Title: Introduction to quantum gravity - Part 8

Date: Feb 08, 2006 08:00 PM

URL: http://pirsa.org/06020014

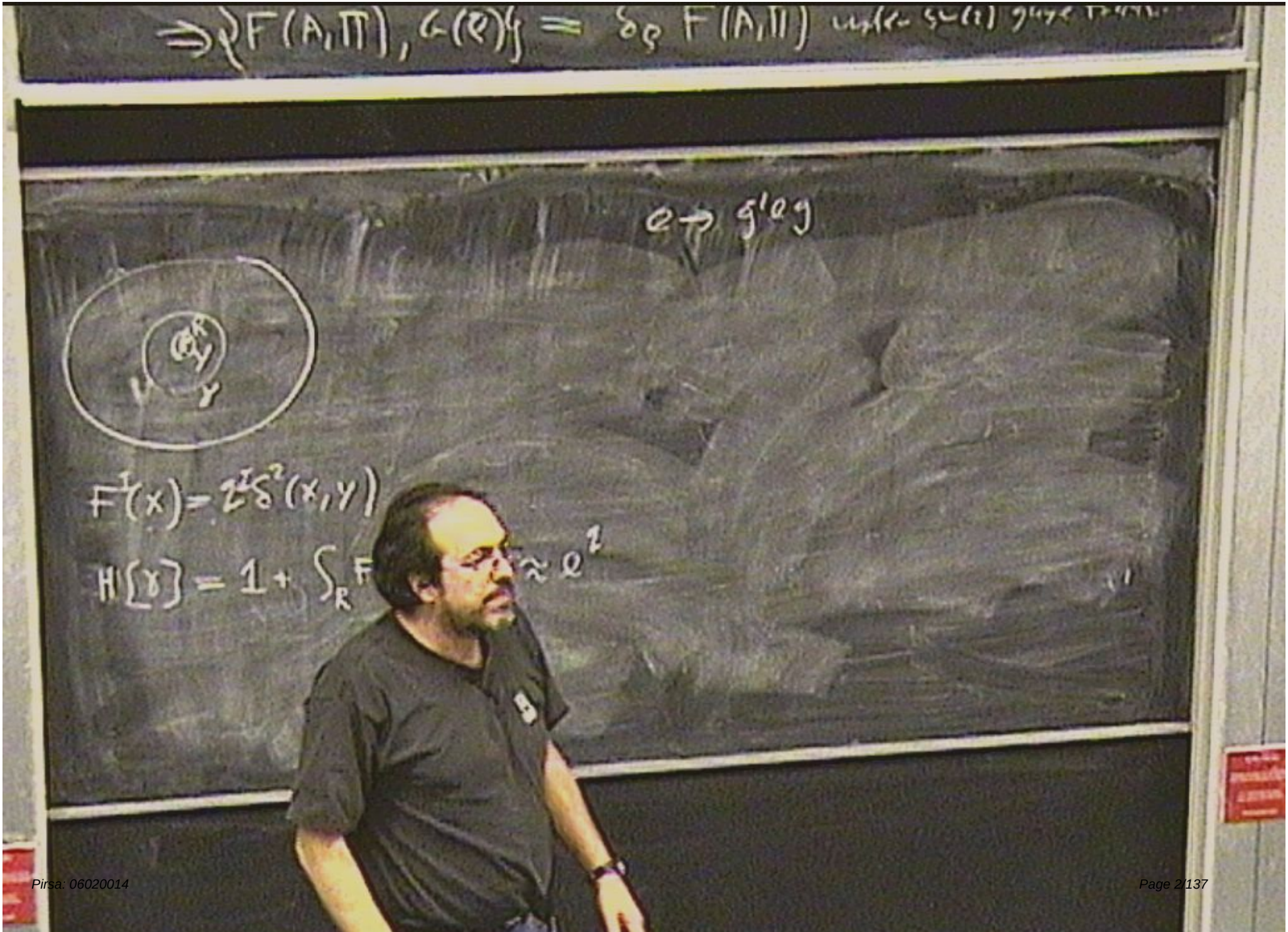
Abstract: This is an introduction to background independent quantum theories of gravity, with a focus on loop quantum gravity and related approaches.

Basic texts:

-Quantum Gravity, by Carlo Rovelli, Cambridge University Press 2005 -Quantum gravityy with a positive cosmological constant, Lee Smolin, hep-th/0209079
-Invitation to loop quantum gravity, Lee Smolin, hep-th/0408048 -Gauge fields, knots and gravity, JC Baez, JP Muniain

Prerequisites:

-undergraduate quantum mechanics
-basics of classical gauge field theories
-basic general relativity
-hamiltonian and lagrangian mechanics
-basics of lie algebras

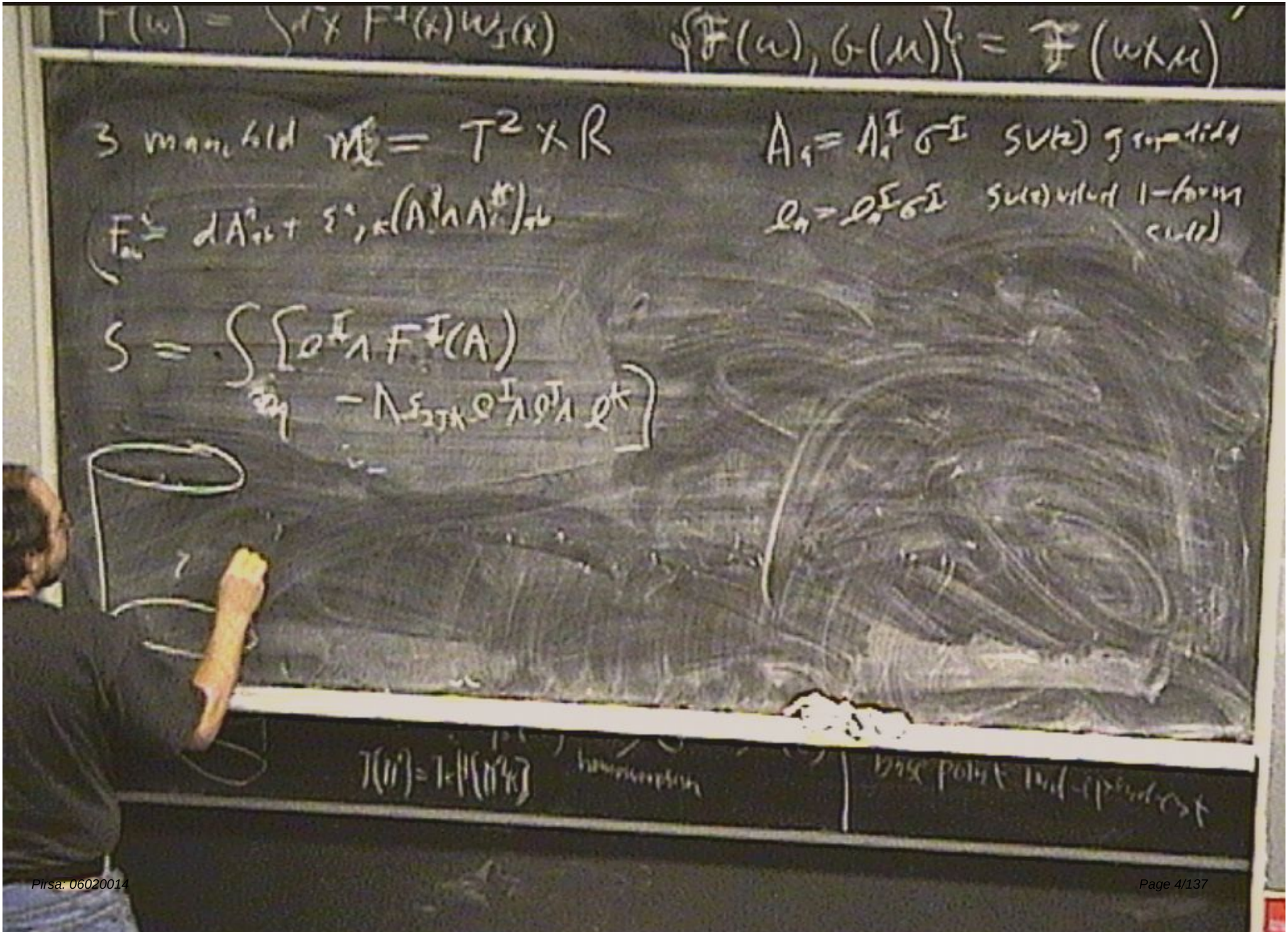

$\Rightarrow \{F(A,\Pi), G(\varphi)\} = \delta_\varphi F(A,\Pi)$
$\varphi \to g^{-1}\varphi g$
$F^{-1}(x) = Z^{-1}\delta^2(x,y)$
$H[\gamma] = 1 + \int_R F \approx e^{\int}$

$\Rightarrow \{F(A,\Pi), G(\varrho)\} = \delta_\varrho F(A,\Pi)$ under SU(1) gauge transf.

$\varrho \to g^{-1} \varrho g$

$F^1(x) = \Sigma^I \delta^2(x,y)$

$H[\gamma] = 1 + \int_R F^1 \sim 1 + t^I \approx e^t$


3 manifold
S =

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab}^i = dA^i_{ab} + \varepsilon^i{}_{jk}(A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int_{\Sigma_\eta} \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right]$$

$T(\Pi^2) = \mathrm{Tr}\,\Pi(\Pi^2 K)$ homomorphism $\qquad$ base point indep. test

$F(\omega) = \int d^3x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab} = dA^a_{ab} + \varepsilon^a{}_{jk} (A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form

$S = \int_{\Sigma_\eta} \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right]$

$J(\Pi) = \mathrm{Tr}\, H(\Pi^2 K)$ homomorphism $\qquad$ base point indep. cst

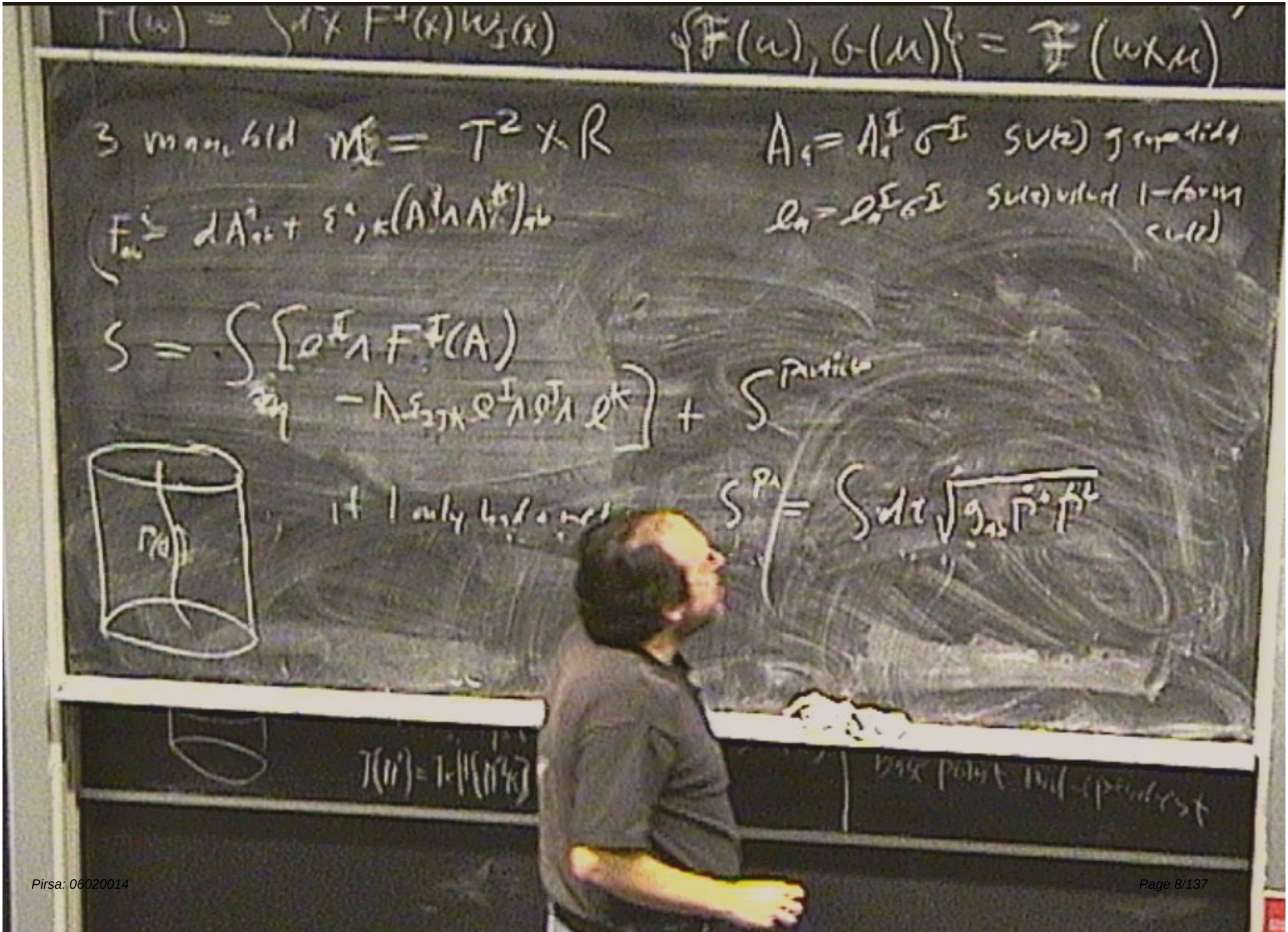
$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab} = dA^i_{ab} + \varepsilon^i{}_{jk} (A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particles}}$$

$J(\Pi^i) = \mathrm{Tr}\, \Pi^i [\Pi^i, \Pi^k]$ homomorphism $\qquad$ base point multiplicity


3 manifold $M_3 = T^2 \times R$
$A_i = A_i^I \sigma^I$ SU(2) gauge field
$\phi_n = \phi_n^I \sigma^I$ SU(2) valued 1-form
$S = \int [\phi^I \wedge F^I(A) - \Lambda \varepsilon_{IJK} \phi^I \wedge \phi^J \wedge \phi^K] + S^{particle}$
$S^{pt} = \int ds \sqrt{g_{ab} \dot{r}^a \dot{r}^b}$
$r(s)$

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab}^i = dA^i_{ab} + \varepsilon^i{}_{jk}(A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$

if I only had a vertex $\qquad S^{pa} = \int d\tau \sqrt{g_{ab}\, \dot{\Gamma}^a \dot{\Gamma}^b}$

$T(\Gamma) = \mathrm{Tr}\, H[\Gamma^a R]$ homomorphism $\qquad$ base point mod-(pinch-cut)

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab}^i = dA^i_{ab} + \varepsilon^i{}_{jk}(A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(1)

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$

$$S = \int dt\, \dot{\Gamma}(t)$$

$T(\Gamma) = \mathrm{Tr}\, H(\Gamma^2 K)$ homomorphism $\qquad$ base point independent

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

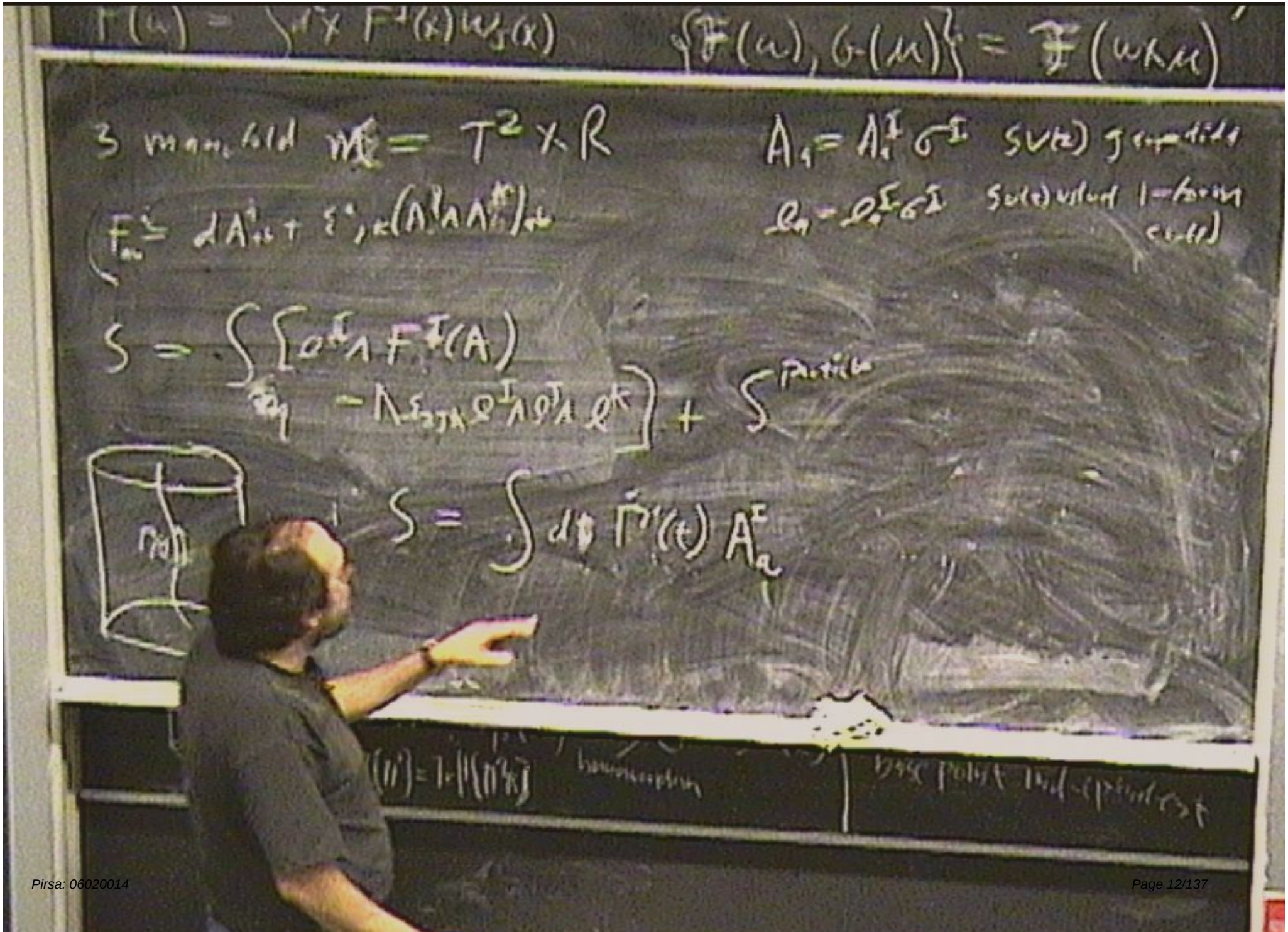
3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

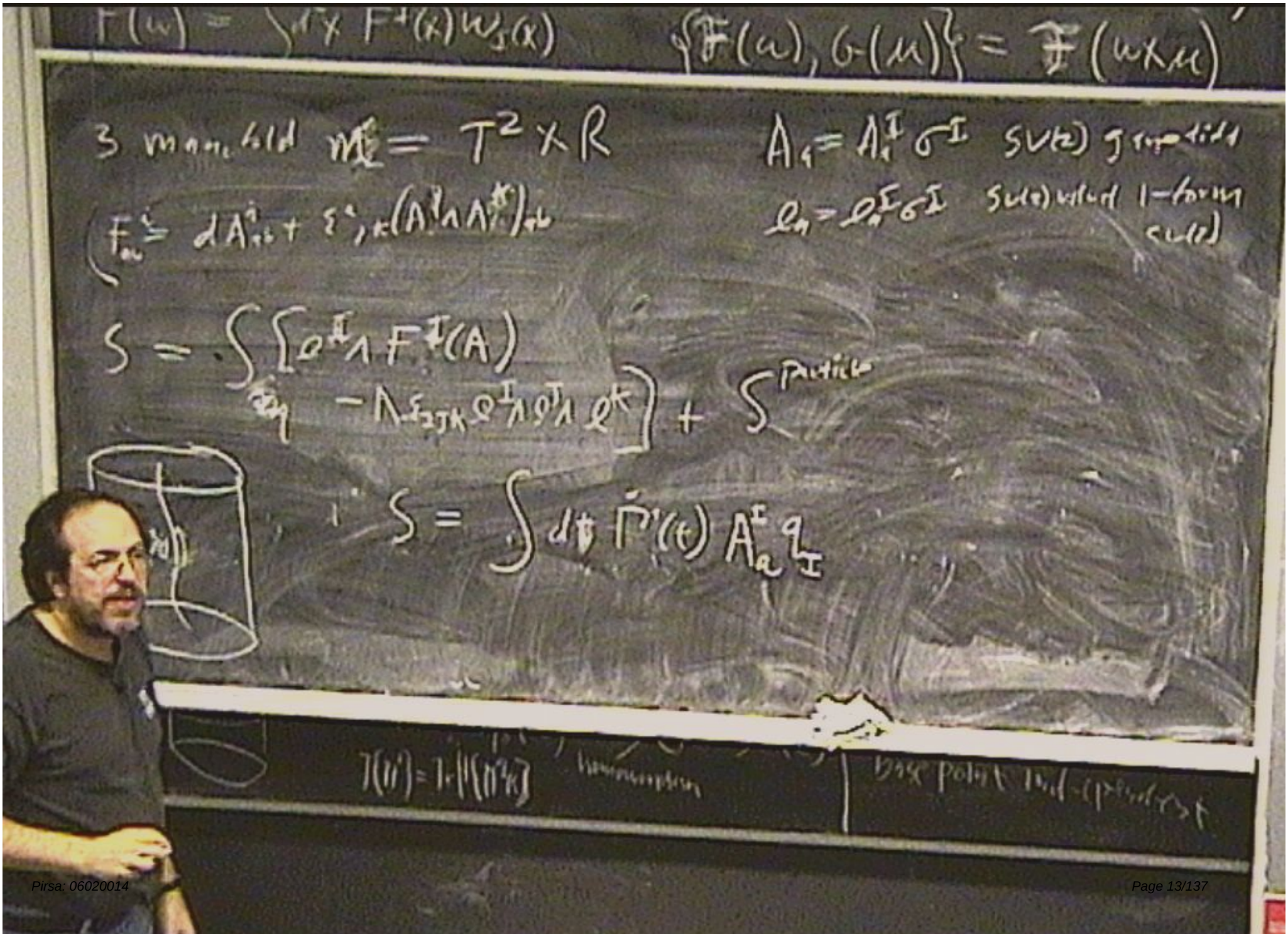
$F_{ab} = dA^a_{ab} + \varepsilon^a{}_{,k}(A^i \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$

$$S = \int dt\, \dot{\Gamma}^a(t)\, A_a$$

$T(\Gamma) = \mathrm{Tr}\, H[\Gamma^a \times]$ homomorphism $\qquad$ base point mid-point rest


3 manifold
SU(2) gauge fields
SU(2) valued 1-form
S = ∫ [e^I ∧ F^I(A)
S = ∫ dt Γ^I(t) A^I_a


3 manifold
SU(2) gauge field
SU(2) valued 1-form
S = ∫ dt Γ̇^I(t) A^I_a q_I

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab} = dA^a_{ab} + \varepsilon^a{}_{,k}(A^i \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$

$$S = \int dt\, \dot{\Gamma}^a(t)\, A_a^I\, q_I(t)$$

$J(\Gamma) = \mathrm{Tr}\, H(\Gamma^a K)$ homomorphism $\qquad$ base point indep. of Γ

$F(\omega) = \int d^3x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab} = dA_{ab} + \varepsilon^a{}_{jk} (A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$

$$S = \int dt\, \dot{\Gamma}(t) \left[A_a^I q_I(t) + e_a^I P_I \right]$$

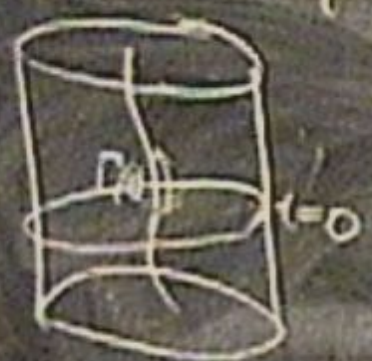
$T(\Gamma) = \mathrm{Tr}\, H(\Gamma^a k)$ homomorphism $\qquad$ base point indep. test

$F(\omega) = \int d^2x\, F^I(x)\, \omega_I(x)$ $\qquad \{F(\omega), G(\mu)\} = F(\omega \times \mu)$

3 manifold $M = T^2 \times R$ $\qquad A_a = A_a^I \sigma^I$ SU(2) gauge field

$F_{ab}^i = dA^i_{ab} + \varepsilon^i{}_{jk}(A^j \wedge A^k)_{ab}$ $\qquad e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$$S = \int_{\Sigma_\eta} \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$



$$S = \int dt\, \dot{\Gamma}^a(t) \left[A_a^I\, q_I(t) + e_a^I\, P_I \right]$$

$\mathcal{T}(M) = \mathrm{Tr}\, H(M^2 K)$ homomorphism

base point mid-(periodic) $\to t$

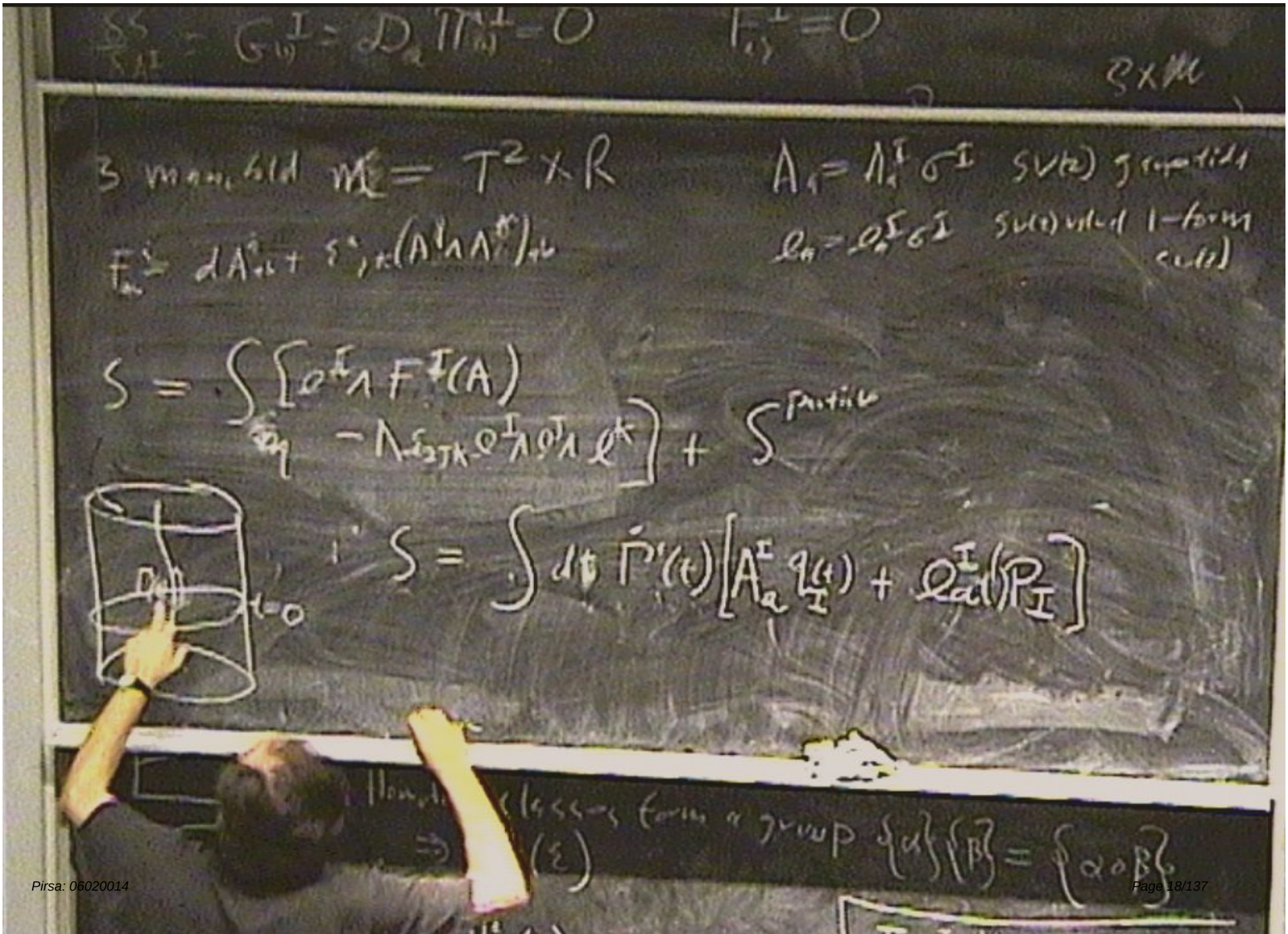
$\Rightarrow \{F(A,\Pi), G(\rho)\} = \delta_\rho F(A,\Pi)$

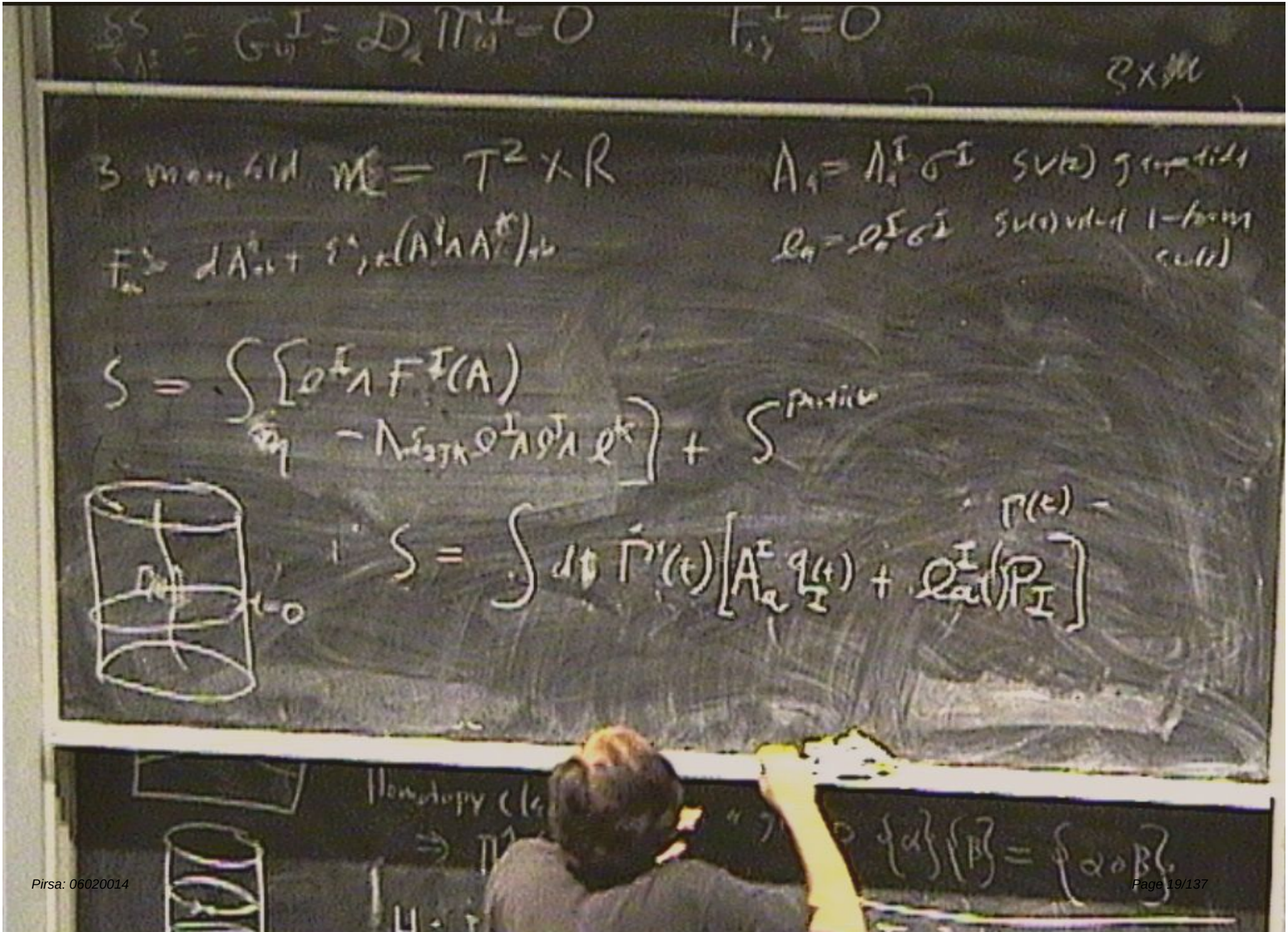
$e \to g' e g$

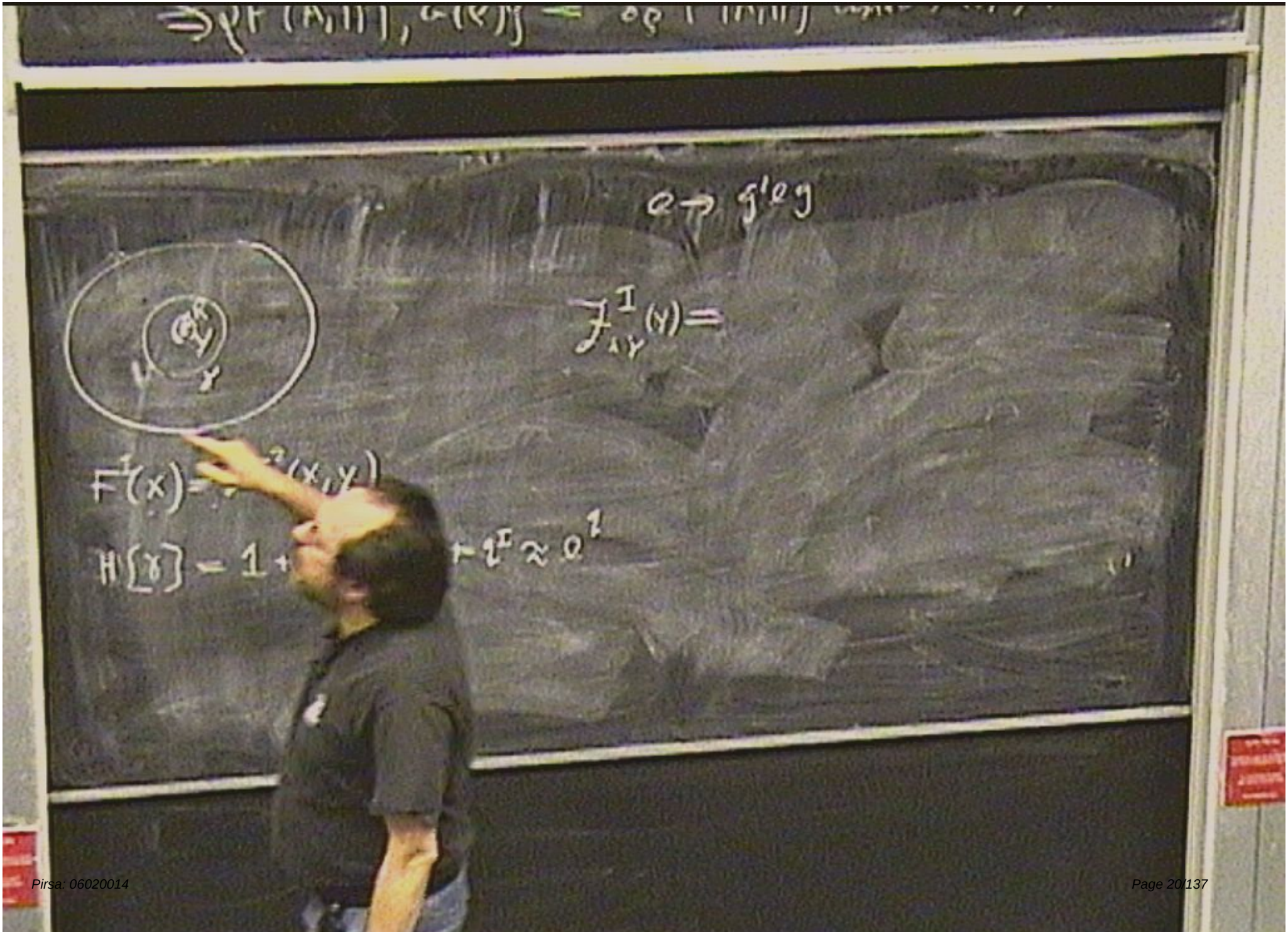
$\mathcal{F}^I_{xy}(y) =$

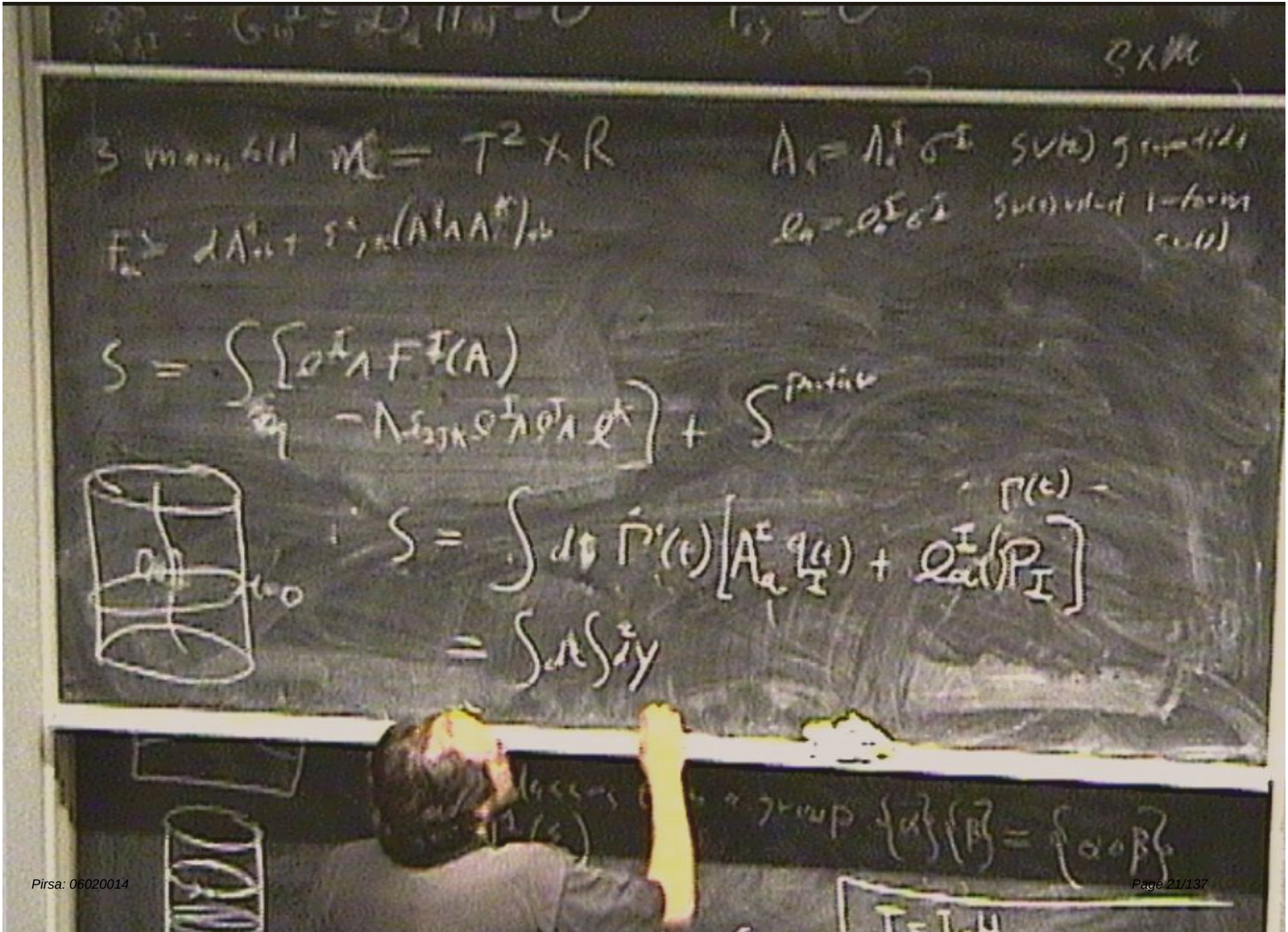
$F^I(x) = t^I \delta^2(x,y)$

$H[\gamma] = 1 + \int_R F \approx 1 + t^I \approx e^t$

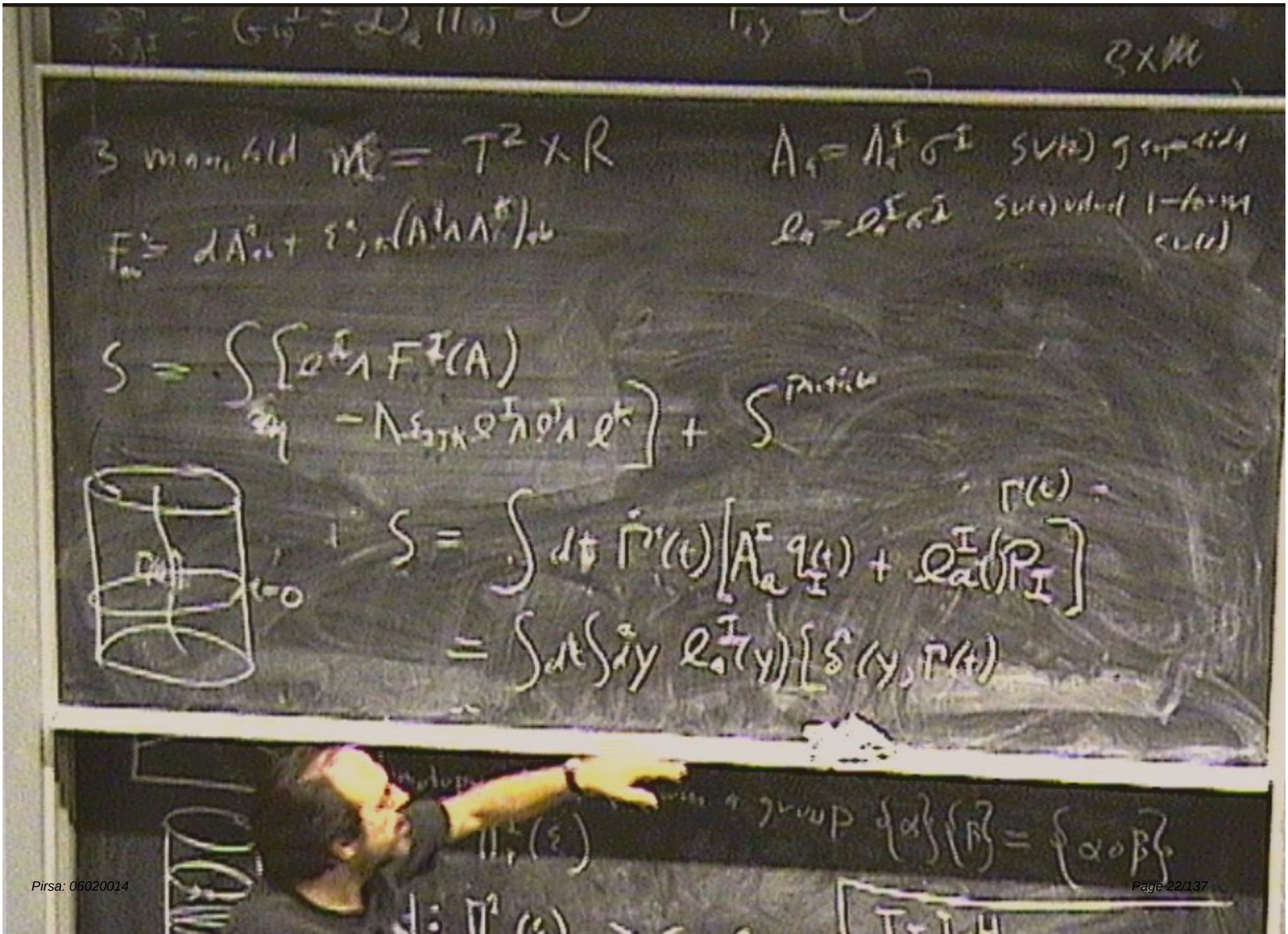

3 manifold $M = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{particle}$
$S = \int dt\, \dot{\Gamma}^i(t) [A_a^I q_I(t) + e_a^I(t) P_I]$
$t=0$
classes form a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$

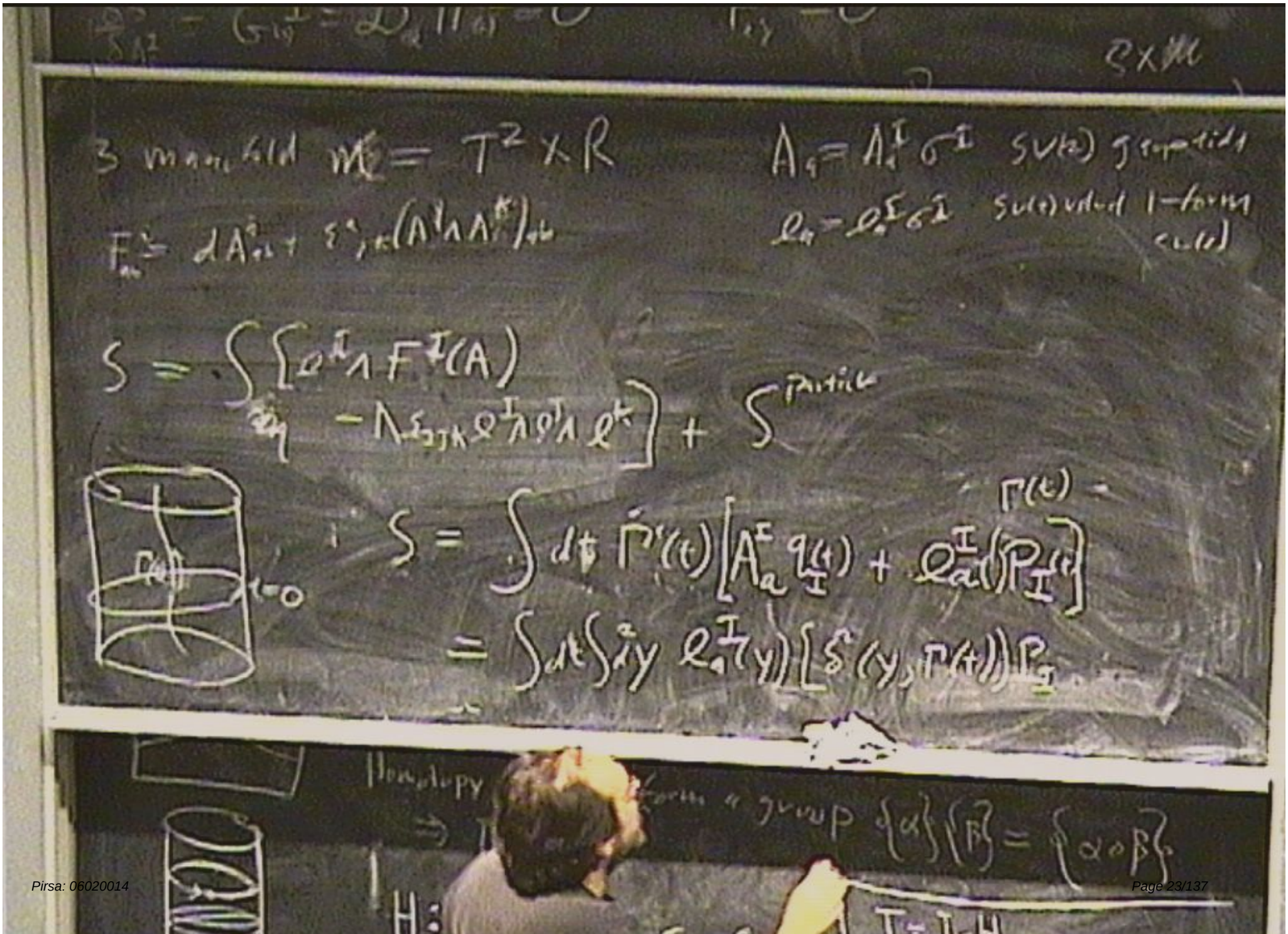

3 manifold $M = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{particle}$
$S = \int dt\, \dot{\Gamma}^I(t) [A_a^I q_I(t) + e_a^I(t) P_I]$
$\Gamma(t)$
$t=0$
Homotopy cl
$\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$





Pirsa: 06020014


3 manifold $\mathcal{M} = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$F_{ab} = dA_{ab} + \epsilon^a{}_{jk}(A^j \wedge A^k)_{ab}$
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{\text{particle}}$
$S = \int dt\, \dot{\Gamma}^a(t) [A_a^I q_I(t) + e_a^I(\Gamma(t)) P_I]$
$= \int dt \int d^2y\, e_a^I(y) [\delta(y, \Gamma(t))$
a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$


3 manifold $\mathcal{M} = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{particle}$
$S = \int dt\, \dot{\Gamma}^a(t) [A_a^I q_I(t) + e_a^I(\Gamma(t)) P_I(t)]$
$= \int dt \int d^2y\, e_a^I(y) [\delta^a(y, \Gamma(t)) P_I$
Homotopy
$\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$

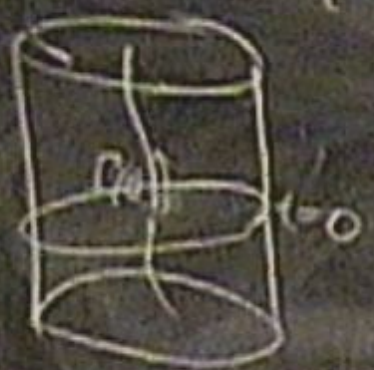
3 manifold $M = T^2 \times R$

$A_a = A_a^I \sigma^I$ SU(2) g superfield

$e_a = e_a^I \sigma^I$ SU(2) valued 1-form SU(2)

$F_{ab}^i = dA^i_{ab} + \varepsilon^i{}_{jk}(A^j \wedge A^k)_{ab}$

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \varepsilon_{IJK} e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$



$$S = \int dt\, \Gamma^I(t) \left[A_a^I q_I(t) + e_a^I(\Gamma(t)) P_I(t) \right]$$

