Title: D-brane ground stakes, multicentered black holes, DT/GW correspondence, and the OSV conjecture

Date: Feb 07, 2006 02:00 PM

URL: http://pirsa.org/06020010

Abstract:

D-brane ground states, multicentered black holes, DT/GW correspondence, and the OSV conjecture

Frederik Denef

University of Leuven

Perimeter, February 7, 2006

work in progress with G. Moore

D-brane ground states, multicentered black holes, DT/GW correspondence, and the OSV conjecture

or: why OSV is probably right

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Outline

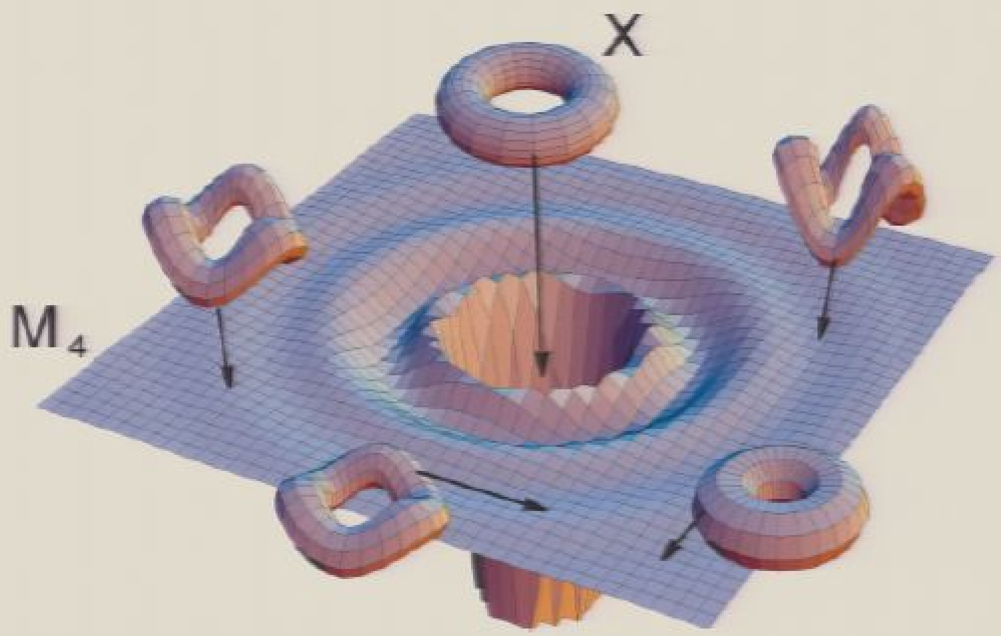
D-brane - 4d sugra correspondence

OSV at small ϕ^0

OSV in general

D-brane - 4d sugra correspondence

Setting



- IIA on Calabi-Yau X $\rightsquigarrow$ 4d $\mathcal{N} = 2$ supergravity $+(h^{1,1} + 1)$ gauge fields

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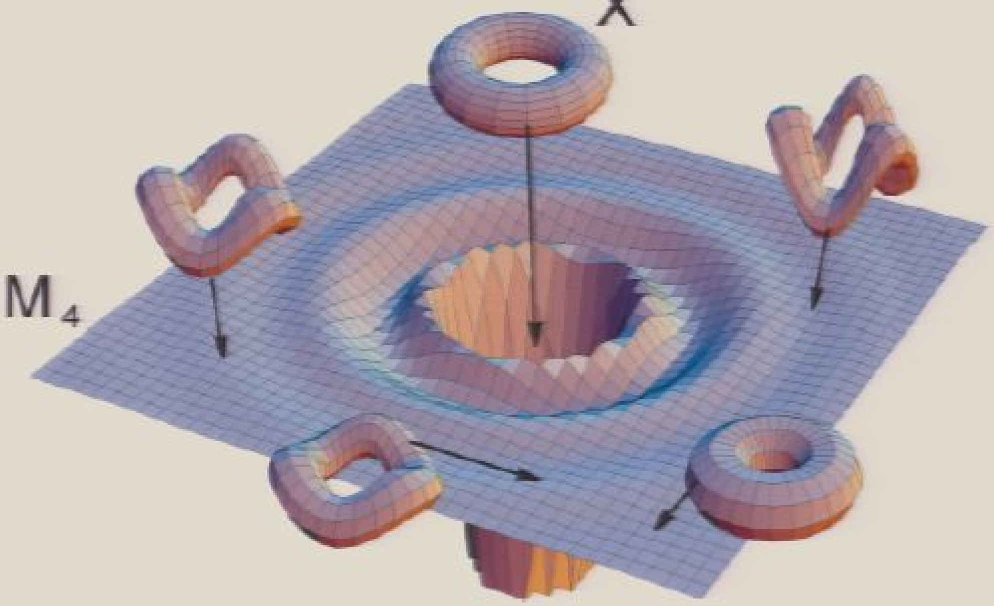

