

Title: Overview of adiabatic quantum computation

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Abstract:

Overview of adiabatic quantum computation

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Outline

- Quantum mechanical computers
- Quantum computation and Hamiltonian dynamics
- The adiabatic theorem
- Adiabatic optimization
 - ▶ Examples of success
 - ▶ Example of failure
 - ▶ Random satisfiability problems
- Universal quantum computation

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A brief history

- Manin, Feynman, early 1980s: Quantum computers should be good at simulating quantum systems
- Deutsch, 1985: Formal model of quantum computers
- Deutsch, Jozsa, Bernstein, Vazirani, Simon, late 1980s/early 1990s: Examples of problems where quantum computers outperform classical ones
- Shor 1994: Efficient quantum algorithms for factoring and discrete log

Quantum bits

- One qubit: $\mathcal{H} = \mathbb{C}^2$

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

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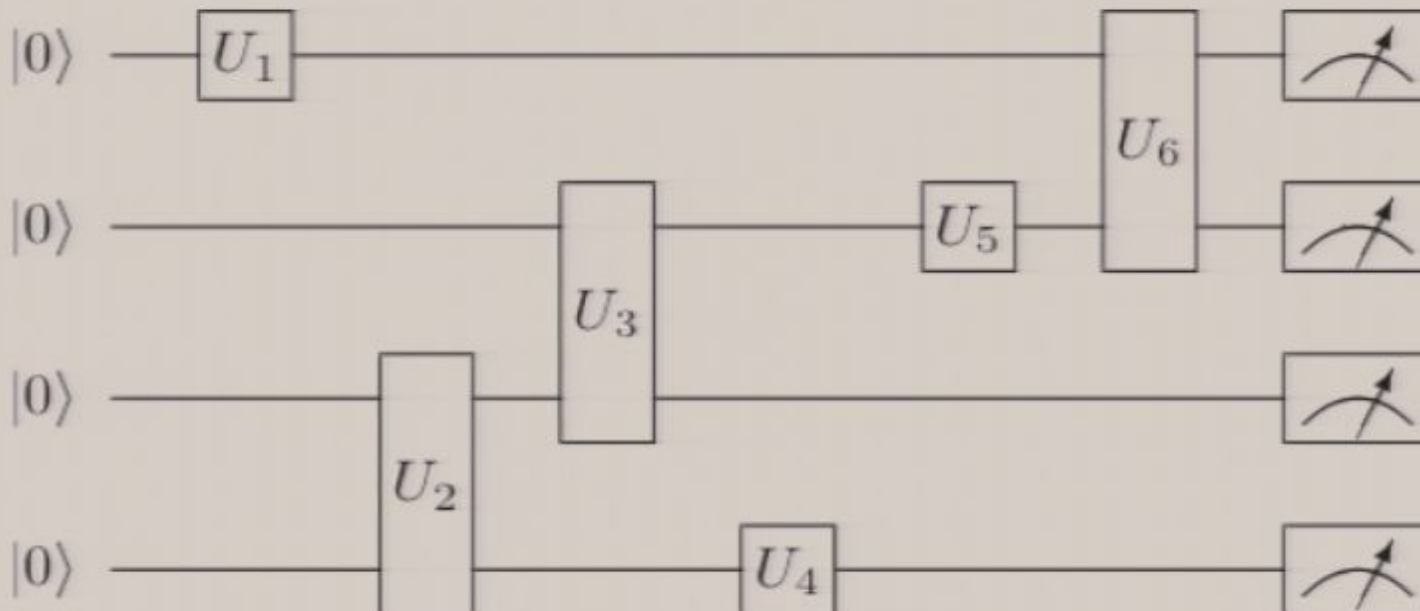
- n qubits: $\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_n$

$$|\psi\rangle = \sum_{z \in \{0,1\}^n} \alpha_z |z_1\rangle \otimes |z_2\rangle \otimes \dots \otimes |z_n\rangle$$

$$\sum_{z \in \{0,1\}^n} |\alpha_z|^2 = 1$$

Quantum circuits

- Prepare n qubits in the state $|0 \dots 0\rangle$
- Apply a sequence of $\text{poly}(n)$ unitary operations acting on one or two qubits at a time
- Measure in the computational basis to get the result