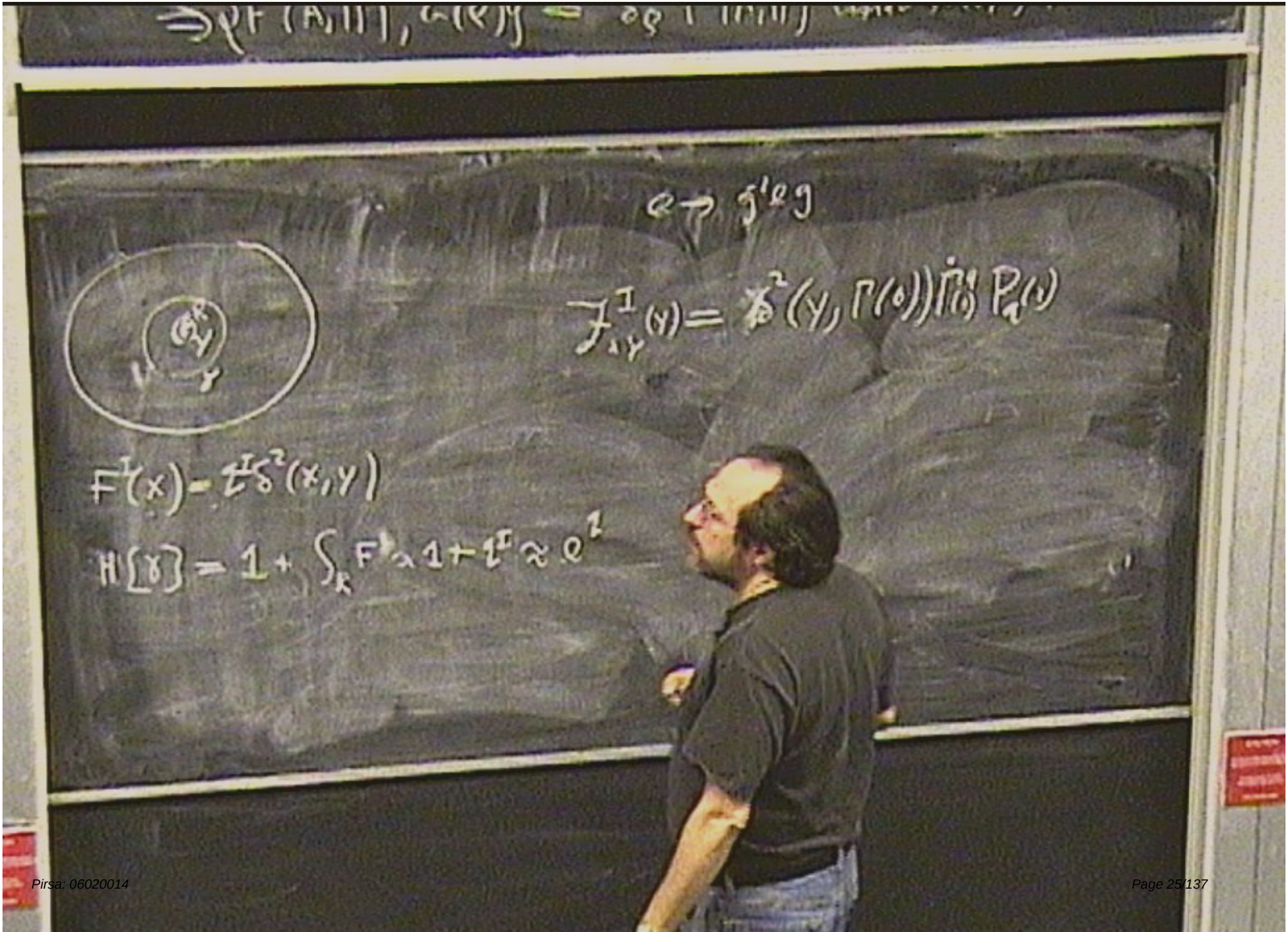
$$= \int dt \int d^2y\, e_a^I(y) \left[\delta^2(y, \Gamma(t)) P_I(t) + A_a^I(y) \left[\delta^2(y, \Gamma(t)) q_I(t) \right. \right.$$

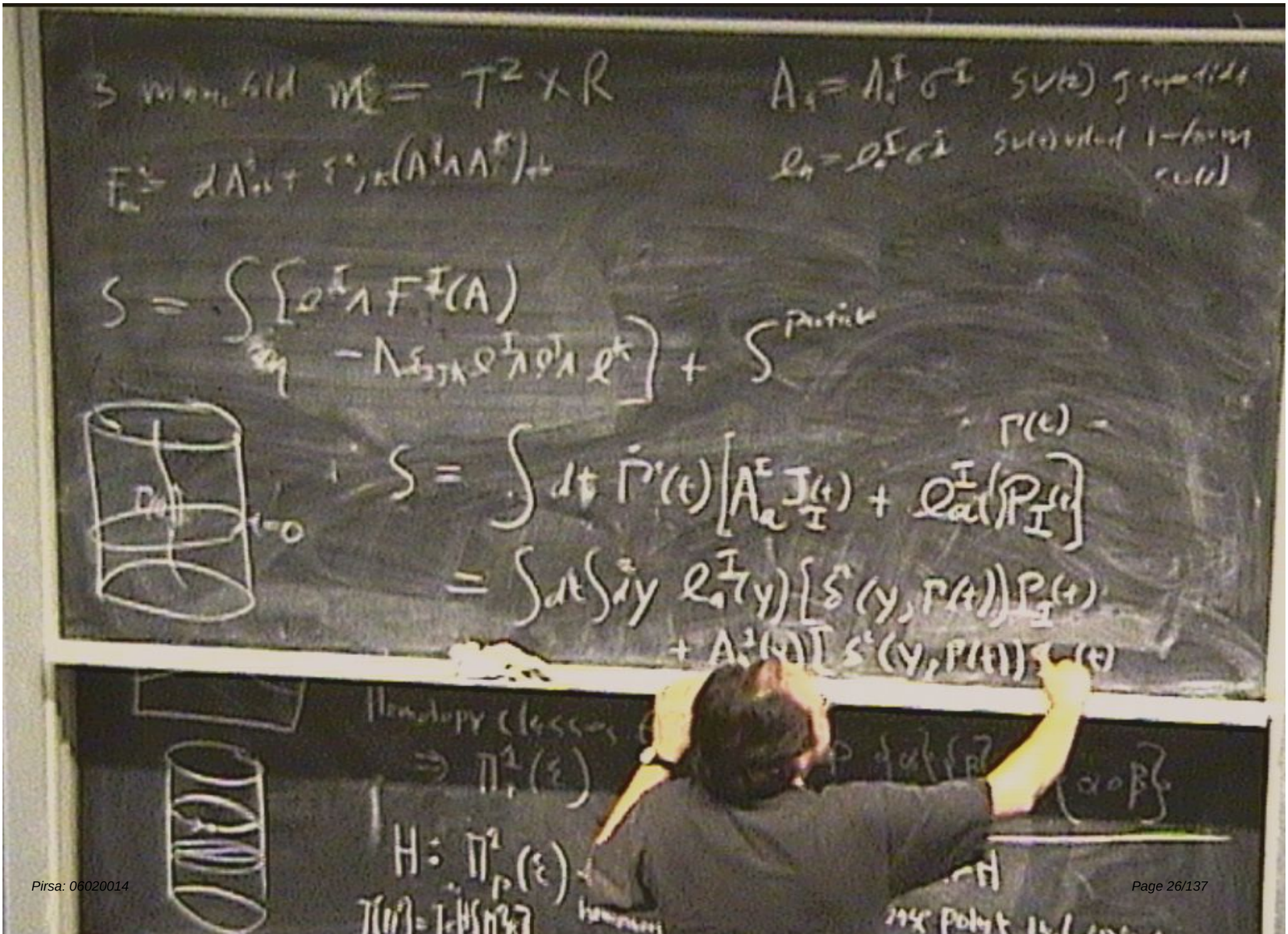
Homotopy classes form a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$

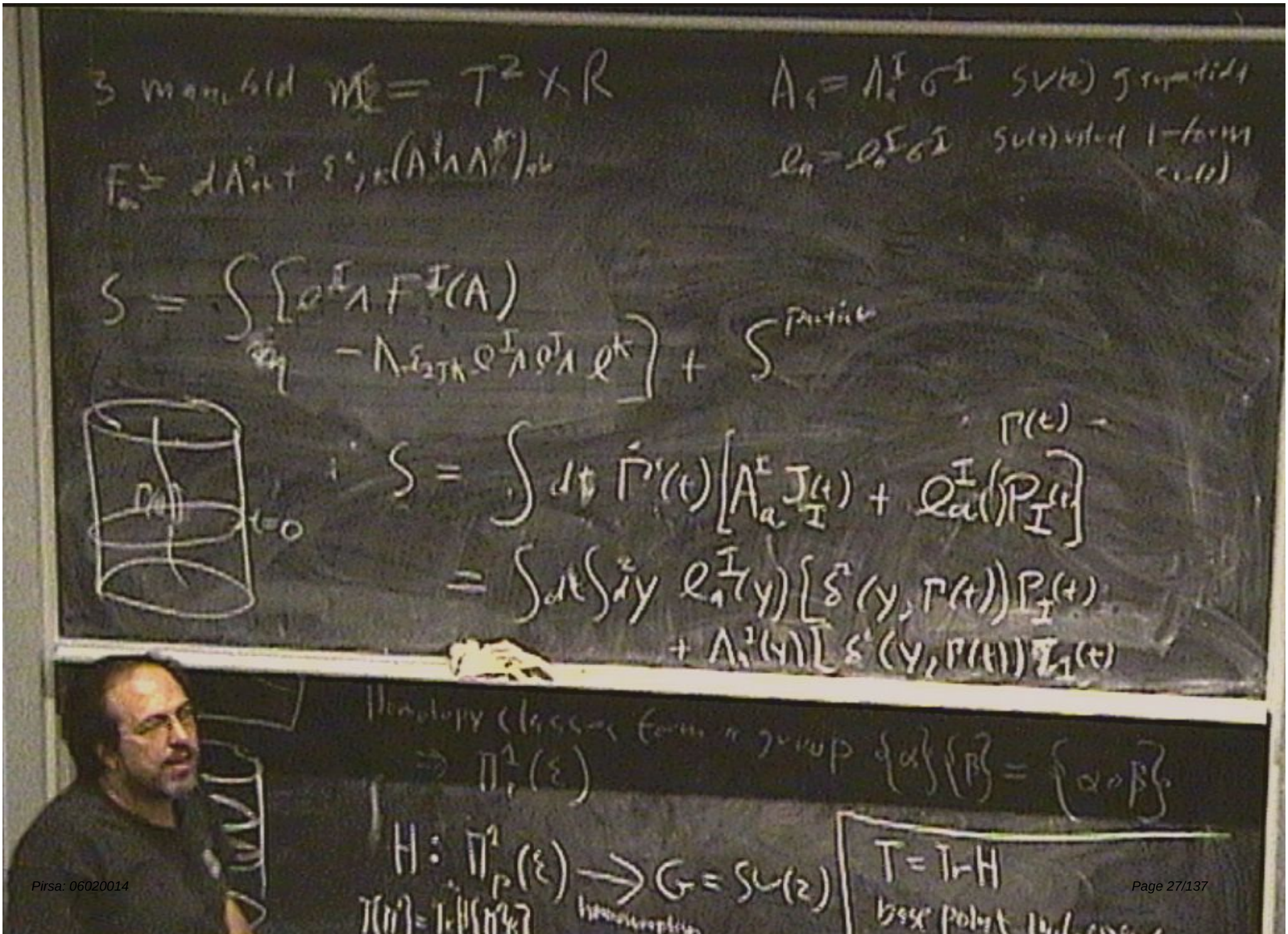
$\Rightarrow \Pi_1^1(\Sigma)$

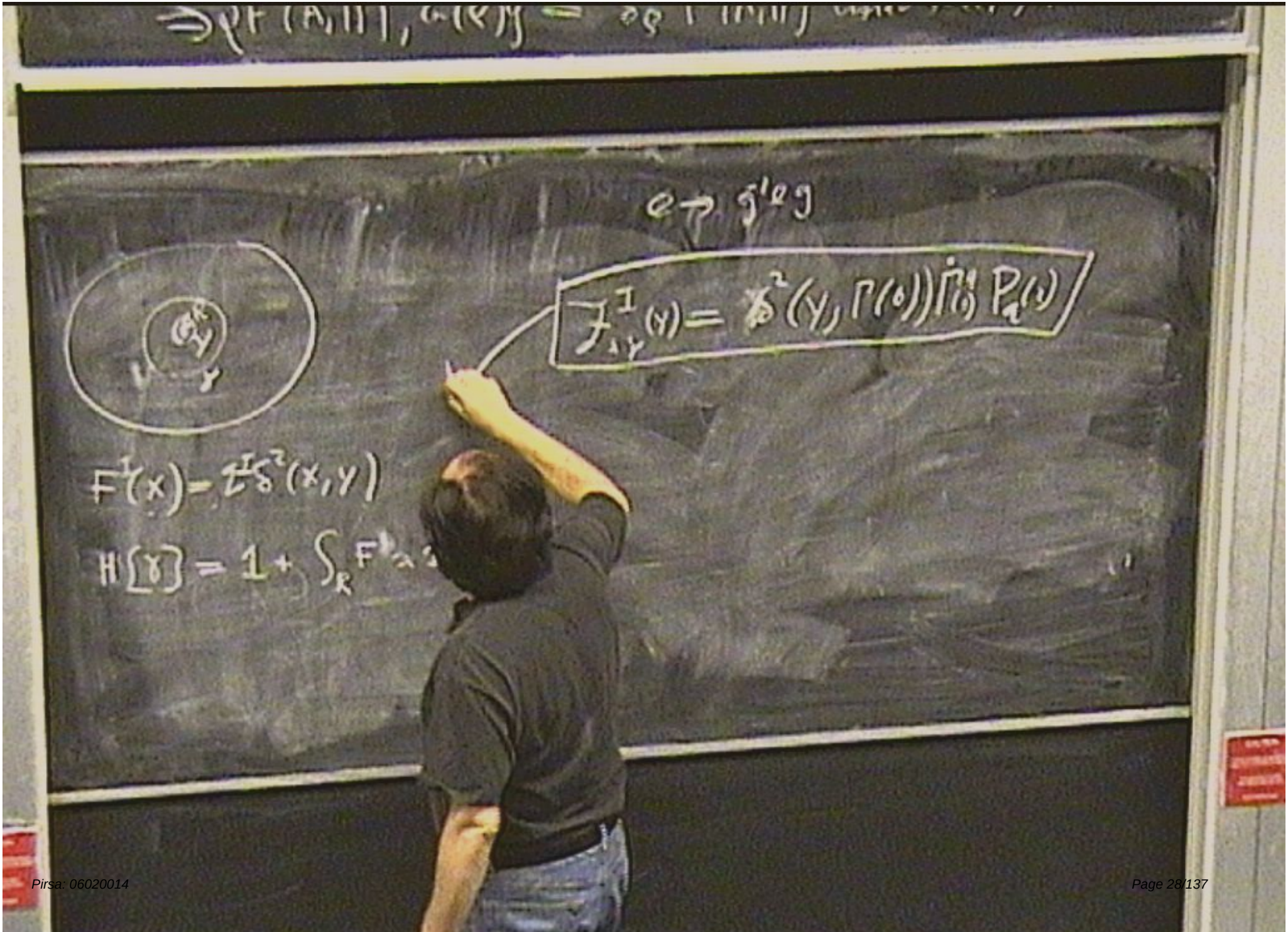
$H : \Pi^1(\Sigma) \to$

$I = I_0 H$


$e \to g^{-1} e g$
$\mathcal{F}^I_{xy}(y) = \delta^2(y, \Gamma(0)) \dot{\Gamma}^a(0) P_a(0)$
$F^I(x) = t^I \delta^2(x, y)$
$H[\gamma] = 1 + \int_R F^I \approx 1 + t^I \approx e^t$


3 manifold $M = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{particle}$
$S = \int dt\, \dot{\Gamma}^a(t) [A_a^I J_I(t) + e_a^I(P_I(t)]$
$\Gamma(t)$
$= \int dt \int d^2y\, e_a^I(y) [\delta^2(y, \Gamma(t)) P_I(t) + A_a^I(y) [\delta^2(y, \Gamma(t)) \delta_I(t)$
Homotopy classes
$\Rightarrow \Pi_1^1(\xi)$
$H: \Pi_p^1(\xi)$


3 manifold $\mathcal{M} = T^2 \times R$
$A_a = A_a^I \sigma^I$ SU(2) gauge field
$e_a = e_a^I \sigma^I$ SU(2) valued 1-form
$F_{ab} = dA^i_{ab} + \epsilon^i_{jk}(A^j \wedge A^k)_{ab}$
$S = \int [e^I \wedge F^I(A) - \Lambda \epsilon_{IJK} e^I \wedge e^J \wedge e^K] + S^{particle}$
$S = \int dt\, \dot{\Gamma}^i(t) [A_a^I J_I(t) + e_a^I(\Gamma(t)) P_I(t)]$
$= \int dt \int d^2y\, e_a^I(y) [\delta^2(y, \Gamma(t)) P_I(t) + A_a^I(y) [\delta^2(y, \Gamma(t)) J_I(t)$
$t=0$
Homotopy classes form a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$
$\Rightarrow \Pi^1_p(\mathcal{E})$
$H : \Pi^1_p(\mathcal{E}) \longrightarrow G = SU(2)$
$T = \mathrm{Tr}\, H$


$e \to g^{-1} e g$
$\mathcal{F}^I_{xy}(y) = \delta^2(y, \Gamma(0))\dot{\Gamma}^q(0) P_q(0)$
$F^I(x) = \mathcal{F}^I \delta^2(x, y)$
$H[\gamma] = 1 + \int_R F$

$\Rightarrow \{F(A,\Pi), G(\rho)\} = \delta_\rho F(A,\Pi)$

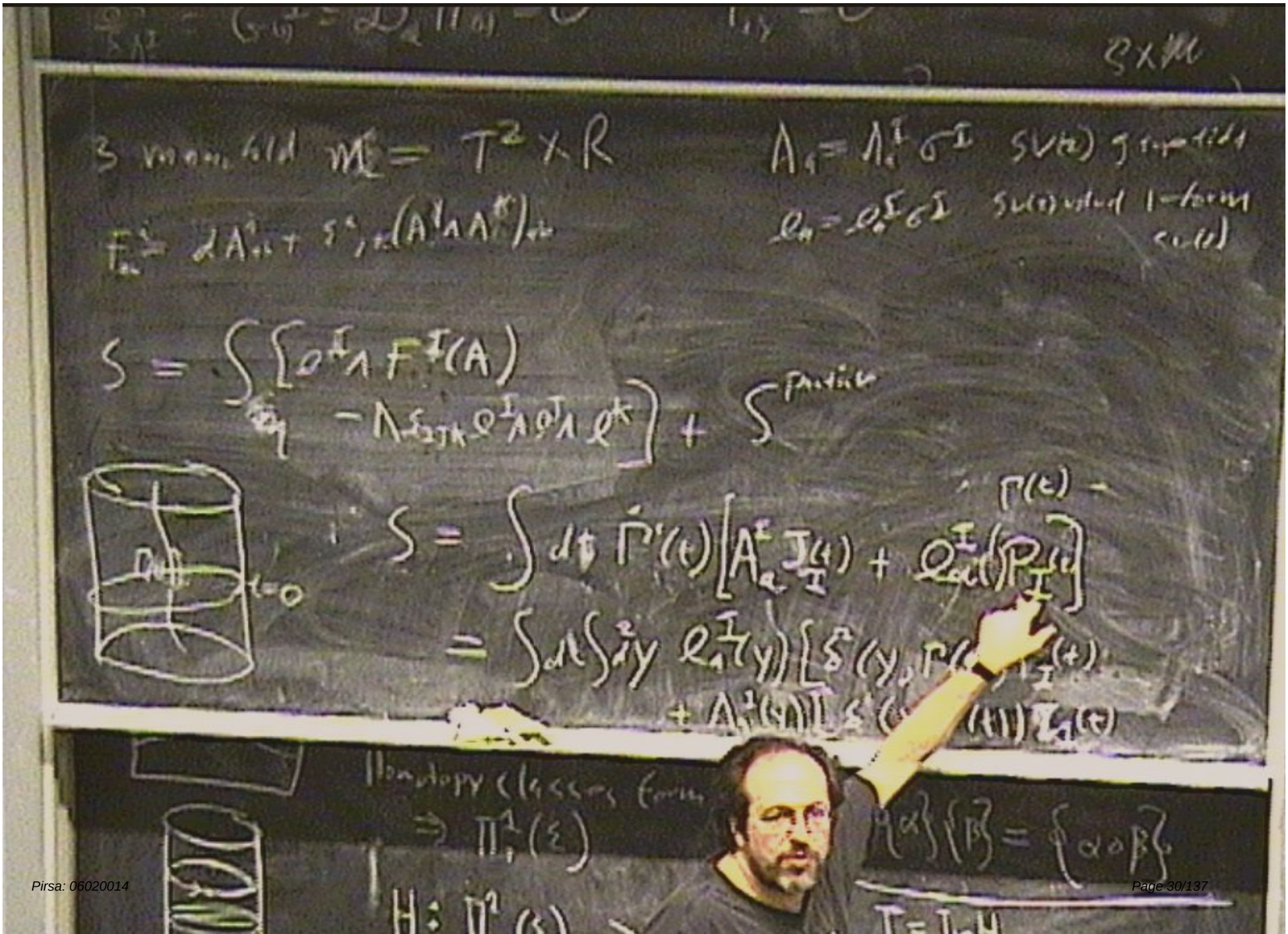
$e \to g' e g$

$$\boxed{\mathcal{F}^I_{xy}(y) = \delta^2(y, \Gamma(0))\, \dot{\Gamma}^q(0)\, P_q(0)}$$

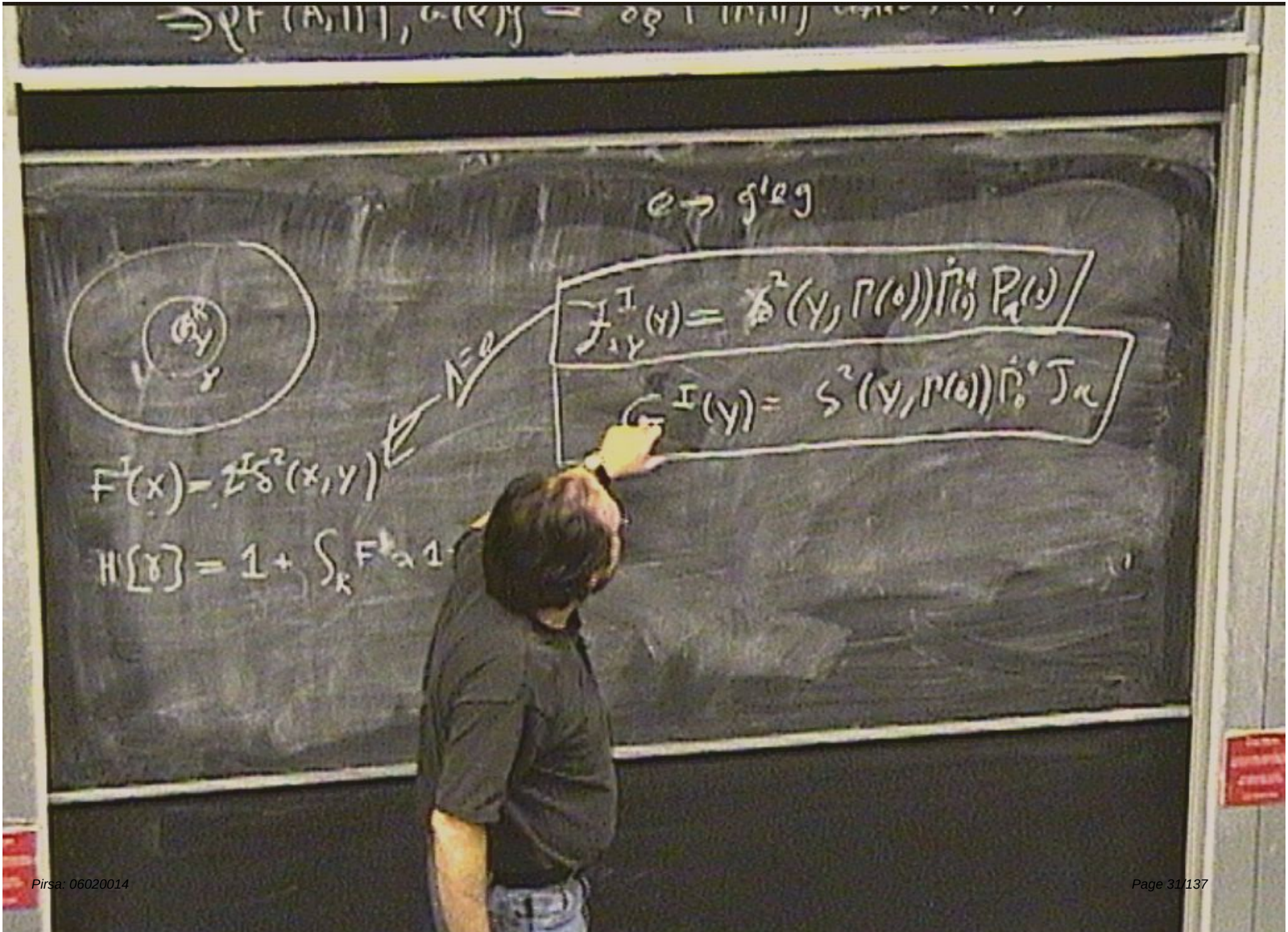
$\Lambda = 0$

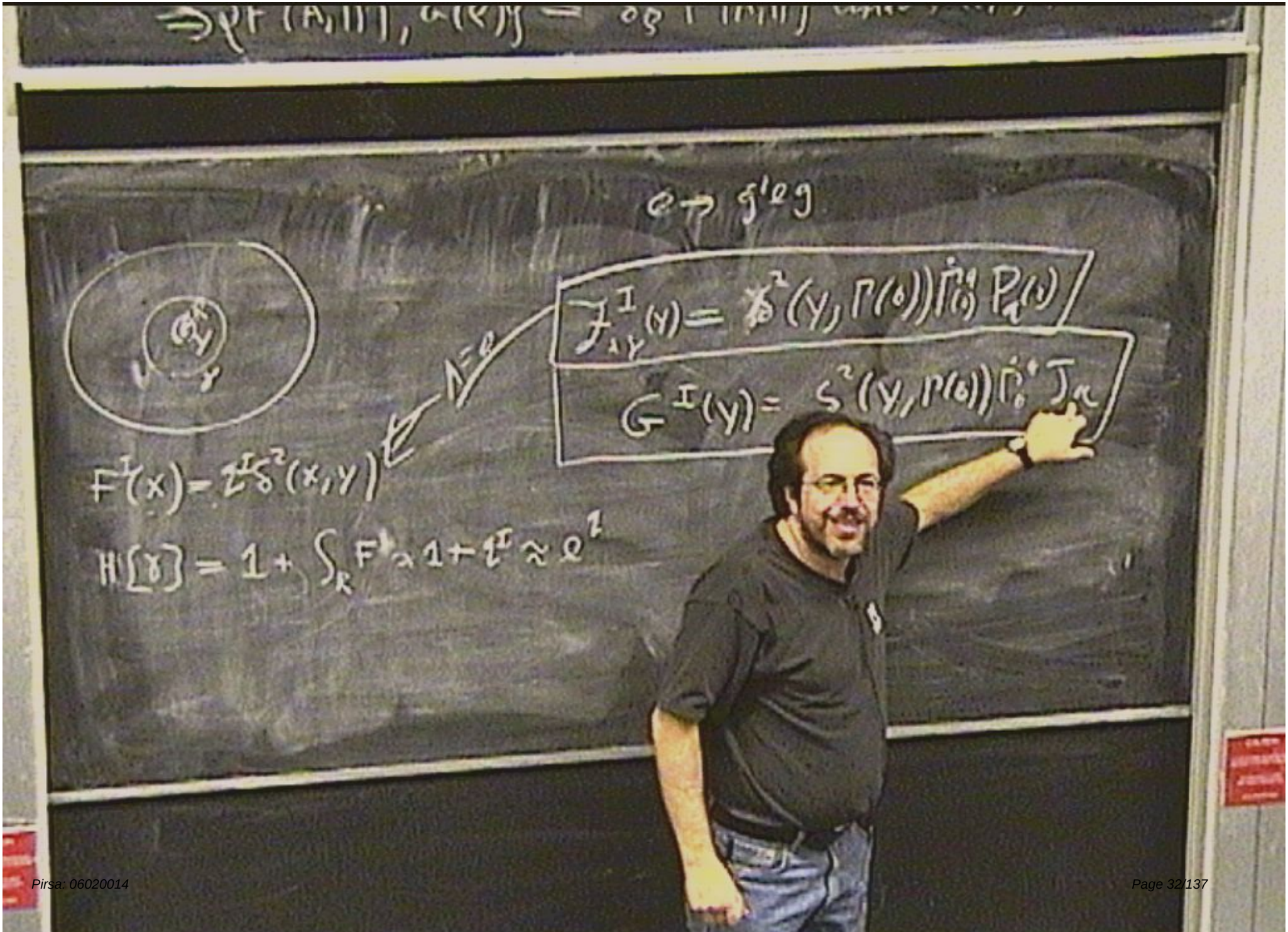
$$F^I(x) = t^I \delta^2(x,y)$$

$$H[\gamma] = 1 + \int_R F \approx 1 + t^I \approx e^t$$

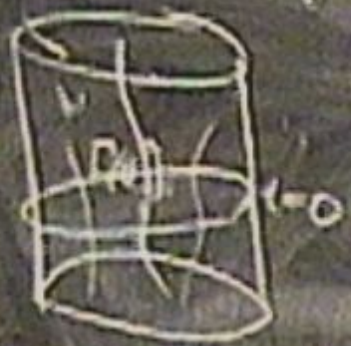


Pirsa: 06020014


e → g'eg
F^I(x) = 2^I δ^2(x,y)
H[γ] = 1 + ∫_R F^I ∝ 1


$\mathcal{J}^I_{\lambda\gamma}(\gamma) = \delta^2(\gamma, \Gamma(\sigma))\dot{\Gamma}^\lambda_\sigma P_\alpha(\gamma)$
$G^I(\gamma) = \delta^2(\gamma, \Gamma(\sigma))\dot{\Gamma}^\alpha_\sigma J_\alpha$
$F^I(x) = Z^I \delta^2(x, \gamma)$
$H[\delta] = 1 + \int_k F^I \sim 1 + \ell^I \approx e^I$

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \epsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$



$$S = \int dt\, \dot{\Gamma}^i(t) \left[A^I_a J_I(t) + e^I_a(t) P_I(t) \right]$$

$$= \int dt \int d^2y\, e^I_a(y) \left[\delta^2(y, \Gamma(t)) P_I(t) + A^I_a(y) \left[\delta^2(y, \Gamma(t)) J_I(t) \right. \right.$$

Homotopy classes form a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$

$\Rightarrow \Pi^1_p(\Sigma)$

$H : \Pi^1_p(\Sigma) \longrightarrow G = SU(2)$

homomorphism

$T(\Pi^1) = T \cdot H(\Pi^1 K)$

$T = T \cdot H$

base point independent

$\Rightarrow \{F(A,\Pi), G(\rho)\} = \delta_\rho F(A,\Pi)$ under SU(2) gauge transf...

$e \to g^{-1} e g$

$$\boxed{\mathcal{J}^I_{xy}(y) = \delta^2(y, \Gamma(0))\, \dot{\Gamma}^a_0\, P_a(0)}$$

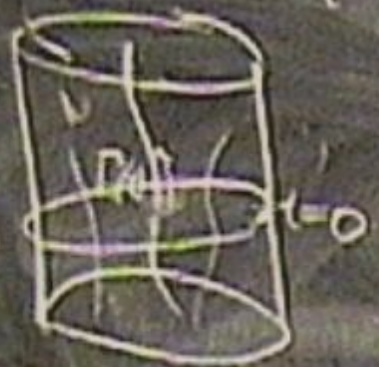
$$\boxed{G^I(y) = \delta^2(y, \Gamma(0))\, \dot{\Gamma}^a_0\, J_a}$$

$\Lambda = 0$

$F^I(x) = q^I \delta^2(x,y)$

$H[\gamma] = 1 + \int_R F^I \approx 1 + q^I \approx e^q$

$$S = \int \left[e^I \wedge F^I(A) - \Lambda \epsilon_{IJK}\, e^I \wedge e^J \wedge e^K \right] + S^{\text{particle}}$$



$$S = \int dt\, \dot{\Gamma}^a(t) \left[A^I_a J_I(t) + e^I_a(\Gamma(t)) P_I(t) \right]$$

$$= \int dt \int d^2y\, e^I_a(y) \left[\delta^2(y, \Gamma(t)) P_I(t) + A^I_a(y) \left[\delta^2(y, \Gamma(t)) \right] J_I(t) \right.$$

Homotopy classes form a group $\{\alpha\}\{\beta\} = \{\alpha \circ \beta\}$

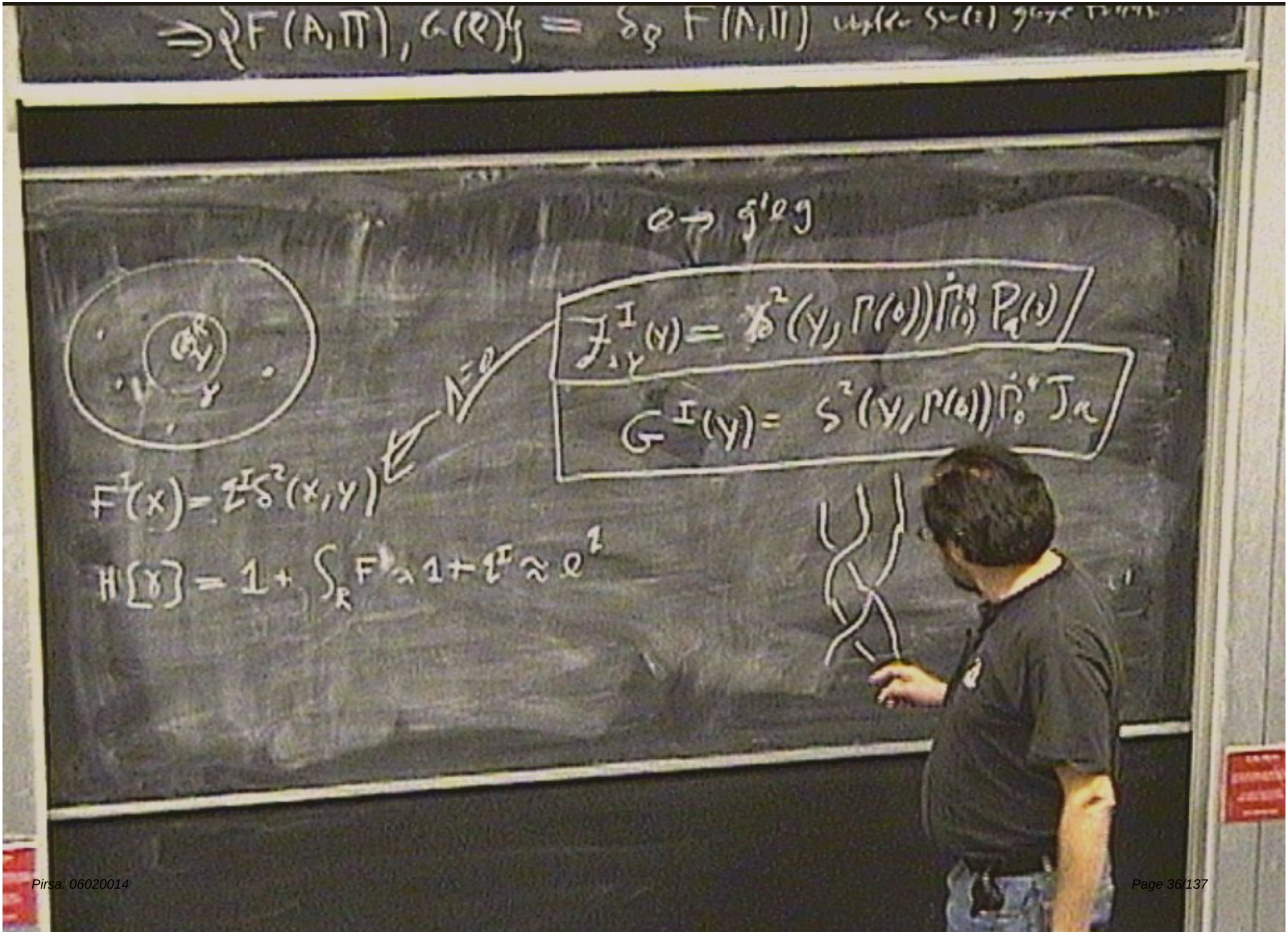
$\Rightarrow \Pi^1_p(\Sigma)$

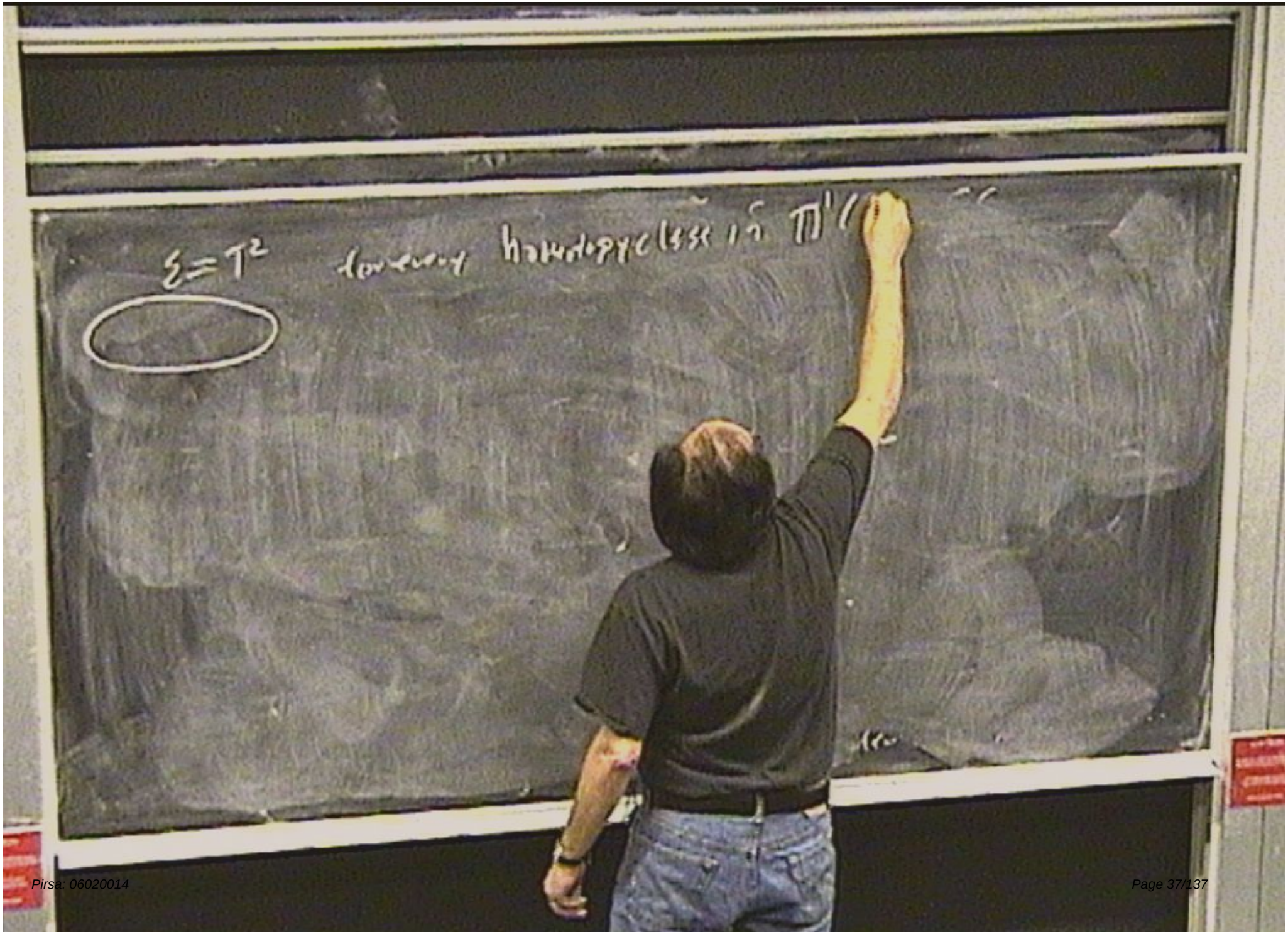
$$H : \Pi^1_p(\Sigma) \xrightarrow[\text{homomorphism}]{} G = SU(2)$$

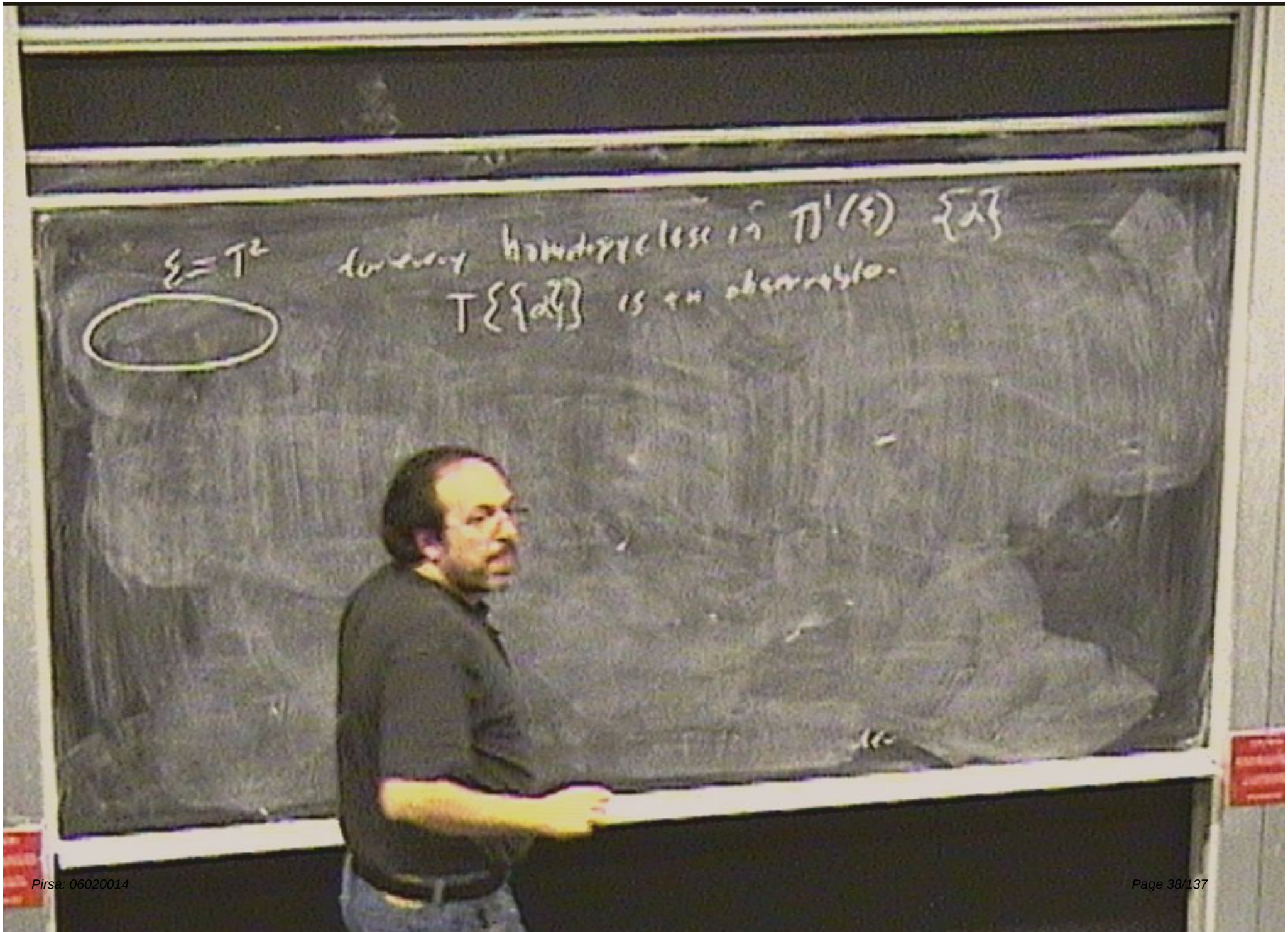
$J(\Pi^1) = \mathrm{Tr}\, H(\Pi^1 K)$

$T = \mathrm{Tr}\, H$

base point independent


$\Rightarrow \{F(A,\Pi), G(\rho)\} = \delta_\rho F(A,\Pi)$
$e \to g^{-1} e g$
$\mathcal{J}^I_{x,y}(y) = \delta^2(y, \Gamma(0)) \dot{\Gamma}^a_0 P_a(0)$
$G^I(y) = \delta^2(y, \Gamma(0)) \dot{\Gamma}^a_0 J_a$
$\Lambda = 0$
$F^I(x) = \xi^I \delta^2(x,y)$
$H[\gamma] = 1 + \int_R F \approx 1 + \xi^I \approx e^{\xi}$

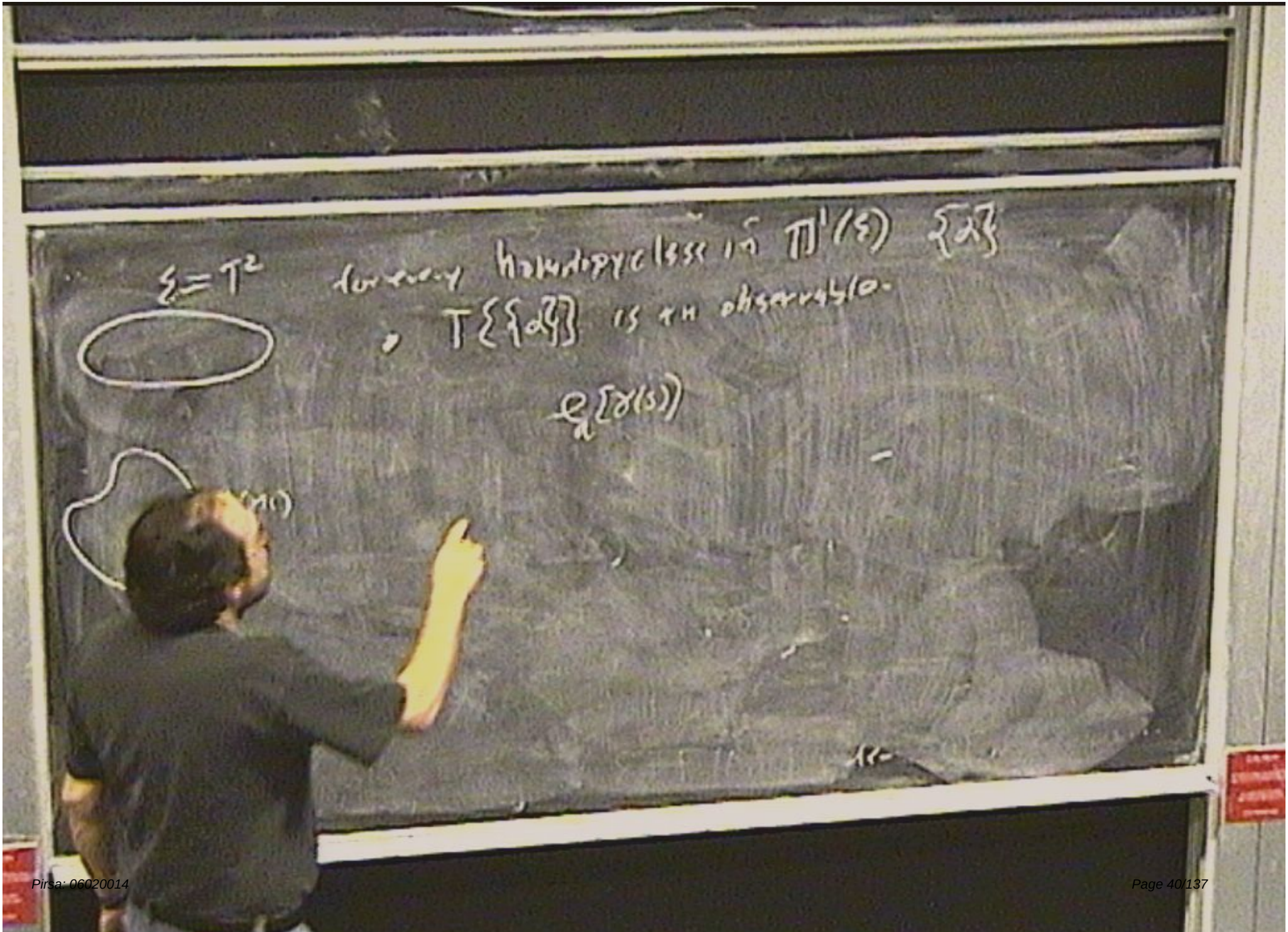

$\mathcal{E}=T^2$


$\Sigma = T^2$
$T\{\{\alpha_i\}\}$ is an observable.

$\Sigma = T^2$ fundamental homotopy class in $\pi^1(\Sigma)$ $\{\lambda\}$

• $T\{\{\alpha\}\}$ is an observable.

γ


$\Sigma = T^2$
$T\{\{\alpha_i\}\}$ is an observable.