- IIA on Calabi-Yau X $\rightsquigarrow$ 4d $\mathcal{N} = 2$ supergravity $+(h^{1,1} + 1)$ gauge fields
- D6-D4-D2-D0 BPS bound st. (D-branes + gauge flux) $\rightsquigarrow$ BPS black holes with magn. and el. charges (p^0, p^A, q_A, q_0)

Wrapped D6-brane charges

D6-branes wrapped on X represented as sheaf E has induced charge vector $Q \in H^*(X)$ given by

$$Q = \mathrm{ch}(E)\sqrt{\widehat{A}} = \mathrm{ch}(E)(1 + \frac{c_2(X)}{24}).$$

In components $(p^0, p^A, q_A, q_0) =$ (D6,D4,D2,D0)-charge:

$$p^0 = Q|_{H^0}, \quad p^A D_A = Q|_{H^2}, \quad q_A = \int_X D_A \wedge Q, \quad q_0 = \int_X Q,$$

where $\{D_A\}$ is an integral basis of $H^2(X)$.

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Special case: ideal sheaf $\mathcal{I}$, $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, with $\beta \equiv [Y] \in H_2(X)$:

$$p^0 = 1, \quad p^A = 0, \quad q_A = -\beta_A + \frac{c_{2,A}}{24}, \quad q_0 = -\chi(\mathcal{O}_Y).$$

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Note: typically dim $H^2(P) \gg$ dim $H^2(X)$, so same charge can have many different flux realizations.

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BH entropy = horizon area/4 $= \pi \min_{B+iJ} |Z(Q, B+iJ)|^2 / 8J^3$

[FKS]

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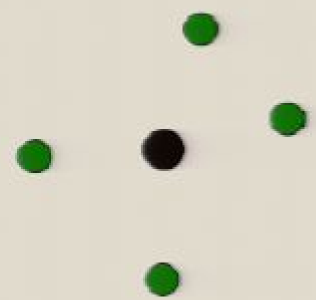

Example: pure D4 wrapped on P has charge vector $Q = P + (P^3 + c_2 \cdot P)/24$ and at $B = 0$,

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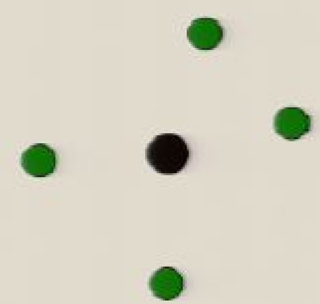

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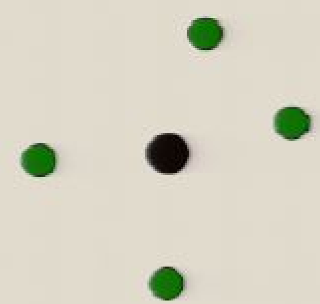

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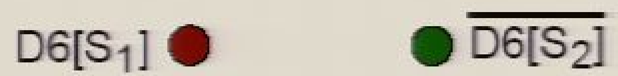
So $Z = 0$ at some large J if P is large and very ample. $\Rightarrow$ No single centered solution.

$D4$ as $D6 - \overline{D6}$ bound state



Consider D6 with flux $F = S_1 \in H^2(X, \mathbb{Z})$ and anti-D6 with flux $F = S_2$. For certain values of the background moduli $B + iJ$, there exists a 2-centered bound state of these charges.

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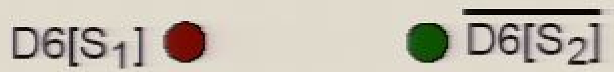

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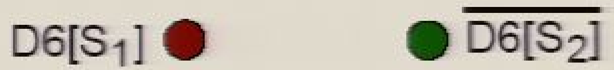

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D6[S_1] $\qquad\qquad$ $\overline{\text{D6[}S_2\text{]}}$

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Note: $S = \frac{P}{2} \bmod 1 = \frac{c_1(P)}{2} \bmod 1$. (as it should [FW,MM])

Size, Stability and Spin: general 2-centered case

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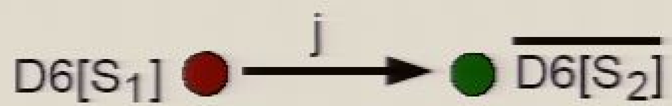