Three major questions

- How can we build a quantum computer?
(Implementations)
- How useful is an imperfect quantum computer?
(Fault tolerance)
- What can we do with a perfect quantum computer?
(Algorithms)

Hamiltonian dynamics

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

In the circuit model, we say a unitary operation can be implemented efficiently if it can be realized (approximately) by a short sequence of one- and two-qubit gates.

What Hamiltonian dynamics can be implemented efficiently?

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- Hamiltonians we can directly realize in the laboratory

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- Hamiltonians we can efficiently simulate using quantum circuits

Simulating Hamiltonian dynamics

Definition. A Hamiltonian H acting on n qubits can be *efficiently simulated* if for any error $\epsilon > 0$ and time $t > 0$ there is a quantum circuit U consisting of $\text{poly}(n, t, 1/\epsilon)$ gates such that $\|U - e^{-iHt}\| < \epsilon$.

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Basic idea: Lie product formula

$$e^{-i(H_1 + \dots + H_k)t} = (e^{-iH_1 t/r} \dots e^{-iH_k t/r})^r + O(kt^2 \max\{\|H_j\|^2\}/r)$$

Sparse Hamiltonians

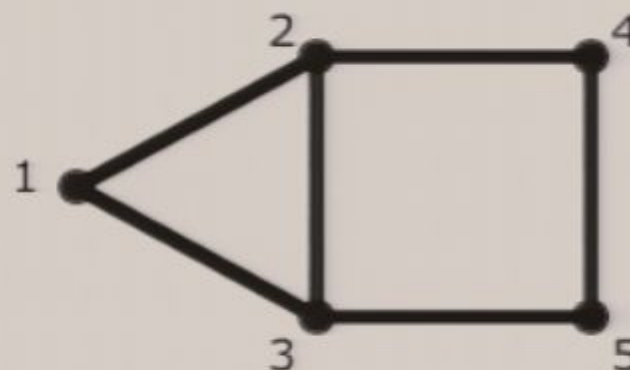
Theorem. Suppose that for any fixed a , we can efficiently compute all the nonzero values of $\langle a | H | b \rangle$. (In particular, there must be only polynomially many such values.) Then H can be simulated efficiently. [Aharonov & Ta-Shma 2003, Childs et al. 2003, Ahokas et al. 2005]

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Basic idea: Color the interaction graph with a small number of colors and simulate each color separately

