

Title: Introduction to quantum gravity - Part 8

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Abstract: This is an introduction to background independent quantum theories of gravity, with a focus on loop quantum gravity and related approaches.

Basic texts:

-Quantum Gravity, by Carlo Rovelli, Cambridge University Press 2005 -Quantum gravity with a positive cosmological constant, Lee Smolin, hep-th/0209079

-Invitation to loop quantum gravity, Lee Smolin, hep-th/0408048 -Gauge fields, knots and gravity, JC Baez, JP Muniain

Prerequisites:

-undergraduate quantum mechanics

-basics of classical gauge field theories

-basic general relativity

-hamiltonian and lagrangian mechanics

-basics of lie algebras

$$\Rightarrow \langle F(A, \Pi), G(\varrho) \rangle = \delta_{\varrho} F(A, \Pi) \text{ with } G(\varrho) \text{ given by}$$

$$\varrho \rightarrow g'(\varrho)$$



$$F^I(x) = Z^I S^2(x, y)$$

$$H[x] = 1 + \int_{\mathbb{R}} F \approx \varrho^2$$



$$\Rightarrow \{F(A, \Pi), G(\rho)\} = \delta_{\rho} F(A, \Pi) \text{ with } \rho = \rho(x, y)$$

$$\rho \rightarrow \delta \rho$$



$$F^1(x) = \int_{\mathcal{R}} \delta^2(x, y)$$

$$H[x] = 1 + \int_{\mathcal{R}} F^1 \approx 1 + \int_{\mathcal{R}} \delta^2 e^2$$

$$\Gamma(\omega) = \int \gamma F^T(x) \omega_I(x)$$

$$\langle \mathbb{F}(\omega), \mathbb{G}(\mu) \rangle = \mathbb{F}(\omega \times \mu)$$

3 manifold $M_3 = T^2 \times \mathbb{R}$

$$A_1 = A_1^T \sigma^I \text{ SU(2) } \gamma \text{ surface}$$

$$F_{loc} \approx dA_{loc} + \epsilon^2 \kappa (A^T \wedge A)_{loc}$$

$$Q_1 \approx Q_1^T \sigma^I \text{ SU(2) with 1-forms (with)}$$

$$S = \int \left[Q^T \wedge F^I(A) - \Lambda \epsilon_{IJK} Q^I \wedge Q^J \wedge Q^K \right]$$



$$\mathcal{I}(U) = 1 - \mathcal{I}(U^2)$$

homomorphism

base point independent

$$F(w) = \int dx F^2(x) w_2(x)$$

$$\{F(w), G(\mu)\} = F(w \times \mu)$$

3 manifold $M_3 = T^2 \times R$

$$F_{loc} \geq dA_{loc} + \epsilon^2 \kappa (A^T \wedge A^T)_{loc}$$

$$A_1 = A_1^T \sigma^I \text{ SU(2) gauge field}$$

$$A_2 = A_2^T \sigma^I \text{ SU(2) valued 1-form field}$$

$$S = \int \left[\sigma^I \wedge F^I(A) - \Lambda \epsilon_{IJK} \sigma^I \wedge \sigma^J \wedge \sigma^K \right]$$



$$T(w) = T \cdot \int [w^2]$$

homomorphism

big point and product

$$\Gamma(\omega) = \int \gamma F^+(x) \omega_3(x)$$

$$\langle F(\omega), G(\mu) \rangle = \mathbb{F}(\omega \times \mu)$$

3 manifold $M_3 = T^2 \times \mathbb{R}$

$$F_{\text{loc}} \cong dA_{\text{loc}} + \epsilon^2 \kappa (A^i \wedge A^j \wedge A^k)_{\text{loc}}$$

$$A_i = A_i^I \sigma^I \quad \text{SU(2) gauge field}$$

$$e_a = e_a^I \sigma^I \quad \text{SU(2) valued 1-form (vielbein)}$$

$$S = \int \left[\epsilon^{\alpha\beta\gamma} F^I_{\alpha\beta} A^I_{\gamma} - \Lambda \epsilon_{\alpha\beta\gamma} e^{\alpha I} \omega^{\beta J} \omega^{\gamma K} \right]$$



$$\gamma(\mu) = 1 \cdot \|\mu^k\|$$

normalization

base point independent

$$\Gamma(u) = \int \gamma F^+(x) u_{\Gamma}(x)$$

$$\langle \mathbb{F}(u), G(\mu) \rangle = \mathbb{F}(u \times \mu)$$

3 manifold $M_3 = T^2 \times \mathbb{R}$

$A_4 = A_i^I \sigma^I$ SU(2) gauge field

$$F_{ab} = dA_{ab} + \epsilon_{ijk} (A_i^j A_k^i)$$

$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_F$ SU(2) valued 1-form (curv)

$$S = \int \left[\frac{1}{4} \text{tr} F^2(A) - \Lambda \epsilon_{ijk} \text{tr} A^i A^j A^k \right] + \int_{\text{particle}}$$



$$\gamma(\mu) = 1 \cdot \|\mu\|$$

homomorphism

base point independent

$$\Gamma(\omega) = \int \sqrt{-1} F^+(x) \omega_{\Sigma}(x)$$

$$\langle F(\omega), G(\mu) \rangle = \int F(\omega \wedge \mu)$$

3 manifold $M_3 = T^2 \times \mathbb{R}$

$$A_1 = A_1^I \sigma^I \quad \text{SU(2) 1-form}$$

$$F_{loc}^I = dA_{loc}^I + \epsilon^{IJK} (A_{loc}^J \wedge A_{loc}^K)$$

$$e_a = e_a^I \sigma^I \quad \text{SU(2) valued 1-form}$$

$$S = \int \left[\frac{1}{4\pi} \text{Tr} F^I F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$



if 1 only had a net

$$S^{\text{PA}} = \int dt \sqrt{g_{ab}} \dot{x}^a \dot{x}^b$$

$$\Gamma(\mu) = 1 - \|\mu\|^2$$

base point independent

$$\Gamma(\omega) = \int \sqrt{-1} F^{\mu\nu}(x) \omega_{\mu\nu}(x)$$

$$\langle F(\omega), G(\mu) \rangle = \int F(\omega \wedge \mu)$$

3 manifold $M_3 = T^2 \times R$

$A_4 = A_i^I G^I$ SU(2) gauge field

$$F_{\mu\nu} = dA_{\mu\nu} + \epsilon^{ijk} (A_i^I \wedge A_j^k) / r$$

$\mathcal{L}_4 = \frac{1}{2} G^I G^I$ SU(2) valued 1-form (curv)

$$S = \int \left[\frac{1}{4} \text{Tr} F^I F^I(A) - \Lambda \epsilon_{ijk} \text{Tr} A^i A^j A^k \right] + \int_{\text{particle}}$$



if 1 only had a metric

$$S^{\text{PA}} = \int dt \sqrt{g_{ab} \dot{x}^a \dot{x}^b}$$

$$\Gamma(\mu) = \int \mu$$

Integration

high point mid-point

$$\Gamma(\omega) = \int \gamma F^{\pm}(x) \omega_{\pm}(x)$$

$$\{F(\omega), G(\mu)\} = \mathbb{F}(\omega, \mu)$$

3 manifold $M_3 = T^2 \times R$

$$A_4 = A_4^I \sigma^I \quad SU(2) \text{ gauge field}$$

$$F_{\mu\nu}^I = dA_{\mu\nu}^I + \epsilon^{IJK} (A^J \wedge A^K)_{\mu\nu}$$

$$e_a = e_a^I \sigma^I \quad SU(2) \text{ valued 1-form}$$

$$S = \int \left[\frac{1}{2} \text{tr} (F \wedge F) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K \right] + S_{\text{particle}}$$



$$S = \int dt \tilde{H}(t)$$

$$\mathcal{H}(t) = \mathcal{H}(t, \{n^i\})$$

Hamiltonian

base point independent

$$\Gamma(\omega) = \int \gamma F^{-1}(x) \omega_I(x)$$

$$\langle \mathbb{F}(\omega), G(\mu) \rangle = \mathbb{F}(\omega \times \mu)$$

3 manifold $M = T^2 \times \mathbb{R}$

$$A_i = A_i^I \sigma^I \quad \text{SU(2) gauge field}$$

$$e_a = e_a^I \sigma^I \quad \text{SU(2) valued 1-form (vielbein)}$$

$$F_{ab} = dA_{ab} + \epsilon^{ijk} (A_i^j A_k^l)_{ab}$$

$$S = \int \left[\frac{1}{2} \text{tr} F^2(A) - \Lambda \epsilon_{ijk} e^i \wedge e^j \wedge e^k \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) A_a$$

$$\Gamma(\mu) = 1 - \|\mu\|^2$$

normalization

base point independent

$$\Gamma(u) = \int \gamma F^{-1}(x) \omega_S(x)$$

$$\langle F(u), G(u) \rangle = F(u) \cdot G(u)$$

3 manifold $M = T^2 \times \mathbb{R}$

$$A_1 = A_1^T \sigma^1 \text{ SU(2) generators}$$

$$F_{ac} = dA_{bc} + \epsilon_{abc} (A^b A^c)$$

$$E_a = \rho_a^i \sigma^i \text{ SU(2) valued 1-forms}$$

$$S = \int \left[\text{Tr} \left(\frac{1}{2} E^a E^b F_{ab} \right) - \Lambda \text{Tr} \left(\frac{1}{2} A^a A^a \right) \right] + \int \text{particle}$$



$$S = \int dt \Gamma^i(t) A_a^E$$



$$\langle \eta | \eta \rangle = 1 - \langle \eta | \eta \rangle$$

homomorphism

base point independent

$$F(\omega) = \int dx F^+(x) \omega_S(x)$$

$$\{F(\omega), G(\mu)\} = F(\omega \times \mu)$$

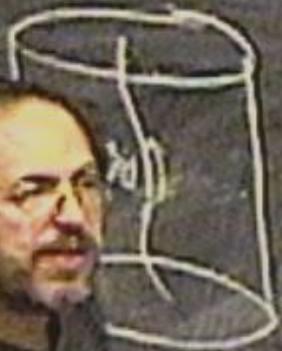
3 manifold $M_3 = T^2 \times R$

$$A_4 = A_4^I \sigma^I \quad SU(2) \text{ gauge field}$$

$$F_{ac} = dA_{bc} + \epsilon_{ijk} (A^i \wedge A^j \wedge A^k)$$

$$L_4 = L_4^I \sigma^I \quad SU(2) \text{ valued 1-form}$$

$$S = \int \left[\frac{1}{2} \text{tr} (F^I F^I) - \Lambda \epsilon_{ijk} \text{tr} (A^i \wedge A^j \wedge A^k) \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) A_a^I q_I$$

$$\Gamma(t) = 1 \cdot \{ \Gamma^i(t) \}$$

homomorphism

base point ind. (pseudotensor)

$$\Gamma(\omega) = \int dx F^+(x) \omega_1(x)$$

$$\{F(\omega), G(\mu)\} = F(\omega \times \mu)$$

3 manifold $M_3 = T^2 \times R$

$$A_4 = A_4^I \sigma^I \quad SU(2) \text{ gauge field}$$

$$F_{\mu\nu} \rightarrow dA_{\mu\nu} + \epsilon_{ijk} (A_i^j A_k^i)_{\mu\nu}$$

$$L_4 \rightarrow \text{Tr} F_{\mu\nu}^2 \quad SU(2) \text{ valued 1-forms (curv)}$$

$$S = \int \left[\text{Tr} A F^I(A) - \Lambda \epsilon_{ijk} \text{Tr} A^i A^j A^k \right] + \int_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) A_a^i q_a(t)$$

$$\Gamma(\mu) = \int \text{Tr} \mu^2$$

Hamiltonian

base point and endpoints

$$\Gamma(\omega) = \int \gamma F^{\pm}(x) \omega_{\pm}(x)$$

$$\{F(\omega), G(\mu)\} = F(\omega, \mu)'$$

3 manifold $M_3 = T^2 \times \mathbb{R}$

$$A_4 = A_a^I \sigma^I \quad \text{SU(2) gauge field}$$

$$F_{ab}^I = dA_{ab}^I + \epsilon^{IJK} (A_a^J A_b^K)_{ab}$$

$$Q_a = Q_a^I \sigma^I \quad \text{SU(2) valued 1-form (1.12)}$$

$$S = \int \left[Q^I \wedge F^I(A) - \Lambda \epsilon_{IJK} Q^I \wedge Q^J \wedge Q^K \right] + S_{\text{particle}}$$



$$S = \int dt \tilde{\Gamma}^I(t) \left[A_a^I q_a^I + Q_a^I P_I \right]$$

$$\Gamma(\mu) = 1 - \|\mu\|^2$$

homomorphism

base point independent

$$\Gamma(\omega) = \int \gamma F^+(x) \omega_S(x)$$

$$\{F(\omega), G(\mu)\} = F(\omega \times \mu)$$

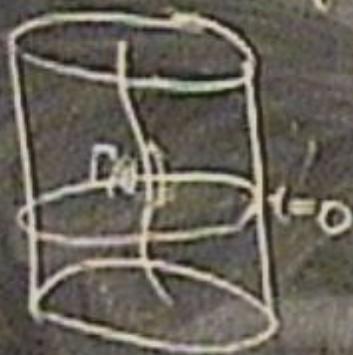
3 manifold $M_3 = T^2 \times \mathbb{R}$

$$A_a = A_a^I \sigma^I \quad \text{SU(2) gauge field}$$

$$F_{ab}^I = dA_{ab}^I + \epsilon^{IJK} (A_a^J \wedge A_b^K)$$

$$Q_a = Q_a^I \sigma^I \quad \text{SU(2) valued 1-form}$$

$$S = \int \left[Q^I \wedge F^I(A) - \Lambda \epsilon_{IJK} Q^I \wedge Q^J \wedge Q^K \right] + S^{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I q_a^I + Q_a^I P_I \right]$$

$$\Gamma(t) = 1 - \|\dot{\Gamma}^I(t)\|^2$$

Hamiltonian

max point and (product)