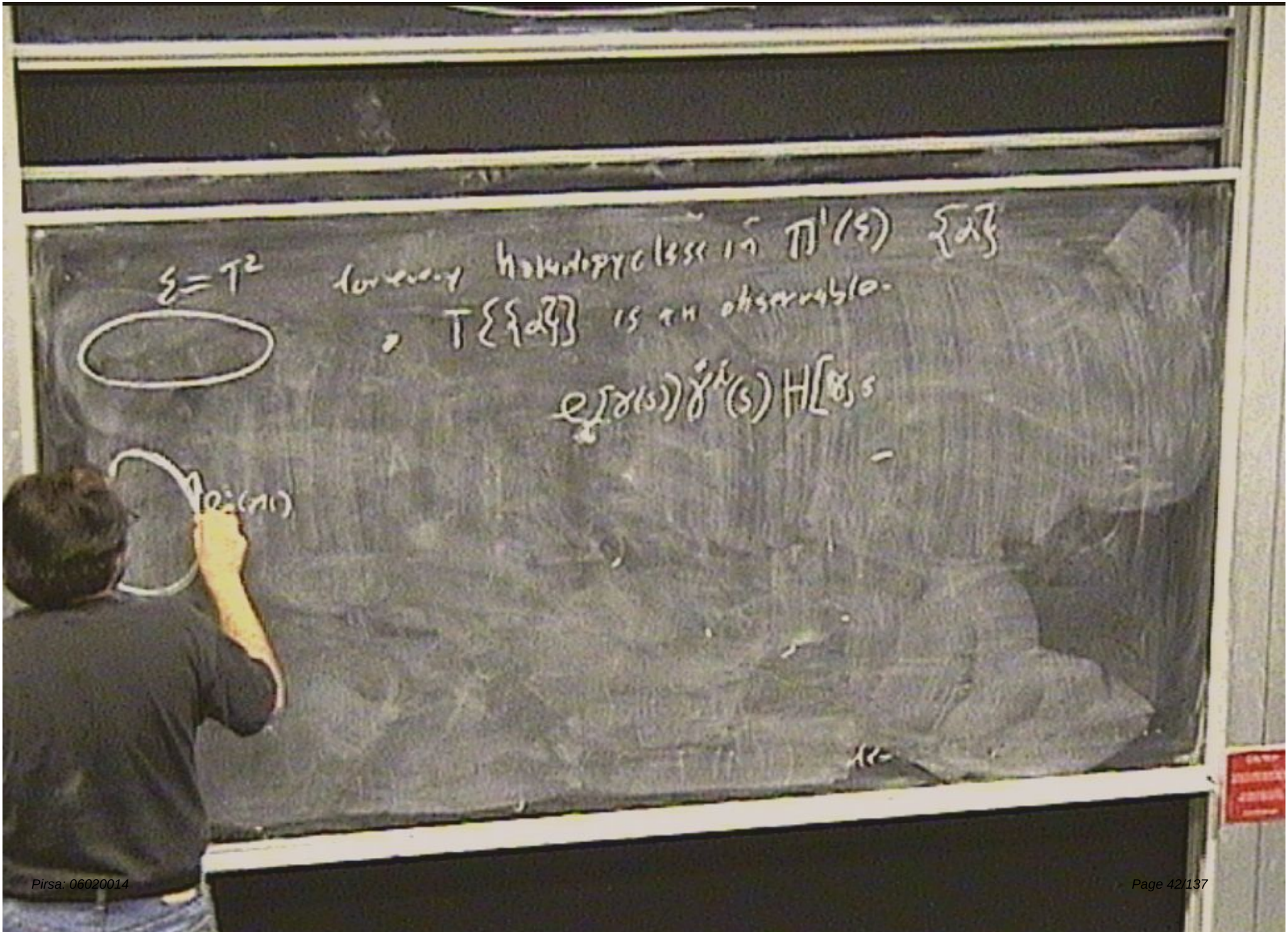
$\Sigma = T^2$ homotopy classes in $\pi^1(\Sigma)$ $\{\alpha\}$

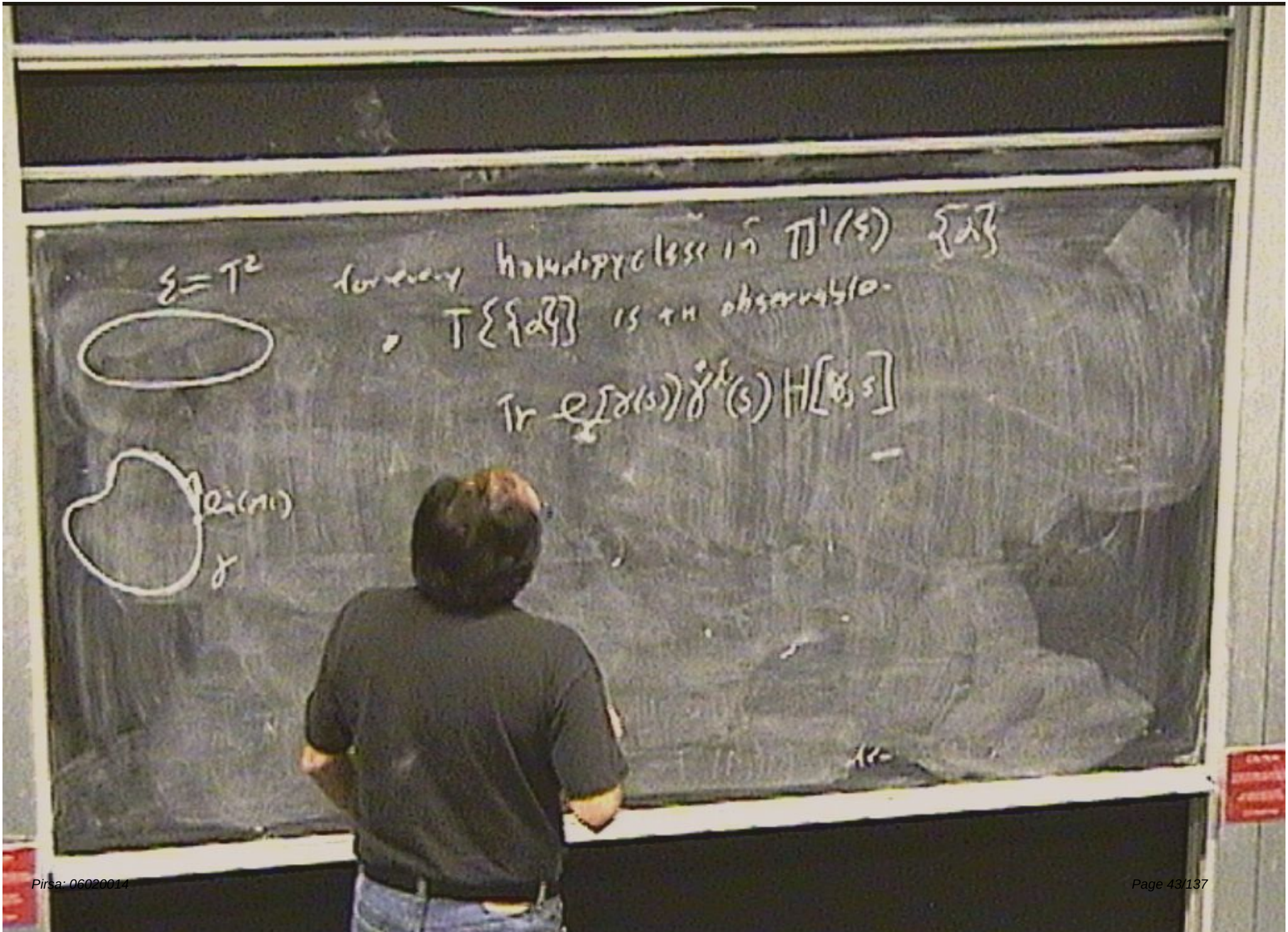
$T\{\{\alpha\}\}$ is an observable.

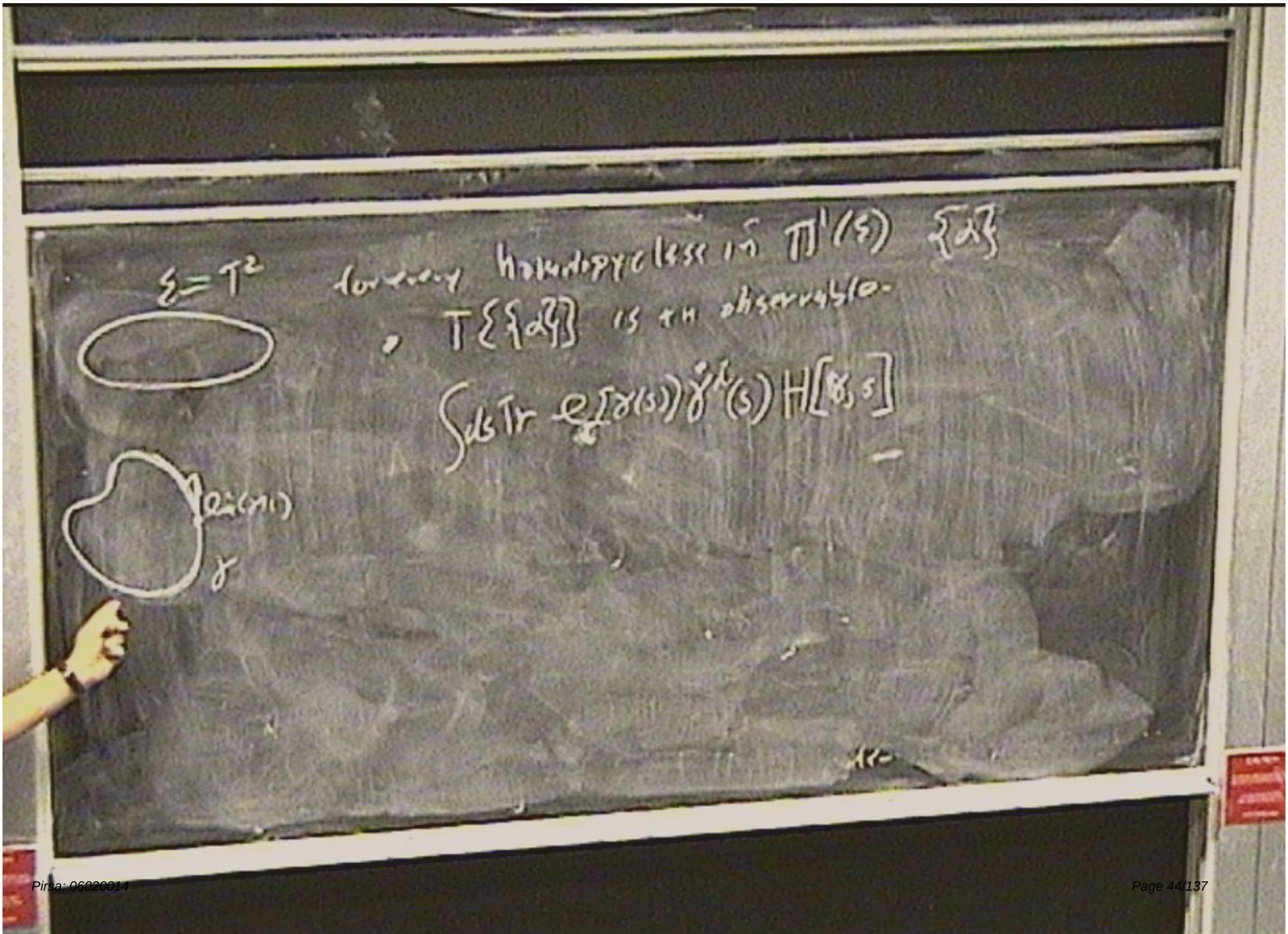
$\rho[\gamma(s)]$

$\rho_i(\pi_1)$

γ


$\Sigma = T^2$
for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$
$T\{\{\alpha\}\}$ is an observable.
$e_\gamma[\gamma(s)]\dot{\gamma}^i(s)H[\gamma, s$

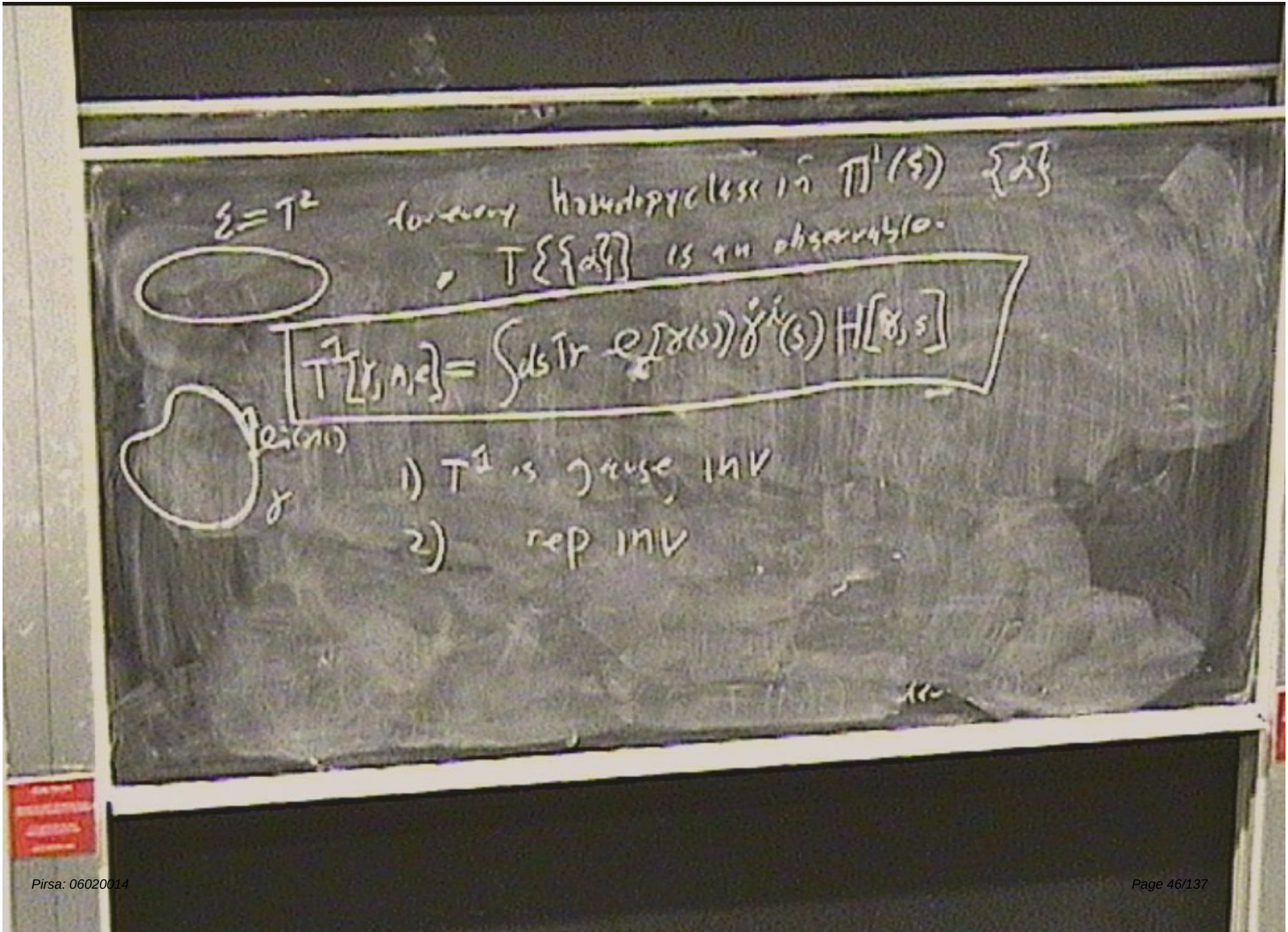

$\Sigma = T^2$
$T[\{\alpha\}]$ is an observable.
$\mathrm{Tr}\, \mathcal{P}[\gamma(s)]\dot{\gamma}^{\mu}(s)\, H[\gamma, s]$
γ


$\Sigma = T^2$
for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$
$T\{\{\alpha\}\}$ is an observable.
$\int ds \, \mathrm{Tr}\, e_{\gamma}[\gamma(s)] \dot{\gamma}^i(s) H[\gamma, s]$
γ

$\Sigma = T^2$ for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$

$T\{\{\alpha\}\}$ is an observable.

$$T^1[\gamma, n, e] = \int ds \, \mathrm{Tr}\, e_i[\gamma(s)]\, \dot{\gamma}^i(s)\, H[\gamma, s]$$


$\Sigma = T^2$
for every homotopy class in $\pi^1(S)$ $\{\alpha\}$
$T\{\{\alpha\}\}$ is an observable.
$T^1[\gamma, n, e] = \int ds \, \mathrm{Tr}\, e[\gamma(s)]\dot{\gamma}^i(s) H[\gamma, s]$
$e^{i\theta(n)}$
γ
1) T^1 is gauge inv
2) rep inv

$\Sigma = T^2$ for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$

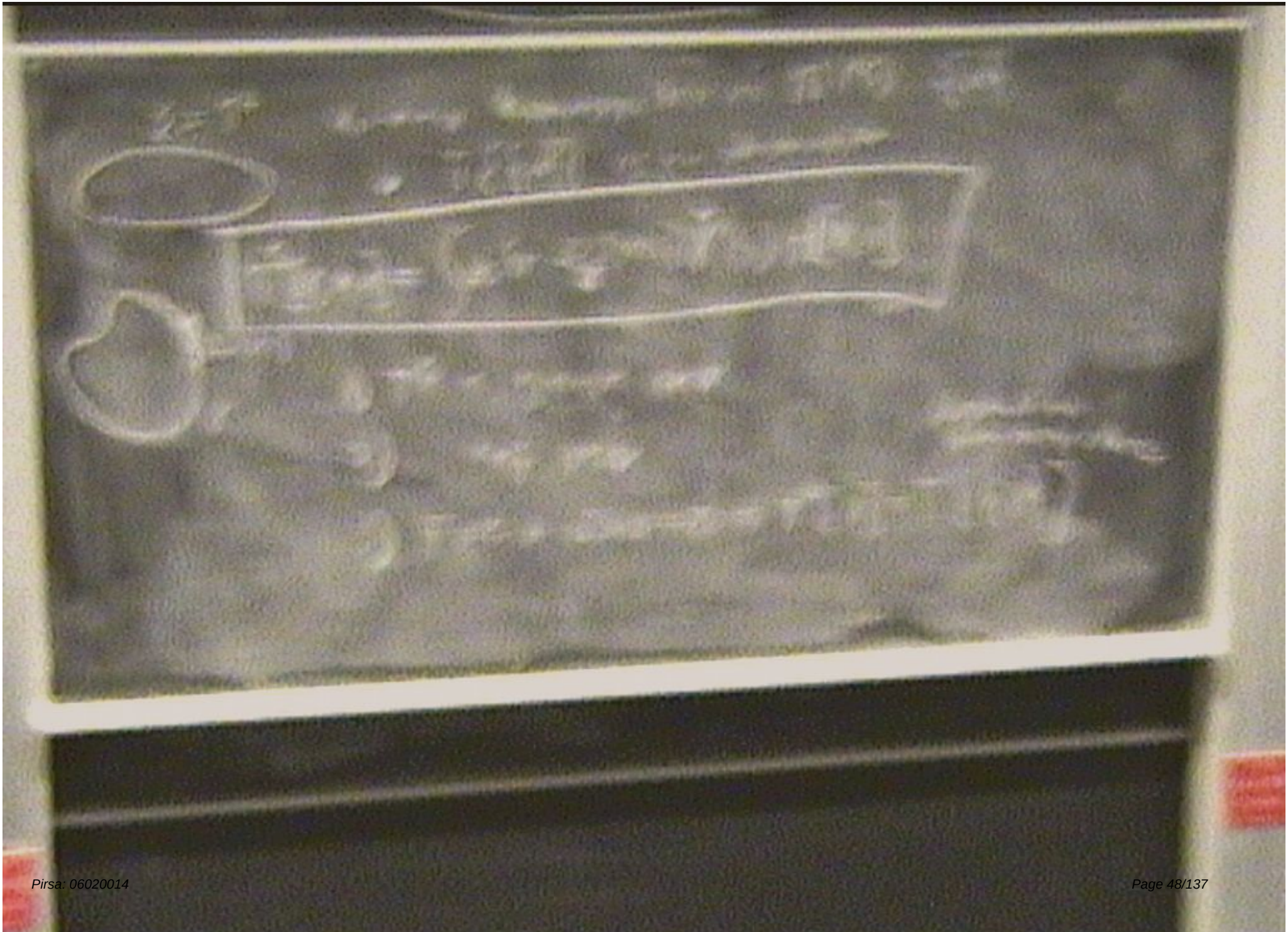
- $T\{\{\alpha\}\}$ is an observable.

$$\boxed{T^1[\gamma, n, e] = \int ds \, \mathrm{Tr}\, e_\alpha[\gamma(s)]\, \dot{\gamma}^i(s)\, H[\gamma, s]}$$

1) T^1 is gauge inv

2) rep inv

3) $F = 0$ & $\mathcal{D} \wedge e = 0$



$\Sigma = T^2$ for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$

$\cdot$ $T\{\{\alpha\}\}$ is an observable.

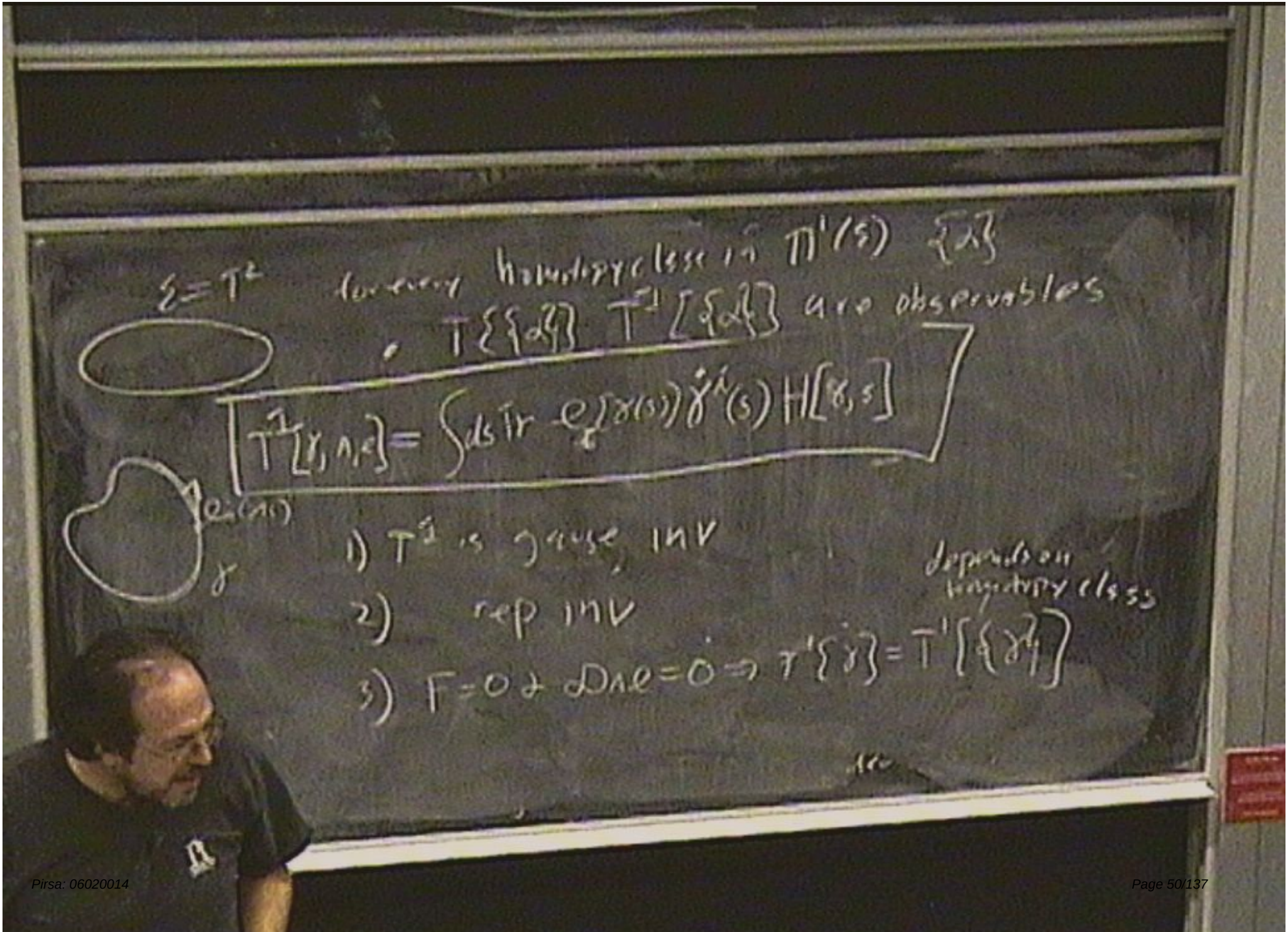
$$\boxed{T^1[\gamma, n, e] = \int ds \, \mathrm{Tr}\, e_\gamma[\gamma(s)]\, \dot{\gamma}^i(s)\, H[\gamma, s]}$$

1) T^1 is gauge inv

2) rep inv

3) $F = 0$ & $\mathcal{D} \wedge e = 0 \Rightarrow T^1\{\gamma\} = T^1[\{\gamma\}]$

depends on homotopy class


$\Sigma = T^2$
for every homotopy class in $\pi^1(\Sigma)$ $\{\alpha\}$
$T\{\{\alpha\}\}$ $T^{-1}[\{\alpha\}]$ are observables
$T^{-1}[\gamma, A, e] = \int ds \, \mathrm{Tr}\, e_\gamma[\gamma(s)] \dot{\gamma}^i(s) H[\gamma, s]$
$e_i(\gamma(s))$
γ
1) T^{-1} is gauge inv
2) rep inv
3) $F = 0$ & $\mathcal{D} \wedge e = 0 \Rightarrow T^{-1}\{\dot{\gamma}\} = T^{-1}[\{\gamma\}]$
depends on homotopy class

$$\{\mathcal{F}(\rho), \mathcal{F}(\mu)\} = \int d^2y\, dz\, \rho_{ij}(y)\, \mu^j(z) \{ f_{i2}(y), \ldots \}$$

$$\partial A + \Lambda A$$

F-Nerve F-Nerve $= \rho\kappa(\partial e + \Lambda e) = \rho\kappa(\partial \Pi + \Lambda \Pi)$

$= \Lambda G(\rho\kappa\mu)$ (check!!!)

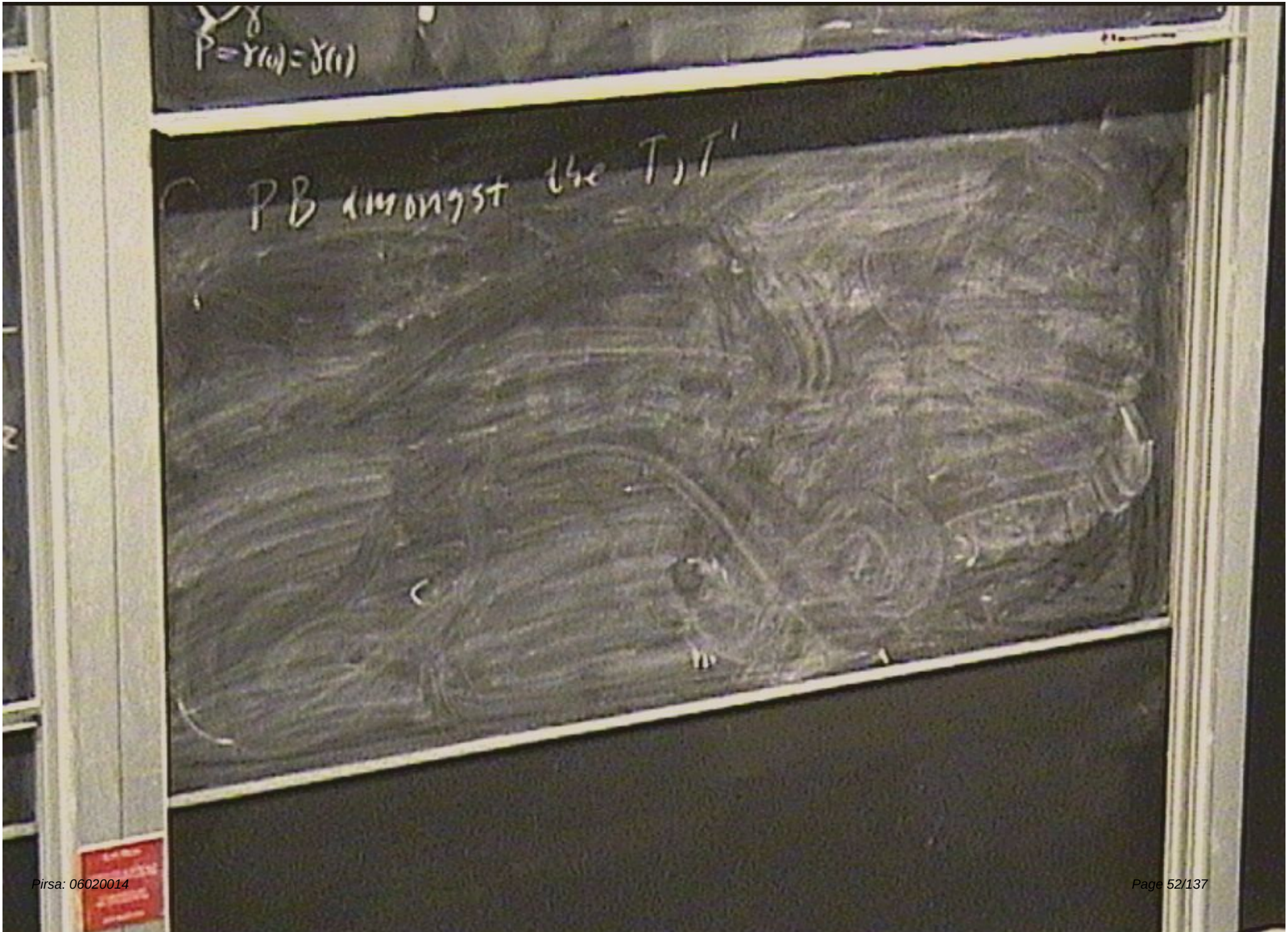
Hint:

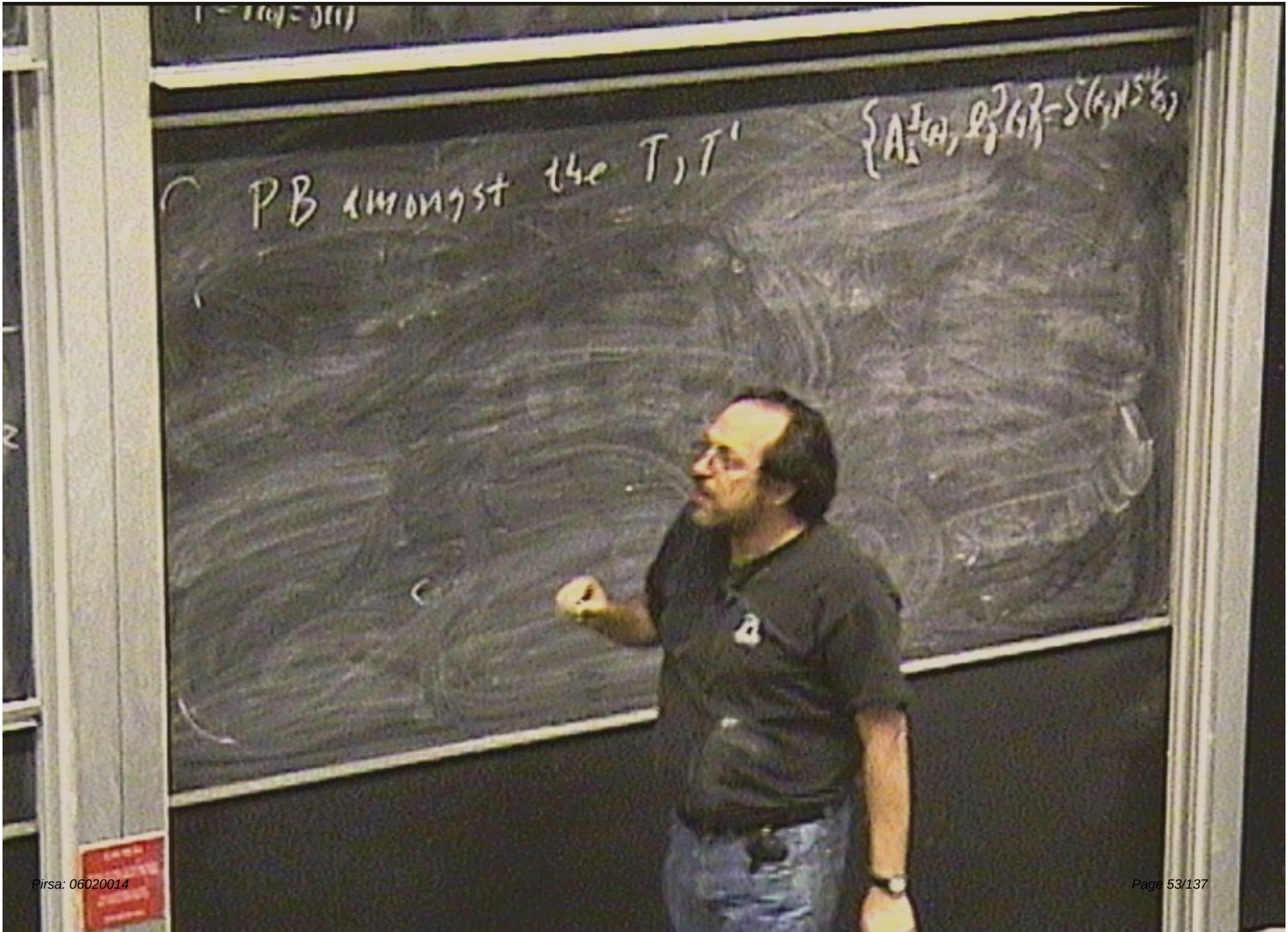
$$\frac{\delta S}{\delta e^I_i} = F^I - \Lambda e^J \wedge e^K \varepsilon^I_{JK} = 0 \overset{\text{rollback}}{\Rightarrow} \varepsilon \equiv \tilde{\mathcal{F}}^I$$

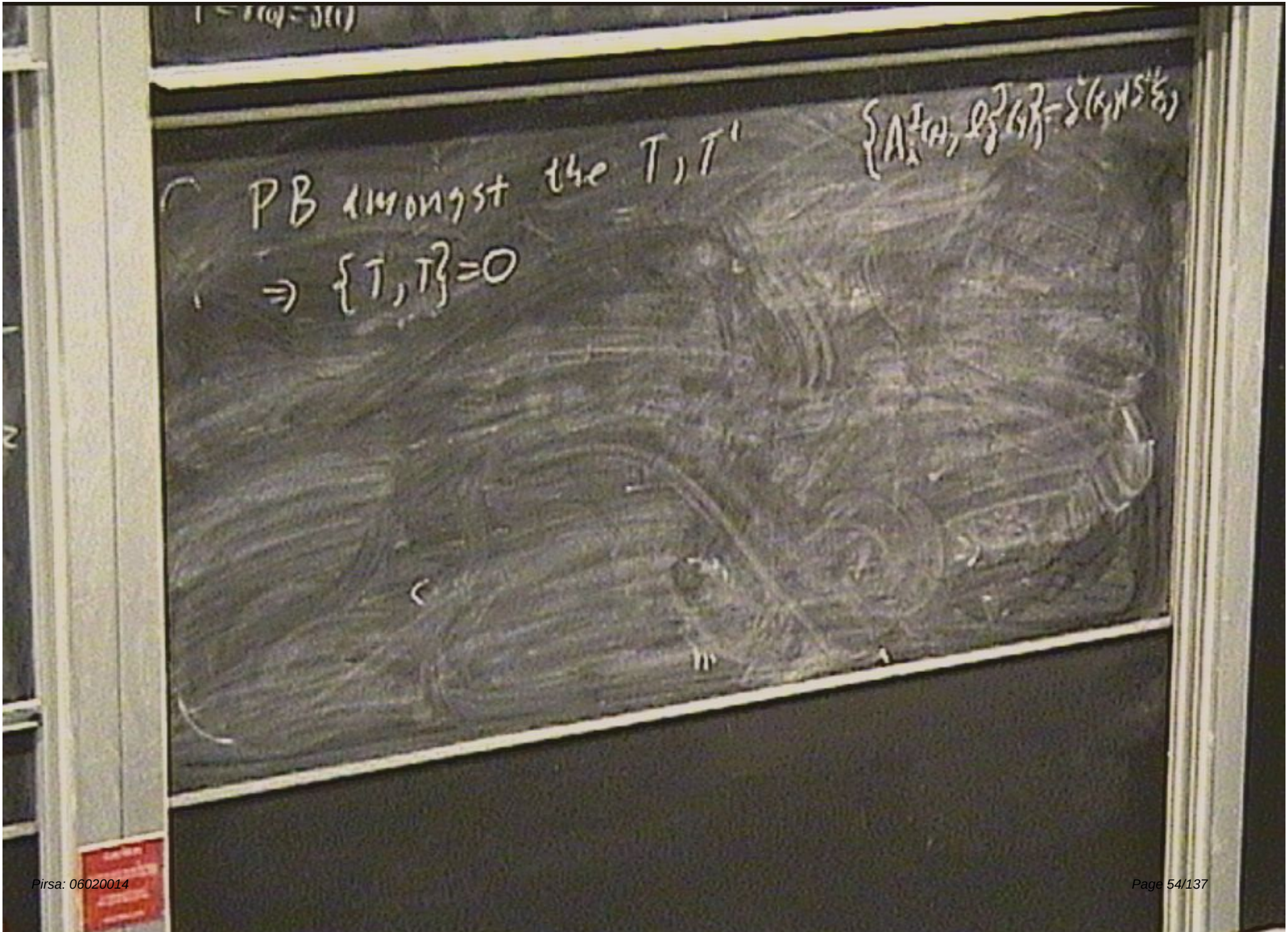
$$\frac{\delta S}{\delta \Lambda^I_i} = \mathcal{D}_\Lambda e = 0 \equiv G^I$$

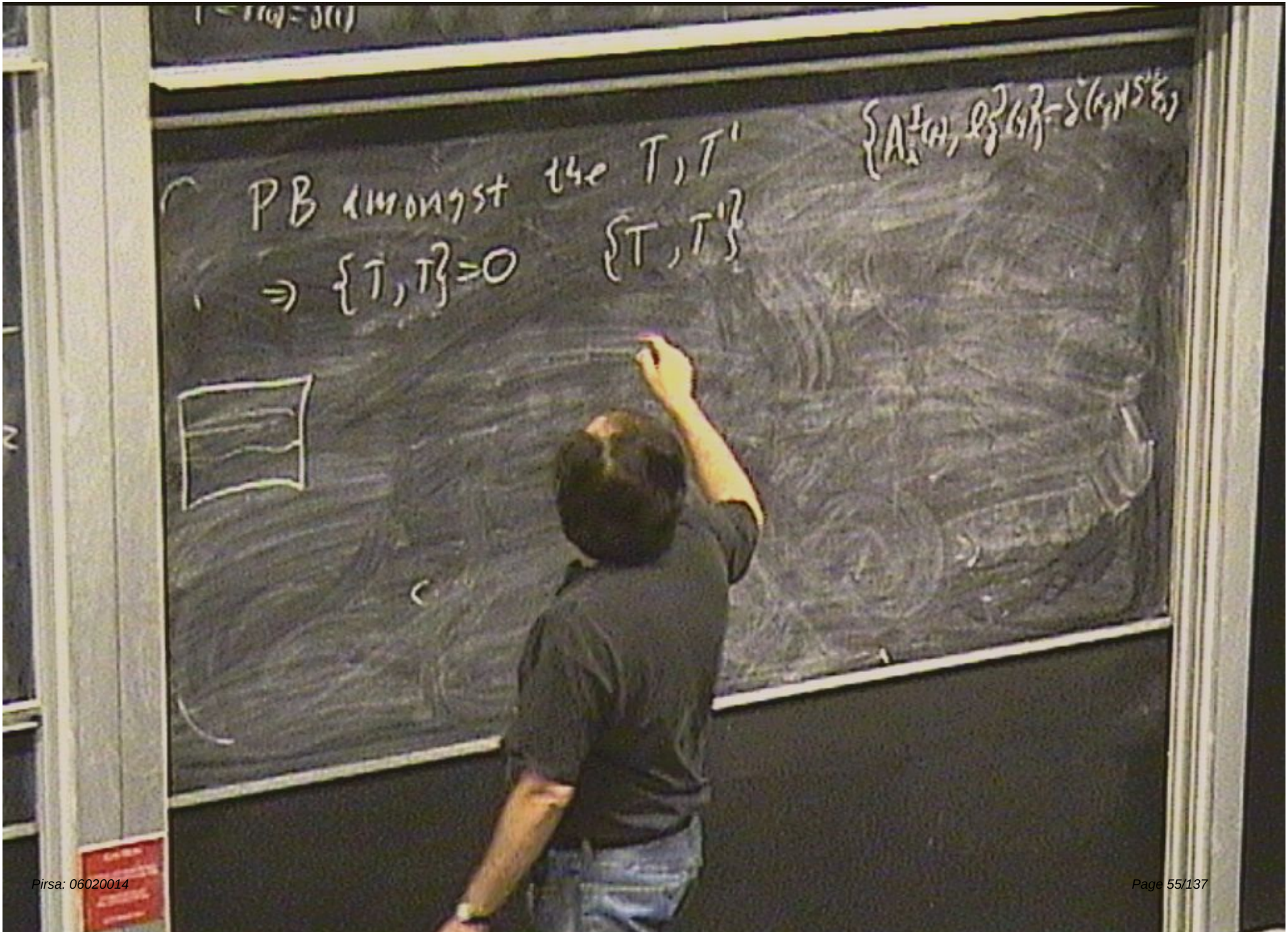
$$\boxed{\dot{A}^I_i - \mathcal{D}_i \Lambda^I_0 - \Lambda e^J_0 \wedge e^K_i \varepsilon_{IJK} = 0}$$

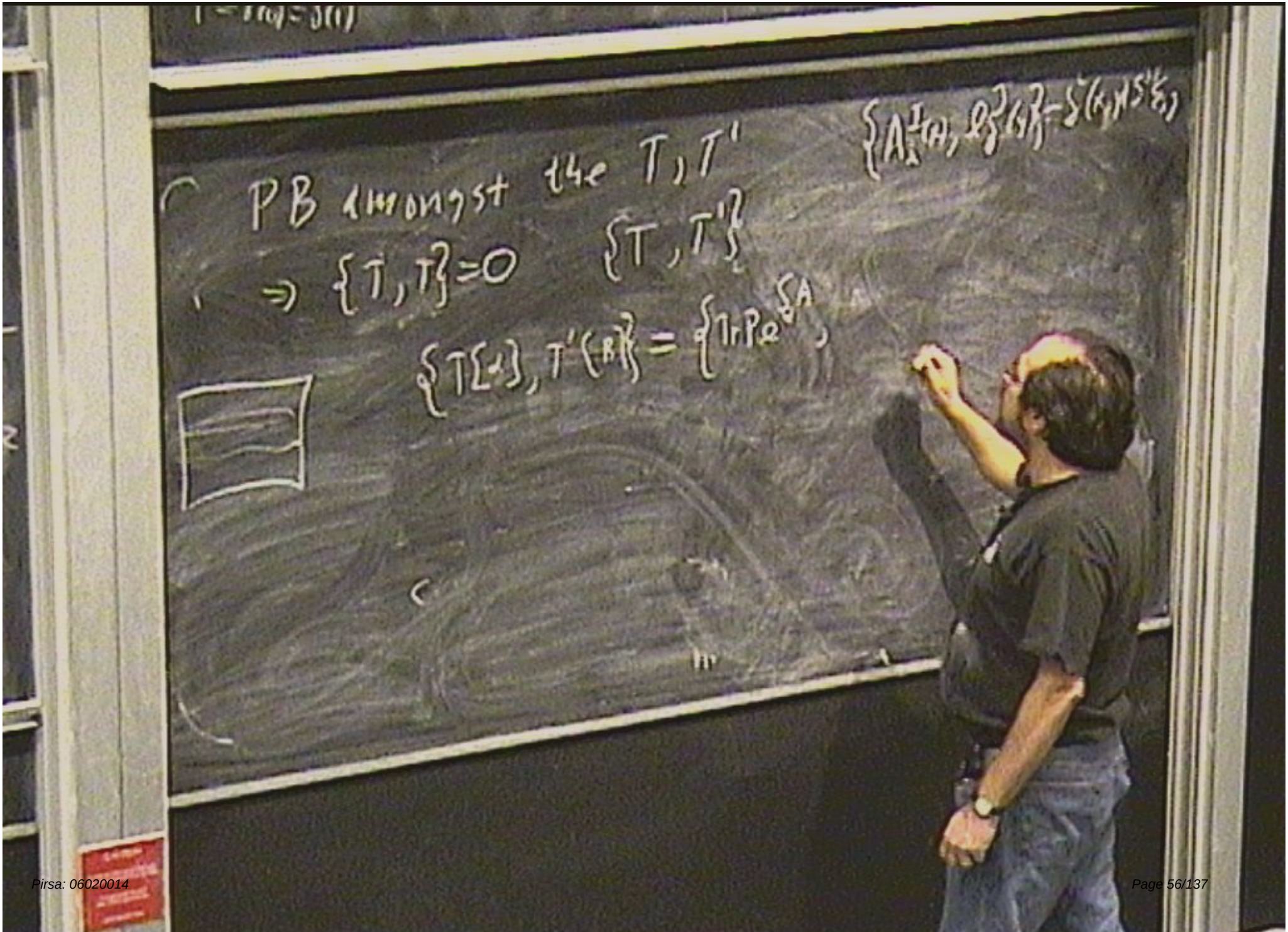
$$\boxed{\dot{e}^I_i + \varepsilon^I_{JK} \Lambda^J_0 e^K_i = 0}$$

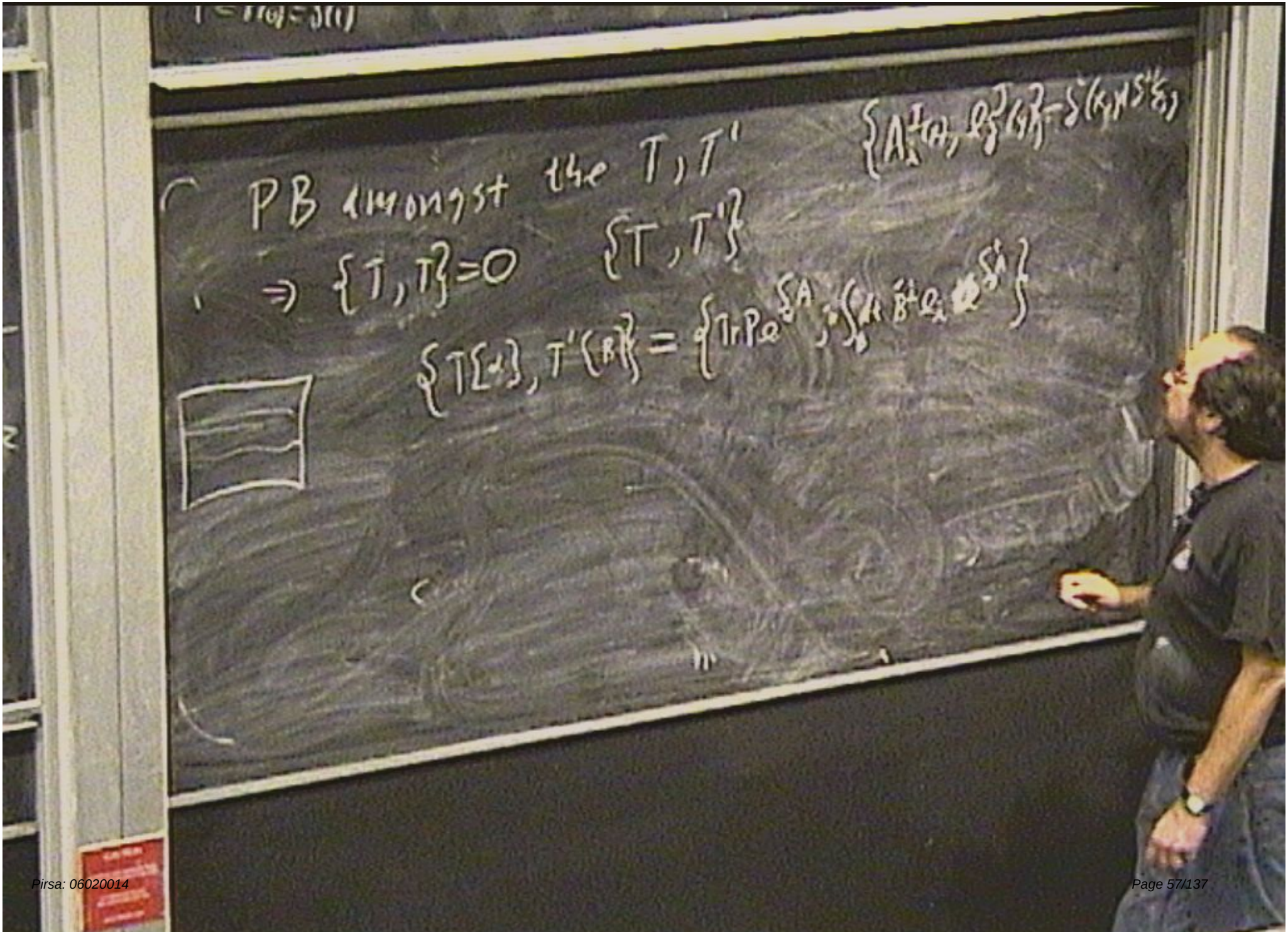

P = γ(0) = γ(1)
PB amongst the T, T'

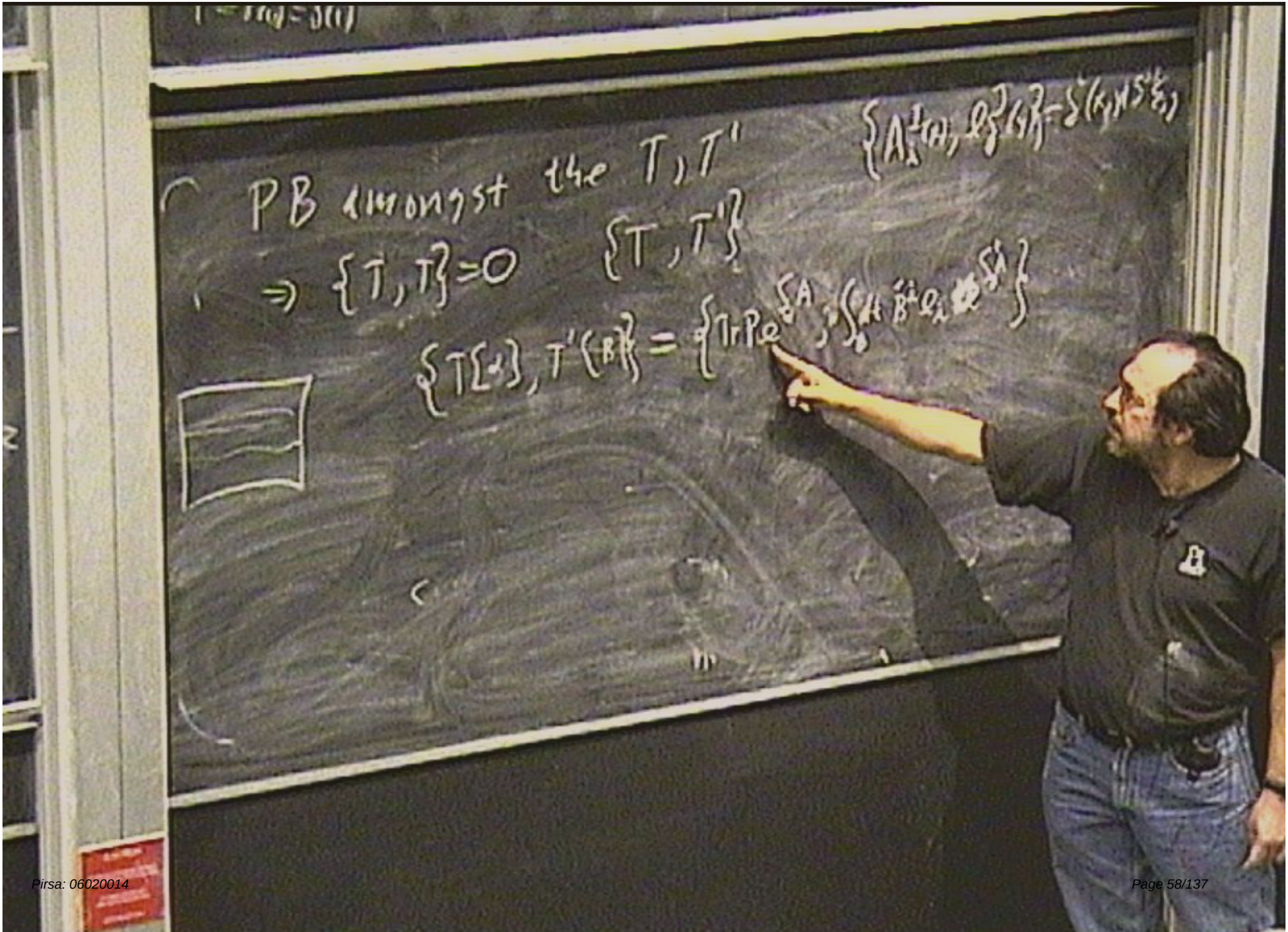

PB amongst the T, T'


PB amongst the T, T'
⇒ {T, T} = 0


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}
{T[α], T'[β]} = {TrPe^{δA},