$$H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

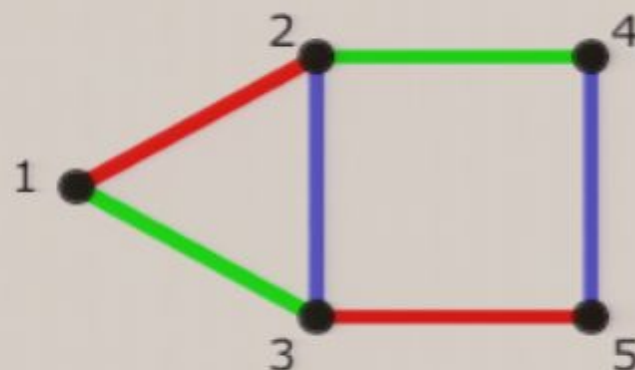


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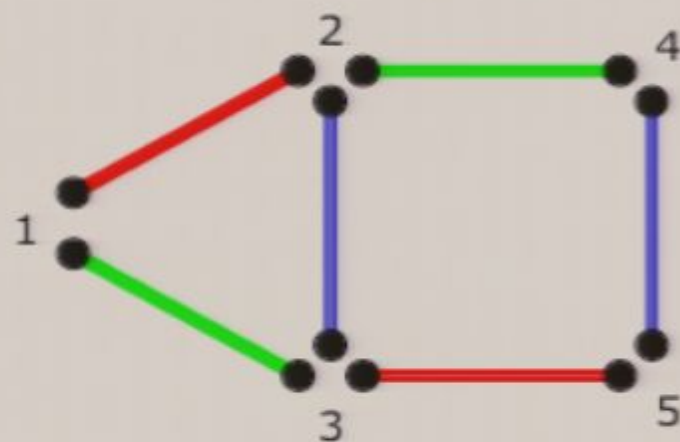


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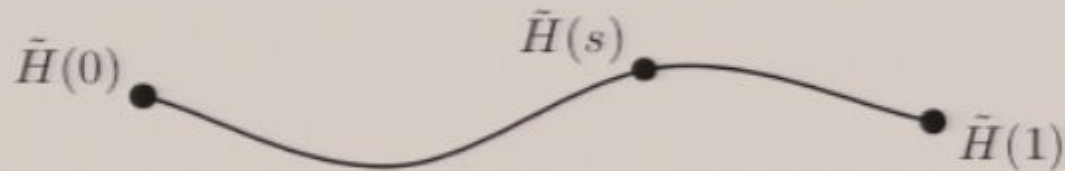
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The adiabatic theorem

Let $\tilde{H}(s)$ be a smoothly varying Hamiltonian for $s \in [0, 1]$

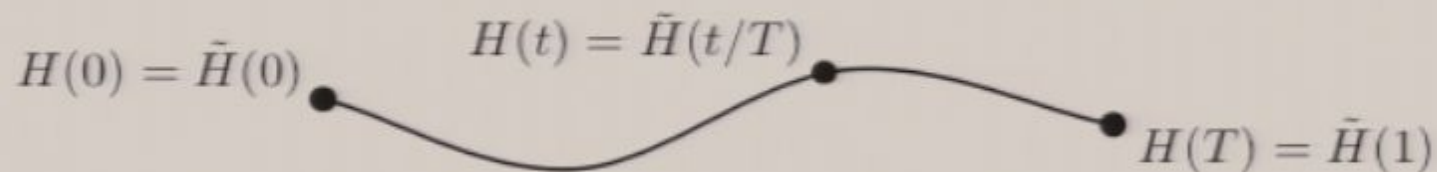


$$\tilde{H}(s) = \sum_{j=0}^{D-1} E_j(s) |E_j(s)\rangle \langle E_j(s)|$$

where $E_0(s) < E_1(s) \leq E_2(s) \leq \dots \leq E_{D-1}(s)$

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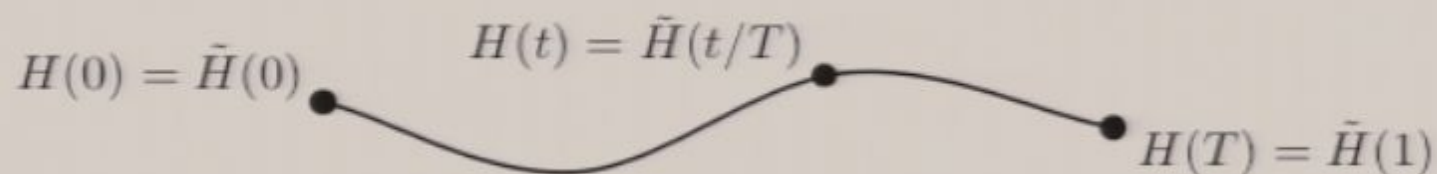
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For large T , $|\psi(T)\rangle \approx |E_0(1)\rangle$. But how large must it be?

Approximately adiabatic evolution

The total run time required for adiabaticity depends on the spectrum of the Hamiltonian.