$\Rightarrow \gamma \in (A, B), \alpha(x) = \text{sgn}(\text{Im}(\gamma))$

$0 \rightarrow g' \circ g$



$$f_{x,y}^I =$$

$$F^I(x) = \mathcal{L}^I S^2(x, y)$$

$$H[x] = 1 + \int_R F^I \approx 1 + \mathcal{L}^I \approx \mathcal{O}^I$$

$$\frac{\delta S}{\delta A^I} = G_{IJ} = D_a \Pi_a^{IJ} = 0 \quad F_{IJ}^I = 0$$

2×4

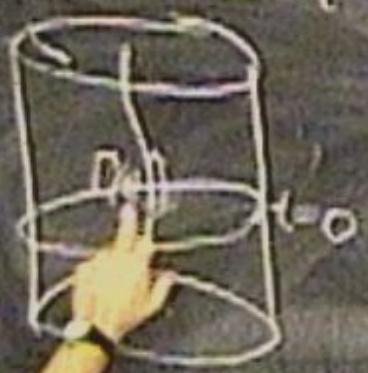
3 manifold $M_3 = T^2 \times R$

$A_a = A_a^I \sigma^I$ (vector) & spatial

$F_{ab} \Rightarrow dA_a^I + \epsilon^I_{JK} (A_a^J A_b^K)$

$Q_a = Q_a^I \sigma^I$ (vector) & 1-form (time)

$$S = \int \left[\sigma^I \wedge F^I(A) - \Lambda_{IJK} \sigma^I \wedge \sigma^J \wedge \sigma^K \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I q_a^I(t) + Q_a^I(p_I) \right]$$

How... classes form a group $\{A\} \{B\} = \{A \circ B\}$

$$G_{uv}^I = D_u \Pi_{v\alpha}^I = 0 \quad F_{12}^I = 0$$

2xM

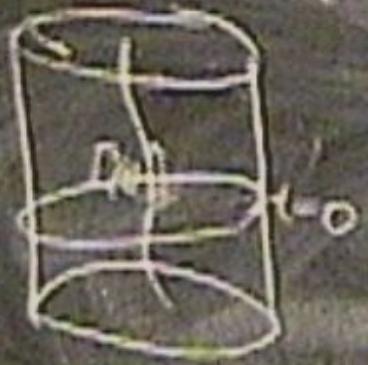
3 manifold $M_3 = T^2 \times R$

$$A_3 = \Lambda_a^I \sigma^I \quad (\text{SU(2) gauge field})$$

$$F_{ab} = dA_{ab} + \epsilon^c{}_{ab} (A^c \wedge A^d)$$

$$Q_a = \rho_a^I \sigma^I \quad (\text{SU(2) valued 1-form})$$

$$S = \int \left[\rho^I \wedge F^I(A) - \Lambda \epsilon_{321} \rho^I \wedge A^J \wedge A^K \right] + \int_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I q_a^I(t) + Q_a^I(p_I) \right]$$

$\Rightarrow \gamma \in (A, B), \alpha \in \mathbb{R} \Rightarrow \dots$

$\alpha \rightarrow g' \circ \gamma$



$$F_{x,y}^{-1}(y) =$$

$$F^{-1}(x) = \dots^2(x,y)$$

$$H[x] = 1 + \dots \rightarrow 2^x \approx 0^x$$

ex 11

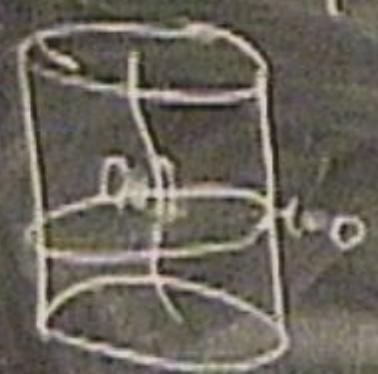
3 manifold $M = T^2 \times \mathbb{R}$

$$A_i = \Lambda_i^j \sigma^j \quad \text{SU(2) gauge field}$$

$$F_{ij} = dA_{ij} + \sigma_{ijk} (A^k A^l - A^l A^k)$$

$$Q_i = \sigma_{ij}^k \sigma^j \quad \text{SU(2) valued 1-form}$$

$$S = \int \left[\sigma^i A F^i(A) - \Lambda_{ijk} \sigma^i A^j A^k \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) \left[A_{ij}^E q_{ij}^I + Q_{ij}^I(P_I) \right]$$

$$= \int_{\mathcal{M}} \mathcal{L} dy$$



2xM

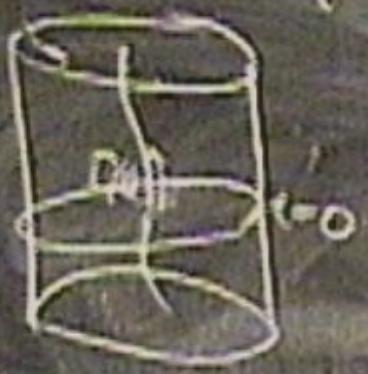
3 manifold $M = T^2 \times R$

$$F_m = dA + \epsilon_{ijk} (A^i A^j A^k)$$

$$A = A_a^I \sigma^I dx^a$$

$$Q = Q_a^I \sigma^I dx^a$$

$$S = \int [\alpha \text{Tr} F^2(A) - \Lambda \epsilon_{ijk} \text{Tr} A^i A^j A^k] + \int \text{matter}$$



$$S = \int dt \dot{\Gamma}^i(t) [A_a^E q_a^I(t) + Q_a^I(p_I)]$$

$$= \int dt \dot{\gamma}^i(t) Q_a^I(\gamma) [S(\gamma, p(t))]$$

$$\frac{\partial}{\partial t} \frac{1}{\sqrt{1-v^2/c^2}} = \dots$$

$$\Gamma_{ij} = \dots$$

ex 11

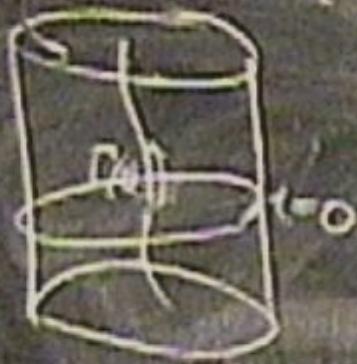
3 manifold $M = T^2 \times R$

$$A_g = A_a^I \sigma^I \quad (SU(2) \text{ gauge field})$$

$$F_a = dA_a^I + \epsilon^{IJK} (A_a^J \wedge A_a^K)$$

$$Q_a = Q_a^I \sigma^I \quad (\text{substituted 1-forms})$$

$$S = \int \left[Q^I \wedge F^I(A) - \Lambda \epsilon_{IJK} Q^I \wedge Q^J \wedge Q^K \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I q_a^I(t) + Q_a^I(p_a^I) \right]$$

$$= \int dt \int dy Q_a^I(y) \left[\delta(y, \Gamma(t)) p_a^I \right]$$

Topology



form a group $\{\alpha\} \{\beta\} = \{\alpha \circ \beta\}$

Q x M

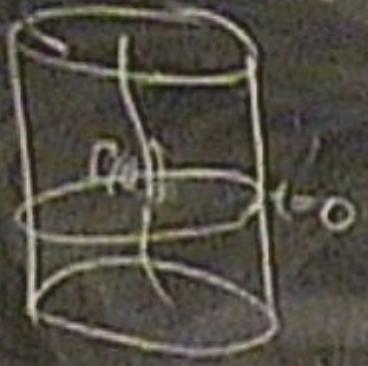
3 manifold $M = T^2 \times \mathbb{R}$

$$F_{\mu\nu} = dA_{\mu\nu} + \epsilon^{\alpha\beta\gamma} (A^{\alpha} \wedge A^{\beta} \wedge A^{\gamma})_{\mu\nu}$$

$A_{\alpha} = A_{\alpha}^I \sigma^I$ SU(2) gauge field

$Q_{\alpha} = Q_{\alpha}^I \sigma^I$ SU(2) valued 1-form

$$S = \int \left[\frac{1}{4} \text{tr} F^2(A) - \Lambda \epsilon_{\alpha\beta\gamma} Q^{\alpha} \wedge Q^{\beta} \wedge Q^{\gamma} \right] + S^{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) \left[A_a^I q_a^I(t) + Q_a^I(t) P_I^I(t) \right]$$

$$= \int dt \int d^2y Q_a^I(t, y) \left[\delta(y, \Gamma(t)) P_I^I(t) + A_a^I(t, y) \delta(y, \Gamma(t)) q_a^I(t) \right]$$

Homotopy classes (from a group $\{g\}$) $\{R\} = \{\alpha \circ \beta\}$
 $\Rightarrow \pi_1^1(\mathbb{R})$

$$H: \pi_1^1(\mathbb{R}) \rightarrow \dots$$

$$\Gamma = \Gamma \circ H$$

$$\Rightarrow \gamma^T(A, B), \gamma^T(C) = \text{sgn}(\det(A)) \dots$$

$$e \rightarrow g' e g$$

$$F_{x,y}^I = \delta^2(y, \Gamma(x)) \dot{\Gamma}(x) P_x(x)$$



$$F^I(x) = \delta^2(x, y)$$

$$H[\delta] = 1 + \int_k F^I \delta^2 e^I$$



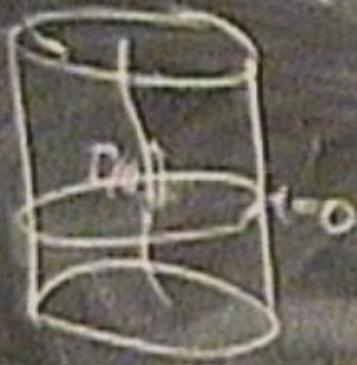
3-manifold $M = T^2 \times R$

$$F_{int} = dA_{int} + \tau_{int}(A_{int} \wedge A_{int})$$

$A_{int} = A_{int}^I \sigma^I$ SU(2) gauge field

$e_{int} = e_{int}^I \sigma^I$ SU(2)-valued 1-form

$$S = \int_{int} \left[e_{int}^I \wedge F_{int}^I(A) - \Lambda_{SU(2)} e_{int}^I \wedge \sigma^I \wedge e_{int}^I \right] + S_{particle}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_{int}^I J_{int}^I(t) + Q_{int}^I(P_{int}^I(t)) \right]$$

$$= \int dt \int_{int} e_{int}^I(t) \left[S(y, P(A)) P_{int}^I(t) + A_{int}^I(t) [S(y, P(A)) \dot{\Gamma}^I(t)] \right]$$



Topology class $\Rightarrow \pi_1(\Sigma)$

$$H = \pi_1(\Sigma)$$

$$\Gamma(\Sigma) = \langle H \rangle$$

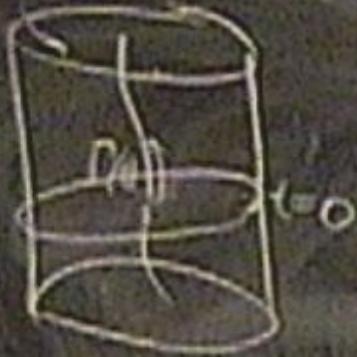
3 manifold $M = T^2 \times \mathbb{R}$

$$F \rightarrow dA + \rho(A \wedge A)$$

$$A = A_a^I \sigma^I \quad \text{SU(2) gauge field}$$

$$e = e_a^I \sigma^I \quad \text{SU(2) valued 1-form}$$

$$S = \int \left[\frac{1}{4\pi} \text{Tr} (F \wedge F) - \frac{1}{8\pi} \text{Tr} (e \wedge \star e) \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^i(t) \left[A_a^I J_I^a(t) + Q_a^I P_I^a(t) \right]$$

$$= \int dt \int d^2y e_a^I(t, y) \left[S^I(y, \Gamma(t)) P_I^a(t) + \Lambda_a^I(t) \tilde{S}^I(y, \Gamma(t)) \tilde{P}_I^a(t) \right]$$

Topology classes form a group $\{\alpha\} \cdot \{\beta\} = \{\alpha + \beta\}$
 $\Rightarrow \pi_1(\Sigma)$

$$H: \pi_1(\Sigma) \rightarrow G = \text{SU}(2)$$

$$T = T - H$$

base point

$$\Rightarrow \gamma \in (A, B], \alpha \in (C, D] = \text{eg } \gamma \text{ and } \alpha$$

$e \rightarrow g' e g$



$$F_{x,y}^1(y) = \delta^2(y, \Gamma(\cdot)) \Gamma(\cdot) P_x(y)$$

$$F^1(x) = \int \delta^2(x, y)$$

$$H[x] = 1 + \int_R F^1(x, y)$$

$\Rightarrow \chi^2(A, B), \chi^2(C) = \dots$

$e \rightarrow g'eg$



$$F_{x,y}^I = \chi^2(y, \Gamma(y)) \Gamma(y) P_x(y)$$

$N=0$

$$F^I(x) = \chi^2(x, y)$$

$$H(y) = 1 + \int_R F^I \approx 1 + \chi^2 \approx e^I$$

3xM

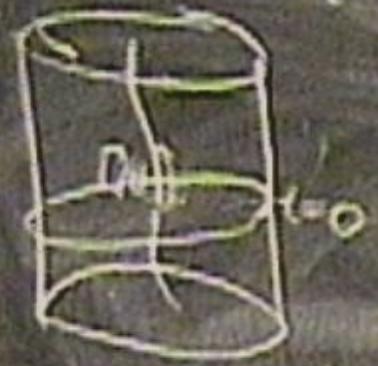
3 manifold $M = T^2 \times \mathbb{R}$

$$A_1 = A_1^I \sigma^I \quad \text{SU(2) gauge field}$$

$$F_{12} = dA_1 + \frac{1}{2} (A_1^I A_1^J) \sigma^I \sigma^J$$

$$Q_1 = Q_1^I \sigma^I \quad \text{su(2) valued 1-form}$$

$$S = \int \left[\frac{1}{4} \text{tr} F^I F^I(A) - \frac{1}{2} \text{tr} Q^I A \sigma^I A \right] + S_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I \frac{J_a^I}{I} + Q_{loc}^I(P_{I^a}^a) \right]$$

$$= \int dt \int dy Q_a^I(y) \left[S(y, P(y), \Gamma(t)) + A^I(y) \Gamma^I(t) \right]$$

