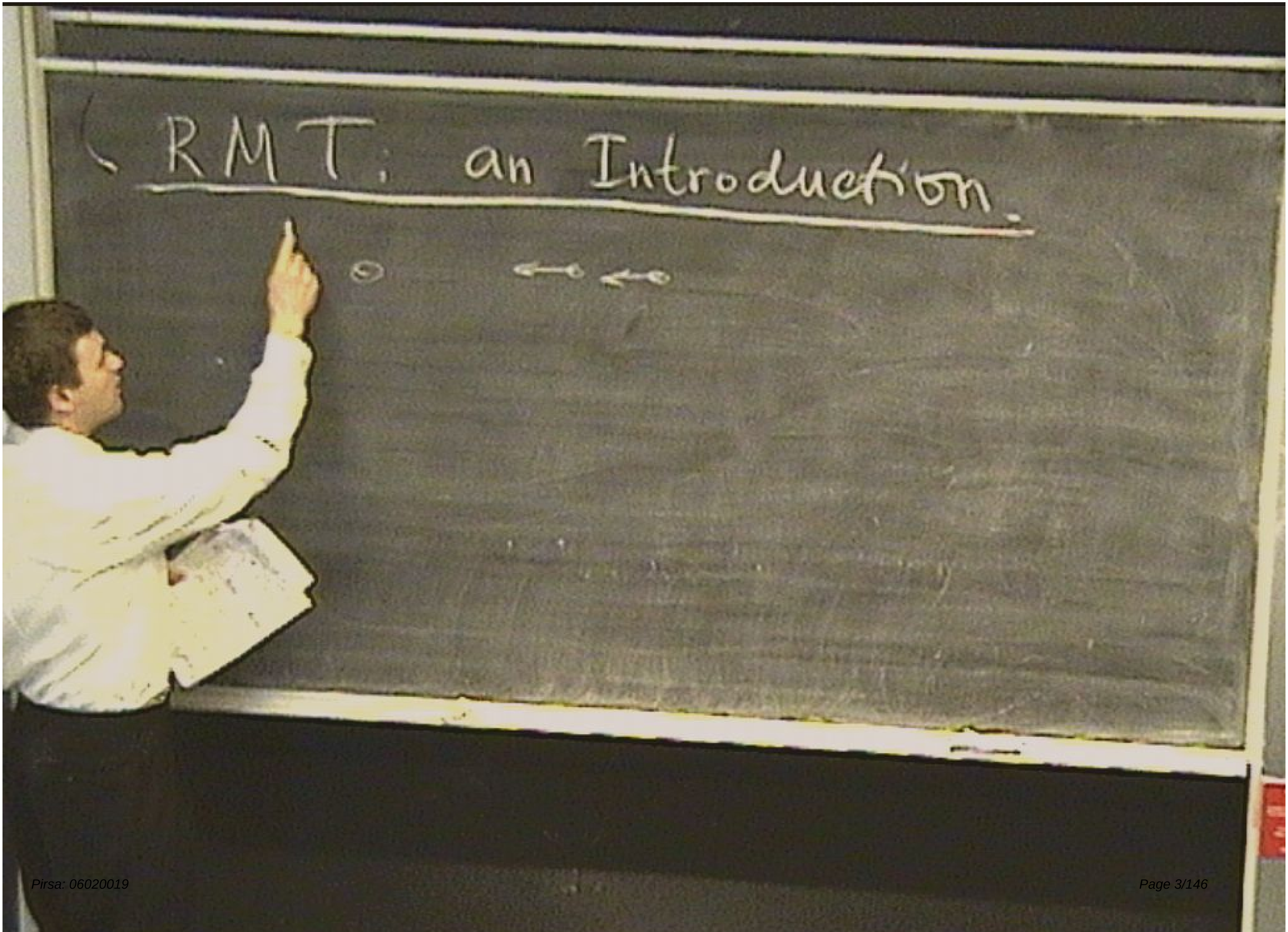
Title: Random Matrices

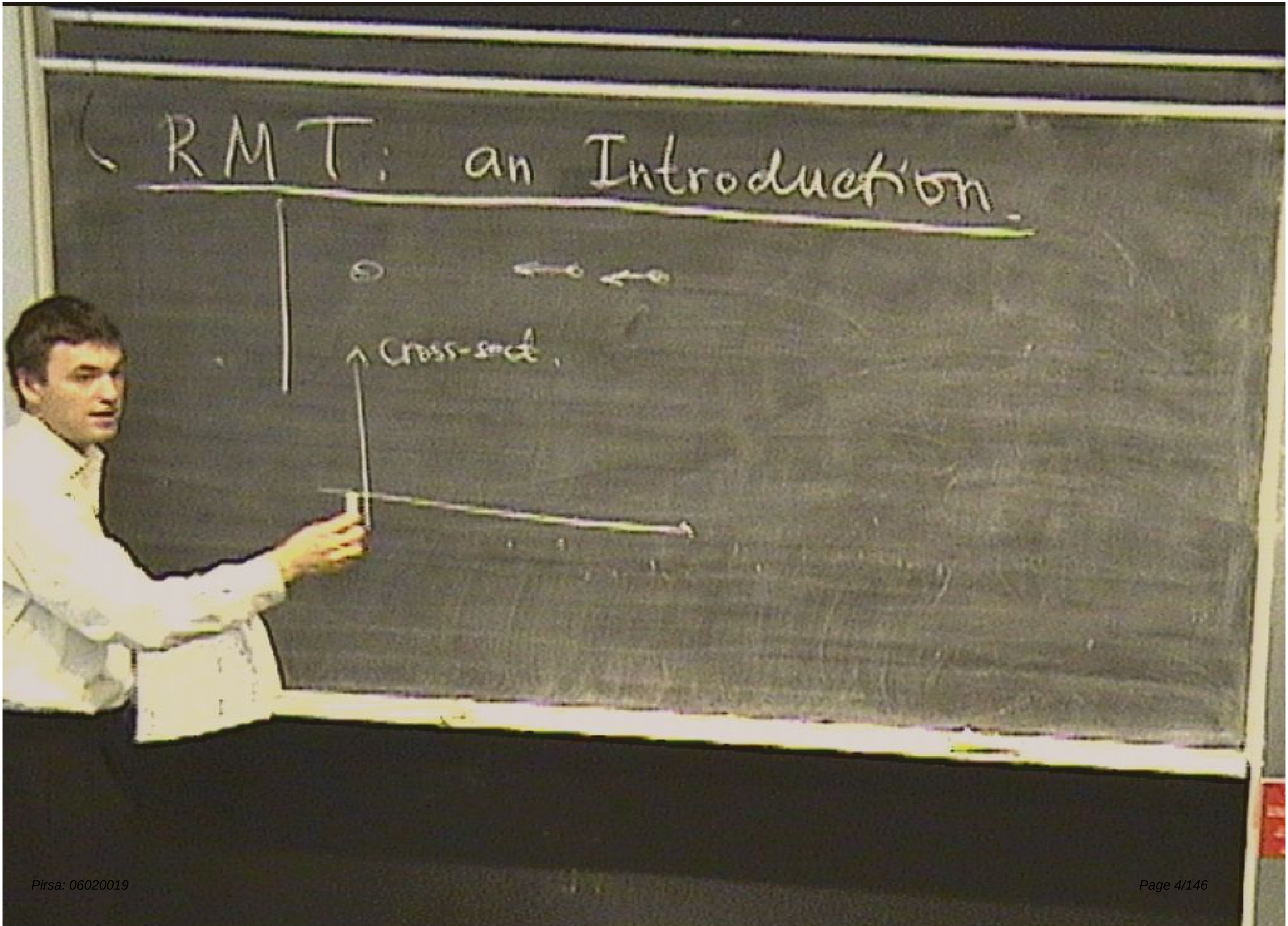
Date: Feb 10, 2006 09:15 AM

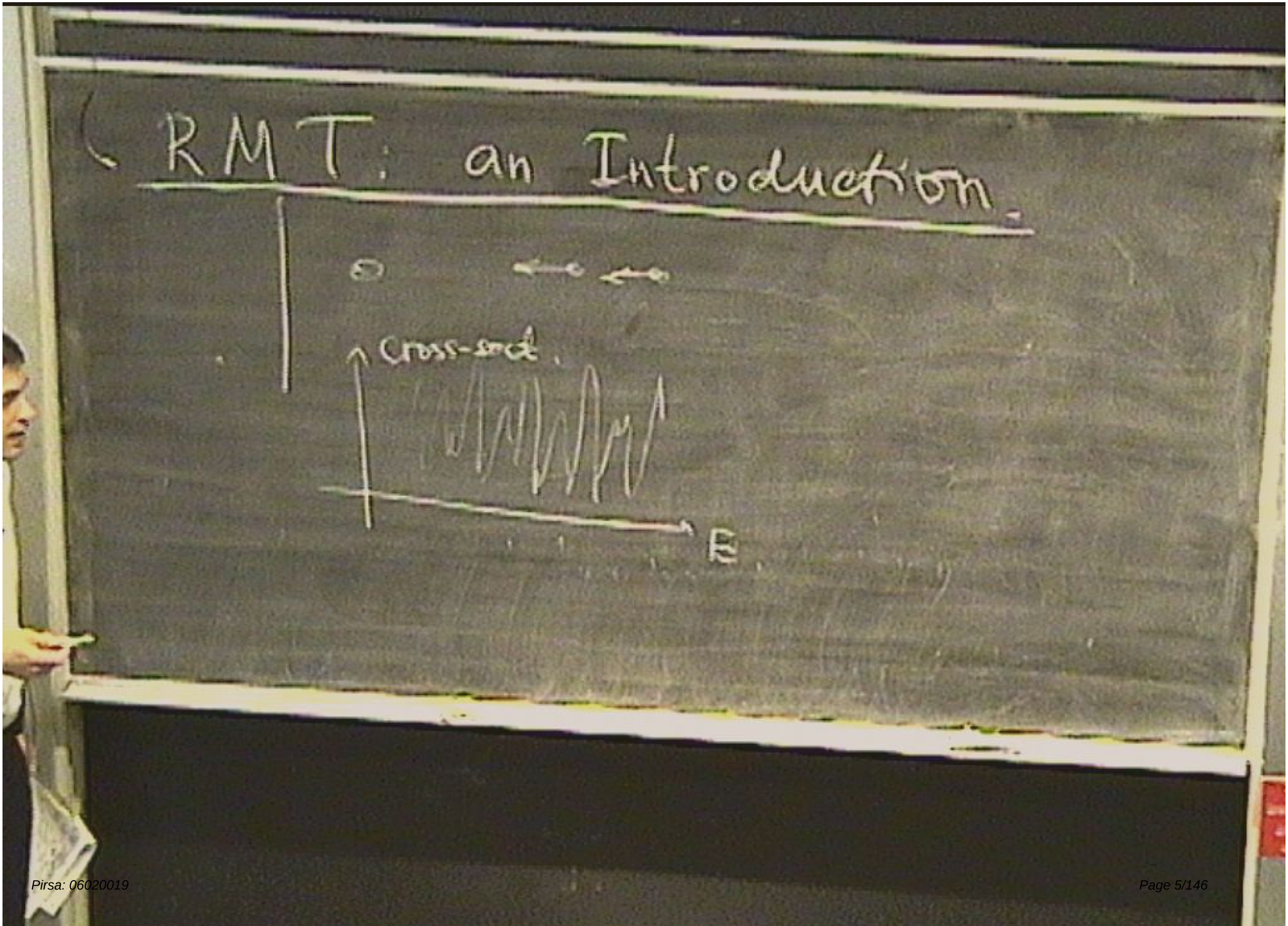
URL: http://pirsa.org/06020019

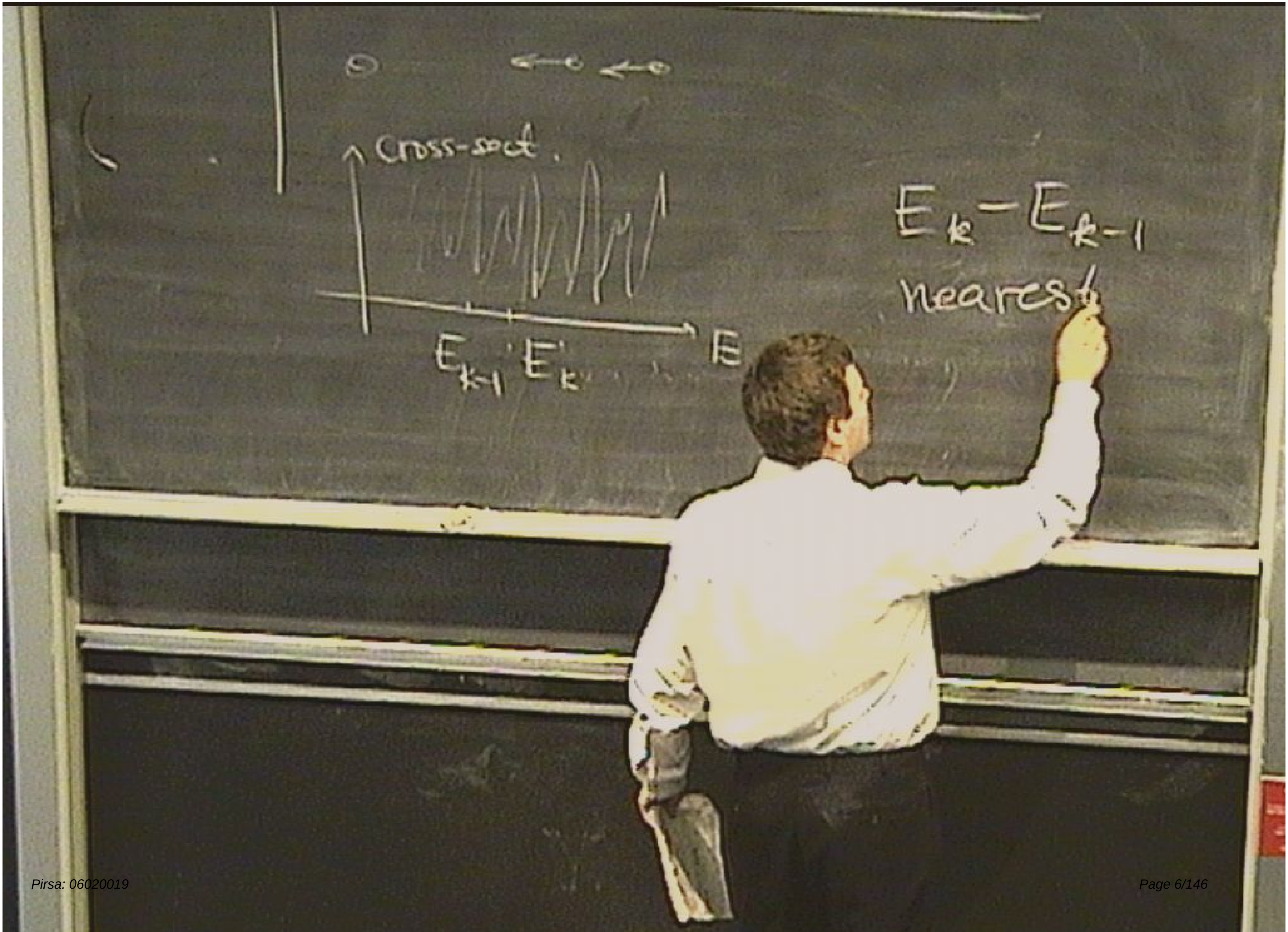
Abstract:

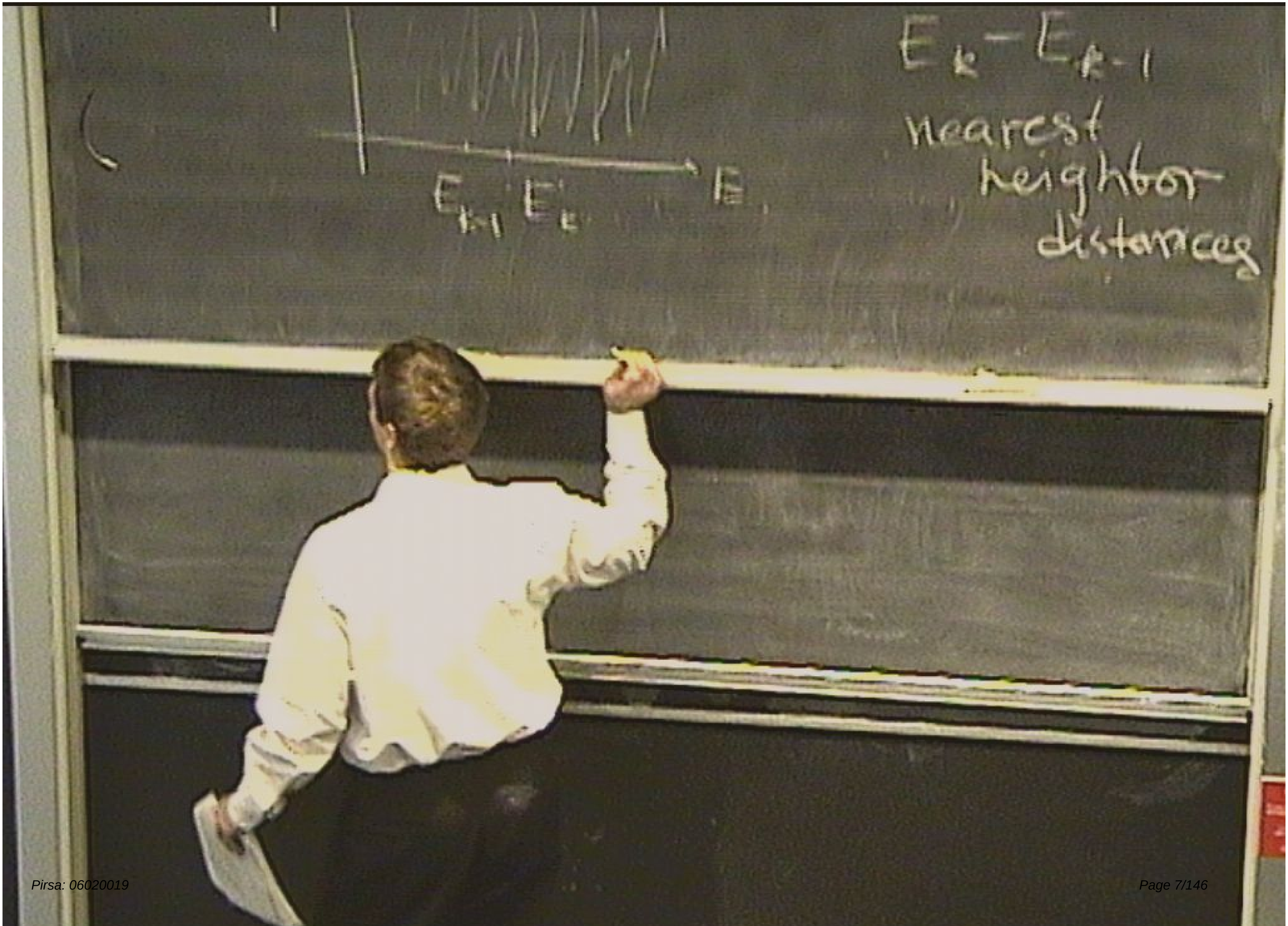

RMT: an Introduction

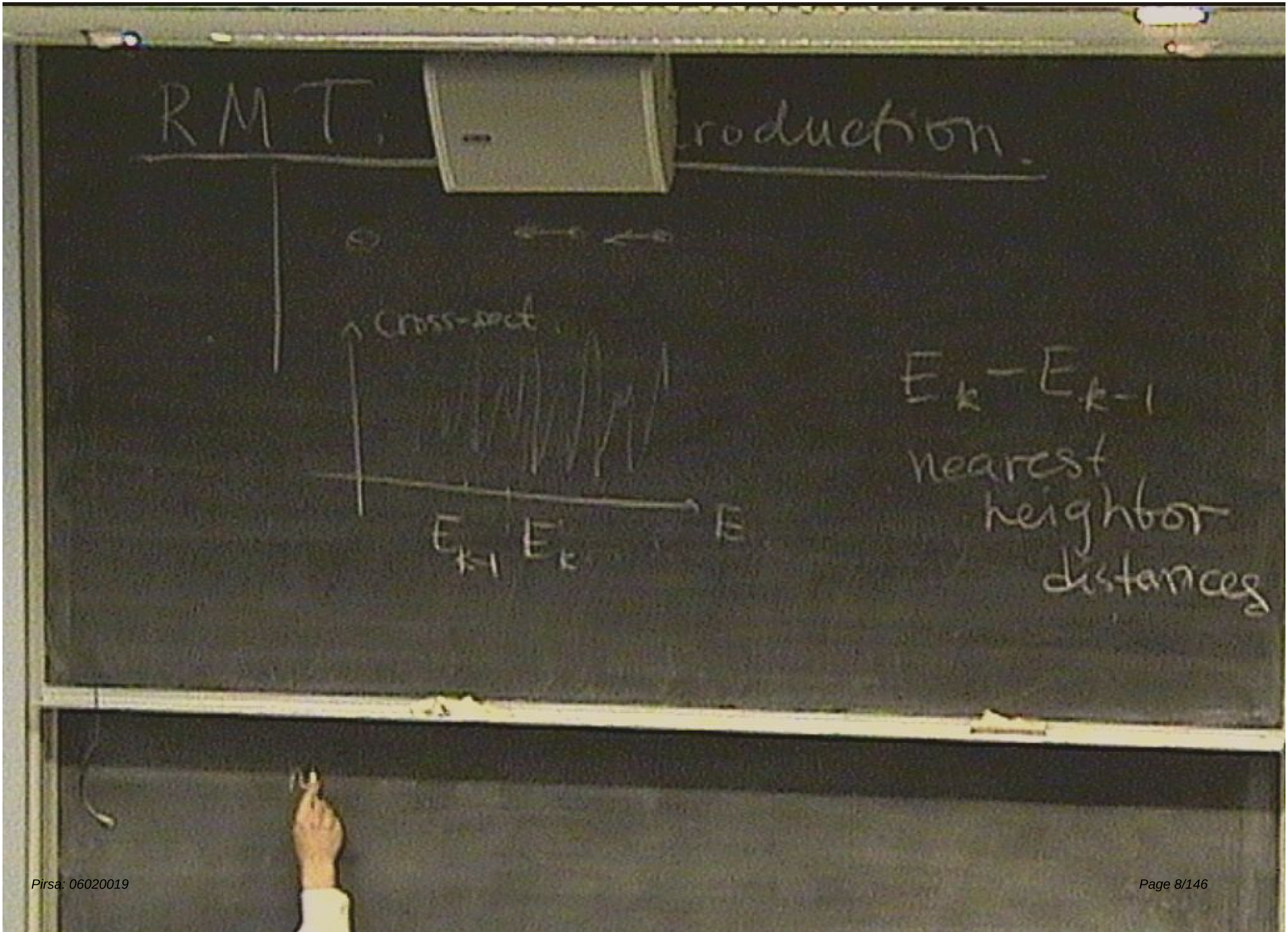

RMT: an Introduction

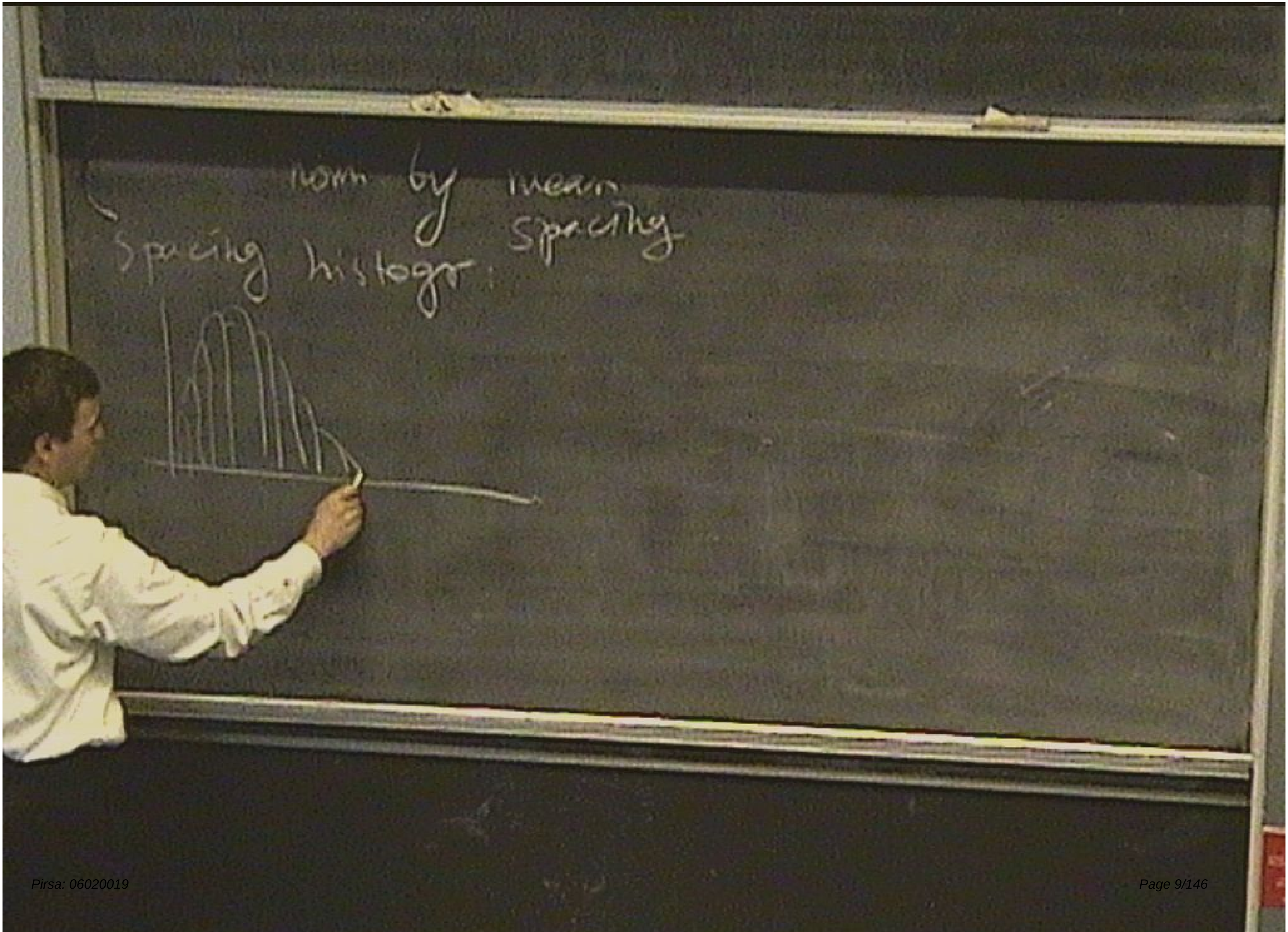

RMT: an Introduction
cross-sect.

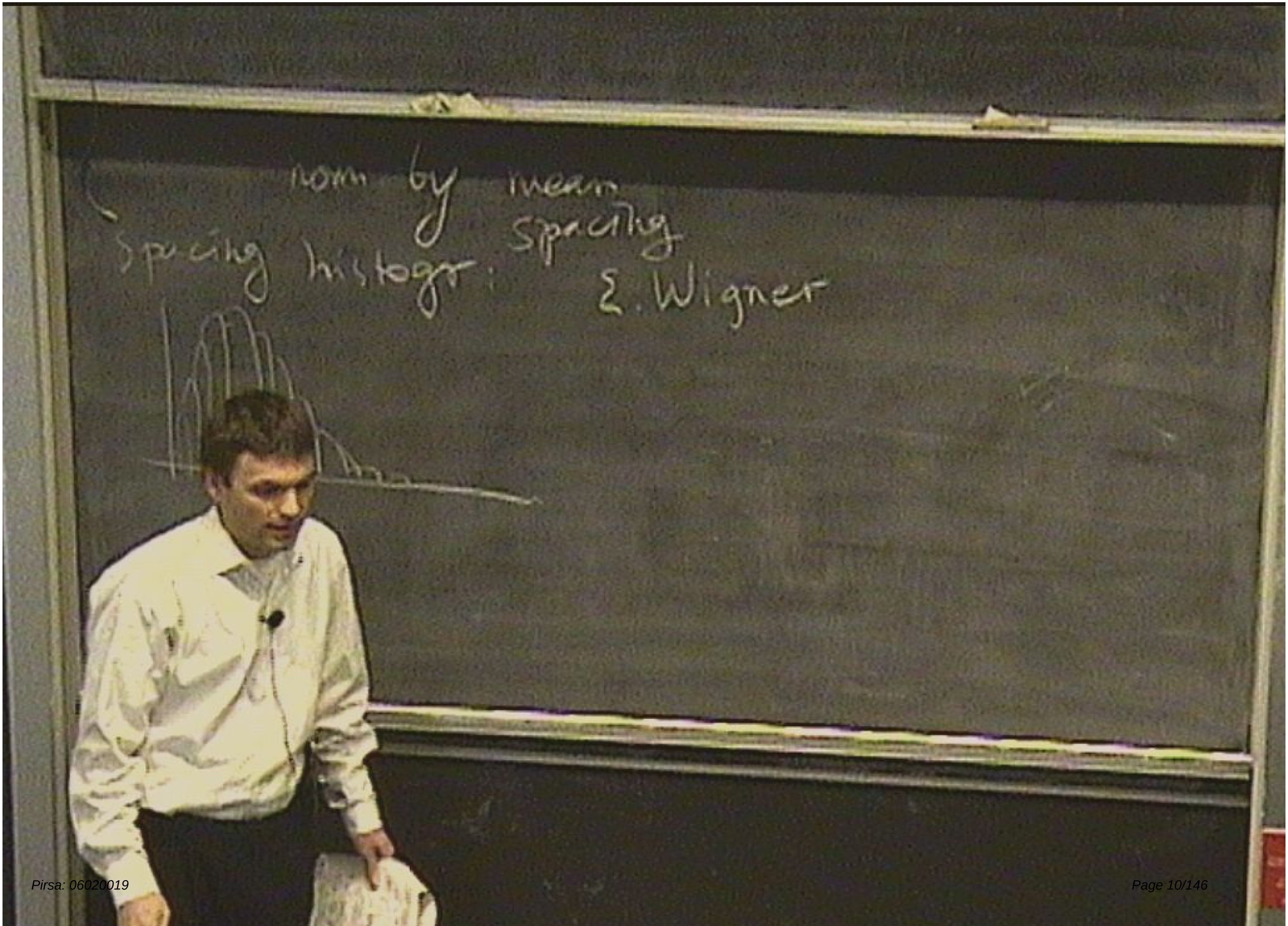

RMT: an Introduction
Cross-sect.
E

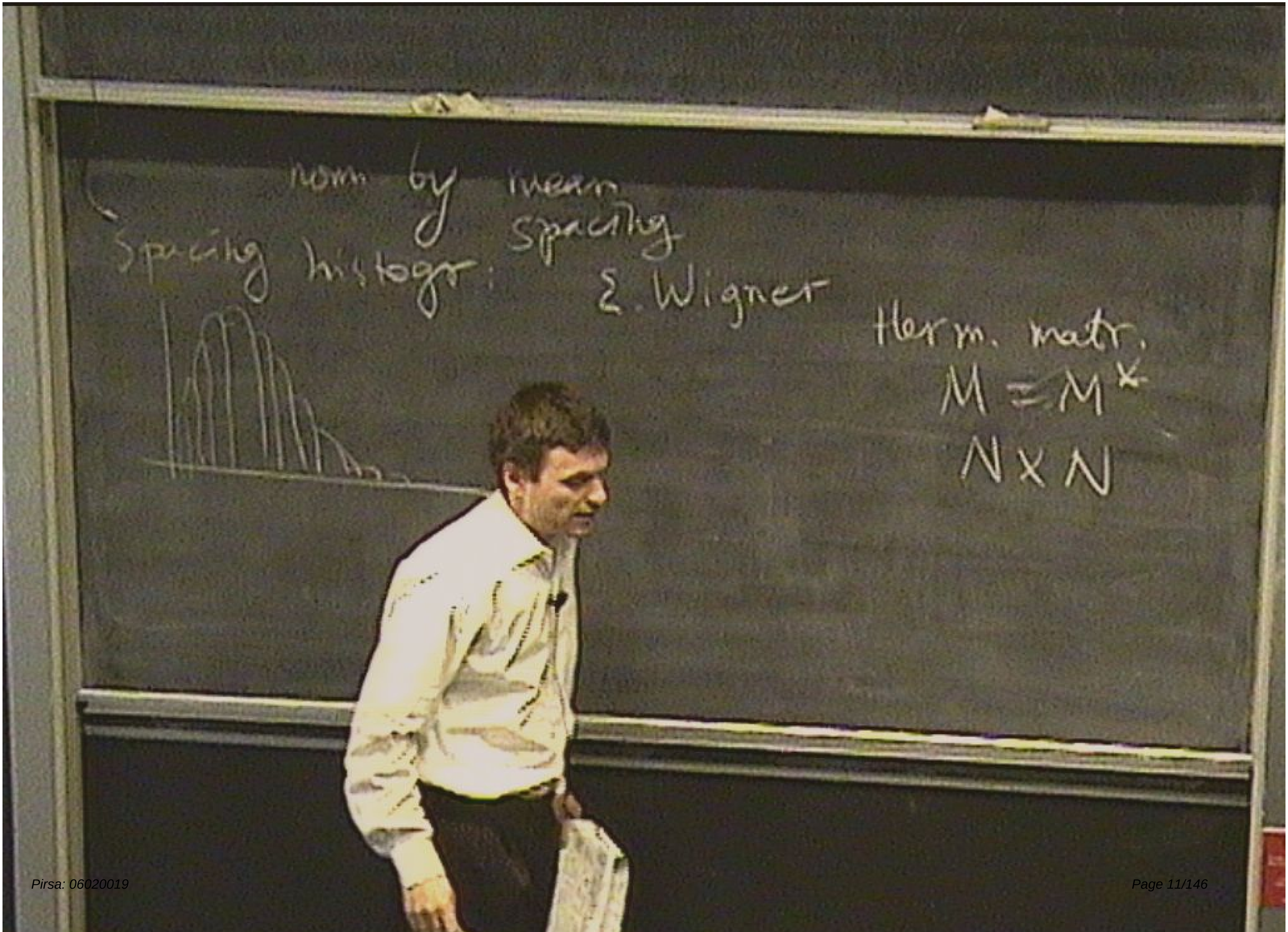

cross-sect.
E_{k-1} E_k
E
$E_k - E_{k-1}$
nearest

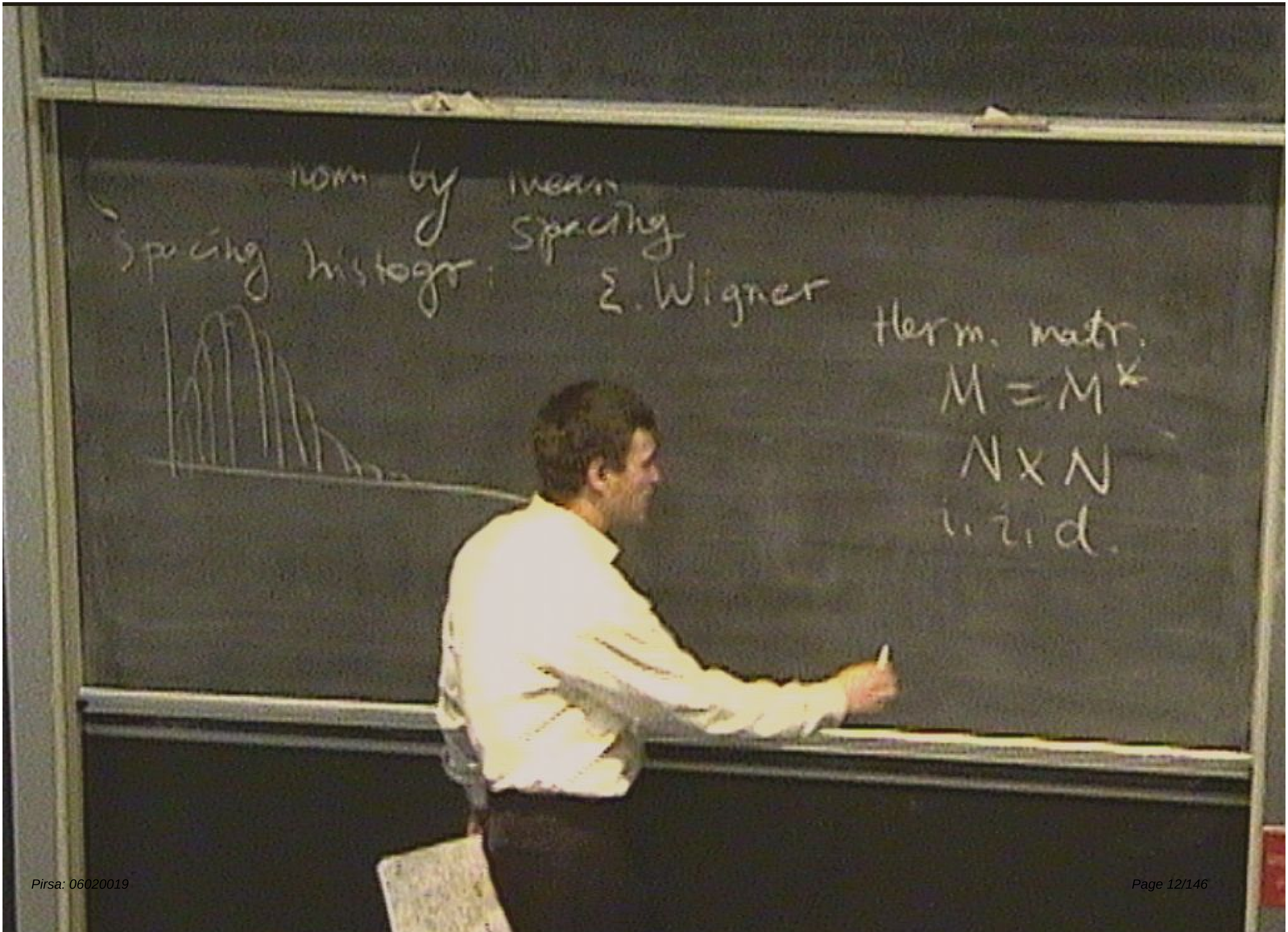

E_k − E_{k-1}
nearest neighbor distances
E_{k-1} E_k
E

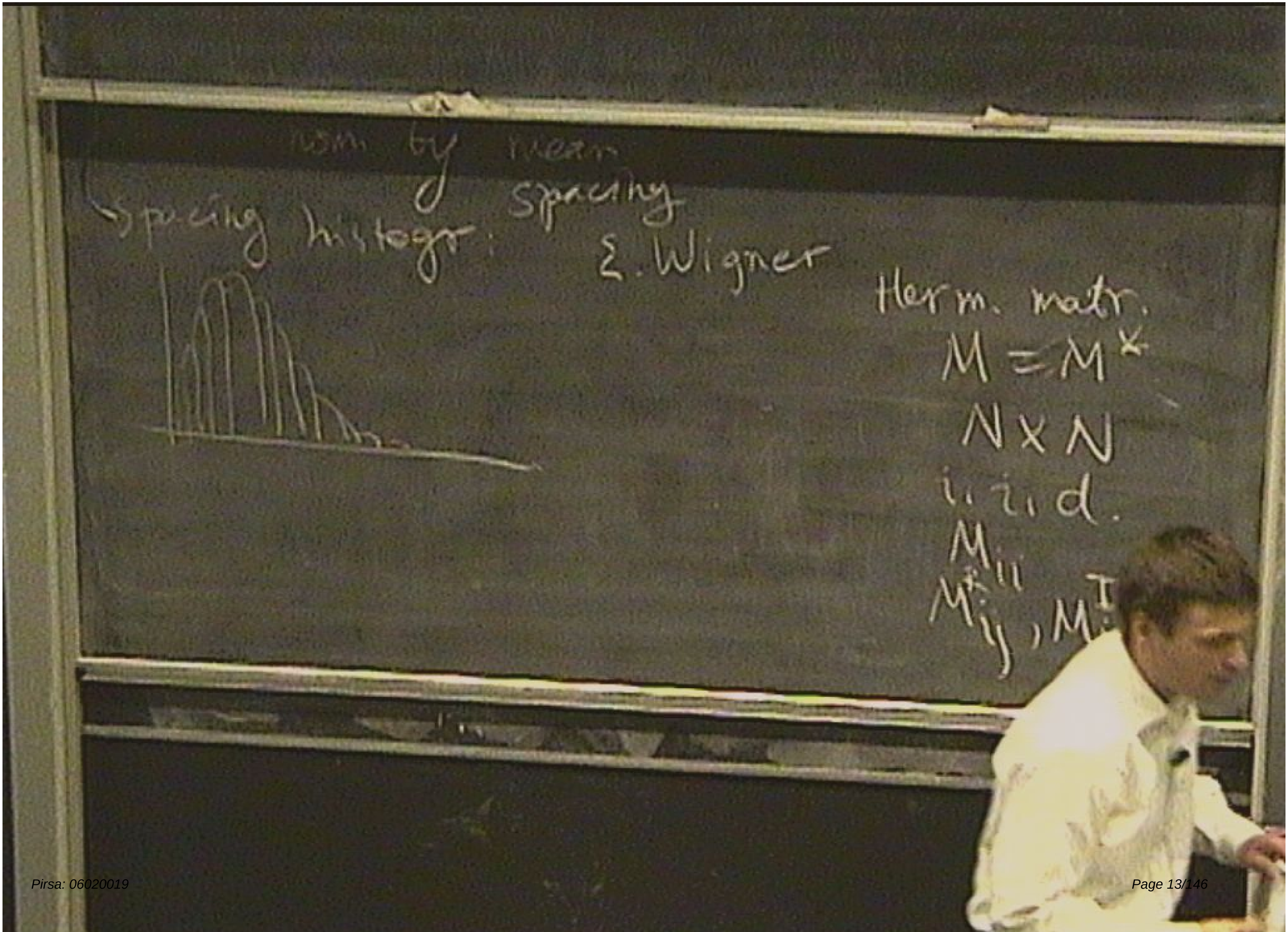

RMT.
roduction
cross-sect
E k-1 E k
E
E k − E k-1
nearest
neighbor
distances

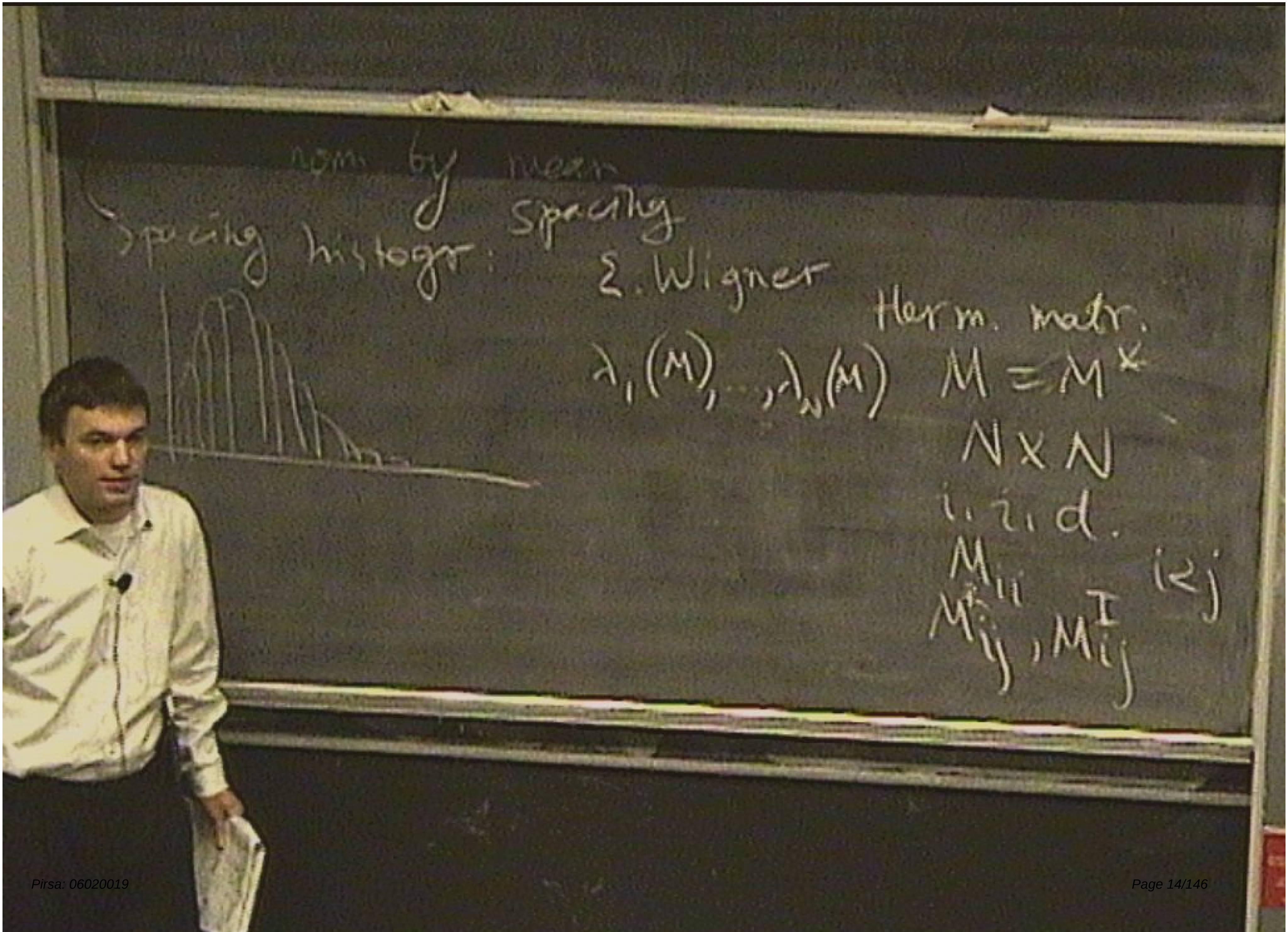

norm by mean spacing
Spacing histogr:

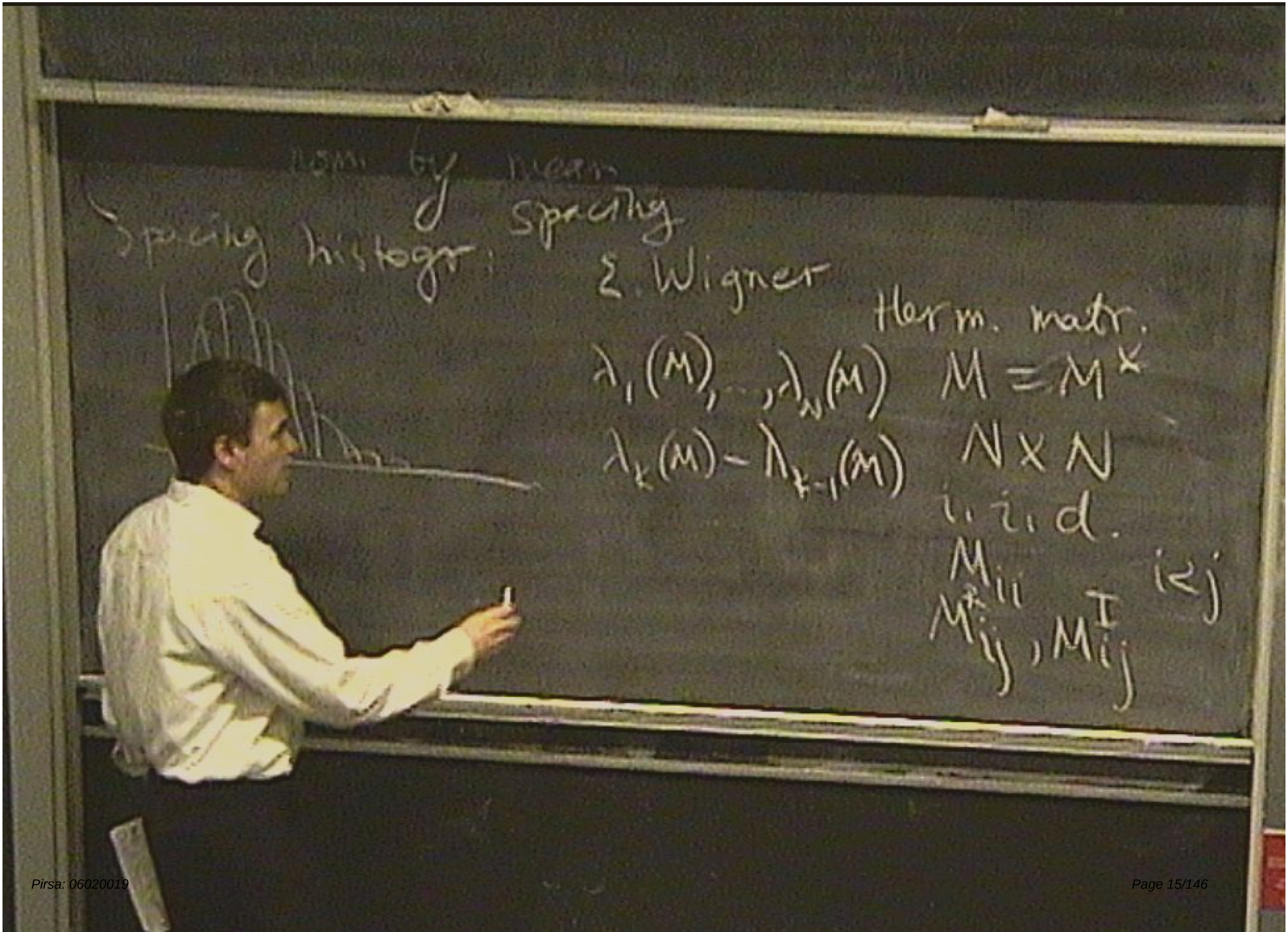

nom by mean spacing
spacing histogr:
E. Wigner


norm by mean spacing
Spacing histogr:
E. Wigner
Herm. matr.
M = M*
N x N


norm by mean spacing
spacing histogr.:
E. Wigner
Herm. matr.
M = M*
N x N
i.i.d.