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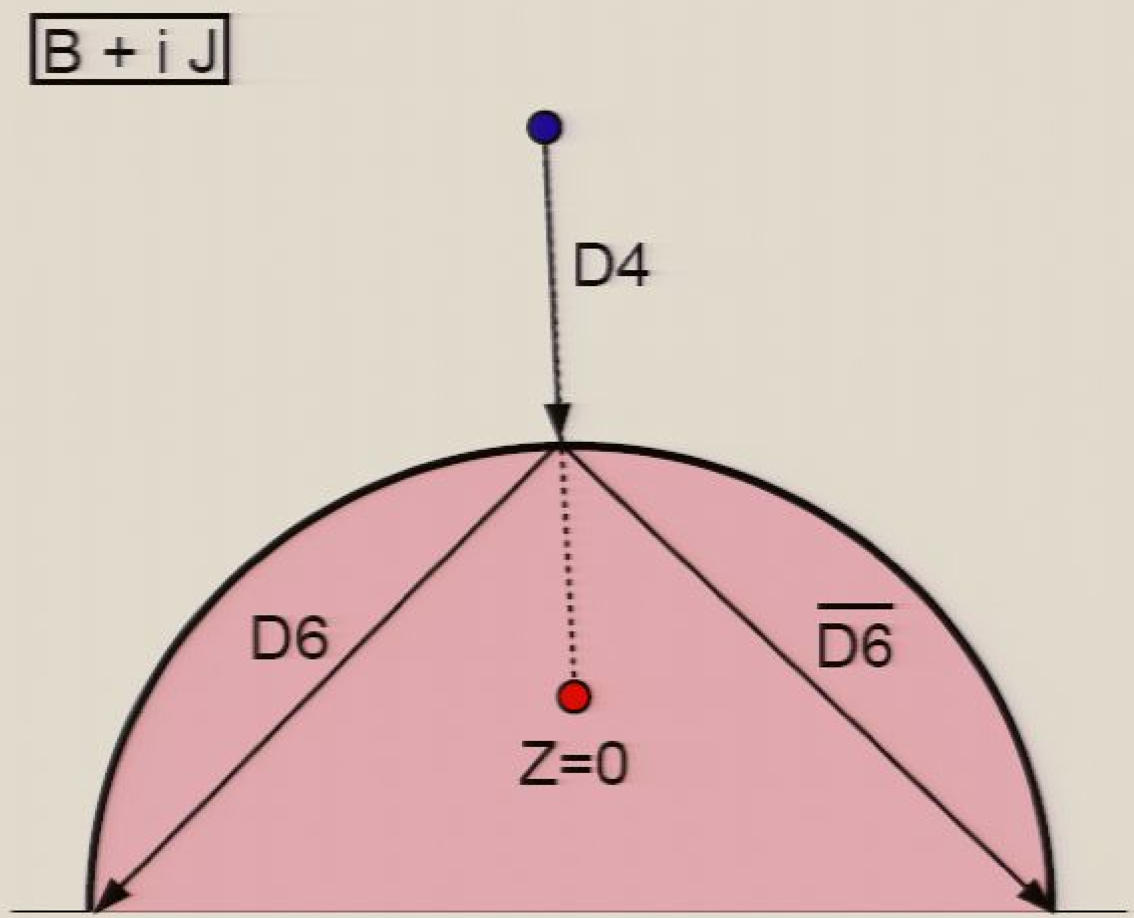
  - along path $J = r\,P$, $B = S$: crosses wall of marginal stability at $r = \frac{1}{2}\sqrt{3 + P \cdot c_2 / P^3}$.

Stability region of $D4 = D6 - \overline{D6}$



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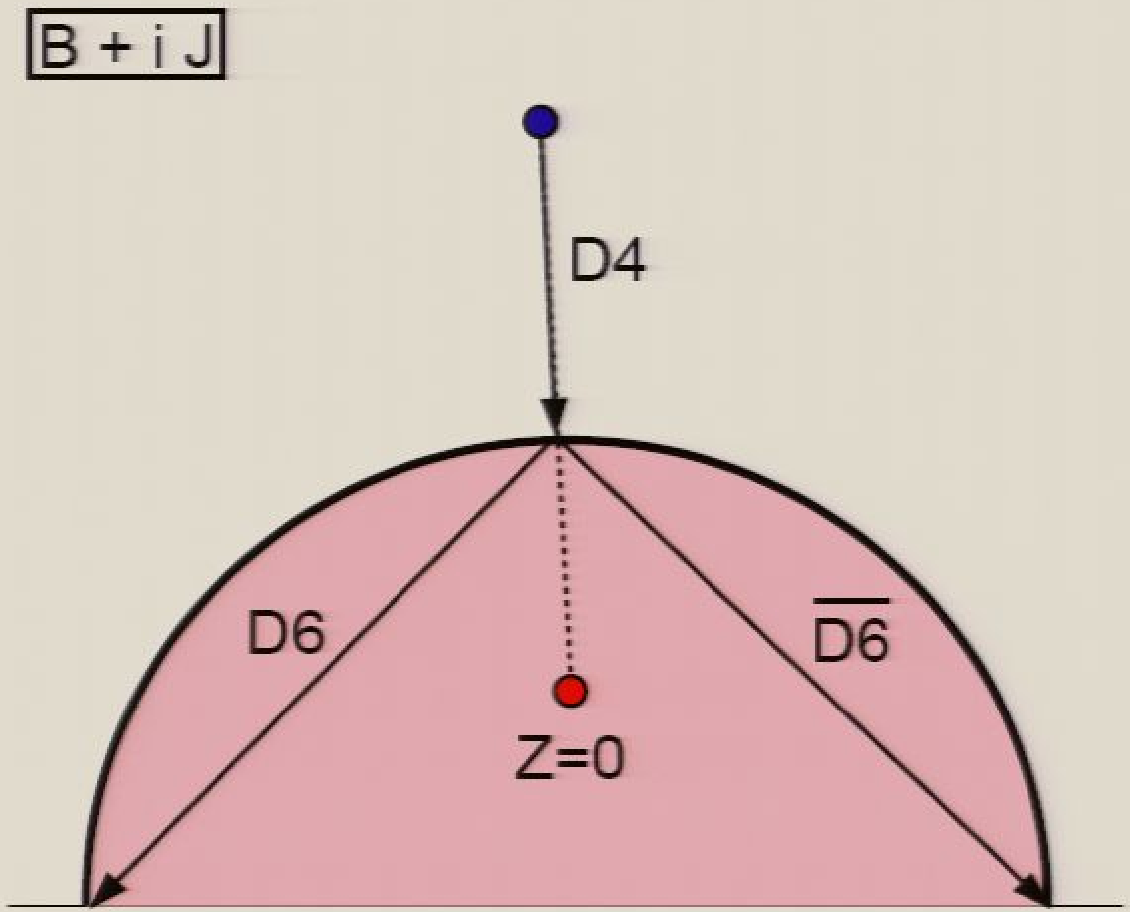