Gap: $\Delta(s) = E_1(s) - E_0(s)$, $\Delta = \min_{s \in [0,1]} \Delta(s)$

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$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \|\dot{\tilde{H}}(s)\|^2$$

Theorem. [Teufel 2003 + perturbation theory]

$$T \geq \frac{4}{\epsilon} \left[\frac{\|\dot{\tilde{H}}(0)\|}{\Delta(0)^2} + \frac{\|\dot{\tilde{H}}(1)\|}{\Delta(1)^2} + \int_0^1 ds \left(10 \frac{\|\dot{\tilde{H}}\|^2}{\Delta^3} + \frac{\|\ddot{\tilde{H}}\|}{\Delta} \right) \right]$$

implies $\|\psi(T)\rangle - |E_0(1)\rangle\| \leq \epsilon$

Satisfiability problems

- Given $h: \{0,1\}^n \rightarrow \{0,1,2,\dots\}$, is there a value of $z \in \{0,1\}^n$ such that $h(z)=0$?
- Alternatively, what z minimizes $h(z)$?
- **Example: 3SAT.** $(z_1 \vee z_2 \vee \bar{z}_3) \wedge \dots \wedge (\bar{z}_{17} \vee z_{37} \vee \bar{z}_{42})$

$$h(z) = \sum_c h_c(z)$$

$$\text{where } h_c(z) = \begin{cases} 0 & \text{clause } c \text{ satisfied by } z \\ 1 & \text{otherwise} \end{cases}$$

Adiabatic optimization

- Define a *problem Hamiltonian* whose ground state encodes the solution:

$$H_P = \sum_{z \in \{0,1\}^n} h(z) |z\rangle\langle z|$$

- Define a beginning Hamiltonian whose ground state is easy to create, for example

$$H_B = - \sum_{j=1}^n \sigma_x^{(j)}$$

- Choose $\tilde{H}(s)$ to interpolate from H_B to H_P , for example

$$\tilde{H}(s) = (1 - s)H_B + s H_P$$

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$$\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$$

$$S_x = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos t/2 \\ \sin t/2 \end{pmatrix}$$

\mathbb{R}^2

$$\begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

-1

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Please mind the gap

Recall rough estimate:

$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \|\dot{\tilde{H}}(s)\|^2$$

For $\tilde{H}(s) = (1-s)H_B + sH_P$,

$$\begin{aligned} \|\dot{\tilde{H}}\| &= \|H_P - H_B\| \\ &\leq \|H_B\| + \|H_P\| \end{aligned}$$

Crucial question: How big is Δ ?

- $\geq 1/\text{poly}(n)$: Efficient quantum algorithm
- $1/\exp(n)$: Inefficient quantum algorithm

Unstructured search

Finding a needle in a haystack: $h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases}$
(here $h: \{0, 1, \dots, N-1\} \rightarrow \{0, 1\}$)

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Query complexity (given black box for h)

- Classically, $\Theta(N)$ queries
- Quantumly, $O(\sqrt{N})$ queries are sufficient to find w
[Grover 1996] $(|z\rangle|a\rangle \mapsto |z\rangle|a \oplus h(z)\rangle)$
- This cannot be improved: $\Omega(\sqrt{N})$ queries are necessary [Bennett et al. 1997]

Example: Adiabatic unstructured search

$$h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases} \Rightarrow H_P = \sum_z h(z) |z\rangle \langle z| = 1 - |w\rangle \langle w|$$