Nontrivial classes form

$$\Rightarrow \Pi_1^1(\mathbb{Z})$$

$$\{ \alpha \} \{ \beta \} = \{ \alpha \beta \}$$

$$H = \Pi^1(\mathbb{Z})$$

$$\Gamma = \Gamma - H$$

$$\Rightarrow \chi^2(A, B), \sigma^2(K) = \sigma^2(A, B) \dots$$

$$e \rightarrow g'eg$$



$$F_{x,y}^I(y) = \delta^2(y, \Gamma(\theta)) \dot{\Gamma}_0^* P_{\theta}(y)$$

$$G^I(y) = \zeta^2(y, \Gamma(\theta)) \dot{\Gamma}_0^* J_{\theta}$$

$$F^I(x) = \int \delta^2(x, y)$$

$$H[\delta] = 1 + \int_k F^I \delta 1$$



$\Rightarrow \chi^2(A, B), \chi^2(C) = \dots$

$e \rightarrow g' e g$

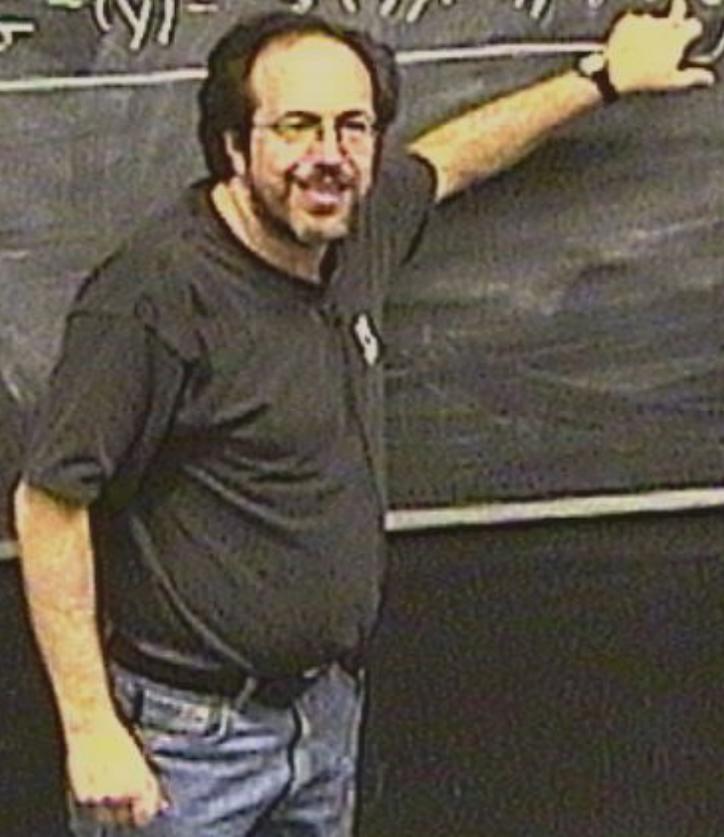


$$F_{x,y}^I(y) = \delta^2(y, \Gamma(\theta)) \dot{\Gamma}_0^y P_q(\theta)$$

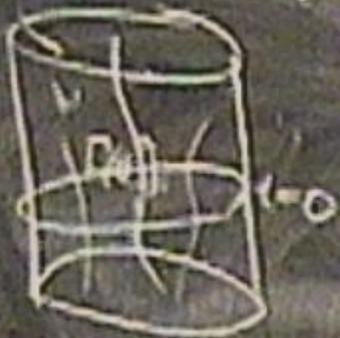
$$G^I(y) = \zeta^2(y, \Gamma(\theta)) \dot{\Gamma}_0^y J_{\kappa}$$

$$F^I(x) = \mathcal{L} \zeta^2(x, y)$$

$$H[\delta] = 1 + \int_R F^I \approx 1 + \mathcal{L} \approx \mathcal{O}^2$$



$$S = \int_{\mathcal{M}} \left[\mathcal{L}^I(A) - \Lambda_{IJ} \mathcal{L}^I(A) \mathcal{L}^J(A) \right] + \int_{\text{particle}}$$



$$S = \int dt \dot{\Gamma}^I(t) \left[A_a^I J_a^I(t) + Q_a^I(t) P_a^I(t) \right]$$

$$= \int dt \int dy \mathcal{L}_a^I(y) \left[\mathcal{L}(y, \Gamma(t)) P_a^I(t) + A_a^I(t) \mathcal{L}(y, \Gamma(t)) \mathcal{L}_a^I(t) \right]$$

Homotopy classes form a group $\{\alpha\} \{\beta\} = \{\alpha \circ \beta\}$
 $\Rightarrow \pi_1^2(\Sigma)$



$$H: \pi_1^2(\Sigma) \rightarrow G = SU(2)$$

$\pi(\alpha) = \Gamma \cdot H(\alpha^k)$ homomorphism

$\Gamma = \Gamma \cdot H$
 base point independent

$$\Rightarrow \{F(A, \Pi), G(\rho)\} = \delta_{\rho} F(A, \Pi) \text{ with } S(\rho) \text{ given by } \dots$$

$\rho \rightarrow g' \rho g$



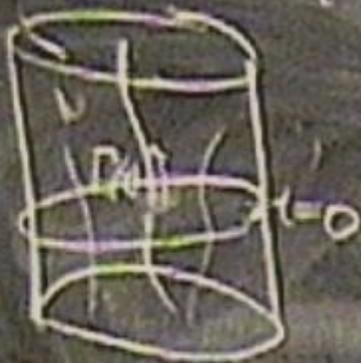
$$F_{xy}^I(y) = S^2(y, \rho(0)) \dot{\rho}'_0 P_a(y)$$

$$G^I(y) = S^2(y, \rho(0)) \dot{\rho}'_0 J_a$$

$$F^I(x) = 2^I S^2(x, y)$$

$$H[\delta] = 1 + \int_R F^I \approx 1 + 2^I \alpha \rho^2$$

$$S = \int_{\mathcal{M}} \left[\omega^I \wedge F^I(A) - \Lambda_{IJK} \omega^I \wedge \omega^J \wedge \omega^K \right] + \int_{\text{Particle}}$$



$$S = \int dt \int \tilde{\Gamma}^I(t) \left[A_a^I J_a^I(t) + Q_a^I(t) P_a^I(t) \right]$$

$$= \int dt \int d^2y \mathcal{L}_1^I(y) \left[\mathcal{L}^I(y, P^I(t)) P_a^I(t) + \Lambda^I(t, \mathcal{M}) \mathcal{L}^I(y, P^I(t)) \mathcal{L}_a^I(t) \right]$$



Homotopy classes form a group $\{\alpha\} \{\beta\} = \{\alpha \circ \beta\}$
 $\Rightarrow \pi_1(\mathcal{E})$

$$H: \pi_1^p(\mathcal{E}) \xrightarrow{\text{homeomorphism}} G = \text{SU}(2)$$

$$\gamma(\pi) = \Gamma \cdot H(\pi^k)$$

$\Gamma = \Gamma \circ H$
 base point independent

$$\Rightarrow \{F(A, \Pi), G(\rho)\} = \delta_g F(A, \Pi) \text{ with } S = (1) \text{ give } \dots$$

$$e \rightarrow g' e g$$



$$F_{,y}^I(y) = \delta^2(y, \Gamma(\rho)) \dot{\Gamma}_0^* P_0^*(y)$$

$$G^I(y) = S^2(y, \rho(\rho)) \dot{\Gamma}_0^* J_A$$

$$F^I(x) = \mathcal{L} S^2(x, y)$$

$$H[\gamma] = 1 + \int_R F^I \approx 1 + \mathcal{L} \approx e^{\mathcal{L}}$$



$$\Sigma = T^2$$



homology homology class in π^1

$$\xi = T^2$$



for every homomorphism in $\Pi'(S)$ $\{\lambda\}$
 $T\{\{\alpha\}\}$ is an observable.

$$\xi = \mathbb{T}^2$$



for every homotopy class in $\pi_1(S) \setminus \{1\}$
• $T\{\xi\}$ is an observable.

$$\xi = T^2$$



for every homotopy class in $\pi^1(\xi)$ $\{a\}$
• $T\{a\}$ is an observable.

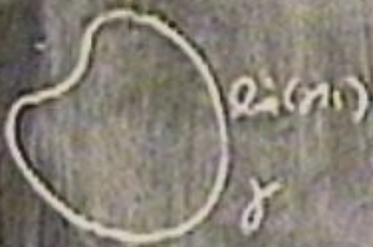
$$e_{\xi}[\alpha(s)]$$

$$\Sigma = T^2$$



for every homotopy class in $\pi_1(\Sigma) \cong \mathbb{Z}$
• $T\{\alpha\}$ is an observable.

$$e(\alpha)$$



α

$$\xi = T^2$$



for every holomorphic $\pi'(s) \in \mathcal{A}$
 $\rightarrow T\{\mathcal{A}\}$ is an observable.

$$e^{\int \gamma(s)} \gamma'(s) H[\gamma, s]$$

$$T\{\mathcal{A}\}$$

$$\xi = T^2$$



for every homotopy class in $\pi^1(S)$ $\{\alpha\}$
• $T\{\alpha\}$ is an observable.

$$\text{Tr } e^{\int \alpha(s)} \dot{\gamma}^k(s) H[\gamma, s]$$



$$\xi = T^2$$



for every homotopy class in $\pi_1(S) \cong \mathbb{Z}$
• $T\{\xi\}$ is an observable.

$$\text{Subtr } e^{i\alpha(s)} \dot{\gamma}^\mu(s) H[\alpha, s]$$



$$\xi = T^2$$

for every homotopy class in $\pi^1(S)$ $\{\alpha\}$
 $\rightarrow T\{\alpha\}$ is an observable.

$$T^2[x, \alpha] = \int ds \text{Tr} e^{i\alpha(s)} \dot{\gamma}^i(s) H[x, s]$$



$$\Sigma = T^2$$



for every homocycle in $\Pi^1(S) \setminus \{\lambda\}$
 $T\{\lambda\}$ is an observable.

$$T^{-1}[y, u] = \int ds \operatorname{tr} e^{\int ds (s)} \dot{\gamma}^i(s) H[x, s]$$



- 1) T^{-1} is gauge inv
- 2) rep inv

$\Sigma = T^2$ for every homotopy class in $\Pi^1(S) \cong \mathbb{Z}$

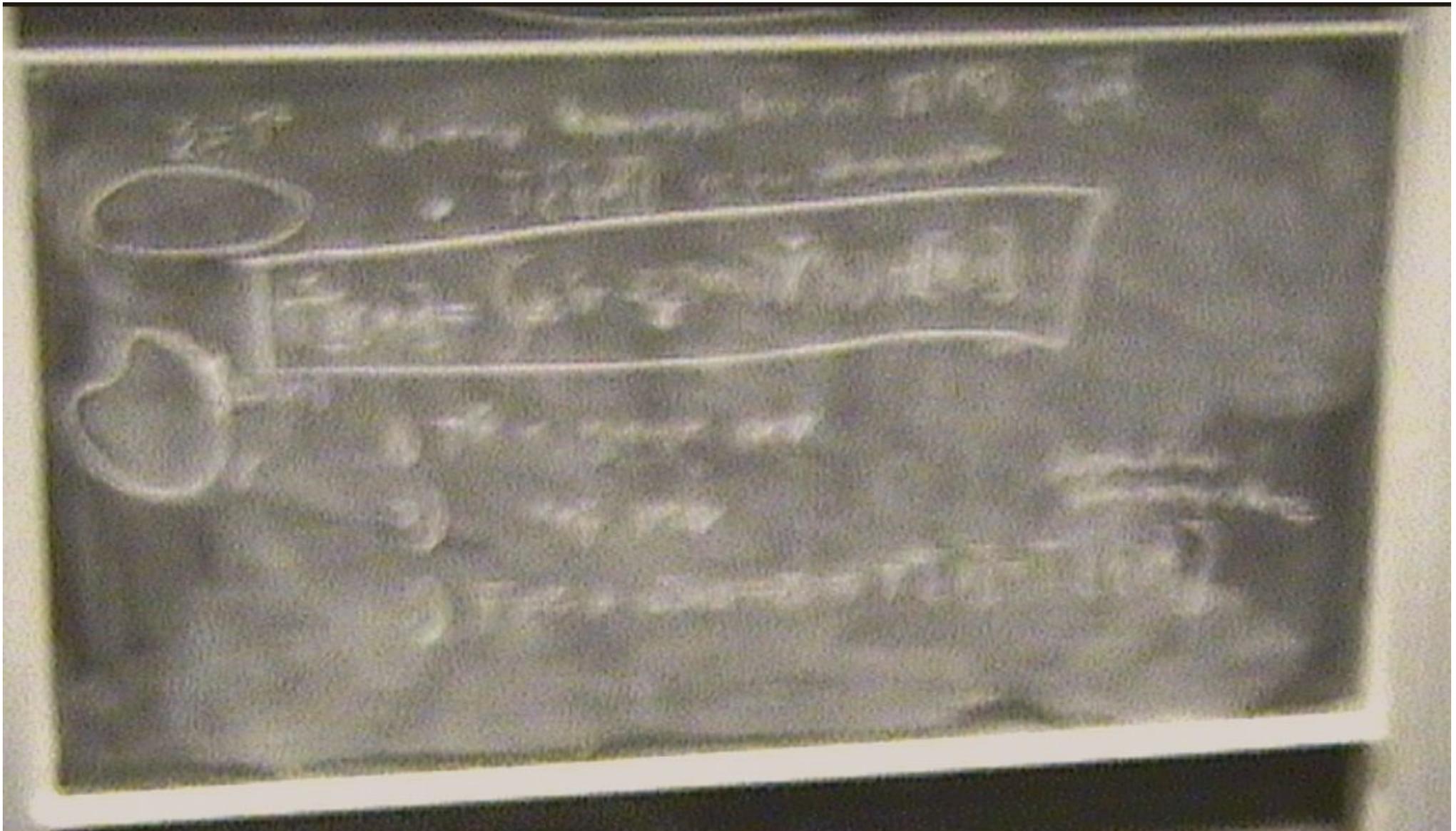
• $T\{\alpha\}$ is an observable.