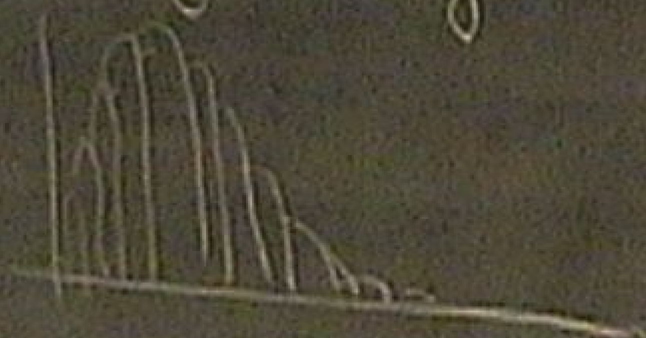

by
spacing
spacing histogr:
E. Wigner
Herm. matr.
M = M*
N x N
i.i.d.


norm by mean spacing
Spacing histogr:
E. Wigner
Herm. matr.
λ1(M), ..., λN(M)
M = M*
N x N
i.i.d.
Mii
i<j
MRij, MIij


norm. by mean spacing
Spacing histogr.:
E. Wigner
Herm. matr.
$\lambda_1(M), \dots, \lambda_N(M)$
$M = M^*$
$\lambda_k(M) - \lambda_{k-1}(M)$
$N \times N$
i.i.d.
M_{ii}
M^R_{ij}, M^I_{ij}
$i<j$

norm. by mean spacing

Spacing histogr.:



2. Wigner

Herm. matr.

$\lambda_1(M) \le \dots \le \lambda_N(M)$ $\quad M = M^*$

$\lambda_k(M) - \lambda_{k-1}(M)$ $\quad N \times N$

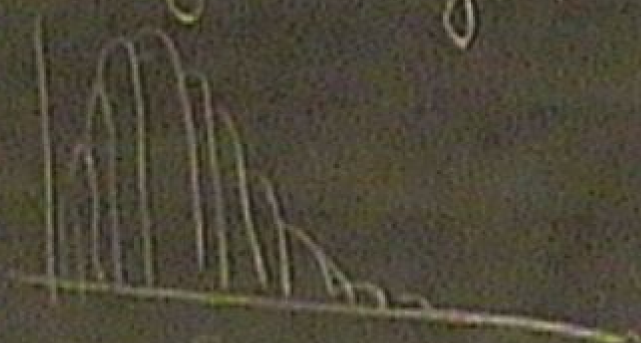
i. i. d.

M_{ii} $\quad i<j$

M^R_{ij}, M^I_{ij}

norm by mean spacing

Spacing histogr.:



same histogram

E. Wigner

Herm. matr.

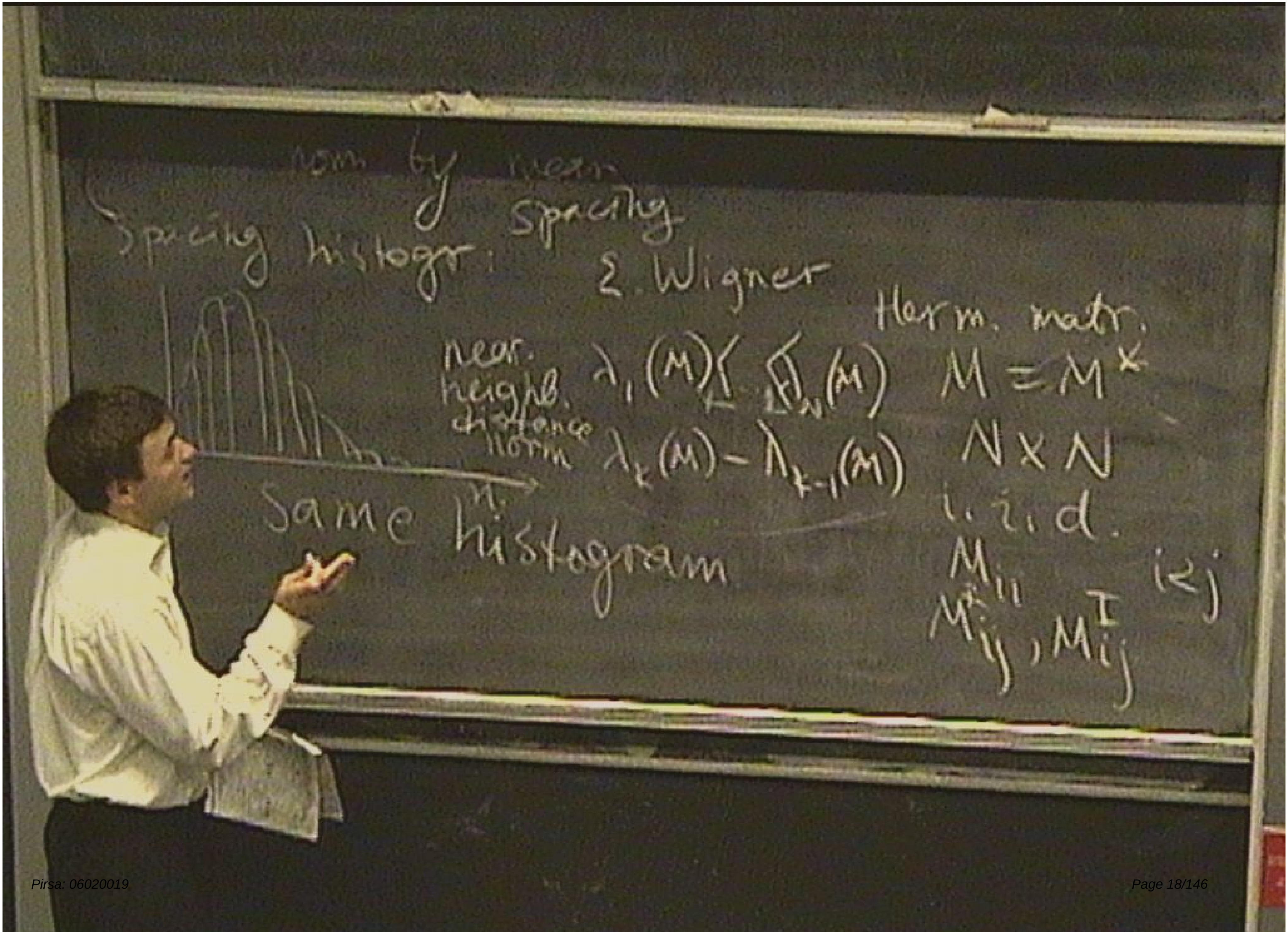
$\lambda_1(M) \le \dots \le \lambda_N(M)$ $\quad M = M^*$

$\lambda_k(M) - \lambda_{k-1}(M)$ $\quad N \times N$

i.i.d.

M_{ii} $\quad i<j$

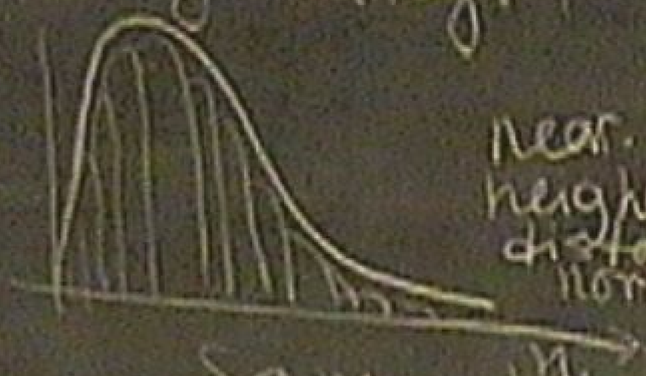
M^R_{ij}, M^I_{ij}


norm by mean spacing
Spacing histogr.:
E. Wigner
Herm. matr.
near. neighb. distance norm
λ1(M) ≤ ... ≤ λN(M)
M = M*
λk(M) − λk-1(M)
N x N
i.i.d.
Mii
Mij, Mij (i<j)
same histogram

norm by mean spacing

Spacing histogr.: E. Wigner

Herm. matr.



near. neighb. distance norm $\lambda_1(M) < \dots < \lambda_N(M)$ $M = M^*$

$\lambda_k(M) - \lambda_{k-1}(M)$ $N \times N$

Same histogram i.i.d.

M_{ii} $i<j$

M^R_{ij}, M^I_{ij}

norm by mean spacing

Spacing histogr.: E. Wigner

Herm. matr.

near. neighb. distance norm

$\lambda_1(M) \le \dots \le \lambda_N(M)$ $\qquad M = M^*$

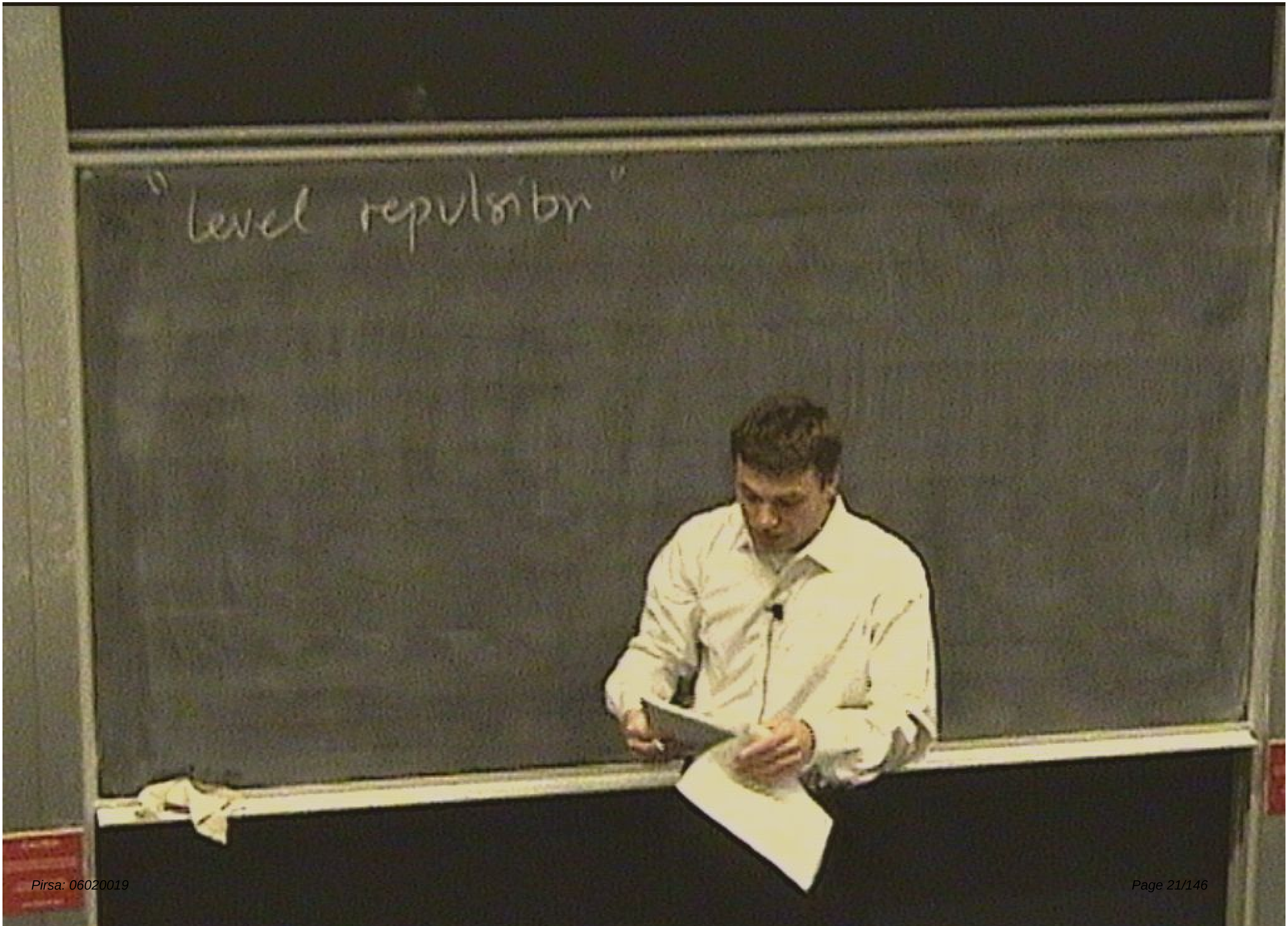
$\lambda_k(M) - \lambda_{k-1}(M)$ $\qquad N \times N$

n. x

same histogram $\qquad$ i.i.d.

$\sim a x e^{-bx^2}$ $\qquad M_{ii}$

$M^R_{ij}, M^I_{ij} \quad i<j$


"level repulsion"

norm by mean spacing

Spacing histogr: E. Wigner

near. neighb. distance $\lambda_1(M) \le \ldots \le \lambda_N(M)$

norm $\lambda_k(M) - \lambda_{k-1}(M)$

Herm. matr. $M = M^*$

$N \times N$

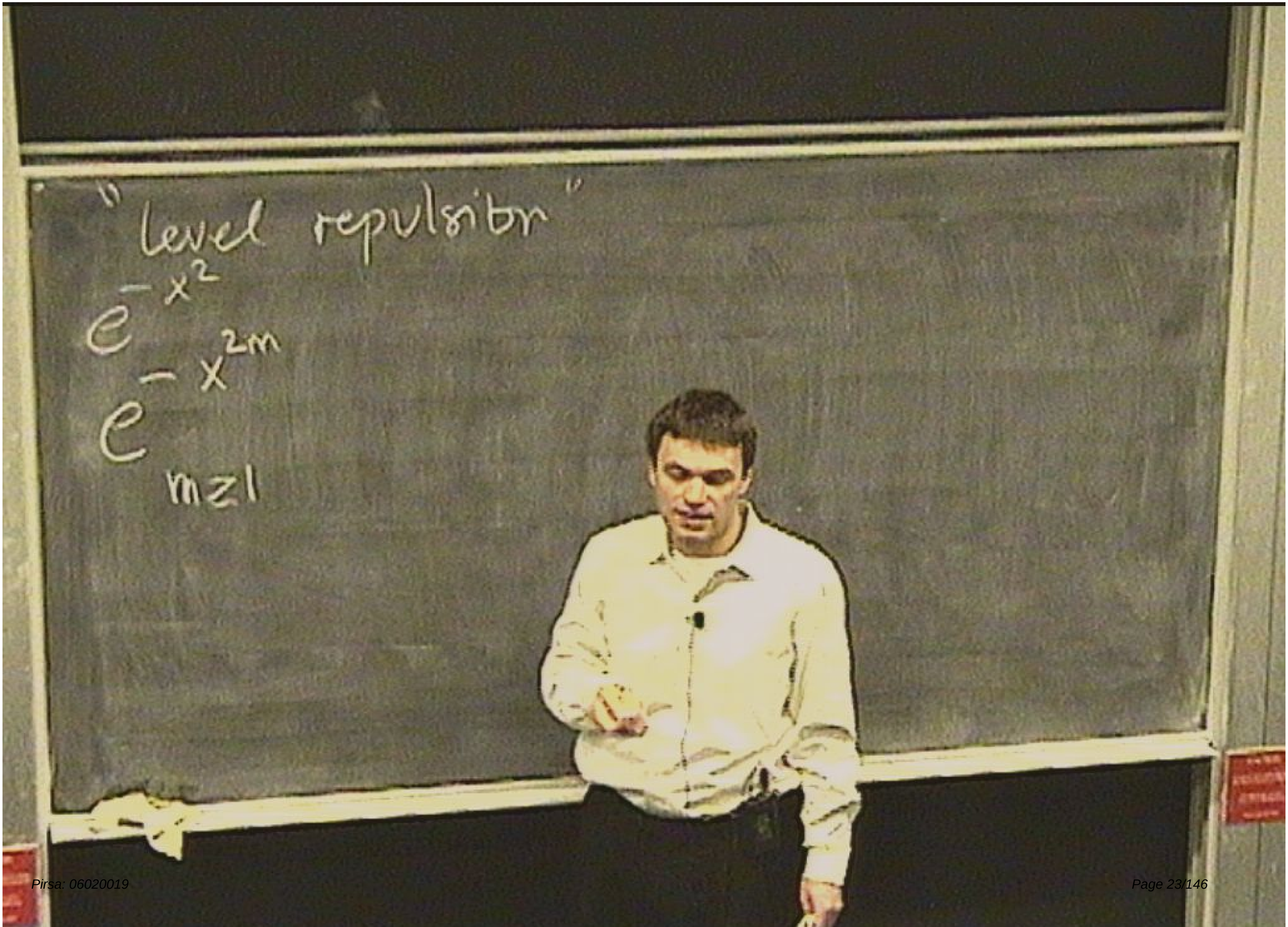
i. i. d.

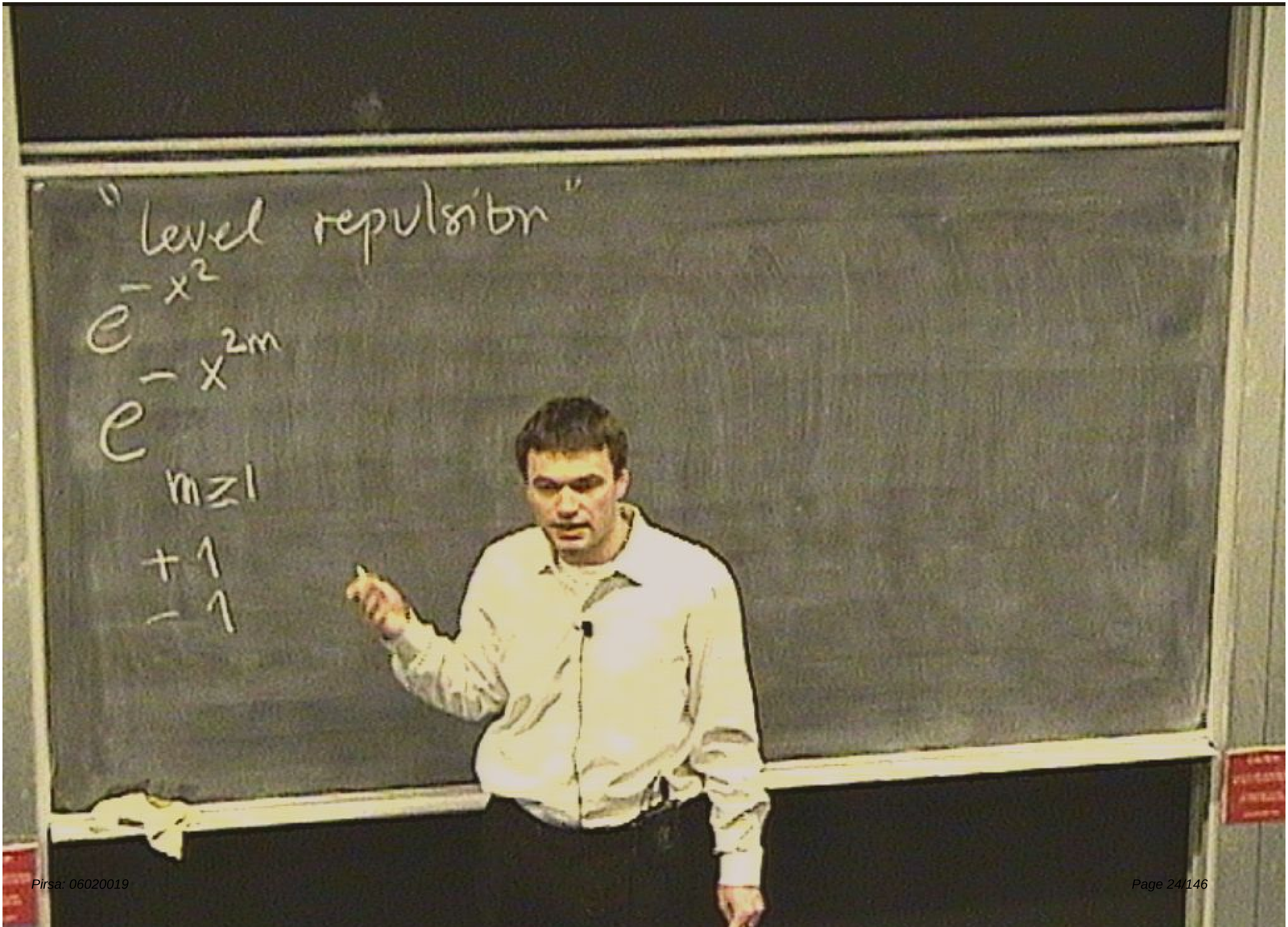
Same histogram

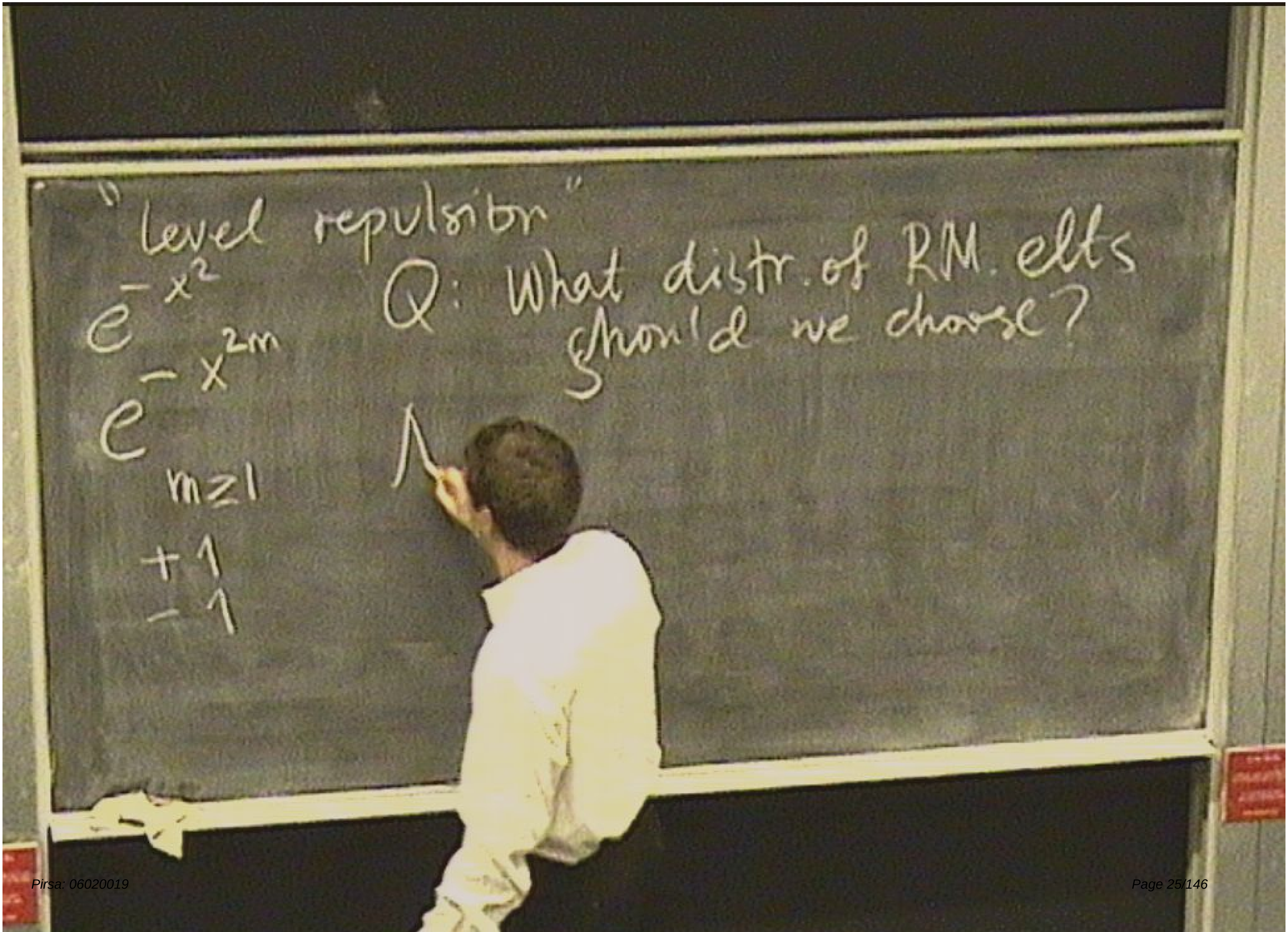
$\sim a x e^{-b x^2}$

M_{ii} $(i<j)$

M^R_{ij}, M^I_{ij}


"level repulsion"
e^{-x^2}
$e^{-x^{2m}}$
$m \geq 1$


"level repulsion"
e^{-x^2}
$e^{-x^{2m}}$
$m \geq 1$
$+1$
-1


"level repulsion"
Q: What distr. of RM elts should we choose?
e^{-x^2}
$e^{-x^{2m}}$
$m \geq 1$
$+1$
-1

"level repulsion"

e^{-x^2}

$e^{-x^{2m}}$

$m \geq 1$

$+1$

-1

Q: What distr. of RM elts should we choose?

A: Up to gen. symm. req., the answer (as $N \to \infty$) is indep. of distribs.

level repulsion

e^{-x^2}

$e^{-x^{2m}}$

$m \geq 1$

$+1$

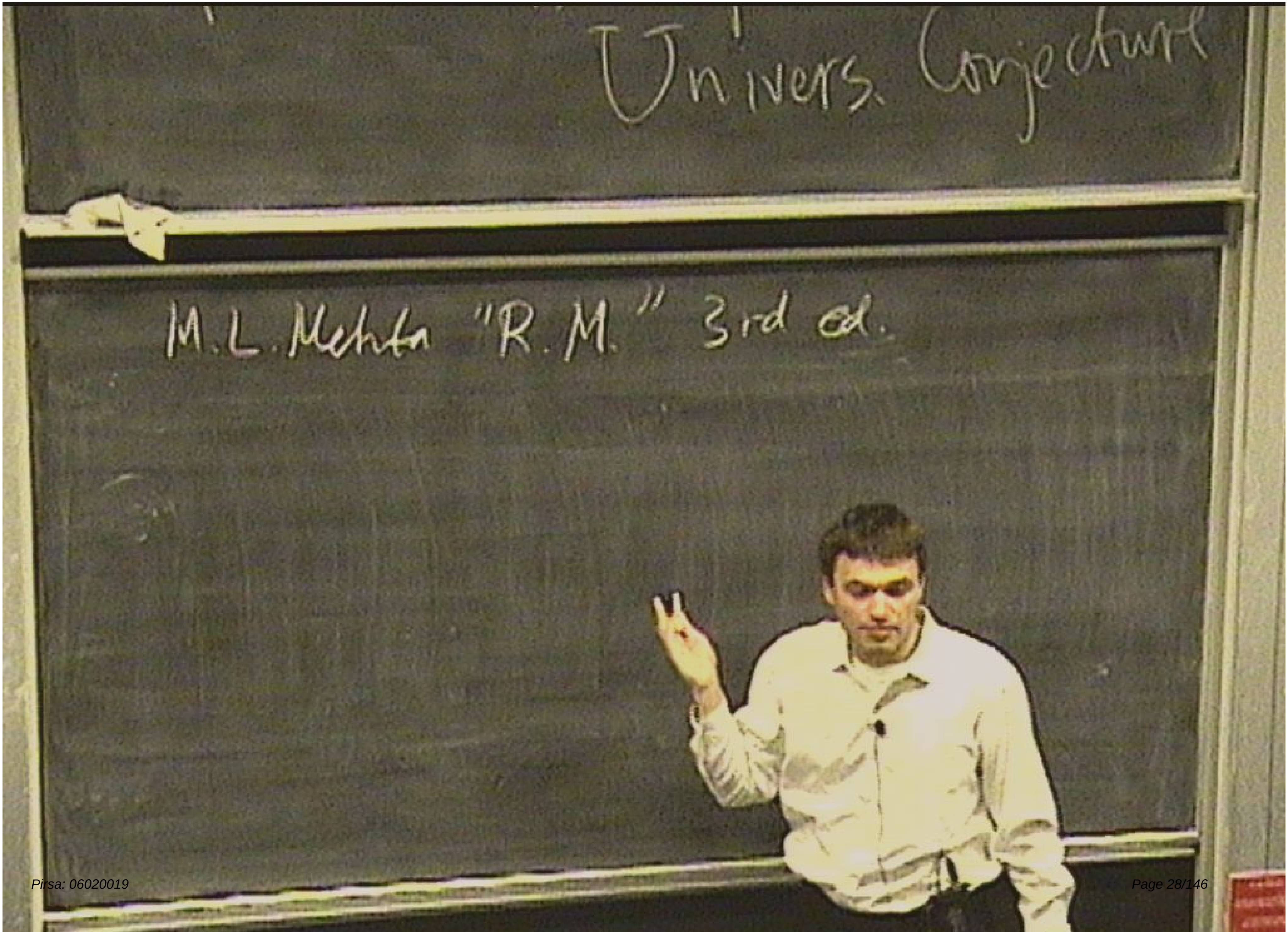
-1

Q: What distr. of RM elts should we choose?

A: Up to gen symm reqs, the answer (as $N \to \infty$) is indep. of distribs.

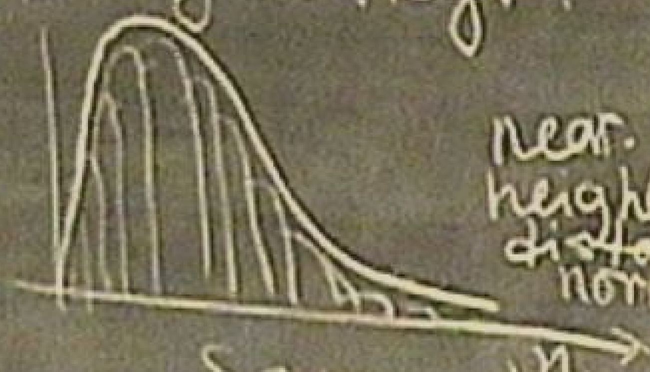
Univers. Conjecture

M. L. Mehta "


Univers. Conjecture
M.L. Mehta "R.M." 3rd ed.

norm by mean spacing

Spacing histogr: E. Wigner



Herm. matr.

near. neighb. distance norm

$\lambda_1(M) \le \dots \le \lambda_N(M)$

$\lambda_k(M) - \lambda_{k-1}(M)$

$M = M^*$

$N \times N$

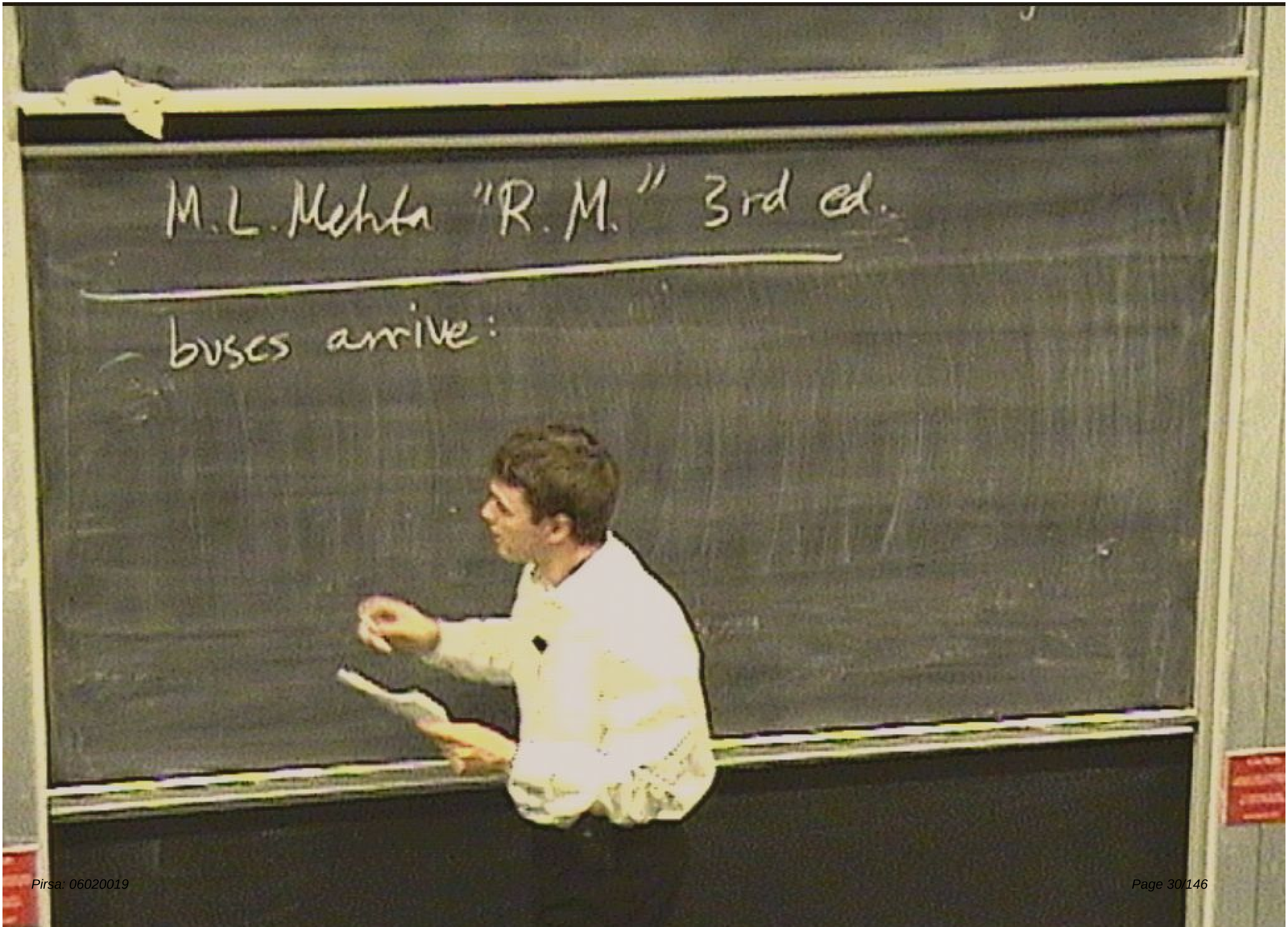
i.i.d.