Start in $|s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle$

$$H_B = 1 - |s\rangle \langle s|$$

$$\begin{aligned} \tilde{H}(s) &= (1-s)H_B \\ &\quad + s H_P \end{aligned}$$

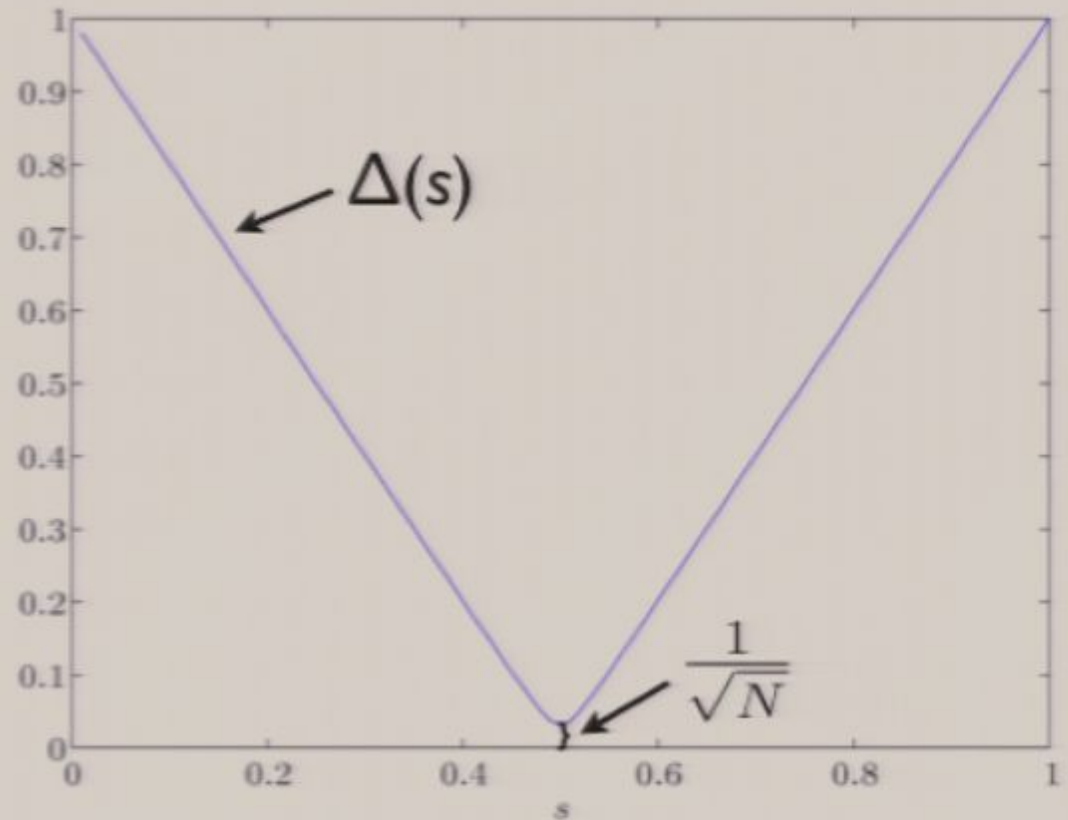
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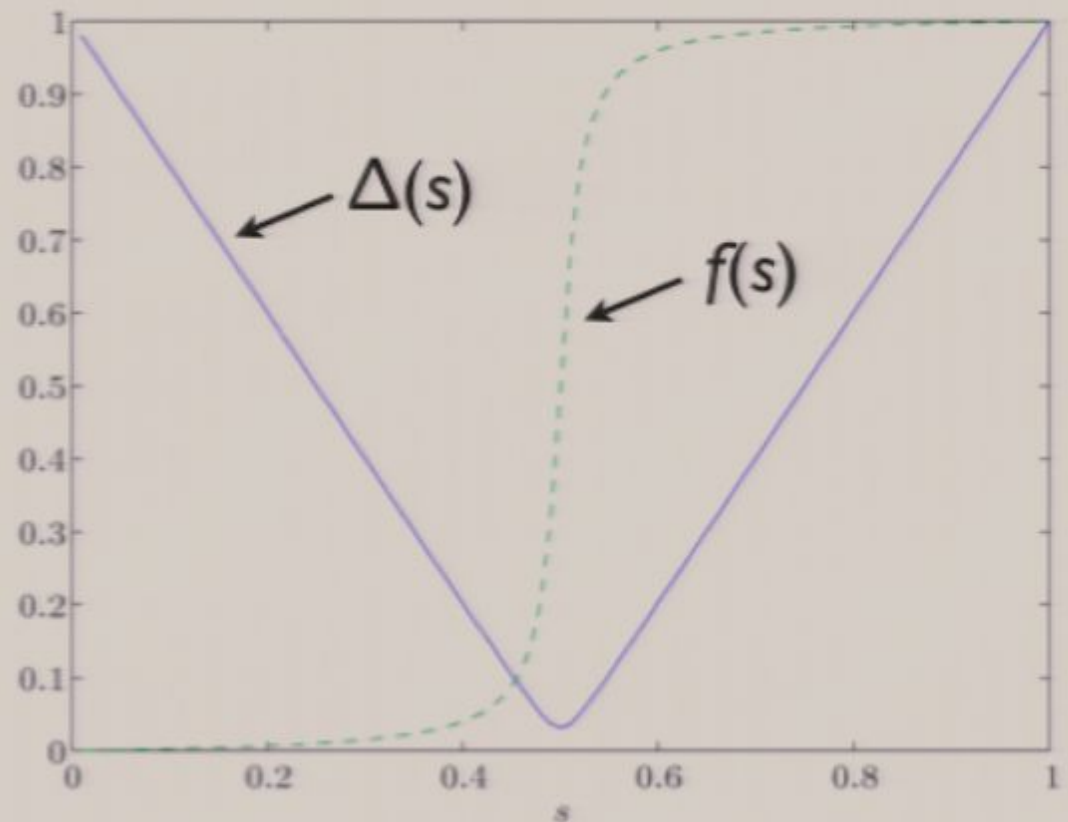
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~~$$\tilde{H}(s) = (1-s)H_B + sH_P$$~~