$$T^{-1}[x, \alpha] = \int ds \operatorname{Tr} e^{\int \alpha(s)} \dot{\gamma}^i(s) H[x, s]$$

1) T^{-1} is gauge inv

2) rep inv

3) $F=0 \wedge D\alpha=0$



$\xi = T^2$ for every homotopy class in $\Pi^1(S)$ $\{\gamma\}$

$\rightarrow T\{\gamma\}$ is an observable.

$$T^2[\gamma, \nu] = \int ds \operatorname{tr} \left(\rho(\gamma(s)) \dot{\gamma}^\mu(s) H[\gamma, s] \right)$$

reim
 γ

- 1) T^2 is gauge INV
- 2) rep INV

depends on
homotopy class

$$3) F=0 \wedge D\nu=0 \Rightarrow T^2\{\gamma\} = T^2\{\gamma\}$$

$$z = T^2$$

for every homology class in $\Pi^1(S)$ $\{\gamma\}$
 $T\{\gamma\}$ $T^{-1}\{\gamma\}$ are observables

$$T^{-1}[Y, A, e] = \int ds \int r e^{i\gamma(s)} \dot{\gamma}^\mu(s) H[\gamma, s]$$

1) T^2 is gauge INV

2) rep INV

3) $F=0 \wedge D_A e=0 \Rightarrow T\{\gamma\} = T^{-1}\{\gamma\}$

depends on
homology class

$$\{Z(s), Z(x)\}$$

$$= \int dy dz \dots \partial A + \Lambda A$$

F-me F-me

$$= \rho(x) (0 + \Lambda e) = \dots \Pi + \Lambda \Pi$$

(check!!!)

$$= AG(\rho(x))$$

Hint:

$$\frac{\delta S}{\delta e^I} = F^I - \Lambda e^I e^J e^K = 0 \xrightarrow{\text{link}} \tilde{F}^I$$



$$\frac{\delta S}{\delta \Lambda^I} = \partial_\Lambda e = 0 \equiv G^I$$

$$\dot{e}_i^I + e_{j+1}^I \Lambda_j^I e_i^I = 0$$

$$\dot{A}_i^I - \Lambda_i A_i^I - \Lambda e_i^J e_i^K = 0$$

$$P = \gamma(t) = \gamma(t)$$

PB amongst the TIT'

$$1 - 1(1) = 0(1)$$

PB amongst the T, T' $\{A_i^T, B_j^T\} = S(k, N, S, k)$



$$T = T(t) = \dots$$

PB amongst the T, T' $\{A^{\mu\nu}, P^{\lambda}\} = \delta^{\mu\lambda} \delta^{\nu\sigma}$

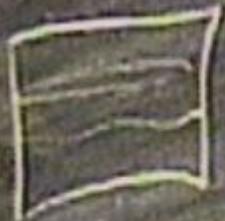
$\Rightarrow \{T, T'\} = 0$

$$1 - T(t) = 0(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^T(t), B_1^T(t)\} = \delta(t, t_1, t_2)$$



$$1 - 1(1) = 0(1)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T[L], T'[R]\} = \{TrPe^{SA}, \dots\}$$

$$\{A_{\alpha}^{\beta}(t), P_{\alpha}^{\beta}(t)\} = \delta_{\alpha\beta} \delta(t, t')$$



$$T = T(t) = \dots$$

PB amongst the T, T' $\{A^I(t), P^J(t)\} = \delta^{IJ}$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T[\alpha], T'[\beta]\} = \left\{ \int dx P \alpha, \int dx \beta \right\}$$



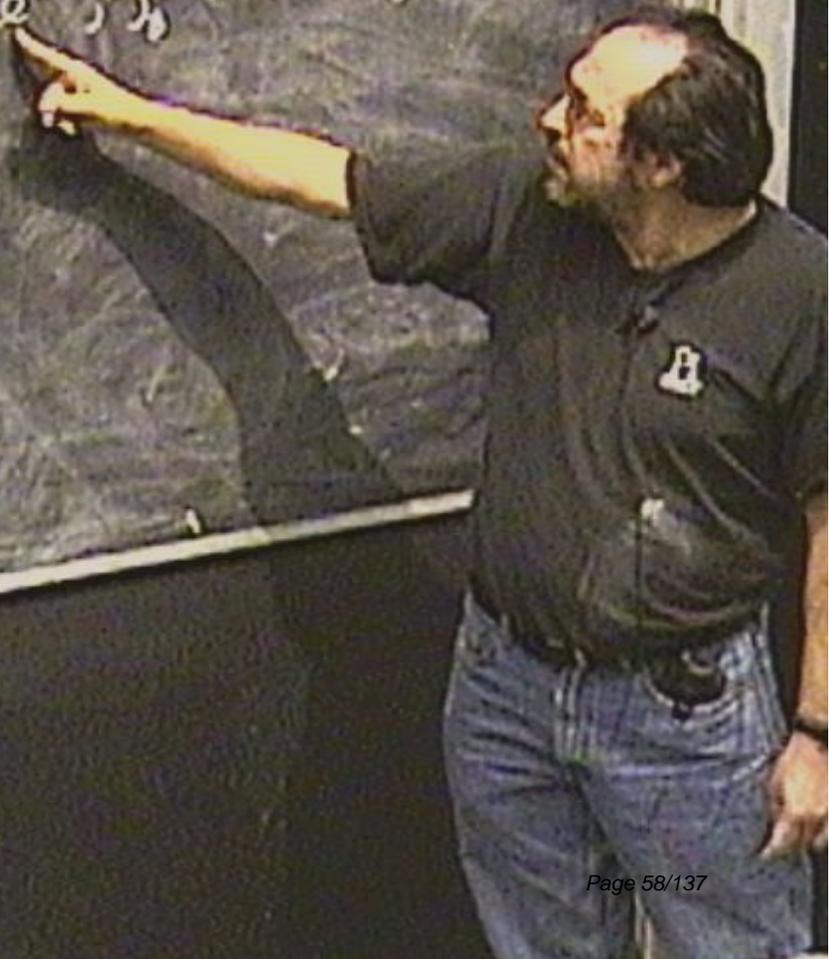
$$1 - \tau(\tau) = \delta(\tau)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T[A], T'[B]\} = \{TrPe^{\int A}, \int B e^{\int A}\}$$

$$\{A, B\} = \delta(x, y) \delta(x', y')$$



$$T = T(x) = D(x)$$

PB amongst the T, T' $\{A^{\mu\nu}, P^{\lambda}\} = \delta^{\mu\lambda} \delta^{\nu\sigma} \epsilon_{\sigma\lambda}$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T(x), T'(y)\} = \left\{ \text{Tr } P e^{\int A}, \int H \right\}$$



$$1 - \gamma(t) = 0(t)$$

$$\{A_i^T(t), B_i^T(t)\} = S^T(x, t, S^T(t))$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T(L), T'(R)\} = \left\{ \text{Tr} P_0 \left\{ \int_A, \int_B \tilde{R}^T \alpha_i \right\} \right\}$$

$$= \int dt \tilde{B}^T(t) \text{Tr} \alpha_i H(t)$$



$$1 - T(t) = 0(t)$$

PB amongst the T, T' $\{A_i^2(t), \rho_i^2(t)\} = \delta(t) \delta^i_j$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T[\Sigma], T'[R]\} = \left\{ \text{Tr} P_{\Sigma}^{SA}, \int_{\Sigma} H \dot{\rho}_i^2 \delta^i_j \right\}$$



$$= \int d(\dot{\rho}_i^2(t)) \text{Tr} P_{\Sigma}^{SA} \left\{ \rho_i^2(R(t)), \text{Tr} P_{\Sigma}^{SA} \right\}$$

$$1 - \gamma(t) = \gamma(t)$$

PB amongst the T, T' $\{A_i^2(t), B_j^2(t)\} = \delta_{ij} S^2 S^2$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{T[\alpha], T'[\beta]\} = \left\{ \text{Tr} P \rho \int A, \int B \dot{B} \dot{C} \int A \right\}$$

$$= - \int d\alpha \dot{B}'(\alpha) \text{Tr} \rho \int H[\alpha] \left\{ \dot{C}_i^2(\beta(\alpha)), \text{Tr} P \rho \int A \right\}$$



$$1 - \gamma(t) = \gamma(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_i^j(t), P_j^k(t)\} = \delta_{ik} \delta_{ij}$$

$$\{T(x), T'(x)\} = \left\{ \text{Tr} P_{\alpha\beta}^{SA}, \int dx \dot{P}^{\alpha\beta} \dot{Q}^{\alpha\beta} SA \right\}$$

$$= \int dx (\dot{P}^{\alpha\beta}(x) \dot{P}_{\alpha\beta}(x) H(x)) \text{Tr} P_{\alpha\beta}^{SA}$$



$\Gamma \sim \Gamma(0) = 0(1)$

$$\{A_i^2(t), B_j^2(t)\} = \delta_{ij} \delta(t-t')$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\begin{aligned} \{T(\Sigma), T'(R)\} &= \left\{ \text{Tr} P_0 \int_{\Sigma} A, \int_{\Sigma} A_i B^i \sigma_i \right\} \\ &= - \int_{\Sigma} A_i B^i(t) \text{Tr} \sigma_i H[\Sigma] \left\{ \underbrace{\int_{\Sigma} \sigma_i^2(B(t))}_{\int_{\Sigma} \text{Tr} P_2 \sigma_i}, \text{Tr} P_0 \int_{\Sigma} A \right\} \\ &\quad \int_{\Sigma} A_i(t) \end{aligned}$$



$\frac{8}{8 A.G.}$

$$\frac{\delta H[\gamma, A]}{\delta A^\alpha(x)}$$

$$\frac{\delta H(\gamma, A)}{\delta A(x)}$$

$$\delta H(\gamma, A) = P \delta \int_{\sigma} A + \delta A$$

H

$$\frac{\delta H(\gamma, A)}{\delta A^{\mu}(x)}$$

$$H[A, SA] = P_{\alpha} \int_{\sigma} A + SA$$

H

$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P \int \delta(A+SA) =$$

$$H(1+A+SA) \dots$$

$$\frac{\delta H(\gamma, A)}{\delta A_0(x)}$$

$$H[A+SA] = P_0 \int_0^1 A+SA = \int dt \delta$$

$$(1+A+SA) \dots$$

$$\frac{\delta H(\gamma, A)}{\delta A_0(x)}$$

$$H[A+SA] = P \int dt \delta A + SA = \int dt \delta$$

$$(1+A+SA) \dots$$

$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P_Q \int_{\mathcal{S}} A+SA = \int dt H(\gamma_{i,0,t}) (SA_i^T \sigma^I \delta^I)$$

$$H(A+SA) \dots$$

$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P_Q \int_{\mathcal{D}} (A+SA) = \int_{\mathcal{D}} dt H(\gamma, \sigma_t) (SA_i^T \sigma^T \delta^i) H(\gamma, t, z)$$

$$H \left((1+A+SA) \dots \right)$$



$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P \int_{\Sigma} A+SA = \int_{\Sigma} dt H(\gamma, \sigma, t) (\delta A_i^j + \sigma^i_j) H[\gamma, t, z]$$

$$H(A+SA)$$

$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P \int dt \int d^3x H(\gamma, A) (SA_i^T \sigma^i \delta^i) H(\gamma, A)$$

$$H(A+SA)$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), \phi(y)\} = \delta^3(x-y)$$

$$\frac{\delta H(\gamma, A)}{\delta A_\mu^\alpha(x)}$$

$$H[A+SA] = P \int_{\mathcal{S}} A + SA = \int dt H(\gamma, \sigma(t)) (\delta A_\mu^\alpha / \delta \sigma^i) H[\gamma, t, \alpha]$$

$$\frac{\delta H}{\delta A_\mu^\alpha(x)} = \int dt H \delta$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), \phi(y)\} = \delta^4(x-y)$$

$$\frac{\delta H(\gamma, A)}{\delta A_i(x)}$$

$$H[A+SA] = P \int_{\sigma} A + SA = \int dt H(\gamma, \sigma) (SA_i^{\mu} / \sigma^{\mu\nu}) H[\gamma, \sigma]$$

$$\frac{\delta H}{\delta A_i(x)} = \int dt H$$

$$H(A+SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\} = \delta^2(x, y)$$

$$\frac{\delta H(\gamma, A)}{\delta A_k^L(x)}$$

$$H[A+SA] = P_Q \int_{\delta} A+SA = \int dt H(\gamma, \sigma, t) (\delta A_k^L \delta^I \delta^j) H[\gamma, t, \sigma]$$

$$\frac{\delta H}{\delta A_k^L(x)} = \int dt H[\sigma] \frac{\delta A_k^L(x)}{\delta A_k^L(x)} H[\sigma]$$

$$H \quad (I + A + SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\} = \delta^d(x-y)$$