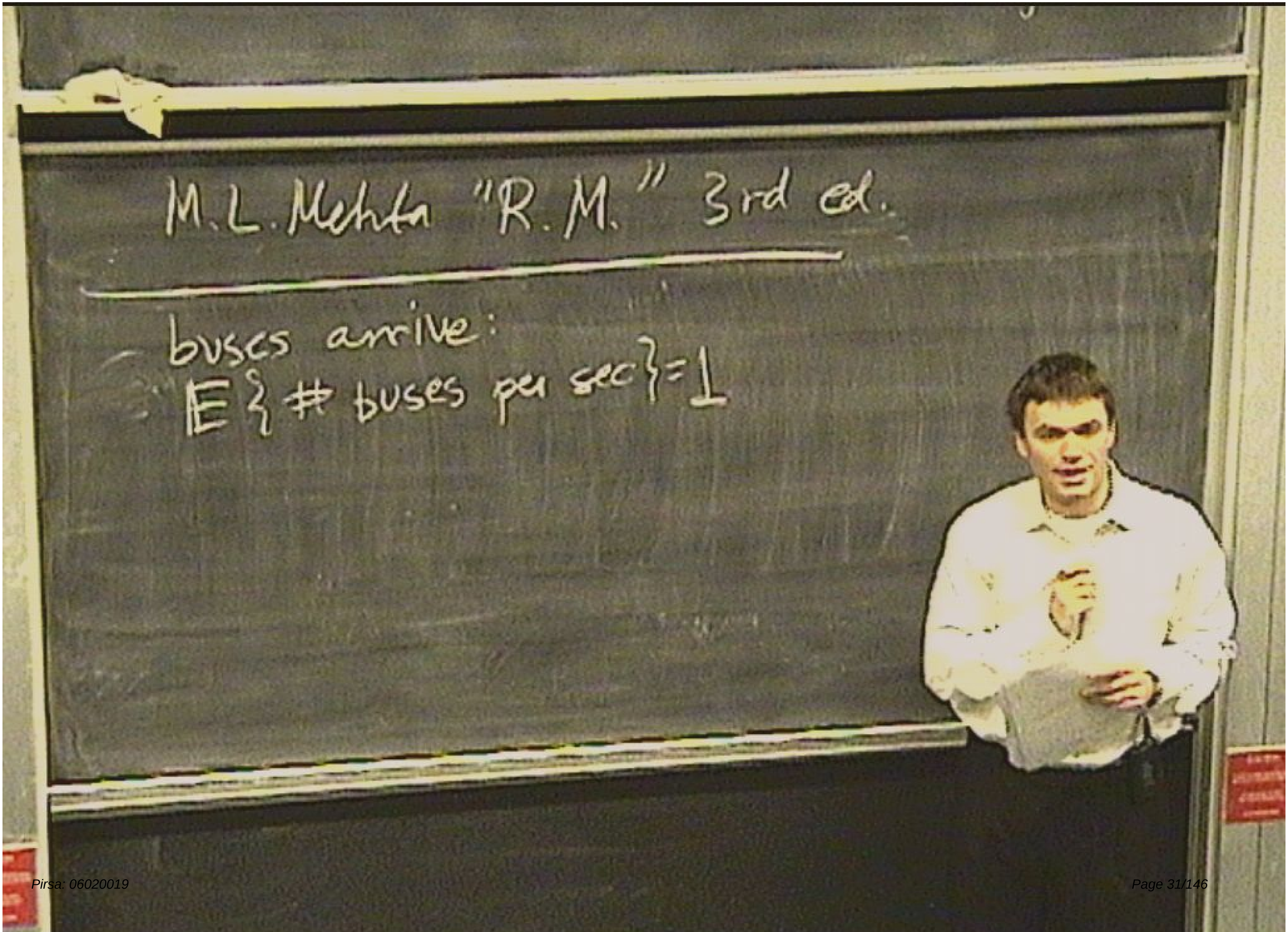
M_{ii}

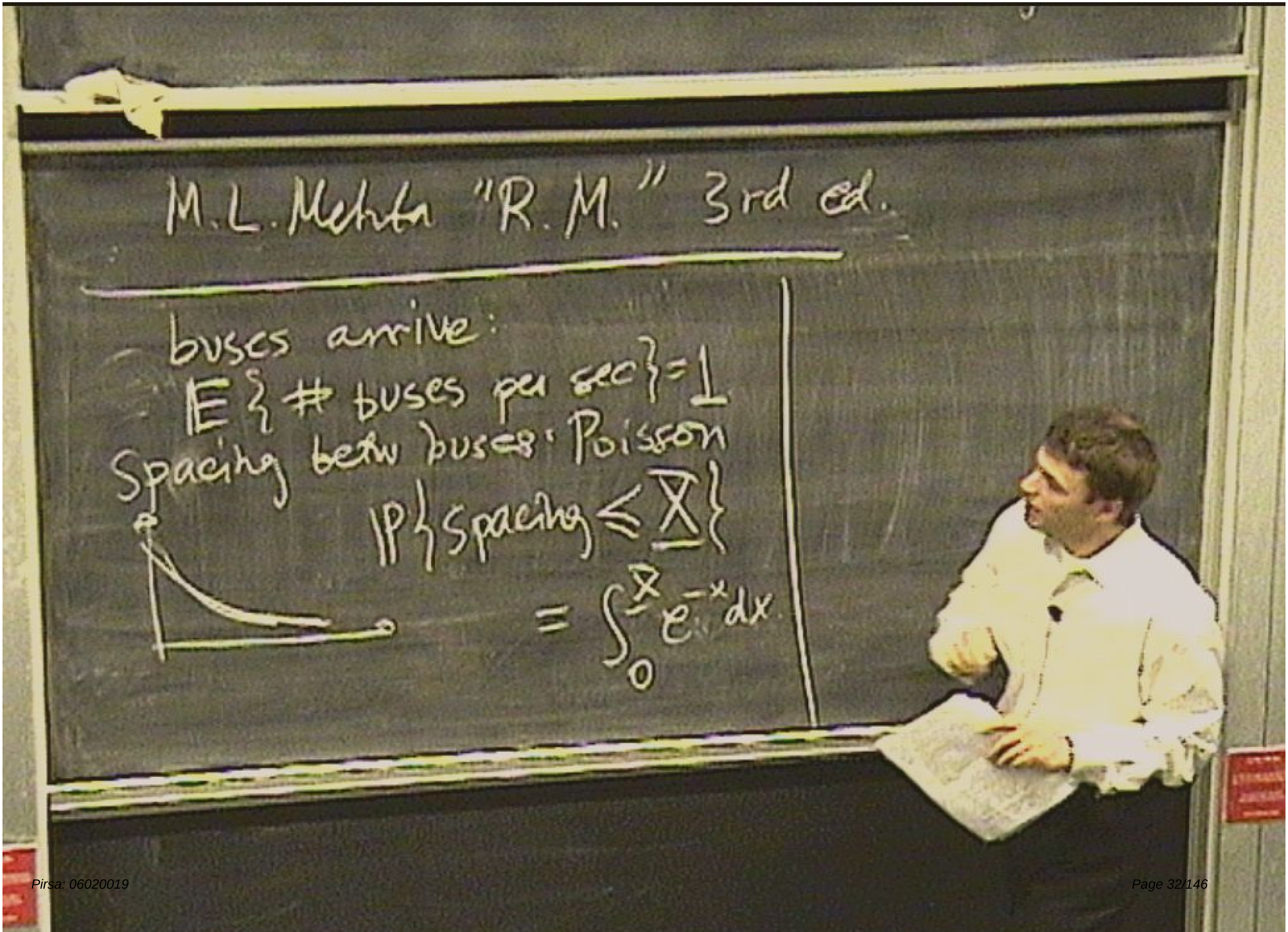
M^R_{ij}, M^I_{ij} $i<j$

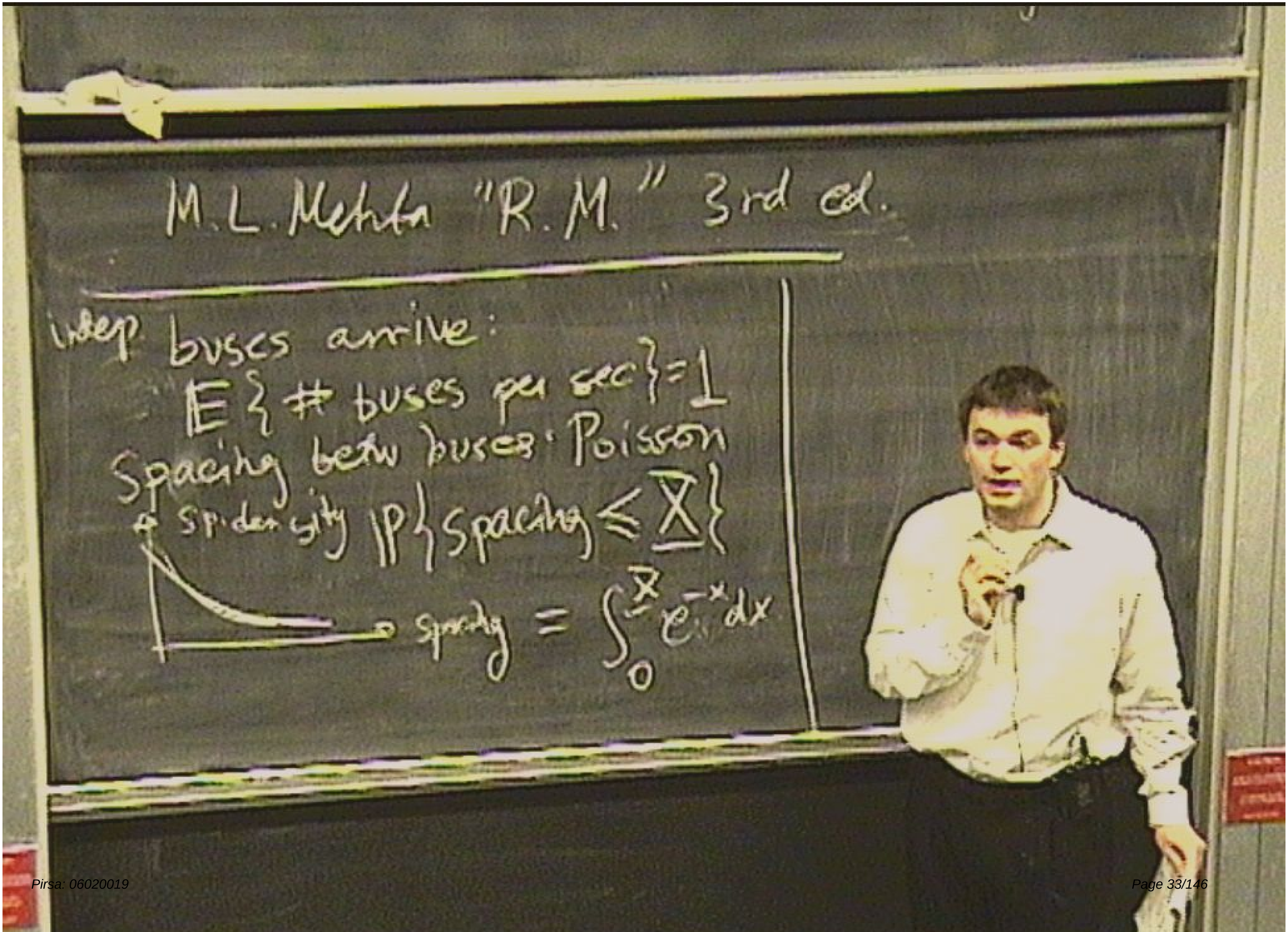
same histogram

$\sim a x e^{-bx^2}$


M.L. Mehta "R.M." 3rd ed.
buses arrive:


M.L. Mehta "R.M." 3rd ed.
buses arrive:
E{# buses per sec}=1


M.L. Mehta "R.M." 3rd ed.
buses arrive:
$\mathbb{E}\{\#\text{ buses per sec}\}=1$
Spacing betw buses: Poisson
$\mathbb{P}\{\text{spacing} \leq X\}$
$= \int_0^X e^{-x}\,dx$


M.L. Mehta "R.M." 3rd ed.
indep. buses arrive:
E{# buses per sec} = 1
Spacing betw buses: Poisson
sp. density
P{spacing ≤ X}
spacing
= ∫_0^X e^{-x} dx

M.L. Mehta "R.M." 3rd ed.

Cuernavaca

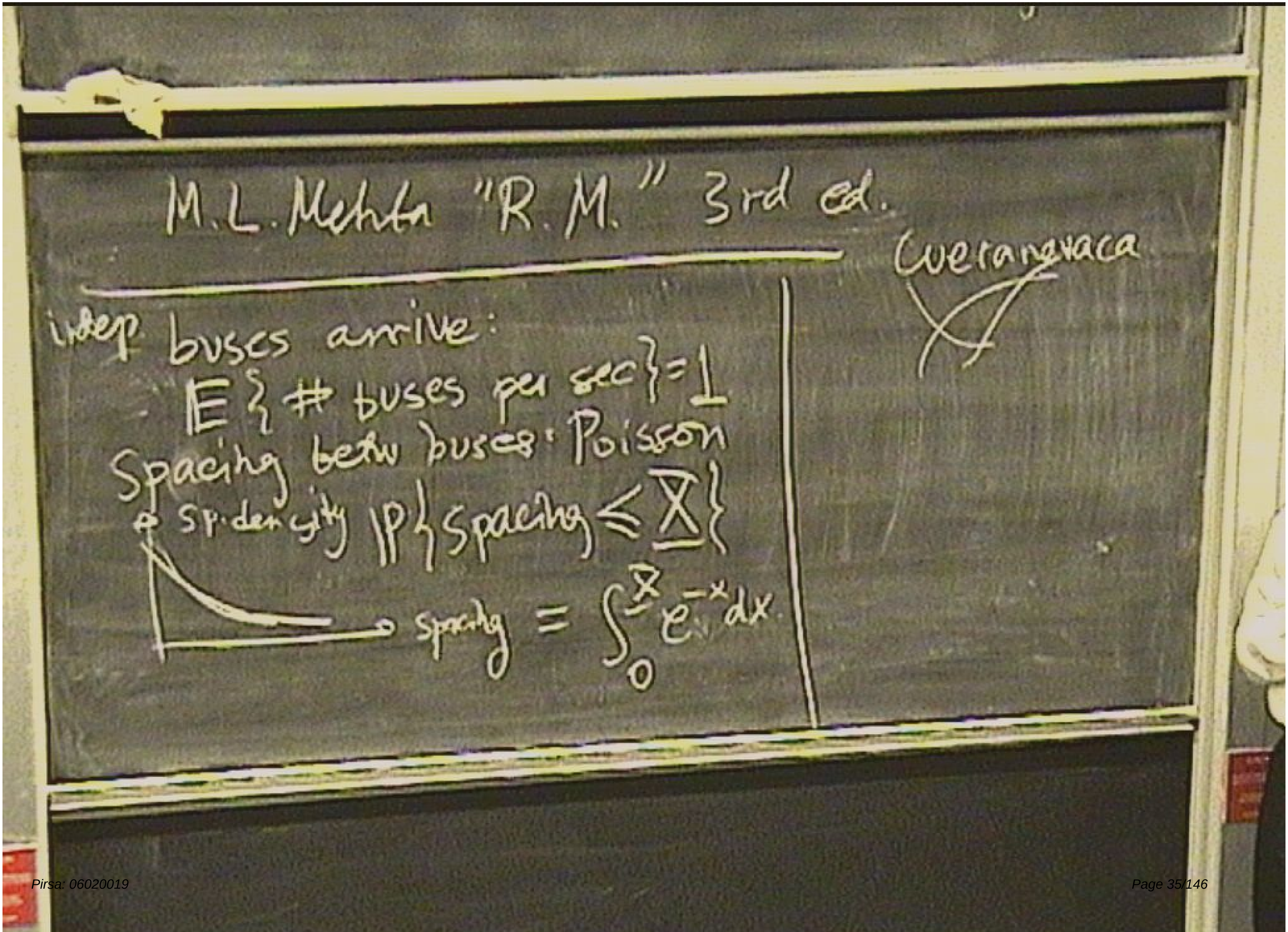
indep buses arrive:

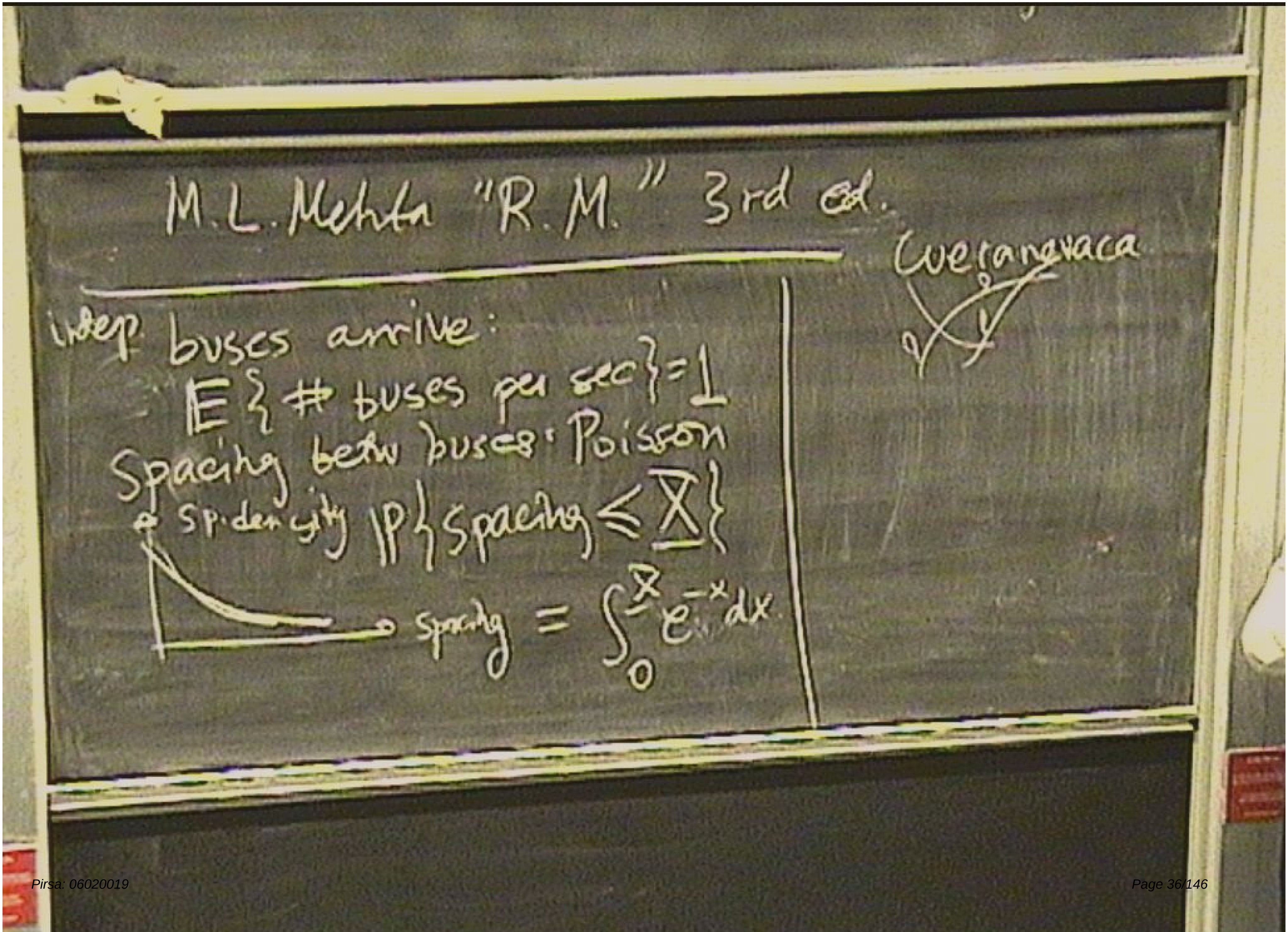
$\mathbb{E}\{\#\text{ buses per sec}\} = 1$

Spacing betw buses: Poisson

sp. density $\quad \mathbb{P}\{\text{spacing} \leq X\}$

spacing $= \int_0^X e^{-x}\,dx.$

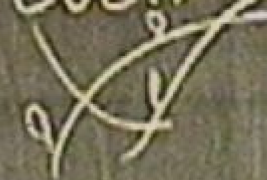

M.L. Mehta "R.M." 3rd ed.
Cuernavaca
indep buses arrive:
$\mathbb{E}\{\#$ buses per sec$\} = 1$
Spacing betw buses: Poisson
sp. density $\mathbb{P}\{$spacing $\leq X\}$
spacing $= \int_0^X e^{-x}\,dx$


M.L. Mehta "R.M." 3rd ed.
Cuernavaca
indep. buses arrive:
$\mathbb{E}\{\#\text{ buses per sec}\} = 1$
Spacing betw buses: Poisson
sp. density
$\mathbb{P}\{\text{spacing} \leq X\}$
spacing
$= \int_0^X e^{-x}\,dx$

M.L. Mehta "R.M." 3rd ed.

Šeba

Cuernavaca

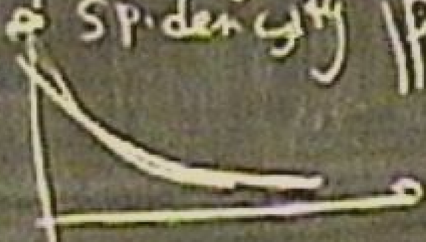


indep buses arrive:

$\mathbb{E}\{\#\text{ buses per sec}\} = 1$

Spacing betw buses: Poisson

sp. density $\mathbb{P}\{\text{spacing} \leq X\}$



spacing $= \int_0^X e^{-x}\,dx.$

M.L. Mehta "R.M." 3rd ed.

Šeba

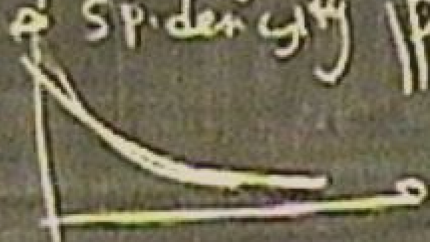
Cuernavaca

indep buses arrive:

$$\mathbb{E}\{\#\text{ buses per sec}\} = 1$$

Spacing betw buses: Poisson

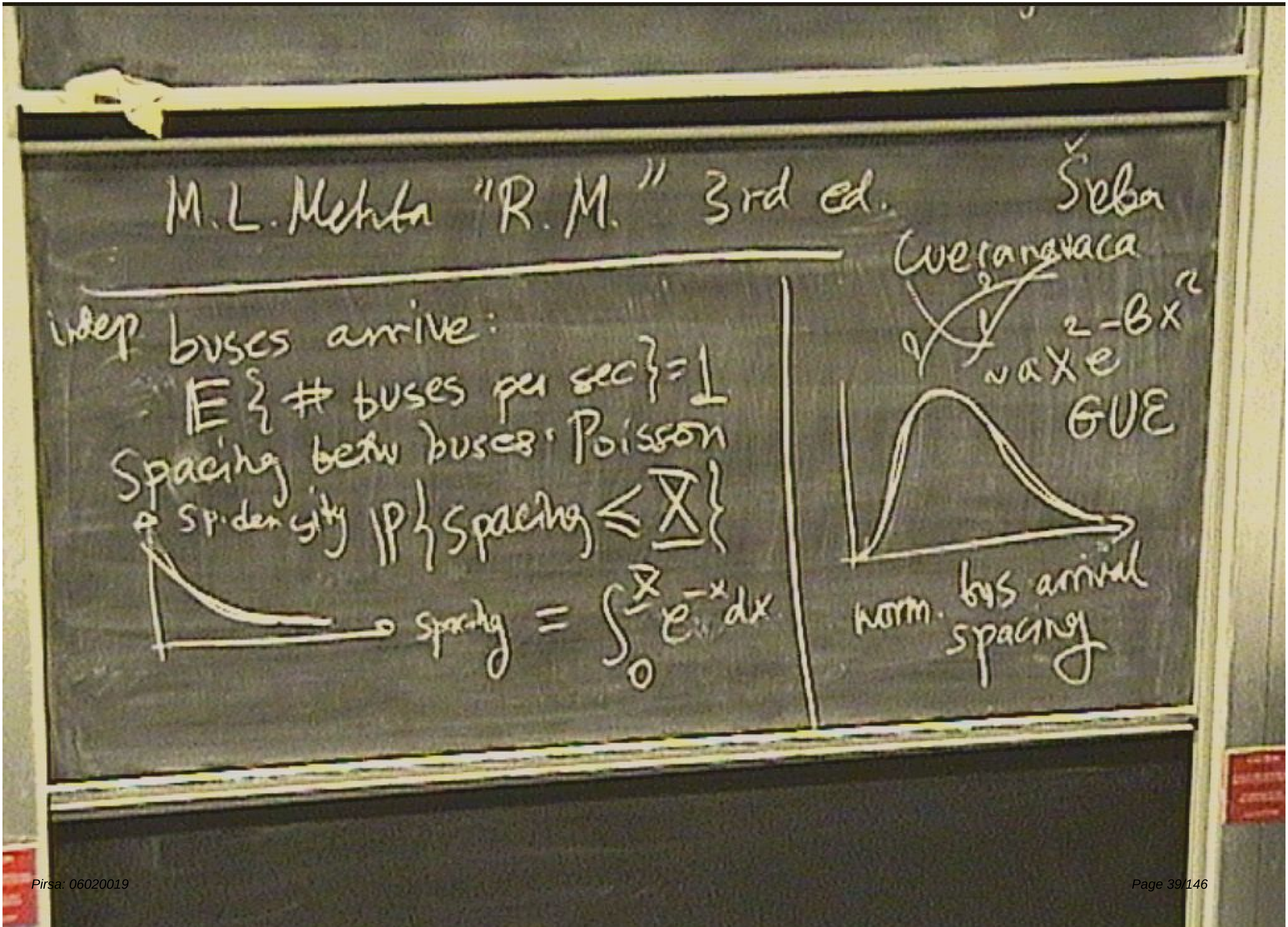
sp. density

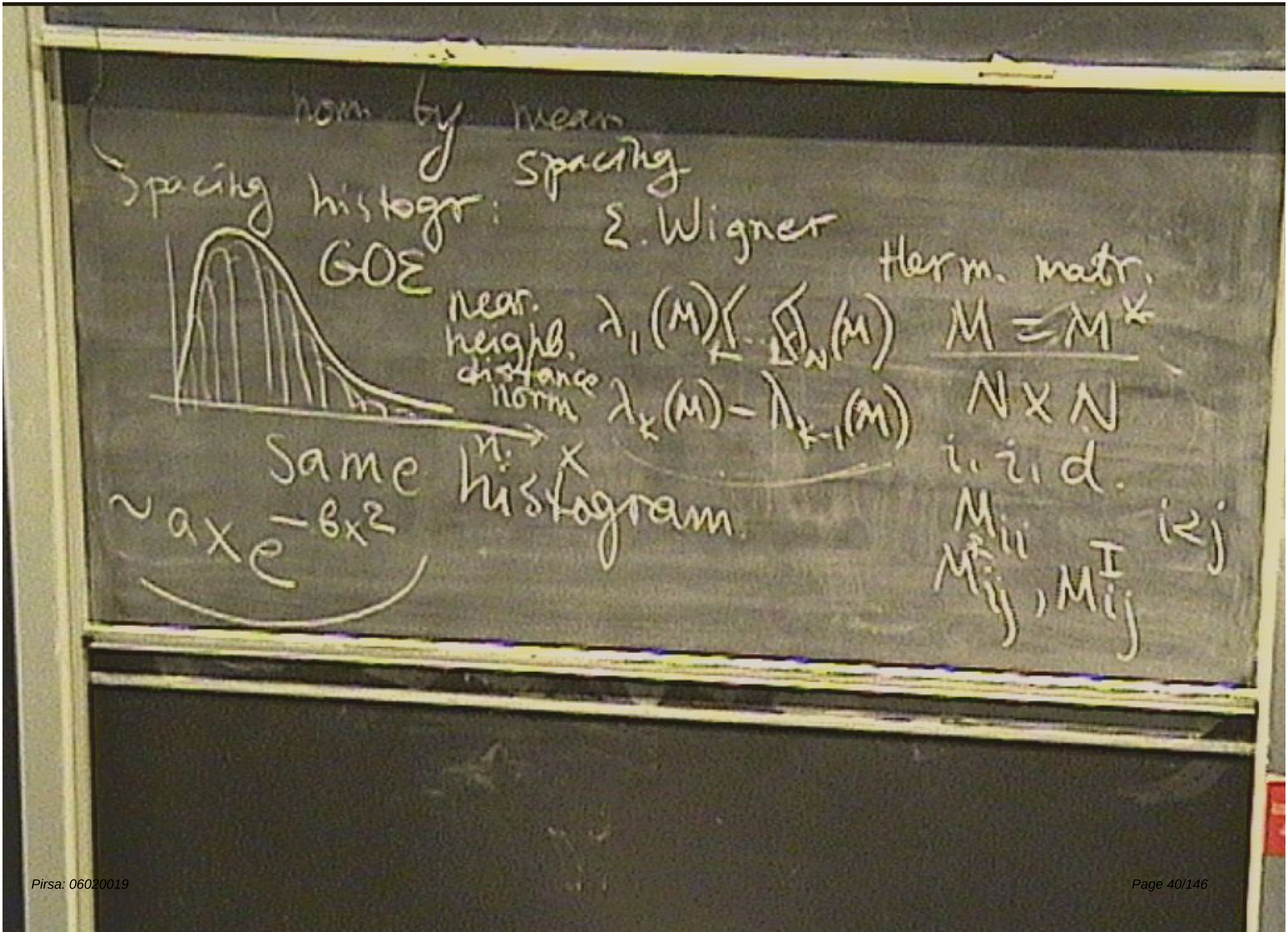


spacing

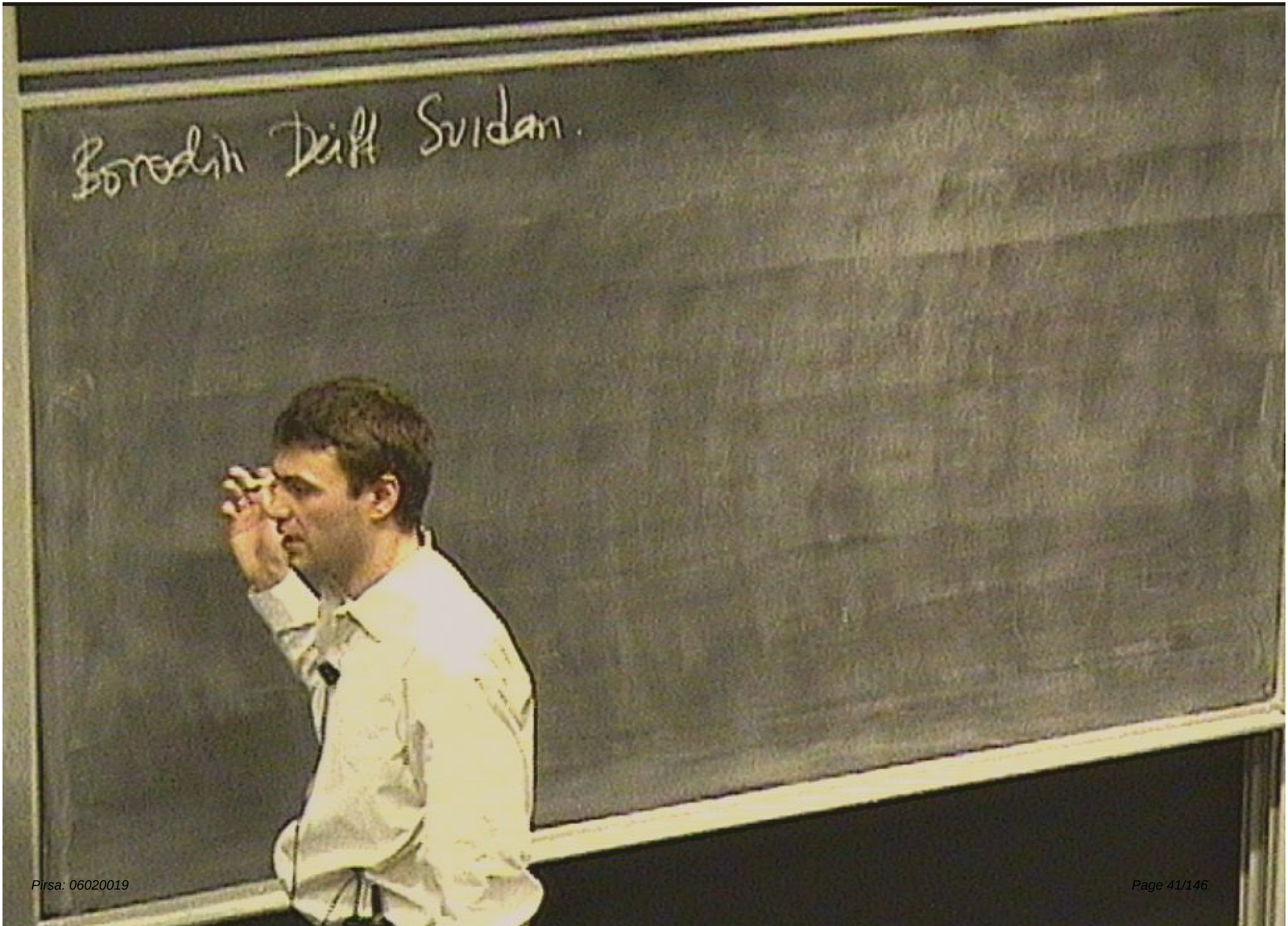
$$\mathbb{P}\{\text{spacing} \le X\} = \int_0^X e^{-x}\,dx$$

norm. bus arrival spacing


M.L. Mehta "R.M." 3rd ed.
Šeba
Cuernavaca
indep buses arrive:
E{# buses per sec} = 1
Spacing betw buses: Poisson
sp. density
P{spacing ≤ X}
spacing
= $\int_0^X e^{-x}dx$
$\sim a x^2 e^{-Bx^2}$
GUE
norm. bus arrival spacing

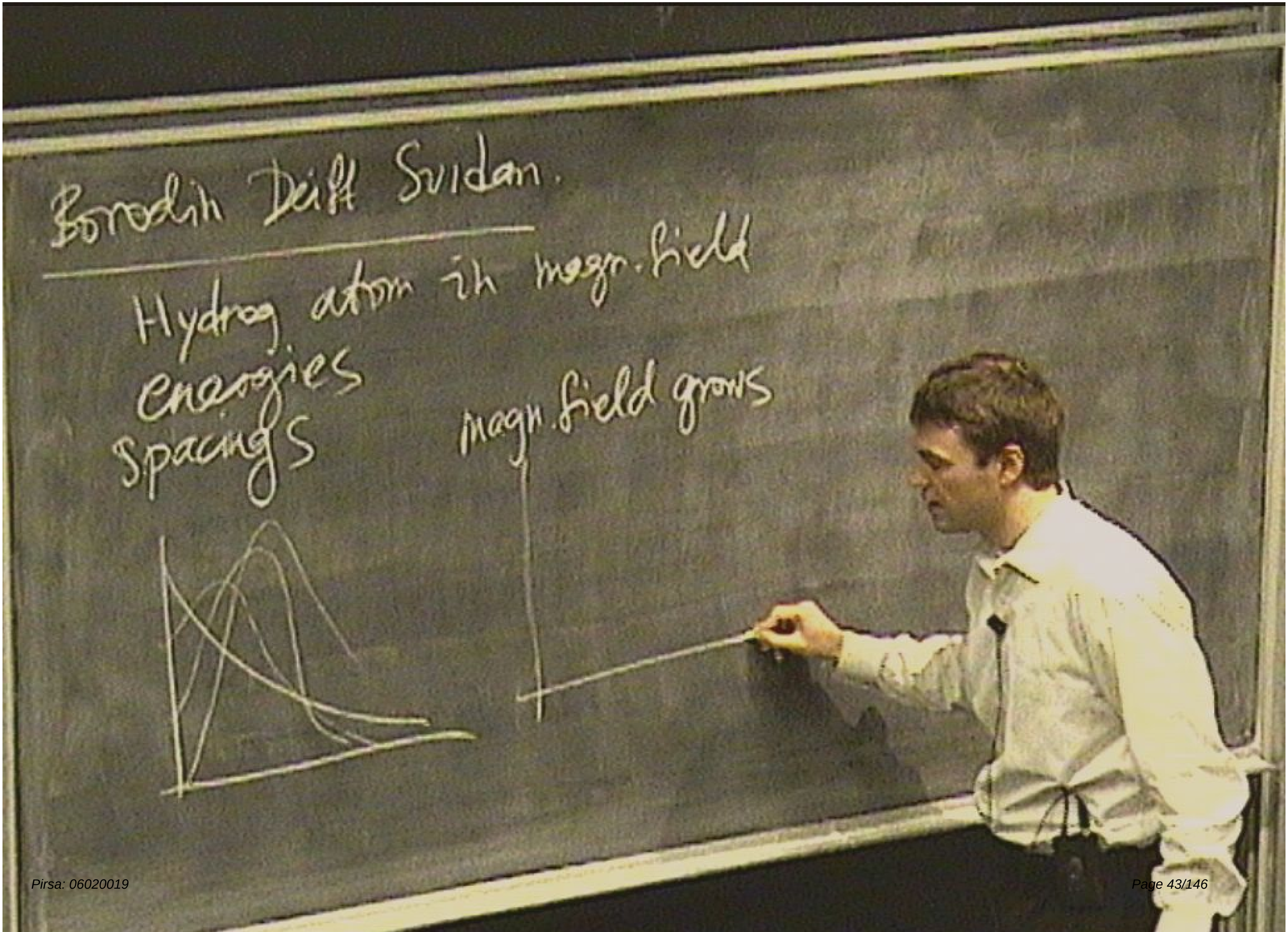


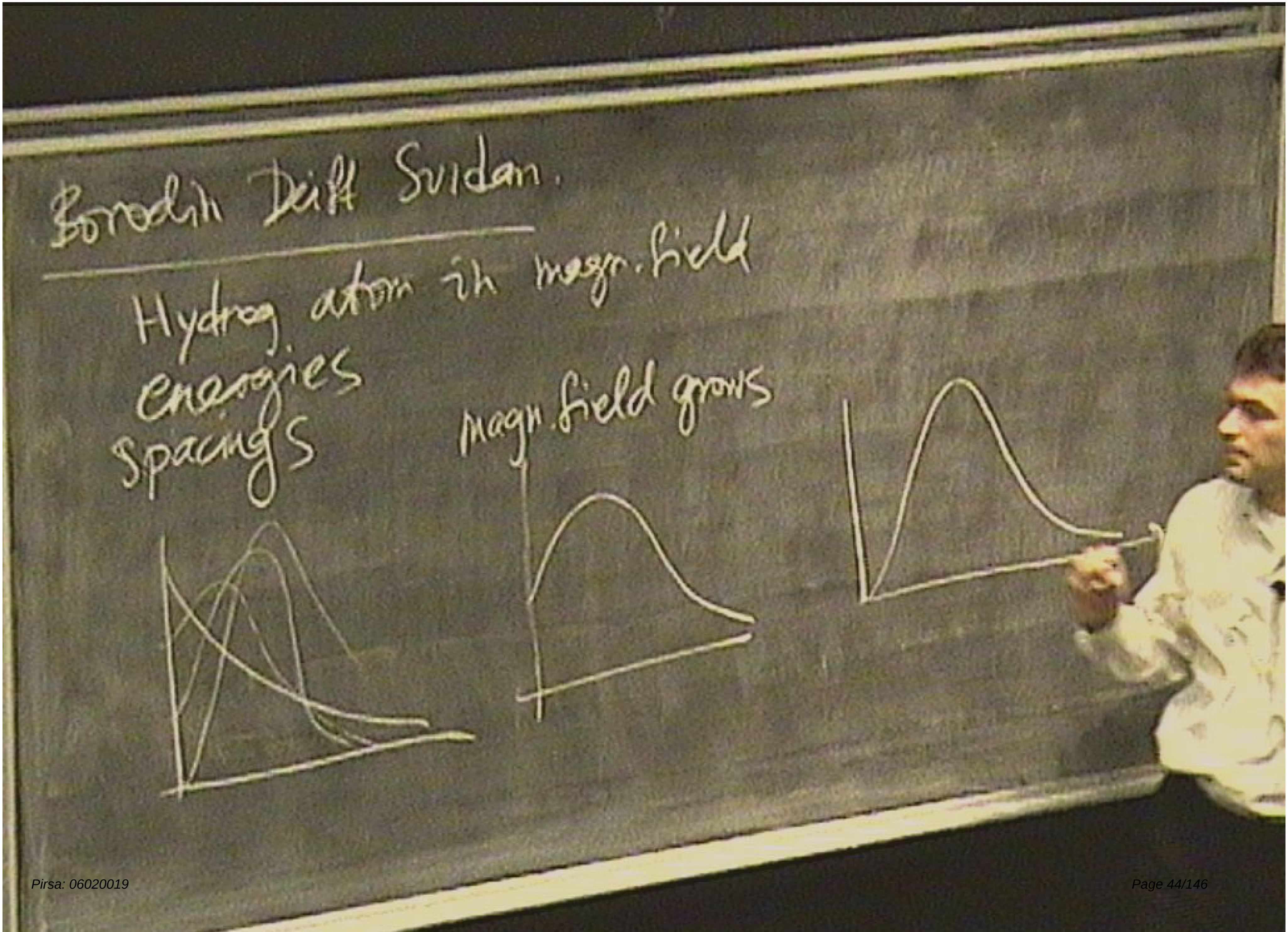
norm by mean spacing
Spacing histogr:
GOE
E. Wigner
Herm. matr.
near. neighb. distance norm
λ1(M) ≤ ... ≤ λN(M)
M = M*
N x N
λk(M) − λk−1(M)
x
Same histogram
i.i.d.
~ a x e^(−bx²)
Mii
i<j
MRij, MIij


Borodin Deift Suidan.

Borodin Deift Suidan.

Hydrog atom in magn. field
energies


Borodin Deift Suidan.
Hydrog atom in magn. field
energies
spacings
magn. field grows


Borodin Deift Suidan.
Hydrog atom in magn. field
energies
spacings
magn. field grows

Borodin Deift Suidan.

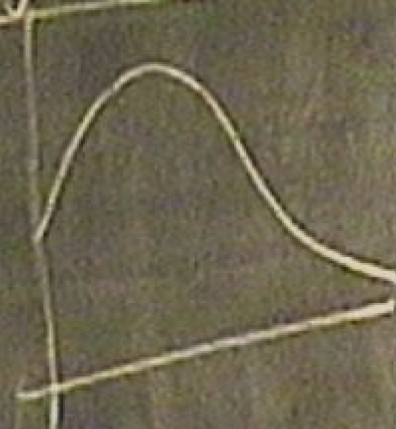
Hydrog atom in magn. field

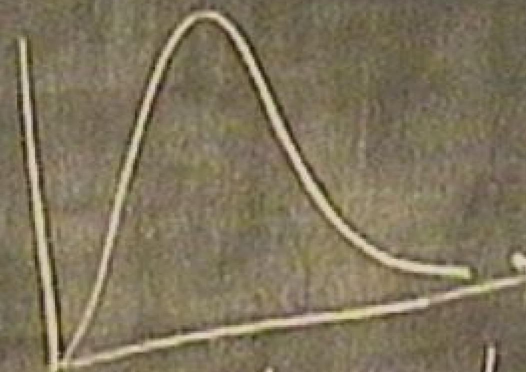
energies

spacings

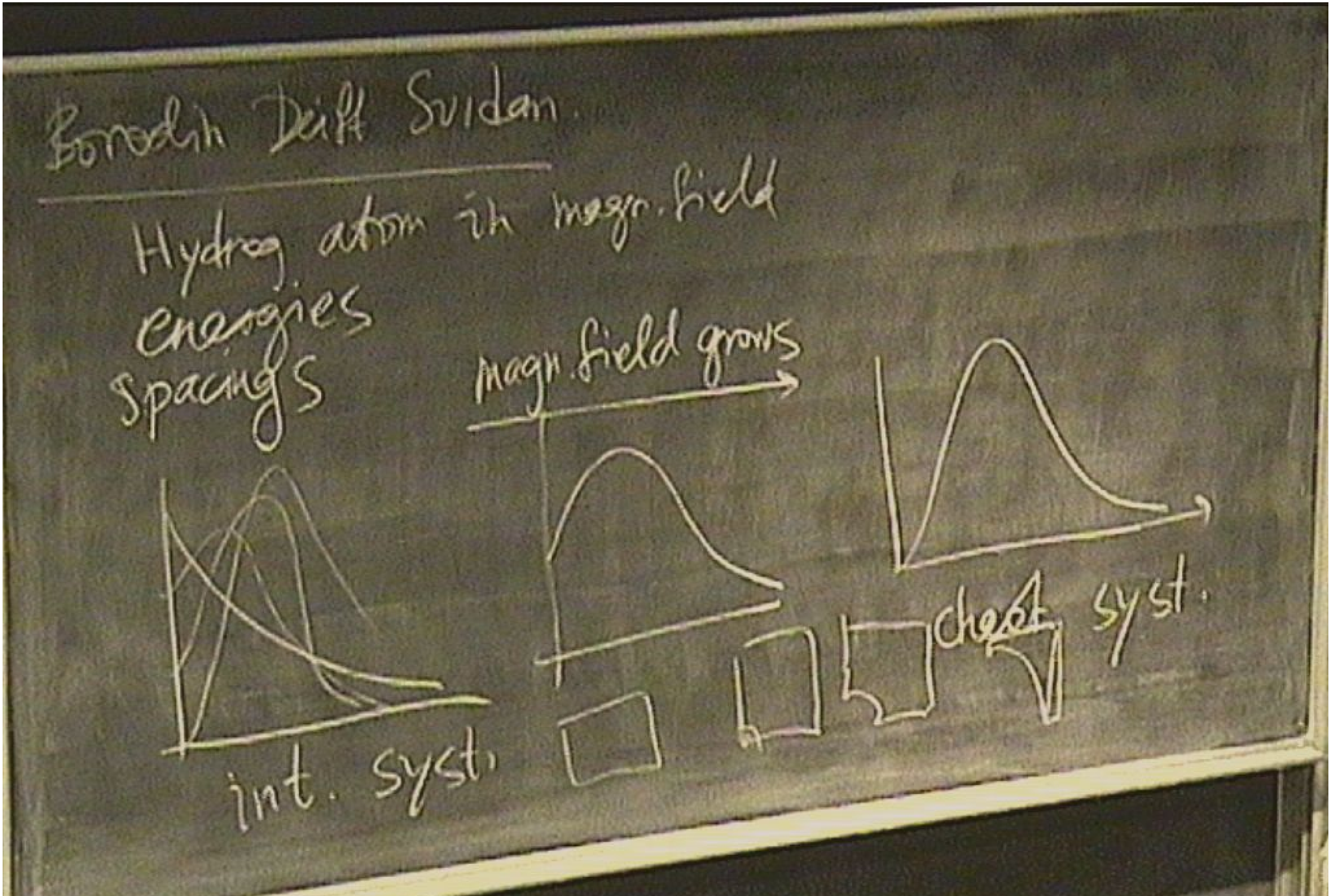
magn. field grows

int. syst.