PB amongst the T, T'
$\Rightarrow \{T, T\} = 0$
$\{T, T'\}$

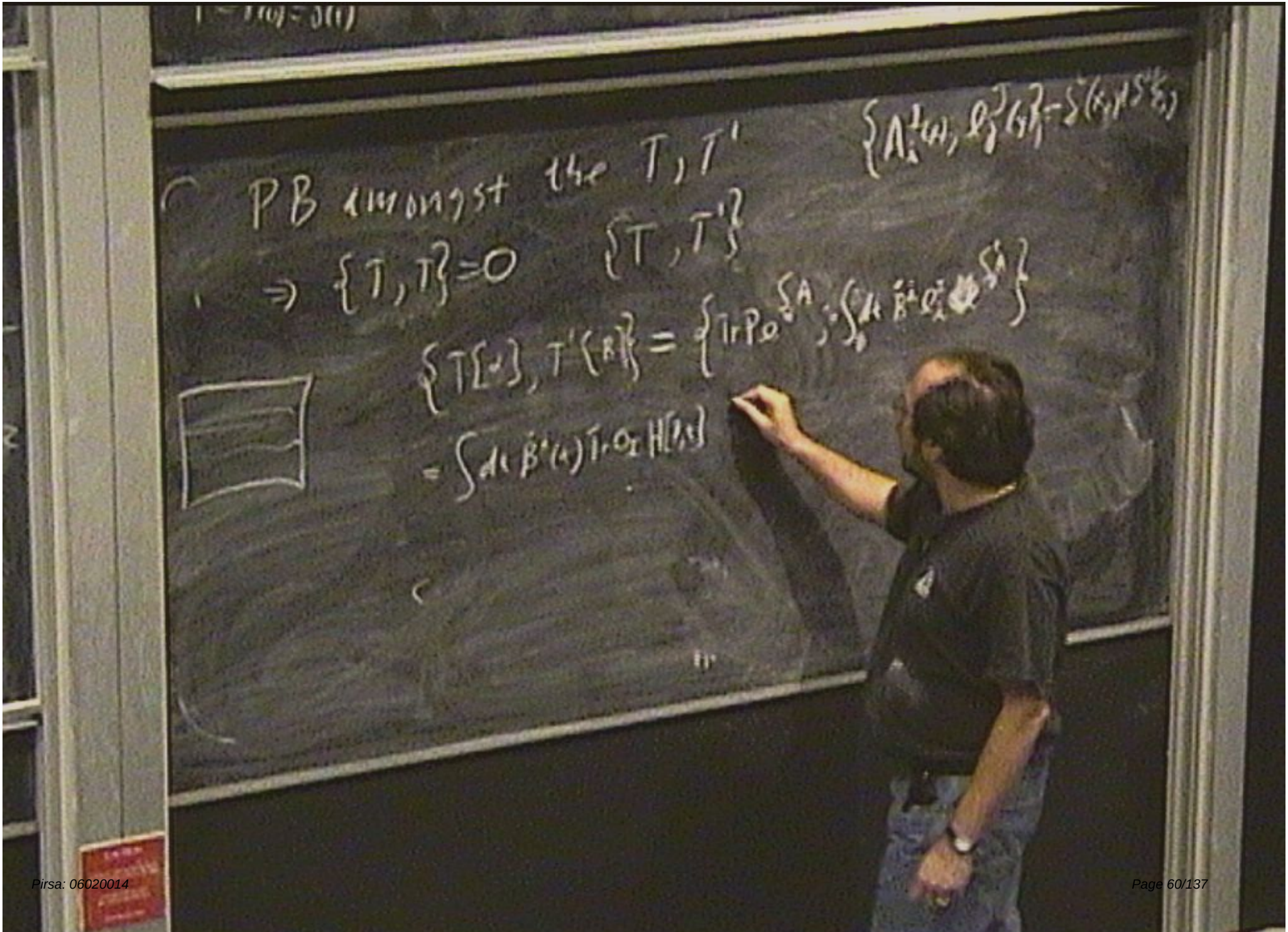

PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}

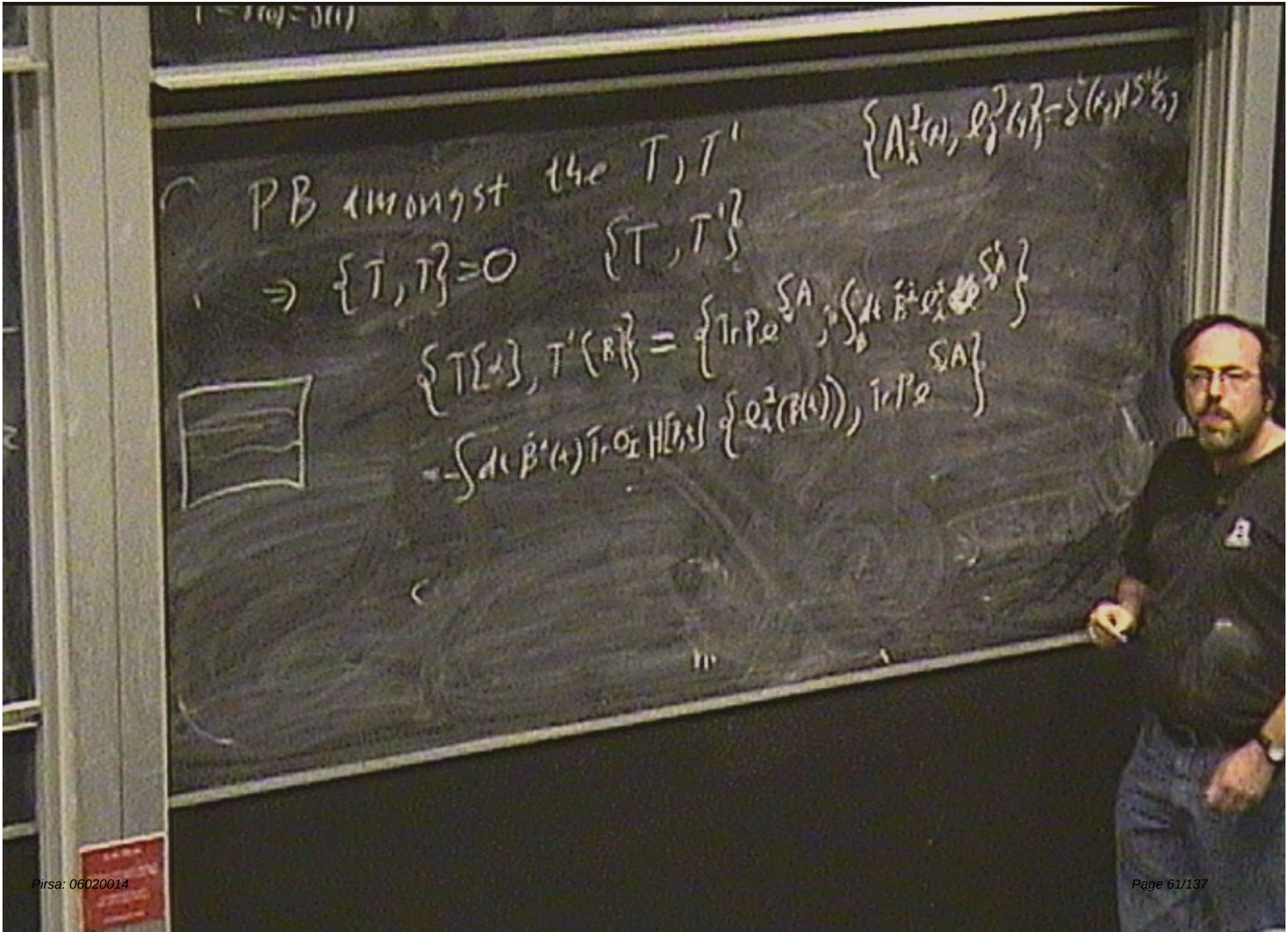
PB amongst the T, T'

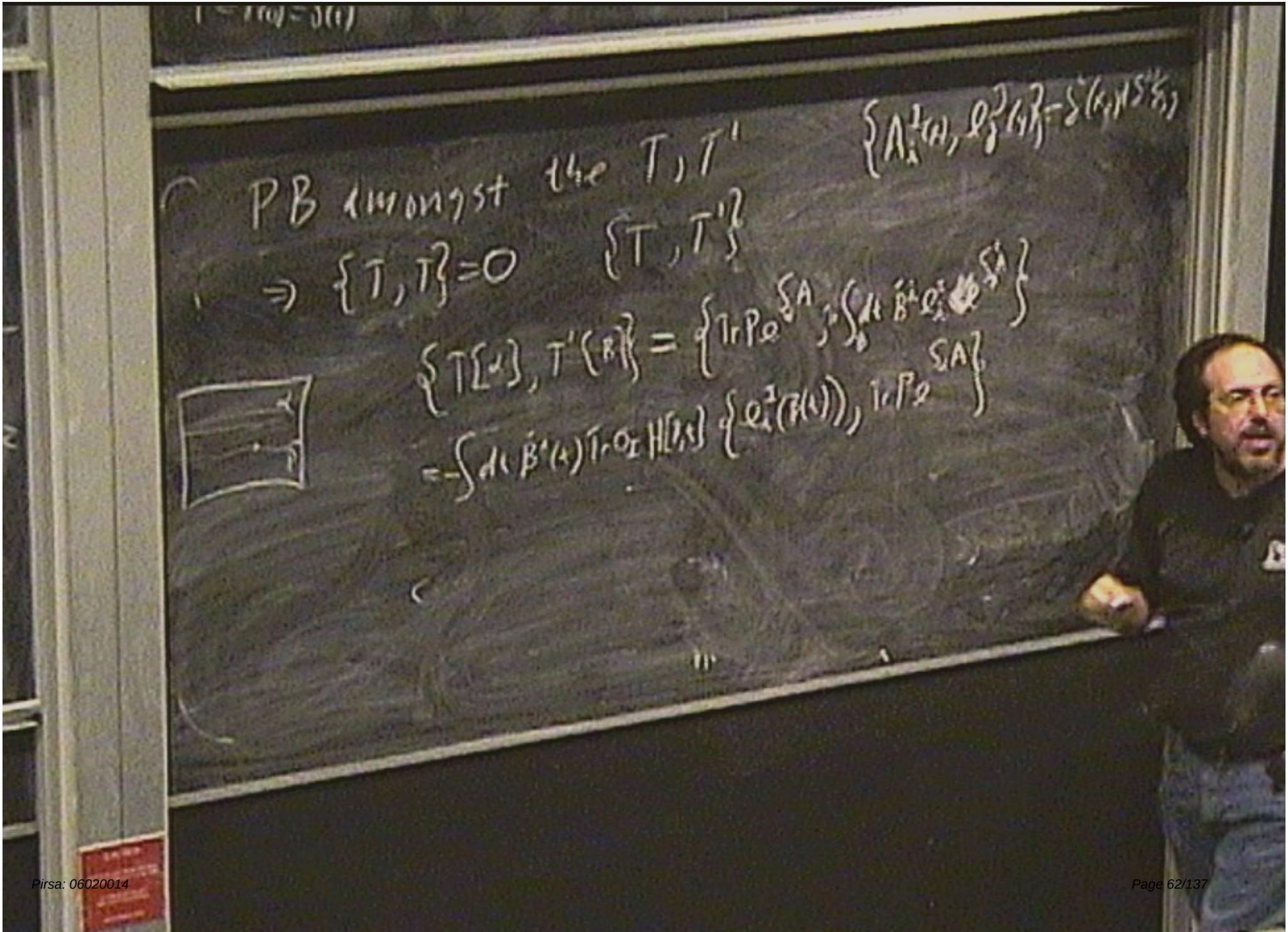
$\Rightarrow \{T, T\} = 0$ $\{T, T'\}$

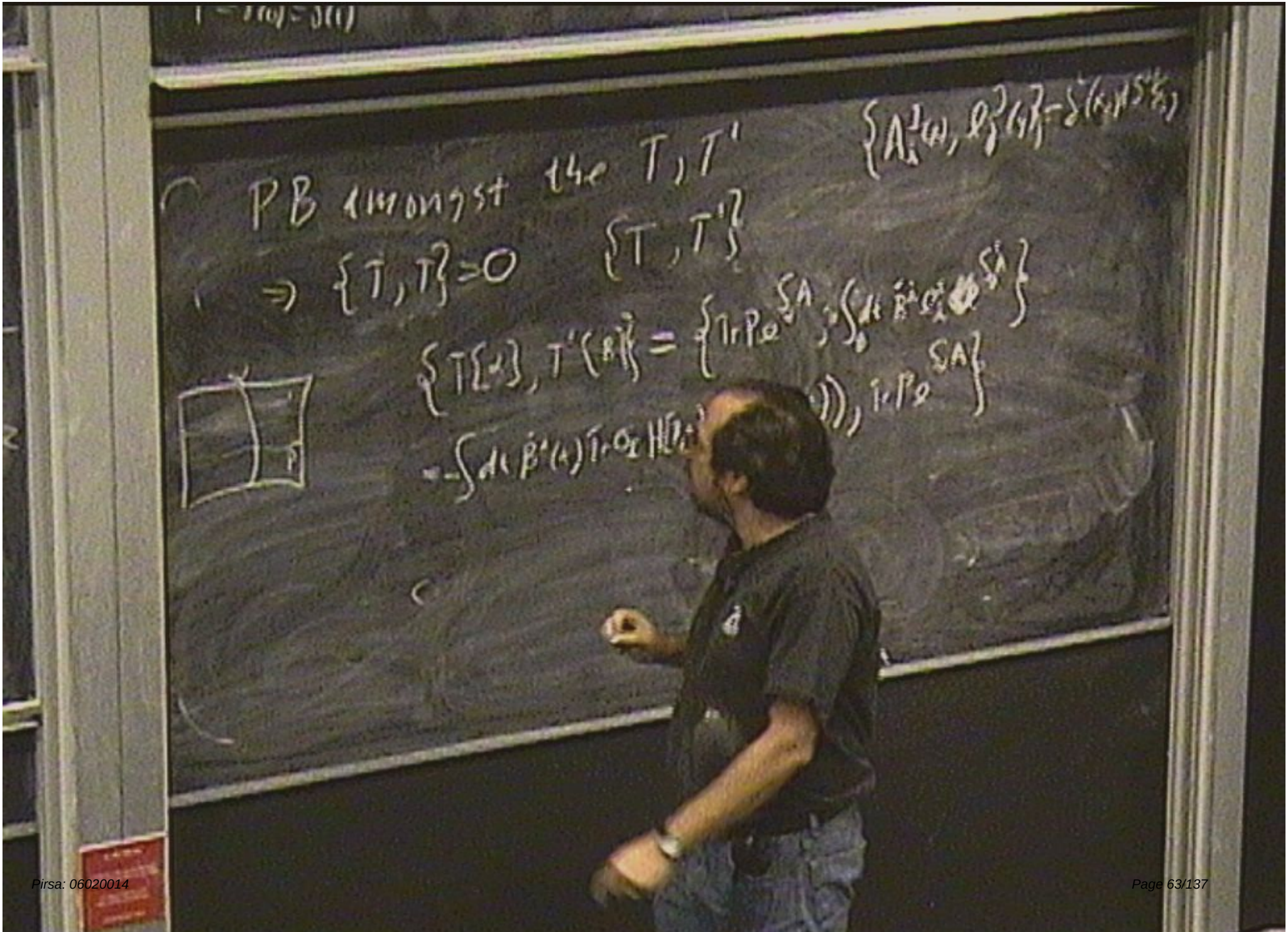
$\{A^I(\sigma), B^J(\sigma')\} = \delta(\sigma, \sigma') \delta^{IJ}$

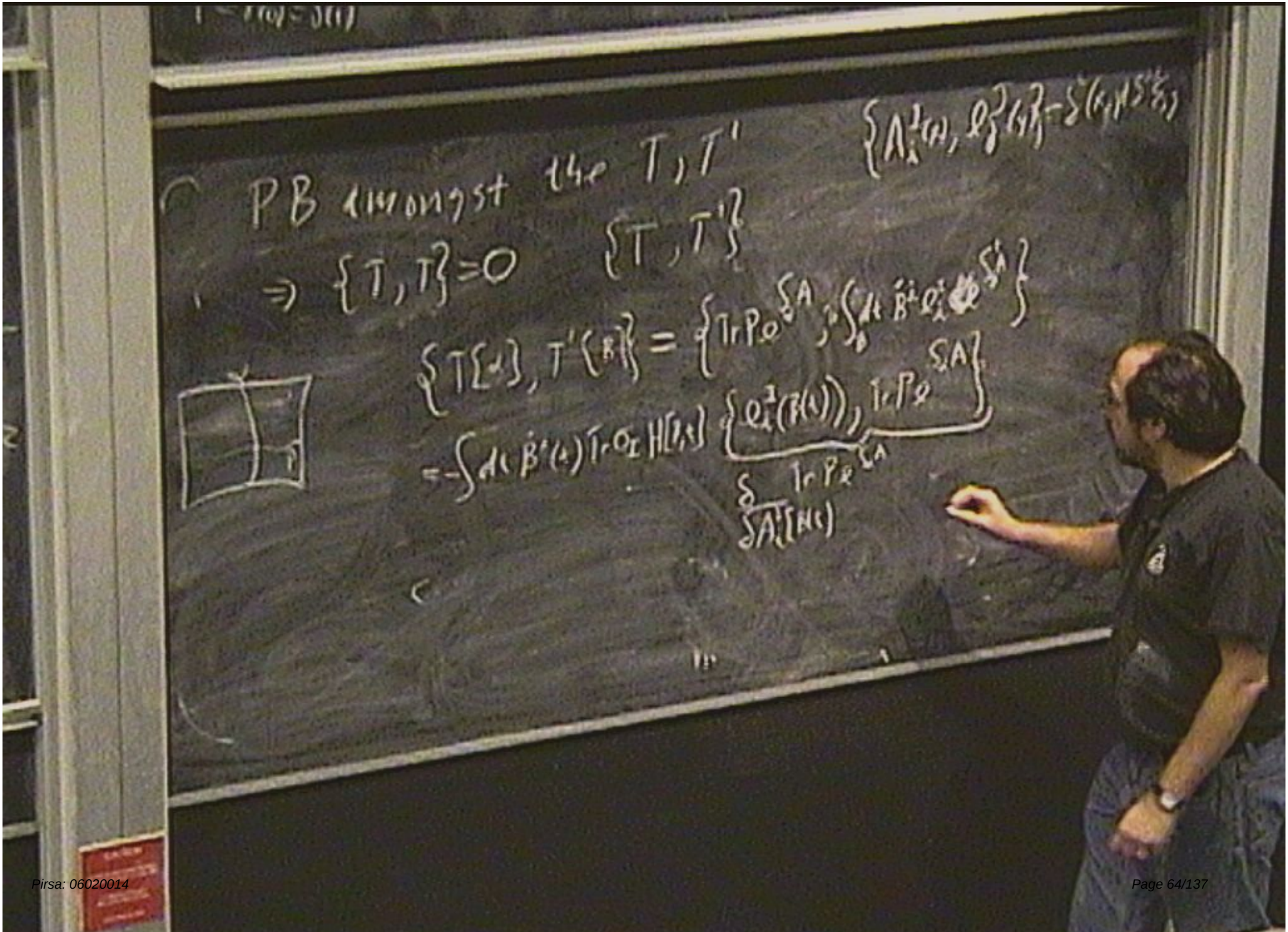
$\{T[\alpha], T'[\beta]\} =$

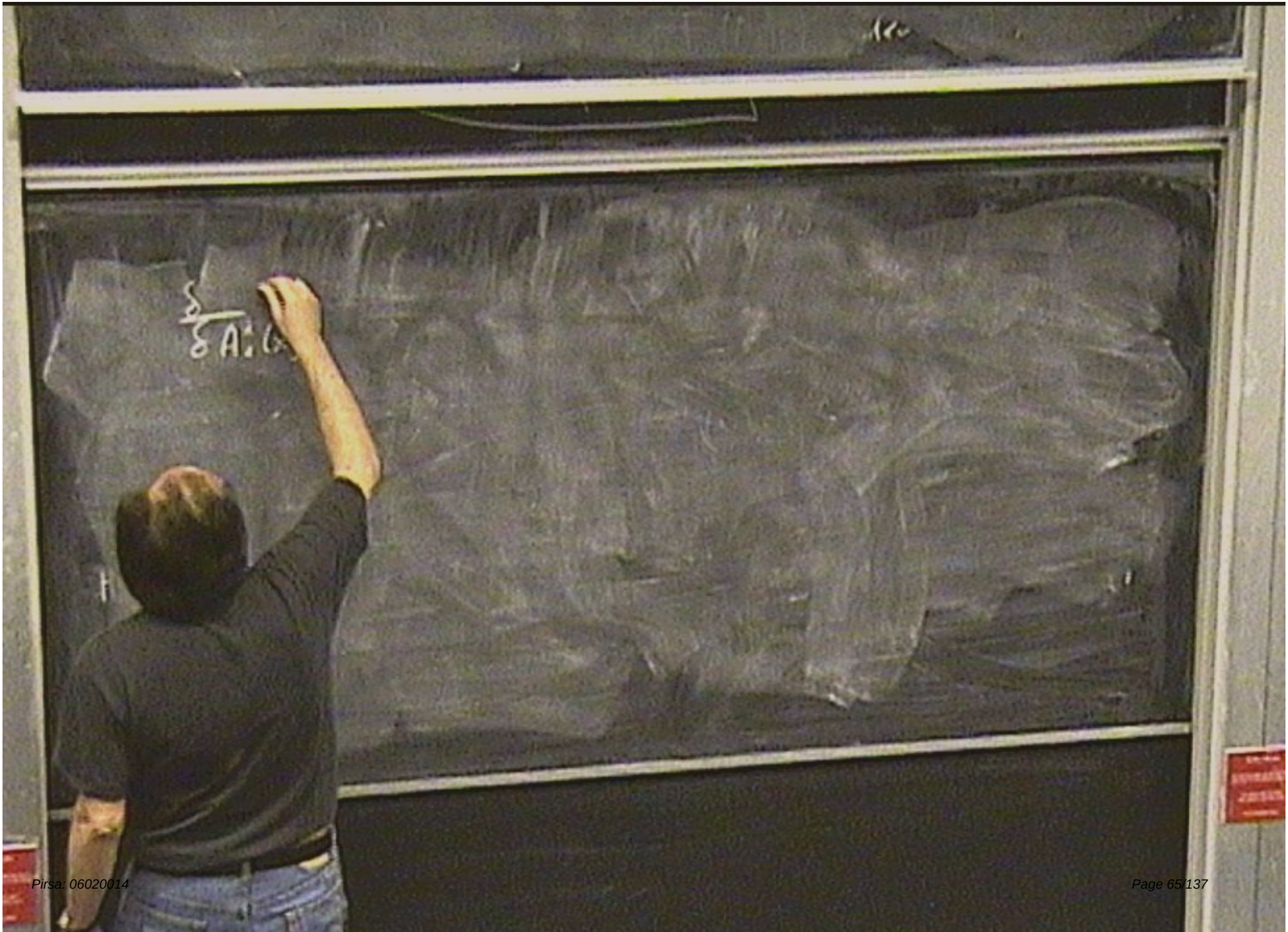

PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}

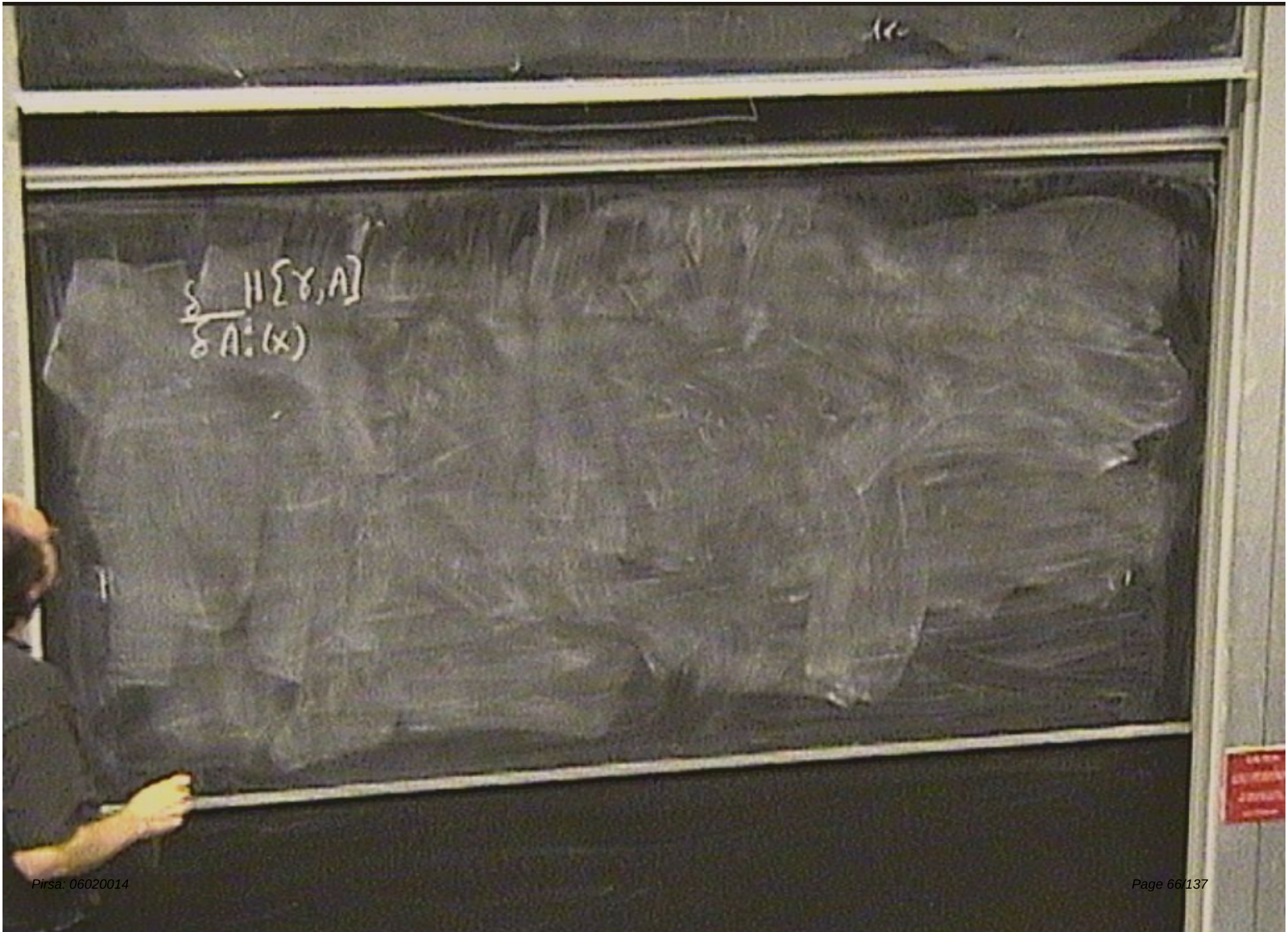

PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}

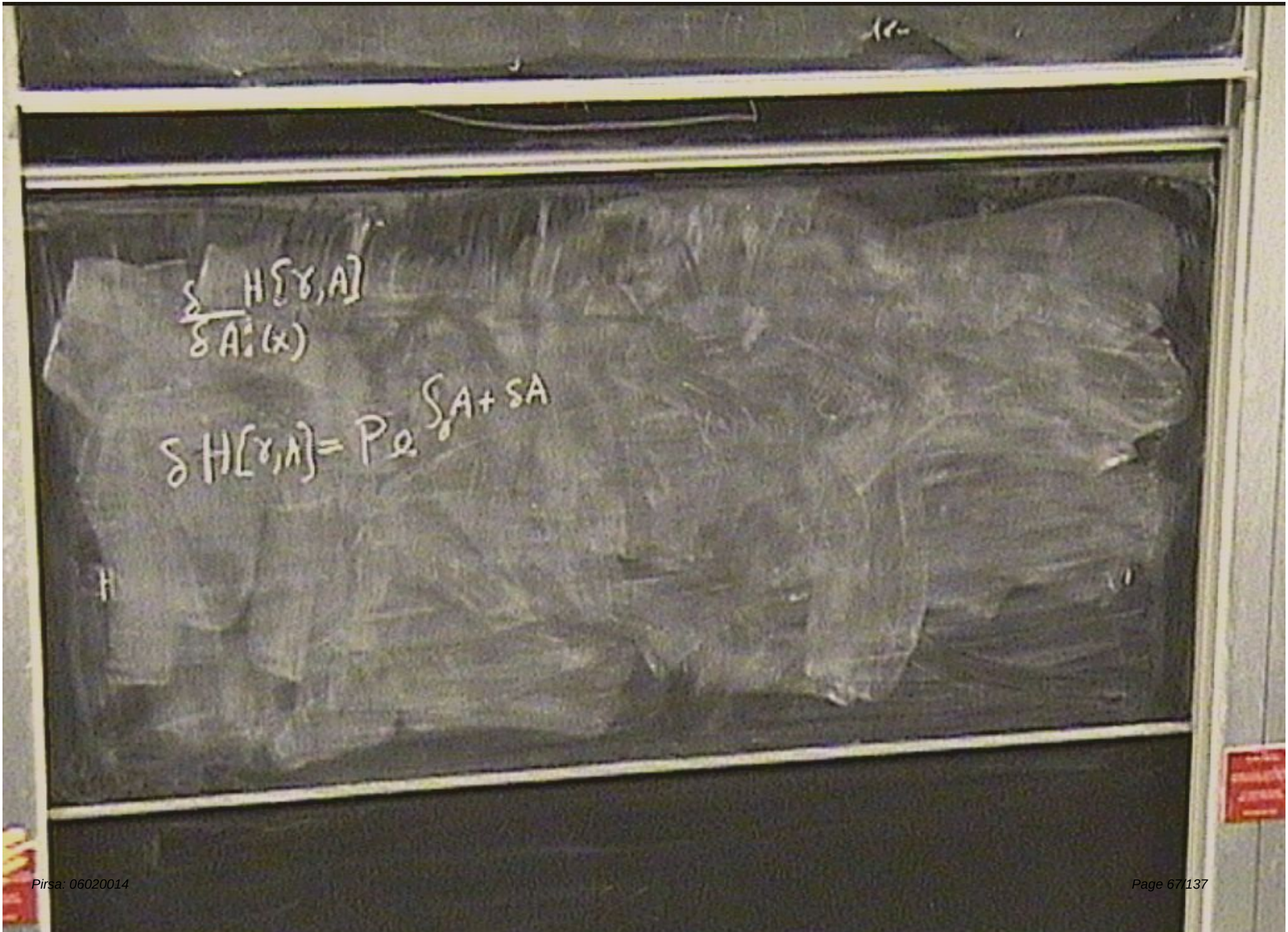

PB amongst the T, T'
$\{T, T\} = 0$
$\{T, T'\}$
$\{T[\alpha], T'[\beta]\} =$

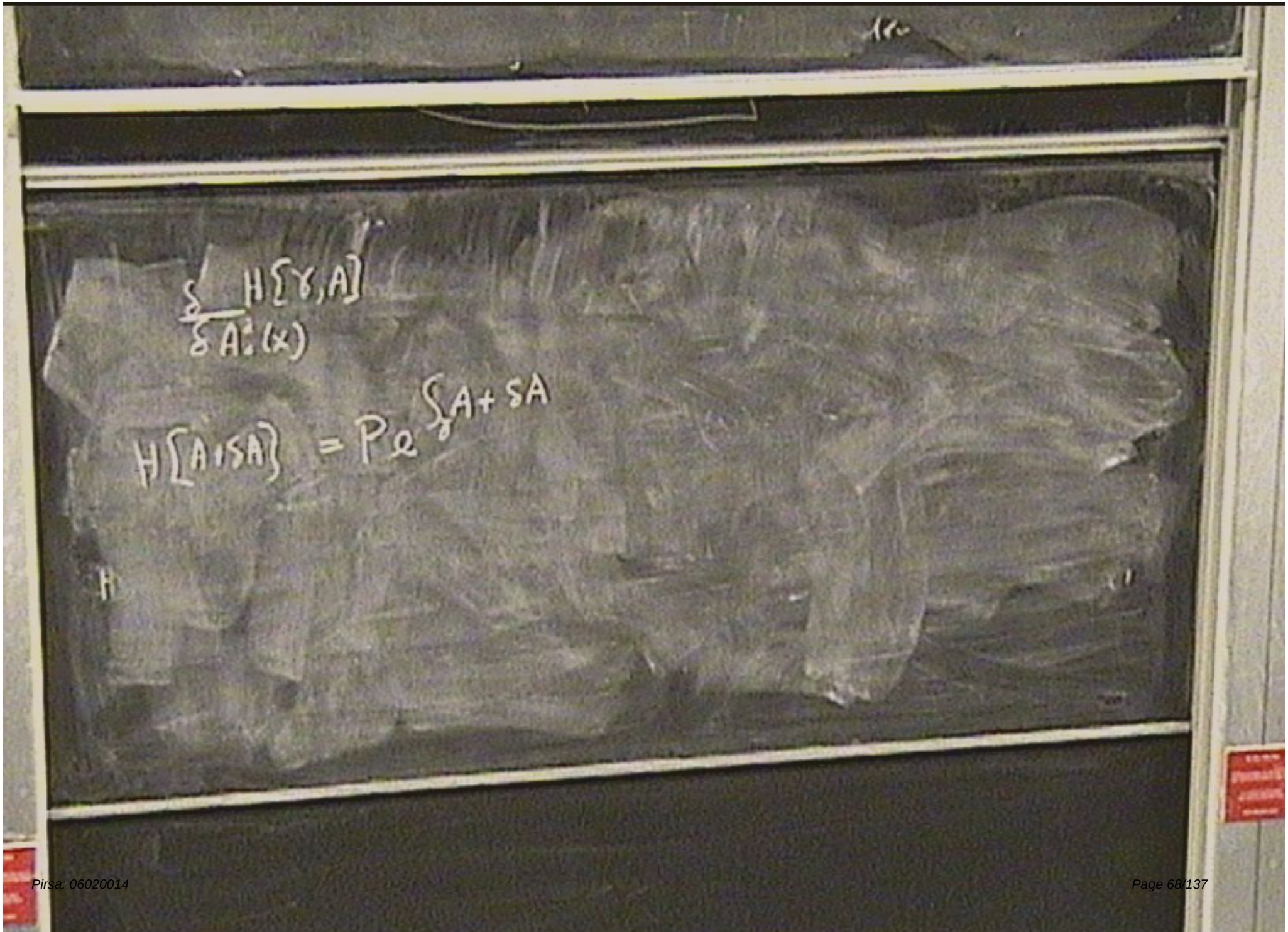

PB amongst the T, T'
$\{T, T\} = 0$
$\{T, T'\}$

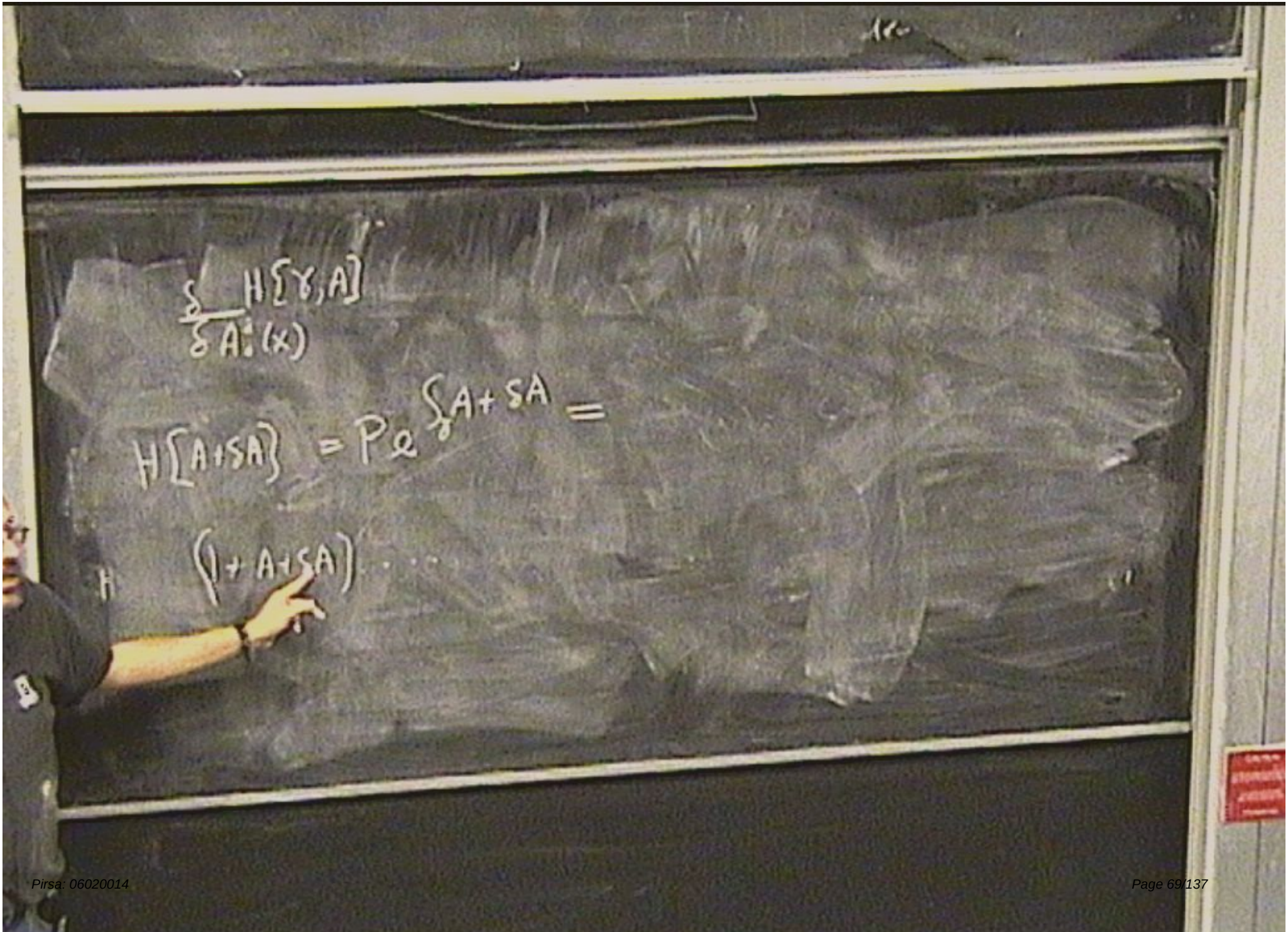

PB amongst the T, T'
$\{T, T\} = 0$
$\{T, T'\}$
$\{\Lambda^1_i(A), \mathcal{Q}^2_j(B)\} = \delta(A,B)\delta_{ij}$
$\{T[\Lambda], T'[B]\} = \{\mathrm{Tr} P_\alpha^{\Sigma_A}, \int_\Sigma d\sigma \dot{B}^i \mathcal{Q}^2_i e^{\Sigma_A}\}$
$= -\int d\sigma \dot{B}^i(\sigma) \mathrm{Tr} Q_\alpha H[\Lambda,\sigma] \{\mathcal{Q}^2_i(B(\sigma)), \mathrm{Tr} P_\alpha^{\Sigma_A}\}$
$\frac{\delta}{\delta A^1_i[B(\sigma)]} \mathrm{Tr} P_\alpha^{\Sigma_A}$

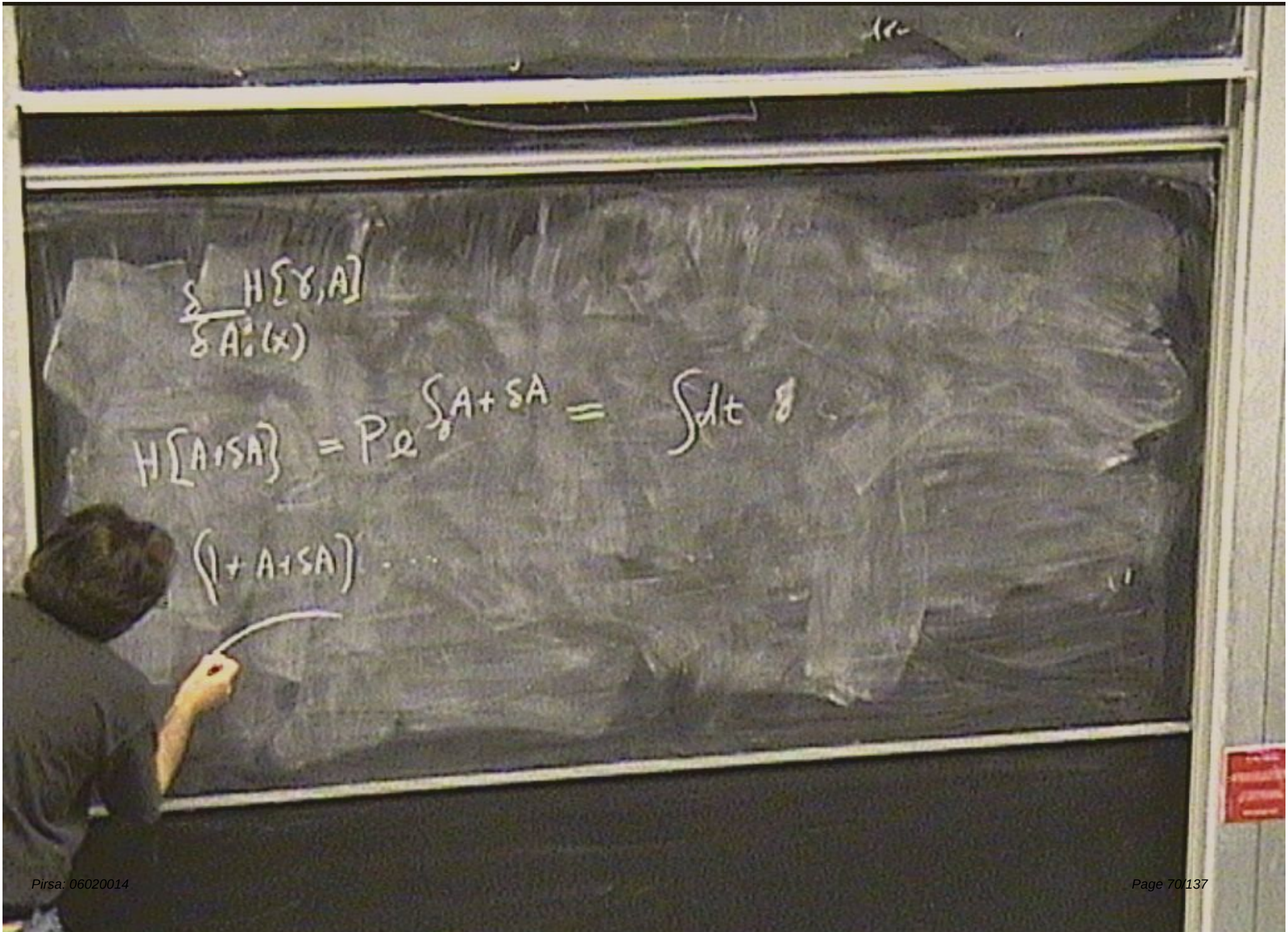


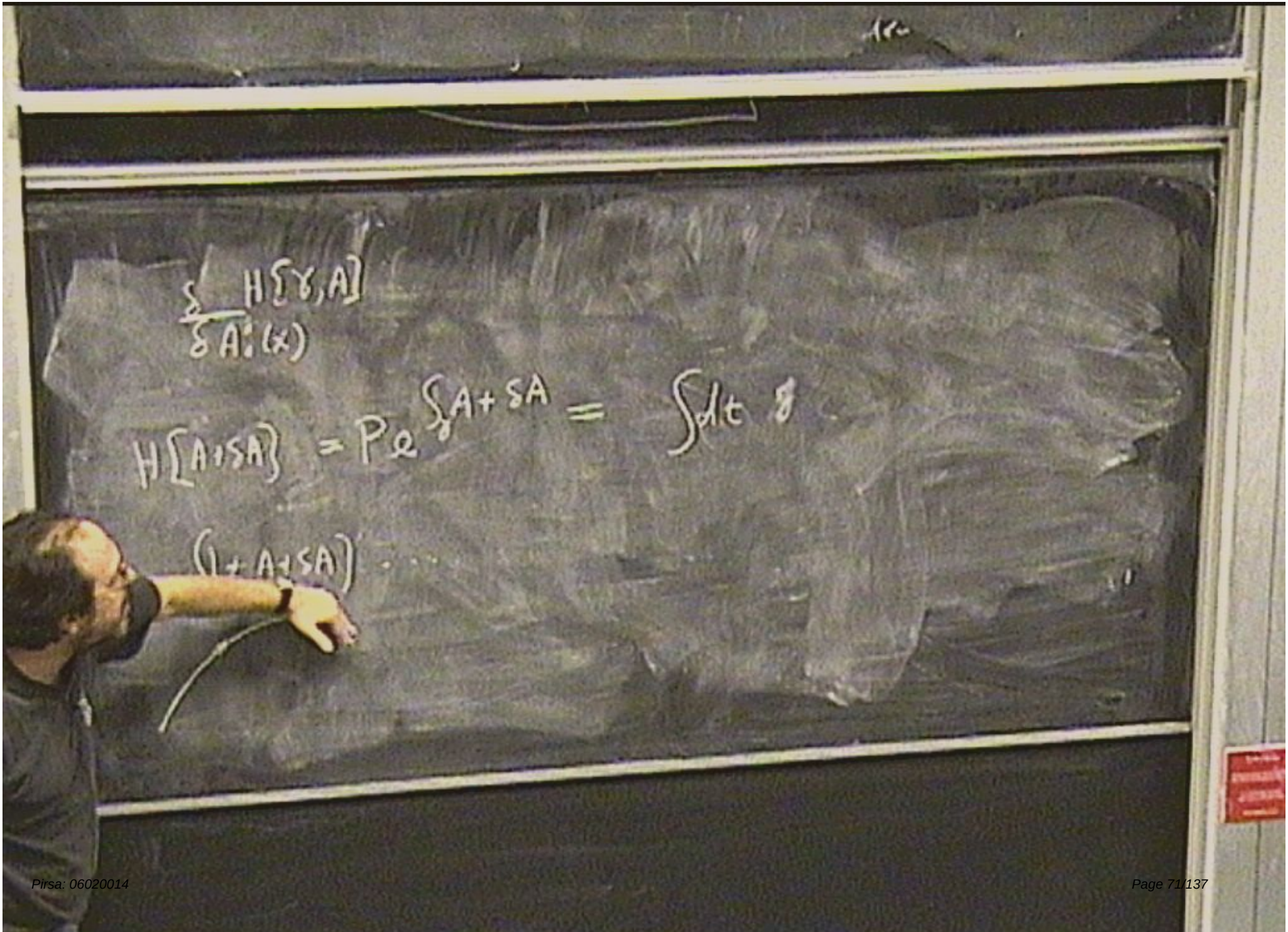

$\frac{\delta}{\delta A_i(x)} H[\gamma, A]$

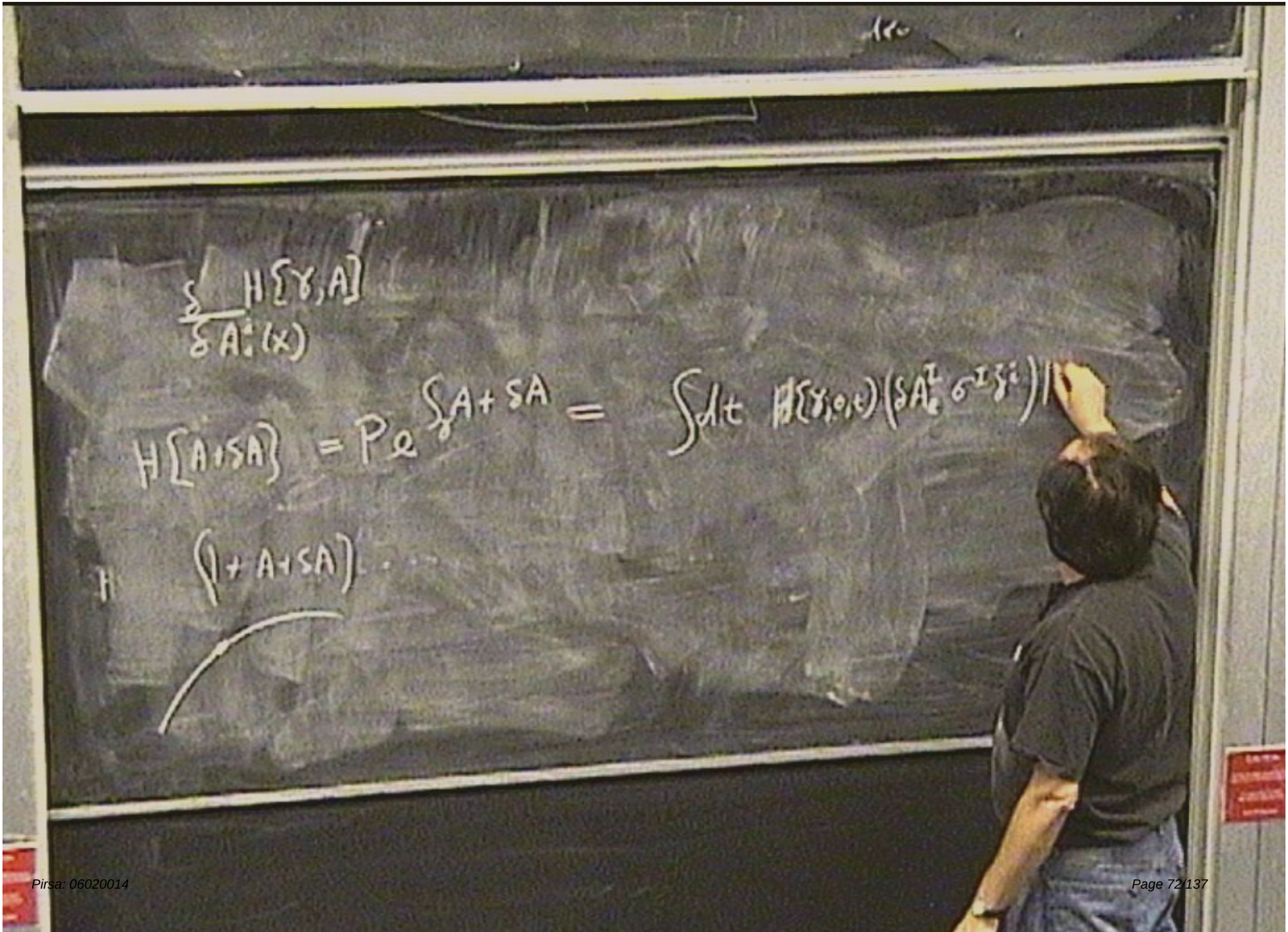

$\frac{\delta H[\gamma, A]}{\delta A_i(x)}$
$\delta H[\gamma, A] = P e^{\int_\gamma A + \delta A}$