$$\frac{\delta H(\gamma, A)}{\delta A^i(x)}$$

$$H[A+SA] = P \int_{\sigma} (A+SA) = \int dt H(\gamma, \sigma(t)) (\delta A^i_{\sigma} \delta^i_{\sigma}) H[\gamma, \sigma]$$

$$\frac{\delta H}{\delta A^i_{\sigma}(x)} = \int dt H[\sigma] \frac{\delta A^i_{\sigma}(t)}{\delta A^i_{\sigma}(x)} H[\sigma]$$

$$H \quad (I + A + SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\} = \delta^d(x-y)$$

$$\frac{\delta H(\gamma, A)}{\delta A_\lambda^\mu(x)} = \int dt H[\mathbb{L}, t] \delta^\mu_\lambda(t) \sigma_I$$

$$H[A + \delta A] = P_Q \int_{\delta} A + \delta A = \int dt H(\gamma, t) (\delta A_\lambda^\mu \delta^\lambda_\mu) H[\mathbb{L}, t]$$

$$\frac{\delta H}{\delta A_\lambda^\mu(x)} = \int dt \frac{\delta H[\mathbb{L}, t]}{\delta A_\lambda^\mu(x)}$$

$$H(A + \delta A)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\}$$

$$\frac{\delta H(\gamma, A)}{\delta A_k^\mu(x)} = \int dt H[\Gamma, t] \delta^\mu(x) \otimes H[\Pi, t] \delta^3(\gamma(t), x)$$

$$H[A + \delta A] = P \int_{\delta}^{A + \delta A} = \int dt H(\gamma, t) (\delta A_k^\mu \delta^\mu \delta^3) H[\gamma, t, z]$$

$$H(A + \delta A)$$



$$\frac{\delta H}{\delta A_k^\mu(x)}$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), \phi(y)\}$$

$$\frac{\delta H(\gamma, A)}{\delta A_k^\mu(x)} = \int dt H[\Gamma, t] \delta^\mu(x) \otimes H[\Pi, t] \delta^\mu(\gamma(t), x)$$

$$H[A+SA] = P \int_{\delta} A+SA = \int dt H(\gamma, t) (\delta A_k^\mu \delta^\mu \delta^\nu) H[\gamma, t, A]$$

$$\frac{\delta H}{\delta A_k^\mu(x)} = \int dt H[\Gamma] \frac{\delta A_k^\mu(x)}{\delta A_k^\mu(x)} H[\Pi]$$

$(I + A+SA)$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), K(y)\} = \delta^4(x-y)$$

$$\frac{\delta H(\gamma, \dot{\gamma}, t)}{\delta A_k^I(x)} = \int dt H[\Gamma, t] \delta^4(x) \sigma_{\Gamma} H[\Pi, t] \delta^3(\gamma(t), x)$$

$$H[A+SA] = P \int_{\delta} A+SA = \int dt H(\gamma, \dot{\gamma}, t) (\delta A_k^I \delta^I \delta^k) H[\gamma, t, \dot{\gamma}]$$

$$\frac{\delta H}{\delta A_k^I(x)} = \int dt H[\Gamma] \frac{\delta A_k^I \delta^I \delta^k H[\Pi]}{\delta A_k^I(x)}$$

$$H(A+SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^4(x-y) = \delta^4(x-y)$$

$$\frac{\delta H(\gamma, \dot{\gamma}, t)}{\delta A_k^I(x)} = \int dt \frac{H[L, \dot{\gamma}] \delta^I(x)}{H[\dot{\gamma}, x]} + \frac{H[\dot{\gamma}, x]}{H[\dot{\gamma}, x]} \delta^I(\gamma(t), x)$$

$$H[A+SA] = P \int_{\delta} A+SA = \int dt H[\gamma, \dot{\gamma}, t] (SA^I \delta^I \delta^I) H[L, \dot{\gamma}, t]$$

$$\frac{\delta H}{\delta A_k^I(x)} = \int dt H[\dot{\gamma}] \frac{\delta SA^I \delta^I \delta^I}{\delta A_k^I(x)} H[L]$$

I (I+A+SA)



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\} - \delta^I(x, y)$$

$$T = T(\alpha) = \beta(\alpha)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_i^T(\alpha), Q_j^T(\alpha)\} = \delta_{ij} S^{\alpha\beta}(\alpha)$$



$$\{T(\alpha), T'(\beta)\} = \left\{ \int \text{Tr} P \alpha^{\beta A}, \int \text{Tr} \beta^{\alpha A} \alpha^{\beta A} \right\}$$

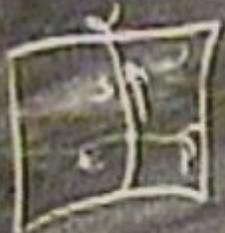
$$= - \int d\alpha \beta^{\alpha}(\alpha) \text{Tr} \sigma_{\alpha} H[\alpha] \left\{ \alpha_i^{\beta}(\beta(\alpha)), \int \text{Tr} P \alpha^{\beta A} \right\}$$



$$T = T(x) = \beta(x)$$

PB amongst the T, T' $\{A_i^T(x), Q_i^T(x)\} = \delta(x, y) \delta_{ij}$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\{T(x), T'(x)\} = \left\{ \int dx' P(x') \dot{Q}(x'), \int dx'' \dot{P}(x'') Q(x'') \right\}$$

$$= - \int dx \dot{P}(x) \int dx' H(x, x') \left\{ Q_i^T(x'), P(x') \right\}$$

$$= - \int dx \dot{P}(x)$$

$$1 - T(t) = D(t)$$

PB amongst the T, T'

$$\{A_1^T(t), \dot{Q}_1^T(t)\} = \delta(x, t) \delta(x')$$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\{T[x], T'[x']\} = \left\{ \int_{\text{left}} \dot{Q}_1^T dx, \int_{\text{right}} \dot{Q}_1^T dx' \right\}$$

$$= - \int dx \dot{B}^T(x) \int_{\text{right}} dx' H(x, x') \left\{ \dot{Q}_1^T(x'), \int_{\text{left}} dx'' \dot{Q}_1^T(x'') \right\}$$

$$= - \int dx \int dx' \dot{B}^T(x)$$

$\Gamma = \Gamma(\eta) = \mathcal{D}(\eta)$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

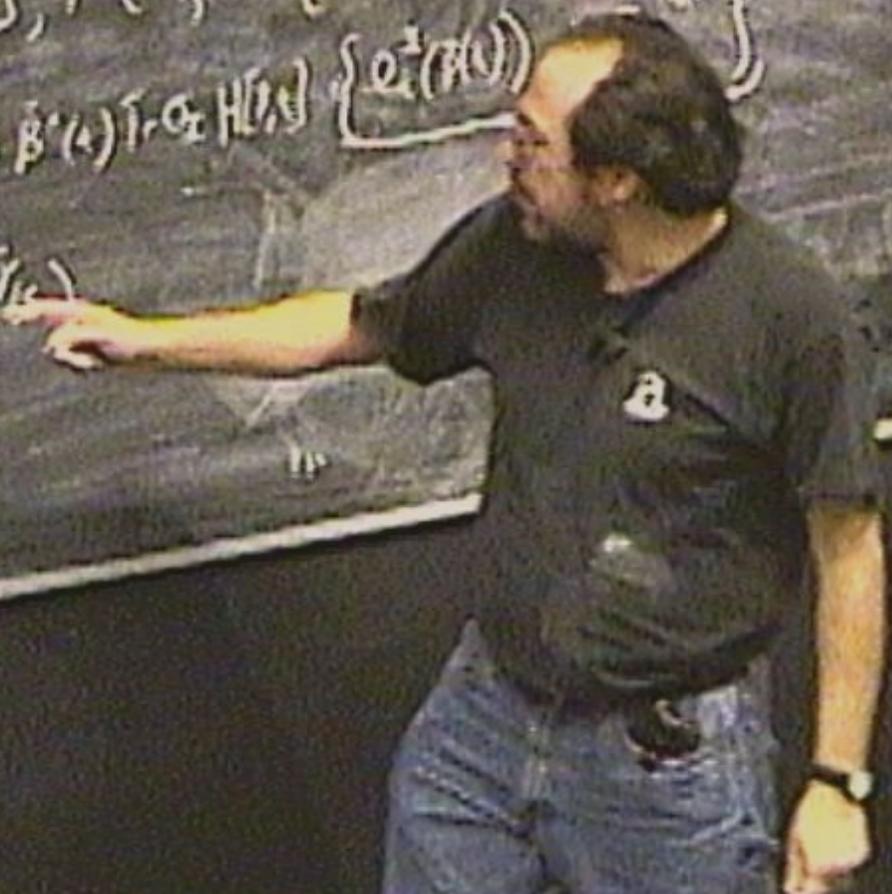
$$\{A_i^j(t), P_j^k(t)\} = \delta_{ik} \delta^{jl}$$



$$\{T[\eta], T'[\eta]\} = \left\{ \text{Tr} P \alpha^i \dot{x}^i, \int_{\Sigma} \left(\dot{x}^i \dot{x}^j \alpha^k \alpha^l \right) \right\}$$

$$= - \int dt \int ds \dot{x}^i(t) \dot{x}^j(t) \left\{ \alpha^i(t) \alpha^j(t) \right\}$$

$$= - \int dt \int ds \dot{x}^i(t) \dot{x}^j(t)$$



$$1 - \Gamma(\omega) = \beta(\omega)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^2(\omega), \rho_2^2(\omega)\} = \delta(\omega_1, \omega_2)$$



$$\{T[\omega], T'[\omega]\} = \left\{ \text{Tr} P \rho^{\omega_A}, \int_{\omega} H \dot{\rho}^{\omega_A} \right\}$$

$$= \int d\omega' \beta'(\omega') \text{Tr} \rho_{\omega'} H[\omega'] \left\{ \rho_{\omega'}^2(\omega'), \text{Tr} P \rho \right\}$$

$$= - \int d\omega \int d\omega' \beta'(\omega') \dot{\rho}'(\omega')$$

$$T = T(t) = \dots$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^2(t), \dots\} = \delta(x, y, z, t)$$



$$\{T[L], T'[R]\} = \left\{ \text{Tr } P_{\alpha}^{SA}, \int_{\Sigma} H \beta^{\alpha} \alpha_{\alpha}^{SA} \right\}$$

$$= - \int dt \int d^3x \beta^{\alpha}(t) \text{Tr } \alpha_{\alpha} H[L] \left\{ \alpha_{\alpha}^2(R[N]), \text{Tr } P_{\alpha}^{SA} \right\}$$

$$= - \int dt \int d^3x \beta^{\alpha}(t) \alpha_{\alpha}^2(s) \delta^3(P[R], \alpha(s))$$



$$1 - \Gamma(\omega) = 0(1)$$

PB amongst the T, T'

$$\{A_1^2(\omega), \rho_1^2(\omega)\} = \delta(\omega, \omega')$$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\{T[\omega], T'[\omega']\} = \left\{ \int_{\Sigma_A} \dot{\rho}_1 \dot{\rho}_2, \int_{\Sigma_B} \dot{\rho}_1 \dot{\rho}_2 \right\}$$

$$= - \int dt \int ds \dot{\rho}_1^a(\omega) \dot{\rho}_2^b(\omega') \delta^2(\omega, \omega')$$

$$= - \int dt \int ds \dot{\rho}_1^a(\omega) \dot{\rho}_2^b(\omega') \delta^2(\omega, \omega')$$

$$\frac{\delta H(\gamma, A)}{\delta A_k(x)} = \int dt \frac{H[\Gamma, t]}{H[\gamma, t]} \delta^+(t) \frac{\delta H(\Pi, t)}{H[\gamma, t]} \delta^3(\gamma(t), x)$$

$$H[A+SA] = P \int \delta A + SA = \int dt H[\gamma, t] (\delta A_k^T \delta^T \delta^k) H[\Gamma, t, A]$$

$$\frac{\delta H}{\delta A_k(x)} = \int dt H[\Gamma] \frac{\delta A_k^T \delta^T \delta^k}{\delta A_k(x)} H[\Pi]$$

$$H (1 + A+SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{H(x), H(y)\} = \delta^4(x-y)$$

$$1 - \gamma(t) = \gamma(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A^{\mu\nu}(t), P^{\alpha\beta}(t)\} = \delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\alpha}$$



$$\{T[E], T'[P]\} = \left\{ \int_{\Sigma_A} T_{\mu\nu} dx^\mu dx^\nu, \int_{\Sigma_B} T'_{\alpha\beta} dx^\alpha dx^\beta \right\}$$

$$= - \int dt \int d^3x \beta^i(t) \dot{\gamma}^j(t) H[\pi^i, \pi^j] \left\{ \alpha^i(t), \pi^j \right\}$$

$$= - \int dt \int d^3x \beta^i(t) \dot{\gamma}^j(t) \xi_{ij}^{\alpha\beta}(\pi^i, \pi^j)$$



$$1 - \epsilon(t) = \delta(t)$$

PB amongst the T, T'

$$\{A_1^2(t), \rho_1^2(t)\} = \delta(t, t') \delta^2(x, x')$$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\{T[E], T'[P]\} = \left\{ \int_{\Sigma_A} \dot{T} \rho_2, \int_{\Sigma_B} \dot{T} \rho_2 \right\}$$

$$= - \int dt \dot{\beta}^i(t) \int_{\Sigma_A} \dot{T} \rho_2 H[\rho_2] \left\{ \rho_2^2(\beta(t)), \int_{\Sigma_A} \dot{T} \rho_2 \right\} \quad \rho_2 = \epsilon, \pi^i$$

$$= - \int dt \int ds \dot{\beta}^i(t) \dot{\alpha}^j(s) \xi_{ij}^2(\beta(t), \alpha(s))$$

$$T = T(\alpha) = \beta(\alpha)$$

PB amongst the T, T' $\{A_1^2(\alpha), \beta(\alpha)\} = \beta(\alpha, \alpha, \alpha)$
 $\Rightarrow \{T, T'\} = 0$ $\{T, T'\}$



