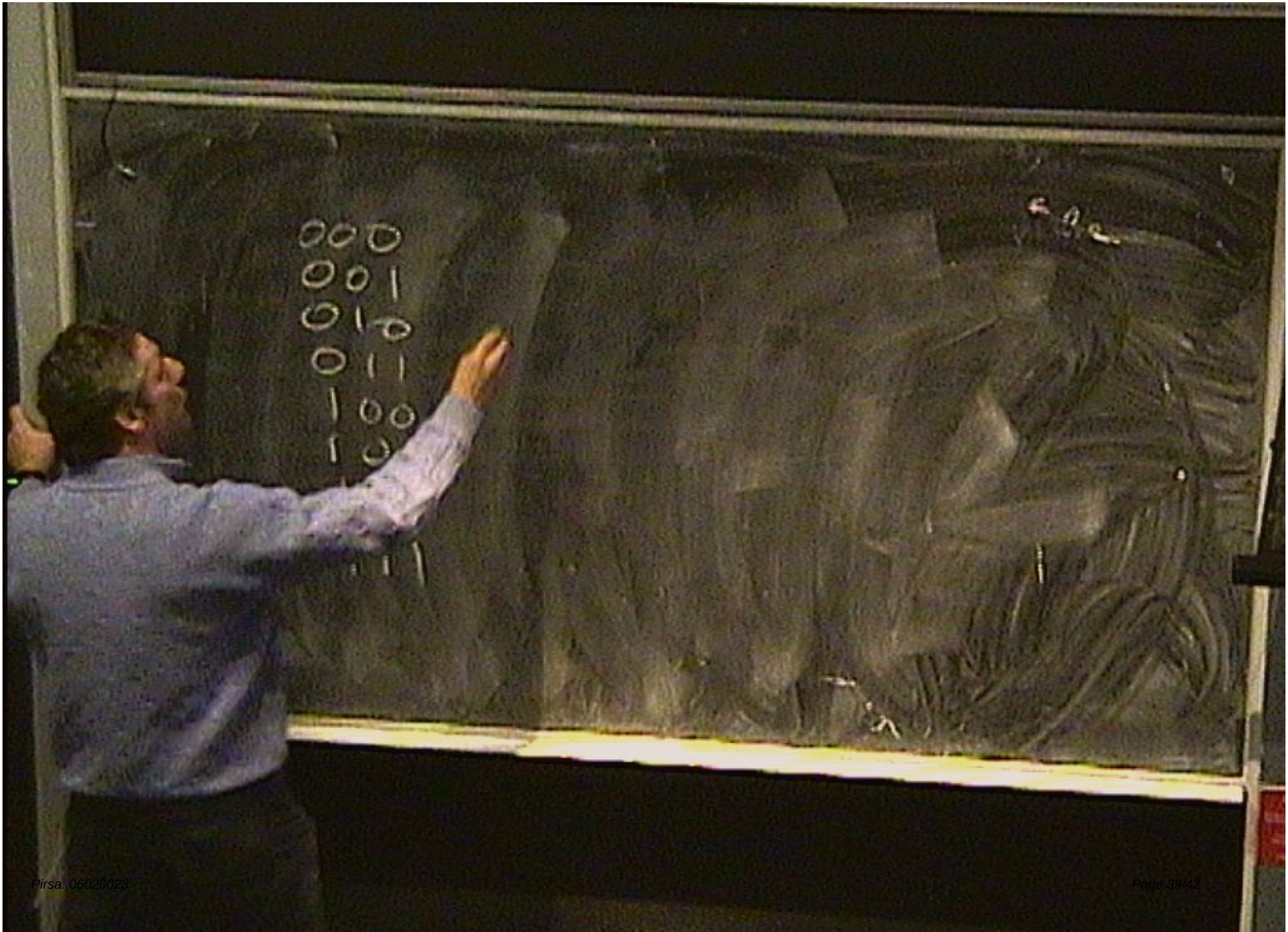
(Since the system is decoupled this is actually the ground state energy of a single qubit.) We then consider the n bit scrambled problem for different values of n. At each n we pick a single random permutation π of $0, \ldots, (2^n - 1)$ and apply it to obtain a cost function $h(\pi^{-1}(z))$ while keeping H_B fixed. The ground state energy divided by n is now plotted for $n = 9, 12, 15$ and 18. From these scrambled problems it is clear that if we let n get large the typical curves will approach a triangle with a discontinuous first derivative at $t = T/2$. For large n, the ground state changes dramatically as t passes through $T/2$. In order to keep the quantum system in the ground state we need to go very slowly near $t = T/2$ and this results in a long required run time.