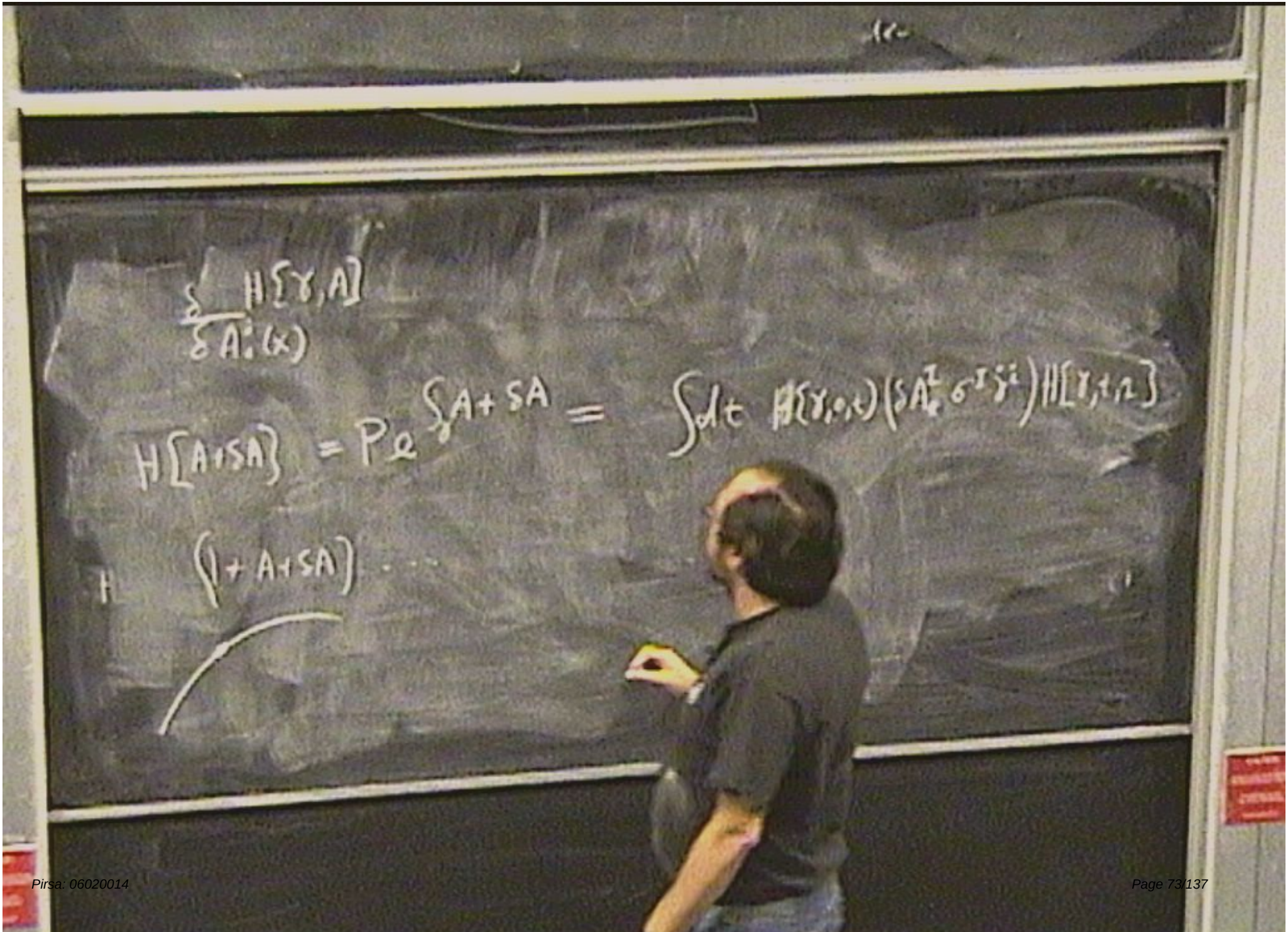

$\frac{\delta H[\gamma, A]}{\delta A_\mu^a(x)}$
$H[A, \delta A] = P e^{\int_\gamma A + \delta A}$

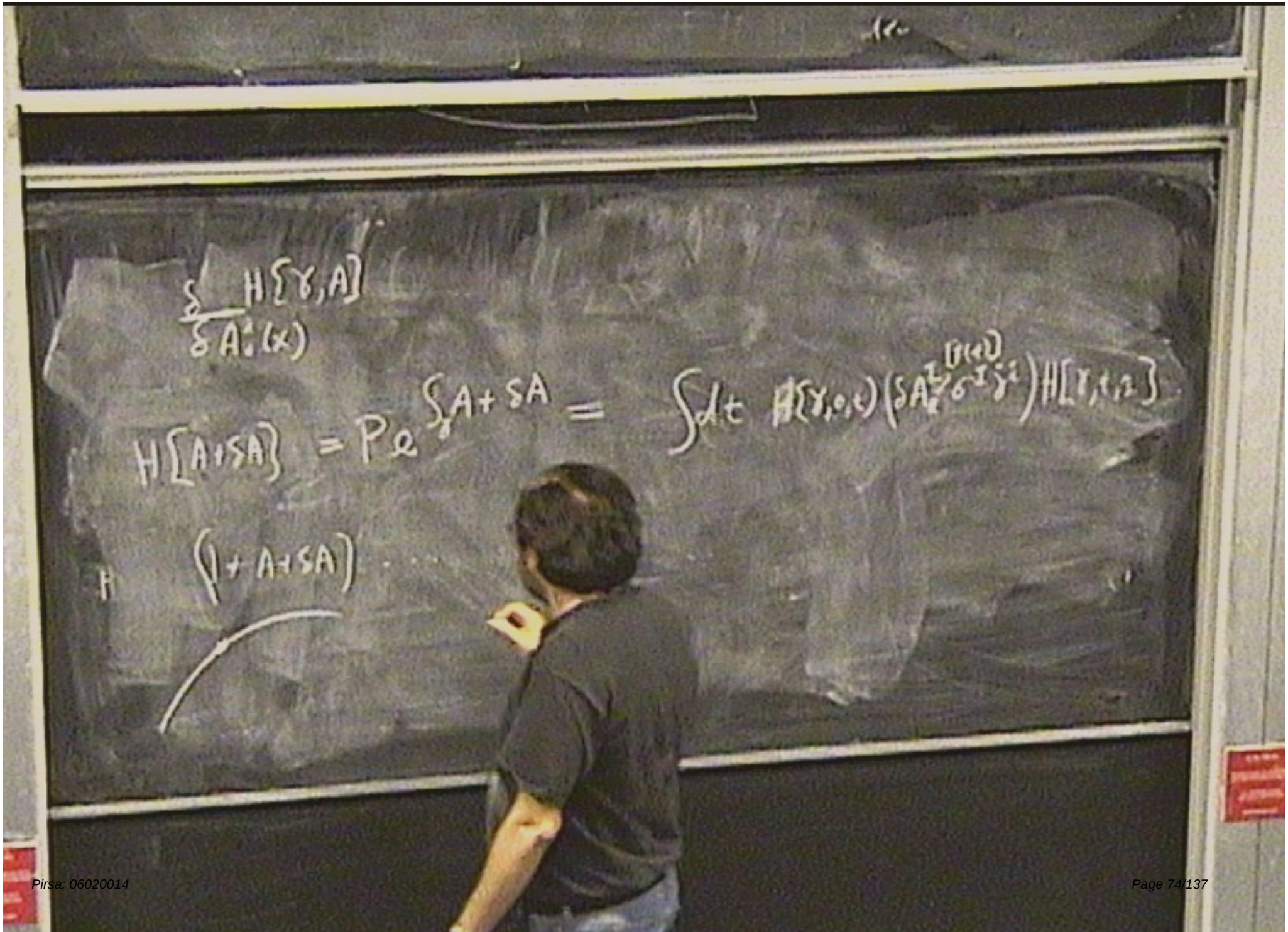

$\frac{\delta H[\gamma, A]}{\delta A_\mu(x)}$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A} =$
$H[A]\,(1 + A + \delta A) \cdots$


$\frac{\delta H[\gamma, A]}{\delta A_\mu^a(x)}$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt \, \gamma$
$(1 + A + \delta A)$


$\frac{\delta H[\gamma, A]}{\delta A_i(x)}$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt\, \gamma$
$(1 + A+\delta A)$


$\frac{\delta H[\gamma, A]}{\delta A_\mu^a(x)}$
$H[A+\delta A] = Pe^{\int_\gamma A+\delta A} = \int dt$
$(1 + A+\delta A)$


$\frac{\delta}{\delta A^i_a(x)} H[\gamma, A]$
$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] (\delta A^i_a \sigma^i \dot{\gamma}^i) H[\gamma, t, 1]$
$(1 + A + \delta A) \ldots$

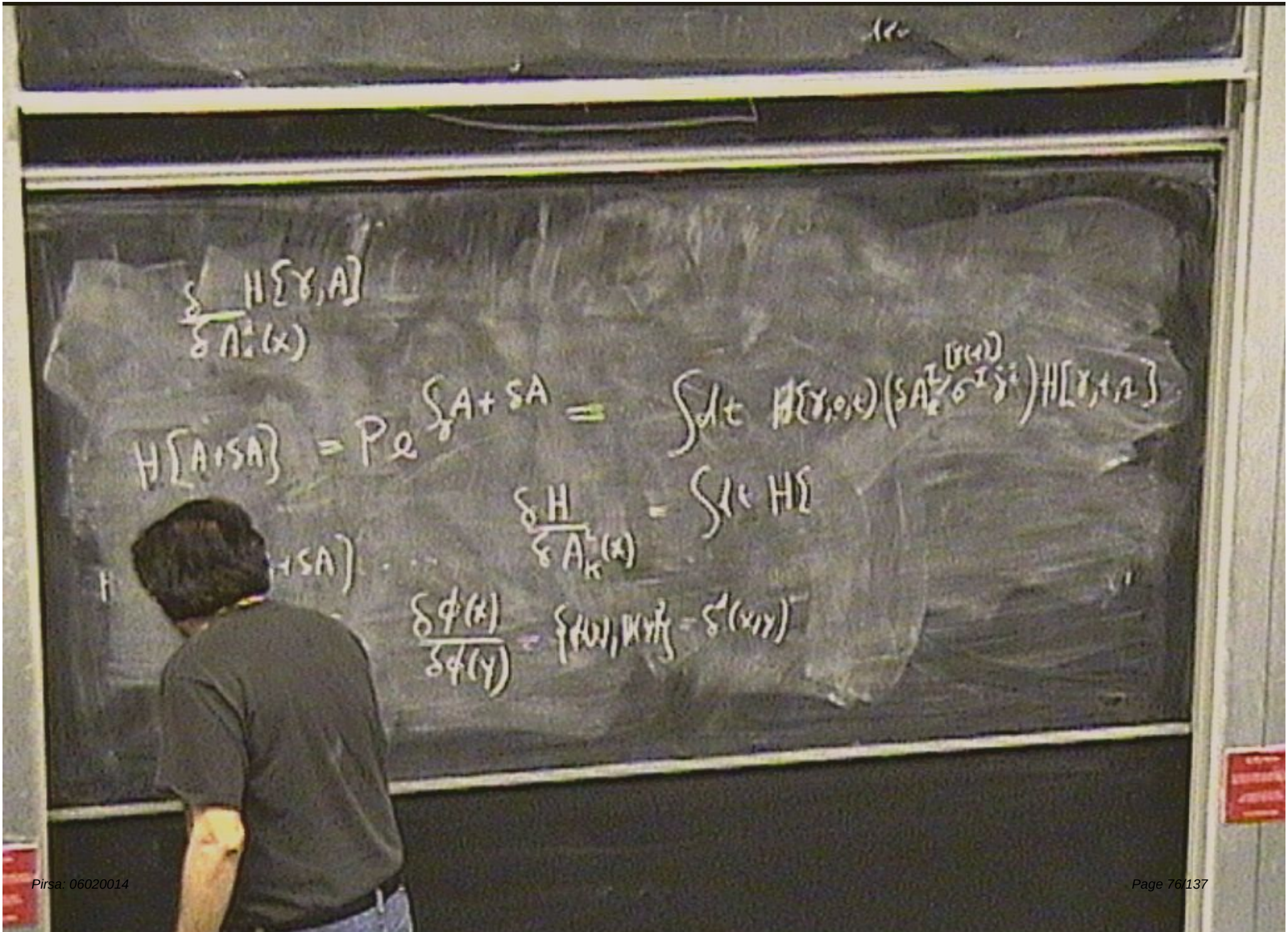

$\frac{\delta}{\delta A^i_\alpha(x)} H[\gamma, A]$
$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] (\delta A^I_\alpha \sigma^I \dot{\gamma}^i) H[\gamma, t, 1]$
$(1 + A + \delta A)$

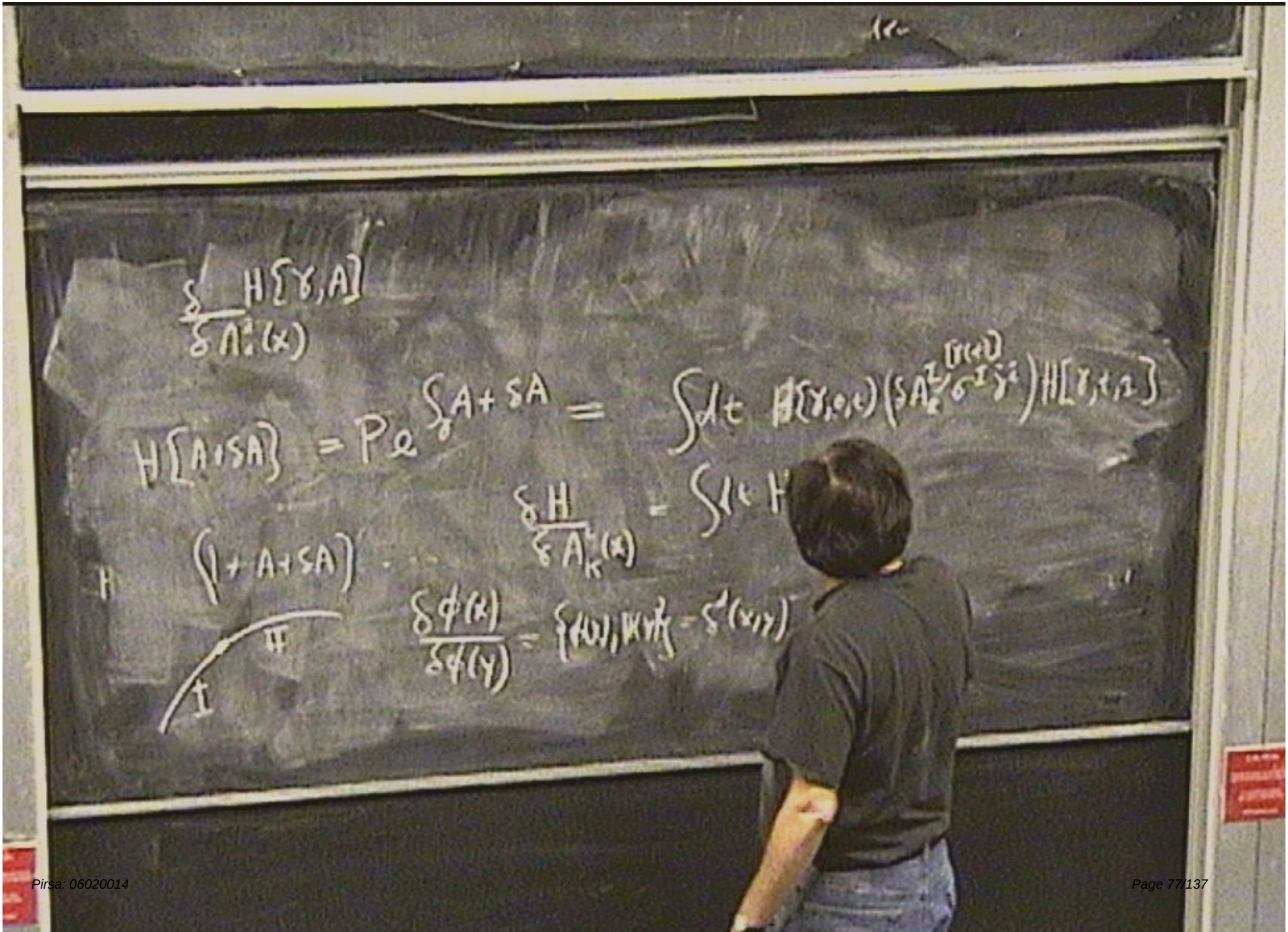
$$\frac{\delta}{\delta A^i_a(x)} H[\gamma, A]$$

$$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] \left(\delta A^i_a \sigma^i \dot{\gamma}^i\right) H[\gamma, t, 1]$$

$$(1 + A + \delta A) \ldots$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \phi(y)\} = \delta^d(x,y)$$


$\frac{\delta}{\delta A_k^a(x)} H[\gamma, A]$
$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] (\delta A_k^I \sigma^I \dot{\gamma}^i) H[\gamma, t, 1]$
$\frac{\delta H}{\delta A_k^a(x)} = \int dt\, H[$
$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \Pi(y)\} = \delta^d(x,y)$


$\frac{\delta}{\delta A_i^a(x)} H[\gamma, A]$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt$
$\frac{\delta H}{\delta A_k^a(x)} = \int dt$
$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^4(x-y)$

$$\frac{\delta}{\delta A_i^a(x)} H[\gamma, A]$$

$$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] \left(\delta A_i^I \sigma^I \dot{\gamma}^i\right) H[\gamma, t, 1]$$

$$\frac{\delta H}{\delta A_k^L(x)} = \int dt\, H[I] \frac{\delta A^I[\gamma(t)]}{\delta A_k^L(x)} H[II]$$

$$H \quad (1 + A + \delta A) \ldots$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \phi(y)\} = \delta^d(x,y)$$

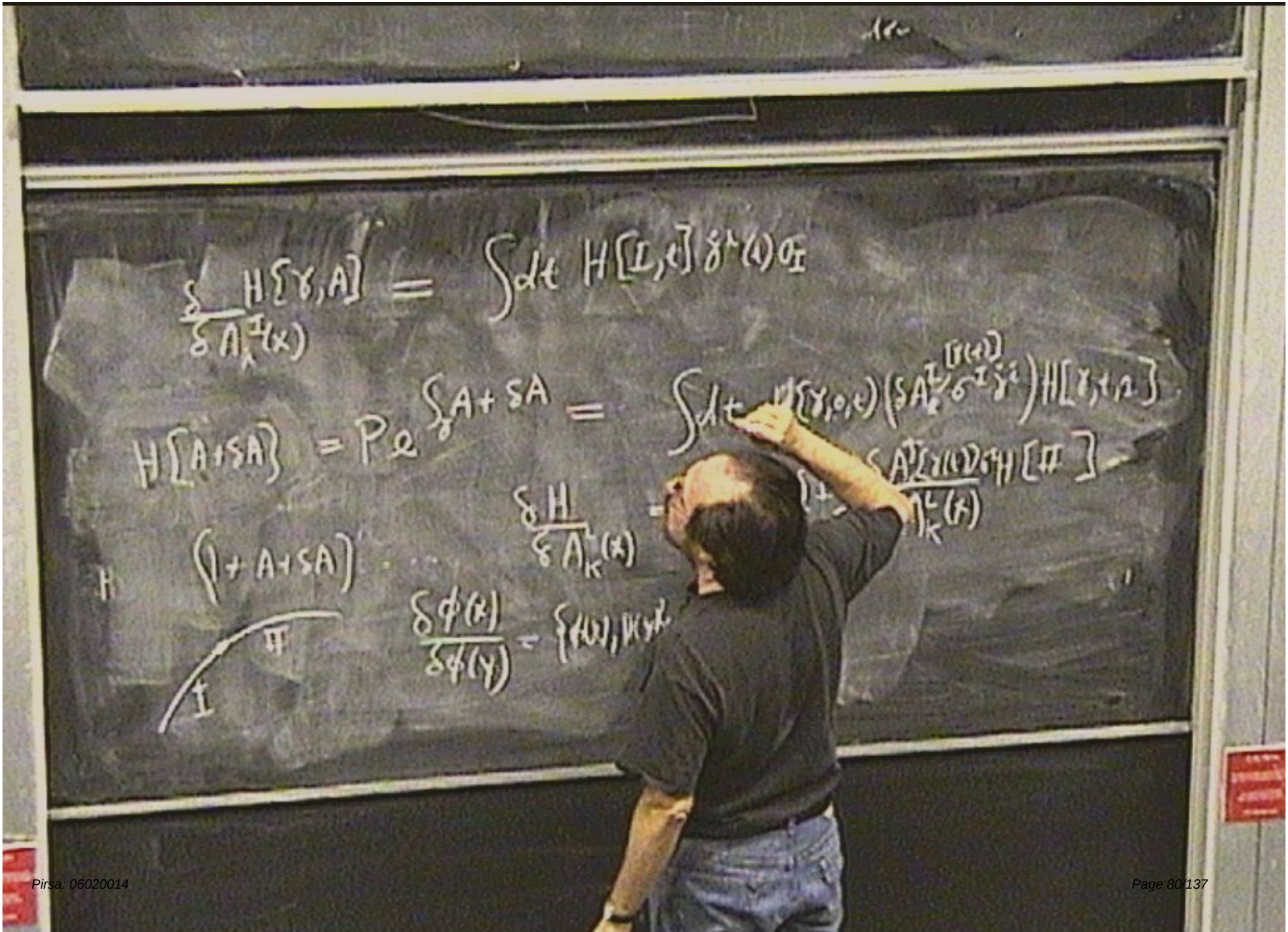
$$\frac{\delta}{\delta A_i^a(x)} H[\gamma, A]$$

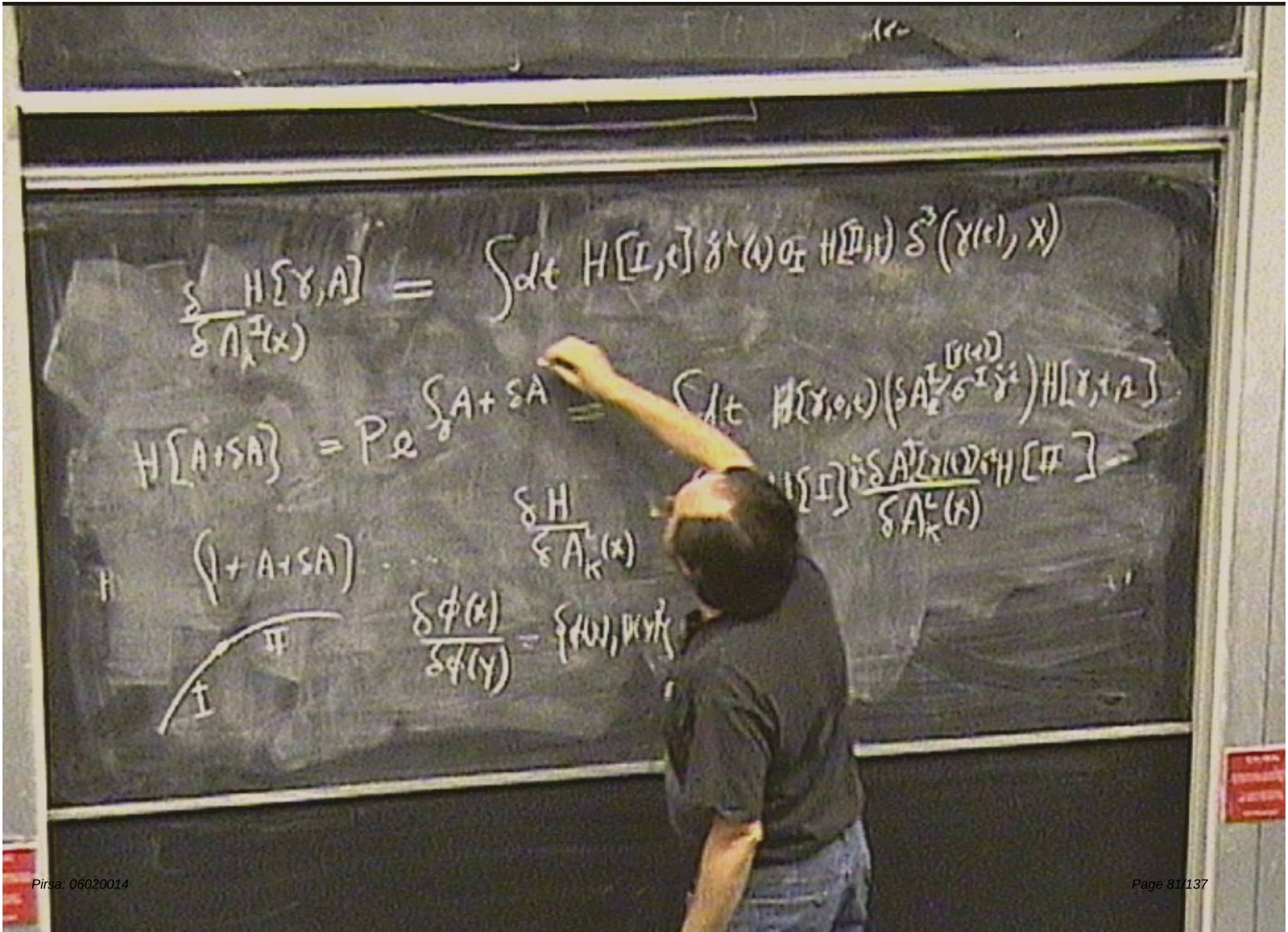
$$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] \left(\delta A_i^I \sigma^I \dot{\gamma}^i\right) H[\gamma, t, 1]$$

$$\frac{\delta H}{\delta A_k^L(x)} = \int dt\, H[I] \frac{\delta A^I[\gamma(t)]}{\delta A_k^L(x)} \sigma^I H[II]$$

$$(1 + A + \delta A) \ldots$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \phi(y)\} = \delta^d(x,y)$$


$\frac{\delta}{\delta A^a_k(x)} H[\gamma, A] = \int dt\, H[\Pi, t]\, \dot{\gamma}^a(t)\, \sigma_\Pi$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt$
$\frac{\delta H}{\delta A^a_k(x)}$
$\frac{\delta \phi(x)}{\delta \phi(y)}$
I
II


$\frac{\delta}{\delta A^a_k(x)} H[\gamma, A] = \int dt\, H[\Pi, t]\, \dot{\gamma}^i(t)\, \sigma^a_I\, H[\Pi, t]\, \delta^3(\gamma(t), x)$
$H[A+\delta A] = P e^{\int_\gamma A+\delta A}$
$\frac{\delta H}{\delta A^a_k(x)}$
$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \}$
I
II

$$\boxed{\frac{\delta}{\delta A_k^I(x)} H[\gamma, A] = \int dt\, H[I,t]\, \dot{\gamma}^k(t)\, \sigma_I\, H[II,t]\, \delta^3(\gamma(t), x)}$$

$$H[A+\delta A] = P e^{\int_\gamma A + \delta A} = \int dt\, H[\gamma, 0, t] \left(\delta A_k^I \sigma^I \dot{\gamma}^k\right) H[\gamma, t, 1]$$

$$\frac{\delta H}{\delta A_k^I(x)} = \int dt\, H[I]\, \frac{\delta A^I[\gamma(t)]\, \sigma^I}{\delta A_k^I(x)}\, H[II]$$

$$(1 + A + \delta A) \ldots$$

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \{\phi(x), \pi(y)\} = \delta^d(x,y)$$

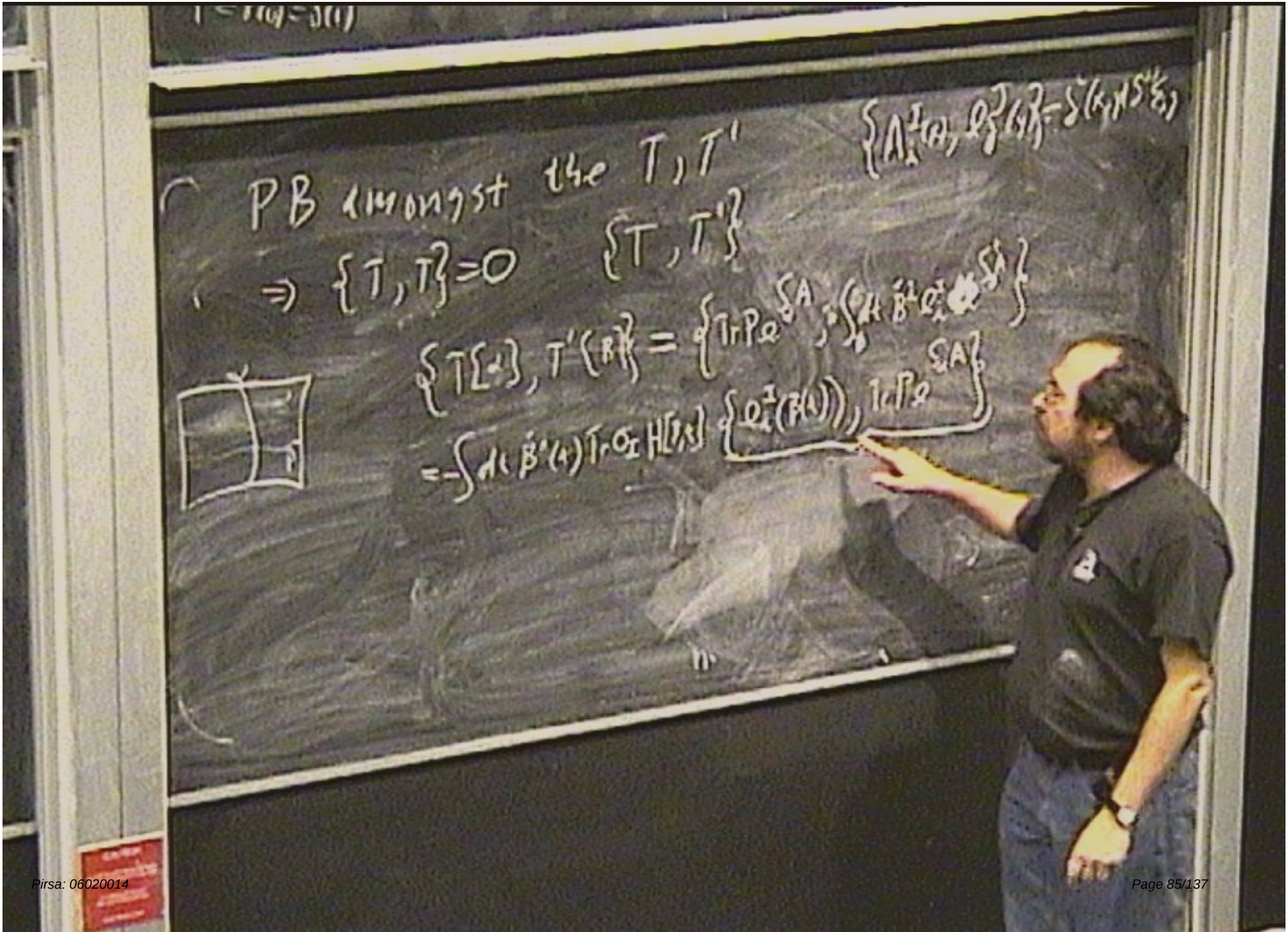
$$\boxed{\{H(\gamma,A),\Pi(x)\}\frac{\delta}{\delta A_k^I(x)} H[\gamma,A] = \int dt\, H[I,t]\,\dot\gamma^i(t)\,\sigma_I\, H[II,t]\,\delta^3(\gamma(t),x)}$$

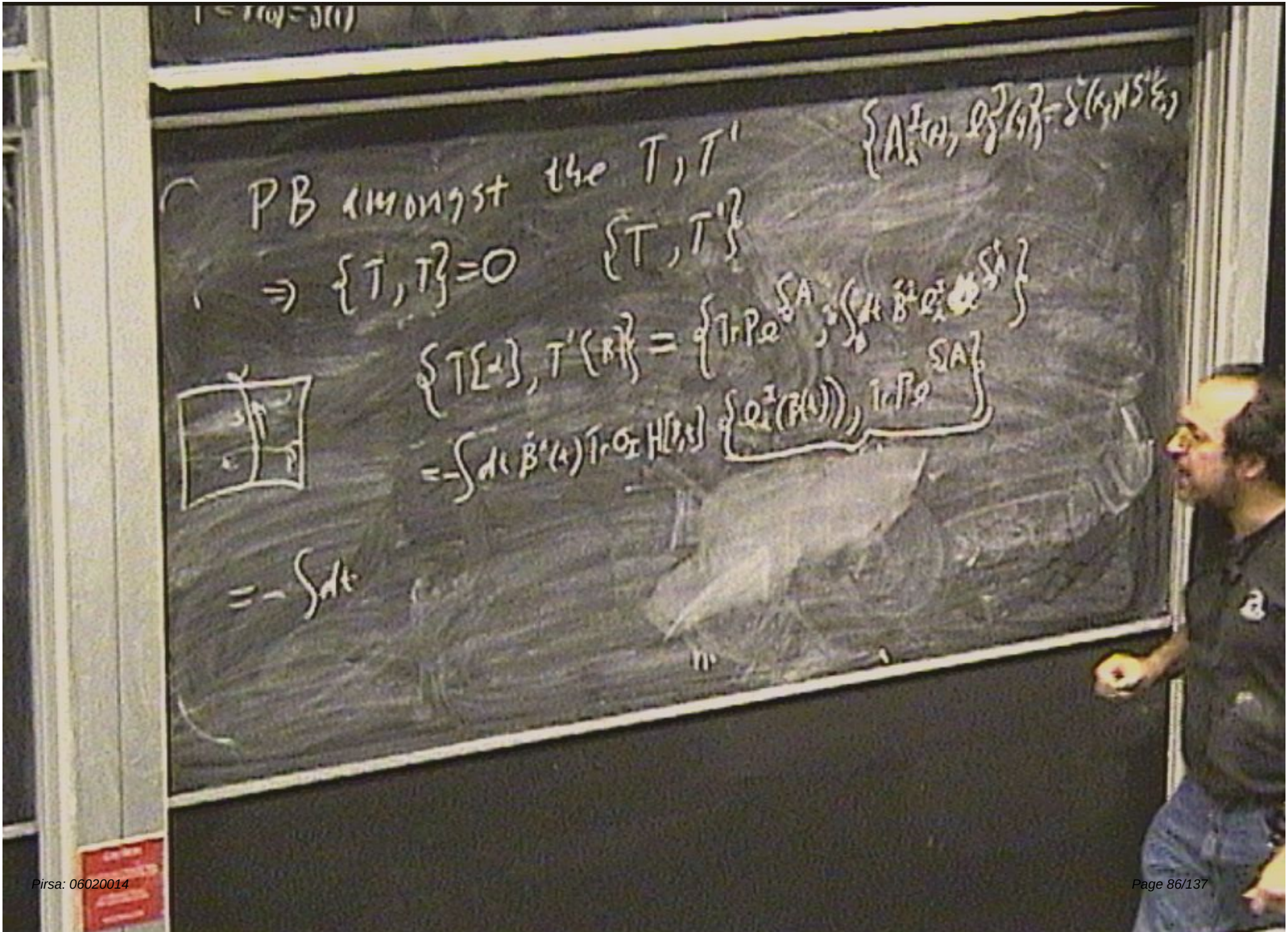
$$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt\, H[\gamma,0,t]\left(\delta A_i^I[\gamma(t)]\,\sigma^I\dot\gamma^i\right) H[\gamma,t,1]$$

$$\frac{\delta H}{\delta A_k^I(x)} = \int dt\, H[I]\,\frac{\delta A^I[\gamma(t)]\,\sigma^I}{\delta A_k^I(x)}\, H[II]$$

$$(1+A+\delta A)\ldots$$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \{\phi(x),\Pi(y)\} = \delta^d(x,y)$$


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}
= −∫dt

$T = T(0) = \delta(1)$

$\{A_x^I(x), e_y^J(y)\} = \delta^{IJ}(x,y)\,\delta^{IJ}(x,y)$

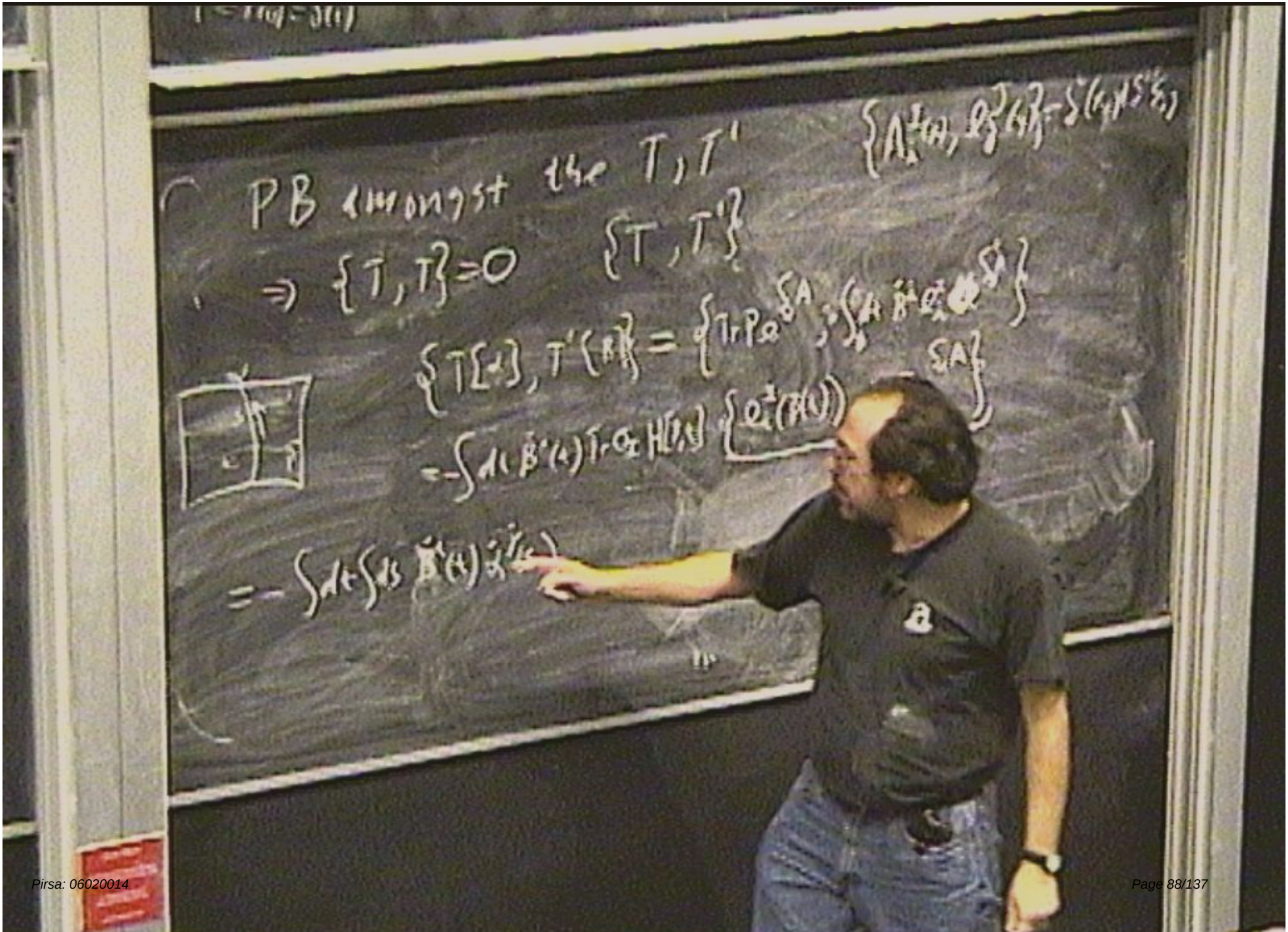
PB amongst the T, T'

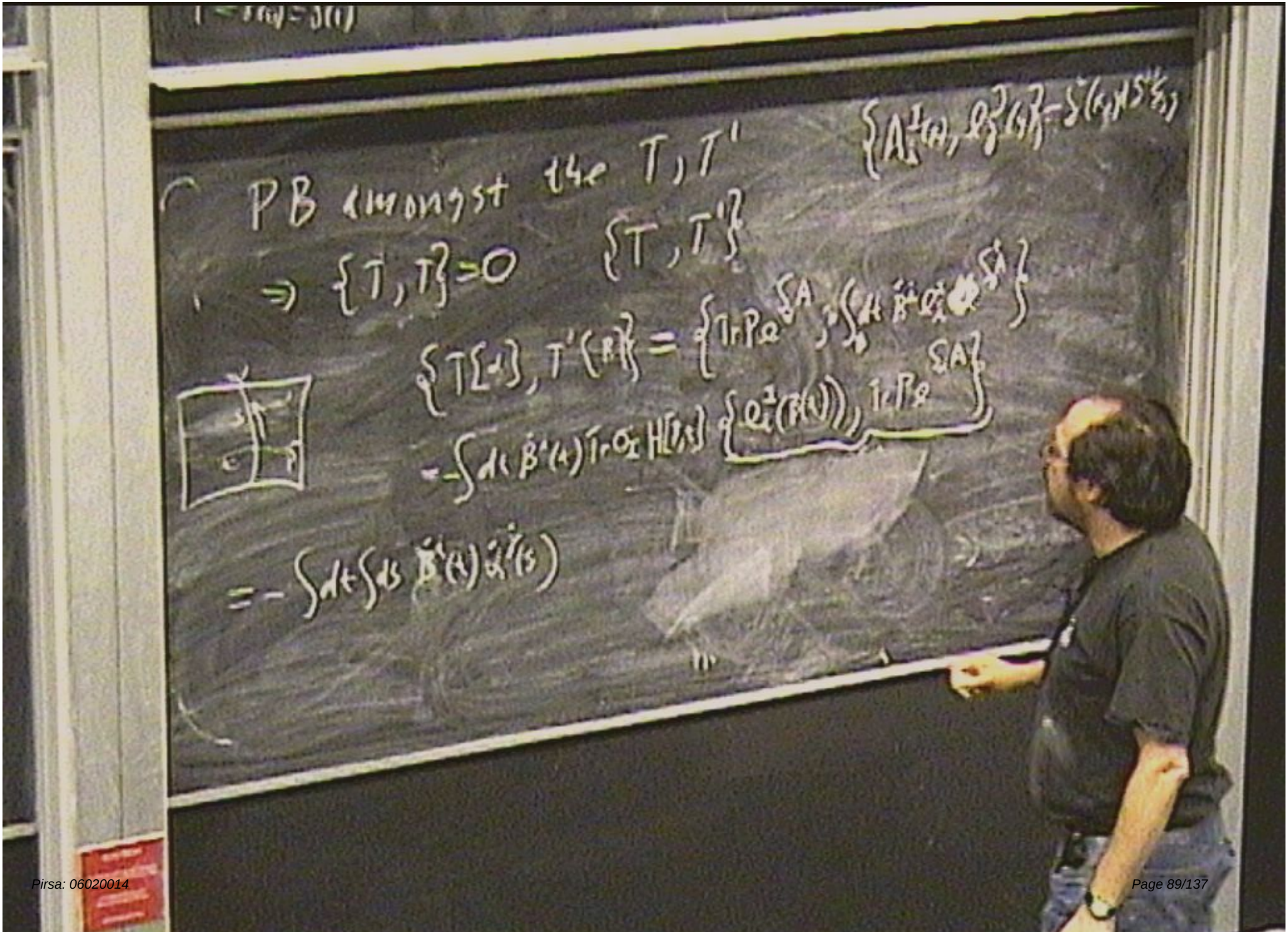
$\Rightarrow \{T, T\} = 0 \qquad \{T, T'\}$

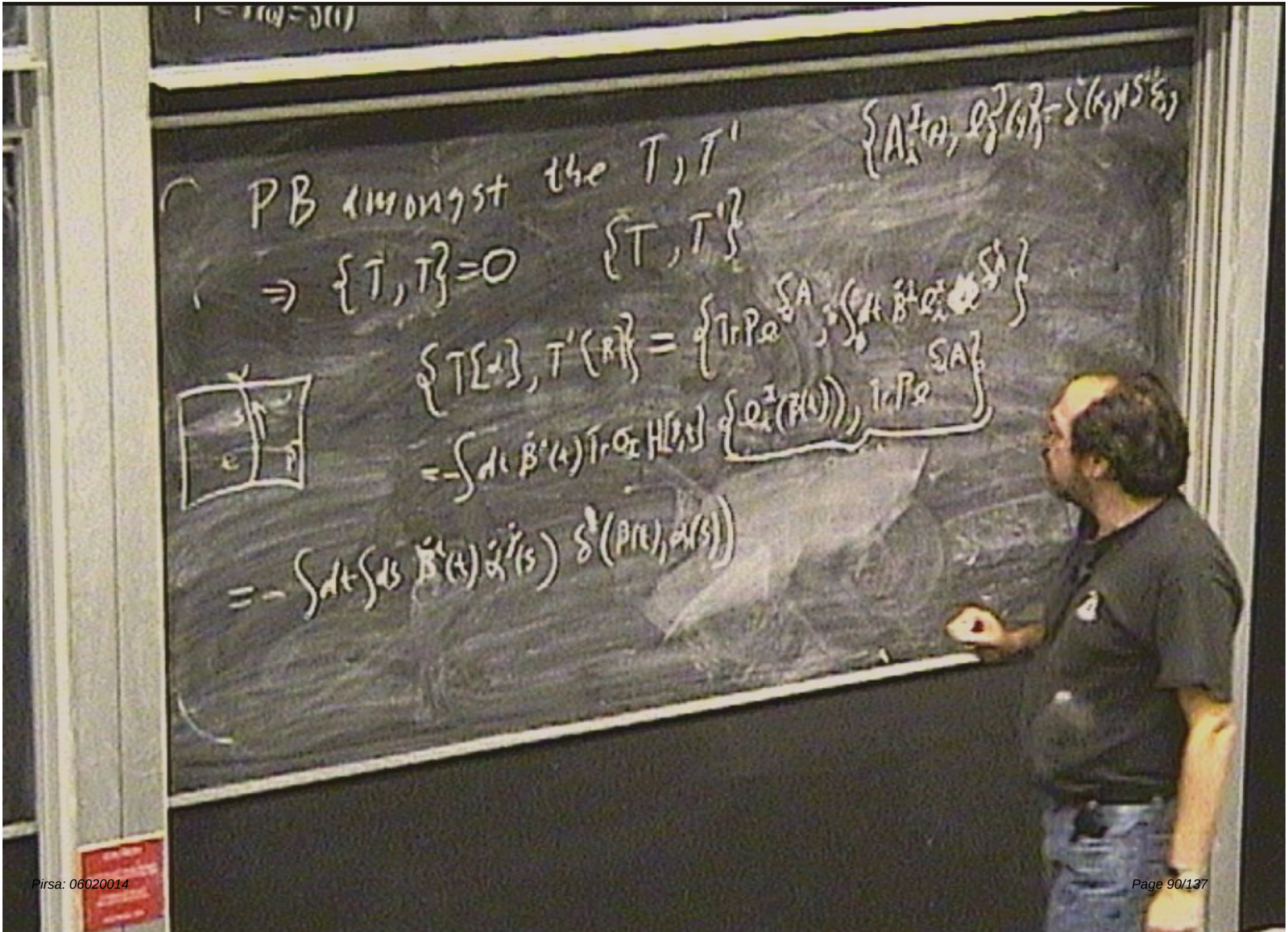
$\{T[\alpha], T'[\beta]\} = \{\mathrm{Tr}\,P e^{\int A}, \int_\beta dt\, \dot{\beta}^I e_t^I e^{\int A}\}$

$= -\int dt\, \dot{\beta}^I(t)\, \mathrm{Tr}\,\sigma_I H[P,t]\, \underbrace{\{e_t^I(\beta(t)), \mathrm{Tr}\,P e^{\int A}\}}$

$= -\int dt \int ds\, \dot{\beta}^I(t)$


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}
{T[α], T'[β]} = {Tr P e^{δA}, ∫ dt β̇^I e_I^J e^{δA}}
= −∫ dt β̇^I(t) Tr σ_I H[β,t] {e_I^J(β(t)), Tr P e^{δA}}
= −∫ dt ∫ ds β̇^I(t) α̇^J(s) δ(β(t), α(s))

$T = T(0) = \delta(1)$

$\{A_i^I(x), e_j^J(y)\} = \delta^2(x,y)\, \delta^{IJ}\, \delta_{ij}$

PB amongst the T, T'

$\Rightarrow \{T, T\} = 0 \qquad \{T, T'\}$

$$\{T[\alpha], T'[\beta]\} = \{\mathrm{Tr}\, P e^{\int A}, \int_B dt\, \dot{\beta}^i e_i^I e^{\int A}\}$$

$$= -\int dt\, \dot{\beta}^i(t)\, \mathrm{Tr}\, \sigma_I H[T,t]\, \underbrace{\{e_i^I(\beta(t)), \mathrm{Tr}\, P e^{\int A}\}}$$

$$= -\int dt \int ds\, \dot{\beta}^i(t)\, \dot{\alpha}^j(s)\, \delta^3(\beta(t), \alpha(s))$$

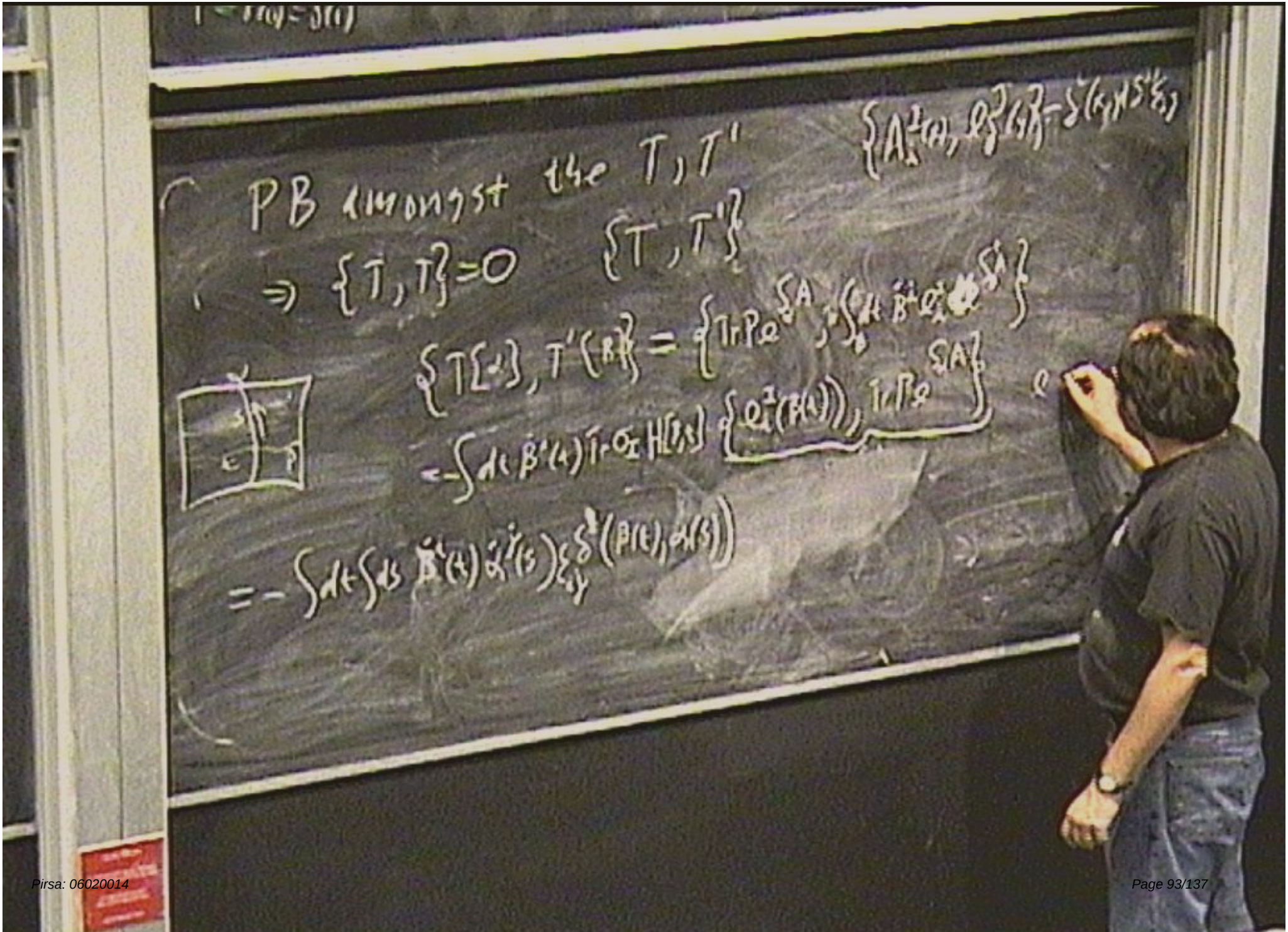
$$\boxed{\{H[\gamma,A],\Pi(x)\} = \frac{\delta}{\delta A_k^I(x)} H[\gamma,A] = \int dt\, \underset{H[\gamma,0,t)}{H[I,t]}\, \dot{\gamma}^i(t)\, \sigma_I\, \underset{H[\gamma,t,1]}{H[II,t]}\, \delta^3(\gamma(t),x)}$$