$$\{T(\alpha), T'(\beta)\} = \left\{ \text{Tr} P e^{S_A}, \int_{\alpha}^{\beta} H(\beta, \alpha) d\alpha \right\}$$

$$= - \int_{\alpha}^{\beta} \left(\int_{\alpha}^{\beta} H(\beta, \alpha) d\alpha \right) \text{Tr} P e^{S_A} d\alpha$$

$$\frac{\delta H(\gamma, A)}{\delta A_k(x)} = \int dt \frac{H(\Gamma, t)}{H(\gamma, t)} \delta^3(\gamma(t), X)$$

$$H[A+SA] = P \int dt H(\gamma, t) (SA^T \sigma^I \delta^i) H[\gamma, t]$$

$$\frac{\delta H}{\delta A_k(x)} = \int dt H(\Gamma) \frac{\delta A^T(t)}{\delta A_k(x)} H[\Gamma]$$

$$H(A+SA)$$



$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{K(x), K(y)\} - \delta^d(x-y)$$

$$T \rightarrow T' = \gamma(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^{\mu\nu}, \alpha_2^{\mu\nu}\} = \delta^{\mu\nu}(\alpha_1^{\mu\nu} \alpha_2^{\mu\nu})$$



$$\{T(x), T'(y)\} = \left\{ \text{Tr} P e^{\int_A}, \int_B \tilde{\alpha}^{\mu\nu} \alpha^{\mu\nu} \right\}$$

$$= - \int dx \tilde{\beta}^{\mu\nu}(x) \text{Tr} \alpha^{\mu\nu} H(x) \left\{ \alpha^{\mu\nu}(y), \text{Tr} P e^{\int_A} \right\}$$

$$= - \left[\int dx + \int dy \tilde{\beta}^{\mu\nu}(x) \alpha^{\mu\nu}(y) \xi_{\mu\nu}^{\mu\nu}(P(x), \alpha(y)) \right] \left[\text{Tr} \alpha^{\mu\nu} H(y) / \text{Tr} H(x) \right]$$

$$T = T(\alpha) = \beta(\alpha)$$

PB amongst the T, T' $\{A_i^2(\alpha), \beta_j^2(\alpha)\} = \delta_{ij}(\alpha, \alpha) \delta_{ij}^2$

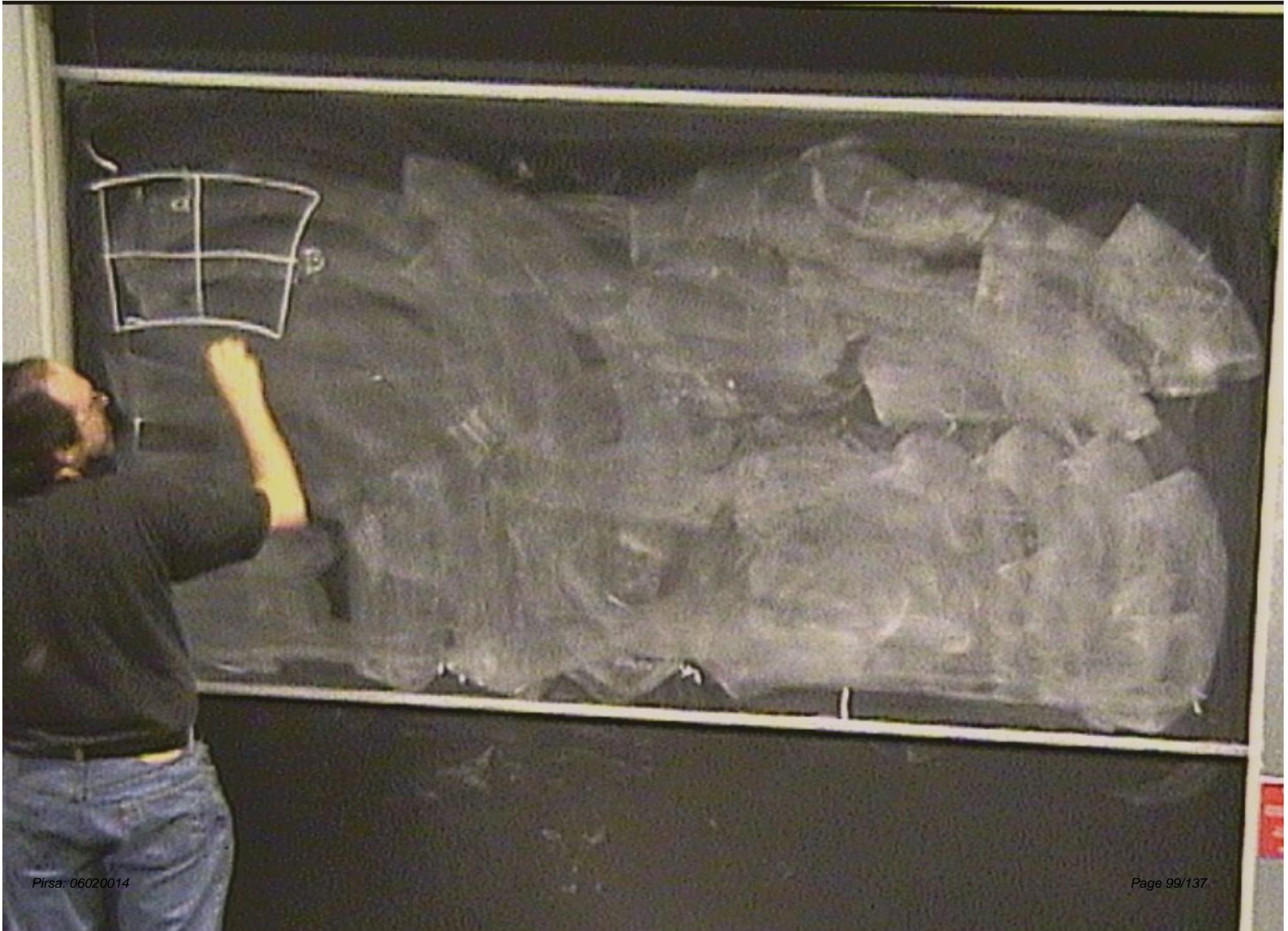
$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



$$\{T(\alpha), T'(\beta)\} = \left\{ \text{Tr} \rho \alpha^A, \int \alpha^B \beta^C \right\}$$

$$= - \int d\alpha \beta'(\alpha) \text{Tr} \rho_\alpha H(\alpha) \left\{ \alpha^A(\alpha), \text{Tr} \rho \alpha^A \right\} \quad \alpha = \xi, \pi$$

$$= - \left[\int d\alpha + \int d\beta \beta'(\alpha) \alpha'(s) \right] \xi \delta^2(\beta(\alpha), \alpha(s)) \left. \text{Tr} \rho_\alpha H(\beta(s)) \text{Tr} H(\beta) \right\}$$

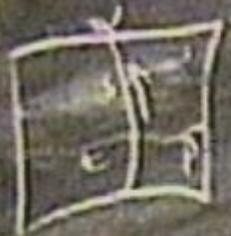


$$I = T(t) = S(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^2(t), B_1^2(t)\} = S(x, y, z)$$

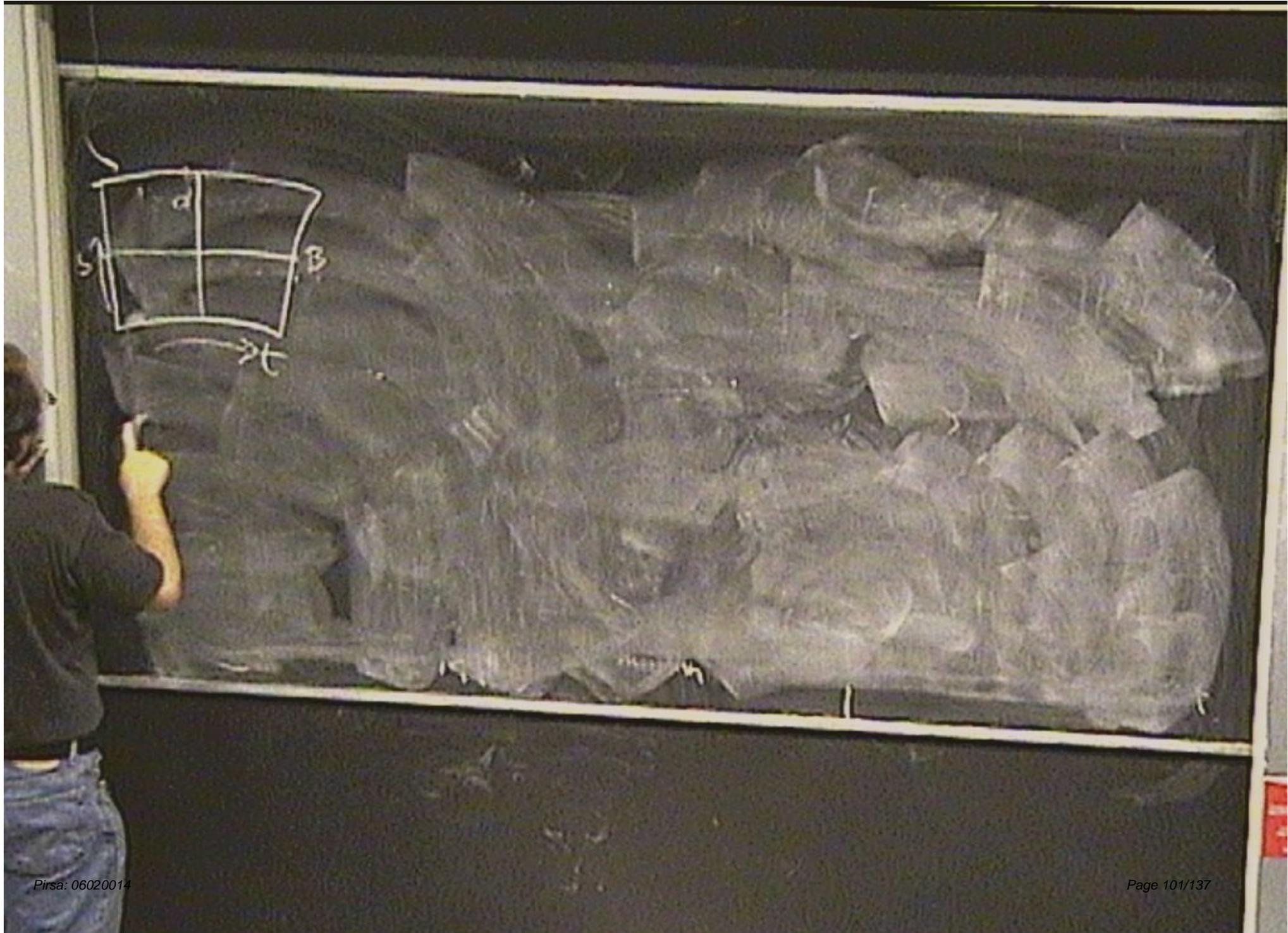


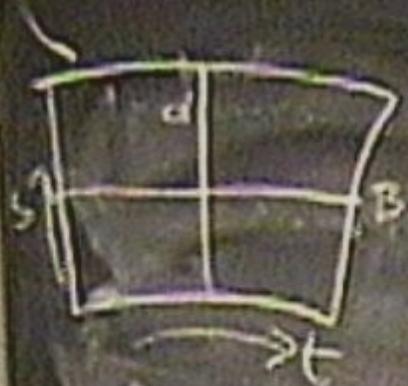
$$\{T(x), T'(y)\} = \left\{ \text{Tr } P_0^{SA}, \left\{ H, B, \dots \right\}^{SA} \right\}$$

$$= -S \left(\text{Tr } H(x), \left\{ Q_1^2(x), \text{Tr } P_0^{SA} \right\} \right) \quad e = \epsilon, \pi$$

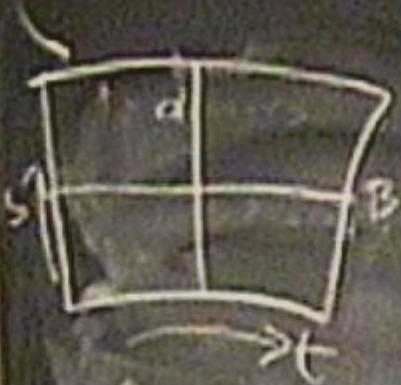
$$= -[S_{dt} + S_{ds}]$$

$$S^*(P(x), Q(y)) \left[\text{Tr } Q_1^2 H(x), \text{Tr } H(x) \right]$$





$$A \rightarrow s=0$$
$$B \rightarrow t=0$$



$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$I = T(t) = J(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^2(t), P_2^2(t)\} = \delta(x, y) \delta(z)$$