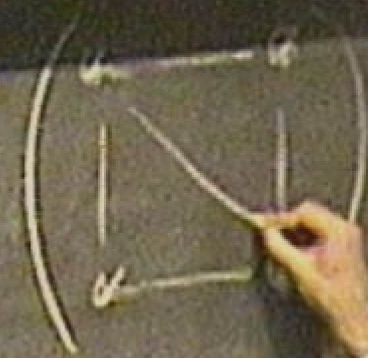


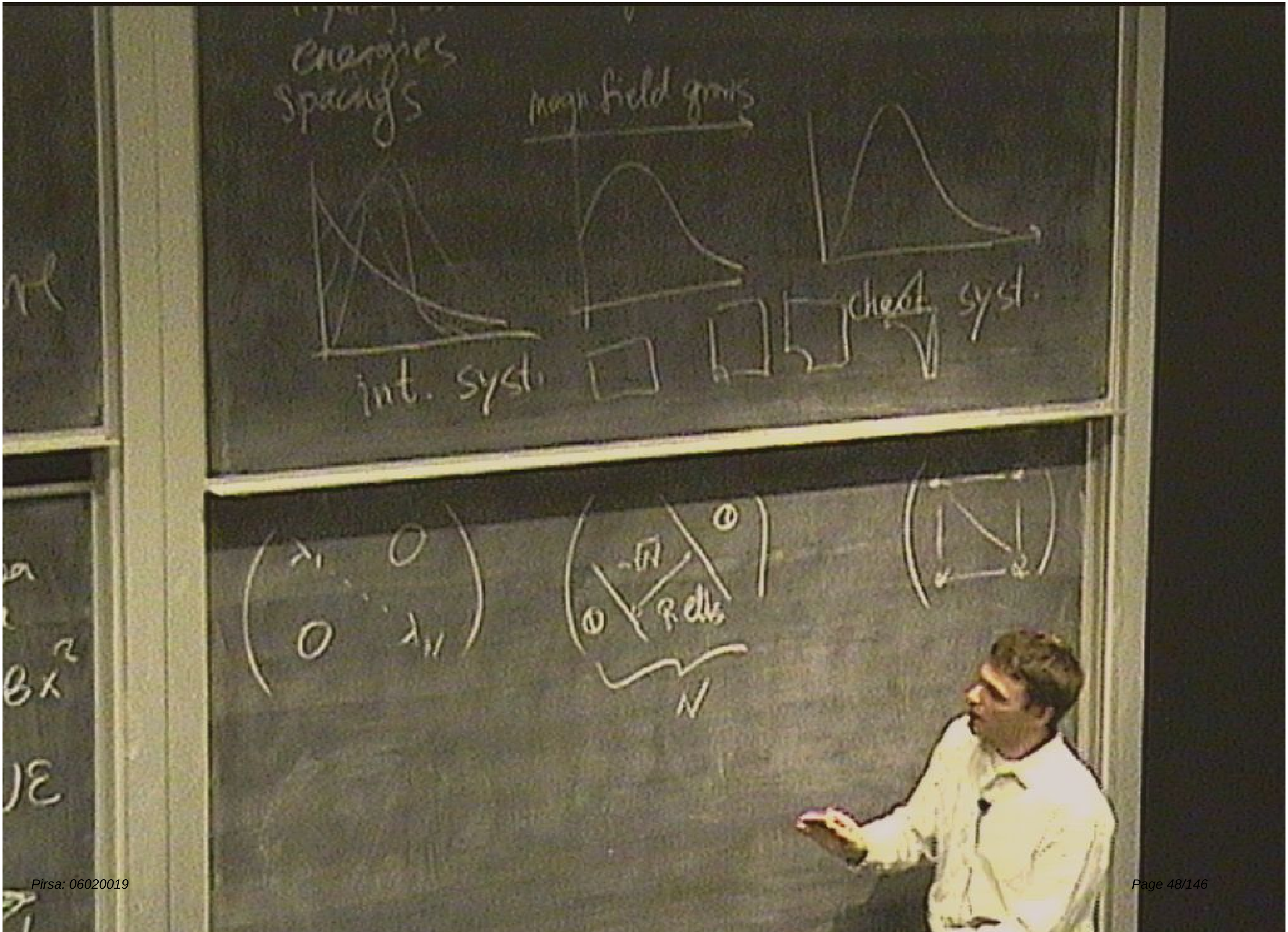


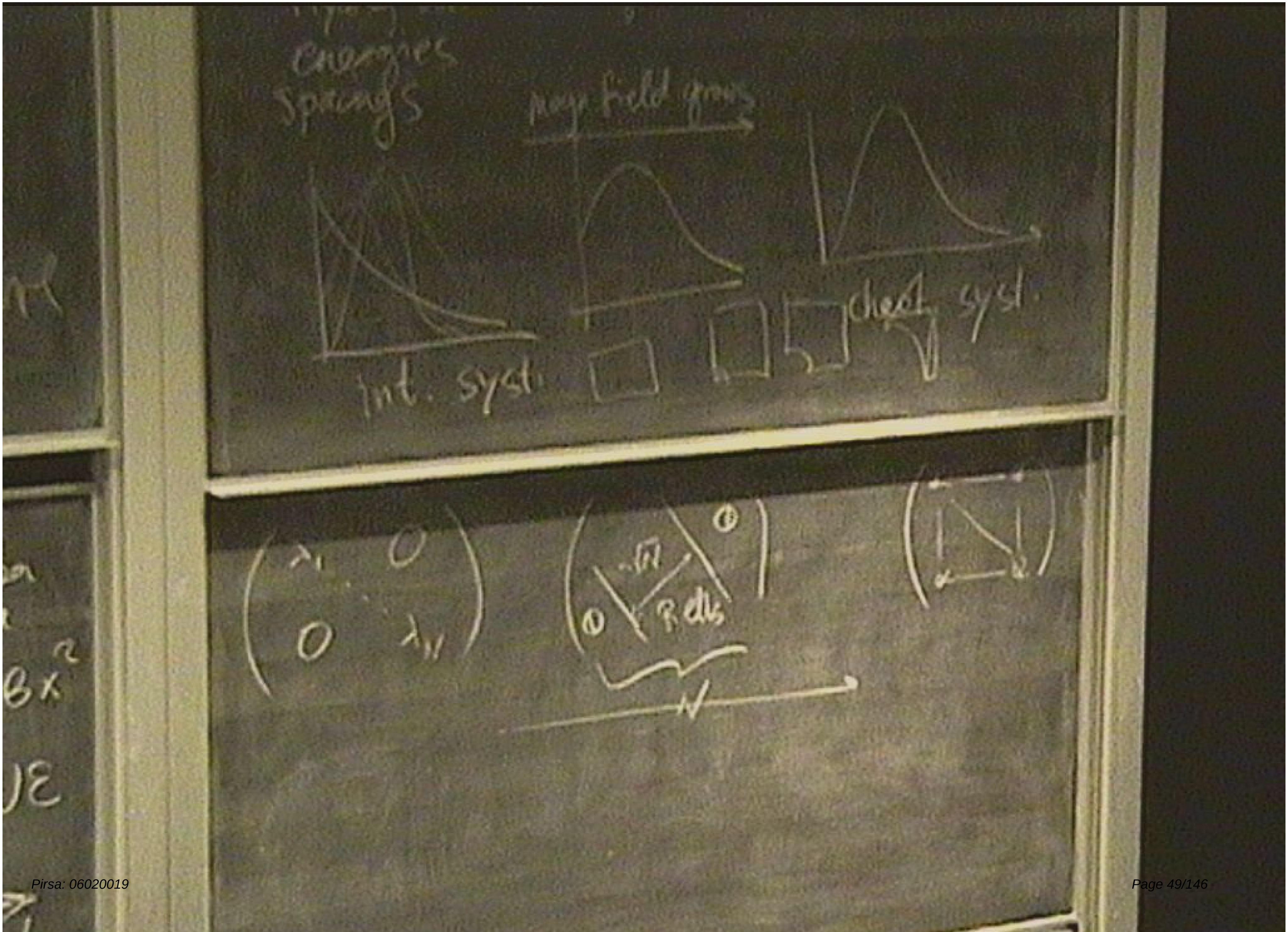
chaot. syst.

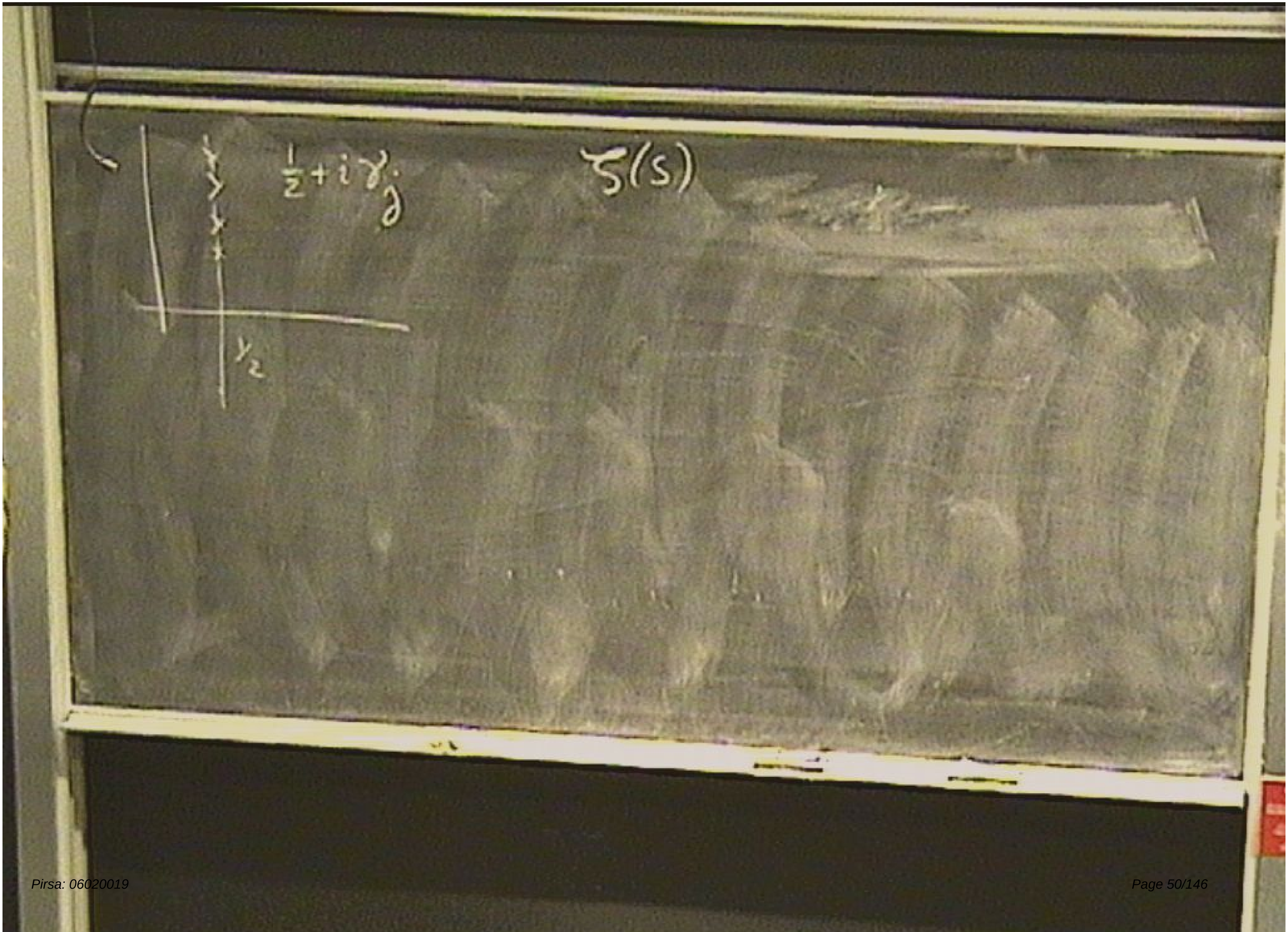

Borodin Drift Suidan
Hydrog atom in magn. field
energies
spacings
magn. field grows
int. syst.
chaot syst.

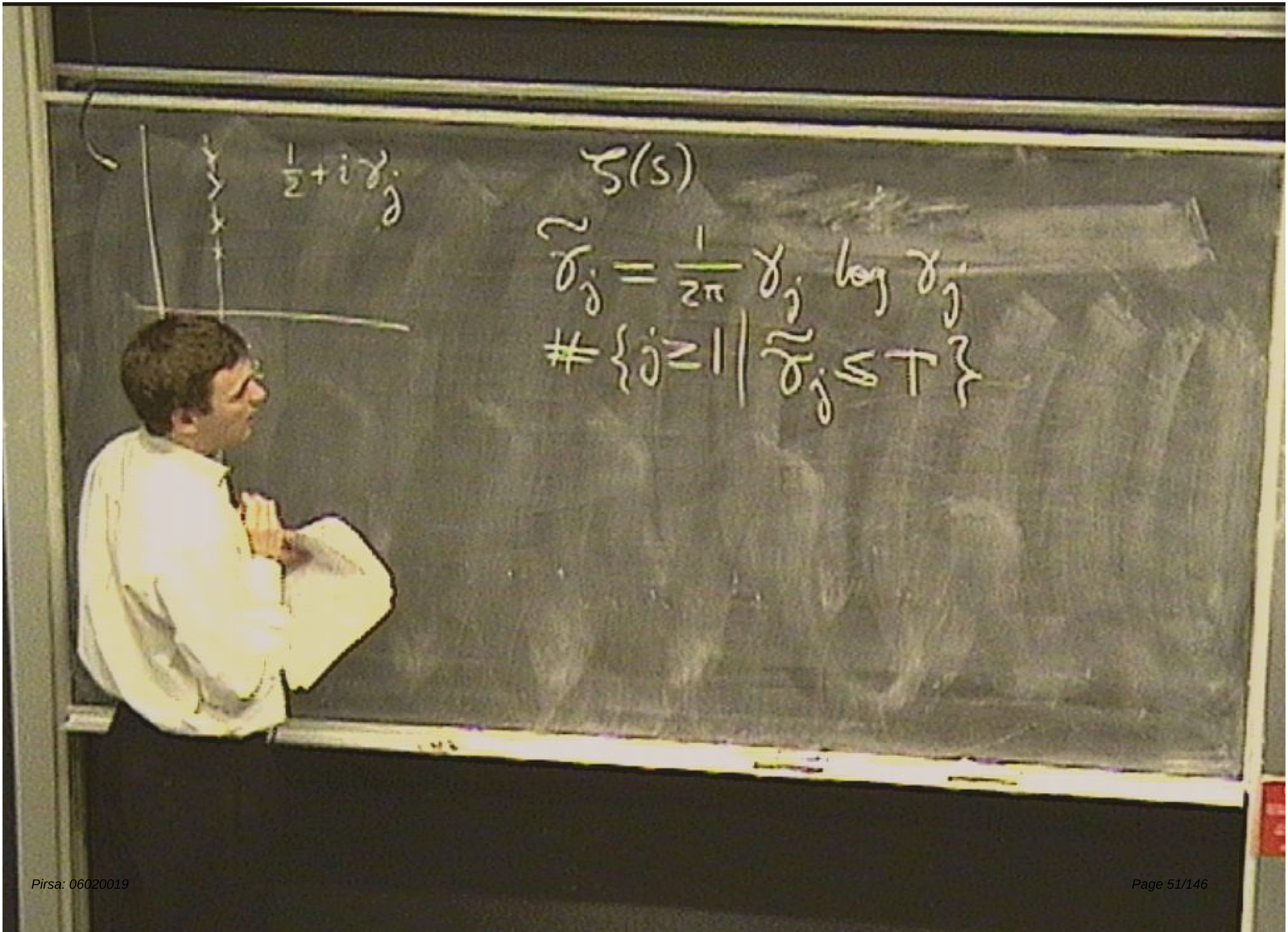
$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

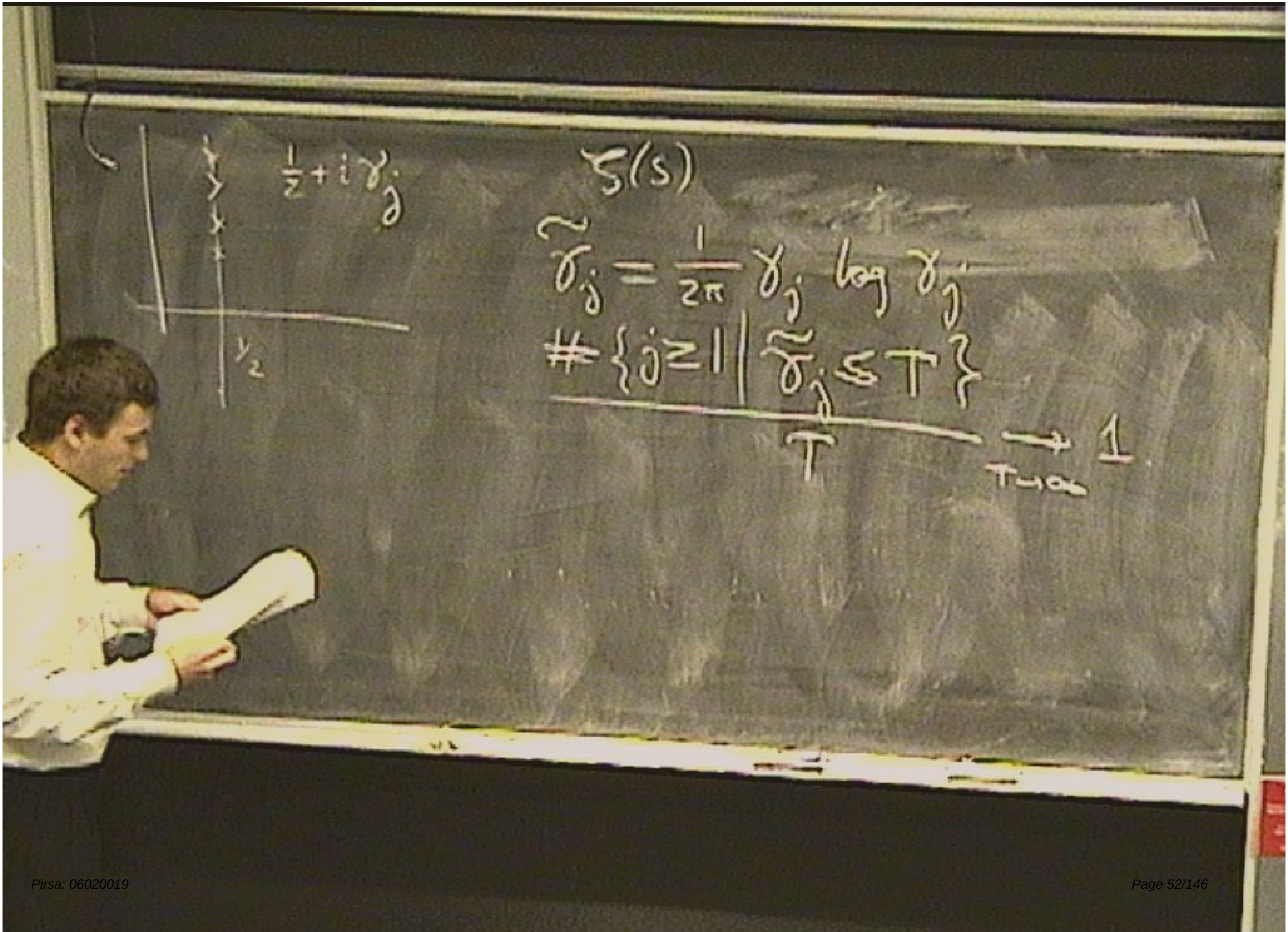



energies
spacings
magn. field goes
int. syst.
chaot. syst.
N

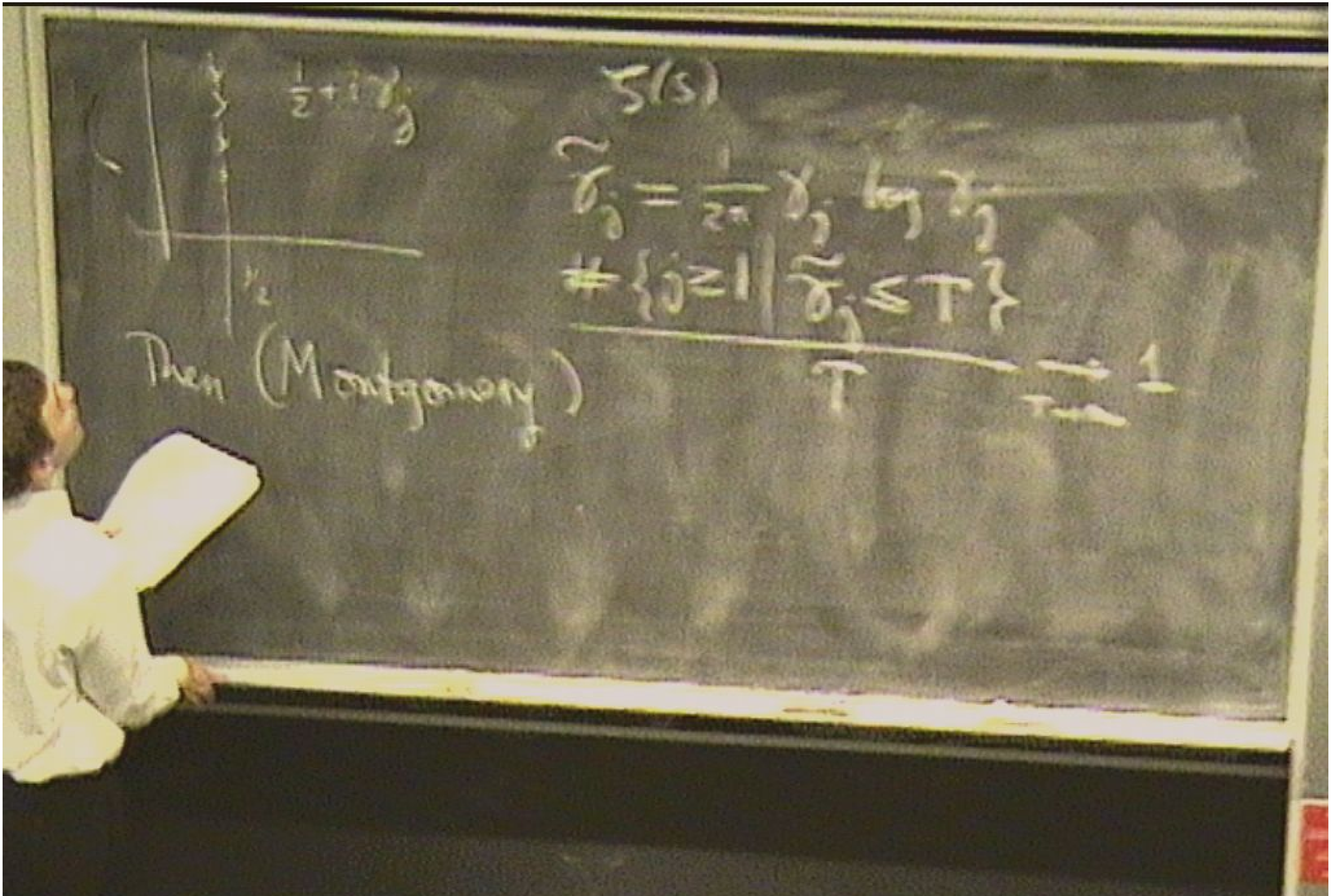

energies
spacings
int. syst.
chaot. syst.
N

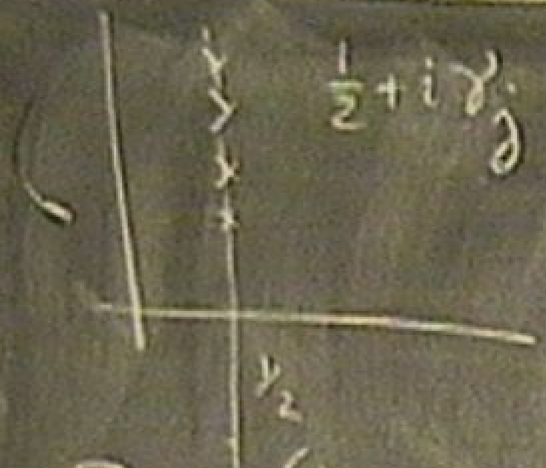

½ + iγⱼ
ζ(s)
½


$\frac{1}{2}+i\gamma_j$
$\zeta(s)$
$\tilde{\gamma}_j = \frac{1}{2\pi}\gamma_j \log \gamma_j$
$\#\{j \geq 1 \mid \tilde{\gamma}_j \leq T\}$



$\frac{1}{2} + i\gamma_j$
$\frac{1}{2}$
$\zeta(s)$
$\tilde{\gamma}_j = \frac{1}{2\pi}\gamma_j \log \gamma_j$
$\frac{\#\{j \geq 1 \mid \tilde{\gamma}_j \leq T\}}{T} \xrightarrow[T \to \infty]{} 1$


$\frac{1}{2}+i\gamma_j$
$\frac{1}{2}$
$\zeta(s)$
$\tilde{\gamma}_j = \frac{1}{2\pi}\gamma_j \log \gamma_j$
$\frac{\#\{j\geq 1 \mid \tilde{\gamma}_j \leq T\}}{T} \xrightarrow[T\to\infty]{} 1$
Then (Montgomery)



$\frac{1}{2} + i\gamma_j$

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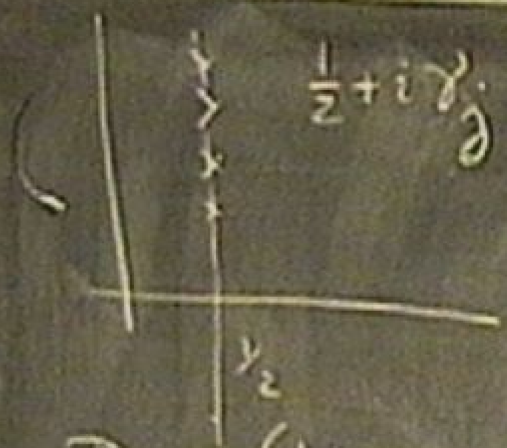
$$M_{ij}^{\ast}, M_{ij}^{I}$$



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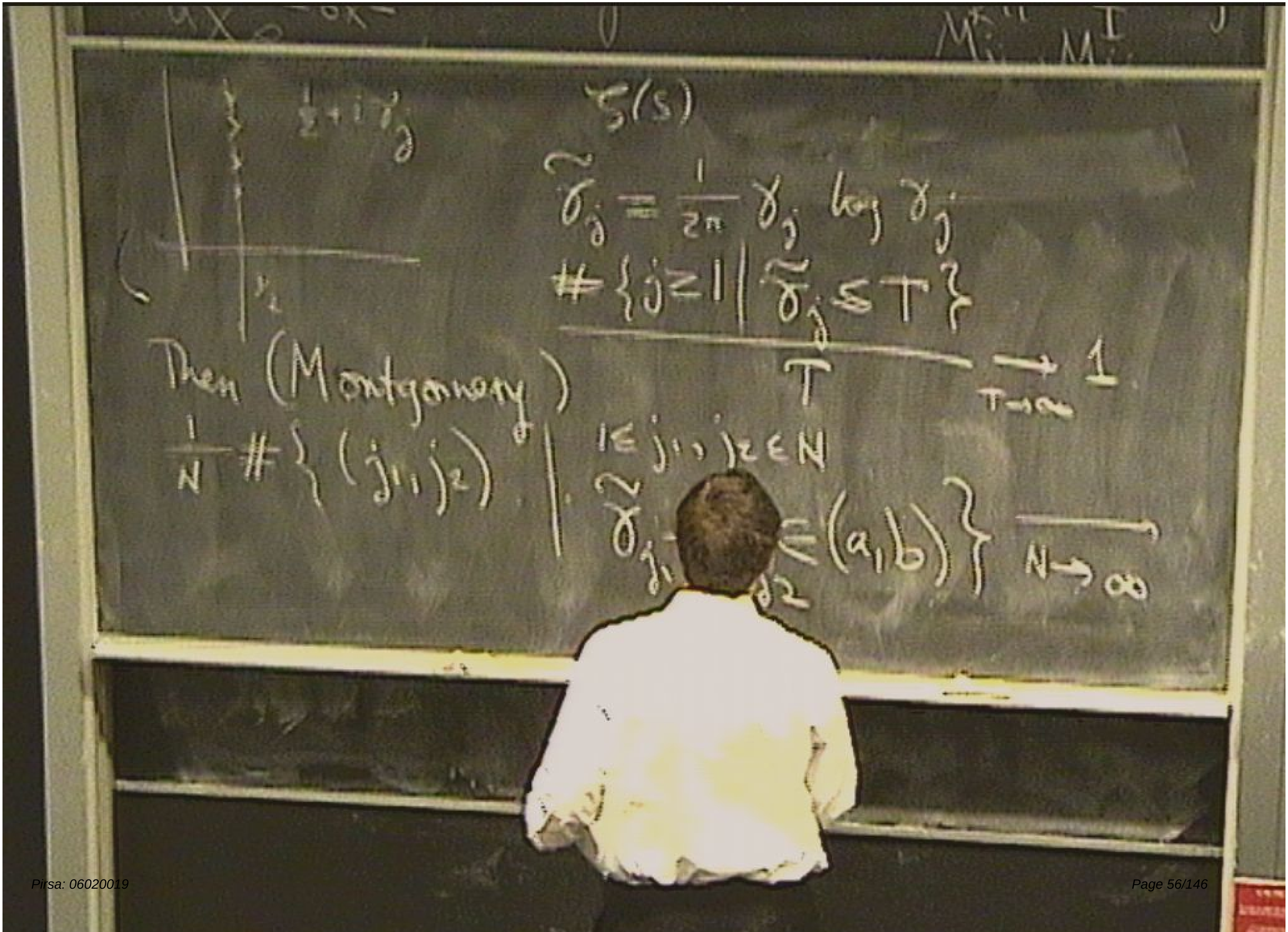
$$\frac{1}{N}\#\left\{(j_1, j_2) \,\middle|\, \begin{array}{l} 1 \leq j_1, j_2 \in N \\ \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b) \end{array}\right\}$$



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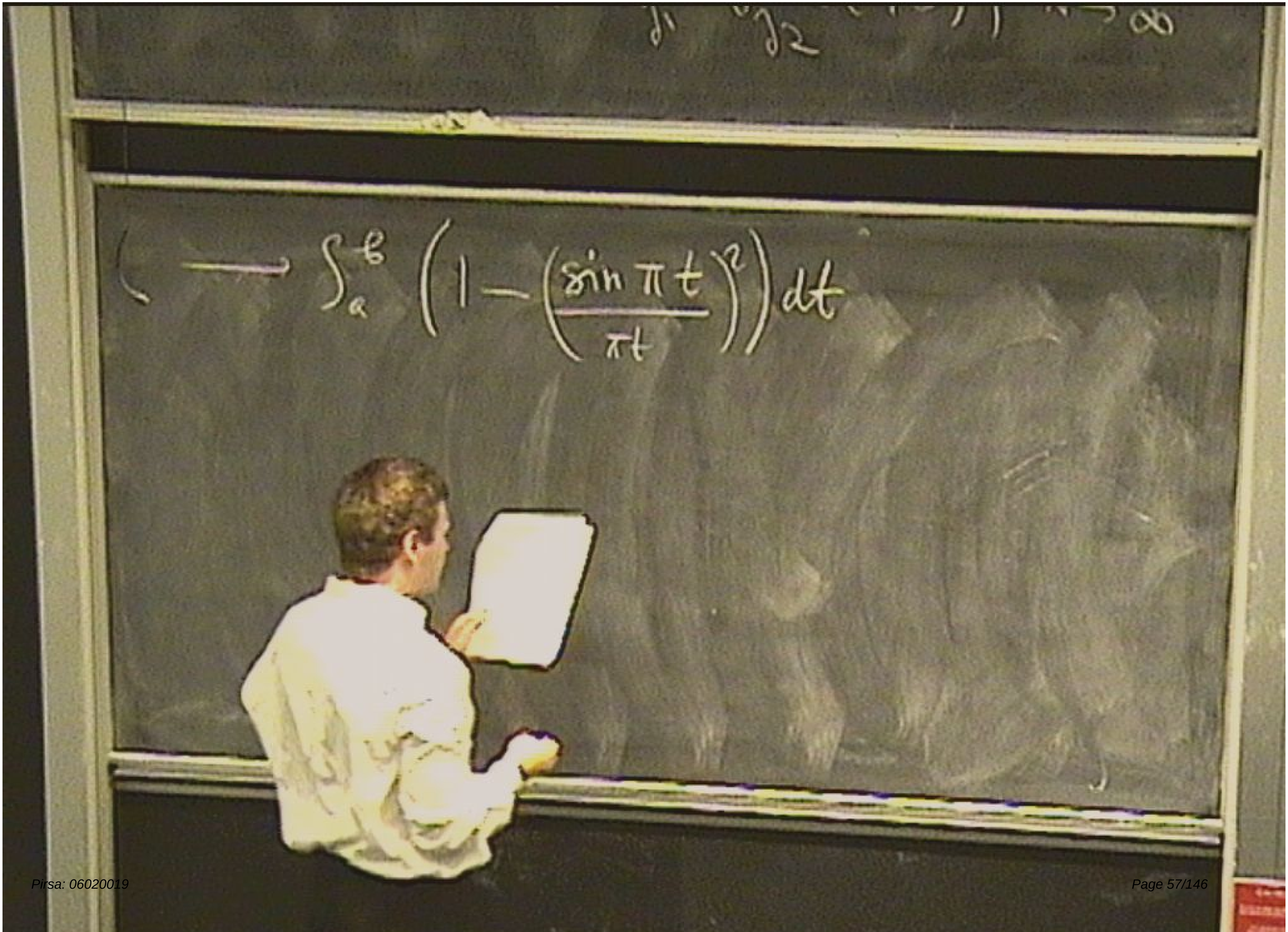
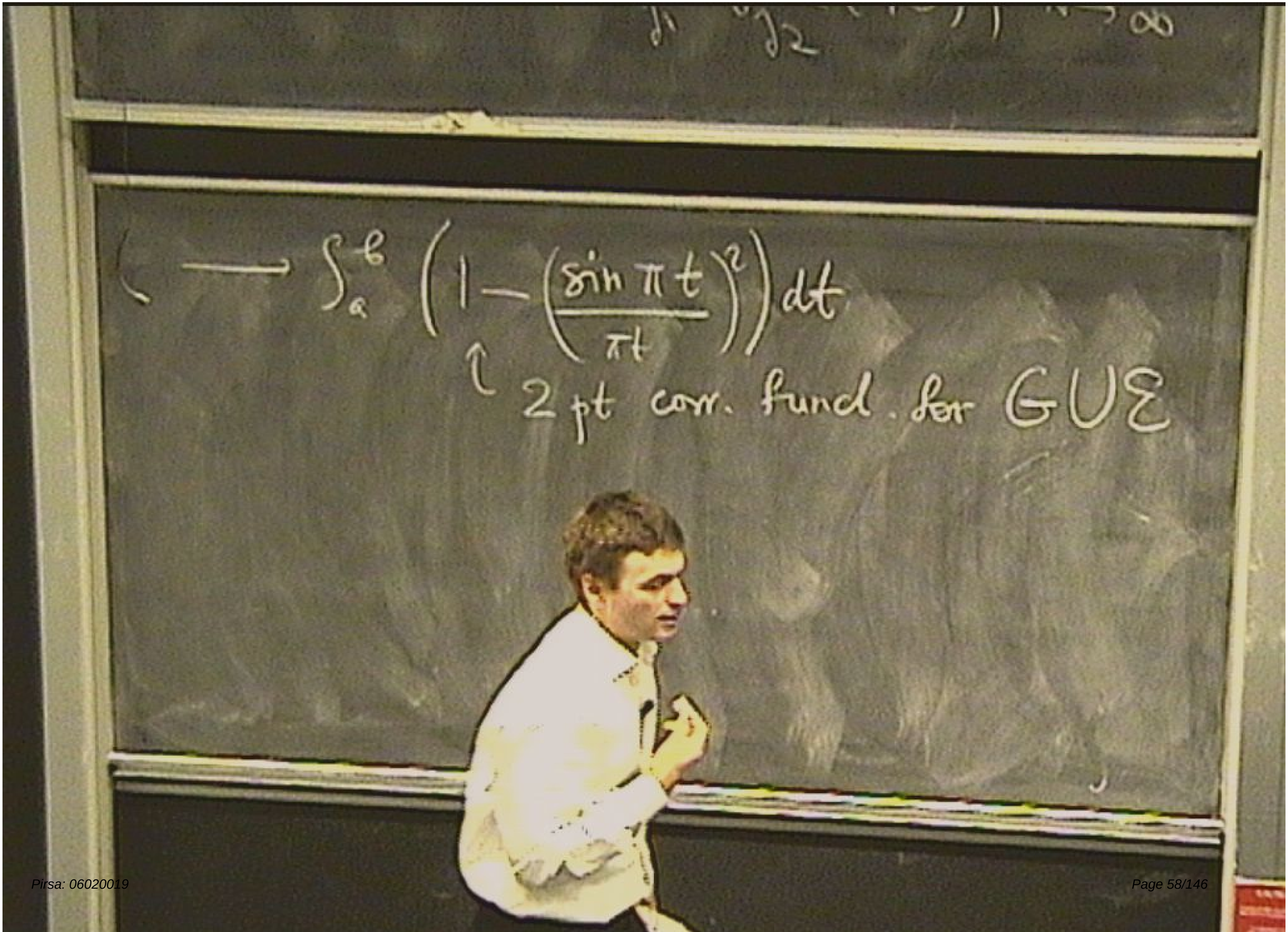

$\longrightarrow \int_a^b \left(1-\left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt$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2 pt corr. fund. for GUE

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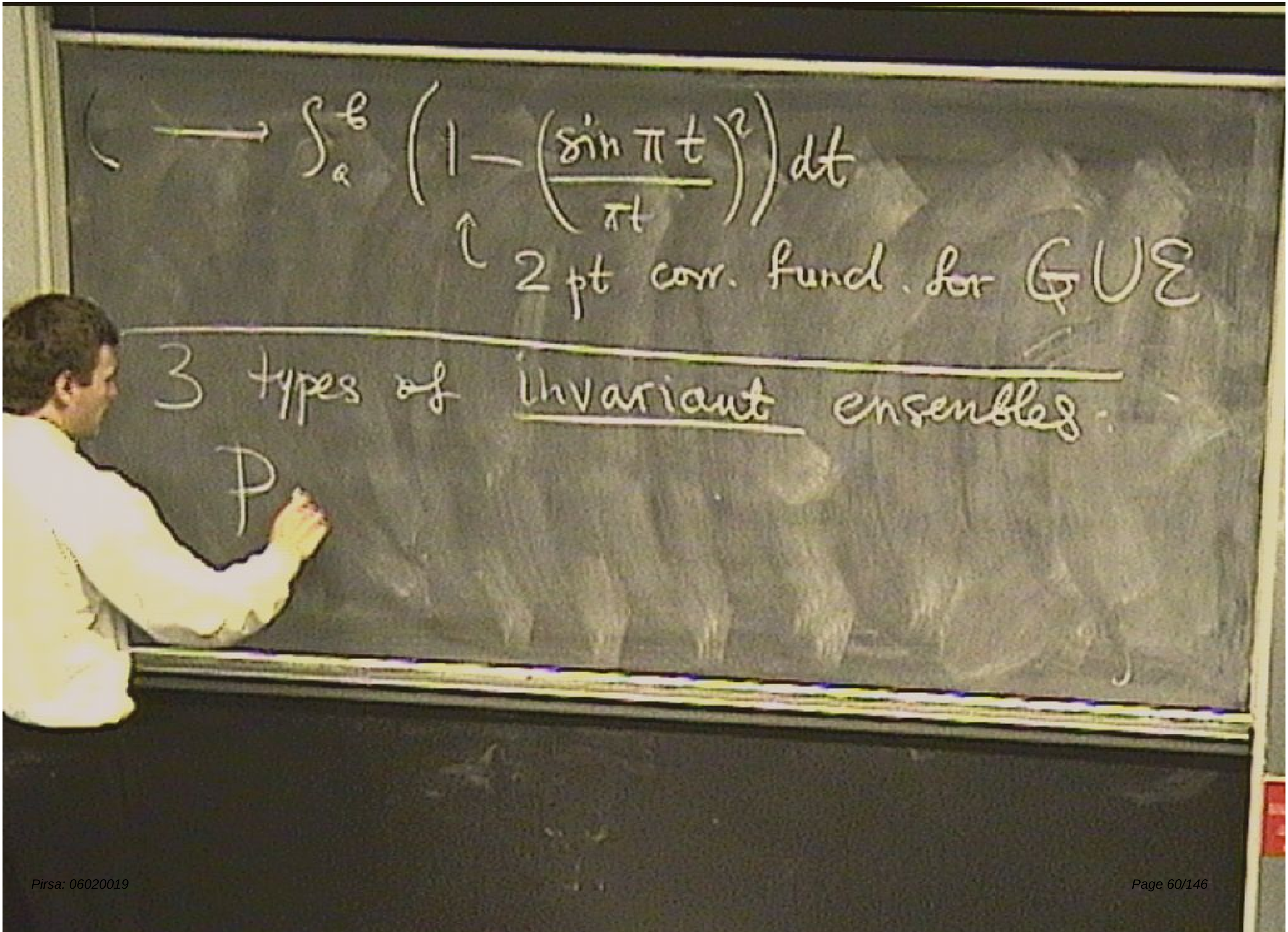
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3 types of invariant ensembles.
P

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3 types of invariant ensembles.

$P(M)$

$$\longrightarrow \int_a^b \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt$$

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---

3 types of <u>invariant</u> ensembles.