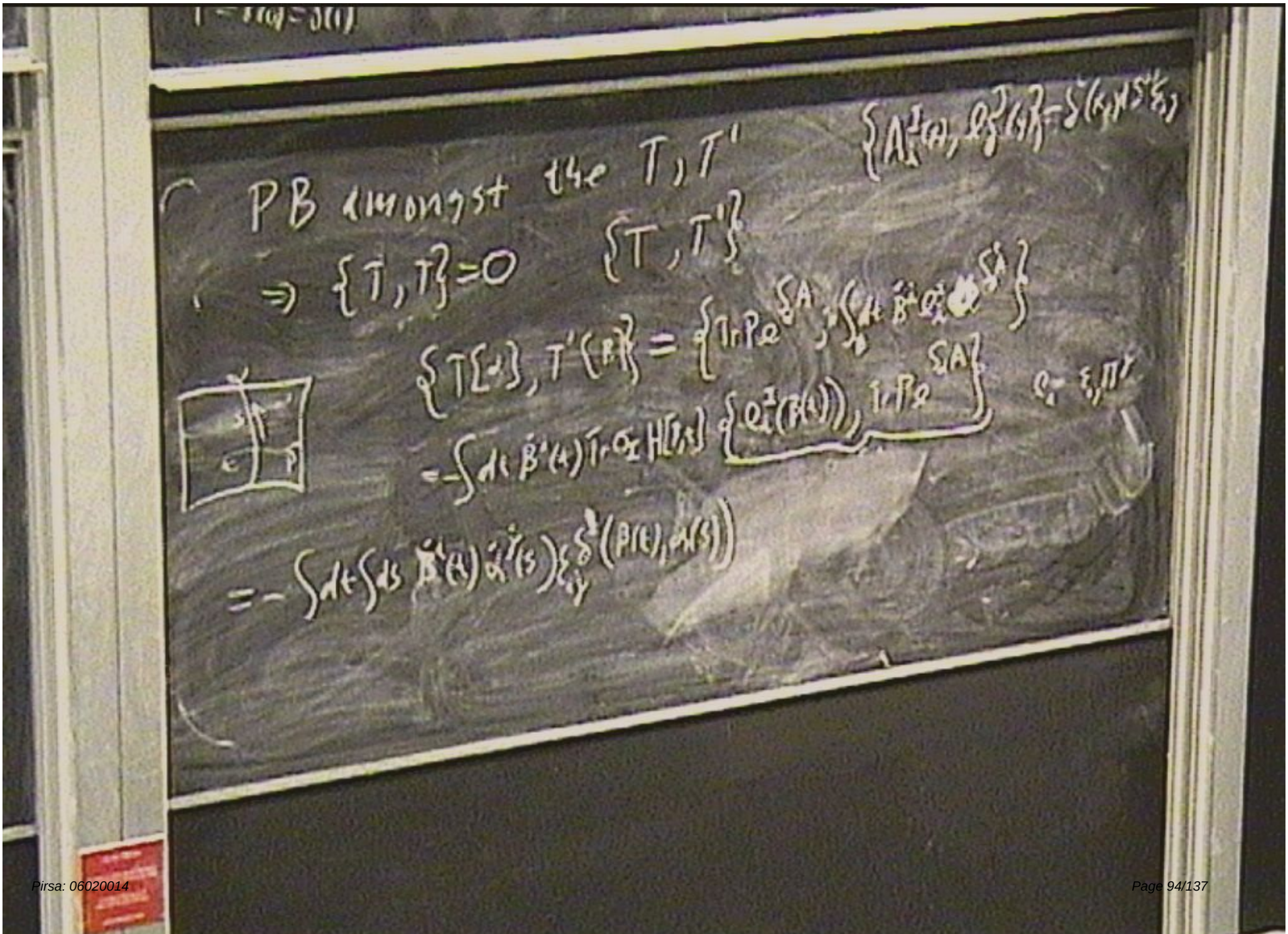
$$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt\, H[\gamma,0,t)\left(\delta A_i^I(\gamma(t))\, \sigma^I \dot{\gamma}^i\right) H[\gamma,t,1]$$

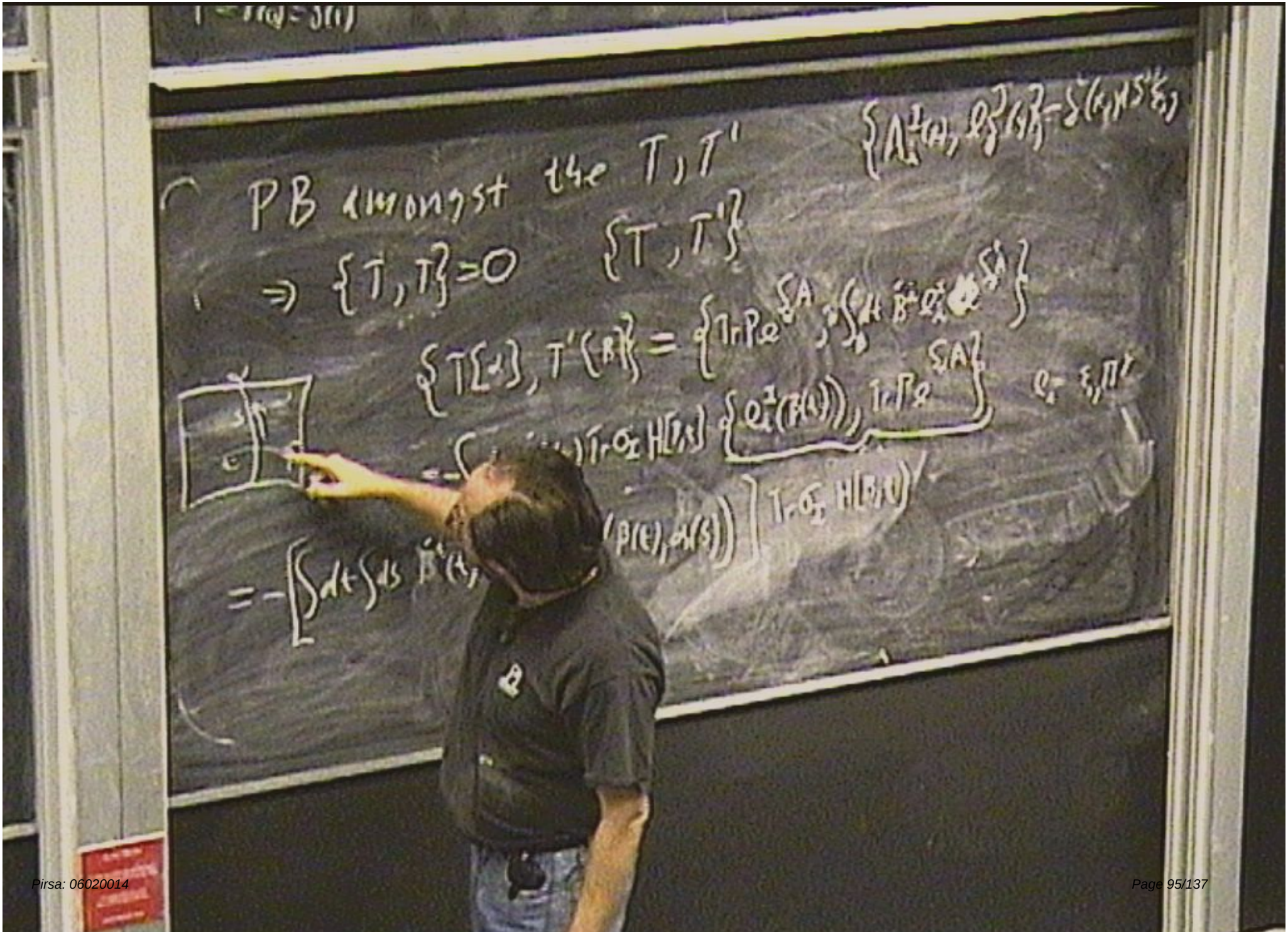
$$\frac{\delta H}{\delta A_k^I(x)} = \int dt\, H[I]\, \frac{\delta A^I(\gamma(t))}{\delta A_k^I(x)}\, \sigma^I H[II]$$

$$(1+A+\delta A) \ldots$$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \{\phi(x),\Pi(y)\} = \delta^d(x,y)$$


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}
{T[α], T'[β]} = {Tr P e^{δA}, ∫ dt β̇^i e^I_i e^{δA}}
= −∫ dt β̇^i(t) Tr σ_I H[T,t] {e^I_i(β(t)), Tr P e^{δA}}
= −∫ dt ∫ ds β̇^i(t) α̇^j(s) ε_{ij} δ^3(β(t), α(s))


PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}

$$\boxed{\{H[\gamma,A],\Pi(x)\}\frac{\delta}{\delta A_k^I(x)} H[\gamma,A] = \int dt\, H[I,t]\,\dot\gamma^i(t)\,\sigma_I\, H[II,t]\,\delta^3(\gamma(t),x)}$$

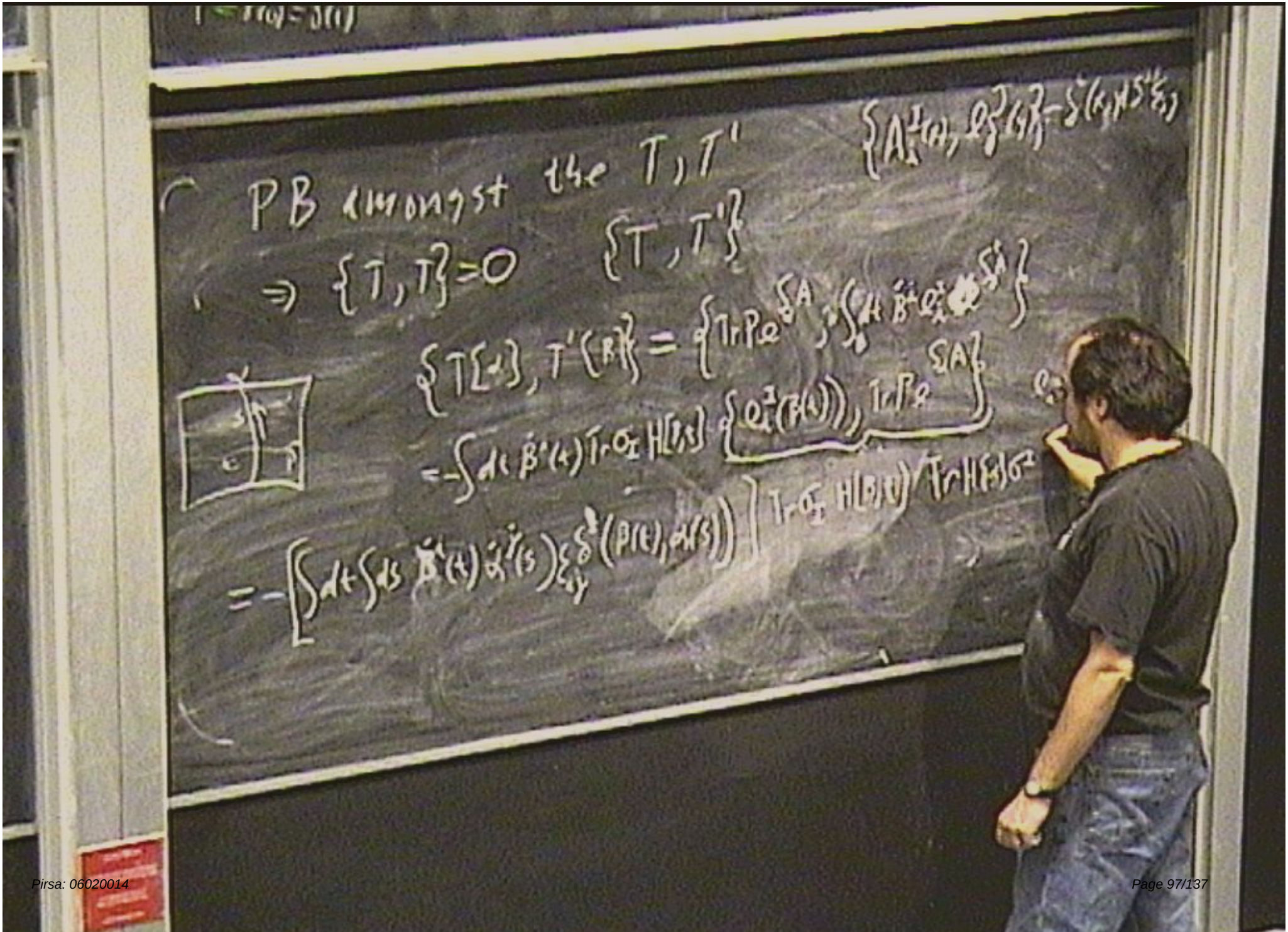
$H[\gamma,0,t)$ $\quad$ $H[\gamma,t,1]$

$$H[A+\delta A] = P e^{\int_\gamma A+\delta A} = \int dt\, H[\gamma,0,t)\left(\delta A^I_i\,\sigma^I\,\dot\gamma^i\right)H[\gamma,t,1]$$

$$\frac{\delta H}{\delta A_k^I(x)} = \int dt\, H[I]\,\frac{\delta A^I_i[\gamma(t)]}{\delta A_k^I(x)}\,\sigma^I H[II]$$

$$(1+A+\delta A)\ldots$$

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \{\phi(x),\Pi(y)\} - \delta^d(x,y)$$


PB amongst the T, T'
$\{T, T'\} = 0$
$\{T[\alpha], T'[\beta]\} = \{\mathrm{Tr}\, P_\alpha^{\delta A}, \int_\beta \dot{B}^{\epsilon}_{x} \sigma^{\epsilon}_{x} \delta^{A}\}$
$= -\int d\epsilon\, \dot{\beta}^{\epsilon}(\epsilon)\, \mathrm{Tr}\, \sigma_\epsilon H[\beta, \epsilon] \{\sigma^{\epsilon}_{x}(\beta(\epsilon)), \mathrm{Tr}\, P_\alpha^{\delta A}\}$
$= -\left[\int d\epsilon \int ds\, \dot{\beta}^{\epsilon}(\epsilon)\, \dot{\alpha}^{\gamma}(s)\, \xi^{\delta}_{xy}(\beta(\epsilon), \alpha(s))\right] \mathrm{Tr}\, \sigma_\epsilon H[\beta(\epsilon)] \mathrm{Tr}\, H[\beta]\, \sigma$

$T = T(q) = \delta(t)$

$\{A_i^I(x), e_j^J(y)\} = \delta^i(x,y)\,\delta^{ij}\,\xi_{ij}$

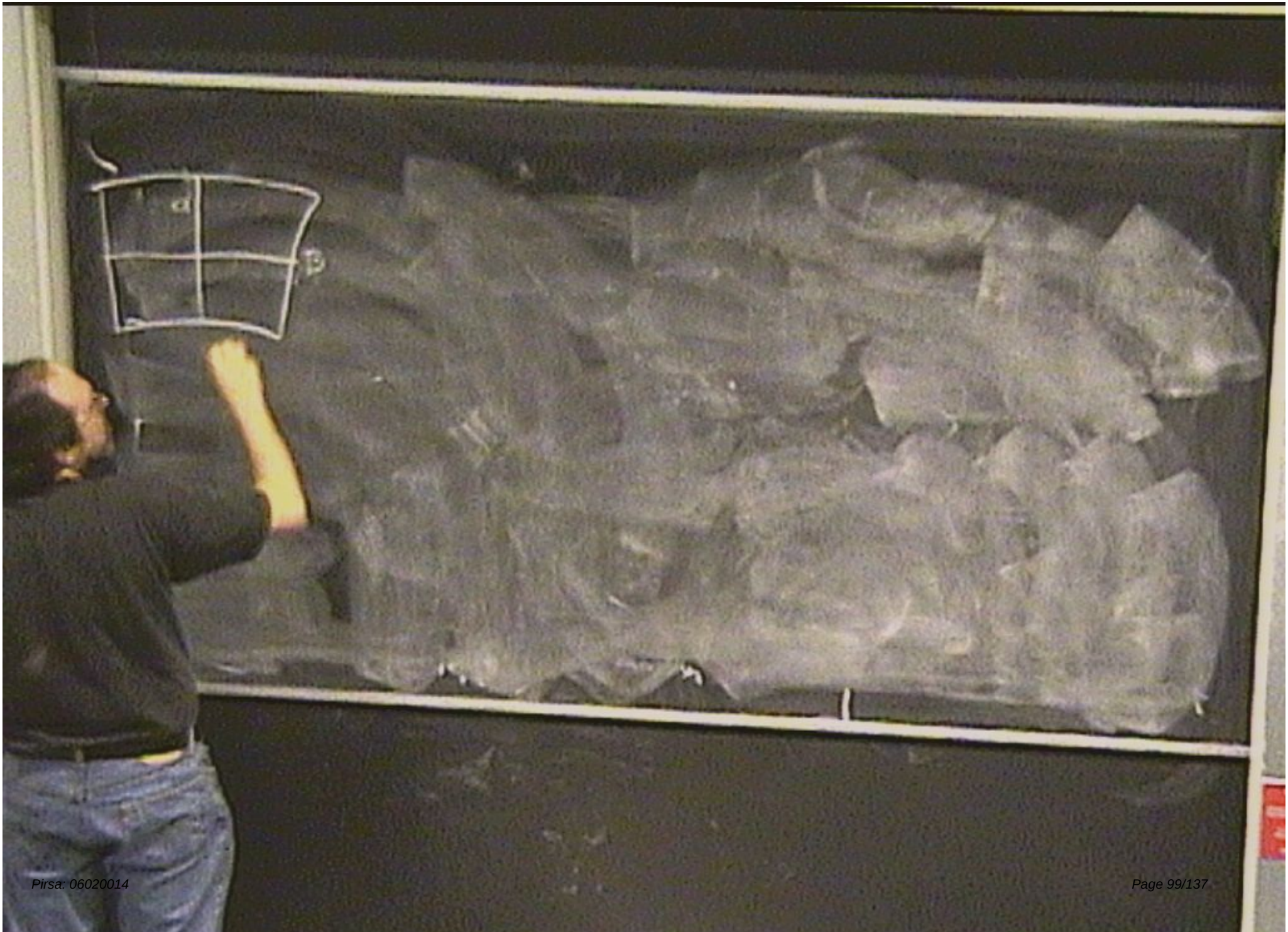
PB amongst the T, T'

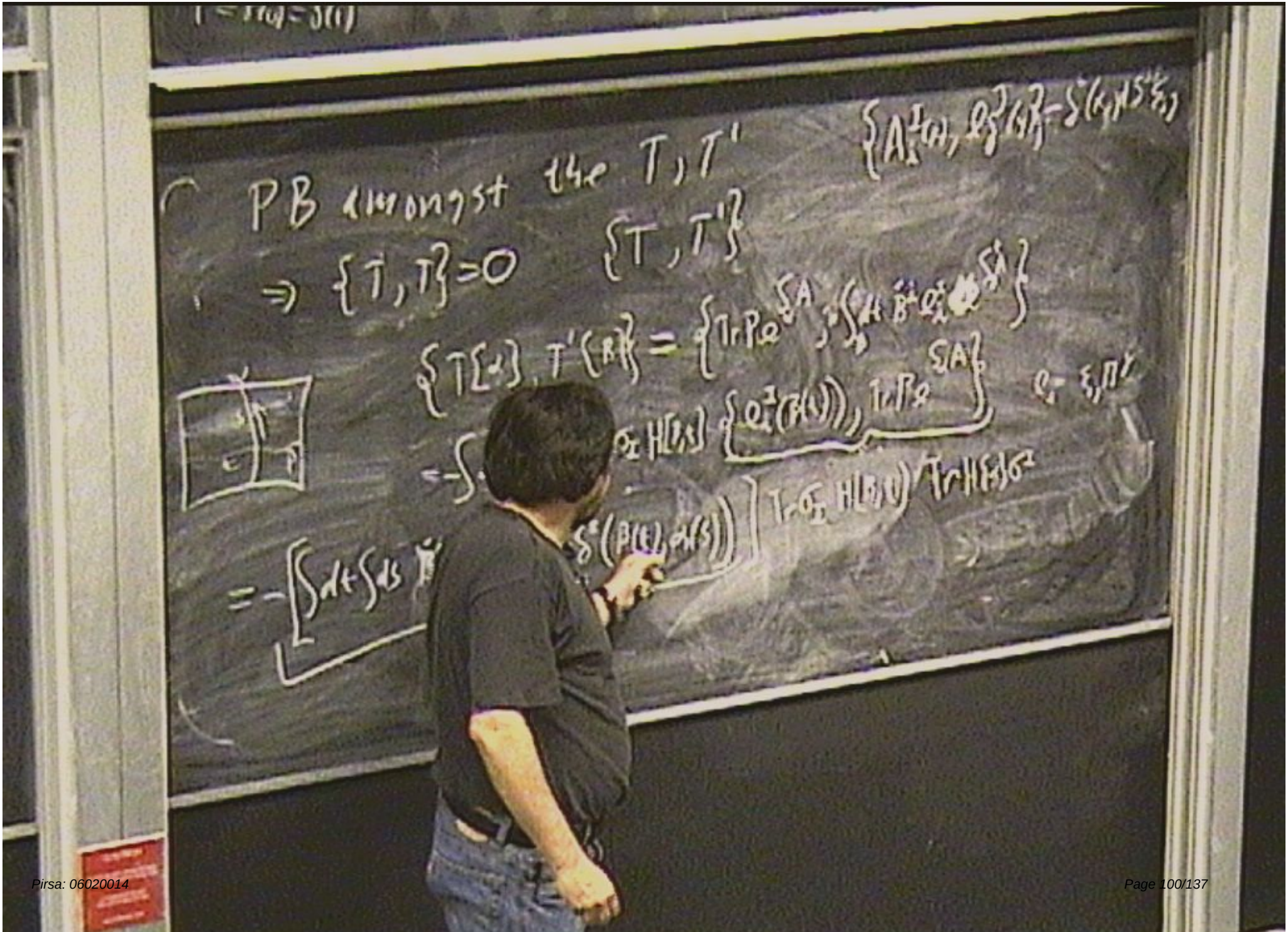
$\Rightarrow \{T, T\} = 0 \qquad \{T, T'\}$

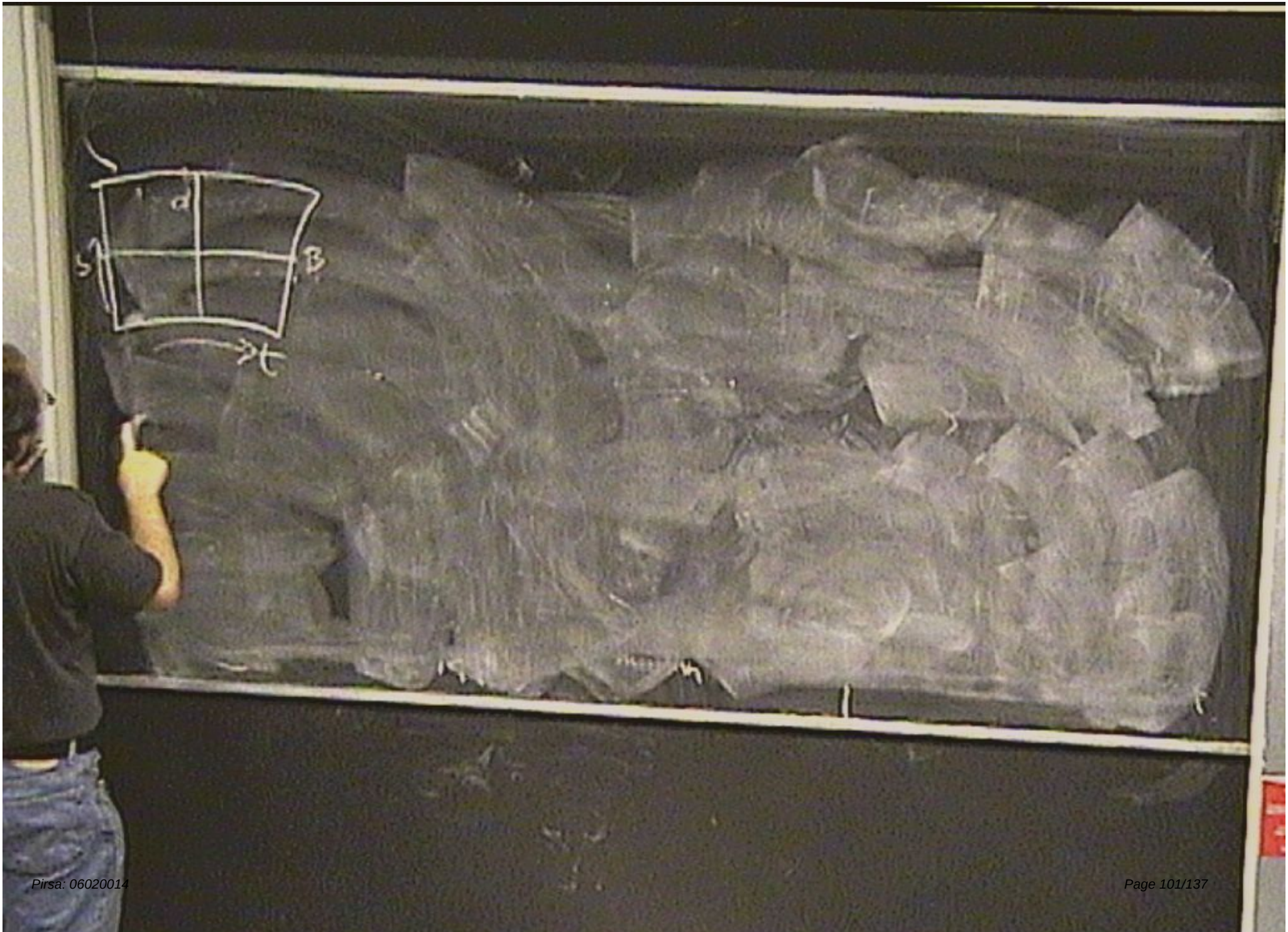
$$\{T[\alpha], T'[\beta]\} = \{\mathrm{Tr}\,P e^{\delta A}, \int_\beta dt\, \dot{\beta}^i e_i^I e^{\delta A}\}$$

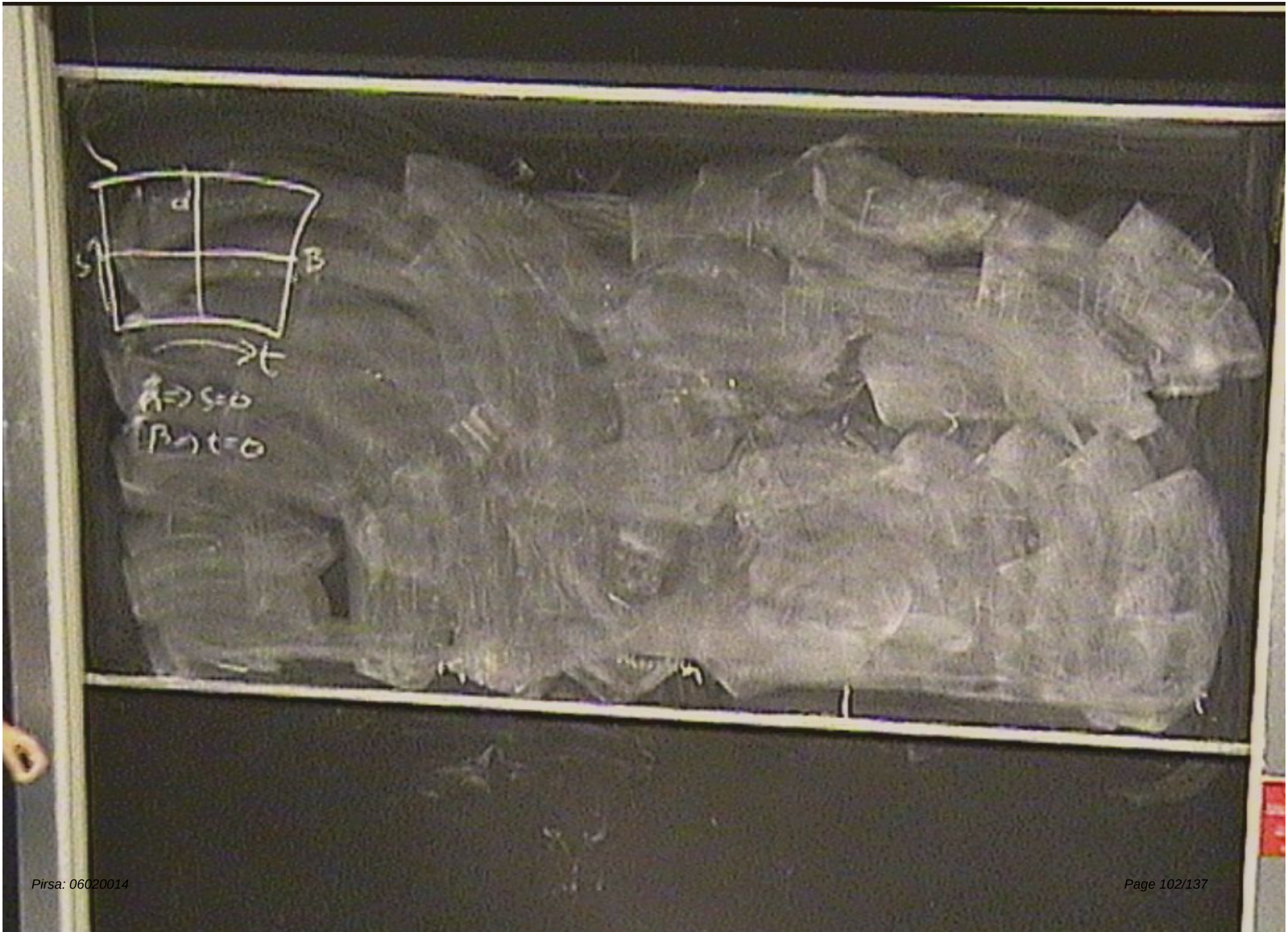
$$= -\int dt\, \dot{\beta}^i(t)\, \mathrm{Tr}\,\sigma_I H[\beta,t]\, \underbrace{\{e_i^I(\beta(t)), \mathrm{Tr}\,P e^{\delta A}\}}_{\mathrm{Tr}\,\sigma_I H[\beta,t]\,\mathrm{Tr}\,H[\alpha]\,\sigma^I} \qquad e_i = \xi_i \pi^i$$

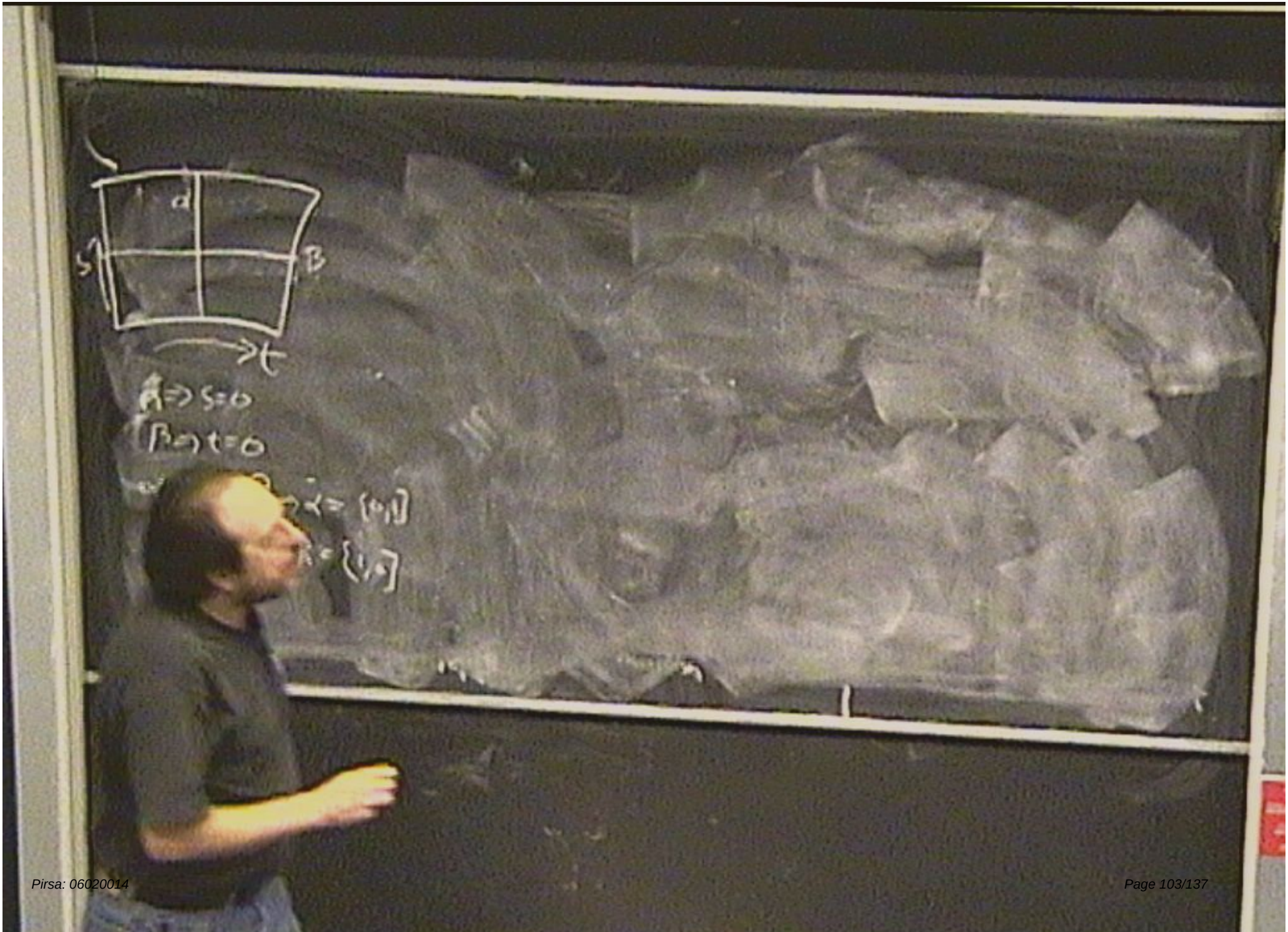
$$= -\underbrace{\left[\int dt \int ds\, \dot{\beta}^i(t)\, \dot{\alpha}^j(s)\, \xi_{ij}\, \delta^3(\beta(t), \alpha(s))\right]}$$