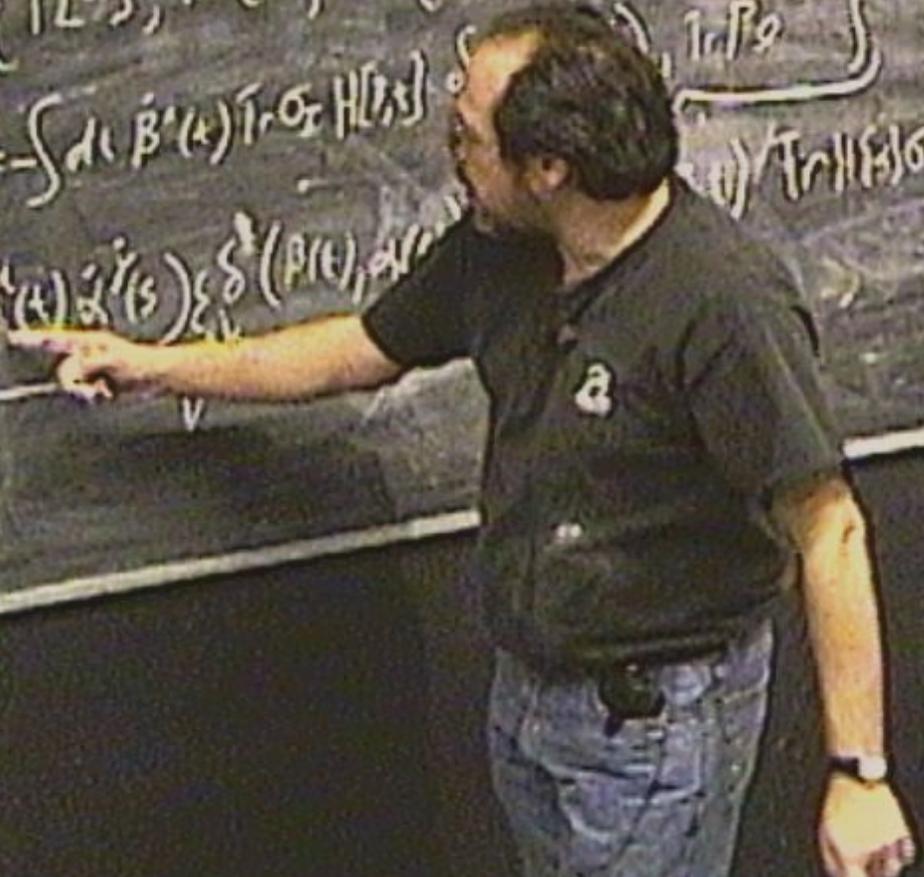
$$\{T(x), T'(y)\} = \left\{ \text{Tr} P_0^{SA}, \left\{ \int dx \dot{\phi}^2 \right\}^{SA} \right\}$$

$$= - \int dx \dot{\phi}^2(x) \text{Tr} \sigma_x H(x)$$

$$\left\{ \text{Tr} P_0^{SA}, \int dx \dot{\phi}^2 \right\} = \epsilon, \pi^2$$

$$= - \left[\int dx + \int dy \right] \dot{\phi}^2(x) \dot{\phi}^2(y) \epsilon \delta(x-y)$$

$\underbrace{\hspace{10em}}_{\int dx \int dy}$



$$T = T(\alpha) = \mathcal{D}(\alpha)$$

PB amongst the T, T' $\{A_x^{\pm 1}(\alpha), B_y^{\pm 1}(\alpha)\} = \delta(\alpha, \alpha' \pm \epsilon)$

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$



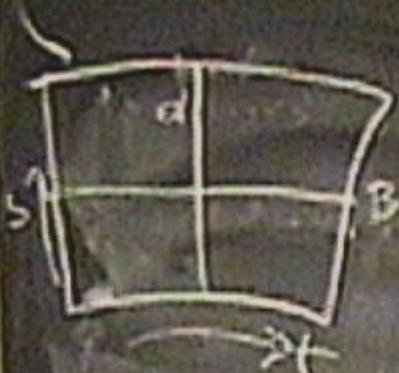
$$\{T(\alpha), T'(\beta)\} = \left\{ \text{Tr} P_{\alpha} \left\{ \int_{\alpha} H \dot{B}^{\pm} \alpha \right\} \right\}, \left\{ \int_{\beta} H \dot{B}^{\pm} \alpha \right\} \right\}$$

$$= - \int d\alpha \dot{B}^{\pm}(\alpha) \text{Tr} \sigma_z H(\alpha) \left\{ \int_{\beta} \dot{B}^{\pm}(\alpha) \right\} \text{Tr} P_{\alpha} \left\{ \int_{\beta} H \dot{B}^{\pm} \alpha \right\} \right\} \quad \epsilon = \epsilon, \pi$$

$$= - \left[\int d\alpha \int d\beta \dot{B}^{\pm}(\alpha) \dot{B}^{\pm}(\beta) \epsilon_{xy} \delta^2(P(\alpha), \alpha(\beta)) \right] \left[\text{Tr} \sigma_z H(\alpha) \right] \left[\text{Tr} H(\beta) \right]$$

$\int d\alpha \int d\beta \pm 1 \delta(\alpha, \alpha') \delta(\alpha, \alpha' \pm \epsilon) = \pm 1$



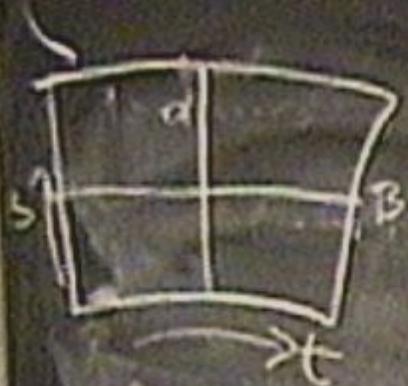


$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$x = [s, t] \Rightarrow \dot{x} = [1, 0]$$

$$B = [1, 0] \quad R = [0, 1]$$



$$\sum_I \text{Tr}(M \sigma^I) \text{Tr}(N \sigma^I)$$

$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$X = [s, t] \Rightarrow X = [1, 1]$$

$$B = [s, 0] \quad R = [1, 0]$$



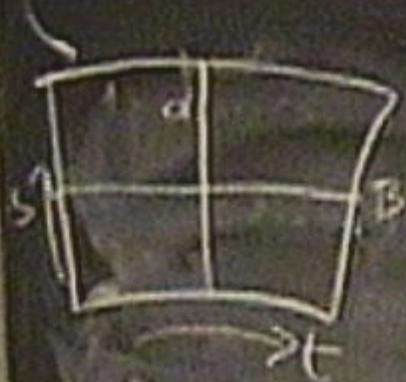
$$\sum_j \text{Tr}(M \sigma_j^T) \text{Tr}(N \sigma_j) = \text{Tr} MN - \text{Tr} MN^T$$

$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \vec{a} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \vec{b} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$



$$\sum_I \text{Tr}(M \sigma_I) \text{Tr}(N \sigma_I) = \text{Tr} MN - \text{Tr} MN^{-1} \quad \text{check}$$

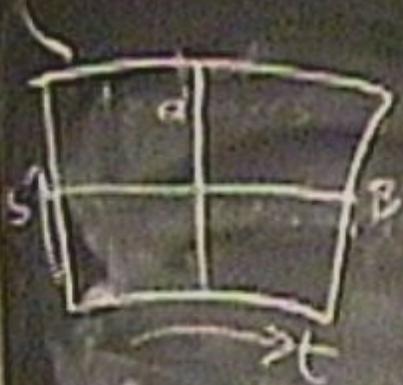
$$A \rightarrow s=0$$

$$B \rightarrow t=0$$

$$x = \begin{bmatrix} s \\ t \end{bmatrix} \rightarrow \tilde{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





$$A \rightarrow s=0$$

$$B \rightarrow t=0$$

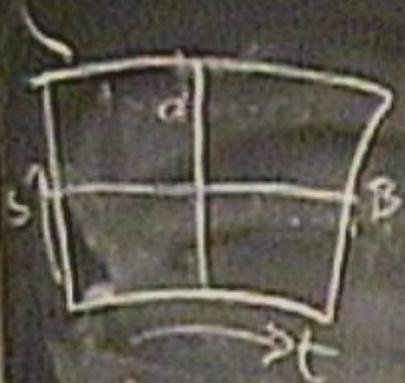
$$\alpha = [y, t] \rightarrow \alpha' = [0, 1]$$

$$B = [s, 0] \quad A = [1, 0]$$

$$\sum_I \text{Tr}(M \sigma^I) \text{Tr}(N \sigma^I) = \text{Tr} MN - \text{Tr} MN^I \quad \text{check!}$$

$$\{T[\alpha], T[\beta]\} = \text{Im}[\alpha, \beta] (T[\alpha \wedge \beta])$$





$$A \Rightarrow s=0$$

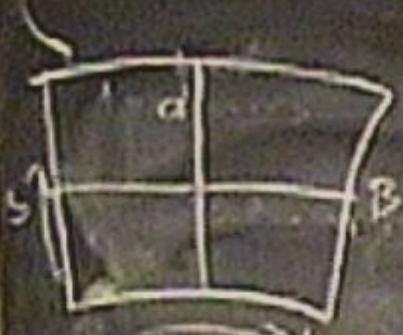
$$B \Rightarrow t=0$$

$$A = [y, t] \Rightarrow A = [1, 1]$$

$$B = [s, 0] \quad R = [1, 0]$$

$$\int_I \text{Tr}(M \sigma^2) \text{Tr}(N \sigma_3) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

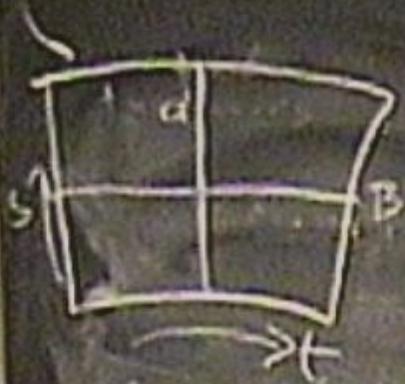
$$\{T[\alpha], T'[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$



$$\sum_I \text{Tr}(M \sigma_I) \text{Tr}(N \sigma_I) = \text{Tr} MN - \text{Tr} MN^T \quad \text{check}$$

$$\{T^A[\alpha], T^B[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$

$$\{T^A[\alpha], T^B[\beta]\} = \text{Int}[\alpha, \beta]$$



$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

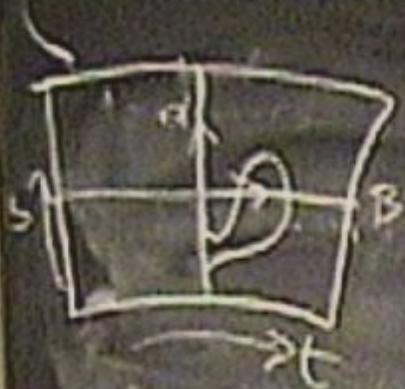
$$x = [s, t] \Rightarrow \dot{x} = [1, 0]$$

$$B = [s, 0] \quad R = [0, 0]$$

$$\sum_I \text{Tr}(M \sigma_I) \text{Tr}(N \sigma_I) = \text{Tr} MN - \text{Tr} MN^T \quad \text{check}$$

$$\{T^A[A], T^B[B]\} = \text{Int}[A, B] (T[A \cdot B] - T[A \cdot B^T])$$

$$\{T^A[A], T^B[B]\} = \text{Int}[A, B] (T^A[x \cdot B] - T^A[x \cdot B^T])$$



$$A \rightarrow s=0$$

$$B \rightarrow t=0$$

$$\alpha = [y, t] \Rightarrow \dot{\alpha} = [1, 1]$$

$$B = [x, 0] \quad R = [1, 1]$$

$$\int_{\Sigma} \text{Tr}(M \sigma^1) \text{Tr}(N \sigma_2) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

$$\{T[\alpha], T[\beta]\} = \int_{\text{Int}[\alpha, \beta]} (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$

$$\{T'[\alpha], T'[\beta]\} = \int_{\text{Int}[\alpha, \beta]} (T'[\alpha \circ \beta] - T'[\alpha \circ \beta'])$$

$$T = T'(t) = 0(t)$$

PB amongst the T, T'

$$\Rightarrow \{T, T'\} = 0 \quad \{T, T'\}$$

$$\{A_1^2(t), P_2^2(t)\} = \delta(x_1, t) \delta(x_2, t)$$

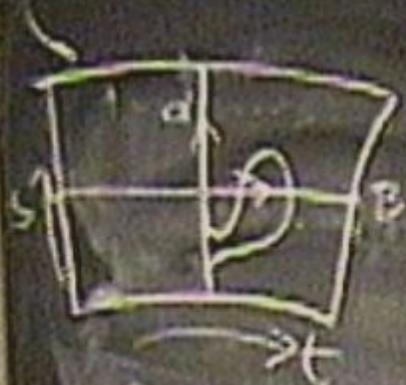
$$\{T(x), T'(x)\} = \left\{ \int dx' P(x') \dot{\phi}(x'), \int dx'' P(x'') \dot{\phi}(x'') \right\}$$



$$= - \int dx' \dot{\phi}(x') \int dx'' P(x'') \{ \dot{\phi}(x''), P(x'') \} \quad e = \pm \pi$$

$$= - \left(\int dx' \int dx'' \dot{\phi}(x') \dot{\phi}(x'') \delta(x' - x'') \right) \int dx'' P(x'') \{ \dot{\phi}(x''), P(x'') \}$$

$$\int dx \dot{\phi}^2 = 1 \quad \delta(x_1) \delta(x_2) = \pm 1$$



$A \rightarrow s=0$
 $B \rightarrow t=0$

$\alpha = [s, t] \Rightarrow \dot{\alpha} = [1, 1]$
 $B = [s, 0] \quad R = [0, 1]$

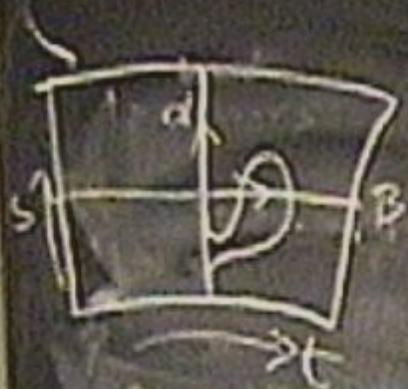
$$\int_{\Sigma} \text{Tr}(M \epsilon^1) \wedge \text{Tr}(N \epsilon_2) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

$$\{T[\alpha], T[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$

$$\{T'[\alpha], T'[\beta]\} = \text{Int}[\alpha, \beta] (T'[\alpha \circ \beta] - T'[\alpha \circ \beta'])$$