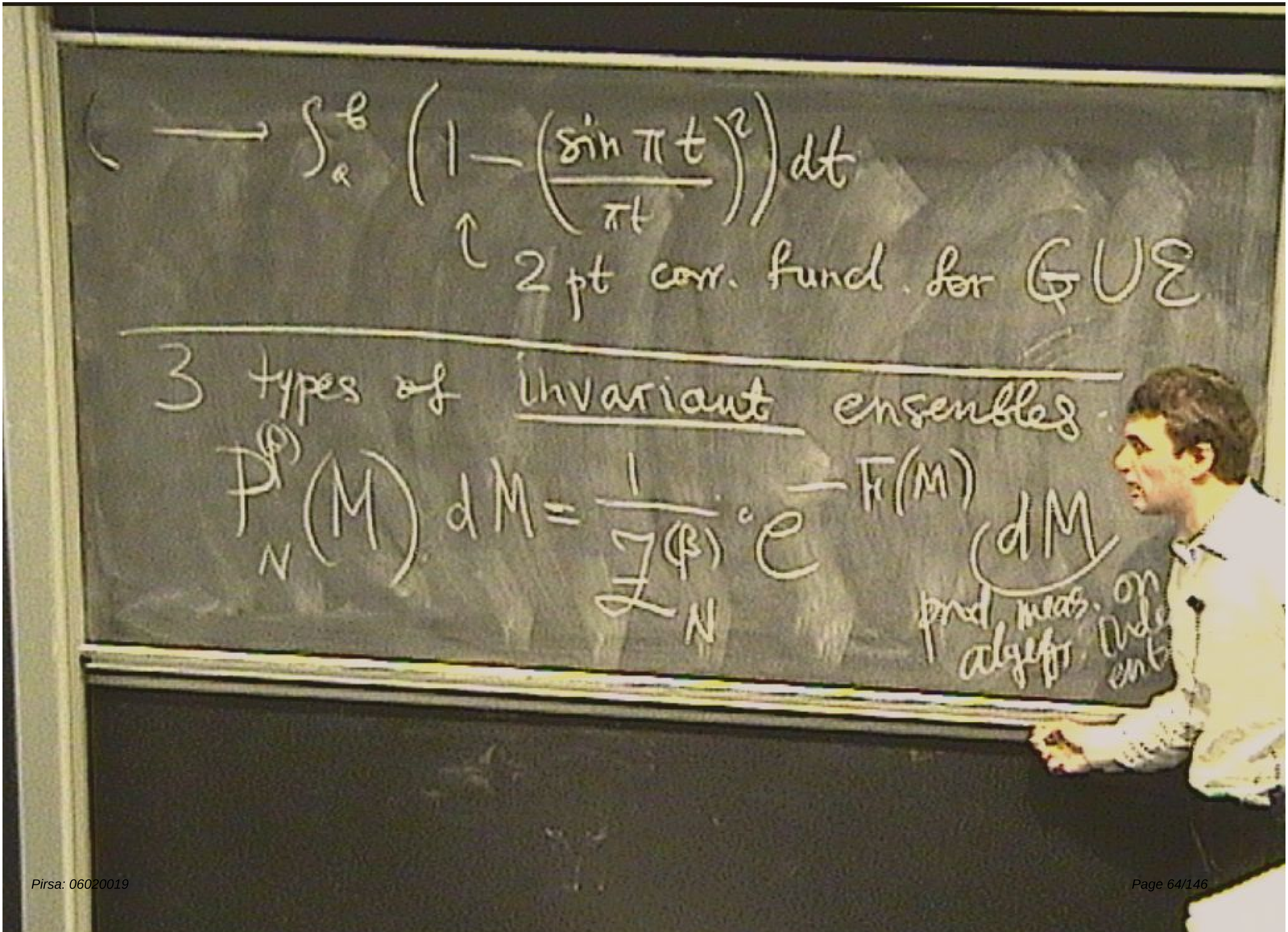
$P^{(\beta)}(M)$

$$\longrightarrow \int_a^b \left(1-\left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt$$

$\uparrow$ 2 pt corr. fund. for GUE

3 types of <u>invariant</u> ensembles:

$$P_N^{(\beta)}(M)\, dM = \frac{1}{Z_N^{(\beta)}} \cdot e^{-F(M)} \, dM$$


$\longrightarrow \int_a^b \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt$
2 pt corr. fund. for GUE
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$P_N^{(\beta)}(M)\, dM = \frac{1}{Z_N^{(\beta)}} \cdot e^{-F(M)} \, (dM$
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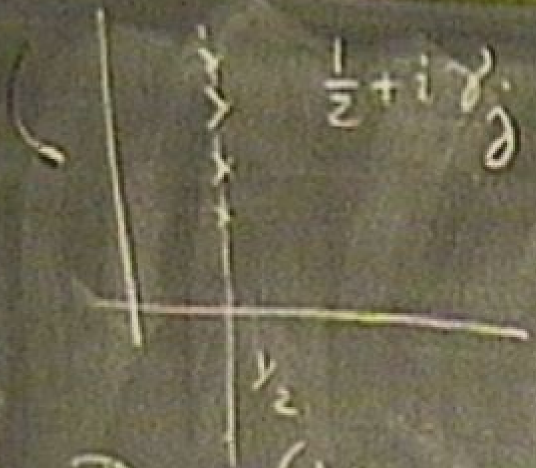

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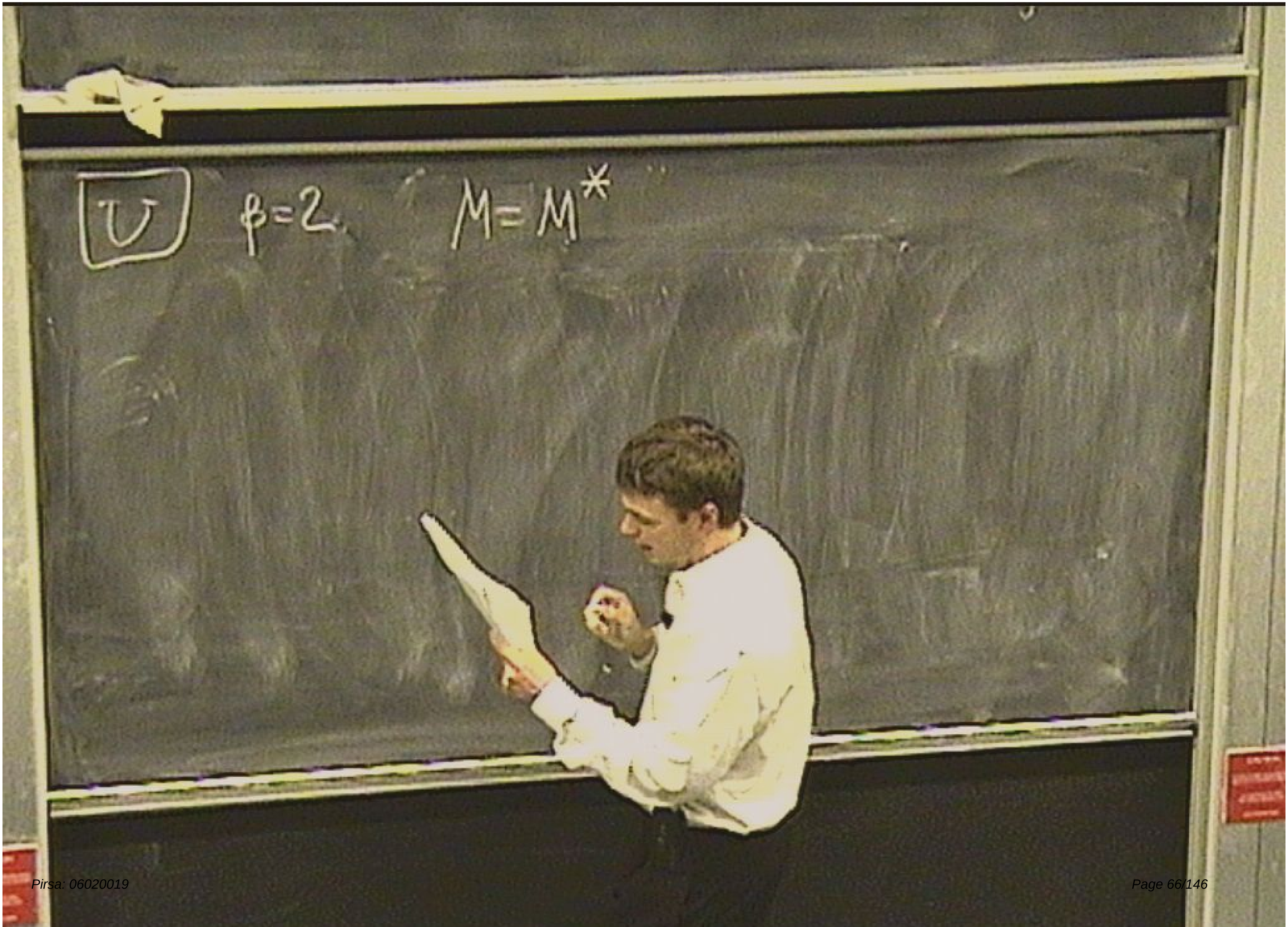
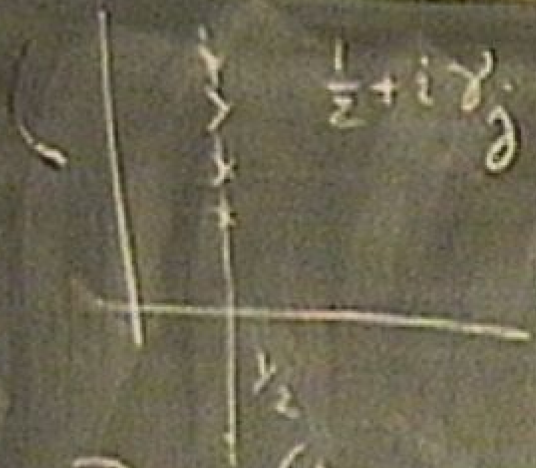

U
β=2
M=M*

$\boxed{U}$ $\beta = 2$. $\quad M = M^*$ ($N \times N$) $\quad P_N^{(\beta)}$ (is inv. under $M \mapsto UMU^*$

$\boxed{U}$ $\beta = 2$. $M = M^*$ $N \times N$ $P_N^{(\beta)}$ (is inv. under

$$M \mapsto U M U^* \quad \forall U : U^* U = I$$



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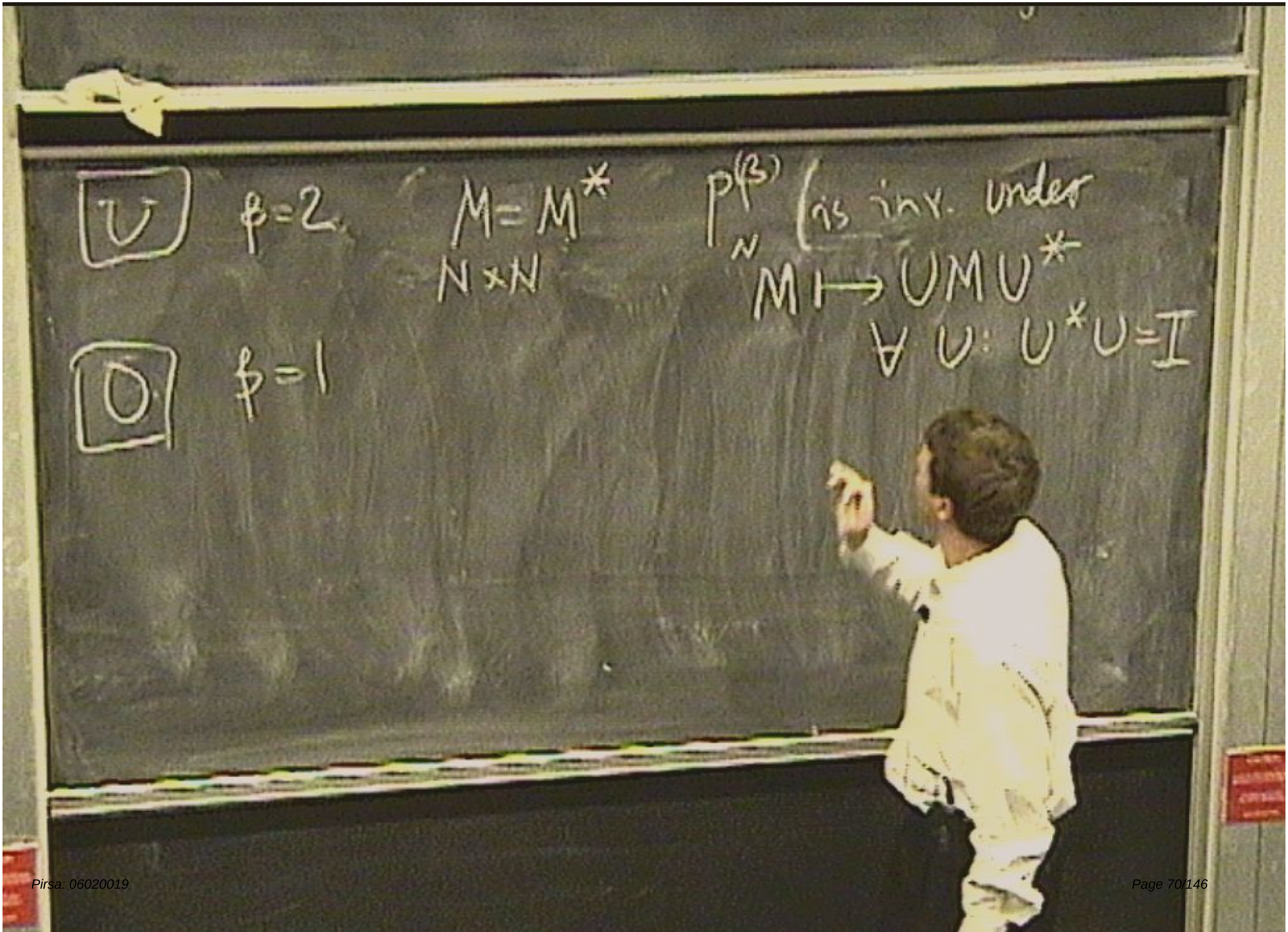
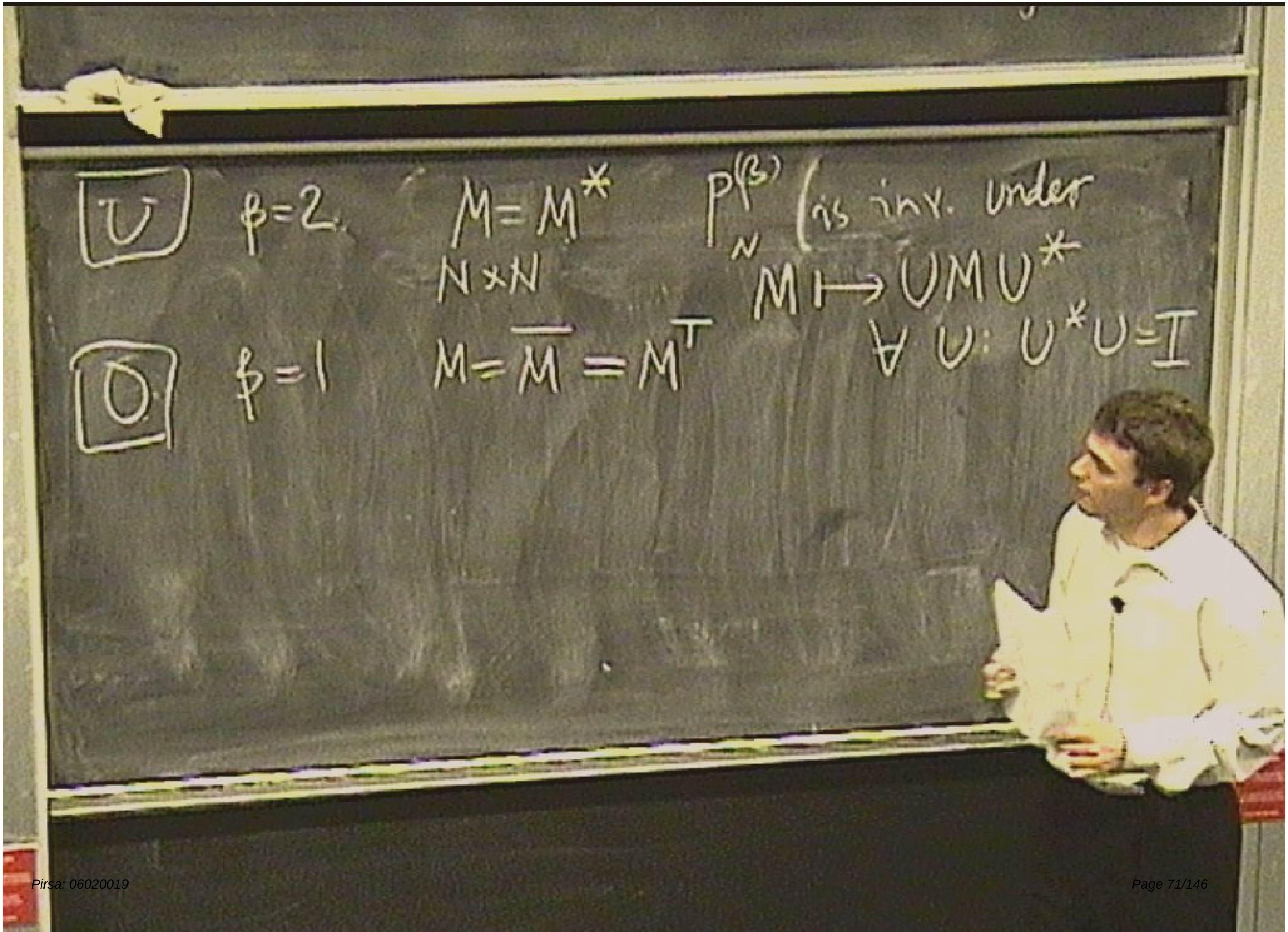
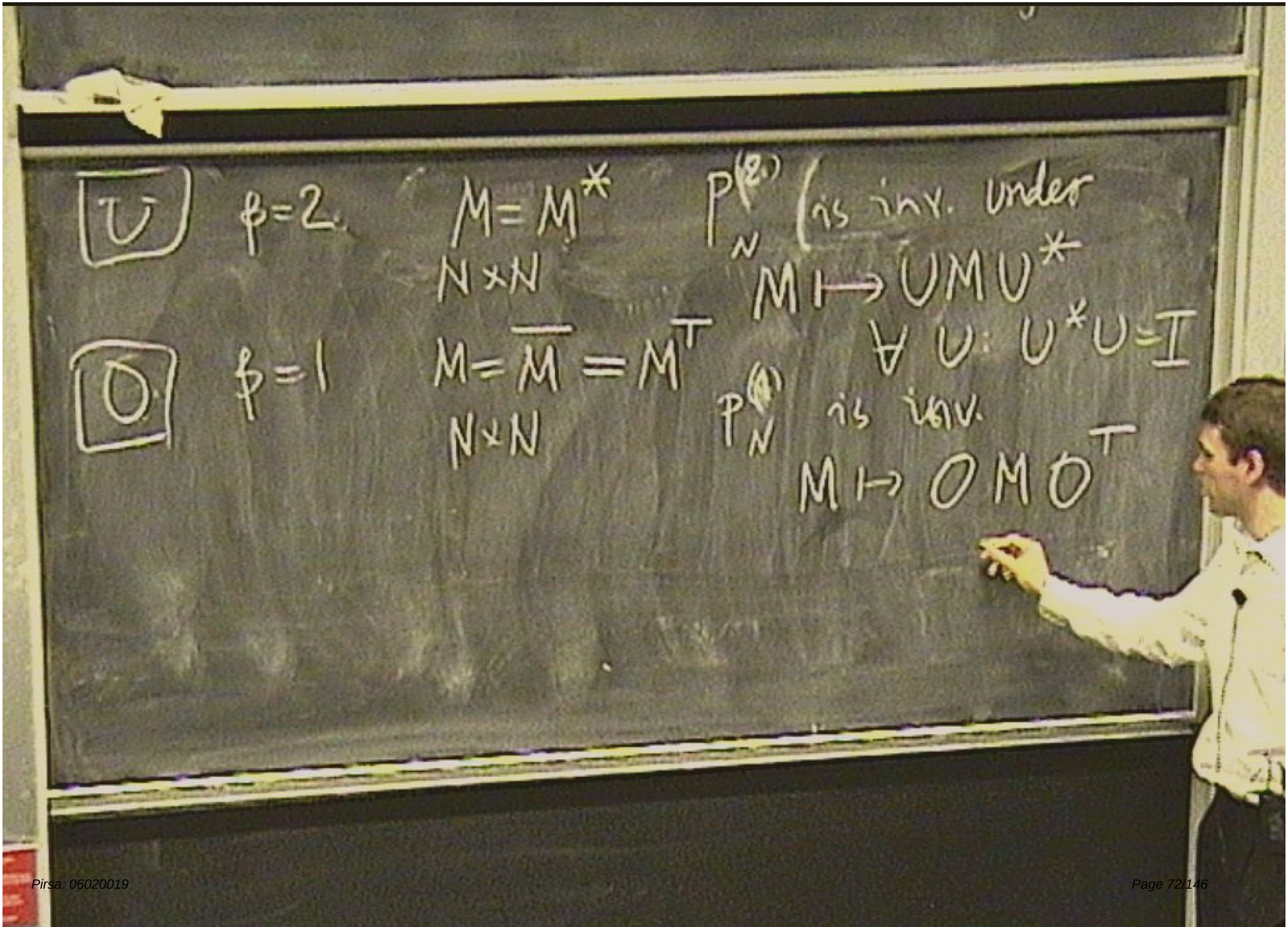

U
β=2
M=M*
N×N
P_N^(β) (is inv. under
M ↦ UMU*
∀ U: U*U=I
O
β=1


U β=2. M=M* P_N^(β) (is inv. under
N×N
M ↦ UMU*
O β=1 M=M̄=M^T
∀U: U*U=I


U β=2 M=M*
N×N
P_N^(2) (is inv. under
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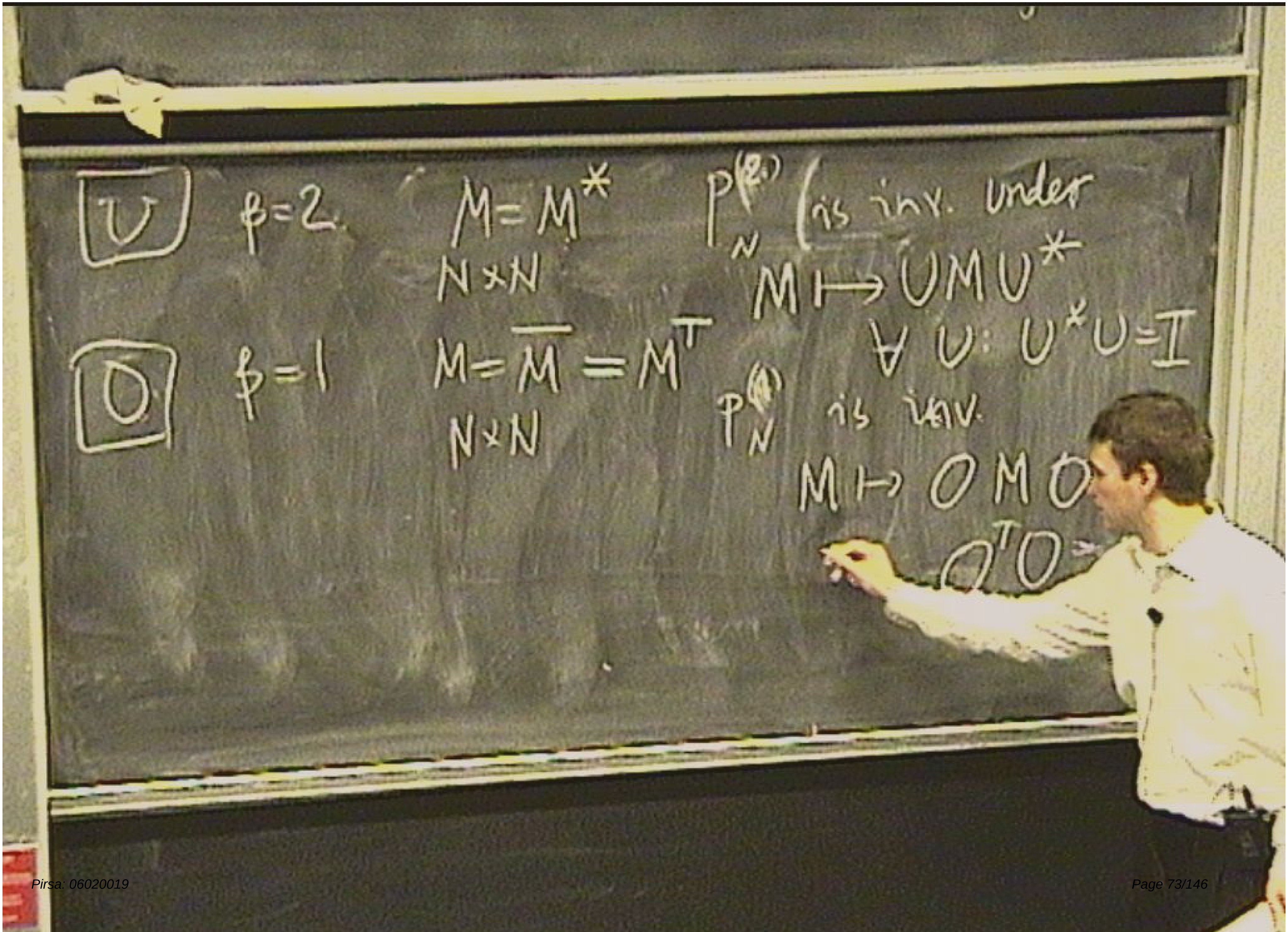
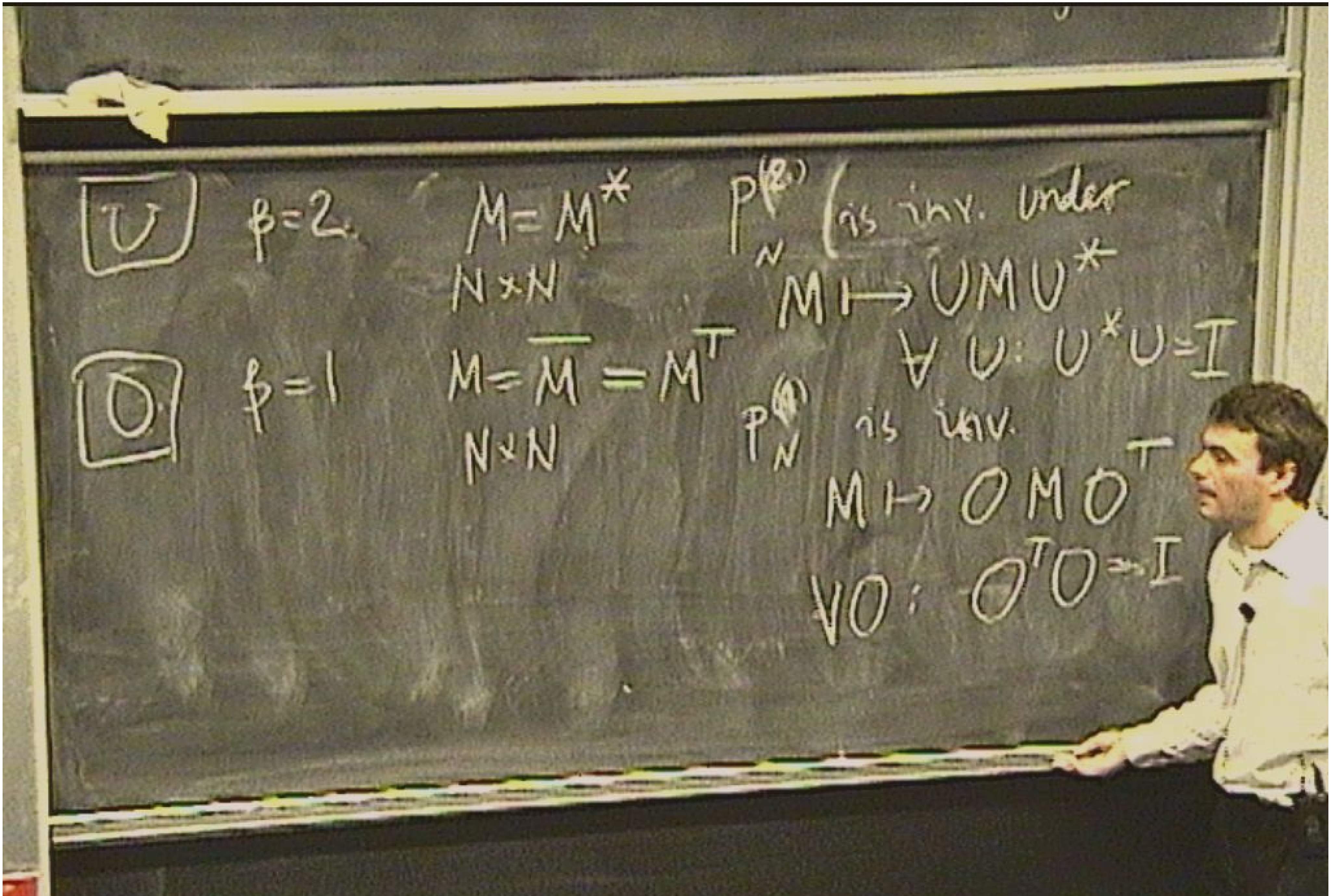
N×N
P_N^(1) is inv.
M ↦ OMO^T


U
β=2
M = M*
N×N
P_N^(2) (is inv. under
M ↦ UMU*
∀ U: U*U = I
O
β=1
M = M̄ = M^T
N×N
P_N^(1) is inv.
M ↦ OMO
O^TO =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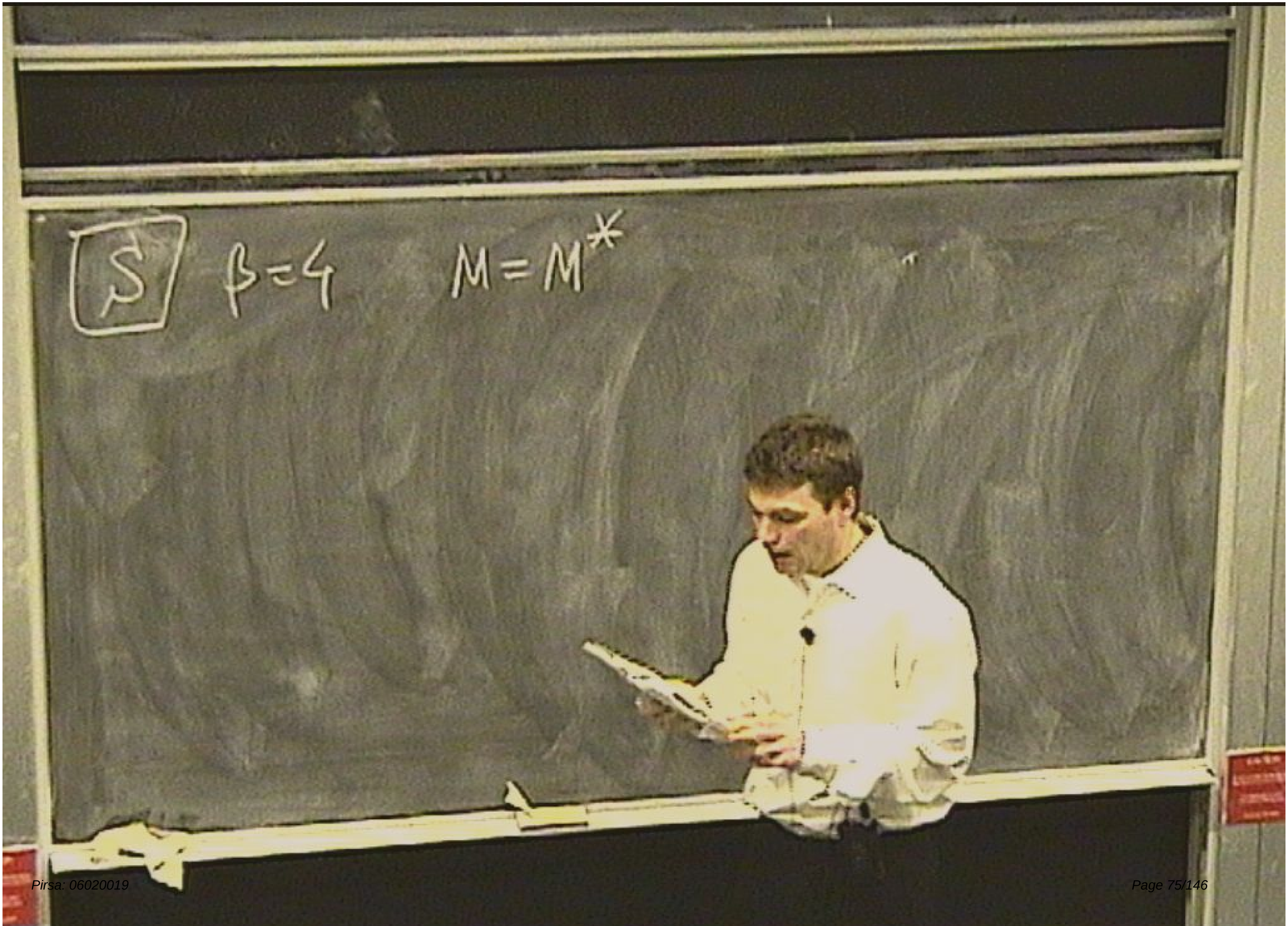
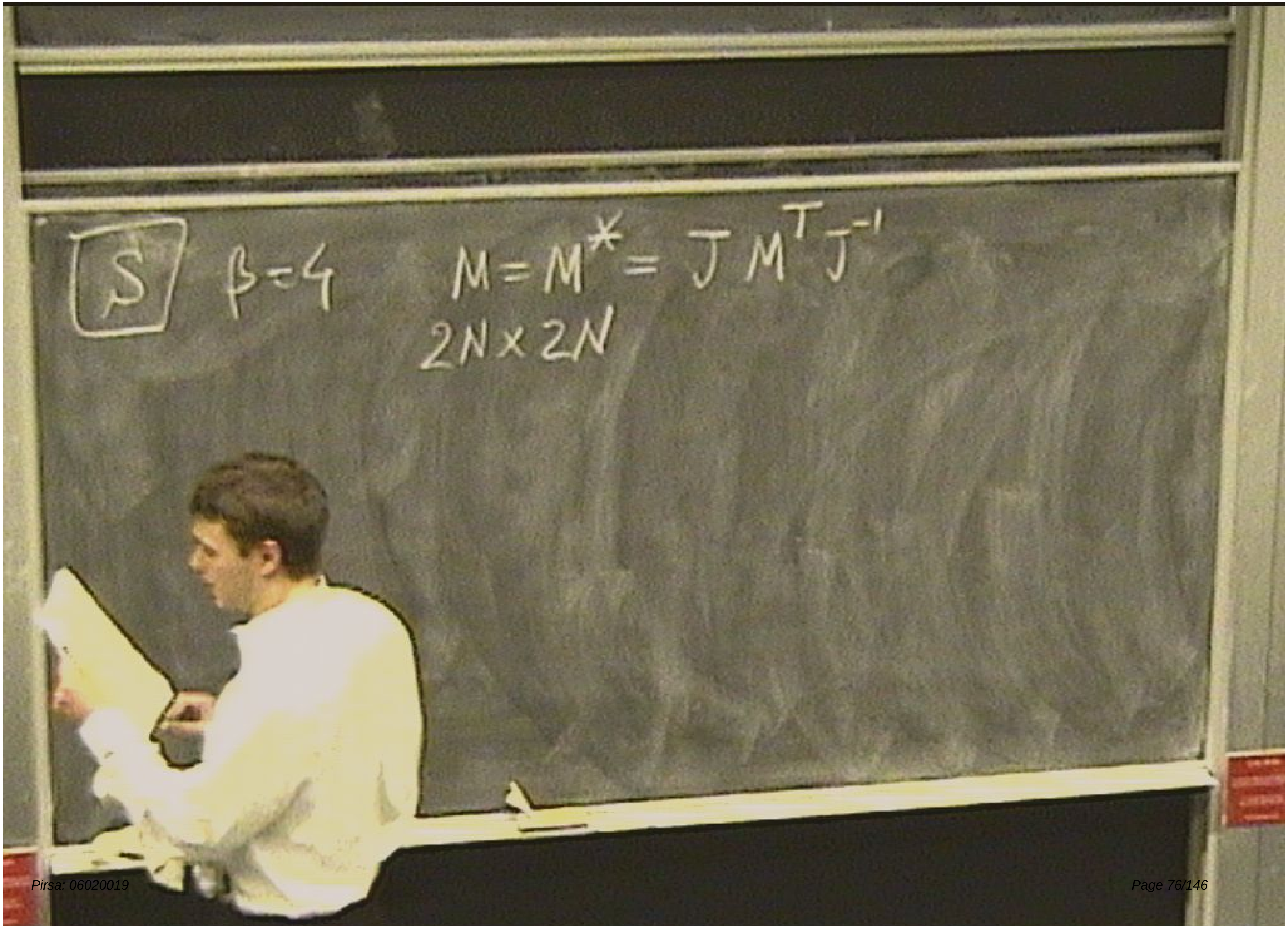
M ↦ OMO^T
∀O: O^T O = I


S
β = 4
M = M*

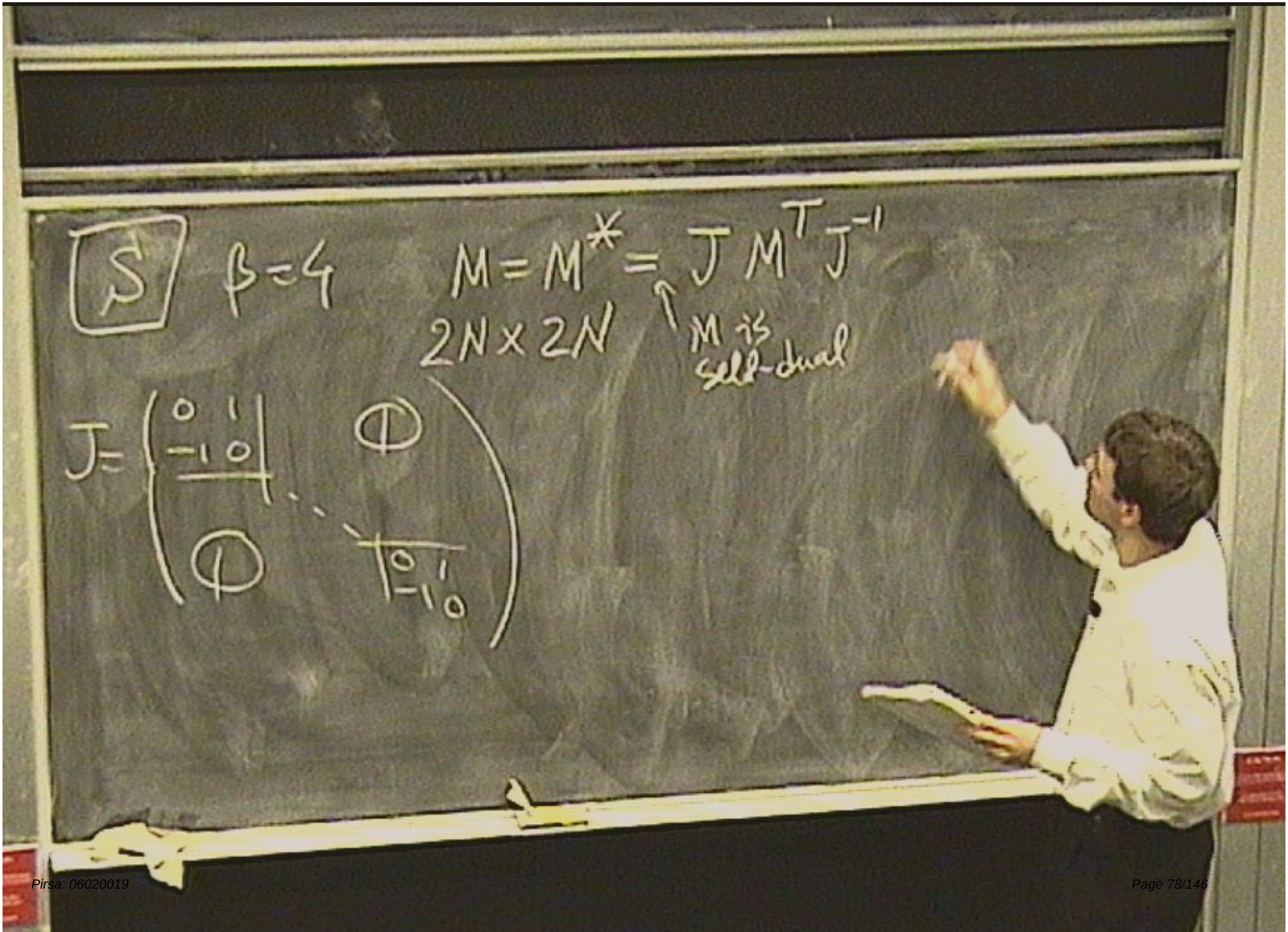

$\boxed{S}$ $\beta=4$ $M=M^*=JM^TJ^{-1}$
$2N\times 2N$

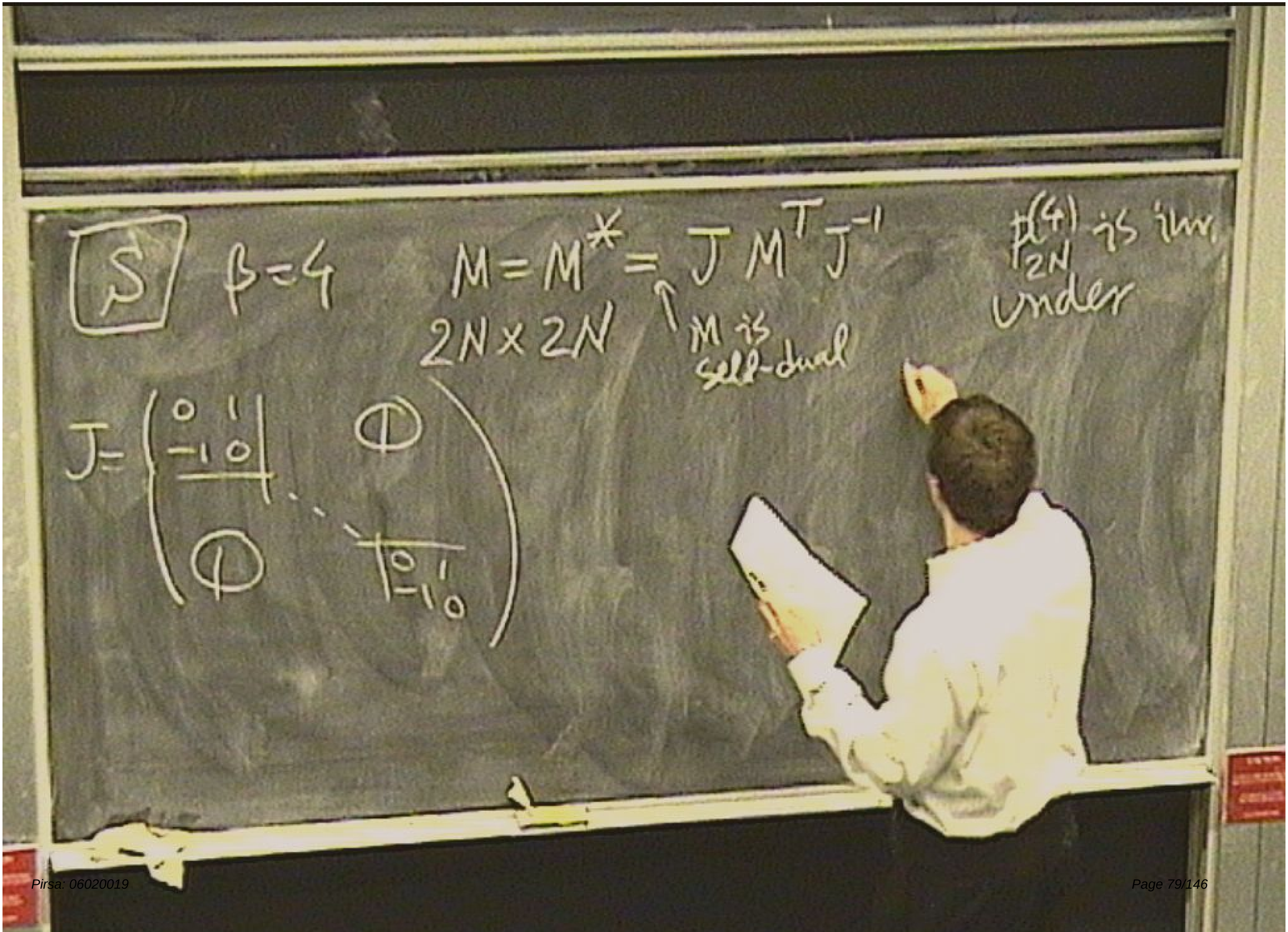
$\boxed{S}$ $\beta = 4$

$$M = M^* = J M^T J^{-1}$$

$2N \times 2N$

$$J = \begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} & & \emptyset \\ & \ddots & \\ \emptyset & & \begin{array}{|cc} \hline 0 & 1 \\ -1 & 0 \end{array} \end{pmatrix}$$


S
β = 4
M = M* = J M^T J^{-1}
2N × 2N
M is self-dual
J =
0 1
-1 0
Φ
Φ
0 1
-1 0


S
β=4
M = M* = J M^T J^-1
2N × 2N
M is self-dual
H_2N^(4) is inv. under
J =
0 1
-1 0
0 1
-1 0

$\boxed{S}$ $\beta = 4$

$$M = M^* = J M^T J^{-1}$$

$2N \times 2N$

M is self-dual

$\mu^{(4)}_{2N}$ is inv. under

$$M \mapsto S M S^*$$

$$S^* S = I$$

$$S J S^* = J$$

$$J = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \Phi \\ & \ddots & \\ \Phi & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

$\boxed{\mathcal{S}}$ $\beta = 4$

$$M = M^* = J M^T J^{-1}$$

$2N \times 2N$

M is self-dual

$$J = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \text{\O} \\ & \ddots & \\ \text{\O} & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

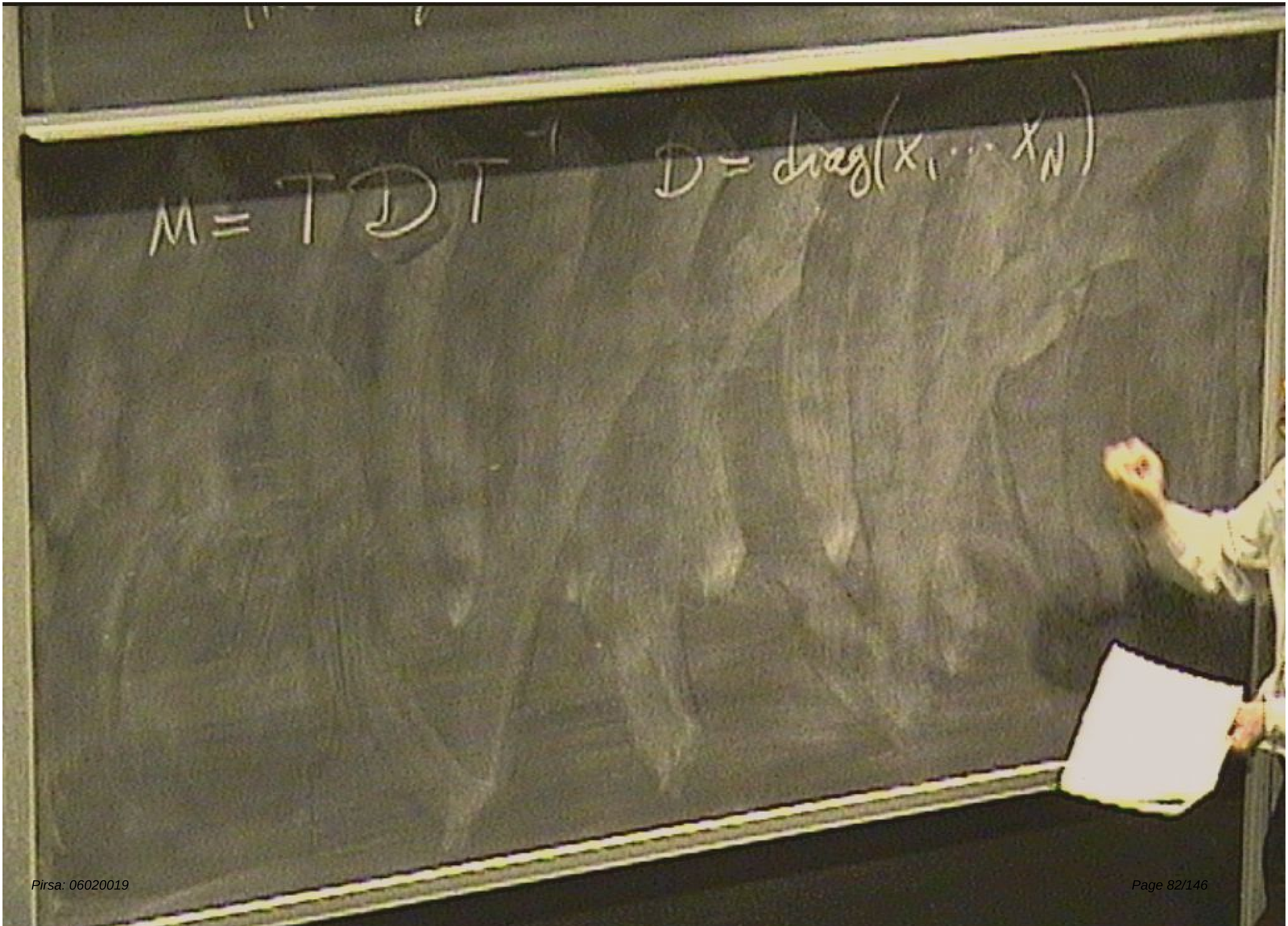
$\mathcal{H}^{(4)}_{2N}$ is inv. under

$$M \mapsto S M S^*$$

$$\forall S: \quad S^* S = I$$

$$S J S^* = J$$

S is sympl.