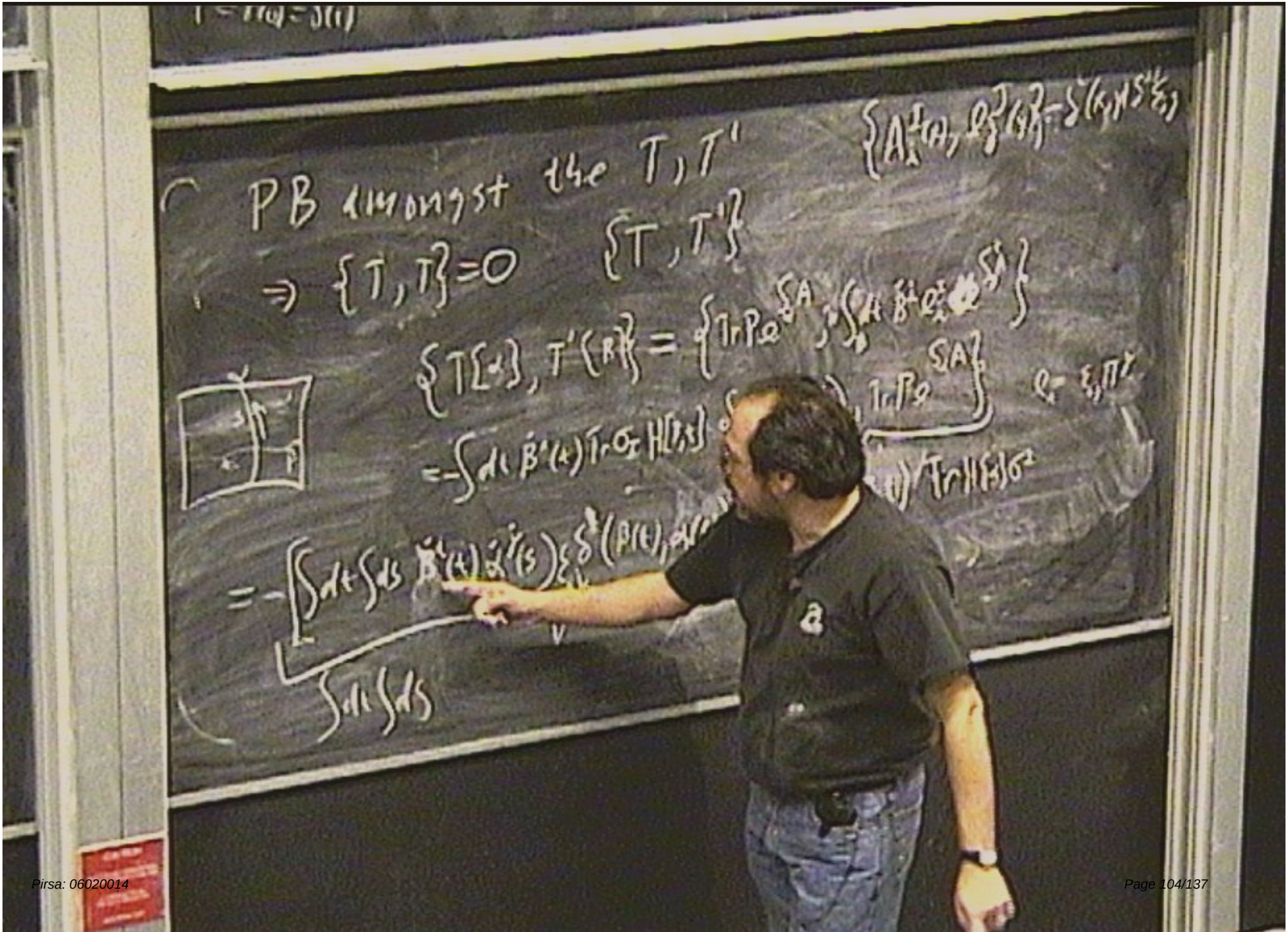

a
b

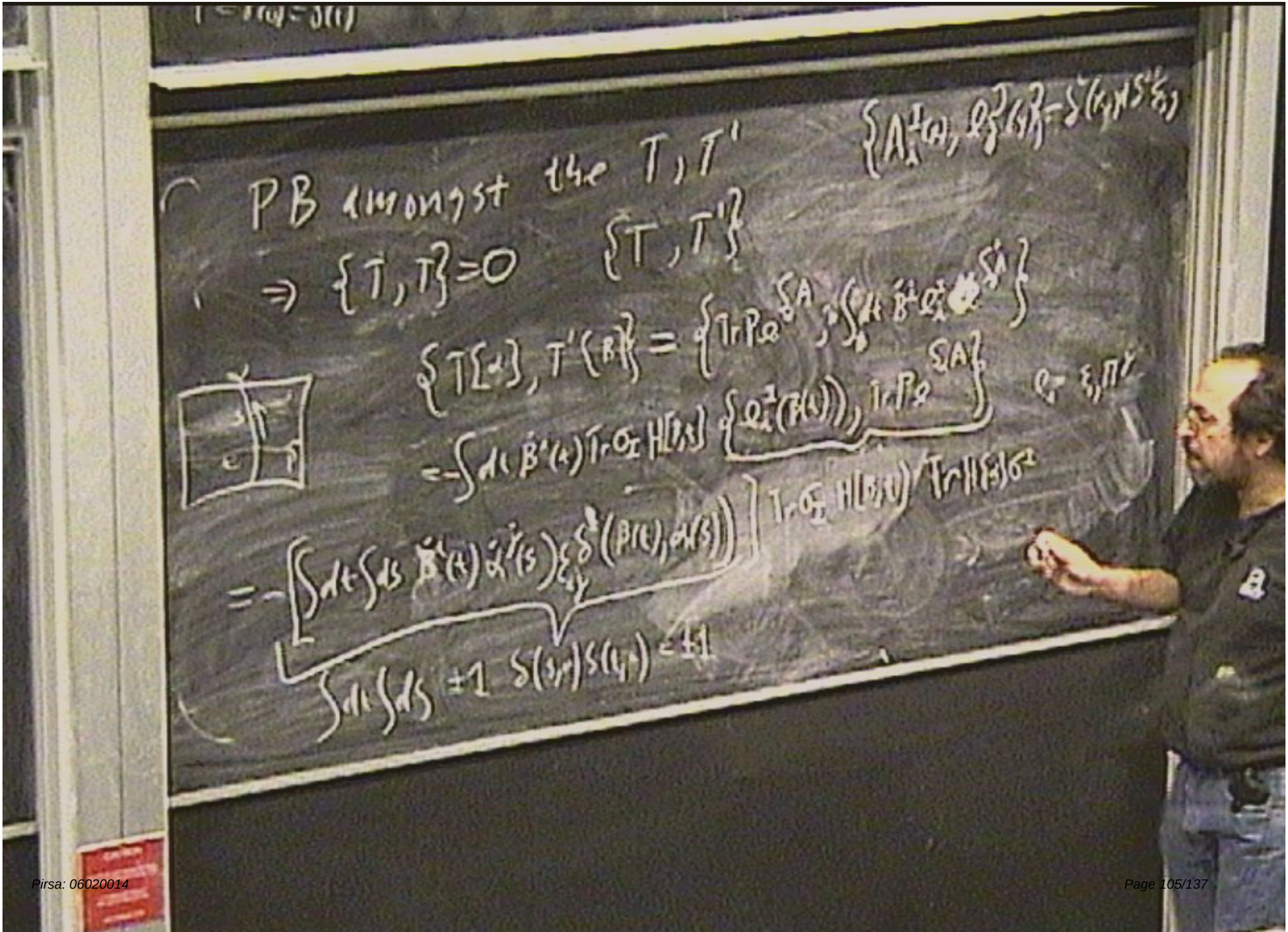

PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}

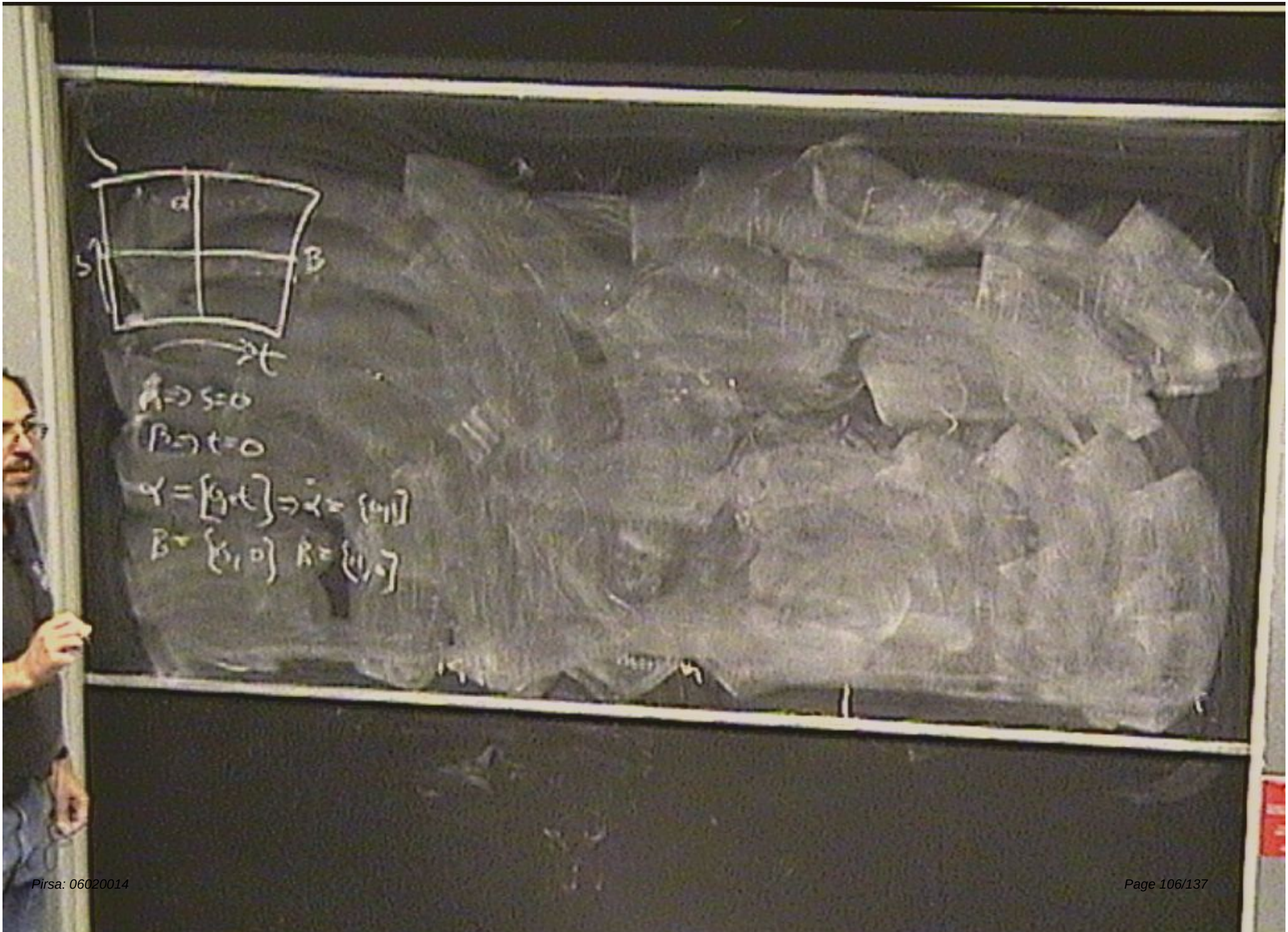

d
B

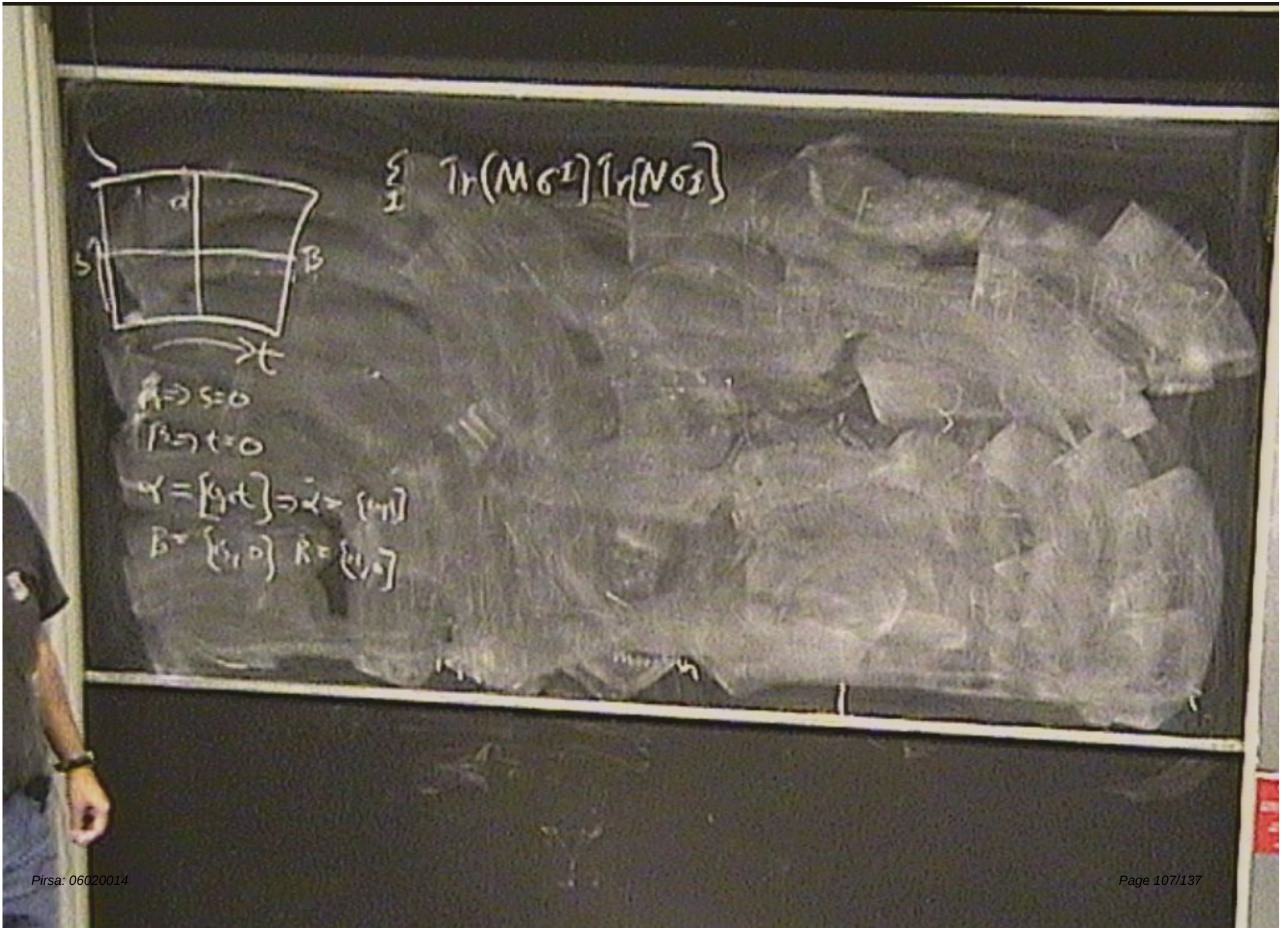

d
B
t
A => s=0
B => t=0

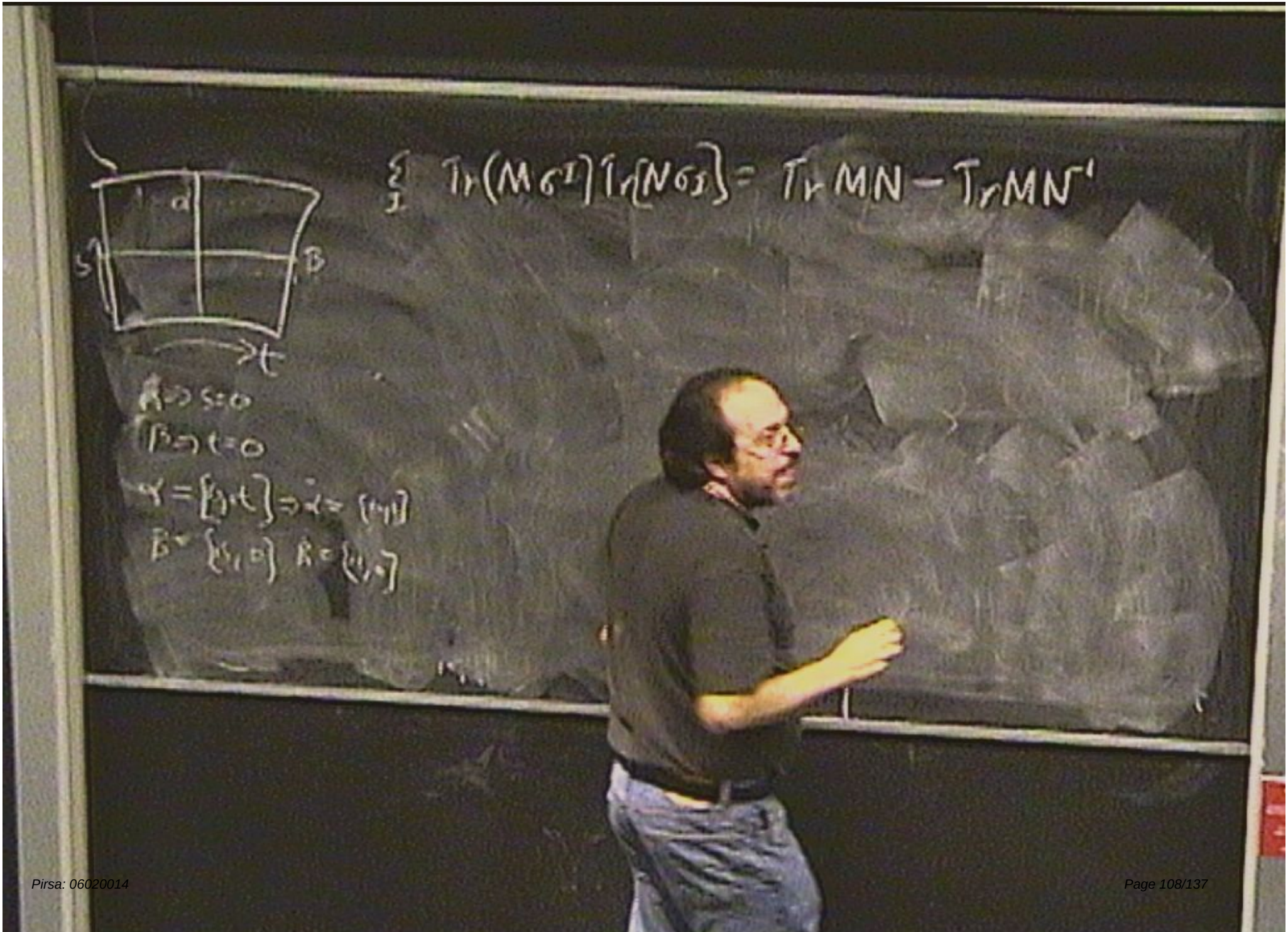

α ⇒ s=0
β ⇒ t=0

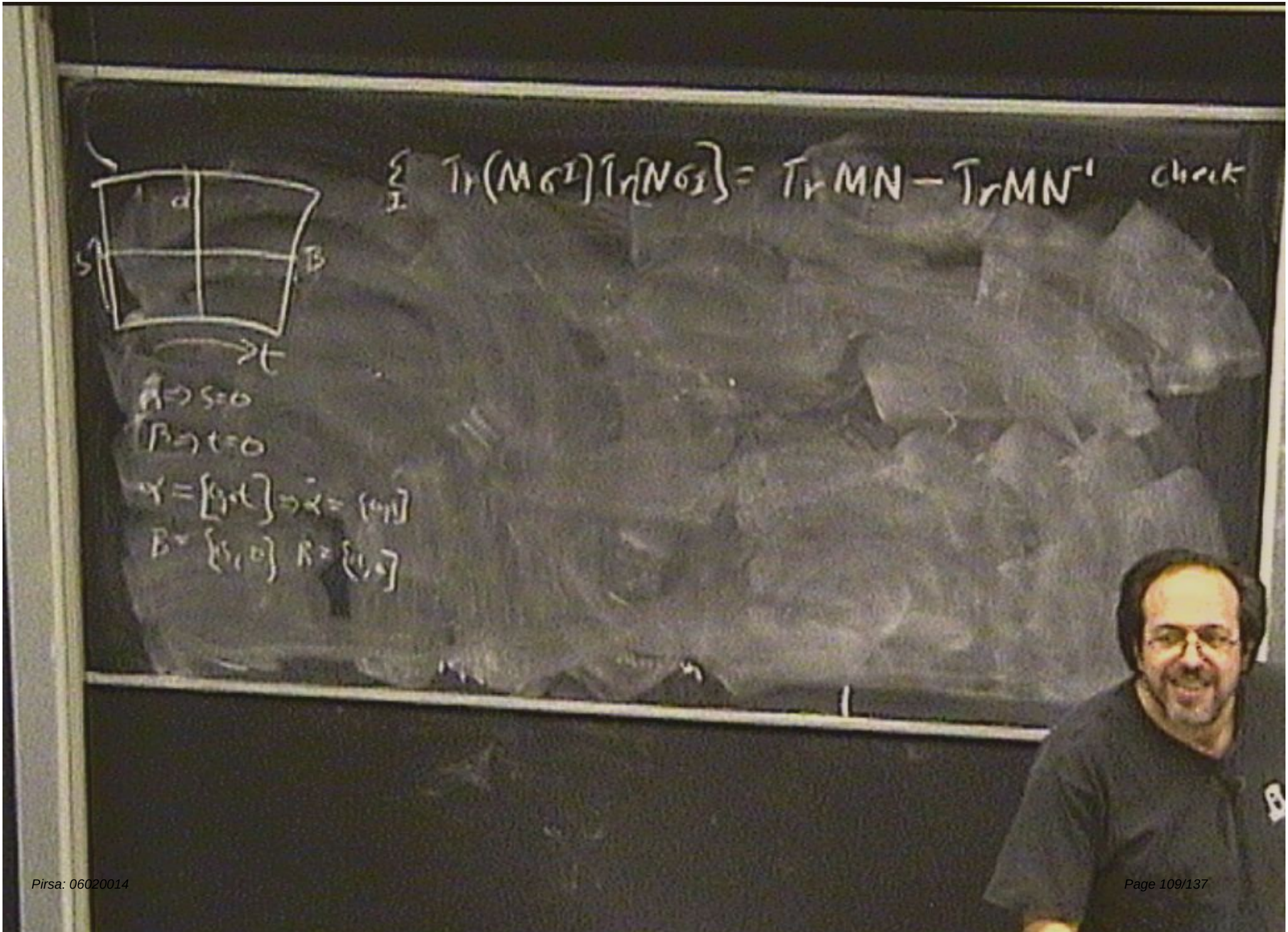

PB amongst the T, T'
{T, T'}
⇒ {T, T} = 0

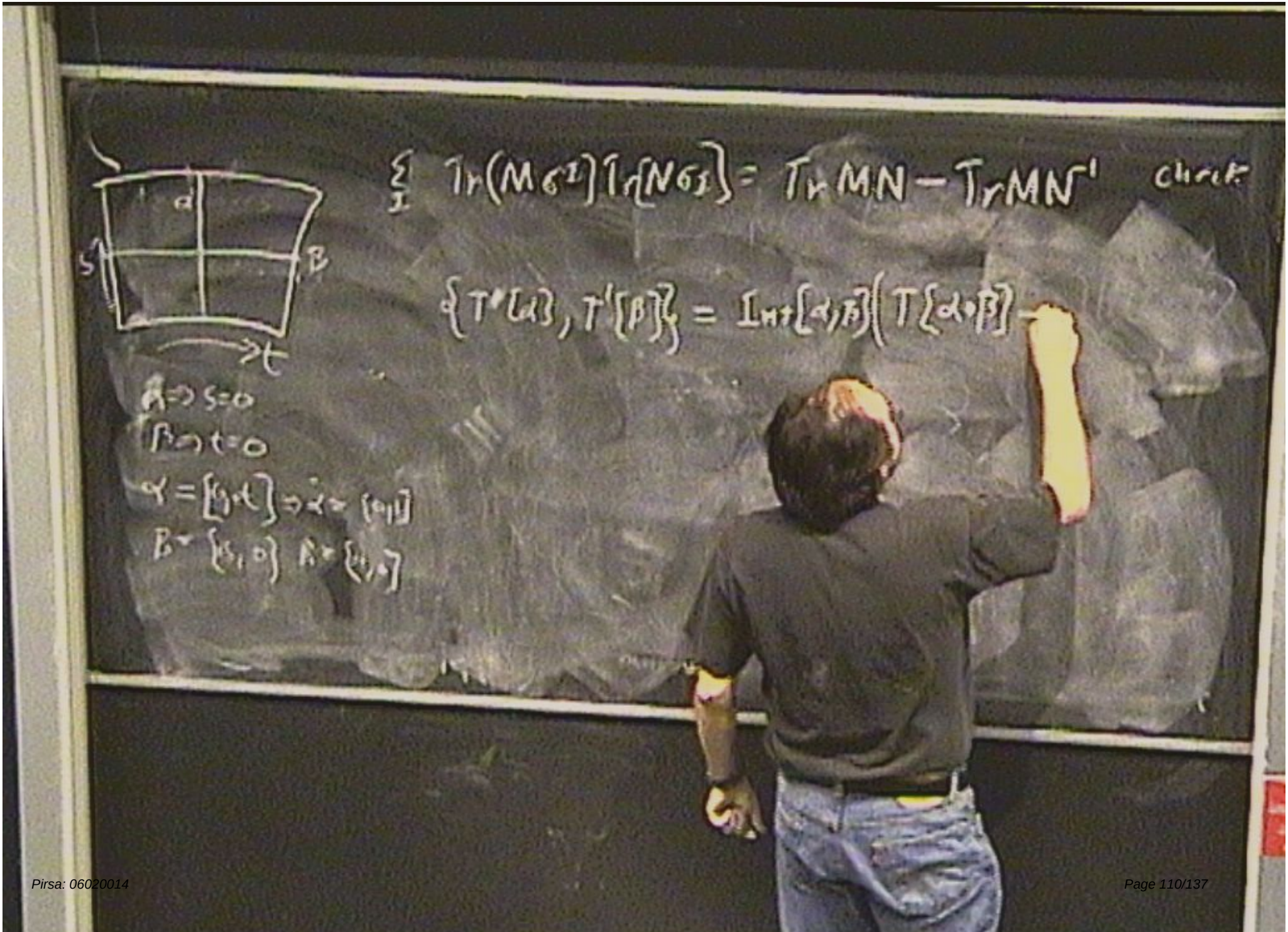

PB amongst the T, T'
⇒ {T, T} = 0
{T, T'}
{T[α], T'[β]} = {Tr P e^{δA}, ∫ dt β̇^i e^I_i e^{δA}}
= −∫ dt β̇^i(t) Tr σ_I H[β,t] {e^I_i(β(t)), Tr P e^{δA}}
= −[∫ dt ∫ ds β̇^i(t) α̇^j(s) ε_{ij} δ(β(t), α(s))] Tr σ_I H[β,t] Tr H[α] σ^I
∫ dt ∫ ds ±1 δ(s,t) δ(t,s) = ±1

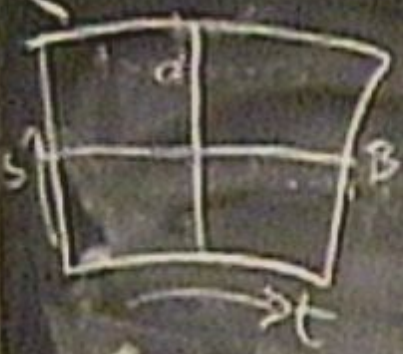

A
B
s
t
A ⇒ s=0
B ⇒ t=0
γ = [s,t] ⇒ γ̇ = (1,1)
B = (x,0) R = (1,y)


Tr(Mσ^I] Tr[Nσ_I]
α
s
B
t
A ⇒ s=0
B ⇒ t=0
α = [0,t] ⇒ α̇ = [0,1]
B = [s,0] Ḃ = [1,0]


$\sum_{ij} \mathrm{Tr}(M\sigma^i)\mathrm{Tr}(N\sigma^j) = \mathrm{Tr}\,MN - \mathrm{Tr}\,M\mathrm{Tr}\,N$
A
B
t
A ⇒ s=0
B ⇒ t=0


$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [0,t] \Rightarrow \dot{\alpha} = [0,1]$
$B = [s,0]$ $R = [1,0]$


$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}MN - \mathrm{Tr}MN^{-1}$
check
$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta](T[\alpha\circ\beta]$
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [0,t] \Rightarrow \dot{\alpha} = [0,1]$
$\beta = [s,0]$



$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}(N\sigma_I) = \mathrm{Tr}MN - \mathrm{Tr}MN^t$ check

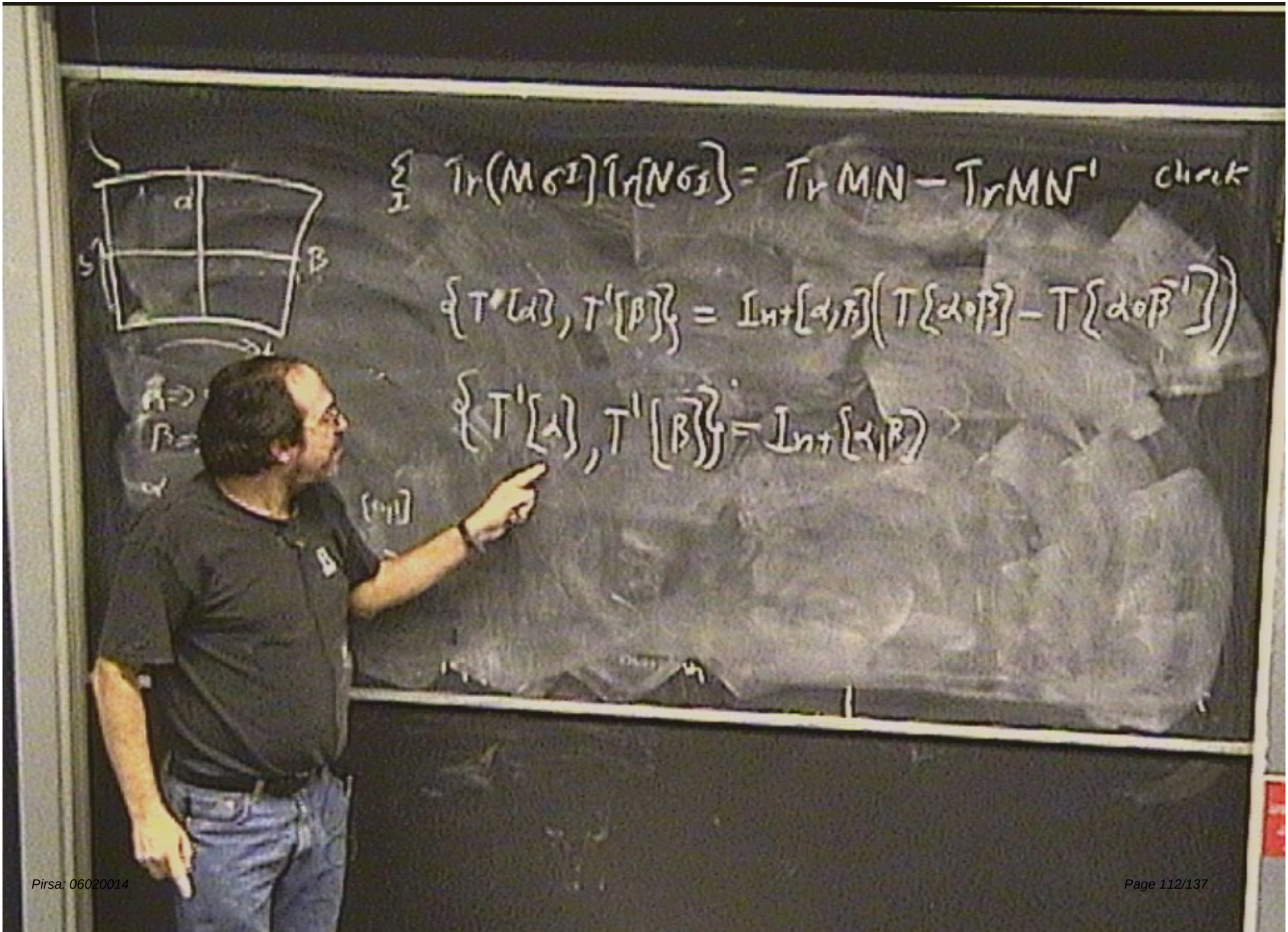
$\{T^a(\alpha), T^b(\beta)\} = \mathrm{Int}(\alpha,\beta)\left(T(\alpha\circ\beta) - T(\alpha\circ\beta^{-1})\right)$

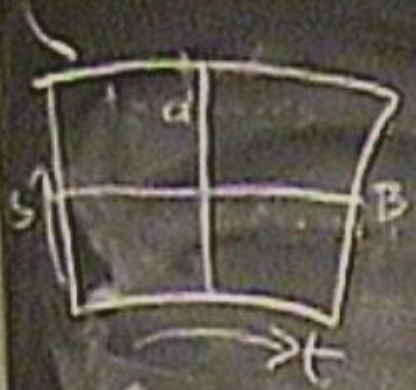
$\alpha \Rightarrow s=0$

$\beta \Rightarrow t=0$

$\gamma = [\beta, \alpha] \Rightarrow \dot{\gamma} = [\dot{\beta}, \alpha]$

$\beta = [\beta, 0]$ $R = [\alpha, 0]$


$\sum_I Tr(M\sigma^I)Tr(N\sigma_I) = TrMN - TrMN'$ check
$\{T'[\alpha], T'[\beta]\} = Int[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$
$\{T'[\alpha], T'[\beta]\} = Int[\alpha,\beta]$
α
β



$$\sum_I \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1} \quad \text{check}$$

$$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$$

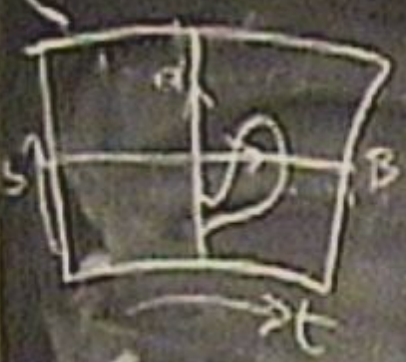
$$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$$

$\alpha \Rightarrow s=0$

$\beta \Rightarrow t=0$

$\alpha = [0,t] \Rightarrow \dot{\alpha} = [0,1]$

$\beta = [s,0]$ $\dot{\beta} = [1,0]$



$$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}(N\sigma_I) = \mathrm{Tr}MN - \mathrm{Tr}MN^t \quad \text{check}$$

$$\{T'[\alpha], T'[\beta]\} = I_{int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$$

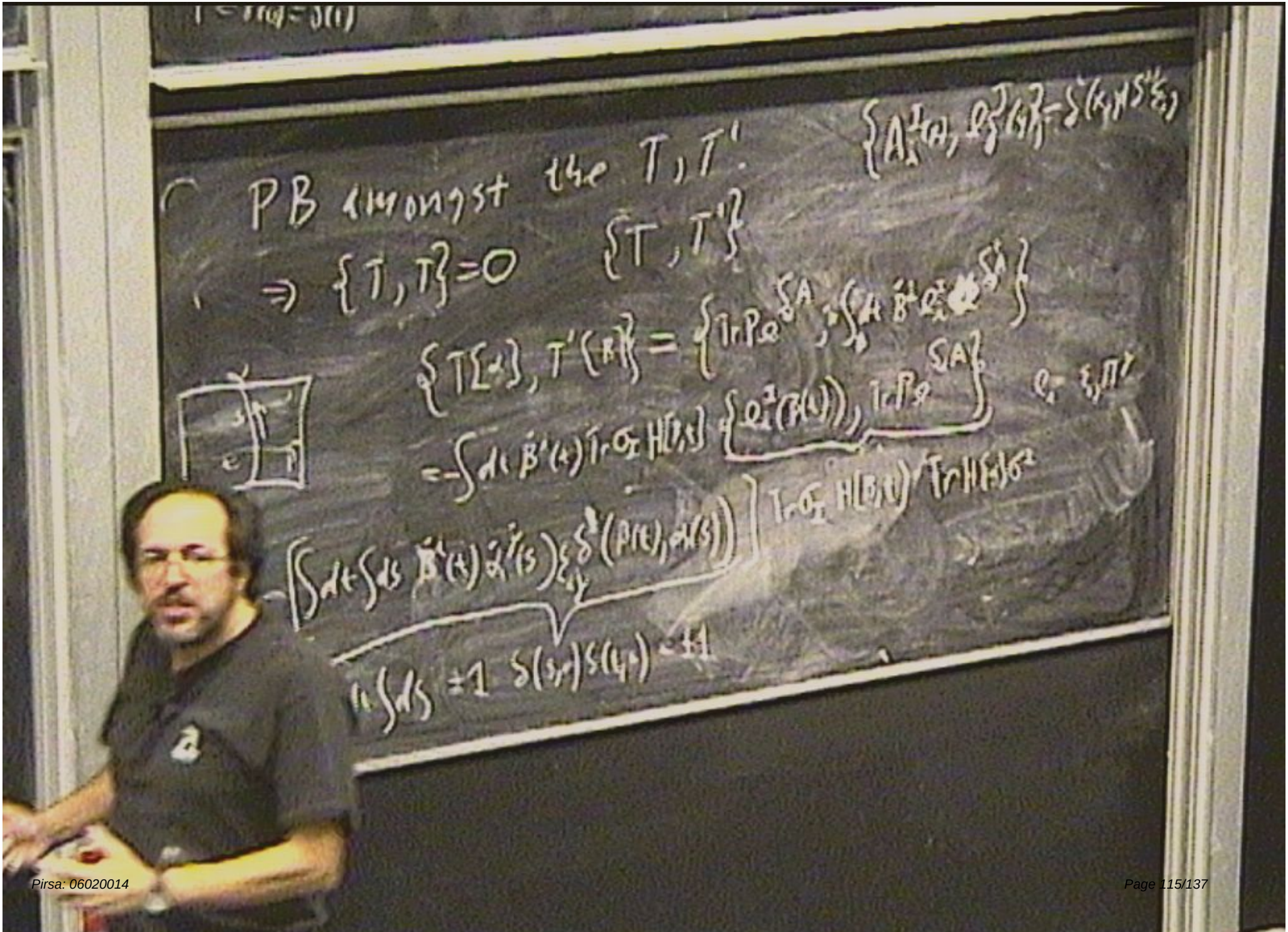
$$\{T'[\alpha], T'[\beta]\} = \underbrace{I_{int}[\alpha,\beta]}_{\text{boundary int}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$$

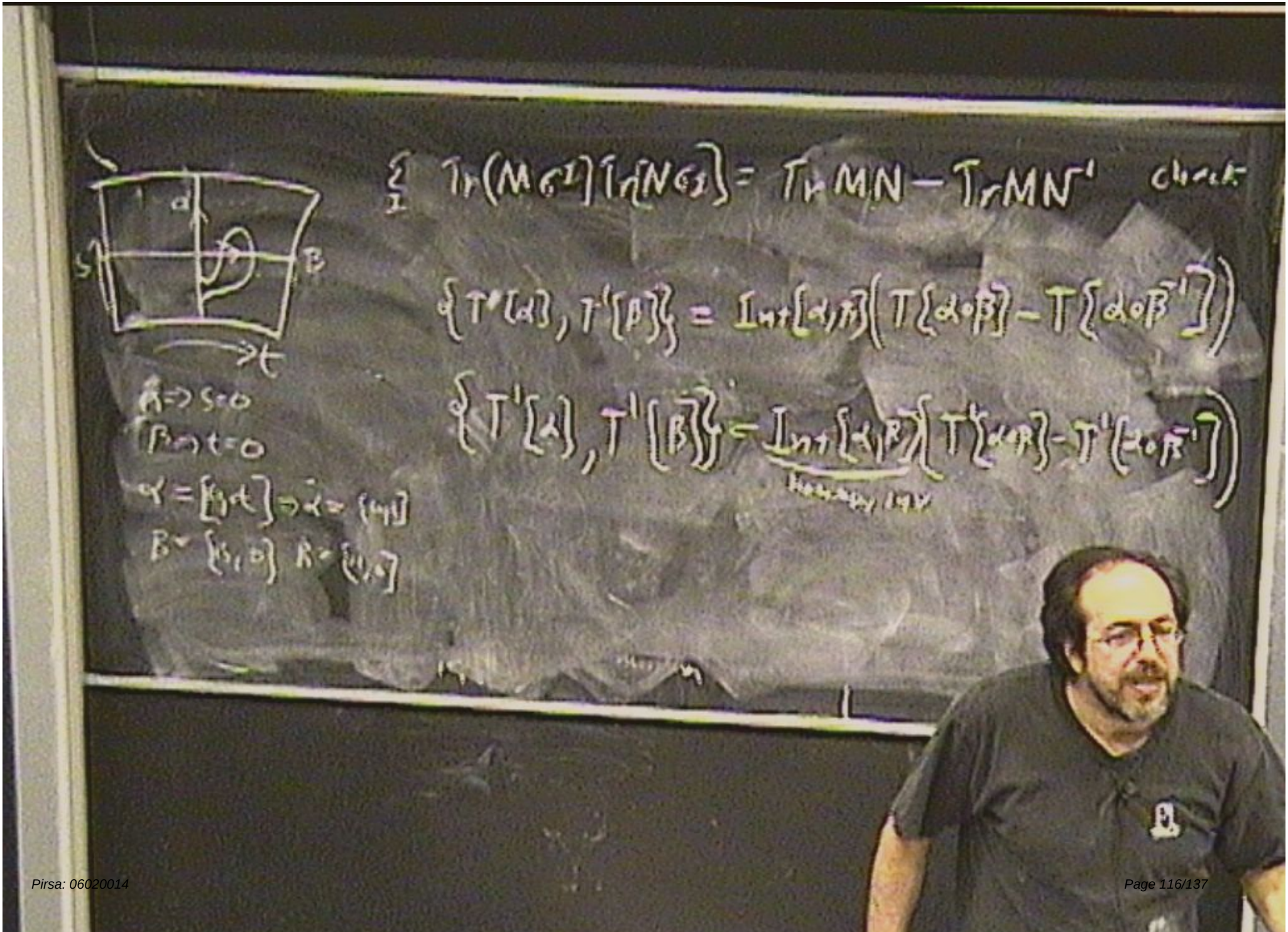
$\alpha \Rightarrow s=0$

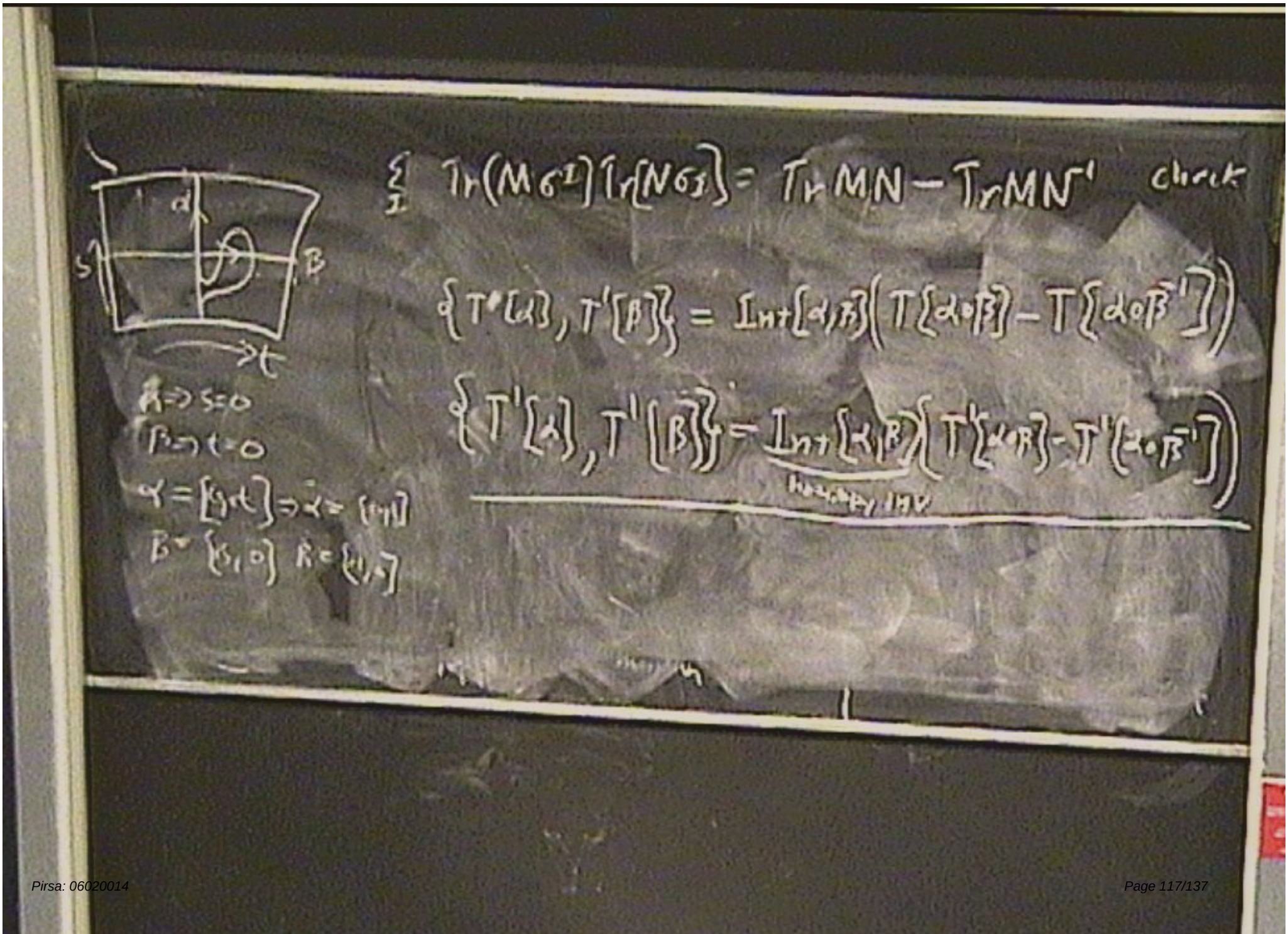
$\beta \Rightarrow t=0$

$\gamma = [s,t] \Rightarrow \dot{\gamma} = [1,1]$

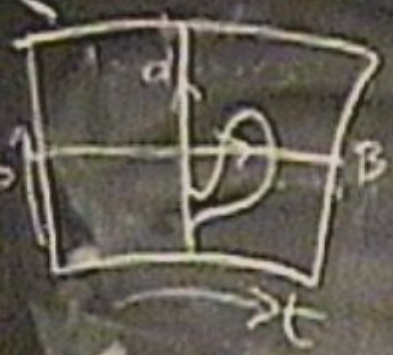
$\beta = [s,0]$ $R = [s,t]$


PB amongst the T, T'.
$\{T, T\} = 0$
$\{T, T'\}$
$\{A^i_x(A), E^j_y(A)\} = \delta(x,y)\delta^{ij}\xi_y$
$\{T[\vec{A}], T'[\vec{B}]\} = \{\int_\Sigma \mathrm{Tr}\, P_a \dot{A}^a_x, \int_\Sigma \dot{B}^b_y E^y_b\}$
$= -\int dx\, \dot{B}^a_x(x)\, \mathrm{Tr}\, \sigma_i H[E,x] \{E^a_x(B(x)), \mathrm{Tr}\, P_a \dot{A}^a_x\}$
$-\left[\int dx \int dy\, \dot{B}^a_x(x) \dot{A}^b_y(y) \xi_y \delta^3(B(x), A(y))\right] \mathrm{Tr}\, \sigma_i H[E,x]$
$\int dy = 1 \quad \delta(x,y)\delta(y,x) = 1$


$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check
$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$
$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [\mathrm{Id}, t] \Rightarrow \dot\alpha = [\mathrm{Id}, 1]$
$\beta = [\mathrm{Id}, 0]$ $\dot\beta = [\mathrm{Id}, 0]$



$\sum_I \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check
$\{T[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$
$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$
homotopy inv
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [\mathrm{const}] \Rightarrow \dot\alpha = [0,1]$
$\beta = [s,0]$ $\dot\beta = [1,0]$



$\alpha \Rightarrow s = 0$

$\beta \Rightarrow t = 0$

$\alpha = [s, t] \Rightarrow \dot{\alpha} = [\dot{s}, \dot{t}]$

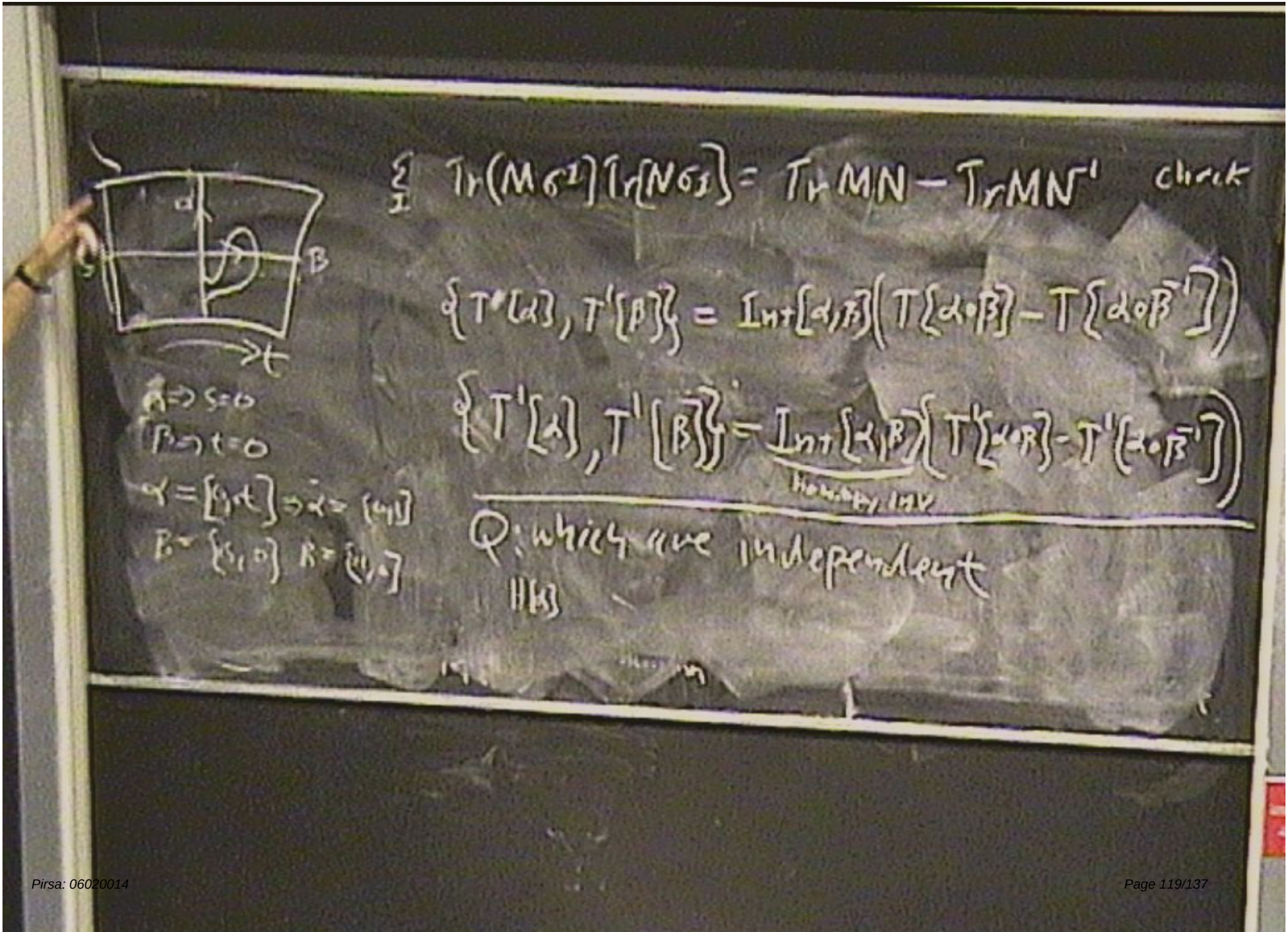
$\beta = [s, 0]$ $R = [1, s]$

$\{ \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}(N\sigma^J) = \mathrm{Tr}MN - \mathrm{Tr}MN^{-1}$ check

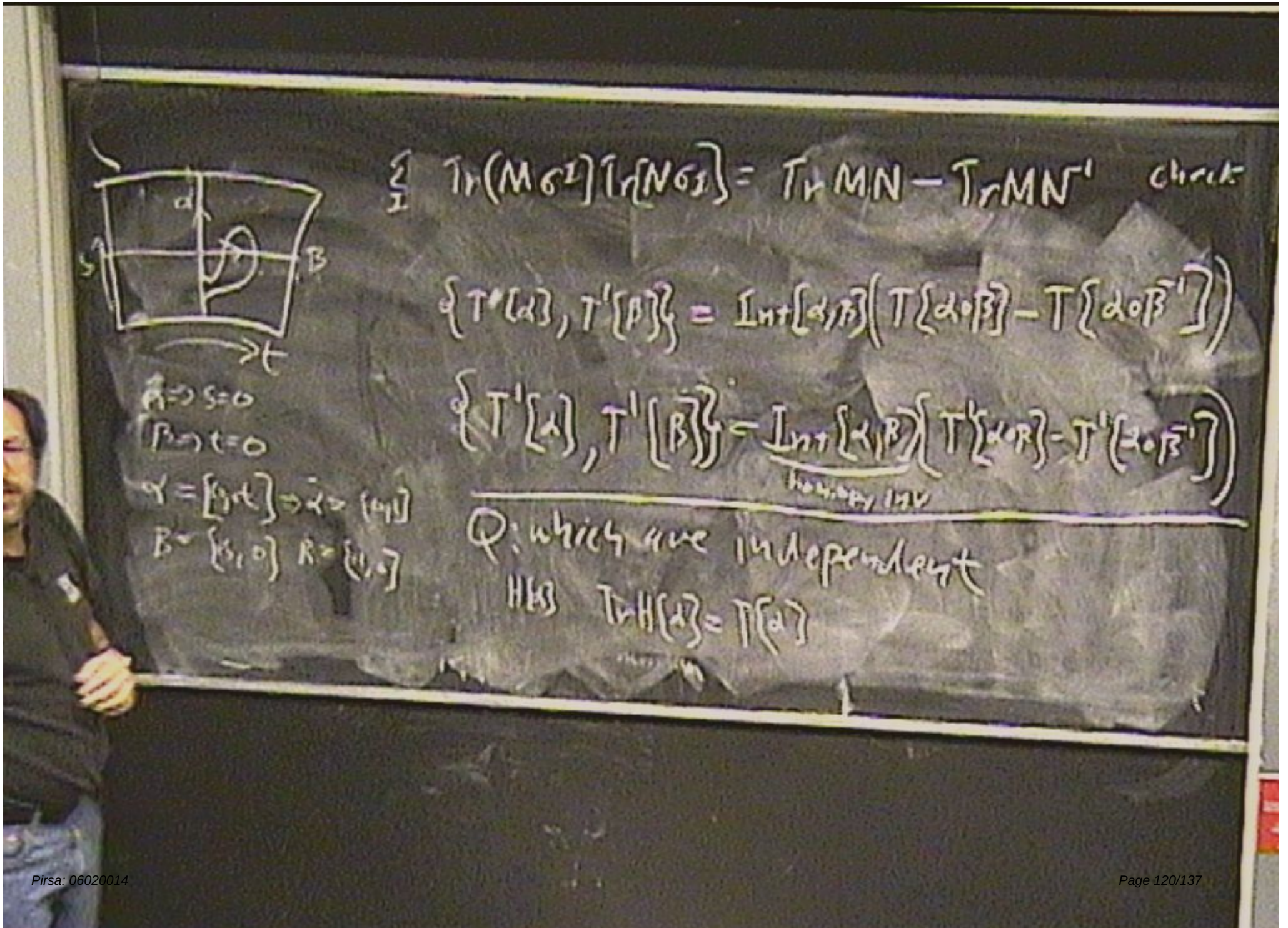
$\{T'[\alpha], T'[\beta]\} = I_{int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$

$\{T'[\alpha], T'[\beta]\} = I_{int}[\alpha,\beta]\left(T'[\alpha\circ R] - T'[\alpha\circ\beta^{-1}]\right)$

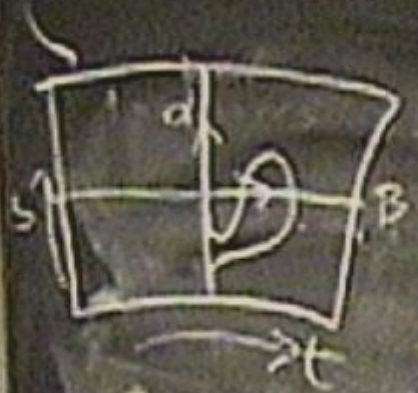
Q: which are independent



$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check
$\{T[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$
$\{T'[\alpha], T'[\beta]\} = \underbrace{\mathrm{Int}[\alpha,\beta]}_{\text{Homotopy inv}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$
Q: which are independent
$H[\alpha]$
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [0,t] \Rightarrow \dot\alpha = [0,1]$
$\beta = [s,0]$ $\dot\beta = [1,0]$



$\sum_I \mathrm{Tr}(M\sigma^I)\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check
$\{T'[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$
$\{T'[\alpha], T'[\beta]\} = \underbrace{\mathrm{Int}[\alpha,\beta]}_{\text{homotopy inv}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$
Q: which are independent
$H[\alpha]$ $\mathrm{Tr}\,H[\alpha] = T[\alpha]$
$\alpha \Rightarrow s=0$
$\beta \Rightarrow t=0$
$\alpha = [s,t] \Rightarrow \dot\alpha = [0,1]$
$B = [s,0]$ $R = [1,0]$



$\sum_I \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check

$\{T[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$

$\{T'[\alpha], T'[\beta]\} = \underbrace{\mathrm{Int}[\alpha,\beta]}_{\text{homotopy inv}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$

$\alpha \Rightarrow s=0$

$\beta \Rightarrow t=0$

$\alpha = [0,t] \Rightarrow \dot\alpha = [0,1]$

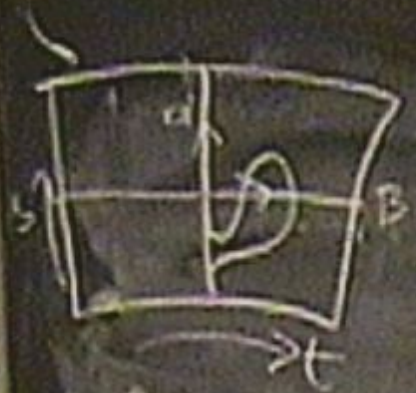
$\beta = [s,0]$ $\dot\beta = [1,0]$

Q: which are independent

$H[\alpha]$ $\mathrm{Tr}\,H[\alpha] = T[\alpha]$

$T[\alpha], T[\alpha^2], T[\alpha^3]$

$T[\alpha\beta]$ $T[\alpha\beta^{-1}]$ $T[\alpha^2\beta]$ ...



$\sum_I \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1}$ check

$\{T[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$

$\{T'[\alpha], T'[\beta]\} = \underbrace{\mathrm{Int}[\alpha,\beta]}_{\text{homotopy inv}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$

$\alpha \Rightarrow s=0$

$\beta \Rightarrow t=0$

$\alpha = [0,t] \Rightarrow \dot\alpha = [0,1]$

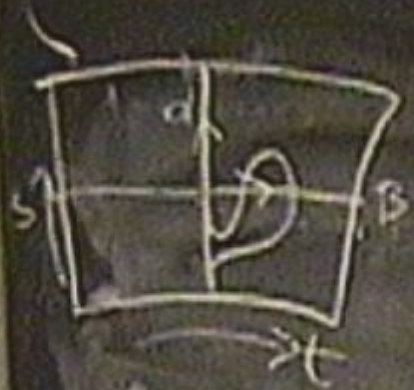
$\beta = [s,0]$ $\dot\beta = [1,0]$

Q: which are independent

$H[\alpha]$ $\mathrm{Tr}\,H[\alpha] = T[\alpha]$

$T[\alpha], T[\alpha^2], T[\alpha^3] \ldots$

$T[\alpha\beta]$ $T[\alpha\beta^{-1}]$ $T[\alpha^2\beta]$ ...