Structure not visible in classical geometric picture of D4 as holomorphic cycle.

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$U(1) \times U(1)$ quiver quantum mechanics with $k = \frac{P^3}{6} + \frac{c_2 \cdot P}{12}$ chiral multiplets Φ^i. (No further matter because pure D6 has no moduli on genuine CY.)

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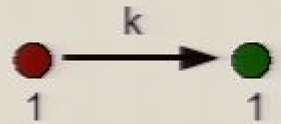

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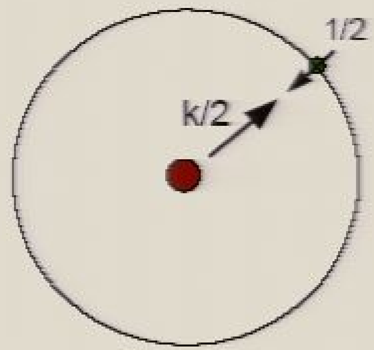

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Size, Stability and Spin: $D6 - \overline{D6}$ system

$$D6[S_1] \bullet \xrightarrow{\;j\;} \bullet\ \overline{D6[S_2]}$$

- Intersection product:

$$-\langle D6[S_1], \overline{D6[S_2]} \rangle = e^{S_1 - S_2} \cdot \widehat{A} = \frac{P^3}{6} + \frac{c_2 \cdot P}{12}$$

  where $P = S_1 - S_2$. Note: if P is class of very ample divisor, this is $\dim H^0(X, \mathcal{L}_P) =$ number of deformations $+ 1$.

- Stability condition $R > 0$, i.e. $\langle Q_1, Q_2 \rangle \, \mathrm{Im}(Z_1 \overline{Z_2}) > 0$:
  - when $J \to \infty$: $\left(\frac{P^3}{6} + \frac{c_2 \cdot P}{12} \right) P \cdot J^2 > 0$. ✓ ok for P very ample.
  - along path $J = r\,P$, $B = S$: crosses wall of marginal stability at $r = \frac{1}{2}\sqrt{3 + P \cdot c_2 / P^3}$.

Size, Stability and Spin: general 2-centered case

- Equilibrium distance between centers

$$R = \frac{\langle Q_1, Q_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \overline{Z_2})}\bigg|_\infty$$

where $\langle Q_1, Q_2 \rangle \equiv Q_1^{mag} \cdot Q_2^{el} - Q_1^{el} \cdot Q_2^{mag}$ = DSZ symplectic intersection product between charges Q_1 and Q_2. In case of two D6 corresponding to sheaves E_1 and E_2 on X:

$$\langle Q_1, Q_2 \rangle = \int \mathrm{ch}(E_1) \wedge \mathrm{ch}(-E_2) \wedge \widehat{A}.$$

- Stability condition: $R > 0$. When approaching wall of marginal stability $\arg Z_1 = \arg Z_2$, $R \to \infty$ and bound state decays. Indeed, spectrum of BPS states is moduli-dependent!

[μ-stab., θ-stab. King, Π-stab. Douglas et al, SLAG stab. Joyce, ...]

- Intrinsic spin stored in electromagnetic field:

$$j = \frac{1}{2} \langle Q_1, Q_2 \rangle.$$

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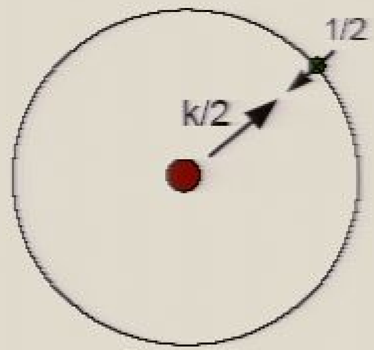

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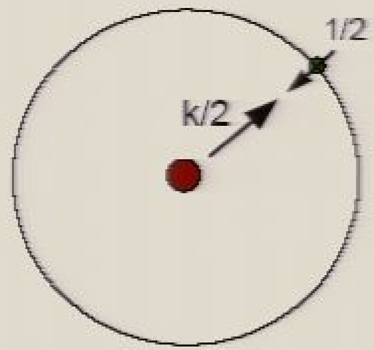