$$\tilde{H}(s) = [1 - f(s)]H_B + f(s)H_P$$



Example: Transverse Ising model

$$H_P = \sum_{j \in \mathbb{Z}_N} \left(1 - \frac{1}{2} \sigma_z^{(j)} \sigma_z^{(j+1)} \right) \quad \text{“agree”}$$

$$H_B = - \sum_{j=1}^n \sigma_x^{(j)} \quad \text{with ground state} \quad |s\rangle = |+\cdots+\rangle$$
$$= \sum_{z \in \{0,1\}^n} |z\rangle$$
$$\tilde{H}(s) = (1-s)H_B + sH_P$$

Diagonalize by fermionization (Jordan-Wigner transformation)

Result: $\Delta \propto \frac{1}{n}$ (at critical point of quantum phase transition)

$$|E_0(s \approx 0)\rangle \approx |+\cdots+\rangle$$

$$|E_0(s \approx 1)\rangle \approx \frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle)$$

[Farhi et al. 2000]

Example: The Fisher problem

$$H_P = \sum_{j \in \mathbb{Z}_N} J_j \left(1 - \frac{1}{2} \sigma_z^{(j)} \sigma_z^{(j+1)} \right) \quad J_j = 1 \text{ or } 2, \text{ chosen randomly}$$

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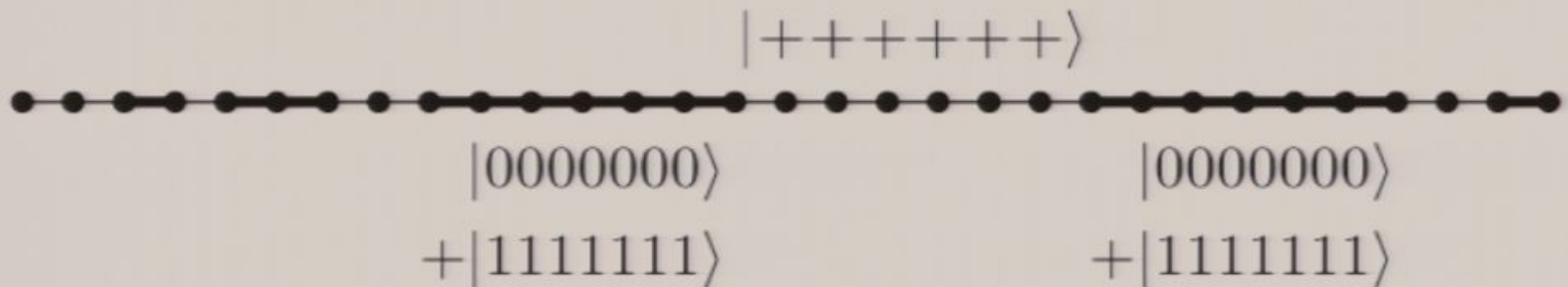
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Random satisfiability problems