$\alpha \Rightarrow s=0$

$\beta \Rightarrow t=0$

$\alpha = [0,t] \Rightarrow \dot{\alpha} = [0,1]$

$\beta = [s,0]$ $\dot{\beta} = [1,0]$

$$\sum_I \mathrm{Tr}(M\sigma^I)\,\mathrm{Tr}[N\sigma_I] = \mathrm{Tr}\,MN - \mathrm{Tr}\,MN^{-1} \quad \text{check}$$

$$\{T[\alpha], T'[\beta]\} = \mathrm{Int}[\alpha,\beta]\left(T[\alpha\circ\beta] - T[\alpha\circ\beta^{-1}]\right)$$

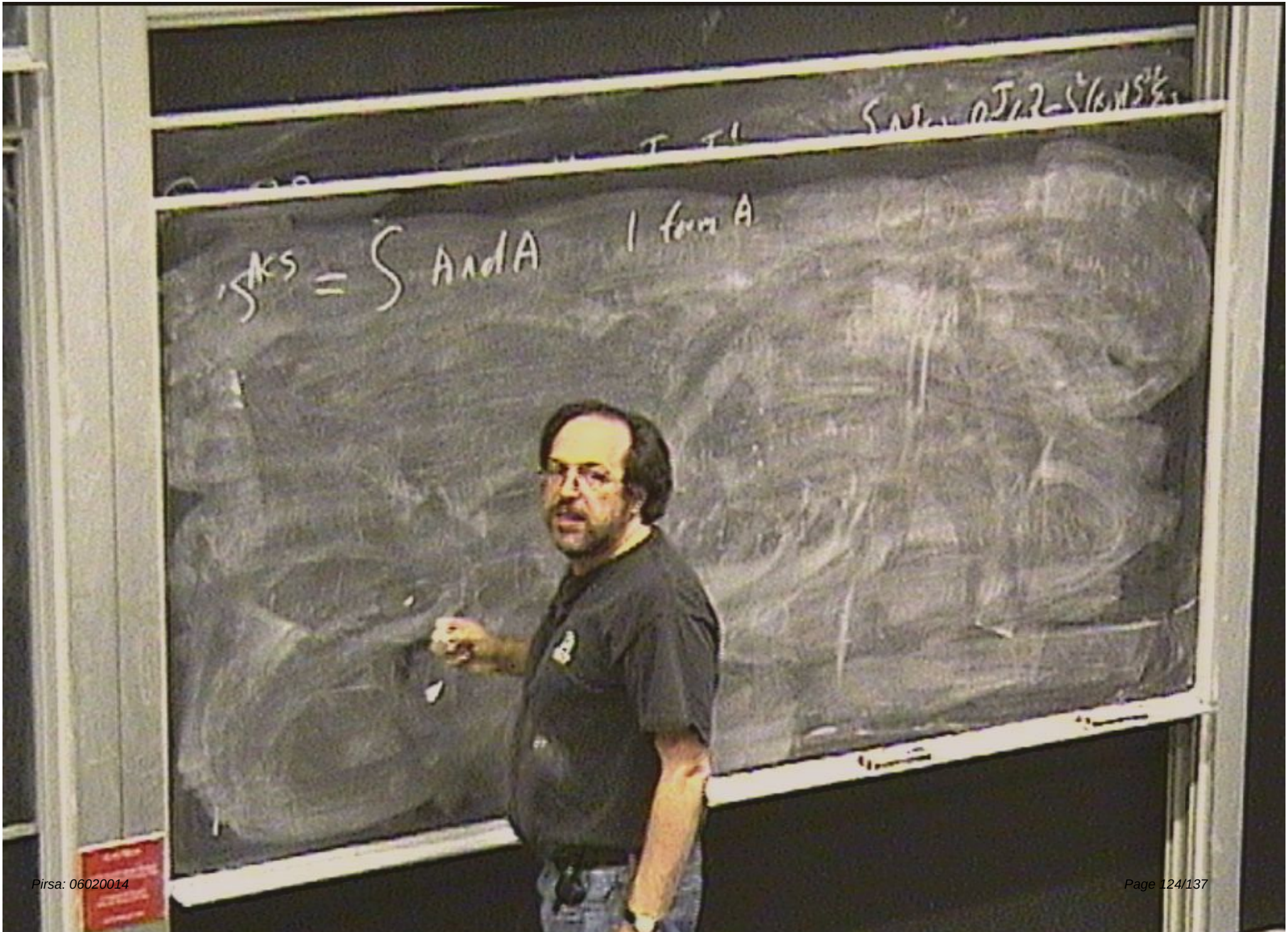
$$\{T'[\alpha], T'[\beta]\} = \underbrace{\mathrm{Int}[\alpha,\beta]}_{\text{homotopy inv}}\left(T'[\alpha\circ\beta] - T'[\alpha\circ\beta^{-1}]\right)$$

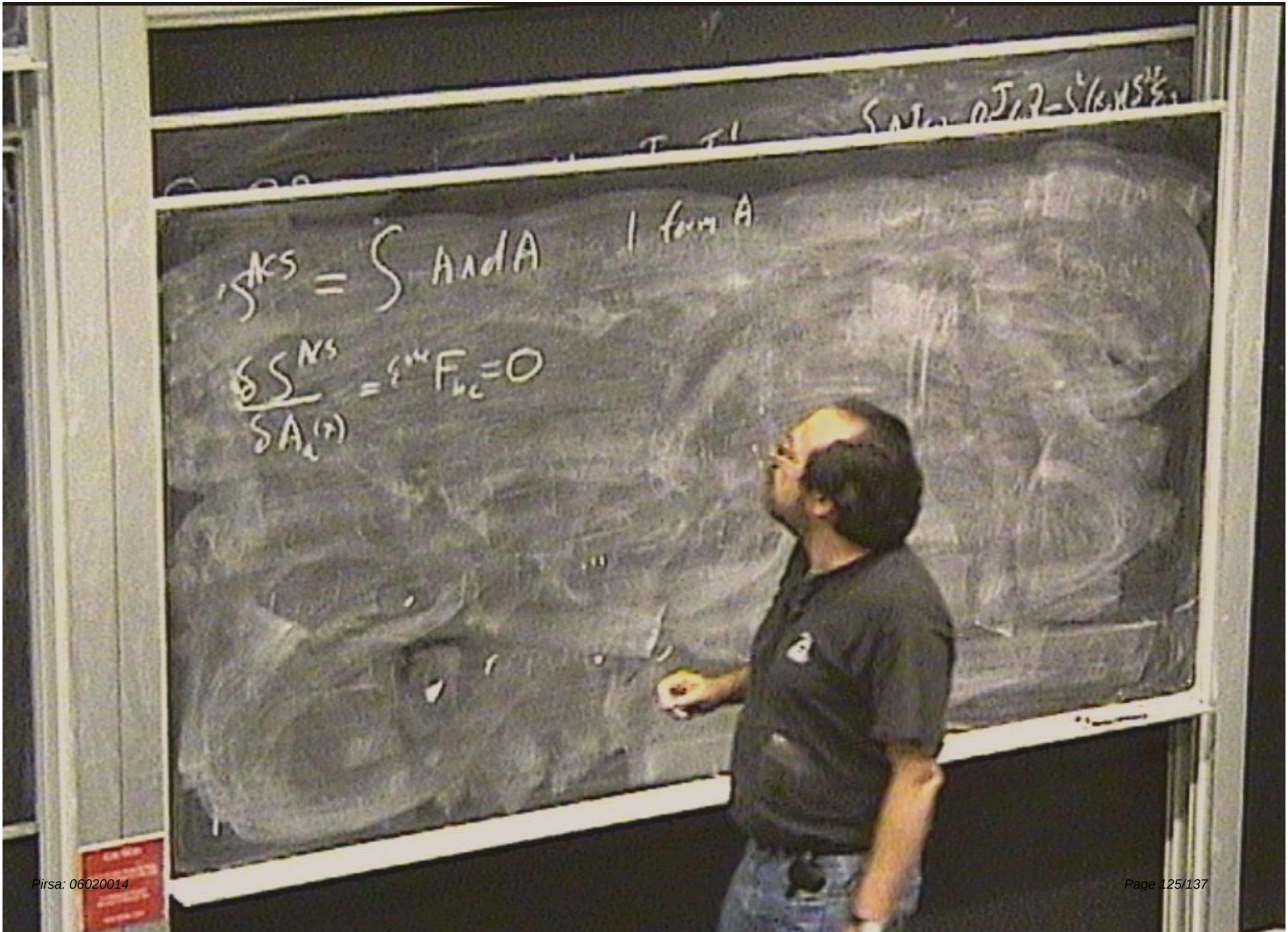
Q: which are independent

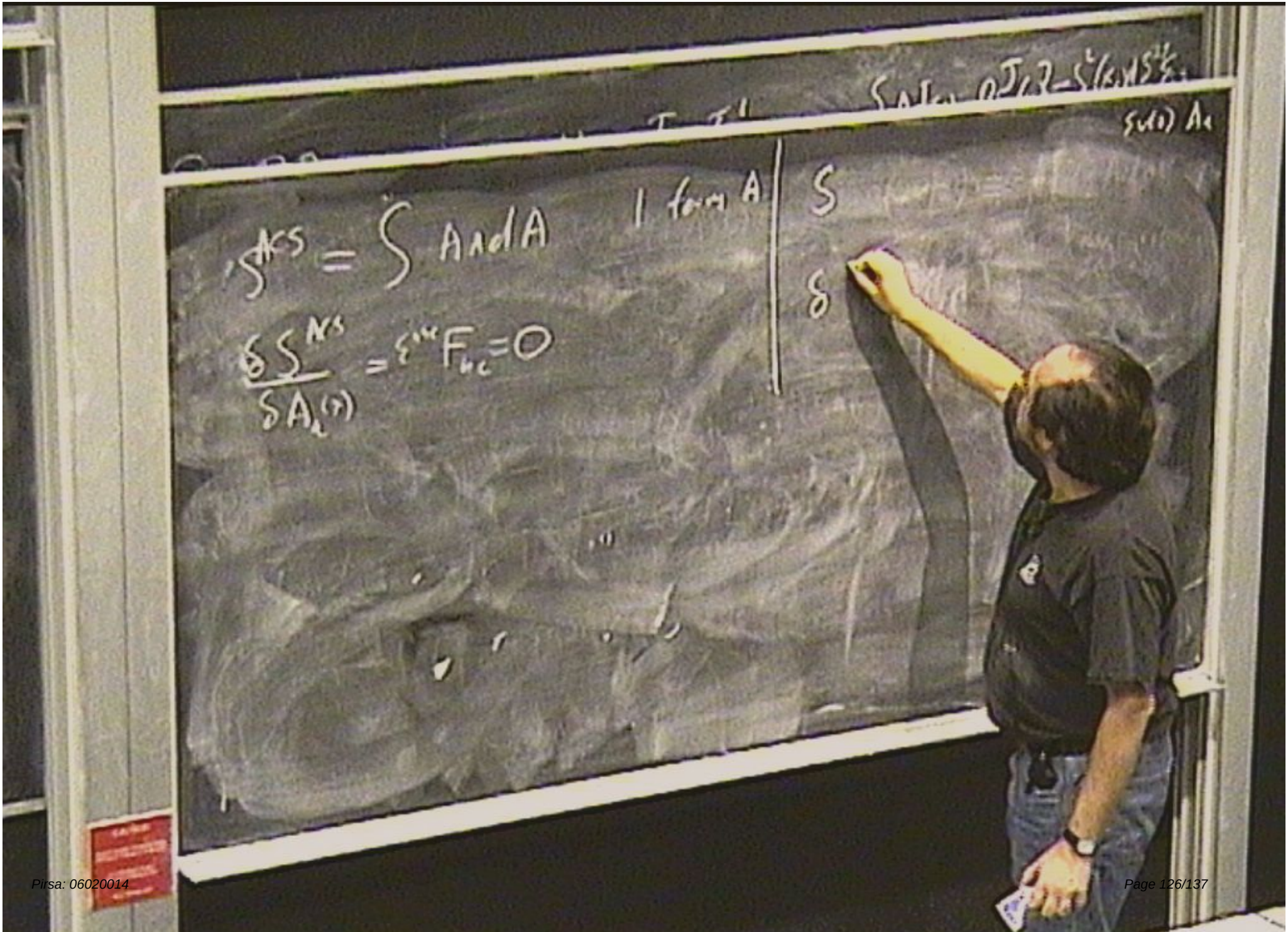
$H[\alpha]$ $\quad \mathrm{Tr}\,H[\alpha] = T[\alpha]$

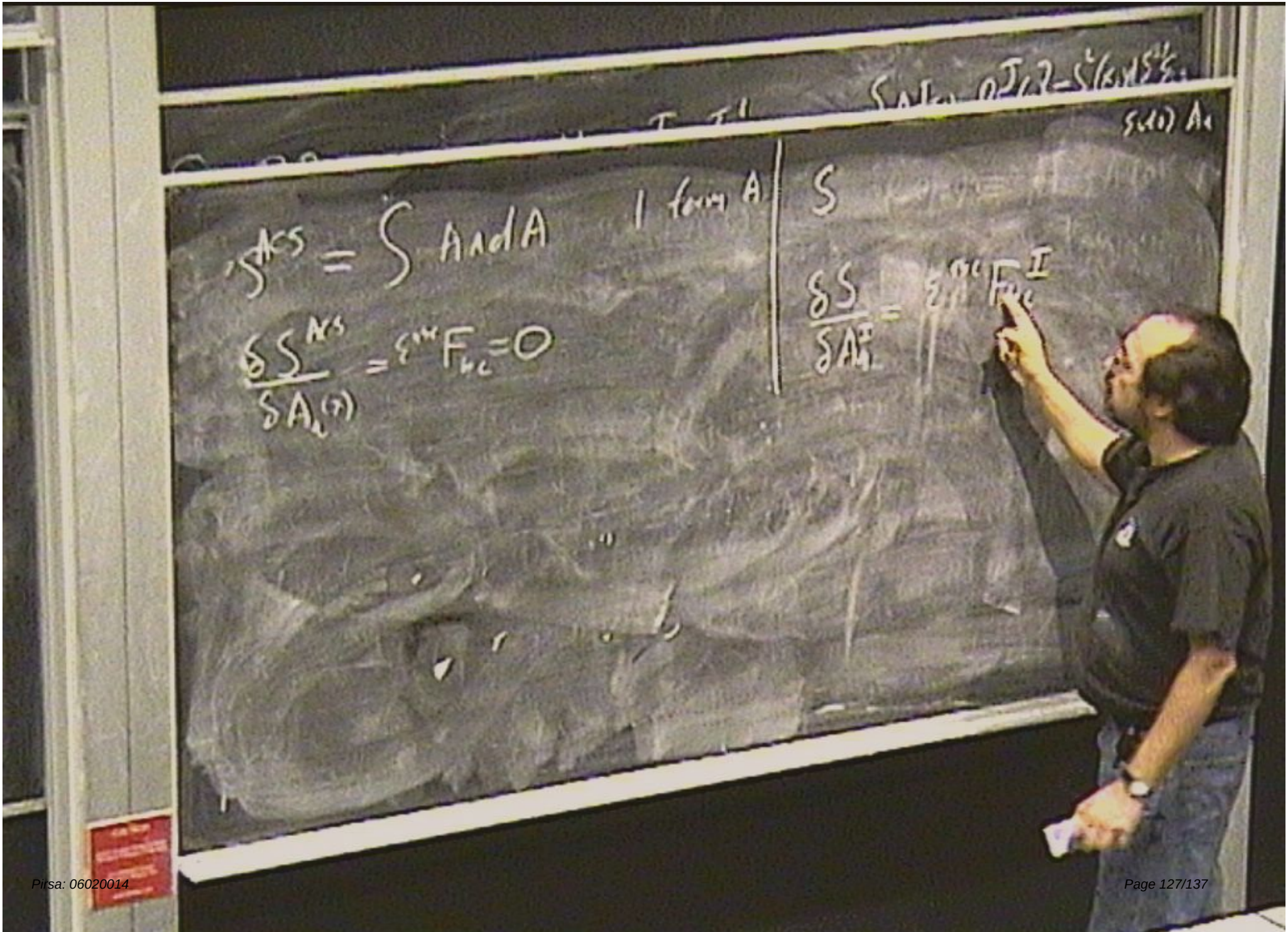
$T[\alpha], T[\alpha^2], T[\alpha^3] \ldots$

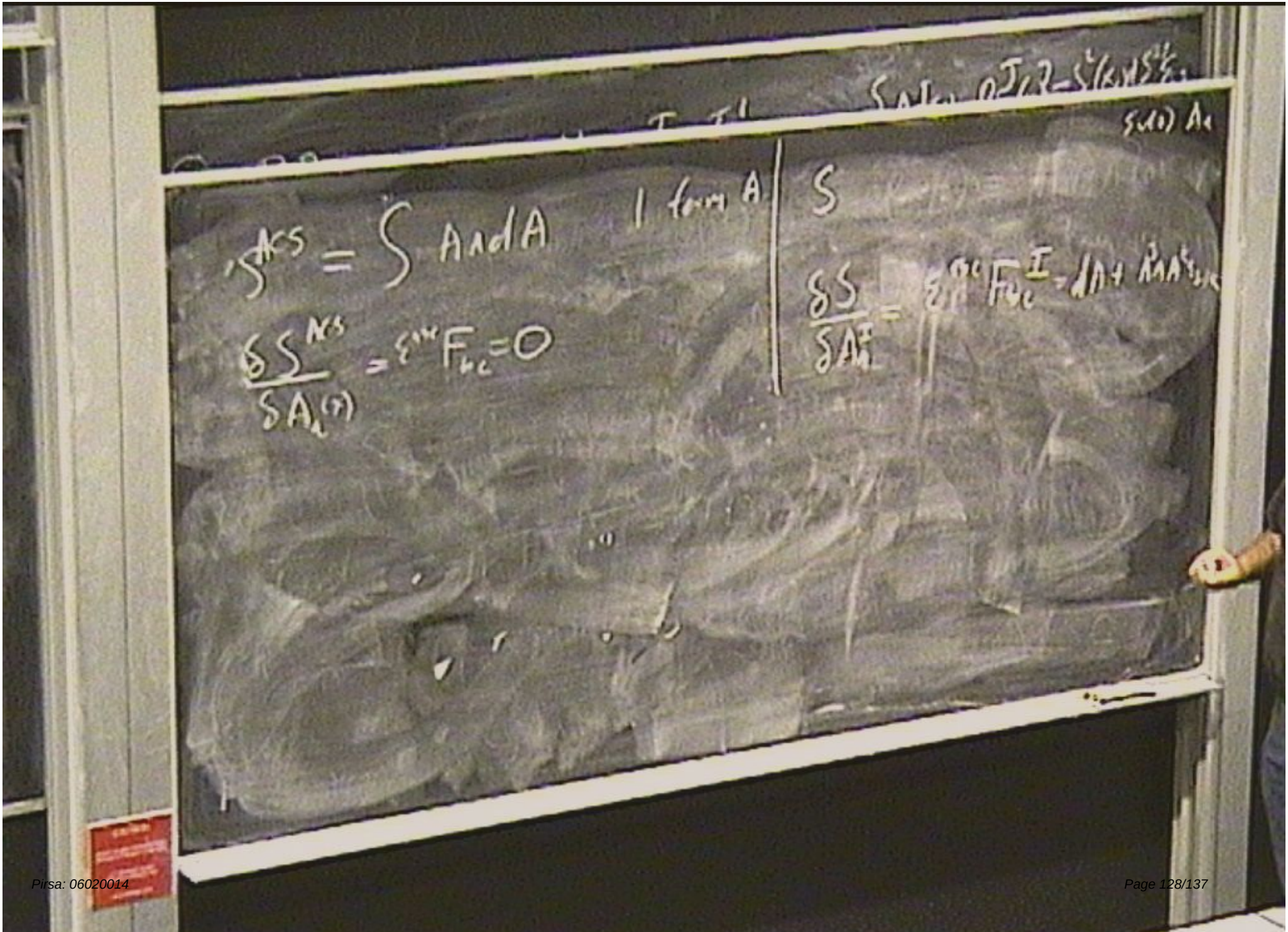
$T[\alpha\beta]$ $T[\alpha\beta^{-1}]$ $T[\alpha^2\beta]$ ...

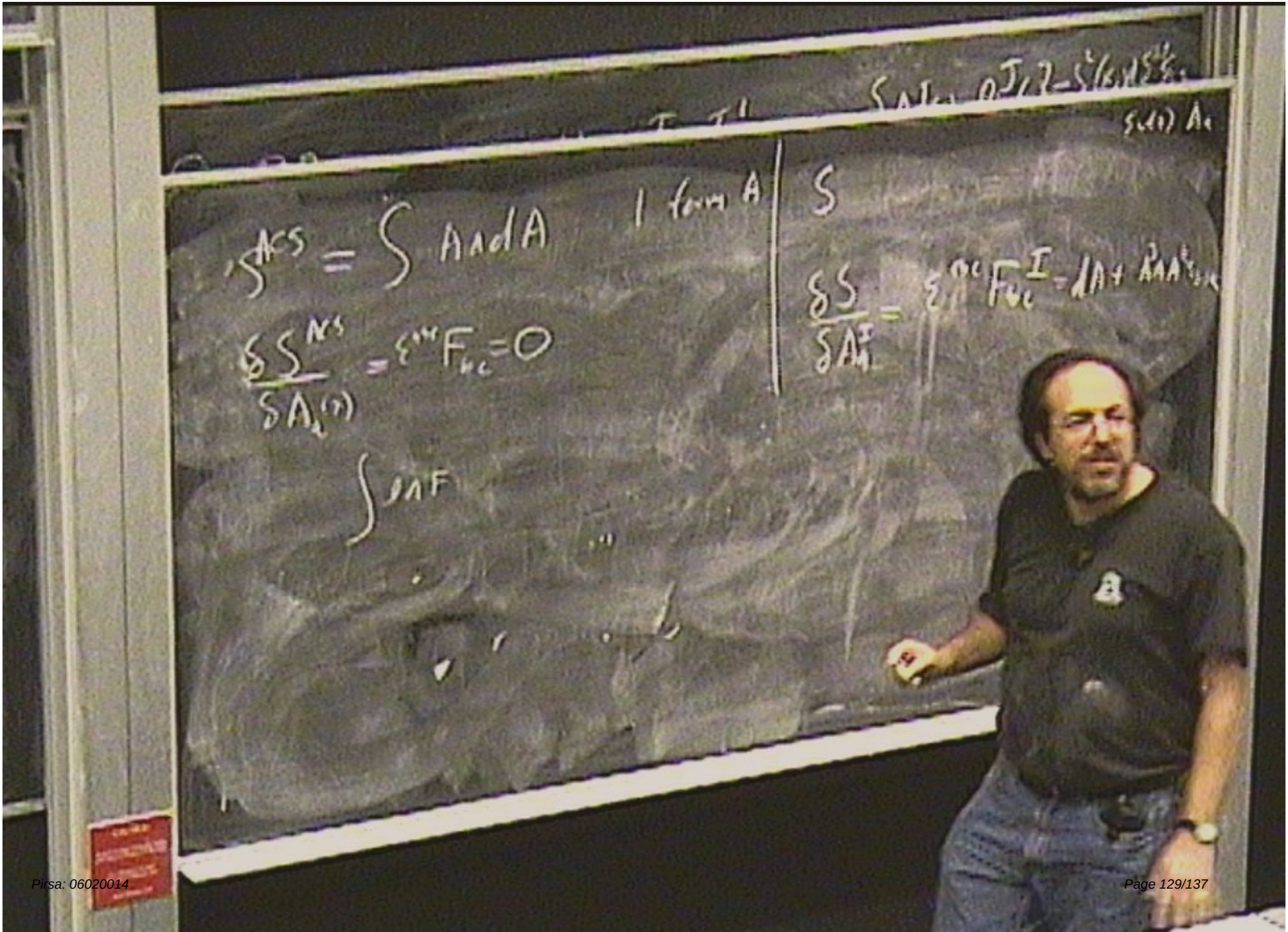

$S^{ACS} = \int A \wedge dA$
1 form A

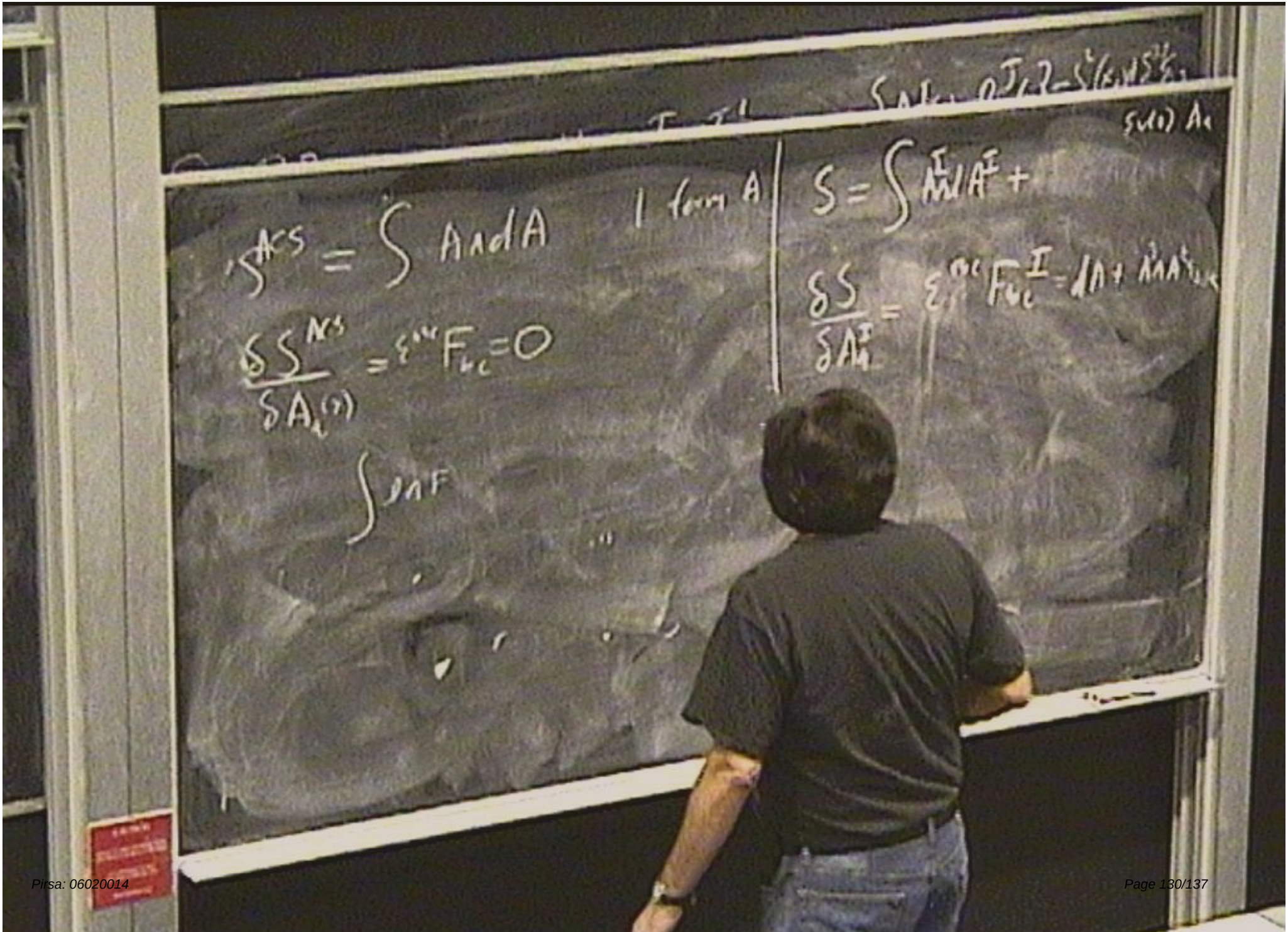

$S^{NCS} = \int A \wedge dA$
form A
$\frac{\delta S^{NCS}}{\delta A_\mu(x)} = \epsilon^{\mu\nu} F_{\mu\nu} = 0$

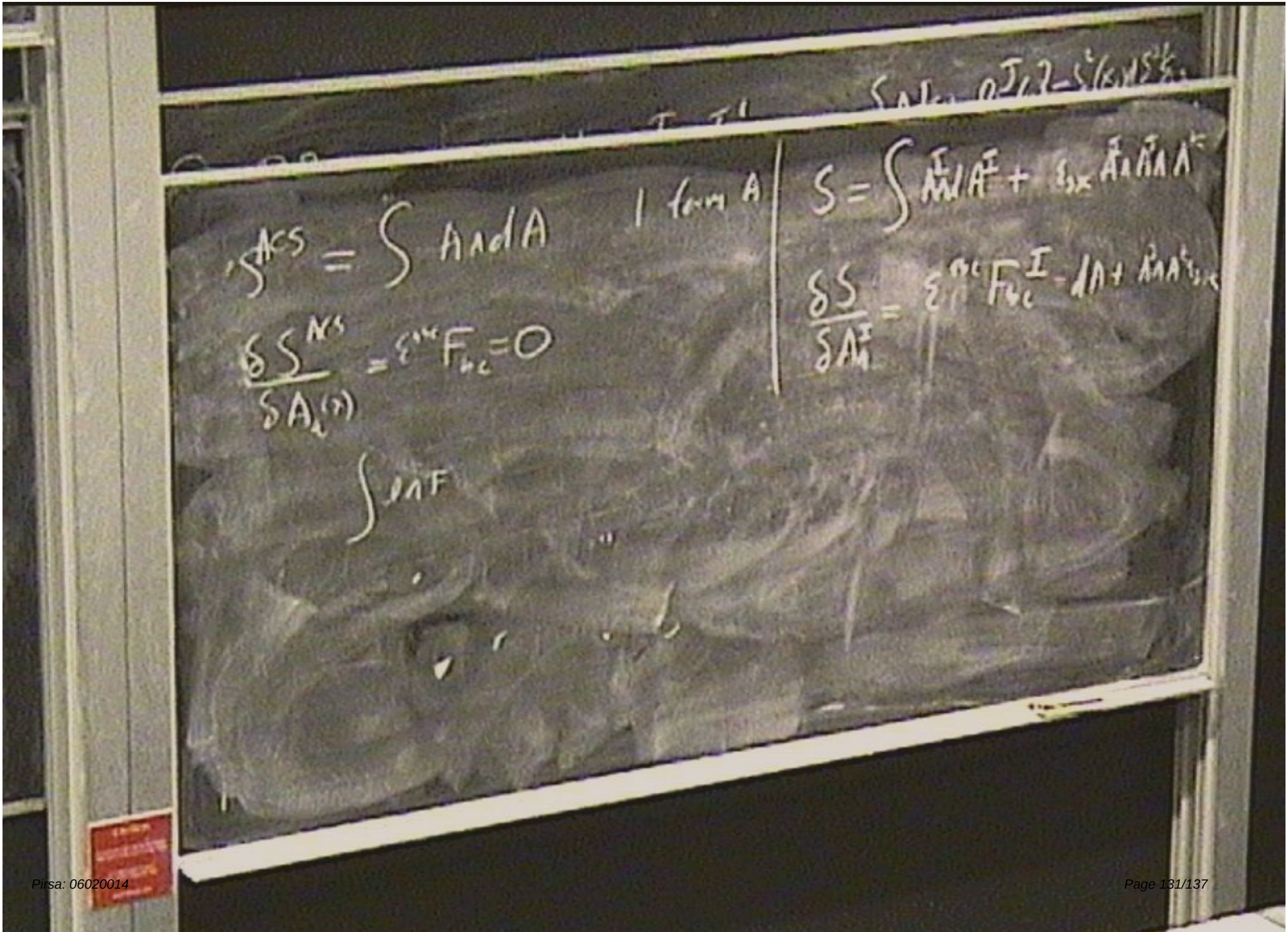

$S^{ACS} = \int A \wedge dA$
1 form A
S
δ
$\frac{\delta S^{ACS}}{\delta A_\mu(x)} = \epsilon^{\mu\nu} F_{\mu\nu} = 0$

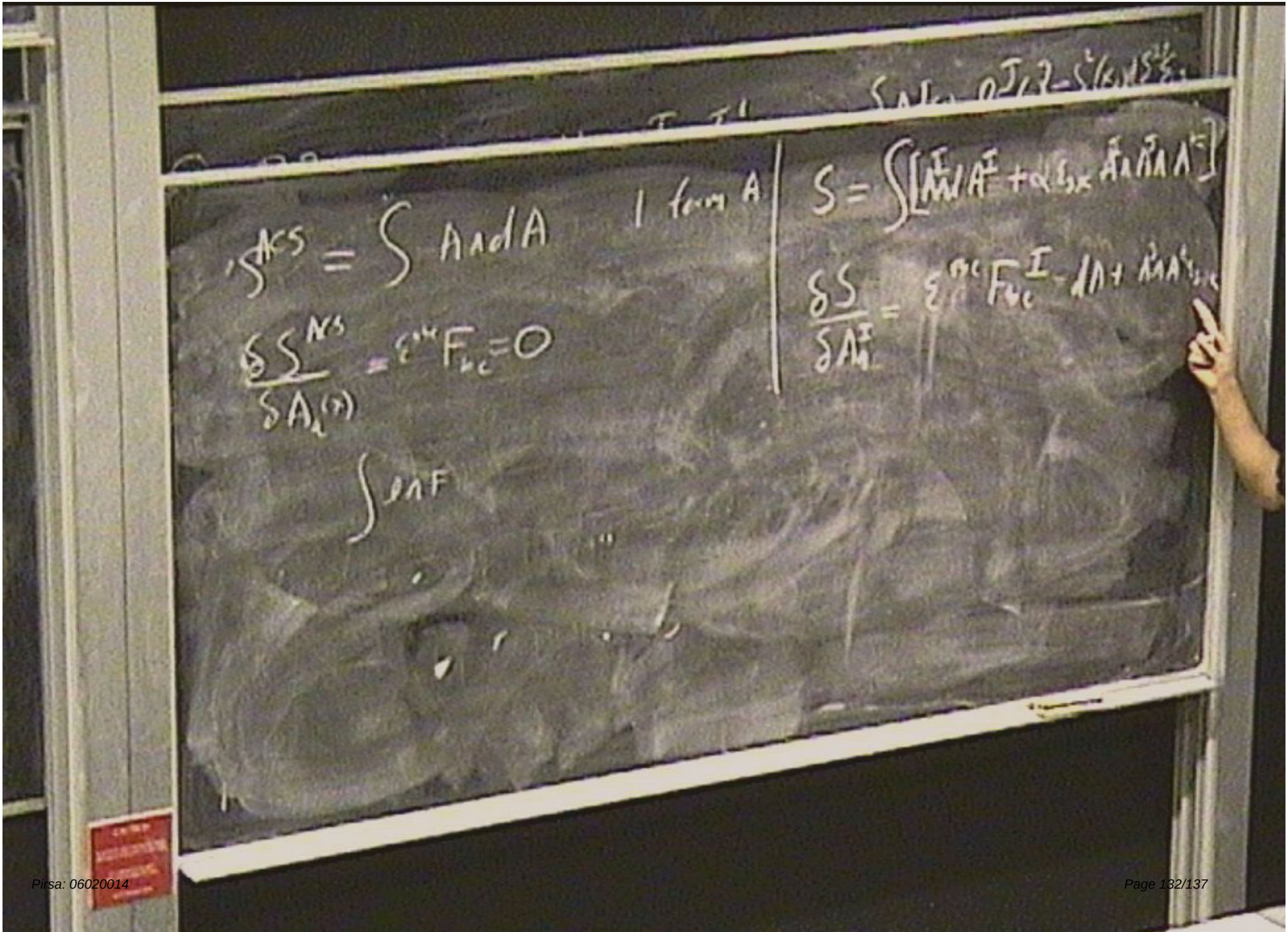

$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_\mu(x)} = \epsilon^{\mu\nu} F_{\mu\nu} = 0$
S
$\frac{\delta S}{\delta A^a_\mu} = \epsilon^{\mu\nu} F^I_{\mu\nu}$

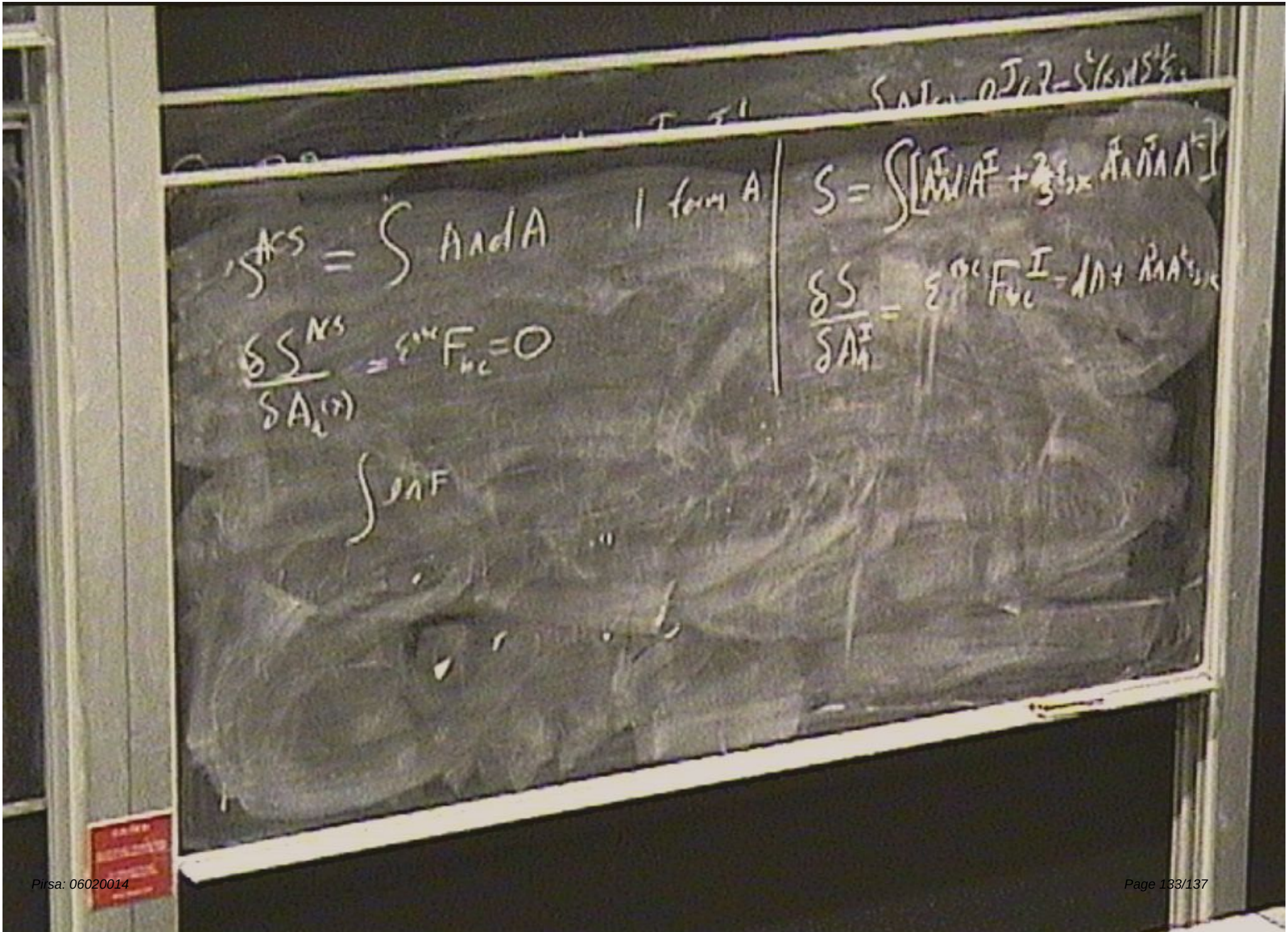

$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_\mu(x)} = \epsilon^{\mu\nu} F_{\nu\mu} = 0$
S
$\frac{\delta S}{\delta A^I_\mu} = \epsilon^{\mu\nu} F^I_{\nu\mu}$

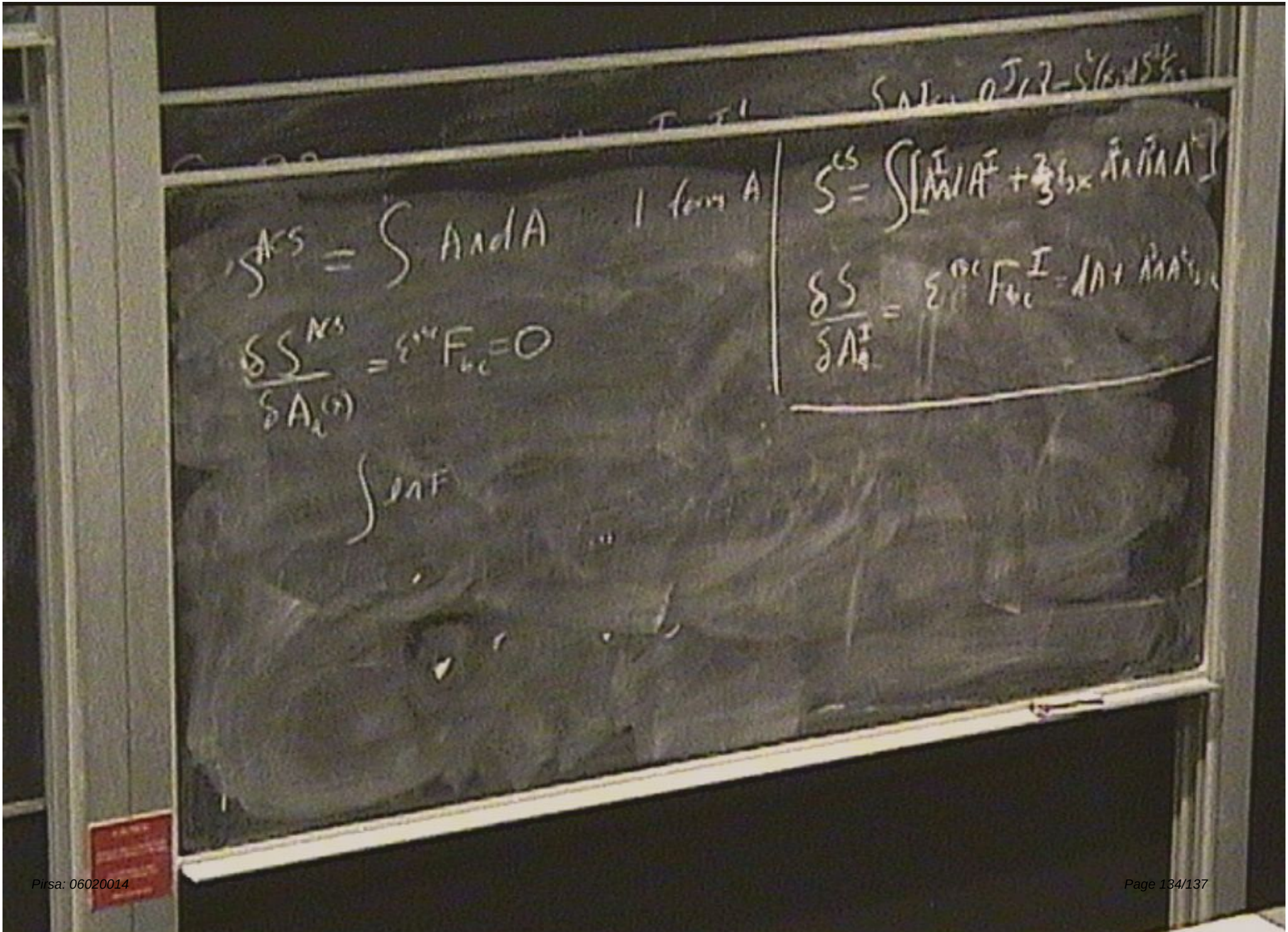

$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_\mu^I(x)} = \epsilon^{\mu\nu} F_{\mu\nu} = 0$
$\int \partial \wedge F$
S
$\frac{\delta S}{\delta A_\mu^I} = \epsilon^{\mu\nu} F_{\mu\nu}^I$

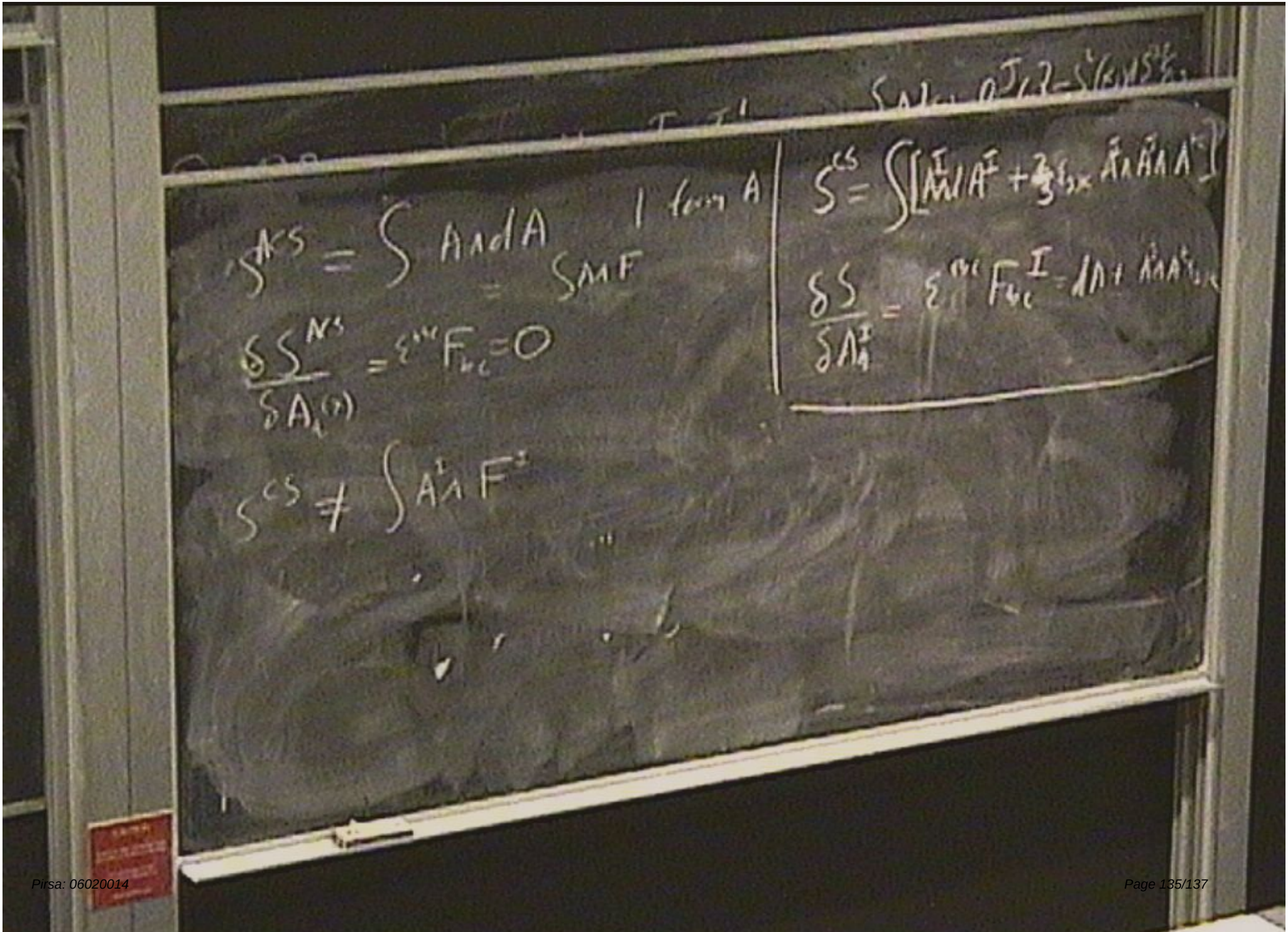

$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_\mu^I(x)} = \epsilon^{\mu\nu} F_{\mu\nu} = 0$
$\int dA F$
$S = \int A^I_\mu \partial A^I +$
$\frac{\delta S}{\delta A^I_\mu} = \epsilon^{\mu\nu} F^I_{\mu\nu} = d\Lambda +$

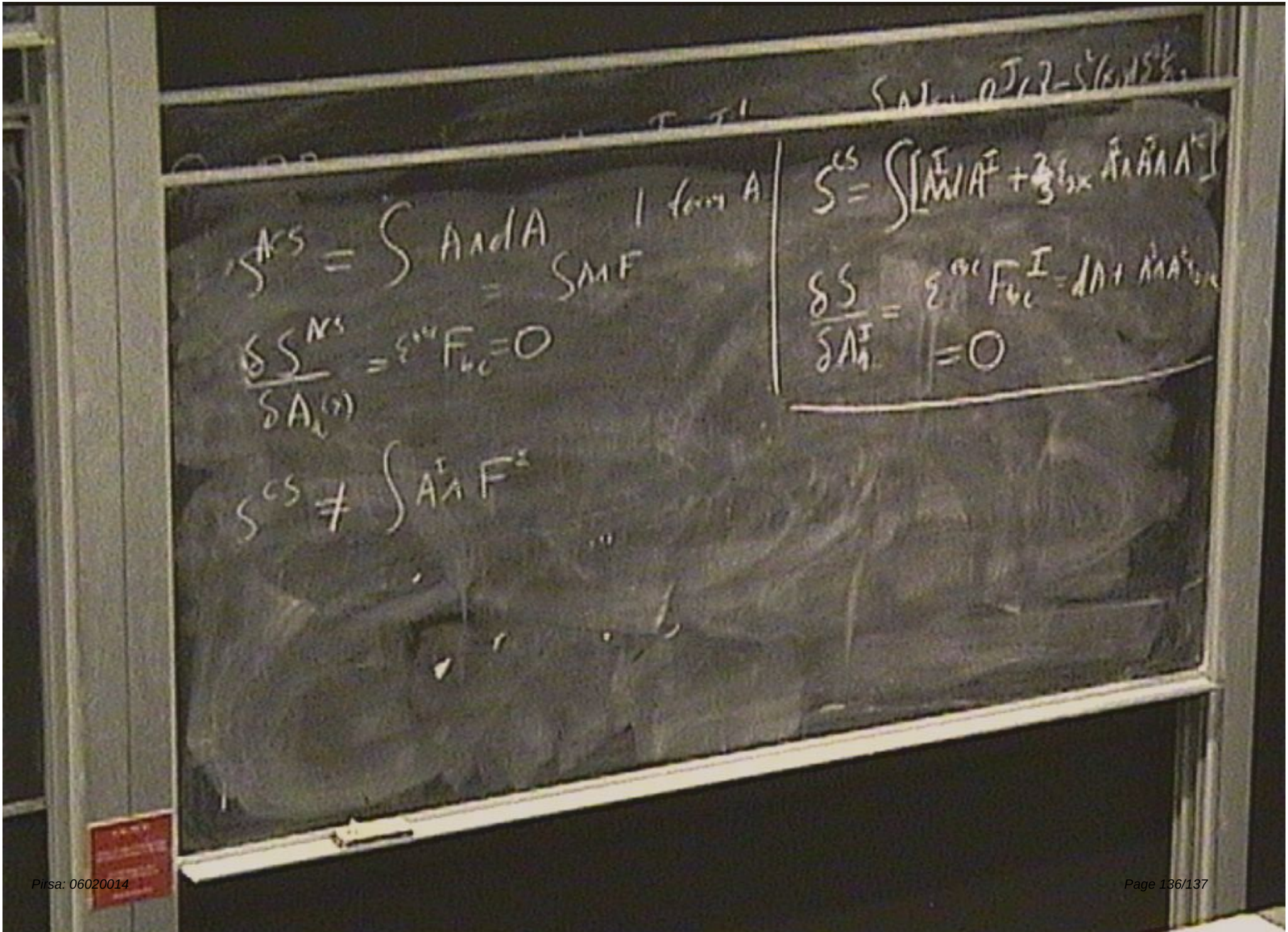

$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_\mu(x)} = \varepsilon^{\mu\nu} F_{\nu\mu} = 0$
$\int \partial \wedge F$
$S = \int$
$\frac{\delta S}{\delta A^I_\mu} = \varepsilon^{\mu\nu} F^I_{\nu\mu} - d\Lambda +$

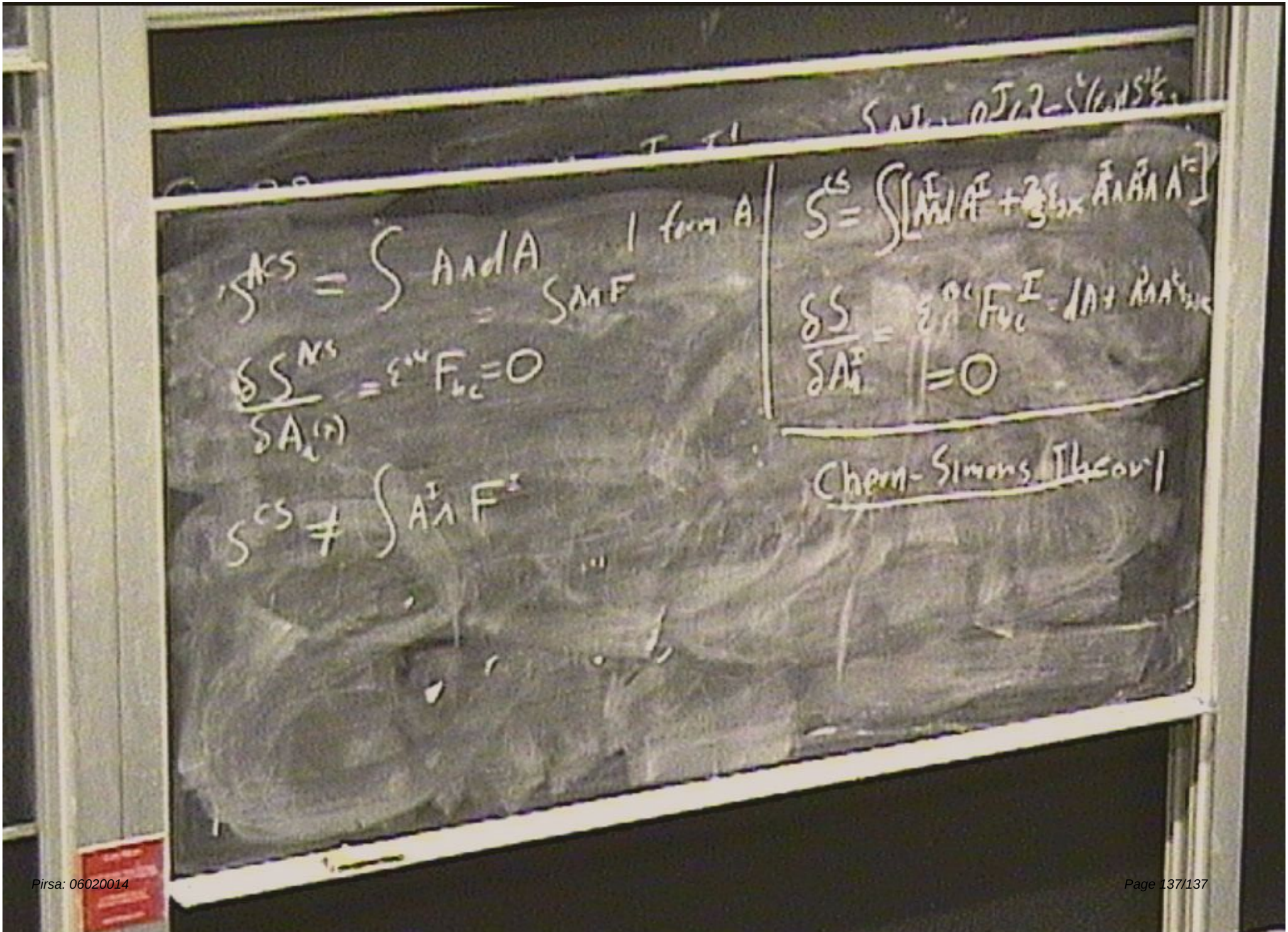

1 form A


$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_a(x)} = \epsilon^{abc} F_{bc} = 0$
$\int \ell \wedge F$
$S = \int [A^I \wedge dA^I + \frac{2}{3}\epsilon_{ijk} A^I \wedge A^J \wedge A^K]$
$\frac{\delta S}{\delta A^I_a} = \epsilon^{abc} F^I_{bc} = dA + A^J \wedge A^K \epsilon_{ijk}$


$S^{ACS} = \int A \wedge dA$
1 form A
$\frac{\delta S^{ACS}}{\delta A_a(x)} = \epsilon^{abc} F_{bc} = 0$
$\int dA F$
$S^{CS} = \int [A^I \wedge dA^I + \frac{2}{3} \epsilon_{abc} A^I \wedge A^J \wedge A^K]$
$\frac{\delta S}{\delta A_a^I} = \epsilon^{abc} F_{bc}^I = dA + A^J \wedge A^K \epsilon_{abc}$


$S^{NS} = \int A \wedge dA$ from A
$= S_{AA}F$
$\frac{\delta S^{NS}}{\delta A_\mu(x)} = \varepsilon^{\mu\nu} F_{\mu\nu} = 0$
$S^{CS} \neq \int A^2 \wedge F^2$
$S^{CS} = \int [A^I \wedge dA^I + \frac{2}{3} f_{IJK} A^I \wedge A^J \wedge A^K]$
$\frac{\delta S}{\delta A_\mu^I} = \varepsilon^{\mu\nu} F_{\mu\nu}^I = dA + A \wedge A$


$S^{NCS} = \int A \wedge dA$ (1 form A)
$= \int A \wedge F$
$\frac{\delta S^{NCS}}{\delta A_\mu^{(I)}} = \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 0$
$S^{CS} \neq \int A^I \wedge F^I$
$S^{CS} = \int [\frac{k}{4\pi} A^I \wedge dA^I + \frac{2}{3} f_{IJK} A^I \wedge A^J \wedge A^K]$
$\frac{\delta S}{\delta A_\mu^I} = \varepsilon^{\mu\nu\rho} F_{\nu\rho}^I = 0$


$S^{NCS} = \int A \wedge dA$
1 form A
$= S_{AM} F$
$\frac{\delta S^{NCS}}{\delta A_\mu^{(I)}} = \varepsilon^{\mu\nu} F_{\mu\nu} = 0$
$S^{CS} \neq \int A^I \wedge F^I$
$S^{CS} = \int [A^I_\mu \partial A^I + \frac{2}{3} g f^{IJK} A^I A^J A^K]$
$\frac{\delta S}{\delta A^I_\mu} = \varepsilon^{\mu\nu} F^I_{\mu\nu} = 0$
Chern-Simons Theory