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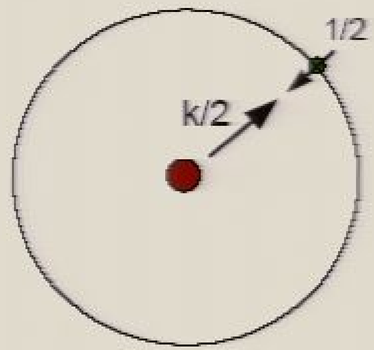

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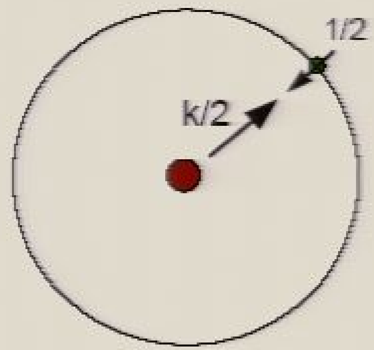

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Ground state has electron spin down and total spin $j = (k-1)/2$ $\Rightarrow$ degeneracy (apart from overall center of mass hypermultiplet factor):

$$d = 2j + 1 = k.$$

$=$ Landau deg. of electron on sphere with k units of magnetic flux.

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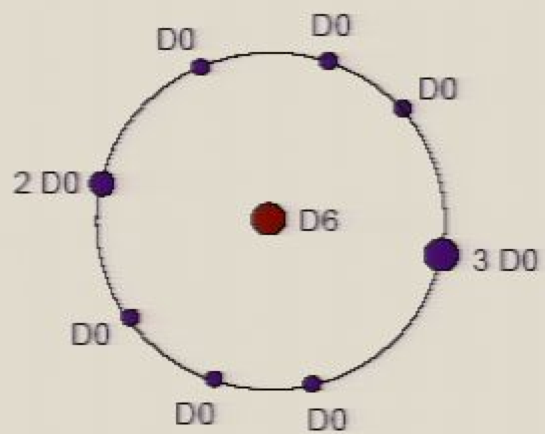

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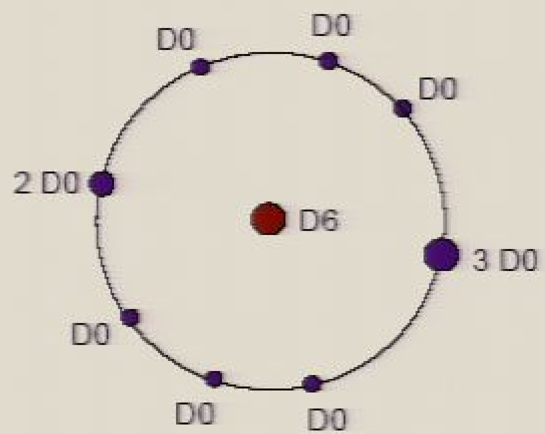

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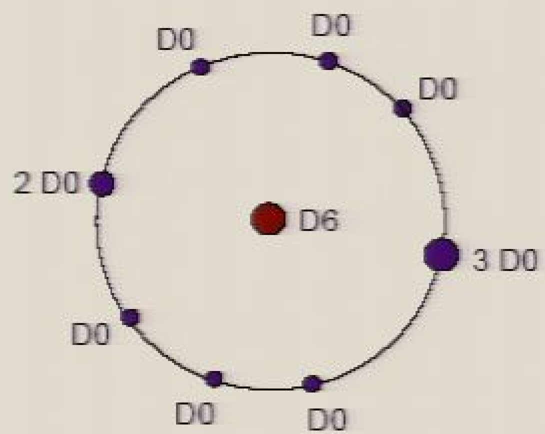

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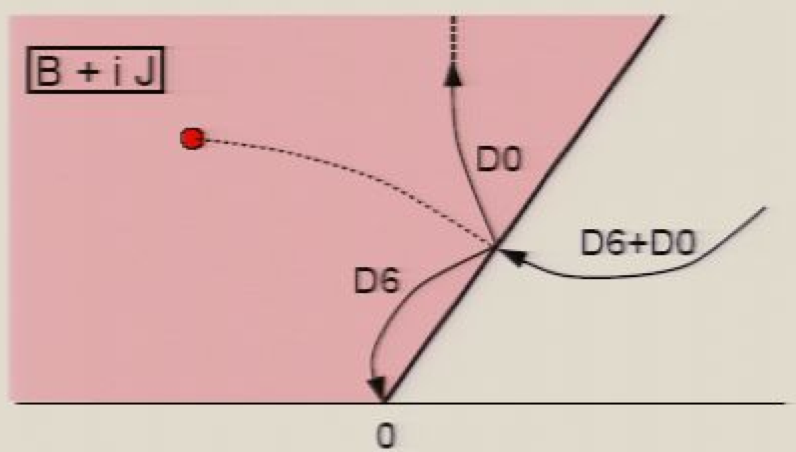