$M = TDT^{-1}$
$D = \mathrm{diag}(x_1 \cdots x_N)$

$M = T D T^{-1} \quad D = \mathrm{diag}(x_1, \cdots, x_N)$

by invariance cond., $F(M)$ is a symm. func. of eigvals

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Imp. class: $F(M) = \mathrm{tr}\, V(M)$

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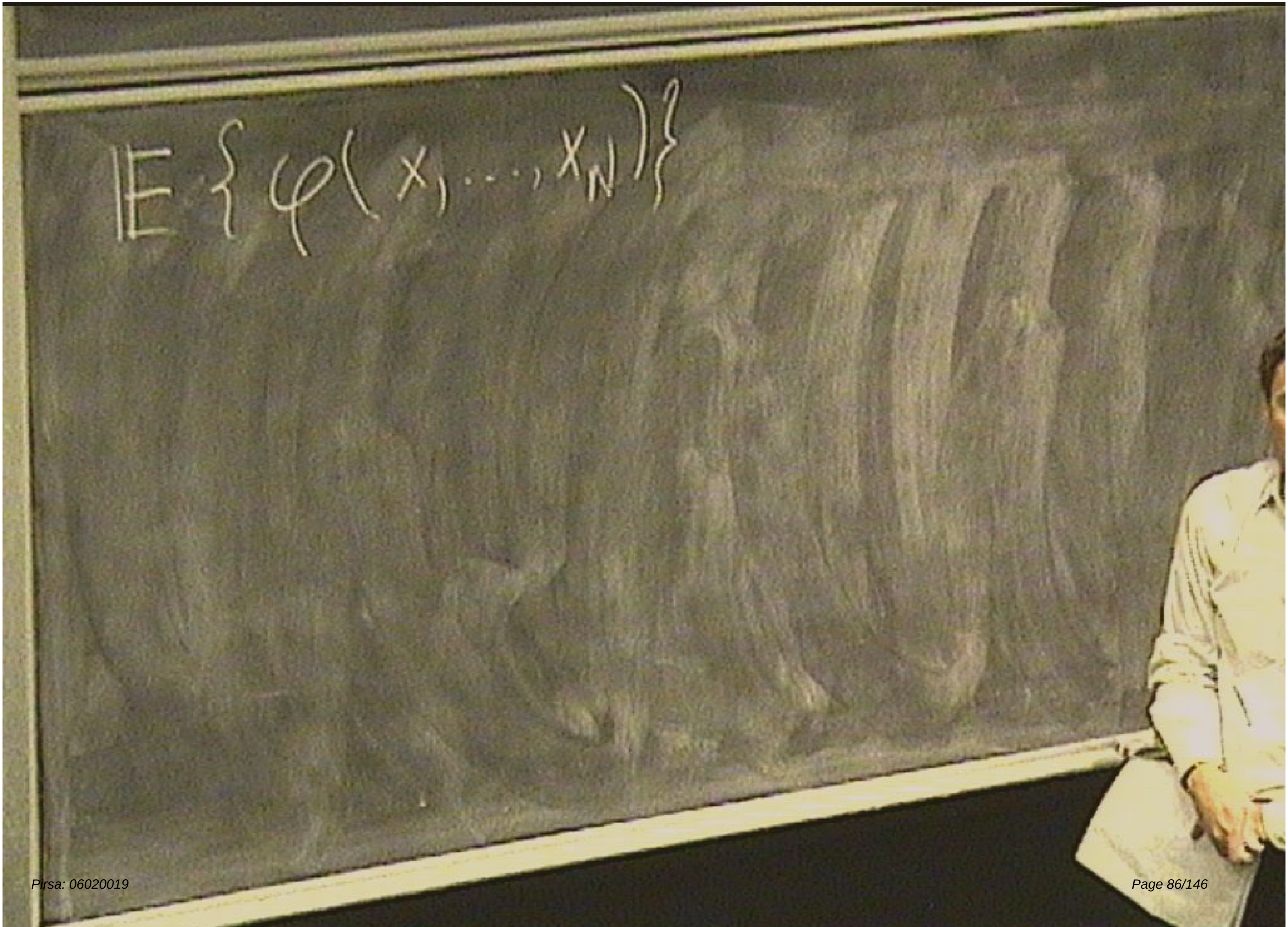
by invariance cond., $F(M)$ is a symm. funct. of eigvals

Imp. class: $F(M) = \text{tr}\, V(M)$

$$V(t) = \varkappa t^{2m} + \ldots$$

$$2m \quad \varkappa_{2m} > 0$$


$\mathbb{E}\{\varphi(x_1,\ldots,x_N)\}$

$$\mathbb{E}\{\varphi(x_1,\dots,x_N)\}$$

$p_N^{(\beta)}$ can be wr. as a prod of dens. on the
$(x_1, \dots x_N)$ and dens. on the

$$\mathbb{E}\{\varphi(x_1,\dots,x_N)\}$$

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$$\mathbb{E}\left\{\varphi(x_1,\dots,x_N)\right\}$$

$p_N^{(\beta)}$ can be wr. as a prod of dens. on $(x_1 \dots x_N)$ and dens. on the "T"

int. out

$$\mathbb{E}\{\varphi(x_1,\dots,x_N)\}$$

$p_N^{(\beta)}(x_1 \dots x_N)$ can be wr. as a prod of dens. on "T" and dens. on the

int. out

BY INVARIANCE

$$\mathbb{E}\{\varphi(x_1,\dots,x_N)\}$$

$P_N^{(\beta)}$ can be wr. as a prod. of dens. on $(x_1 \dots x_N)$ and dens. on the T.

int. out

BY INVARIANCE

$$P_N^{(\beta)}(x_1,\dots,x_N) = Z_N^{-1} \prod_{1\le i<j\le N} |x_i - x_j|^{\beta}$$

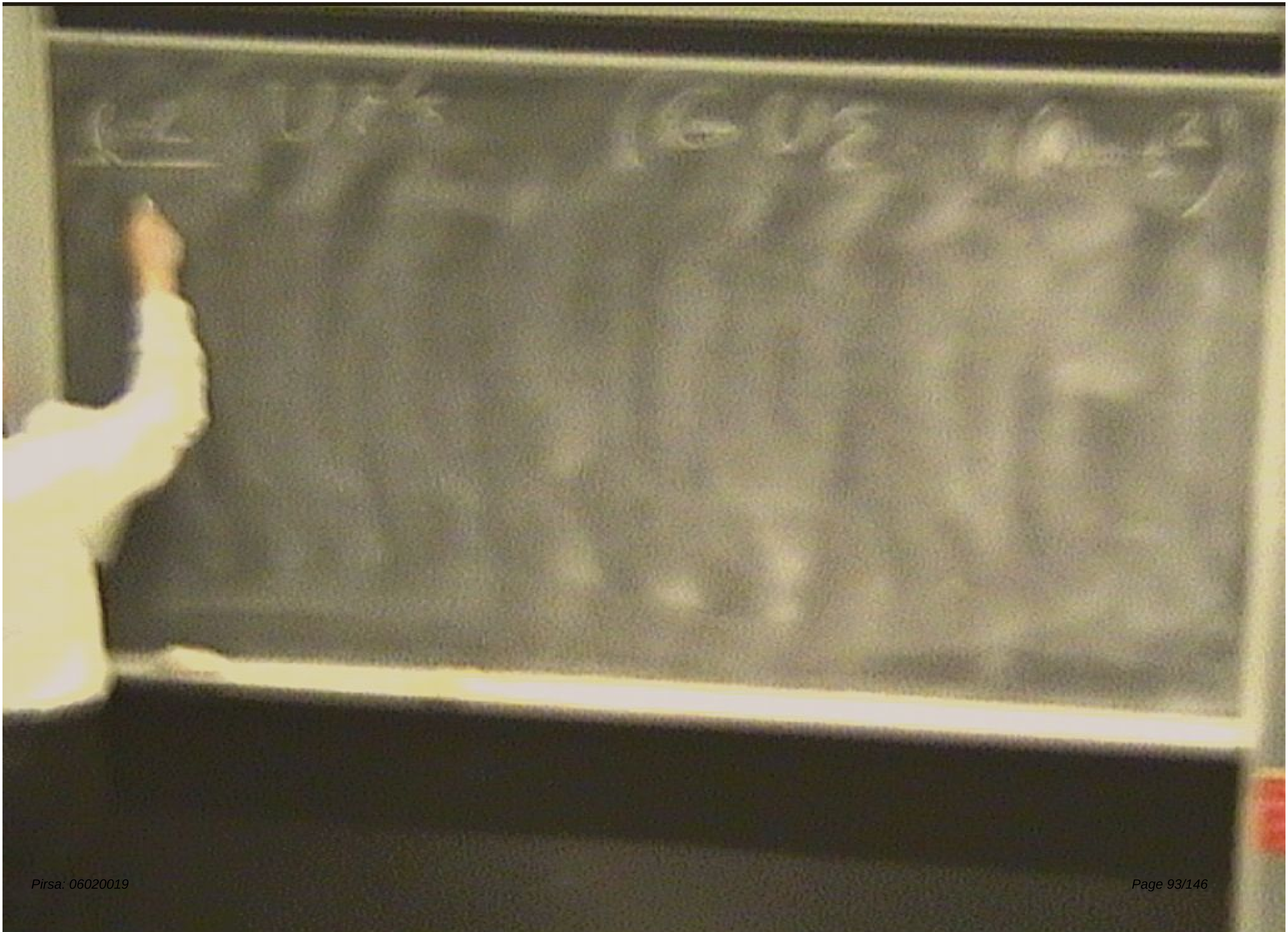
$\mathbb{E}\{\varphi(x_1,\dots,x_N)\}$

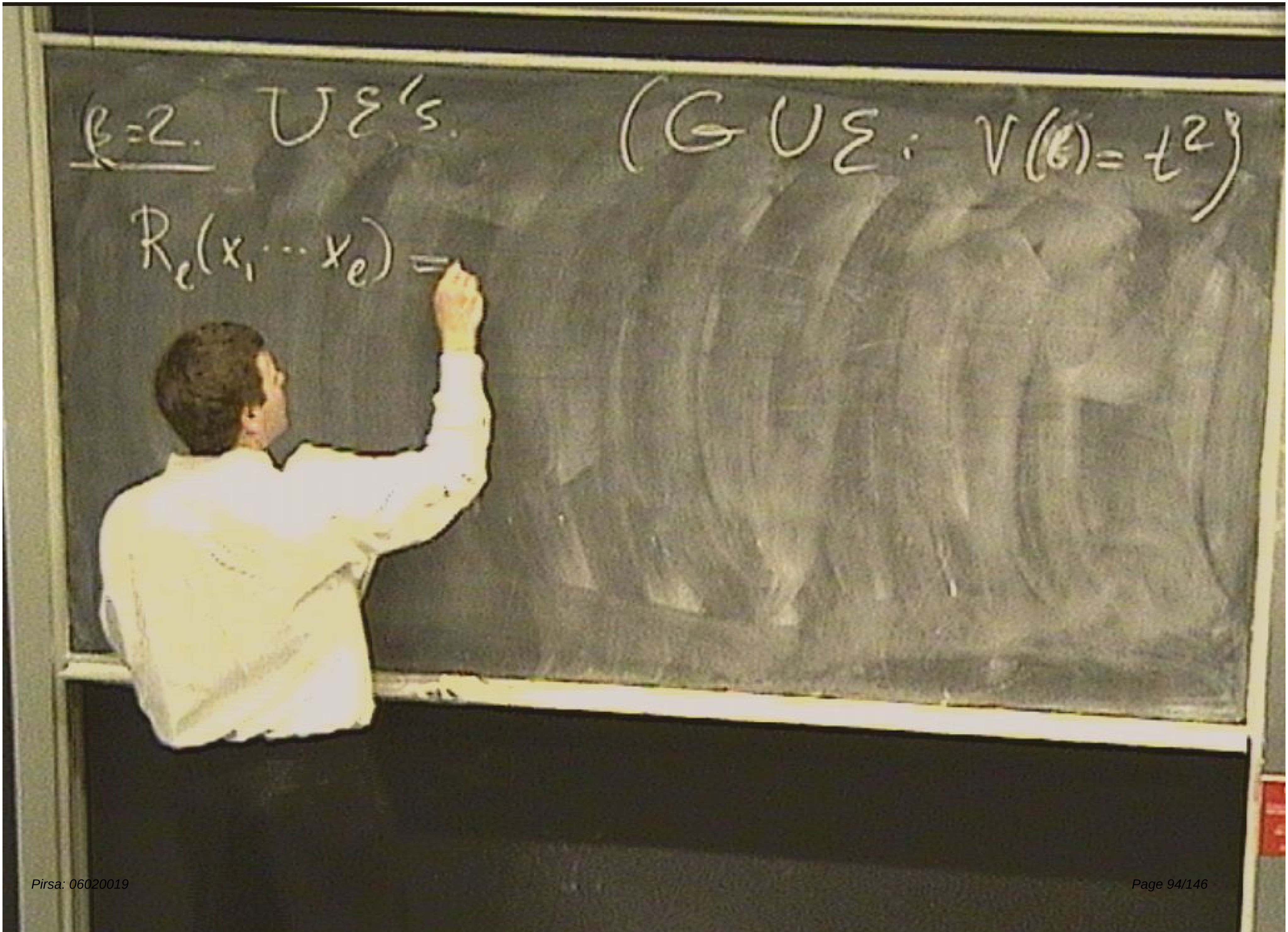
$P_N^{(\beta)}$ can be wr. as a prod of dens. on $(x_1 \dots x_N)$ and dens. on the T.

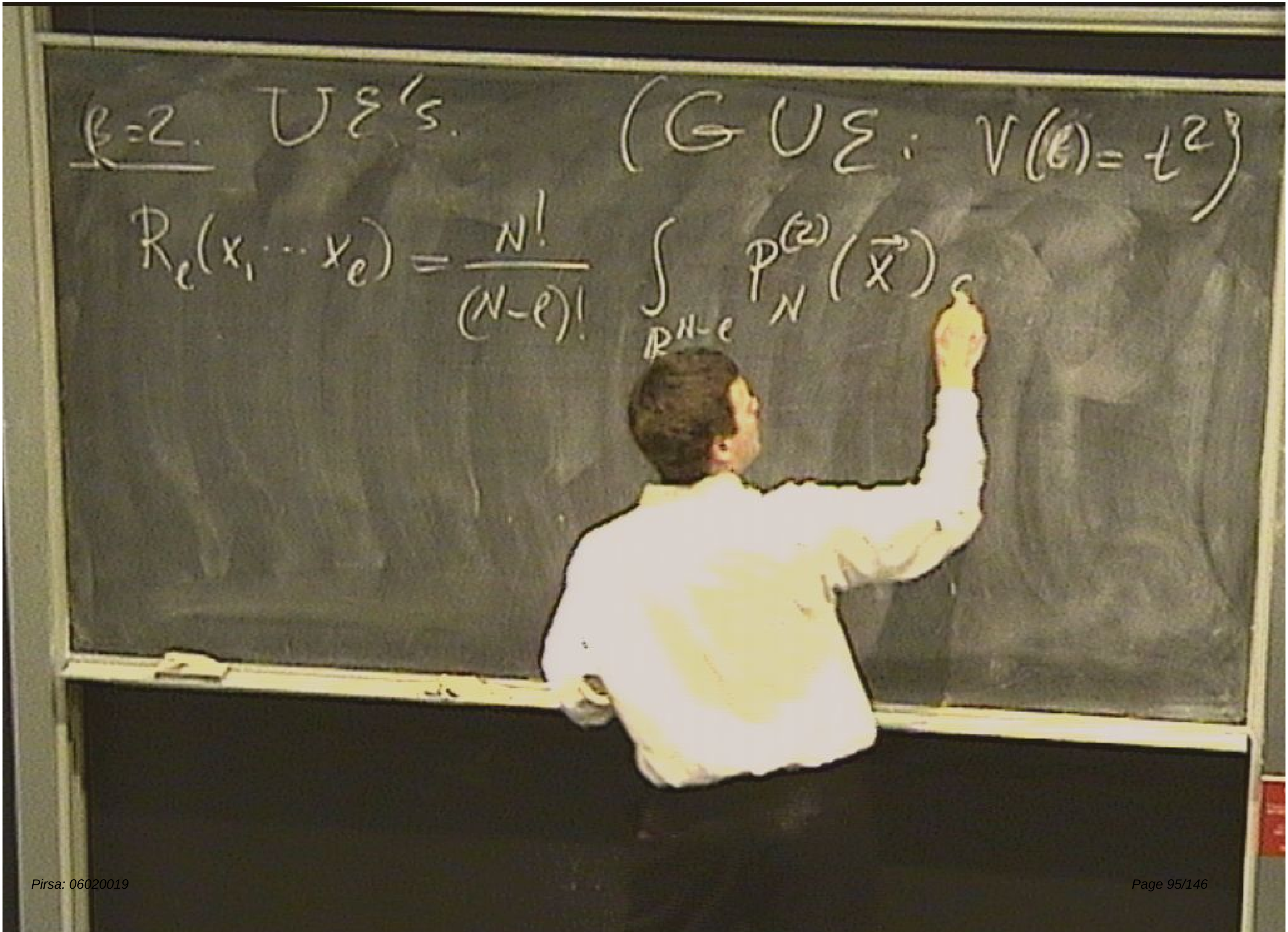
int. out

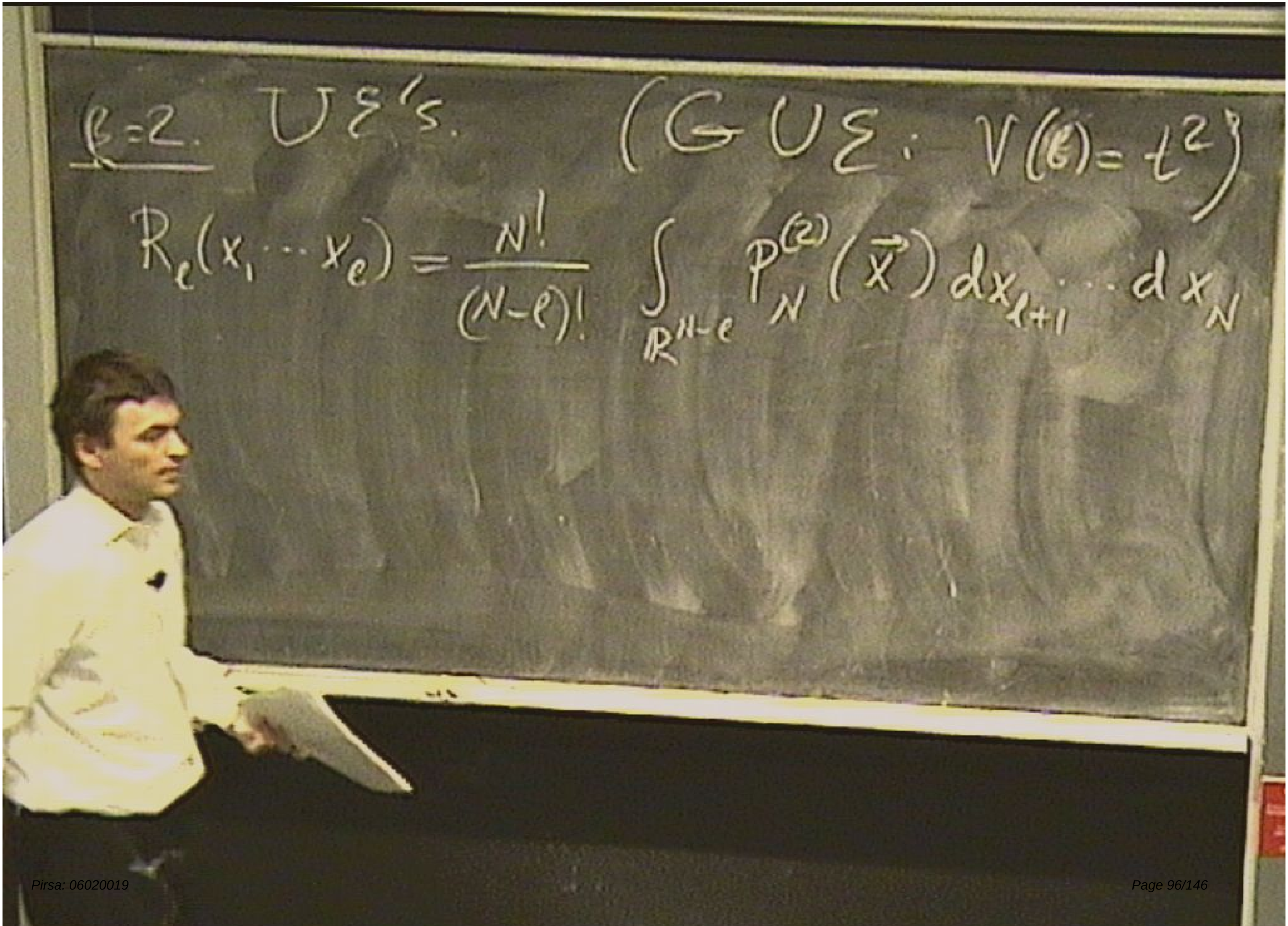
BY INVARIANCE

$$P_N^{(\beta)}(x_1,\dots,x_N)\,dx_1\dots dx_N = Z_N^{-1} \prod_{1\le i<j\le N} |x_i-x_j|^\beta \cdot \prod_{i=1}^{N} e^{-V(x_i)}\,dx_1\dots dx_N$$




β=2. UE's.
(GUE: V(t)=t²)
R_ℓ(x_1 ··· x_ℓ) =


$\beta=2.$ UE's. (GUE: $V(t)=t^2$)
$R_\ell(x_1 \cdots x_\ell) = \frac{N!}{(N-\ell)!} \int_{\mathbb{R}^{N-\ell}} P_N^{(2)}(\vec{x})$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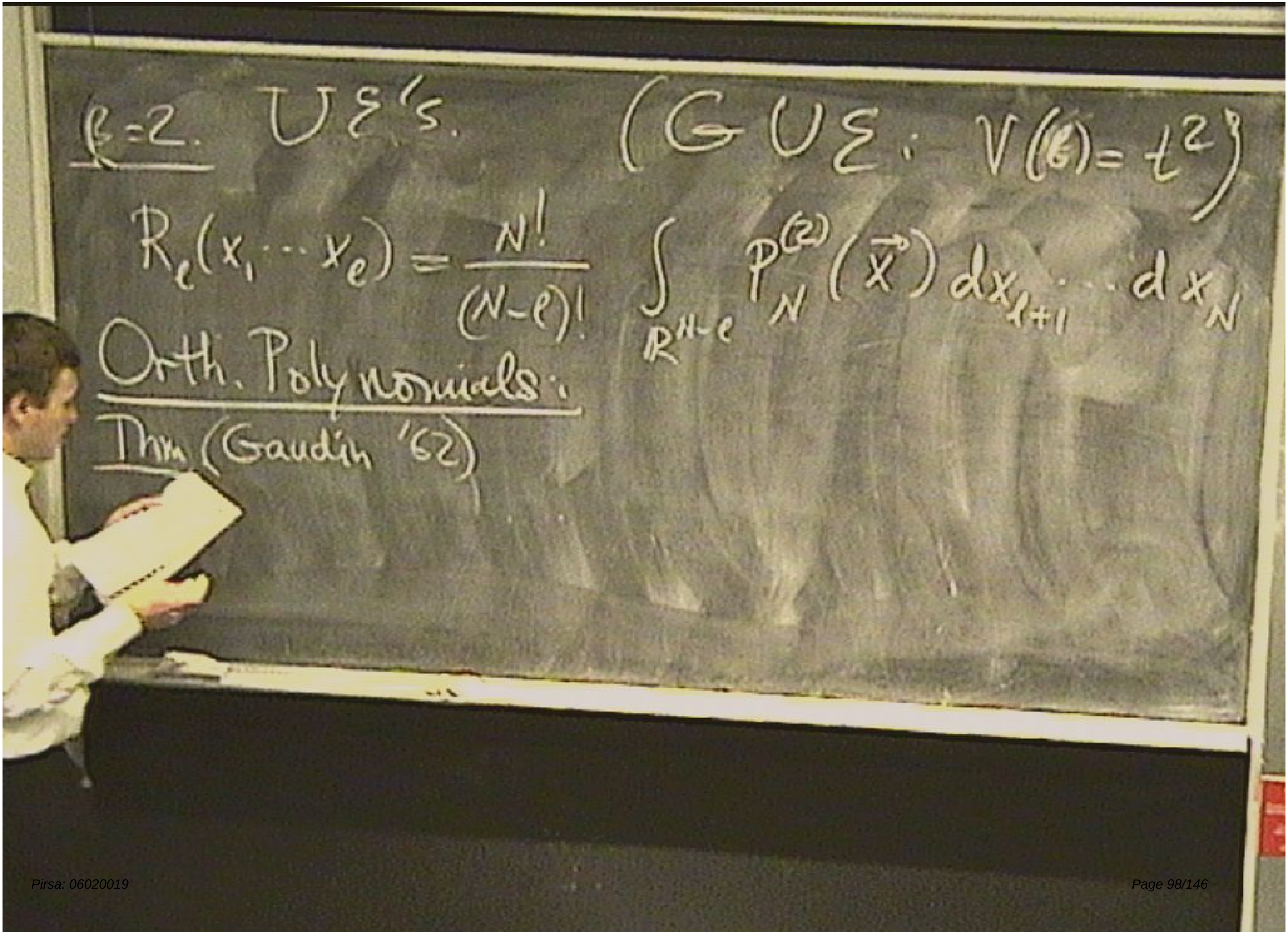
Orth. Polynomials: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Orth. Polynomials:
Thm (Gaudin '62)

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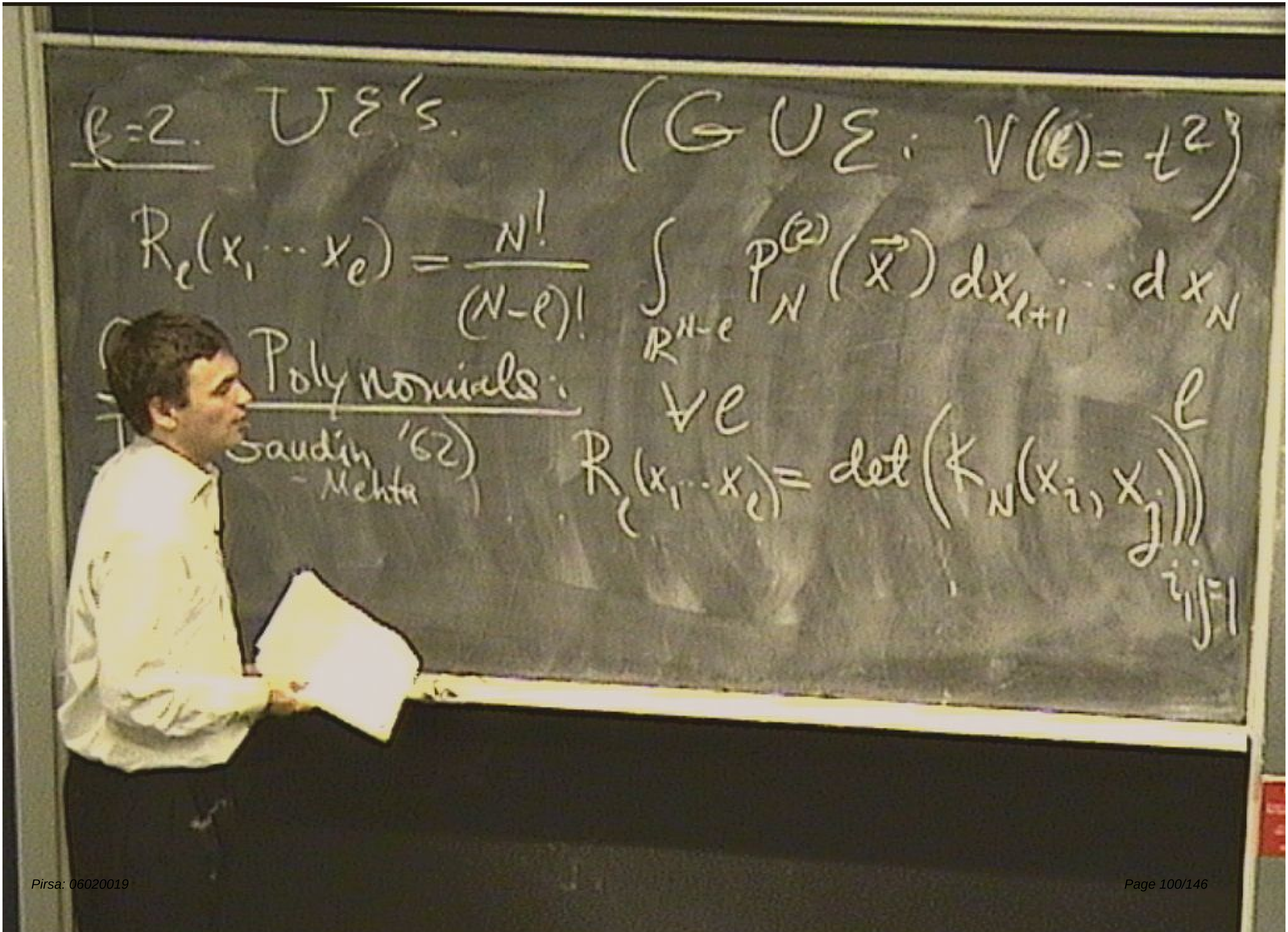
$\forall \ell$

<u>Orth. Polynomials:</u>

<u>Thm</u> (Gaudin '62 - Mehta) $\mathbb{R}$


$\beta=2$. UE's.
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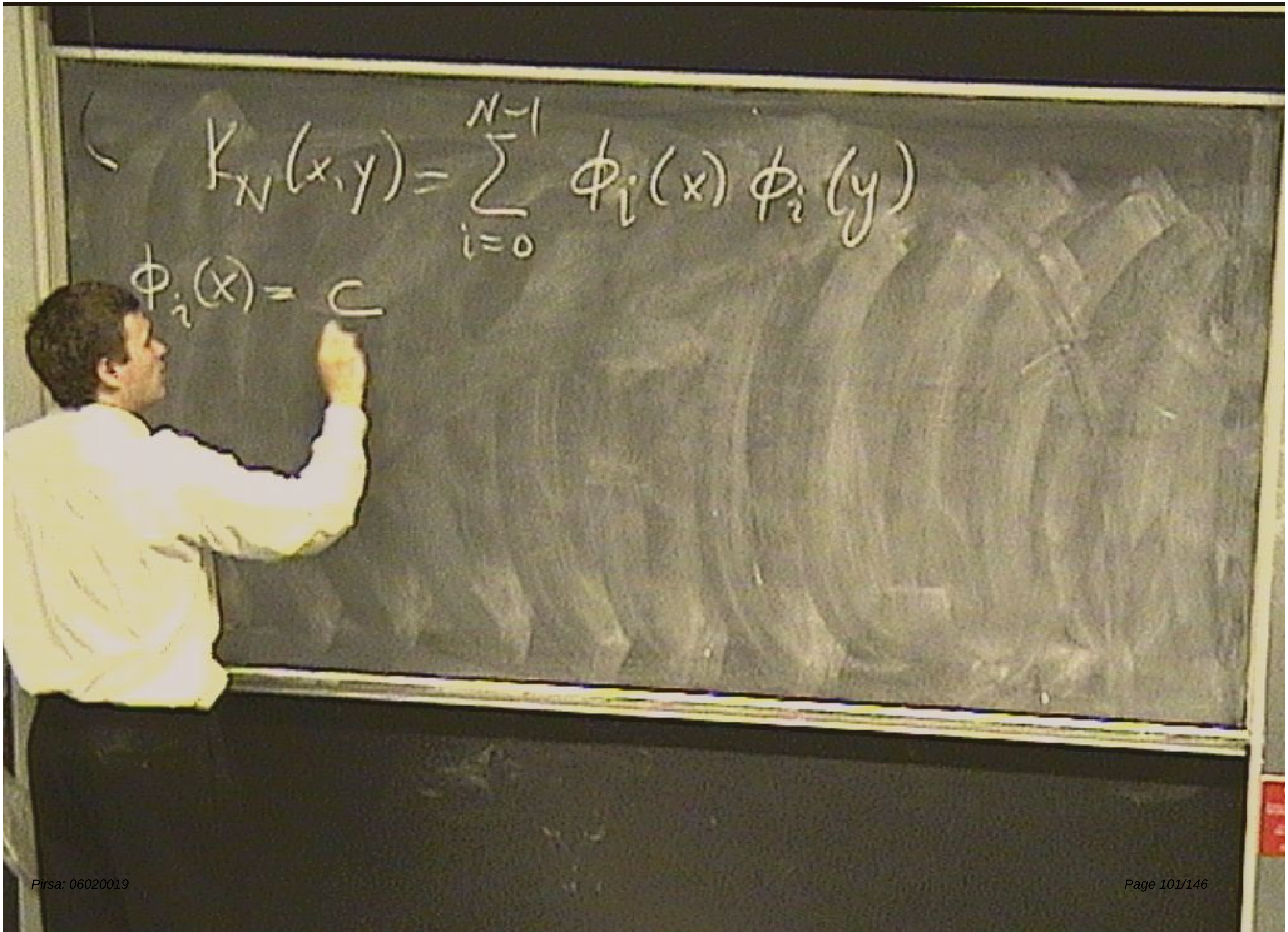
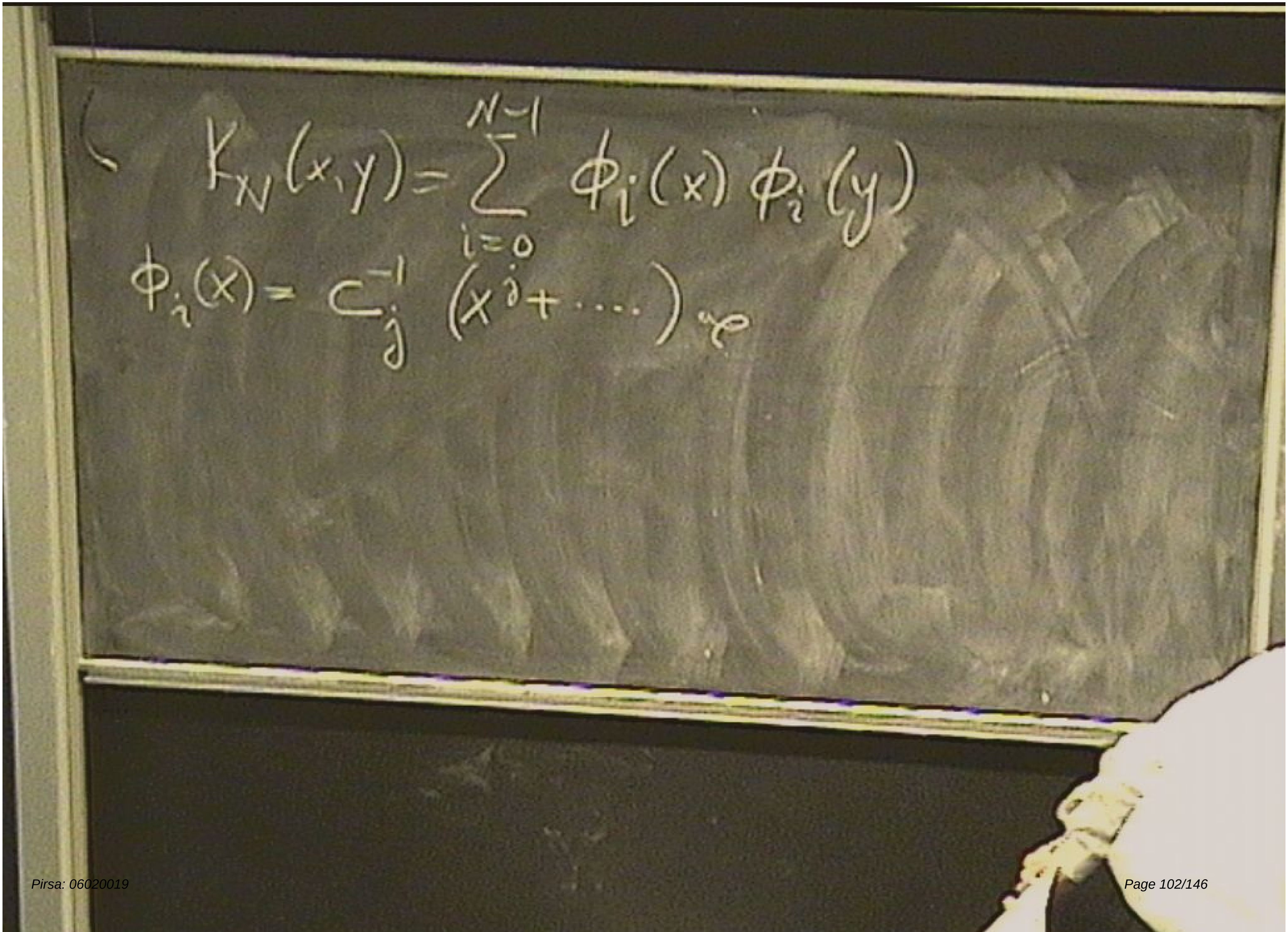
$R_\ell(x_1 \cdots x_\ell) = \frac{N!}{(N-\ell)!} \int_{\mathbb{R}^{N-\ell}} P_N^{(2)}(\vec{x})\, dx_{\ell+1} \cdots dx_N$
Polynomials:
(Gaudin '62)
Mehta
$\forall \ell$
$R_\ell(x_1 \cdots x_\ell) = \det\left(K_N(x_i, x_j)\right)_{i,j=1}^{\ell}$


$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$
$\phi_i(x) = C$


$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$
$\phi_i(x) = c_j^{-1}\,(x^j + \cdots\cdots)\,e$

$$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$$

$$\phi_j(x) = c_j^{-1} \underbrace{\left(x^j + \cdots\right)}_{\pi_j(x)} e^{-\frac{1}{2}V(x)}, \quad x \in \mathbb{R}$$

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$$\int_{-\infty}^{\infty} \phi_j(x)\,\phi_k(x)\,dx =$$

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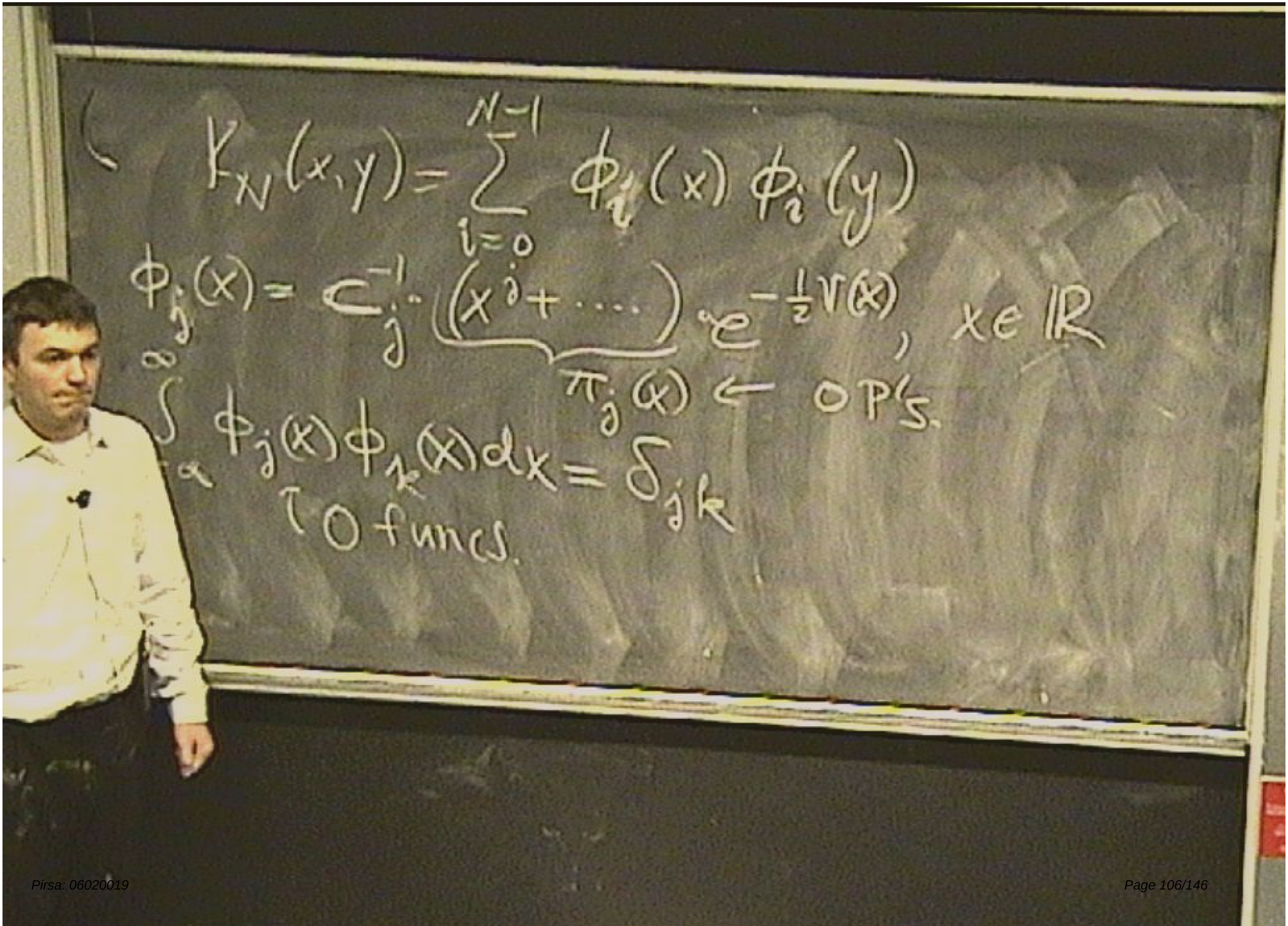
$$\int_{-\infty}^{\infty} \phi_j(x)\,\phi_k(x)\,dx = \delta_{jk}$$


$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$
$\phi_j(x) = c_j^{-1} \cdot \underbrace{(x^j + \dots)}_{\pi_j(x)} e^{-\frac{1}{2}V(x)}, \quad x \in \mathbb{R}$
$\leftarrow$ OP's
$\int_{-\infty}^{\infty} \phi_j(x)\,\phi_k(x)\,dx = \delta_{jk}$
$\uparrow$ O funcs.

$$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$$

$$\phi_j(x) = c_j^{-1}\cdot \underbrace{\left(x^j + \cdots\right)}_{\pi_j(x)}\, e^{-\frac{1}{2}V(x)}, \quad x \in \mathbb{R}$$

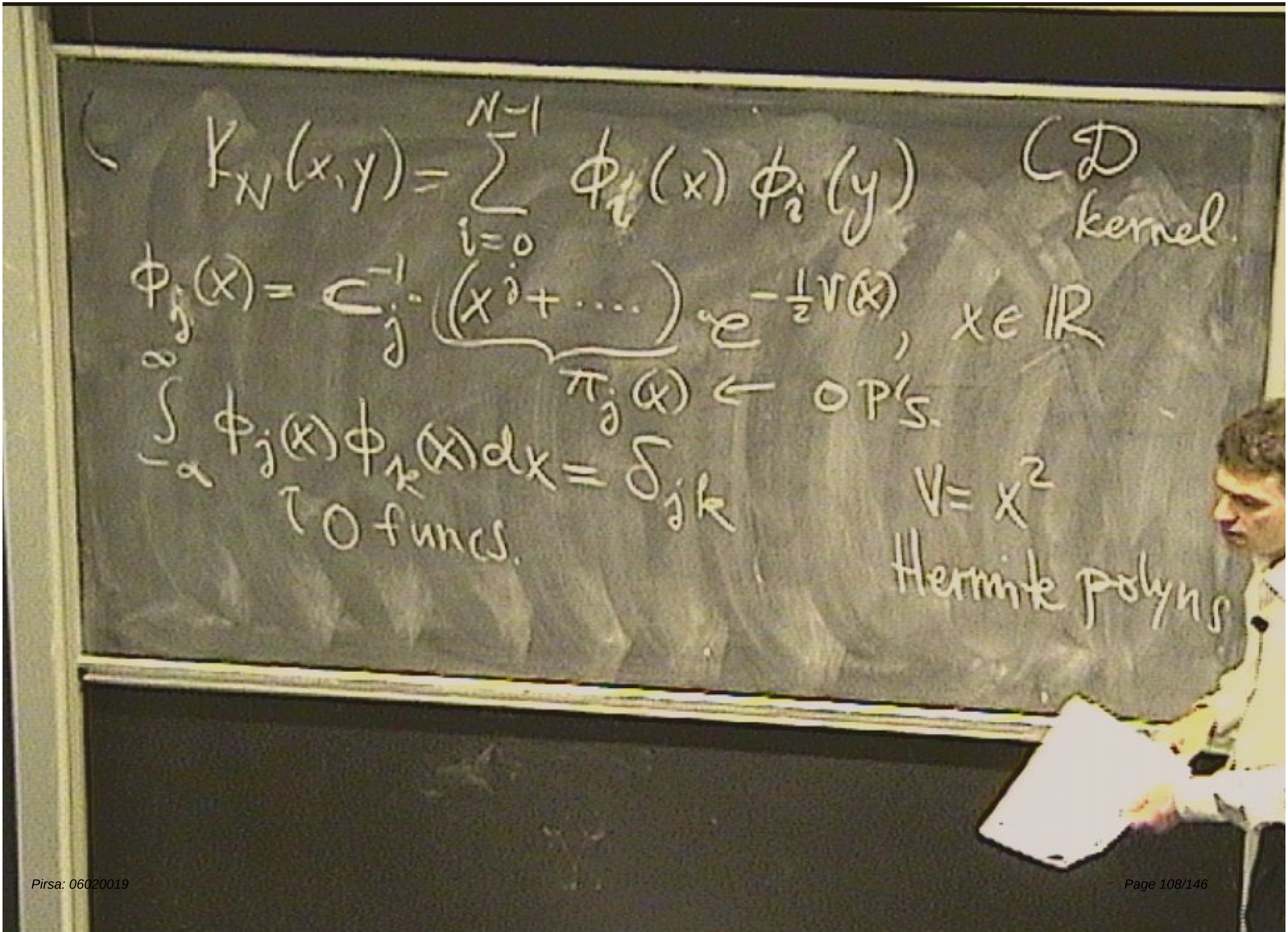
$\pi_j(x) \leftarrow$ OP's.

$$\int_{-\infty}^{\infty} \phi_j(x)\,\phi_k(x)\,dx = \delta_{jk}$$

$\uparrow$ O funcs.