IV. CONCLUSIONS

In this paper we have two main results about the performance of the quantum adiabatic algorithm when used to find the minimum of a classical cost function $h(z)$ with $z = 0, \ldots, N-1$. Theorem 1 says that for any cost function $h(z)$, if the beginning Hamiltonian is a one dimensional projector onto the uniform superposition of all the $|z\rangle$ basis states, the algorithm will not find the minimum of h if T is less then of order $\sqrt{N}$. This is true regardless of how simple it is to classically find the minimum of $h(z)$.

In Theorem 2 we start with any beginning Hamiltonian and classical cost function h. Replacing $h(z)$ by a scrambled version, i.e. $h^{[\pi]}(z) = h(\pi(z))$ with π a permutation of 0 to $N-1$, will make it impossible for the algorithm to find the minimum of $h^{[\pi]}$ in time less than order $\sqrt{N}$ for a typical permutation π. For example suppose we have a cost function $h(z)$ and have chosen H_B so that the quantum algorithm finds the minimum in time of order $\log N$. Still scrambling the cost function results in algorithmic failure.

These results do not imply anything about the more interesting case where H_B and H_P are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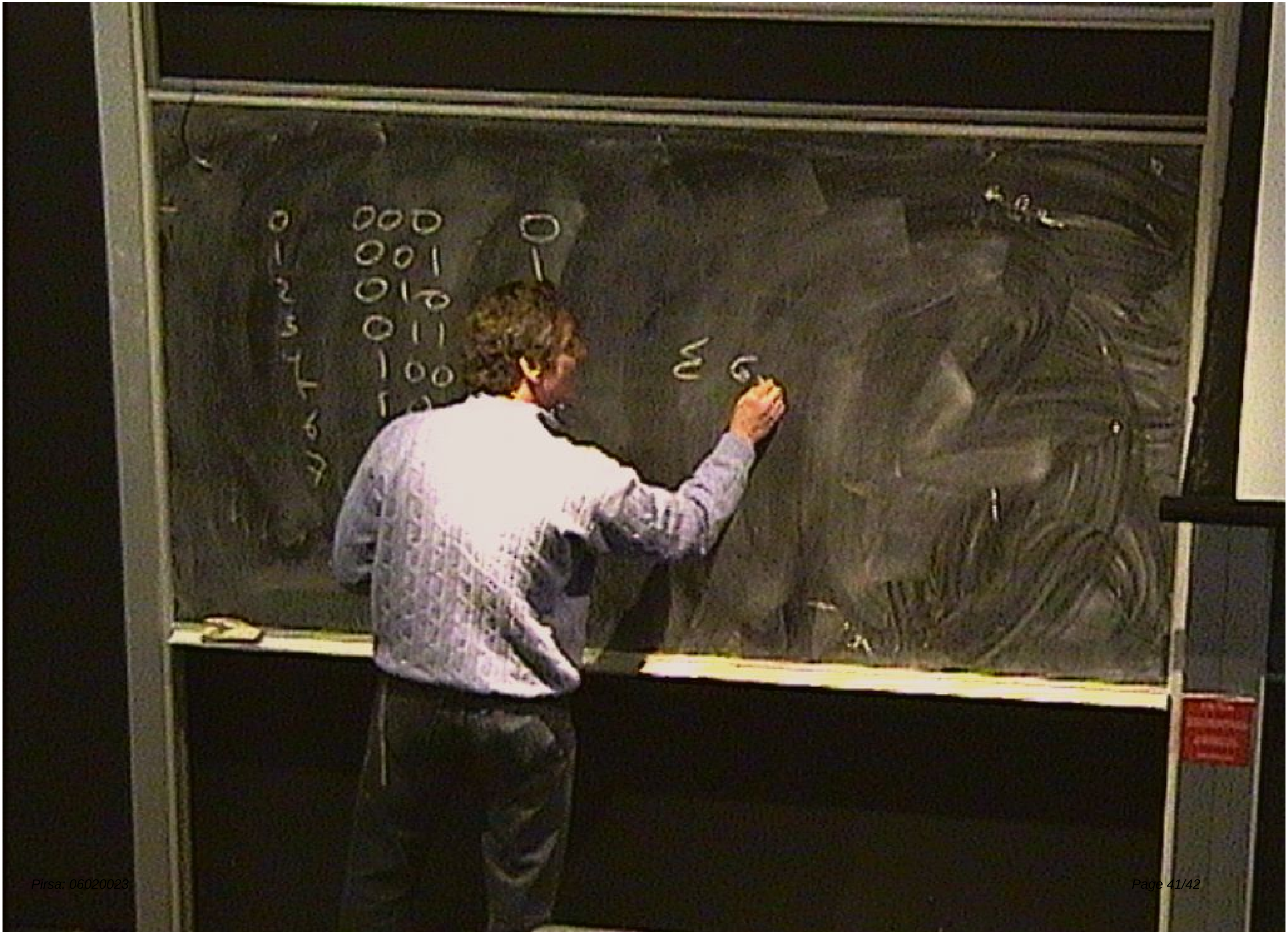
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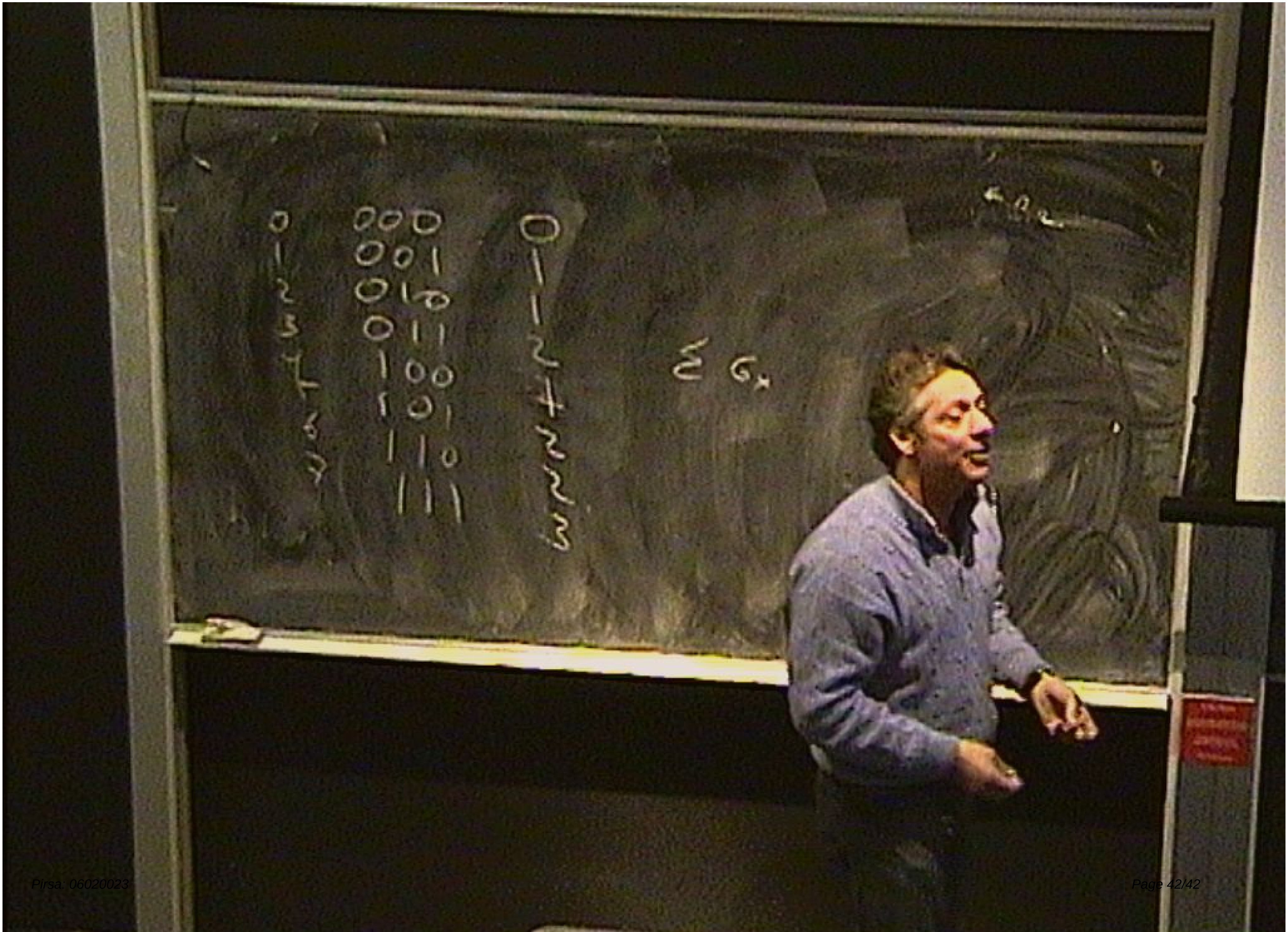
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