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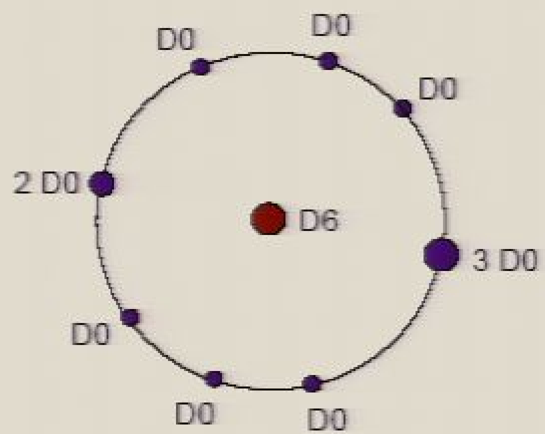

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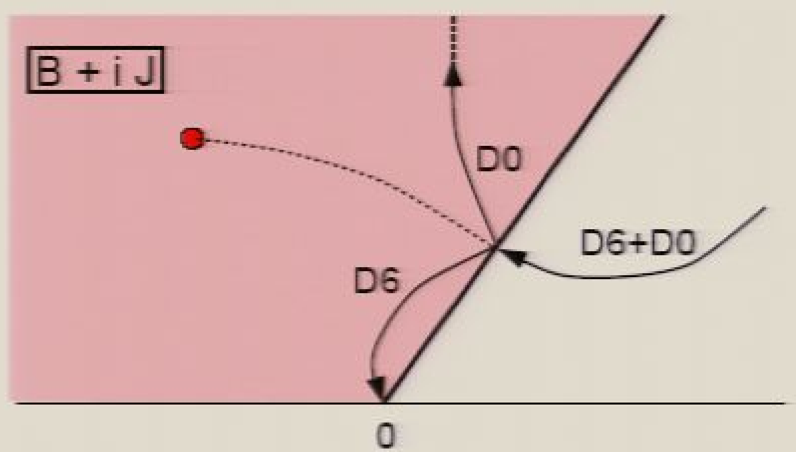

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This gives as generating function for the number of $(D6, ND0)$ BPS states, counted with signs:

$$\mathrm{Tr}\,(-1)^F q^N = \left(\prod_{n=1}^{\infty} (1 - q^n)^n \right)^{-\chi(X)} = M(q)^{-\chi(X)}$$

D6 - N D0: microscopic counting; DT invariants

Donaldson-Thomas invariants "count" ideal sheaves with D6 charge 1, D2 charge $-\beta + c_2/24$ and D0 charge $-n$. We will assume they thus count D6-D2-D0 BPS bound states (in appropriate region of CY moduli space).

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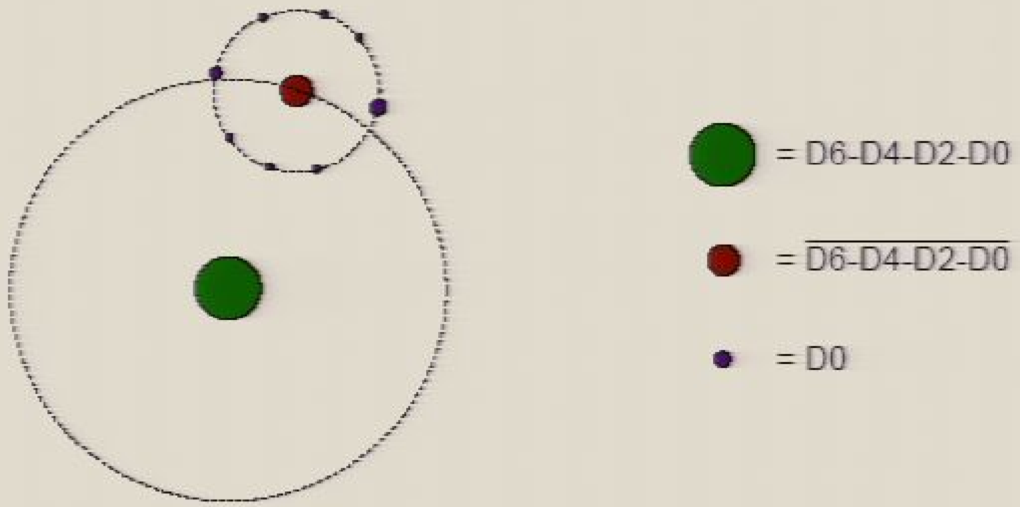