$V = x^2$

Hermite polyns


$K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\,\phi_i(y)$
CD kernel
$\phi_j(x) = c_j^{-1} \cdot \underbrace{(x^j + \cdots)}_{\pi_j(x)} e^{-\frac{1}{2}V(x)}$, $x \in \mathbb{R}$
$\pi_j(x) \leftarrow$ OP's
$\int_{-\infty}^{\infty} \phi_j(x)\phi_k(x)\,dx = \delta_{jk}$
$\uparrow$ O funcs.
$V = x^2$
Hermite polyns

$$p_N^{(2)}(\vec{x}) = \frac{1}{N!} \det\left(K_N(x_i, x_j)\right)_{i,j=1}^{N}$$

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CD f-la: $$K_N(x,y) = b_N \frac{\phi_N(x)\phi_{N-1}(y) - \phi_{N-1}(x)\phi_N(y)}{x-y}$$

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$$x\phi_j(x) =$$

$$p_N^{(2)}(\vec{x}) = \frac{1}{N!} \det\left(K_N(x_i, x_j)\right)_{i,j=1}^{N}$$

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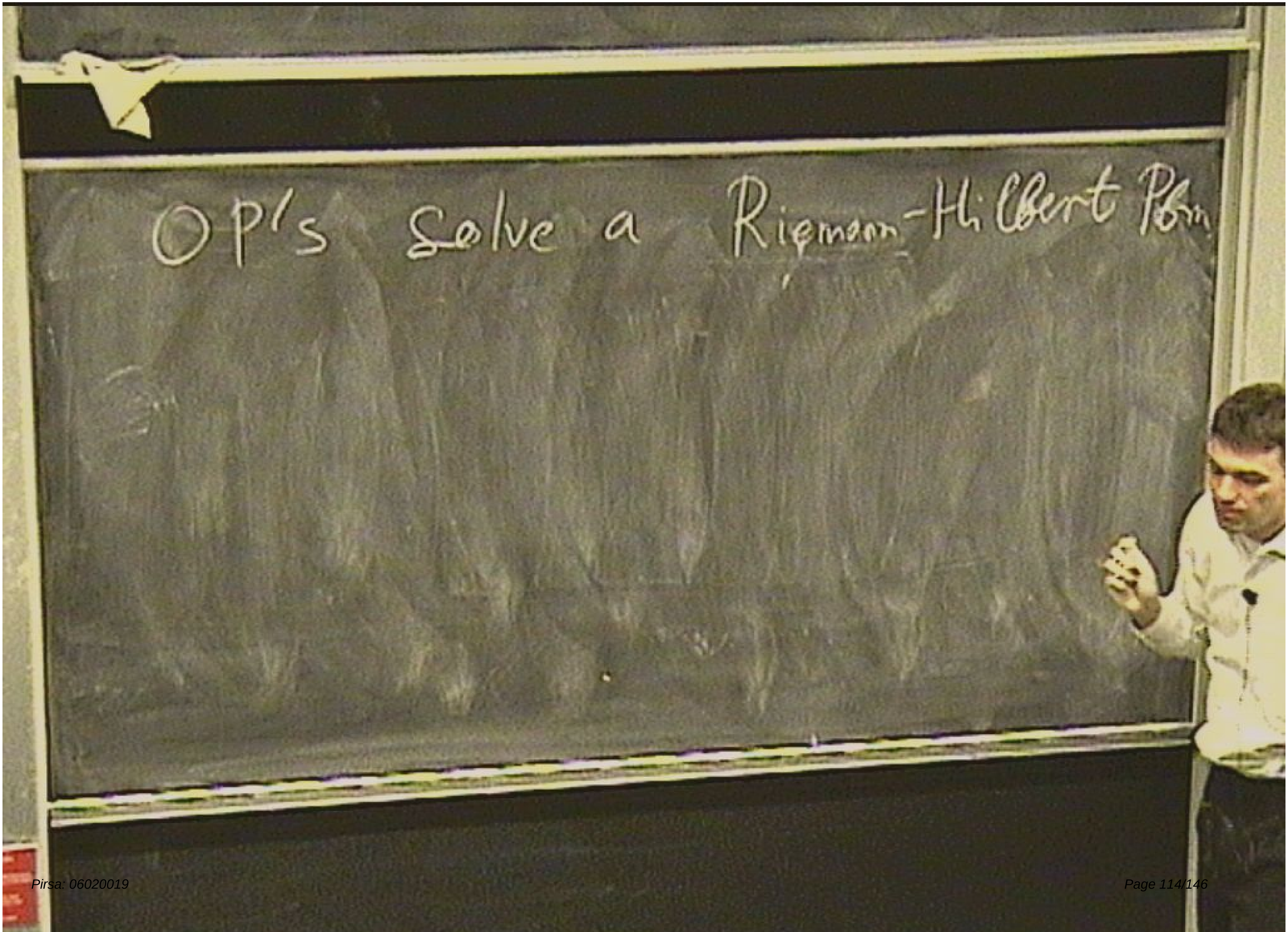
$$x\phi_j(x) = b_j \phi_{j+1}(x) + a_j \phi_j(x) + b_{j-1}\phi_{j-1}(x)$$

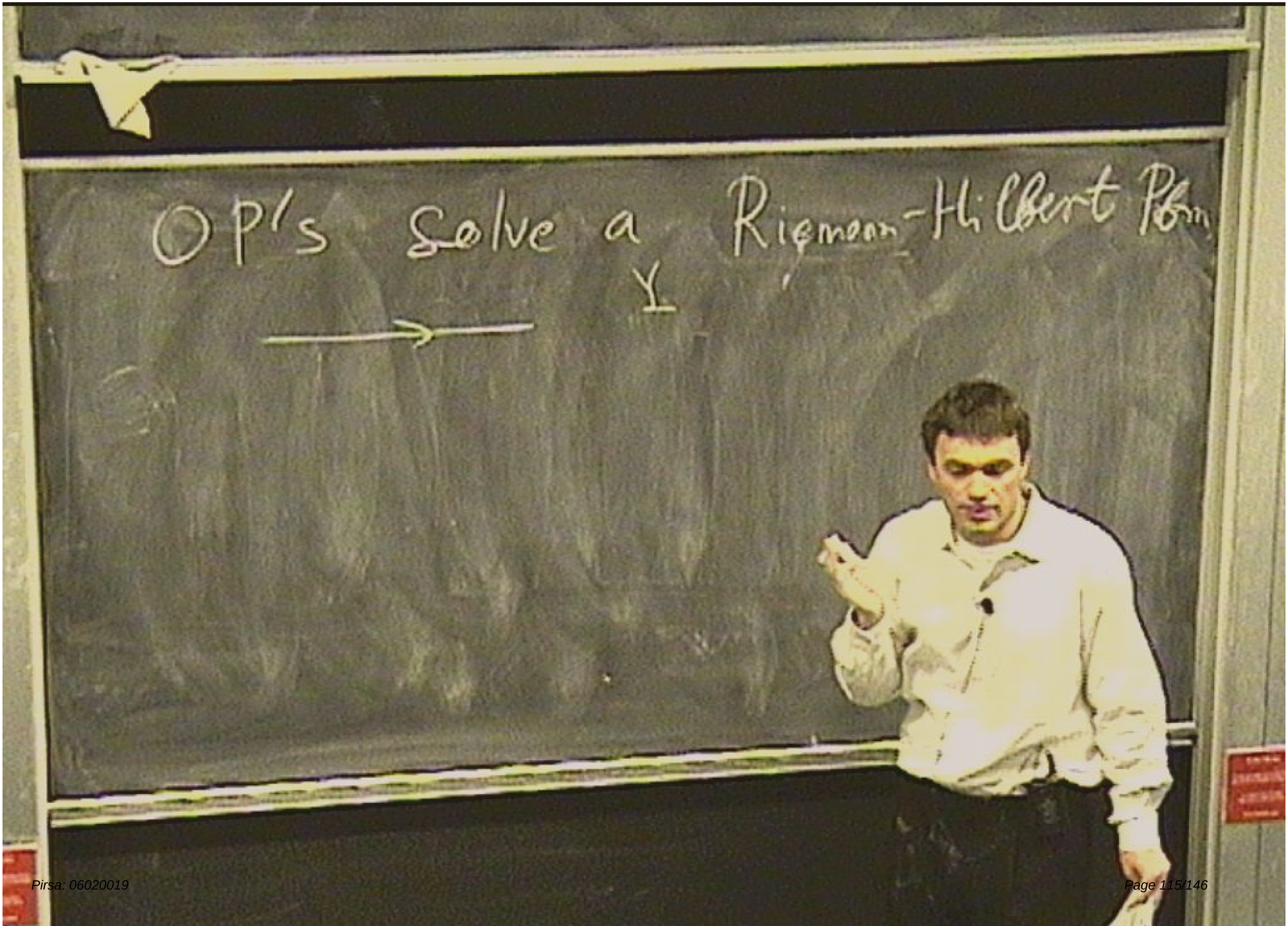
$$p_N^{(2)}(\vec{x}) = \frac{1}{N!} \det\left(K_N(x_i, x_j)\right)_{i,j=1}^{N}$$

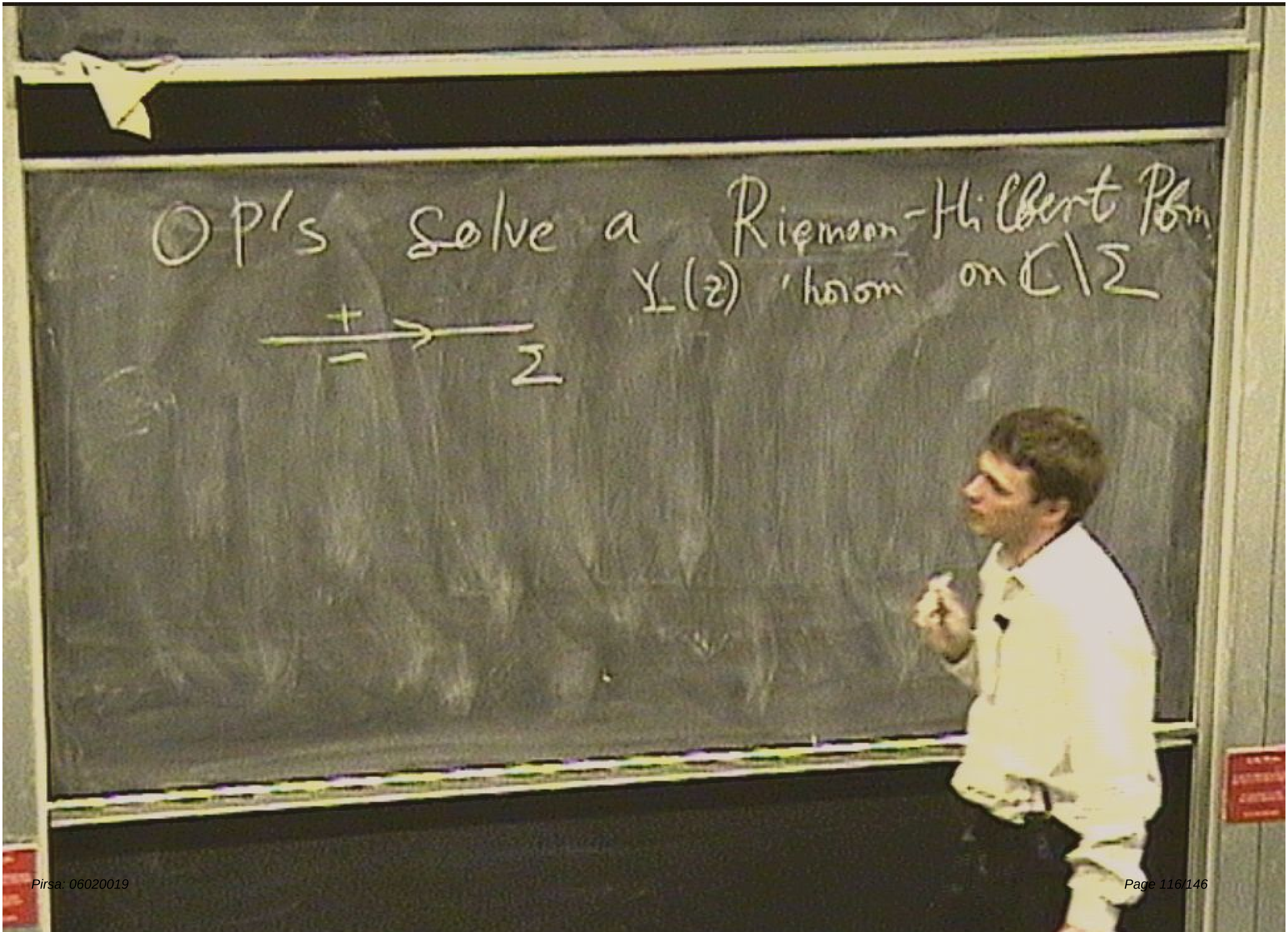
CD f-la: $K_N(x,y) = b_N \dfrac{\phi_N(x)\phi_{N-1}(y) - \phi_{N-1}(x)\phi_N(y)}{x-y}$

$$x\phi_j(x) = b_j\phi_{j+1}(x) + a_j\phi_j(x) + b_{j-1}\phi_{j-1}(x)$$

$N \to \infty$: have to study b_N, ϕ_N, ϕ_{N-1}


OP's Solve a Riemann-Hilbert Pblm


OP's Solve a Riemann-Hilbert Pbm
Y


OP's solve a Riemann-Hilbert Pblm
Y(z) 'holom on ℂ\Σ
+
−
Σ

OP's solve a Riemann-Hilbert Pblm

$Y(z)$ holom on $\mathbb{C} \setminus \Sigma$

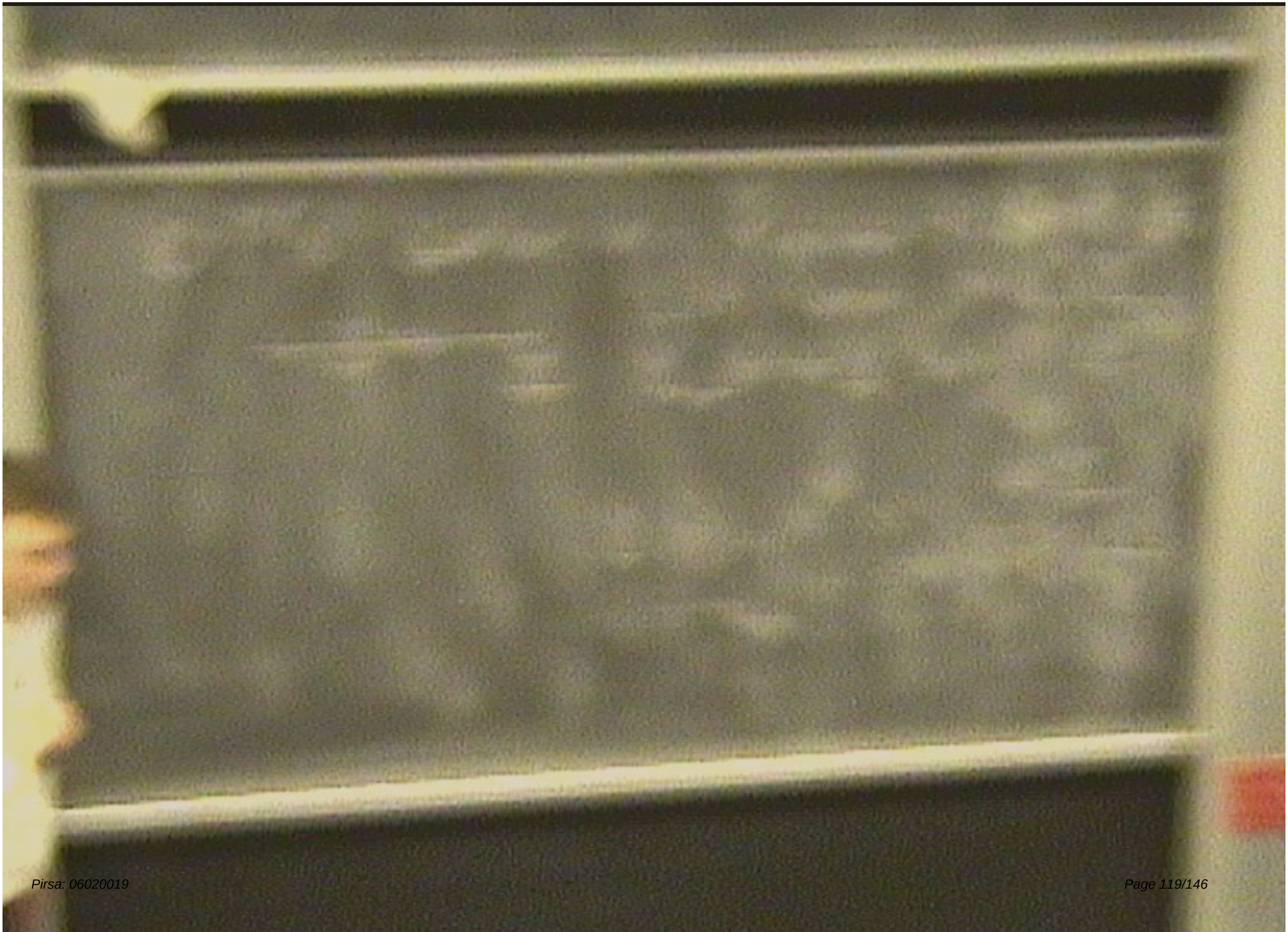
$$Y_+(z) = Y_-(z) \cdot \begin{pmatrix} 1 & e^{-V(x)} \\ 0 & 1 \end{pmatrix}$$

OP's solve a Riemann-Hilbert Pblm

$Y(z)$ holom on $\mathbb{C} \setminus \Sigma$

Σ $\qquad$ $Y_+(z) = Y_-(z) \cdot \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix}$

$z \in \Sigma$



OP's solve a Riemann-Hilbert Pblm

$Y(z)$ holom on $\mathbb{C} \setminus \Sigma$

$$\overset{+}{\underset{-}{\longrightarrow}} \quad \Sigma$$

$$Y_+(z) = Y_-(z) \cdot \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix}$$

$z \in \Sigma$

$$Y(z) \underset{z \to \infty}{=} \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \left(I + O\left(\frac{1}{|z|} \right) \right)$$

OP's solve a Riemann-Hilbert Pbm

$\underline{Y}(z)$ holom on $\mathbb{C} \setminus \Sigma$

$$\underline{Y}_+(z) = \underline{Y}_-(z) \cdot \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix} \quad z \in \Sigma$$

$$\underline{Y}(z) = \begin{pmatrix} \pi_N(z) & * \\ * & * \end{pmatrix}$$

$$\underline{Y}(z) \underset{z \to \infty}{=} \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \left(I + O\left(\frac{1}{|z|}\right) \right)$$

OP's Solve a Riemann-Hilbert Pblm

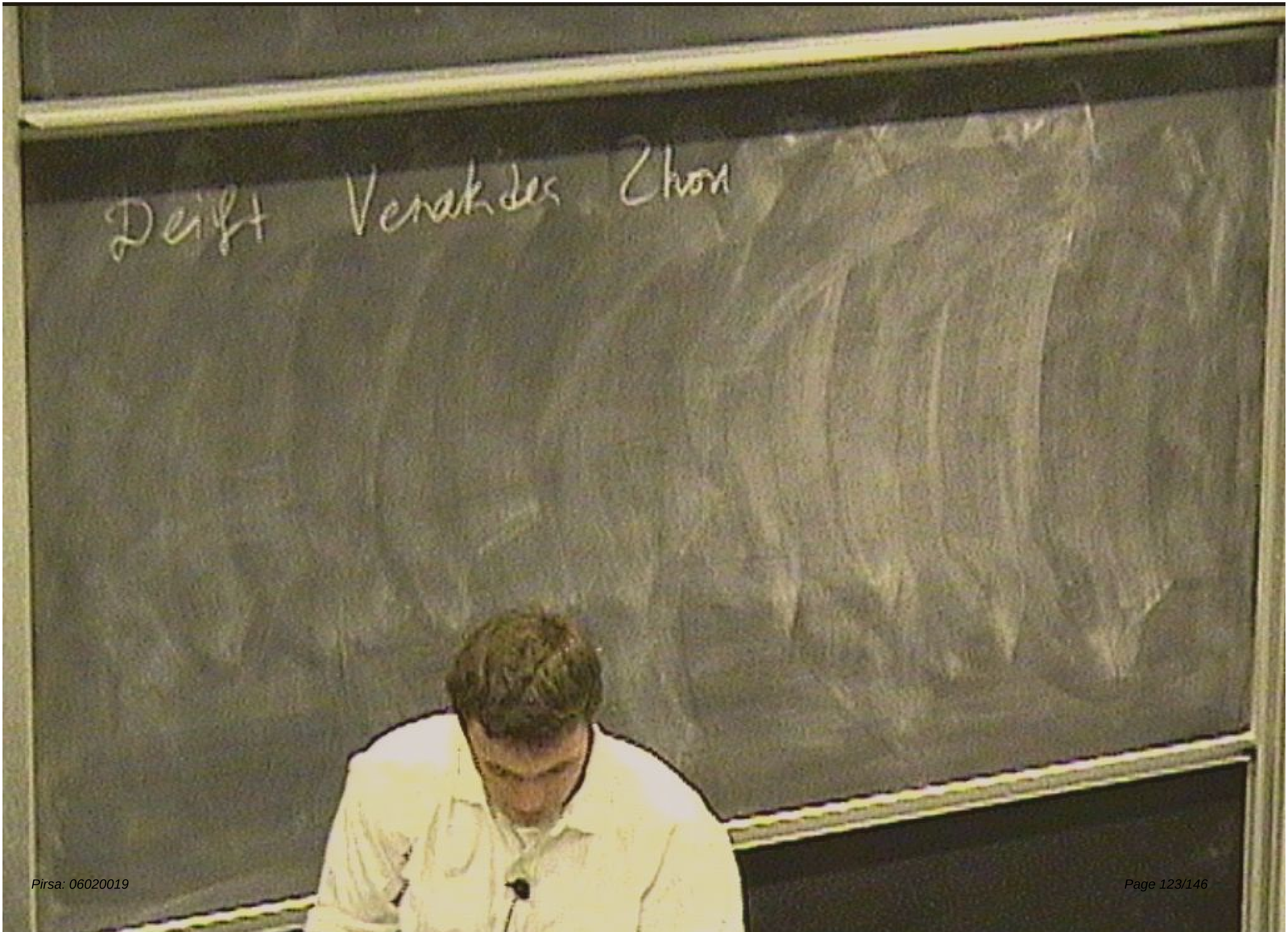
$Y(z)$ holom on $\mathbb{C} \setminus \Sigma$

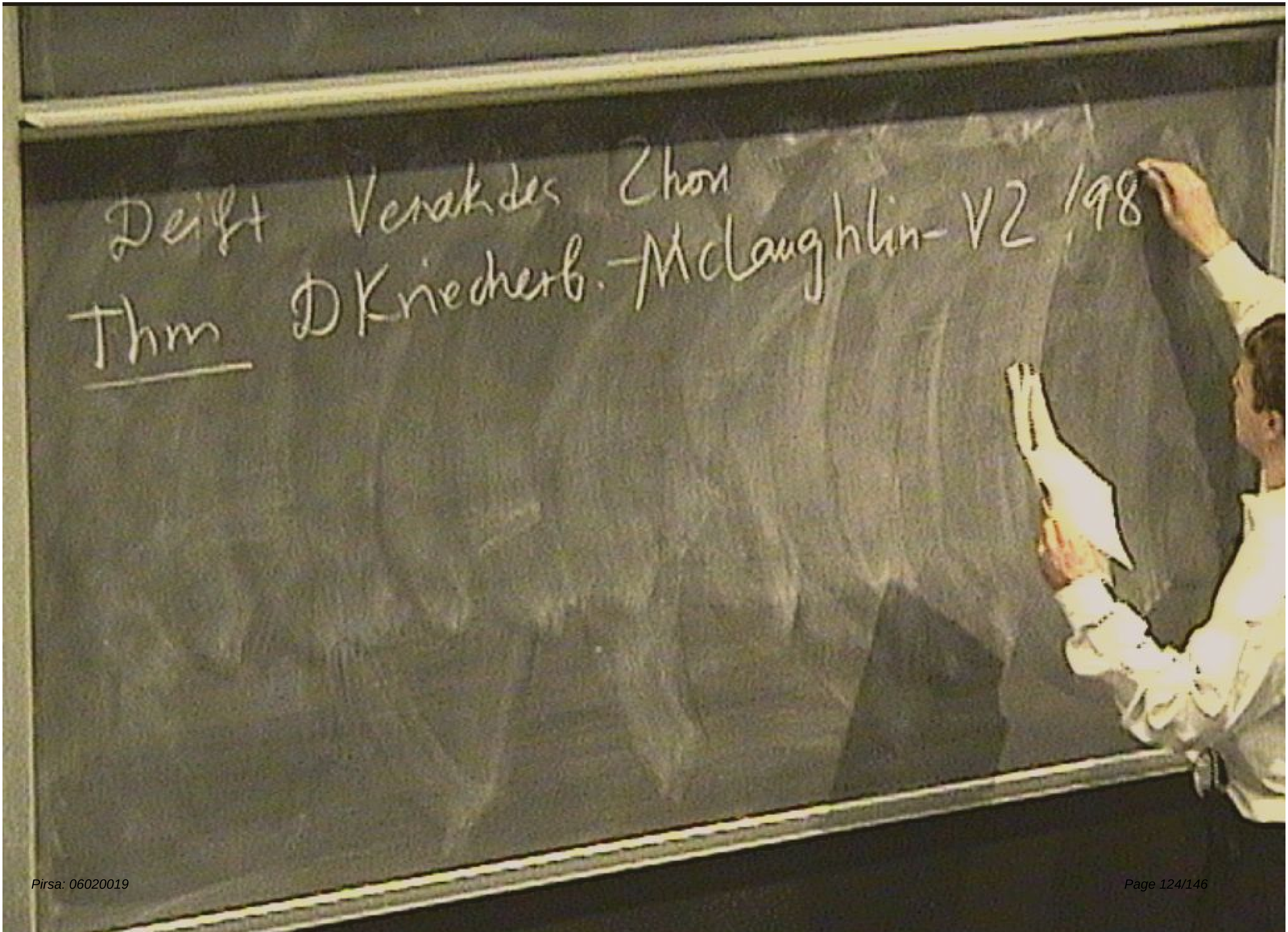
$$Y_+(z) = Y_-(z) \cdot \begin{pmatrix} 1 & e^{-V(x)} \\ 0 & 1 \end{pmatrix} \qquad z \in \Sigma$$

$$Y(z) = \begin{pmatrix} \pi_N(z) & * \\ * & * \end{pmatrix}$$

$$Y(z) = \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix} \left(I + O\left(\frac{1}{|z|} \right) \right) \qquad z \to \infty$$

Fokas-Its-Kitaev '91


Deift Venakides Zhou


Deift Venakides Zhou
Thm DKriecherb.-McLaughlin-VZ 198

Deift Venakides Zhou

<u>Thm</u> (D Kriecherb. - McLaughlin - VZ '98)

For $\forall$ $V(x) = \alpha_{2m} x^{2m} + \cdots$, $\alpha_{2m} > 0$

Deift Venakides Zhou

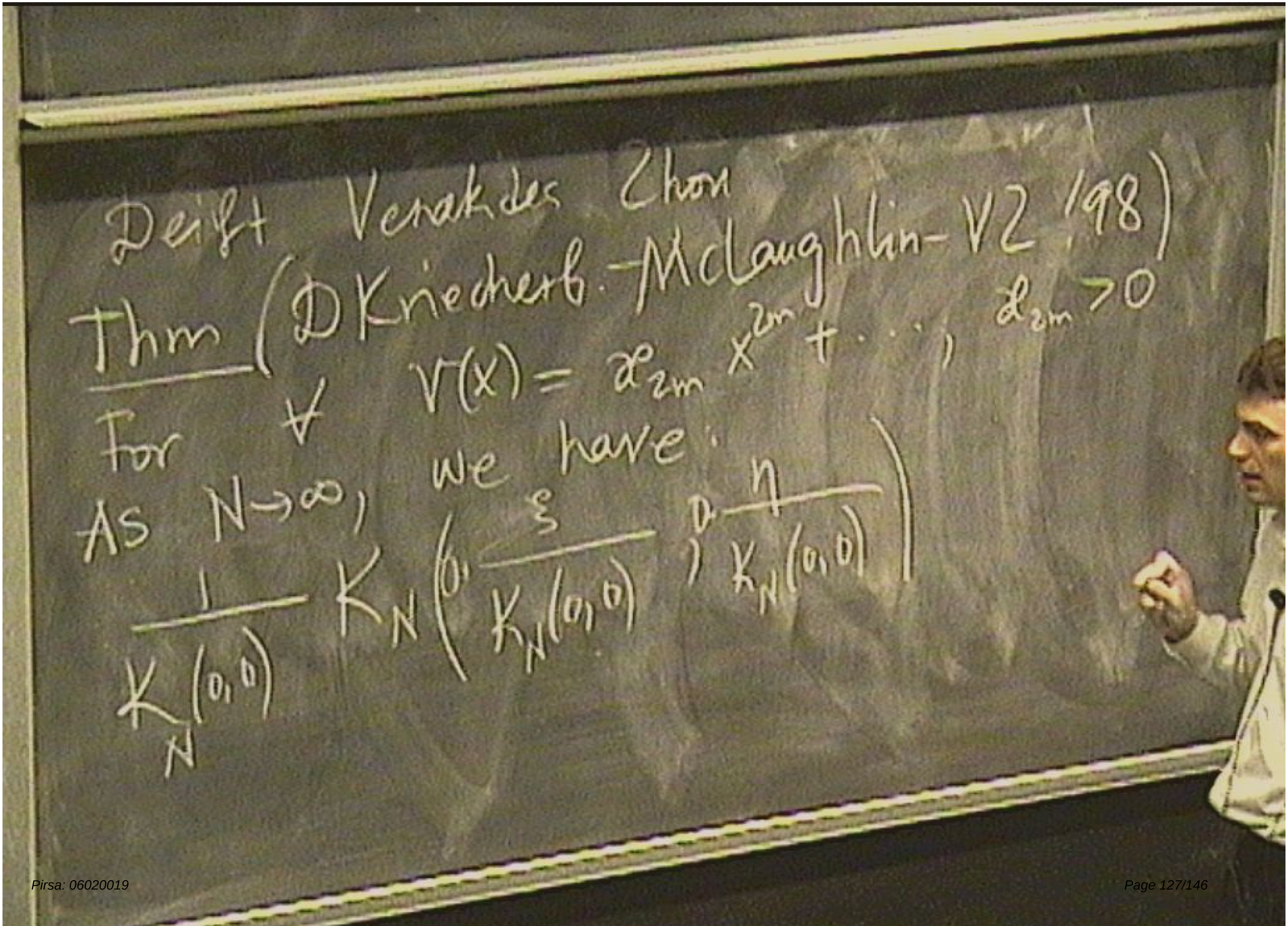
Thm (D Kriecherb. - McLaughlin - VZ '98)

For $\forall$ $V(x) = \varkappa_{2m} x^{2m} + \dots$, $\varkappa_{2m} > 0$

we have:

As $N \to \infty$,

$\frac{1}{K} K_N \Big($


Deift Venakides Zhou
Thm (DKriecherb.-McLaughlin-VZ '98)
For ∀ $V(x) = \alpha_{2m} x^{2m} + \dots$, $\alpha_{2m} > 0$
As $N \to \infty$, we have:
$\frac{1}{K_N(0,0)} K_N\left(\frac{\xi}{K_N(0,0)}, \frac{\eta}{K_N(0,0)}\right)$

Deift Venakides Zhou

Thm (D Kriecherb. - McLaughlin - VZ '98)

For $\forall$ $V(x) = \varkappa_{2m} x^{2m} + \ldots$, $\varkappa_{2m} > 0$

we have:

As $N \to \infty$,

$$\frac{1}{K_N(0,0)} K_N\left(0, \frac{\xi}{K_N(0,0)} ; 0, \frac{\eta}{K_N(0,0)}\right) \xrightarrow[N\to\infty]{}$$

Deift Venakides Zhou

Thm (D Kriecherb. McLaughlin-VZ '98)

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Deift Venakides Zhou

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Thm (D Kriecherb. – McLaughlin – VZ '98)

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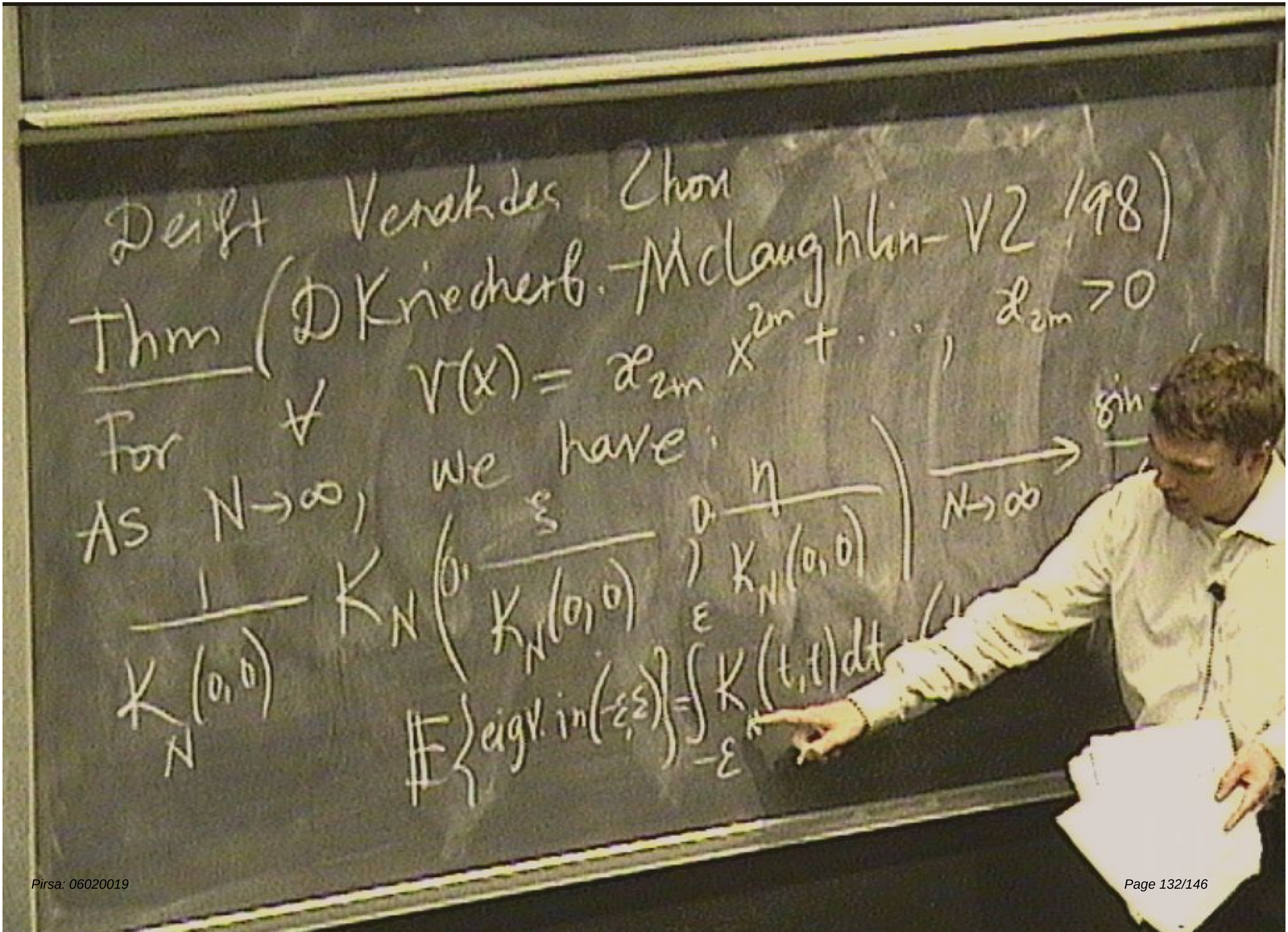
we have:

As $N \to \infty$,

$$\frac{1}{K_N(0,0)} K_N\left(0 + \frac{\xi}{K_N(0,0)}, 0 + \frac{\eta}{K_N(0,0)}\right) \xrightarrow[N\to\infty]{} \frac{\sin \pi(\xi-\eta)}{\pi(\xi-\eta)}$$

$$\mathbb{E}\{\text{eigv. in } (-\varepsilon,\varepsilon)\} = \int_{-\varepsilon}^{\varepsilon} K_N(t,t)\,dt \cdot (1 + o(1)), \quad N \to \infty$$


Deift Venakides Zhou
Thm (DKriecherb. McLaughlin-VZ '98)
For ∀ V(x) = κ2m x^2m + . . . , κ2m > 0
we have:
As N→∞,
1/KN(0,0) KN(0 + ξ/KN(0,0), 0 + η/KN(0,0)) → N→∞
E{eigv. in (-ε,ε)} = ∫ from -ε to ε KN(t,t)dt

Deift Venakides Zh

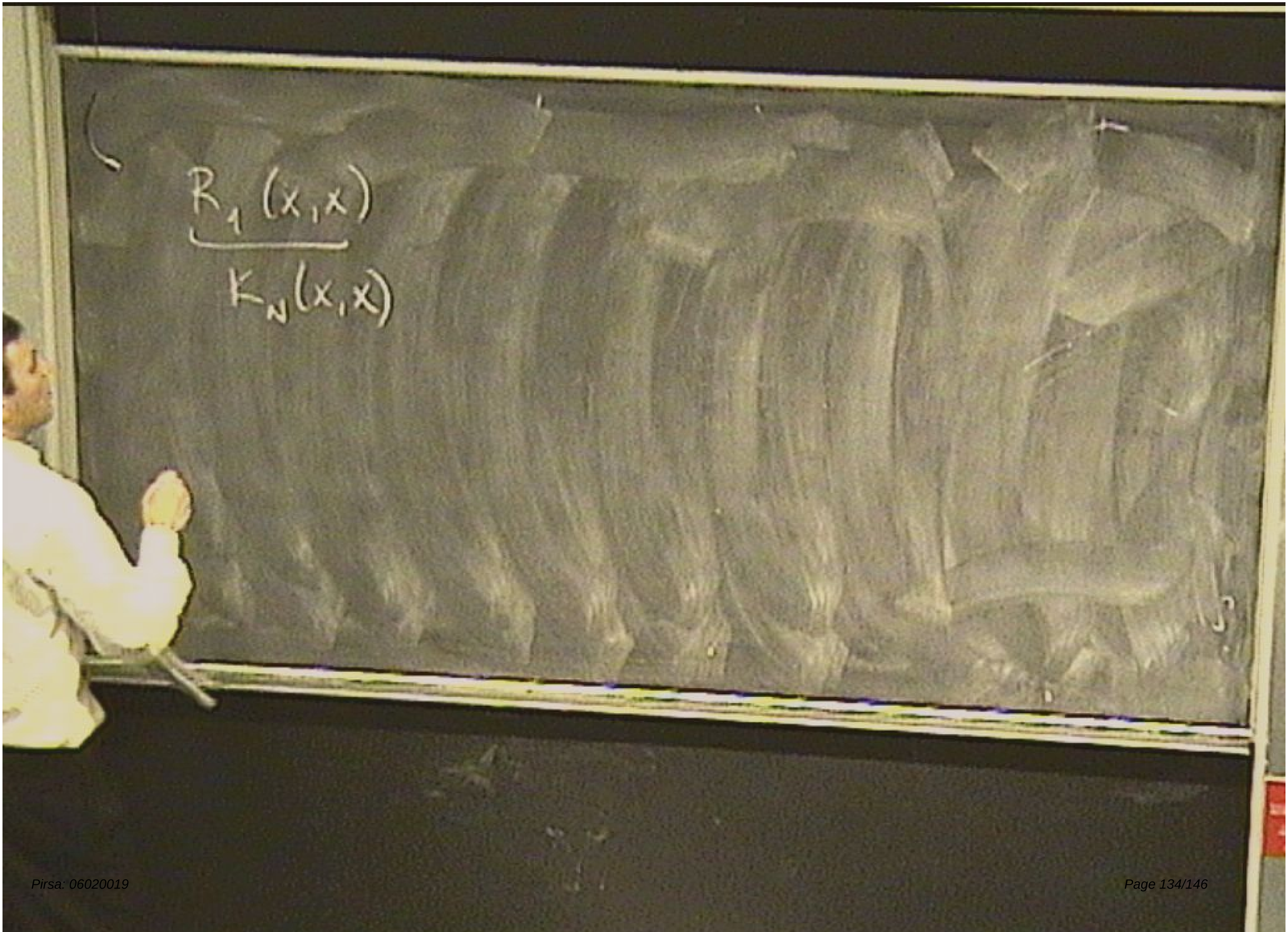
Thm (D Kriecherb. - M ... 2 '98)