Consider random instances of some satisfiability problem (e.g. 3SAT, Exact cover, ...) with a fixed ratio of clauses/bits.

Few clauses: underconstrained. Many solutions, easy to find.

Many clauses: overconstrained. No solutions, easy to find a contradiction.

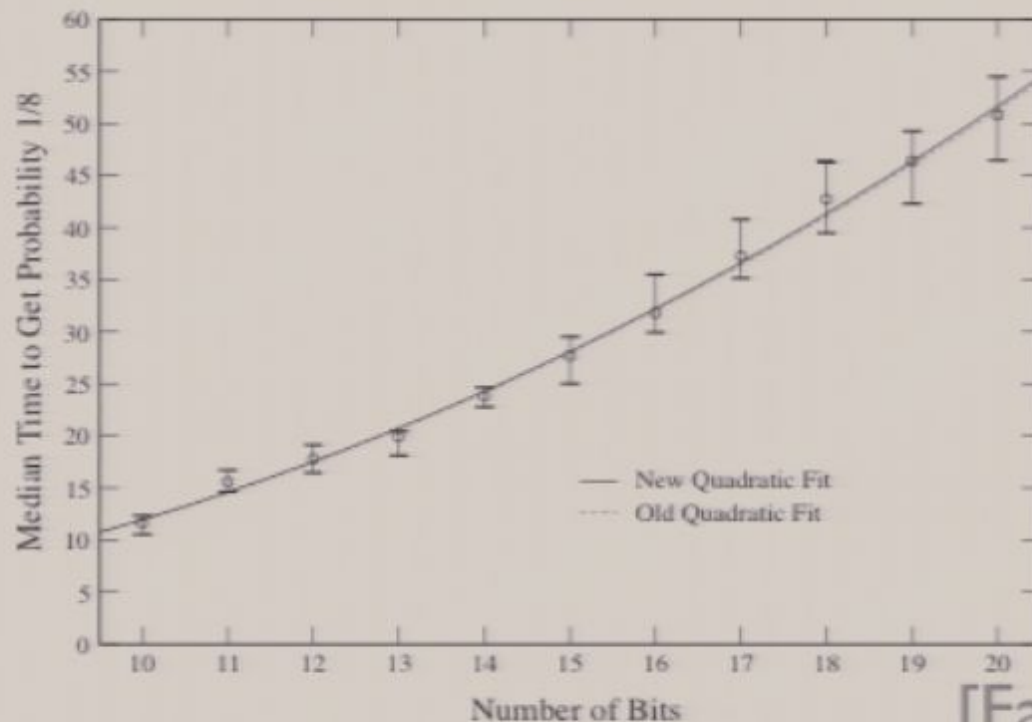
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Simulation results for random exact cover instances with unique satisfying assignments:



Universal quantum computation

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[Feynman 1985]:
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Basic idea [Aharonov et al. 2004]: Use this as $-H_P$.

Final ground state:
$$\frac{1}{\sqrt{k}} \sum_{j=1}^k U_j U_{j-1} \cdots U_1 |0\rangle \otimes |j\rangle$$

H_B enforces correct initial state.

Add energy penalties to stay in an appropriate subspace.

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Note: This is adiabatic, but not adiabatic *optimization*.



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