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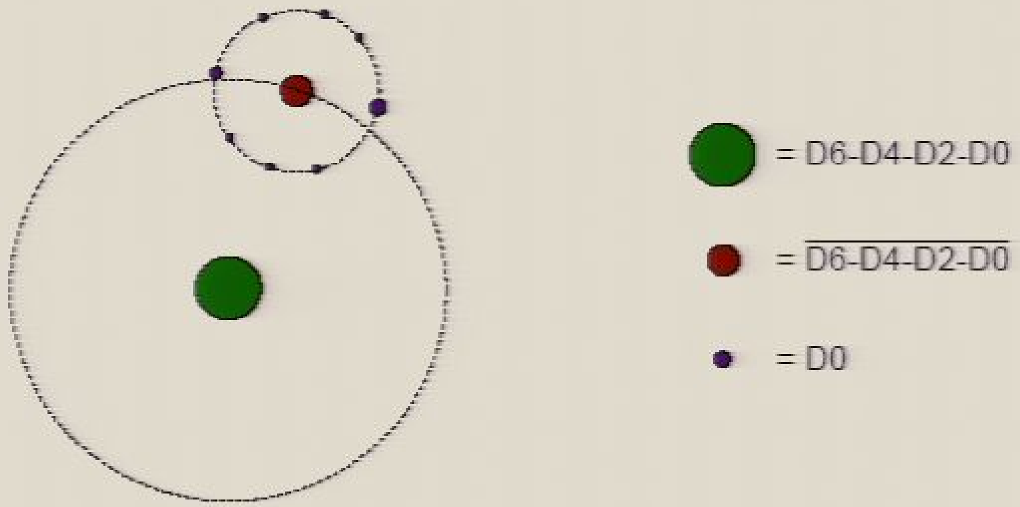

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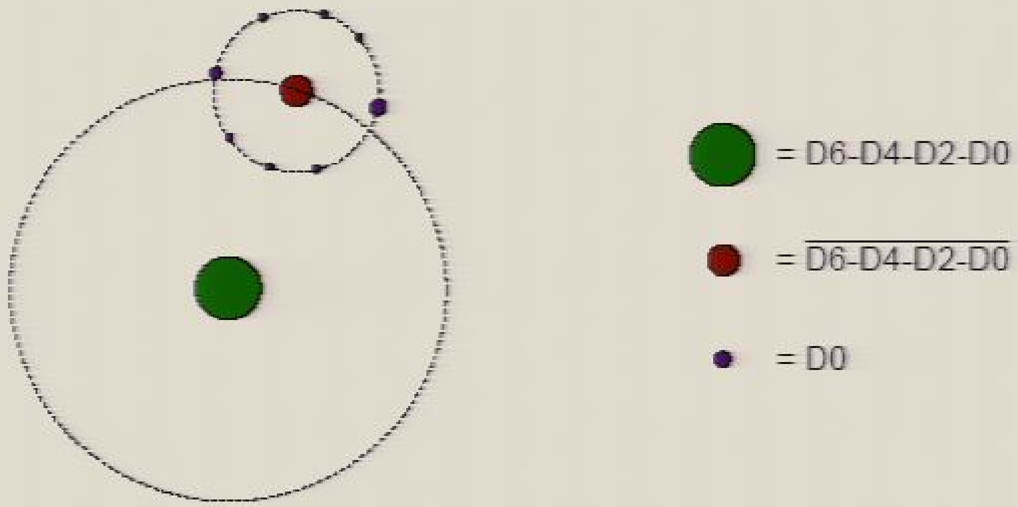

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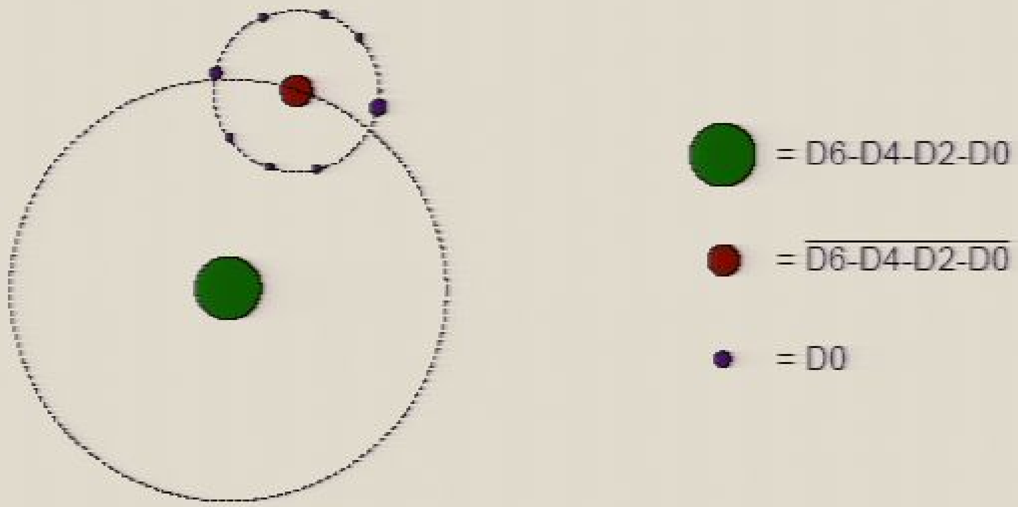