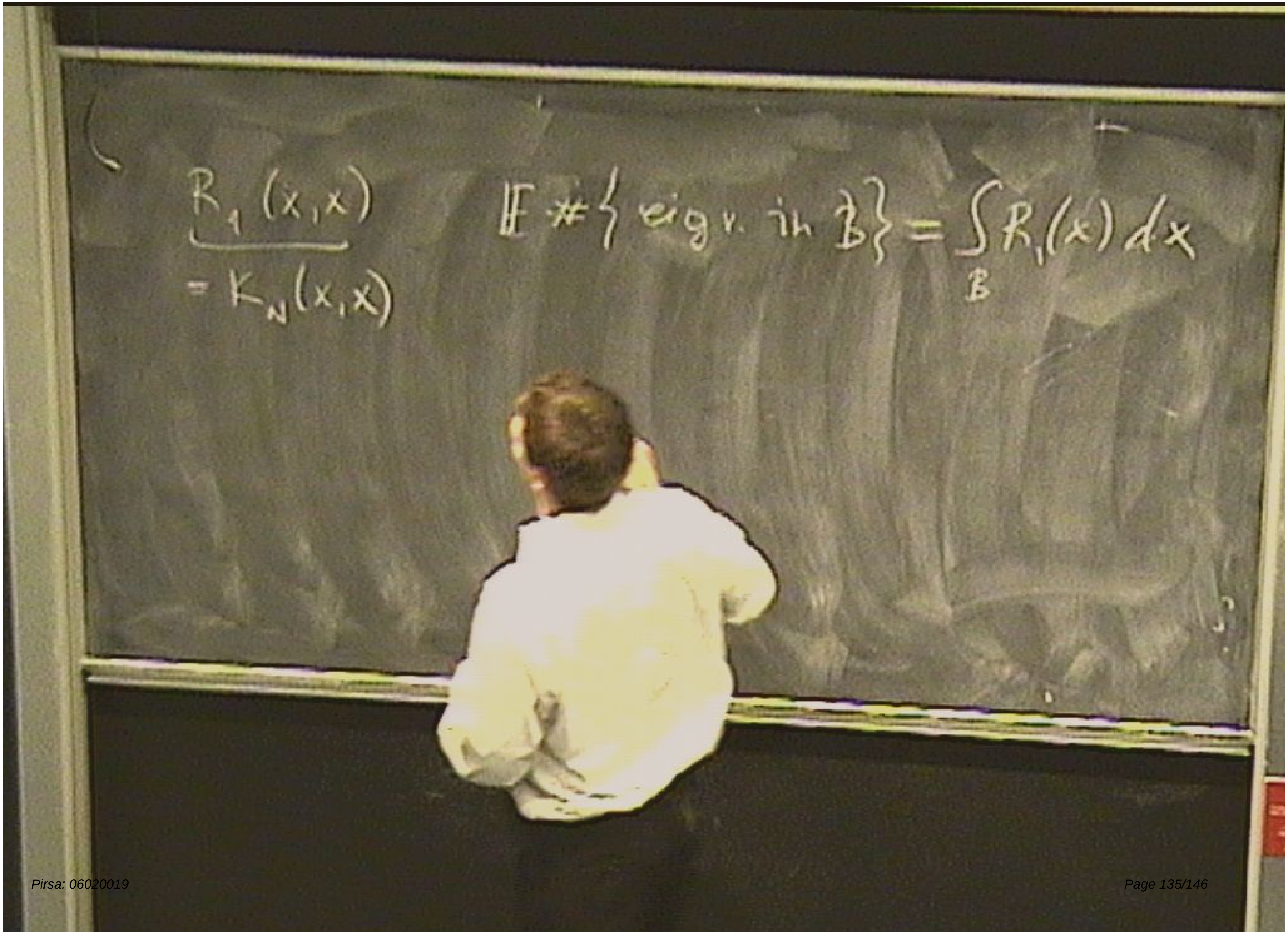
For $\forall$ $V(x) = \varkappa_{2m} x^{2m} + \cdots$, $\varkappa_{2m} > 0$

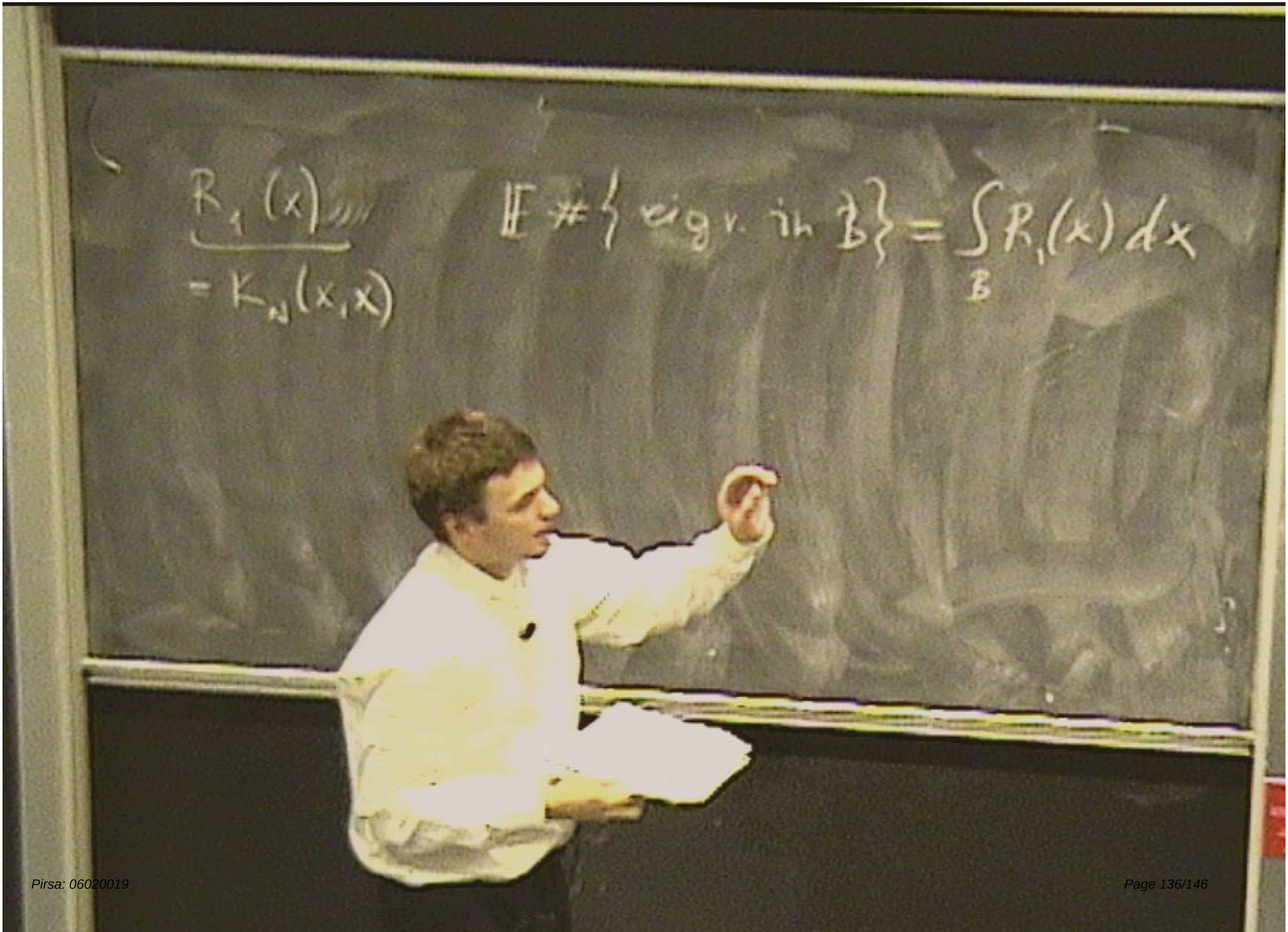
As $N \to \infty$, we have:

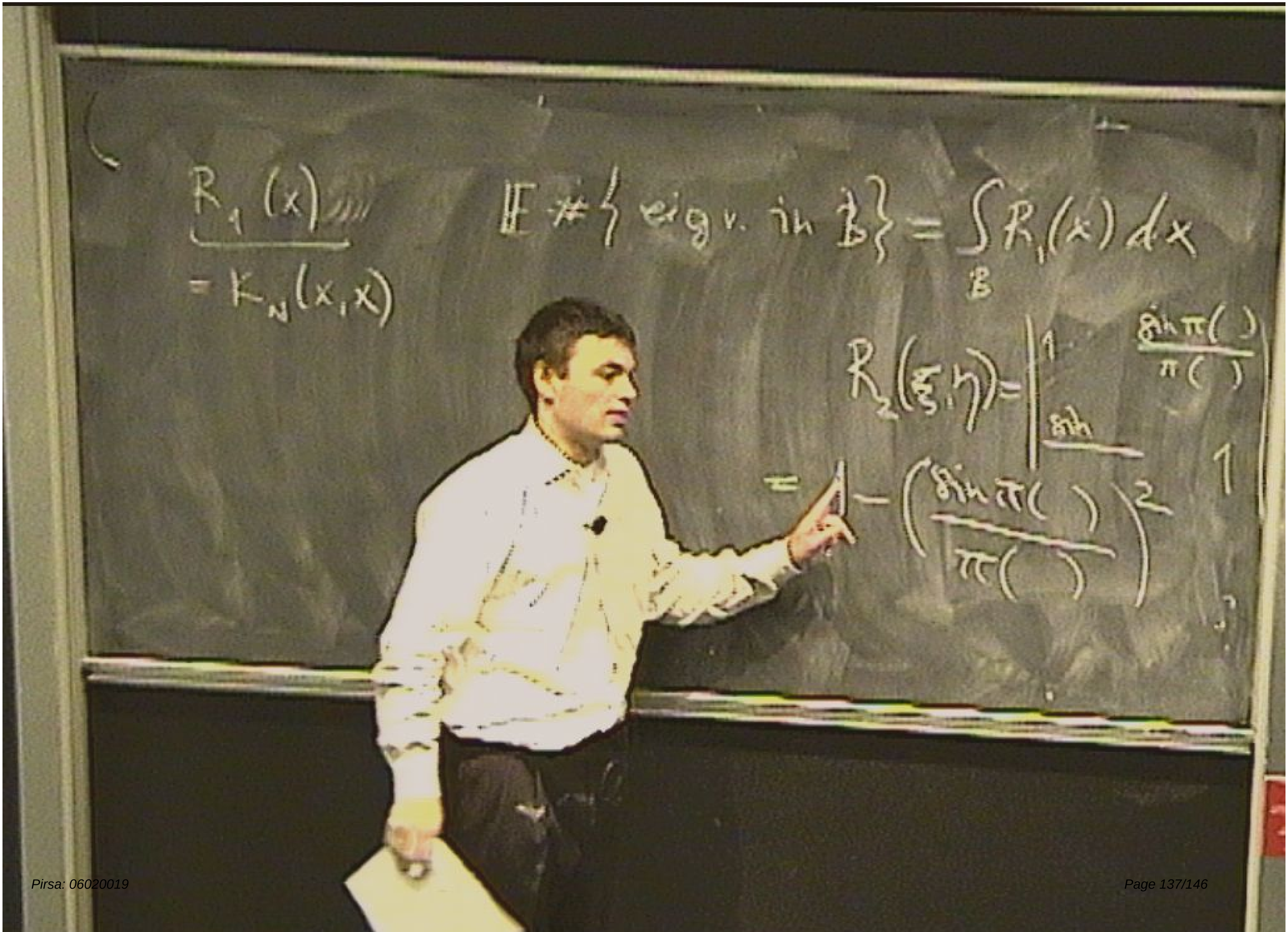
$$\frac{1}{K_N(0,0)} K_N\left(0 + \frac{\xi}{K_N(0,0)}, 0 + \frac{\eta}{K_N(0,0)}\right) \xrightarrow[N\to\infty]{} \frac{\sin \pi(\xi-\eta)}{\pi(\xi-\eta)}$$

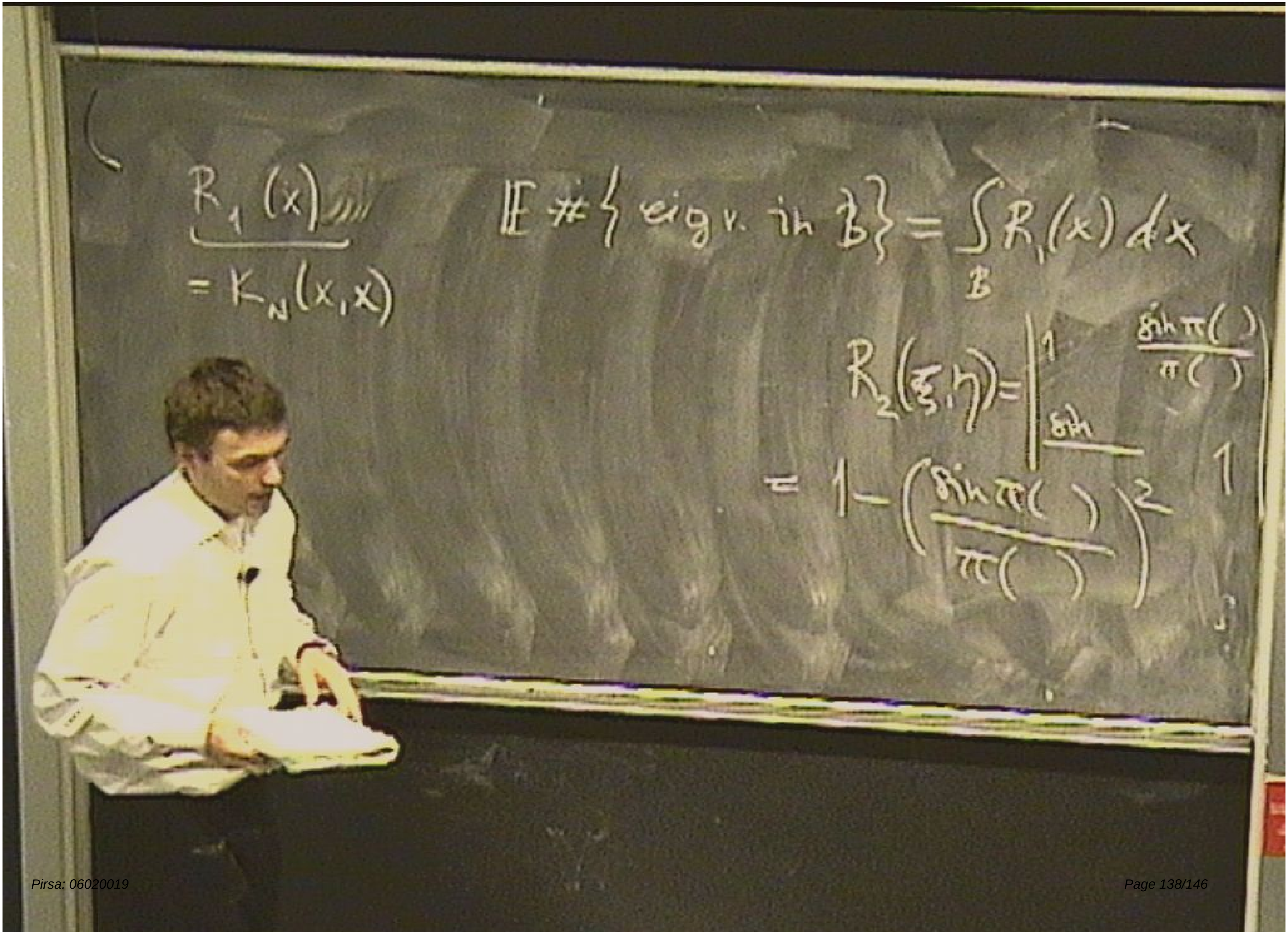
$$\mathbb{E}\{\text{eigv. in } (-\varepsilon,\varepsilon)\} = \int_{-\varepsilon}^{\varepsilon} K_N(t,t)\,dt \cdot (1+o(1)), \quad N \to \infty$$

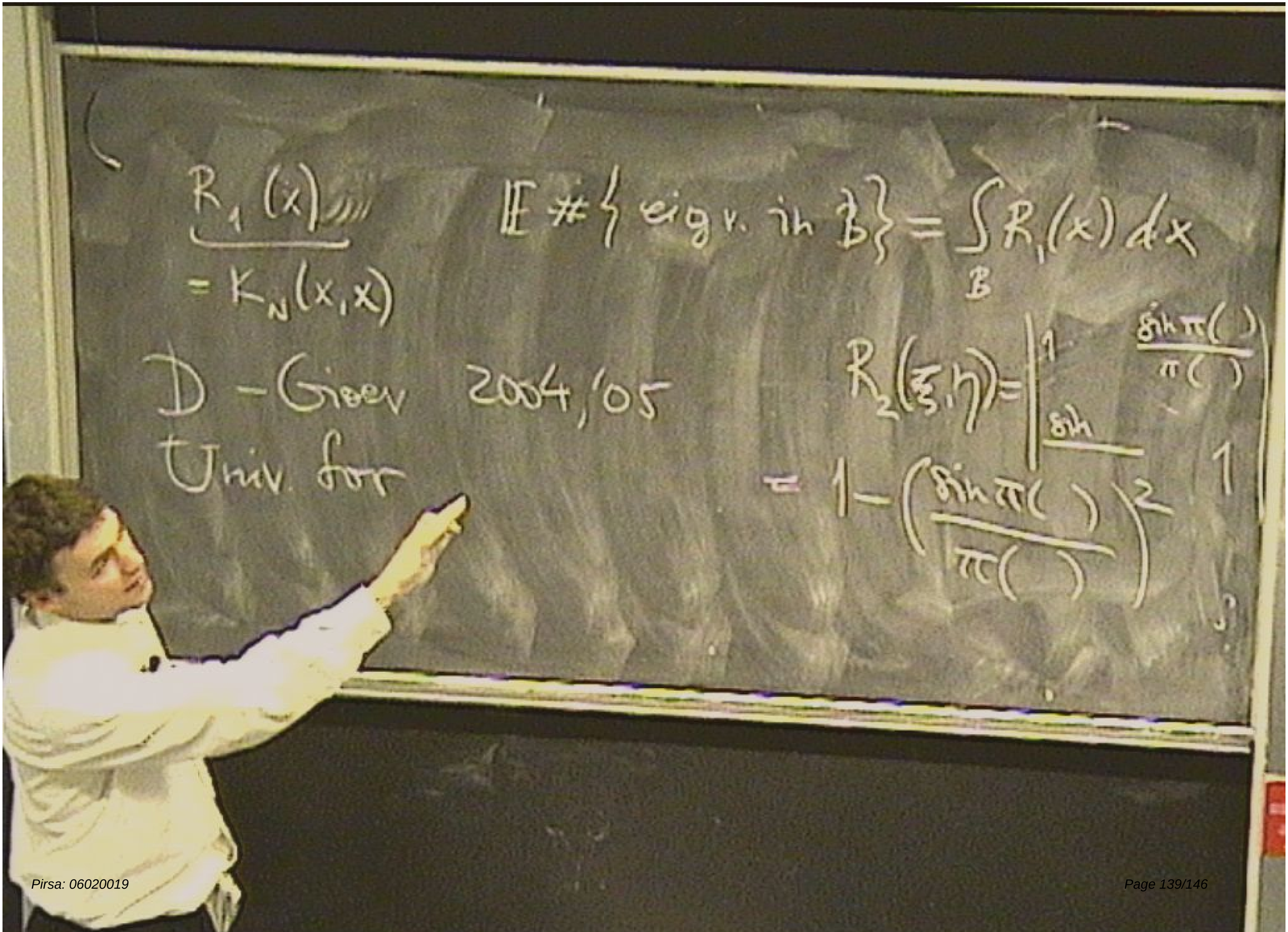

$\underbrace{R_1(x,x)}_{K_N(x,x)}$


$R_1(x,x)$
$= K_N(x,x)$
$\mathbb{E}\,\#\{\text{eigv. in } B\} = \int_B R_1(x)\,dx$


$R_1(x) = K_N(x,x)$
$\mathbb{E}\,\#\{\text{eigv. in } B\} = \int_B R_1(x)\,dx$


$R_1(x)$
$= K_N(x,x)$
$\mathbb{E}\,\#\{\text{eig v. in } B\} = \int_B R_1(x)\,dx$
$R_2(\xi,\eta) = \begin{vmatrix} 1 & \frac{\sin \pi(\ )}{\pi(\ )} \\ \frac{\sin}{} & 1 \end{vmatrix}$
$= 1 - \left(\frac{\sin \pi(\ )}{\pi(\ )}\right)^2$


R₁(x) = K_N(x,x)
𝔼 #{eigv. in B} = ∫_B R₁(x) dx
R₂(ξ,η) = | 1 sinπ( )/π( ) ; sin 1 |
= 1 − (sinπ( )/π( ))²



$R_1(x)$
$= K_N(x,x)$
$\mathbb{E}\,\#\{\text{eigv. in } B\} = \int_B R_1(x)\,dx$
D-Gioev 2004, '05
Univ. for
$R_2(\xi,\eta) = \begin{vmatrix} 1 & \frac{\sin\pi(\ )}{\pi(\ )} \\ \sin & 1 \end{vmatrix}$
$= 1 - \left(\frac{\sin\pi(\ )}{\pi(\ )}\right)^2$

$\underbrace{R_1(x)}$

$= K_N(x,x)$

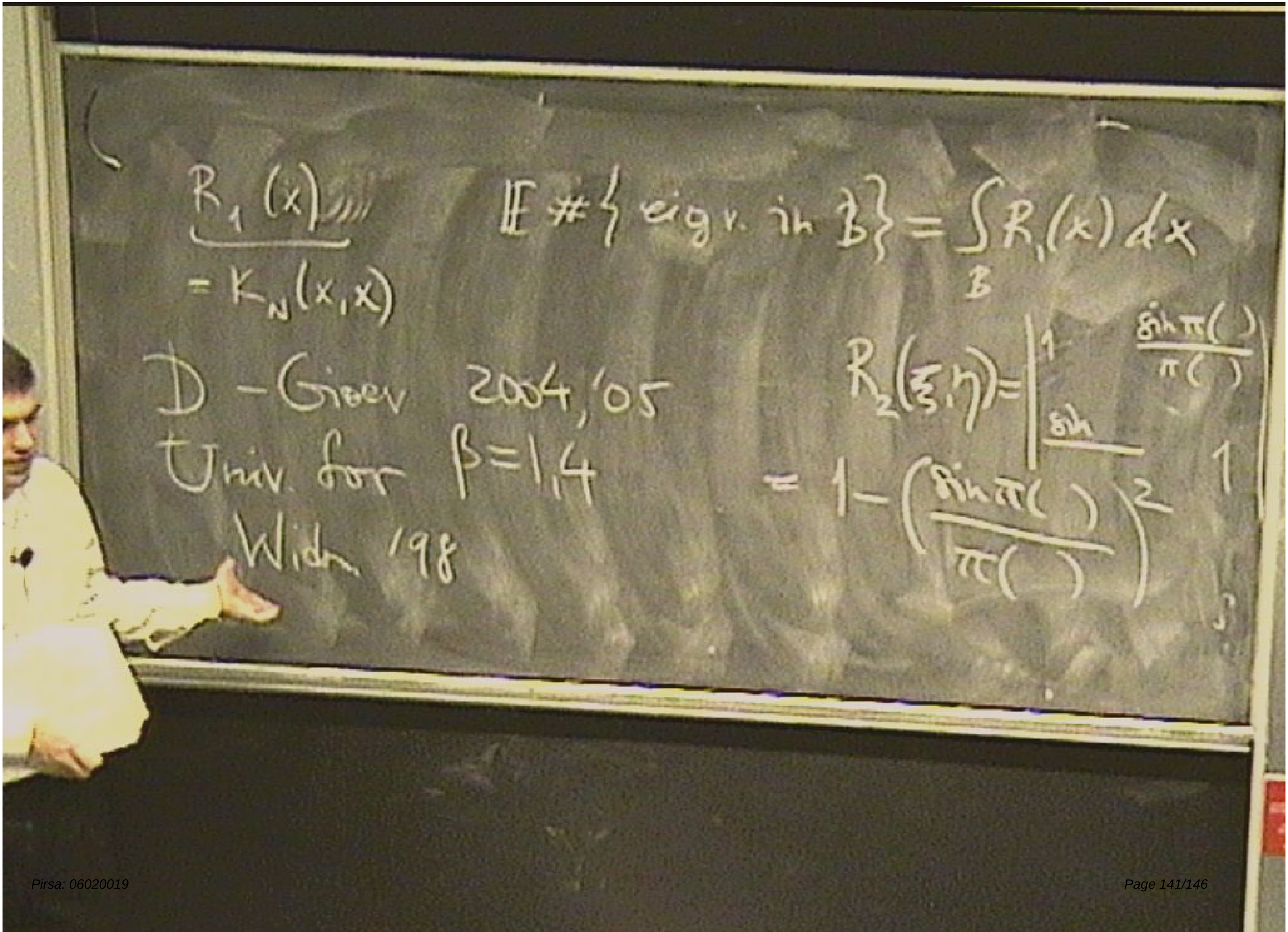
$\mathbb{E}\,\#\{\text{eigv. in } B\} = \int_B R_1(x)\,dx$

D – Gioev 2004, '05

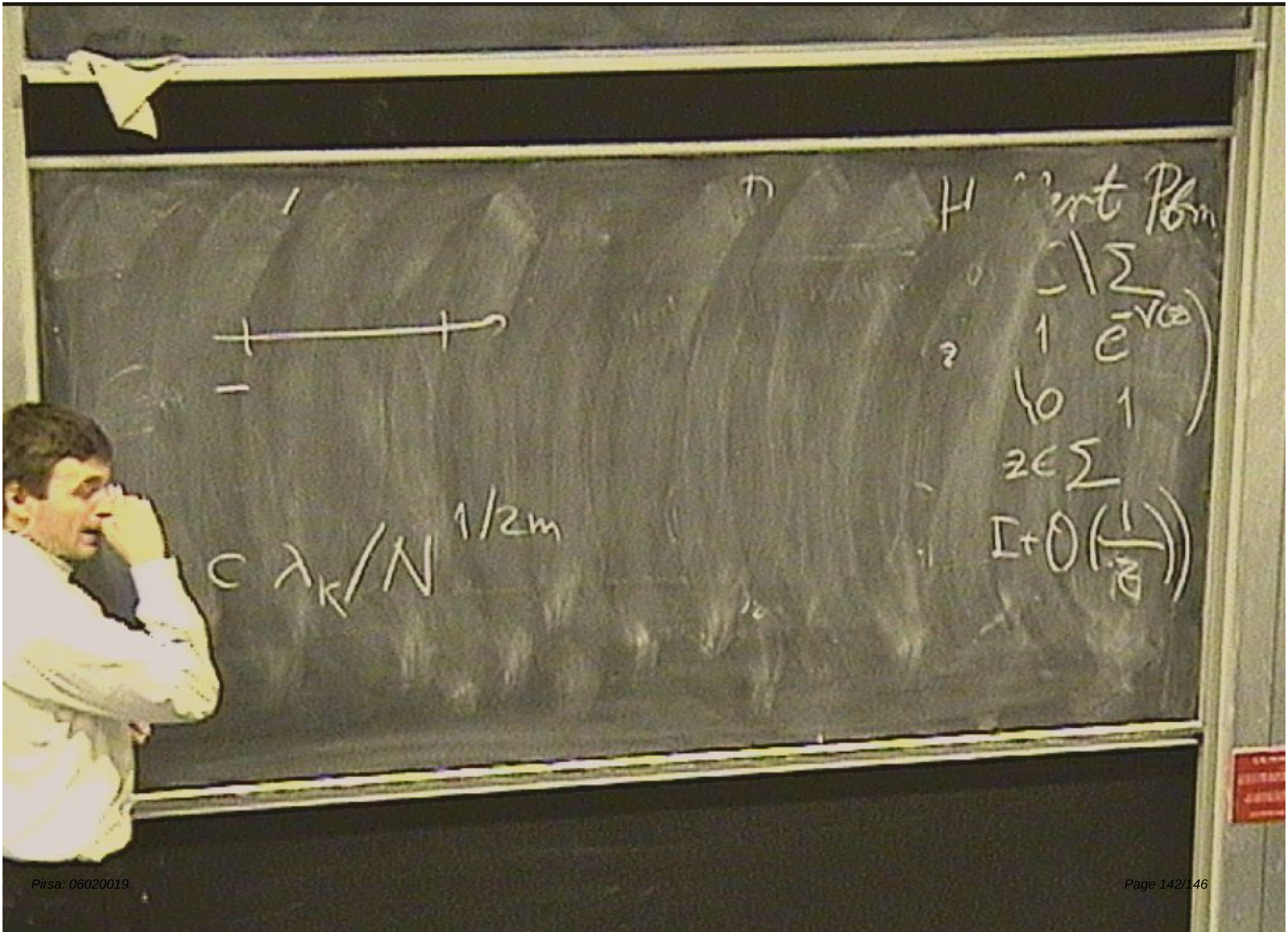
Univ. for $\beta = 1, 4$

$$R_2(\xi,\eta) = \begin{vmatrix} 1 & \frac{\sin\pi(\ )}{\pi(\ )} \\ \frac{\sin}{} & 1 \end{vmatrix}$$

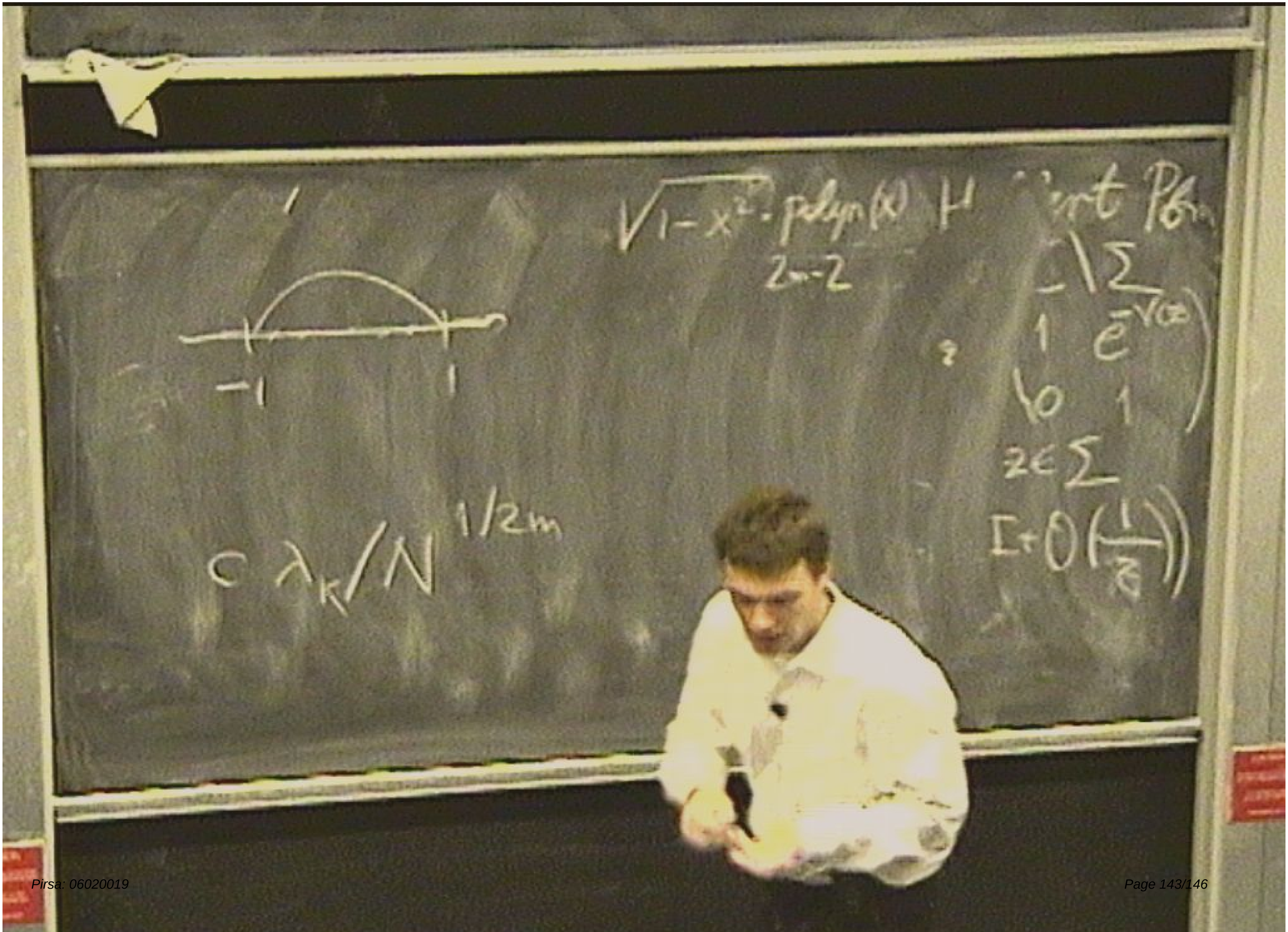
$$= 1 - \left(\frac{\sin\pi(\ )}{\pi(\ )}\right)^2$$

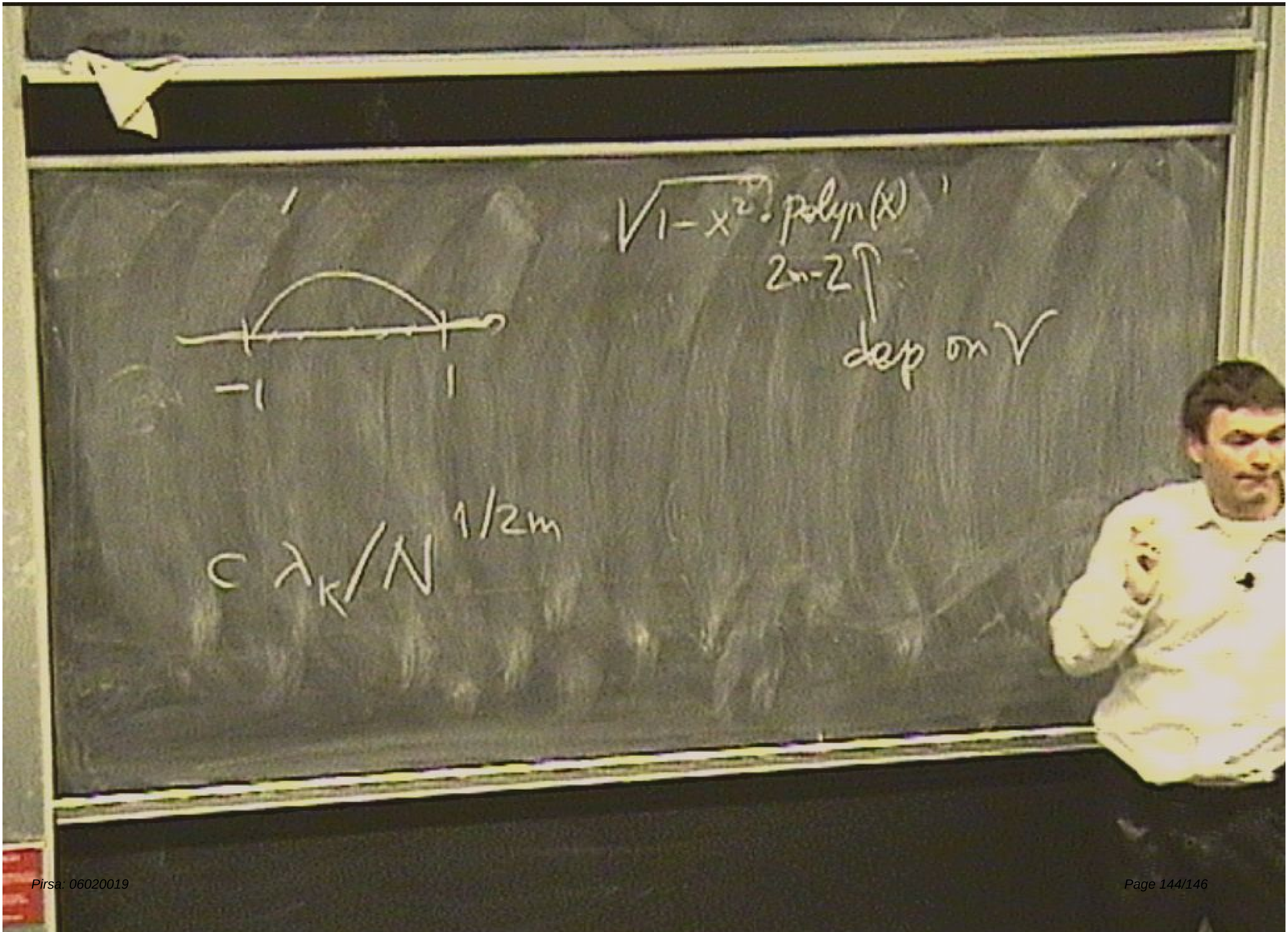


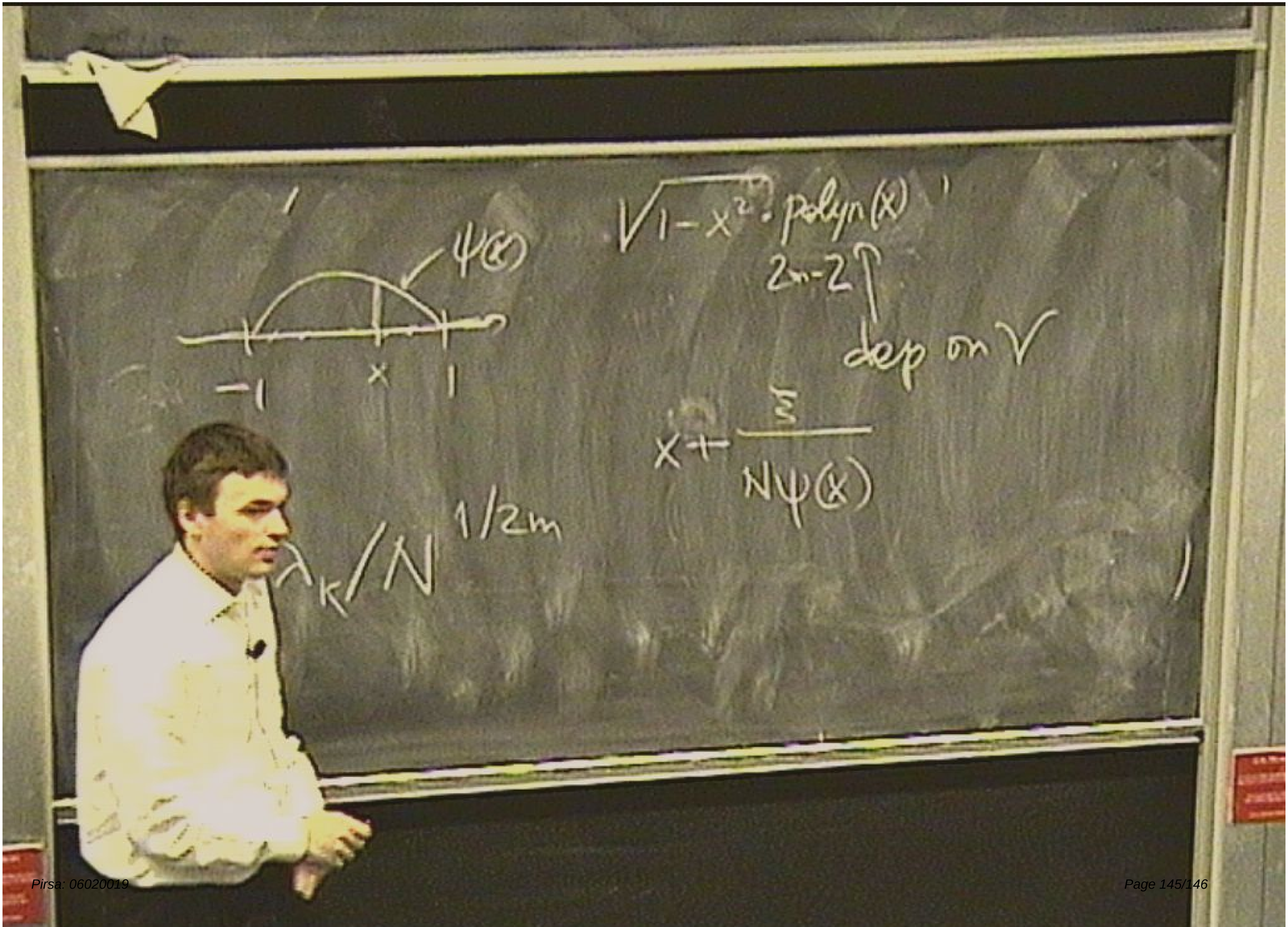
R1 (x)
= KN(x,x)
E # { eigv. in B } = ∫B R1(x) dx
D - Gioev 2004, '05
Univ. for β=1,4
Widom '98
R2(ξ,η) = | 1 sinπ( )/π( ) sin 1 |
= 1 - (sinπ( )/π( ))²

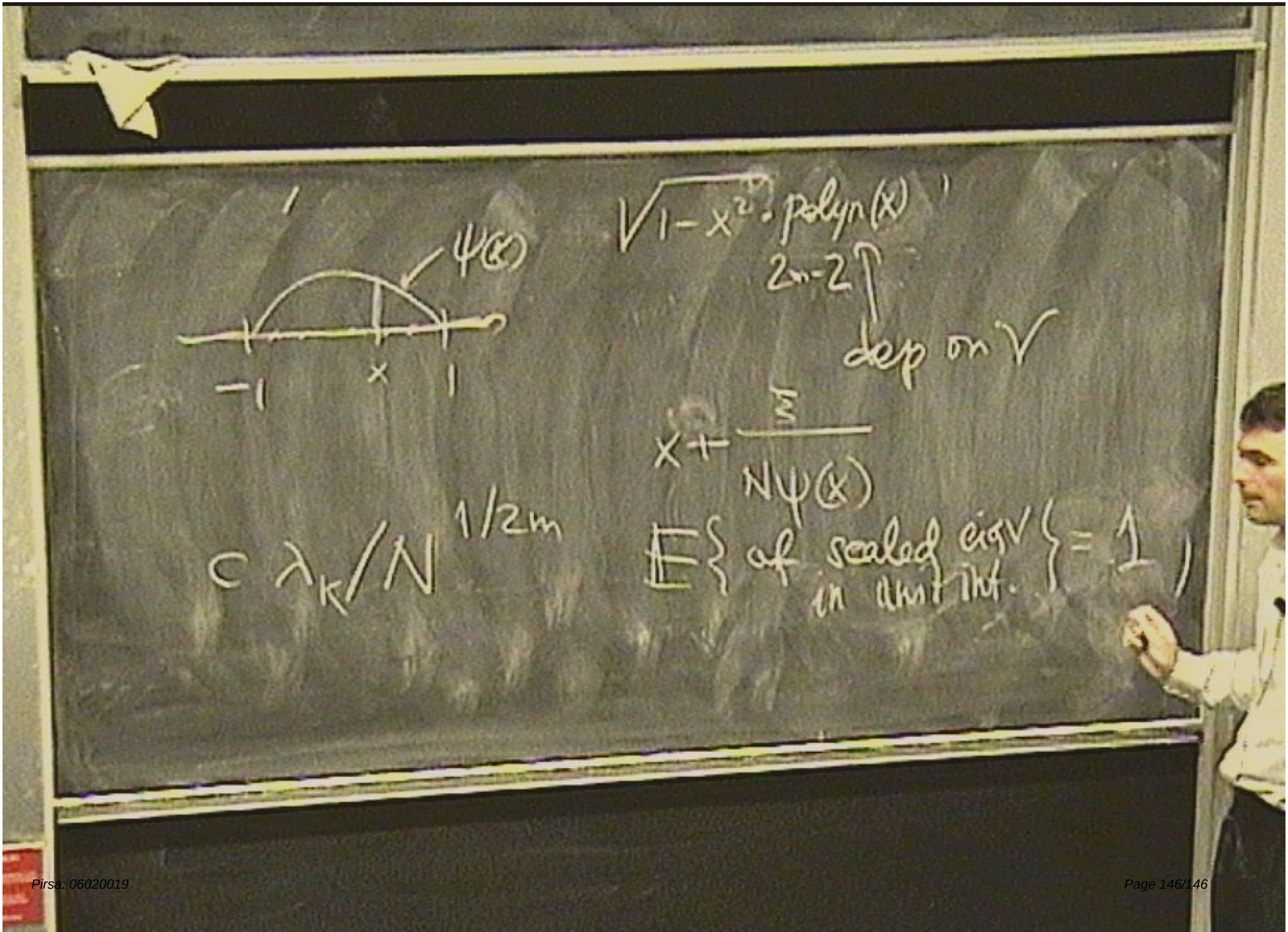


ert Pblm
C \ Σ
1 e^{-V(z)}
0 1
z ∈ Σ
I + O(1/z)
c λ_k/N^{1/2m}


√1−x² · polyn(x)
2m−2
−1
1
z∈Σ
1/2m
c λk/N


√1-x² · polyn(x)
2m-2
dep on V
-1
1
c λk/N 1/2m


ψ(x)
−1
x
1
√1−x² · polyn(x)
2m−2
dep on V
x + ξ / Nψ(x)
λk/N 1/2m


ψ(x)
−1
x
1
√(1−x²)·polyn(x)
2m−2
dep on V
x + Σ / Nψ(x)
c λ_k/N^(1/2m)
E{ of scaled eigv in unit int. } = 1