Hawking 1977





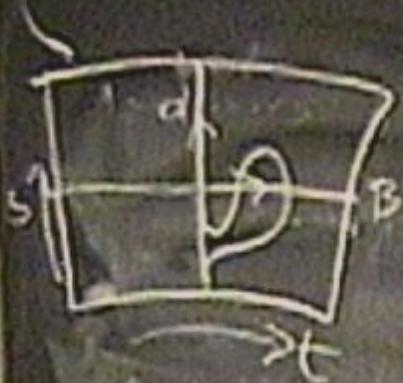
$A \Rightarrow s=0$
 $B \Rightarrow t=0$

$\alpha = [y, t] \Rightarrow \bar{\alpha} = [1, 1]$
 $B = [s, 0] \quad R = [1, 1]$

$$\sum_I \text{Tr}(M \sigma^I) \text{Tr}(N \sigma^I) = \text{Tr} MN - \text{Tr} MN^T \quad \text{check}$$

$$\{T[\alpha], T[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$

$$\{T'[\alpha], T'[\beta]\} = \text{Int}[\alpha, \beta] (T'[\alpha \circ \beta] - T'[\alpha \circ \beta'])$$



$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$\alpha = [s, t] \Rightarrow \dot{\alpha} = [1, 0]$$

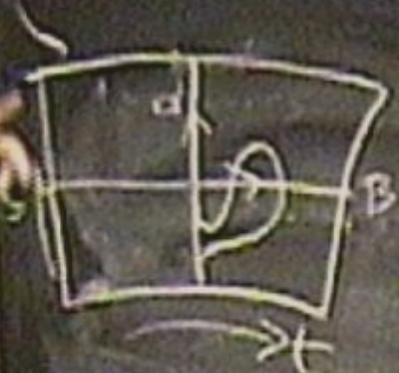
$$B = [s, 0] \quad R = [1, 1]$$

$$\int_I \text{Tr}(M \sigma^1) \text{Tr}(N \sigma^2) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

$$\{T[\alpha], T[B]\} = \int_{\text{Int}[\alpha, B]} (T[\alpha \circ B] - T[\alpha \circ B'])$$

$$\{T'[\alpha], T'[B]\} = \int_{\text{Int}[\alpha, B]} (T'[\alpha \circ B] - T'[\alpha \circ B'])$$

Q: which are independent



$a \rightarrow s=0$
 $B \rightarrow t=0$
 $\alpha = [y, t] \rightarrow \tilde{\alpha} = [y, 1]$
 $\beta = [s, 0] \quad \tilde{\beta} = [1, 0]$

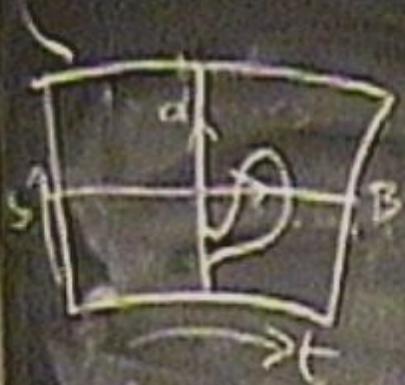
$$\sum_I \text{Tr}(M \sigma^I) \text{Tr}(N \sigma^I) = \text{Tr} MN - \text{Tr} MN^I \quad \text{check}$$

$$\{T[\alpha], T[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \cdot \beta] - T[\alpha \cdot \tilde{\beta}^I])$$

$$\{T'[\tilde{\alpha}], T'[\tilde{\beta}]\} = \text{Int}[\tilde{\alpha}, \tilde{\beta}] (T'[\tilde{\alpha} \cdot \tilde{\beta}] - T'[\tilde{\alpha} \cdot \tilde{\beta}^I])$$

Homotopy INV

Q: which are independent
 HB3



$$A \rightarrow s=0$$

$$B \rightarrow t=0$$

$$A = [0, t] \rightarrow \tilde{A} = [0, 1]$$

$$B = [s, 0] \quad R = [1, 0]$$

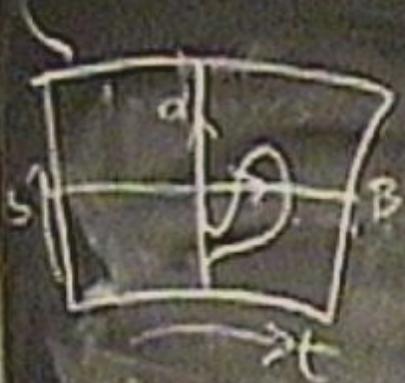
$$\int_I \text{Tr}(M \sigma^2) \text{Tr}(N \sigma_2) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

$$\{T^A[A], T^B[B]\} = \text{Int}[A, B] (T[A \circ B] - T[A \circ B'])$$

$$\{T^A[A], T^B[B]\} = \frac{\text{Int}[A, B]}{\text{hor. by } 12V} (T[A \circ B] - T[A \circ B'])$$

Q: which are independent

$$\text{HES} \quad \text{Tr} H[A] = T[A]$$



$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$\alpha = [y, t] \Rightarrow \dot{\alpha} = [u, 1]$$

$$B = [x, 0] \quad R = [1, 0]$$

$$\int_I \text{Tr}(M \sigma^2) \text{Tr}(N \sigma^2) = \text{Tr} MN - \text{Tr} MN' \quad \text{check}$$

$$\{T[\alpha], T[\beta]\} = \int_{\text{Int}[\alpha, \beta]} (T[\alpha \circ \beta] - T[\alpha \circ \beta'])$$

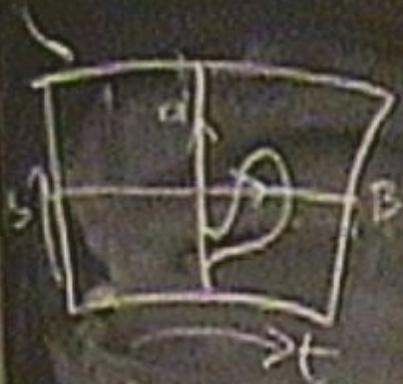
$$\{T'[\alpha], T'[\beta]\} = \int_{\text{Int}[\alpha, \beta]} (T'[\alpha \circ \beta] - T'[\alpha \circ \beta'])$$

Q: which are independent

$$\text{Tr} H[\alpha] = \text{Tr}[\alpha]$$

$$\text{Tr}[\alpha], \text{Tr}[\alpha'], \text{Tr}[\alpha'']$$

$$\text{Tr}[\alpha \circ \beta], \text{Tr}[\alpha \circ \beta'], \text{Tr}[\alpha \circ \beta'']$$



$$A \Rightarrow s=0$$

$$B \Rightarrow t=0$$

$$y = [y, t] \Rightarrow \alpha = [y, t]$$

$$B = [s, 0] \quad R = [0, A]$$

$$\int_I \text{Tr}(M \sigma^2) \text{Tr}(N \sigma_3) = \text{Tr} MN - \text{Tr} MN^{-1} \quad \text{check}$$

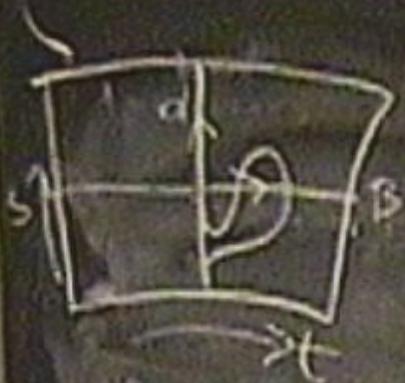
$$\{T[\alpha], T[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta^{-1}])$$

$$\{T'[\alpha], T'[\beta]\} = \frac{\text{Int}[\alpha, \beta]}{\text{hom. by 19V}} (T'[\alpha \circ \beta] - T'[\alpha \circ \beta^{-1}])$$

Q: which are independent

$$\text{III} \quad \text{Tr} H[\alpha] = T[\alpha]$$

$$\begin{matrix} T[\alpha], T[\alpha'], T[\alpha''] \\ T[\alpha\beta], T[\alpha\beta'], T[\alpha'\beta'] \end{matrix}$$



$A \rightarrow s=0$
 $B \rightarrow t=0$

$\gamma = [g, t] \Rightarrow \dot{\alpha} = [1, 0]$
 $B = [s, 0] \quad R = [1, 1]$

$$\int_I \text{Tr}(M \sigma^1) \text{Tr}(N \sigma^2) = \text{Tr} MN - \text{Tr} MN^T \quad \text{check}$$

$$\{T^{\alpha}[\alpha], T^{\beta}[\beta]\} = \text{Int}[\alpha, \beta] (T[\alpha \circ \beta] - T[\alpha \circ \beta^T])$$

$$\{T^{\alpha}[\alpha], T^{\beta}[\beta]\} = \frac{\text{Int}[\alpha, \beta]}{\text{length}(\alpha)} (T[\alpha \circ \beta] - T[\alpha \circ \beta^T])$$

Q: which are independent

$$\text{III} \quad \text{Tr} H[\alpha] = T[\alpha] \left(\begin{array}{l} \pi[\alpha], \pi[\alpha^T], T[\alpha^T] \\ T[\alpha \circ \beta], T[\alpha \circ \beta^T], \pi[\alpha \circ \beta] \end{array} \right)$$

$$\int_{ACS} = \int A_n dA \quad | \text{form A}$$

$$\delta S_{NS} = \int A n dA \quad | \text{form A}$$

$$\frac{\delta S_{NS}}{\delta A_n(\tau)} = \epsilon^{uv} F_{uv} = 0$$



$$S_{ACS} = \int A_n dA$$

$$\frac{\delta S_{NS}}{\delta A_n} = \epsilon^{nc} F_{bc} = 0$$

1 form A | S
 δ

Saha $\partial J_{12} - S^{12} / S^2$
 Saha A_c

$$S_{ACS} = \int A_n dA$$

$$\frac{\delta S_{ACS}}{\delta A_n} = \epsilon^{nk} F_{kl} = 0$$

1 form A

S

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kl}$$

$S_{ACS} = \int A_n dA = \int \epsilon^{nk} F_{kl} dA$

$$S_{ACS} = \int A_n dA$$

$$\frac{\delta S_{NS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

1 form A

S

$$\frac{\delta S}{\delta A_n} =$$

$\epsilon^{nk} F_{kc} = dA + \dots$

$$S_{ACS} = \int A_n dA$$

$$\frac{\delta S_{ACS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$\int \delta A_n F$$

1 form A

S

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kc} = \delta A_n + \dots$$

$S_{ACS} = \int A_n dA$



$$S_{ACS} = \int A_n dA$$

$$\frac{\delta S_{NS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$\int dA F$$

1 form A

$$S = \int \frac{1}{2} F_{\mu\nu} F^{\mu\nu} +$$

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kc} = dA + \dots$$

$S_{ACS} = \int A_n dA$
 $S_{NS} = \int \frac{1}{2} F_{\mu\nu} F^{\mu\nu} +$

$$S_{ACS} = \int A_n dA$$

1 term A

$$\frac{\delta S^{NS}}{\delta A_n^{(2)}} = \epsilon^{abcd} F_{bc} = 0$$

$$\int dA F$$

$$S = \int \frac{F}{4\pi} dA + \epsilon_{abc} \tilde{A}_a \tilde{A}_b \tilde{A}_c$$

$$\frac{\delta S}{\delta A_n^I} = \epsilon^{abcd} F_{bc} \delta A_n^I - dA + \tilde{A}_a \tilde{A}_b \tilde{A}_c$$

$$S_{NS} = \int A_n dA$$

1 form A

$$\frac{\delta S_{NS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$\int \delta A F$$

$$S = \int [\frac{1}{2} \epsilon^{nk} F_{nk} F_{lc} F_{lc}]$$

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kc} \epsilon^{lk} F_{lc} - \dots$$



$$S_{ACS} = \int A_n dA$$

1 form A

$$\frac{\delta S^{NS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$\int J_n F$$

$$S = \int [\frac{1}{2} \epsilon^{nk} F_{nk} F_{lc} + \dots]$$

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kc} - \dots$$

$$S^{CS} = \int A_n dA$$

1 form A

$$\frac{\delta S^{NS}}{\delta A_n^{(2)}} = \epsilon^{nk} F_{kc} = 0$$

$$\int dA F$$

$$S^{CS} = \int \left[\frac{1}{2} \epsilon^{nk} F_{nk} + \frac{1}{2} \epsilon^{nk} \tilde{F}_{nk} \right]$$

$$\frac{\delta S}{\delta A_n^{(2)}} = \epsilon^{nk} F_{kc} = dA + \tilde{F}_{nk}$$

$$S^{NS} = \int A_n dA \quad | \text{form } A$$

$$\frac{\delta S^{NS}}{\delta A_n} = \epsilon^{nbc} F_{bc} = 0$$

$$S^{CS} \neq \int A_n F^n$$

$$S^{CS} = \int \left[\frac{1}{2} \epsilon^{abc} \dot{A}_a \dot{A}_b \dot{A}_c + \dots \right]$$

$$\frac{\delta S}{\delta A_n} = \epsilon^{nbc} F_{bc} = \dots$$

$$S^{NS} = \int A_n dA \quad | \text{form } A$$

$$= S_{MF}$$

$$\frac{\delta S^{NS}}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$S^{CS} \neq \int A_n F^2$$

$$S^{CS} = \int \left[\frac{1}{2} \epsilon^{nk} F_{nk}^2 + \frac{1}{3} \epsilon^{nk} A_n A_k A^c \right]$$

$$\frac{\delta S}{\delta A_n} = \epsilon^{nk} F_{kc} = 0$$

$$S^{CS} = \int A \wedge dA \quad | \text{form } A$$

$$\frac{\delta S^{MS}}{\delta A_\mu^{(n)}} = \epsilon^{\mu\nu} F_{\nu\lambda} = 0$$

$$S^{CS} \neq \int A \wedge F^2$$

$$S^{CS} = \int [A \wedge F + \frac{1}{2} \epsilon_{ijk} A^i A^j A^k]$$

$$\frac{\delta S}{\delta A_i^a} = \epsilon^{ijkl} F_{jk}^a = 0$$

Chern-Simons Theory