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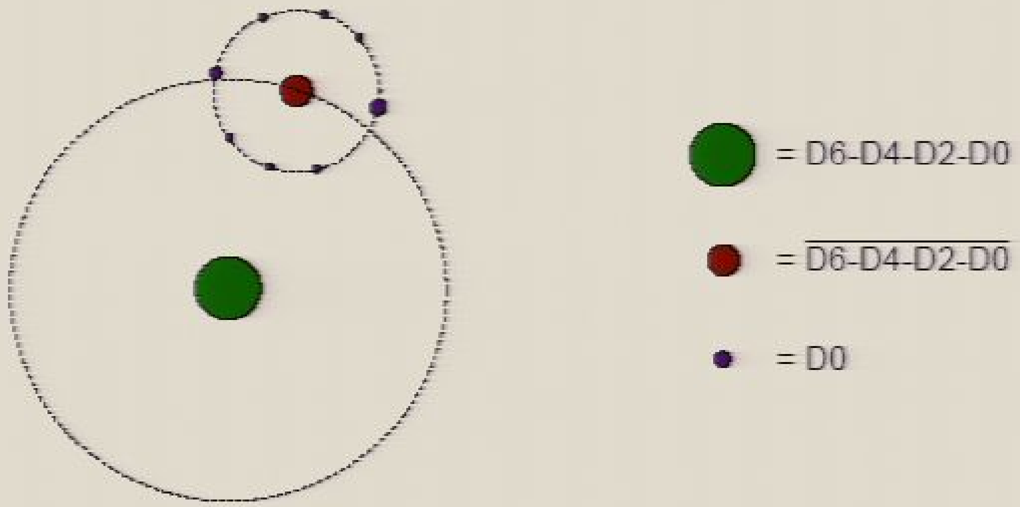

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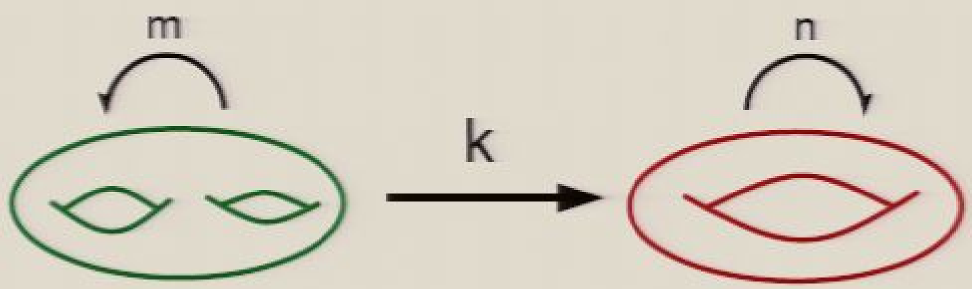
Total degeneracy obtained by summing over all possible configurations with same net charge. If "entropic additions" to original $D6 - \overline{D6}$ are not too big, entropy will still be dominated by configurations as above, rather than single centered one (does not even exist as long as $\chi(P)/24$ contribution to q_0 dominates).

Microscopic description of these configurations

In this regime near MS, quiver QM:



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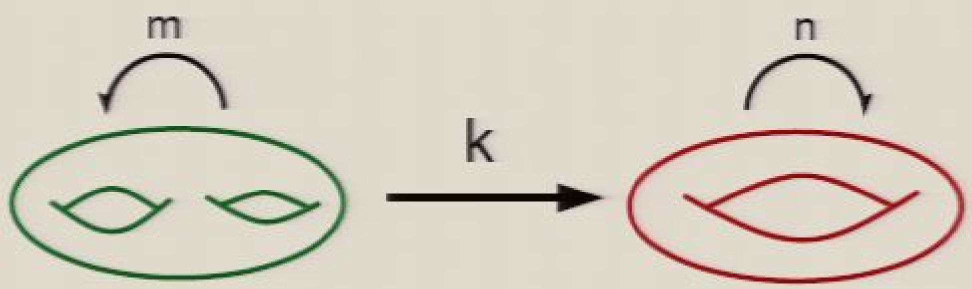

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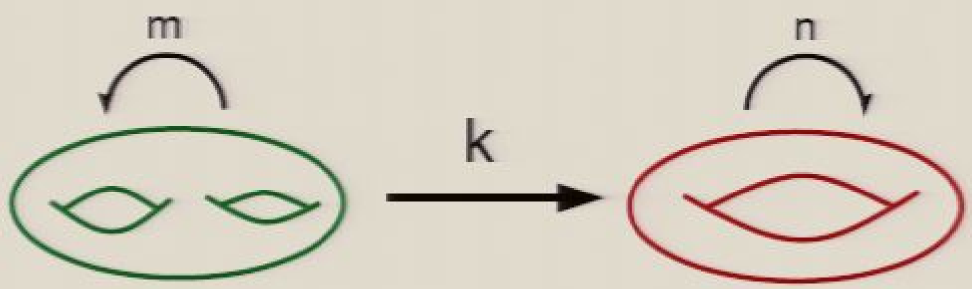

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$$\mathcal{Z}_{osv}(\phi^0, \Phi^A) = \sum_{q_0, q_A} \Omega(q_0, q_2)\, e^{-\pi\phi^0 q_0 - \pi\Phi^A q_A}$$

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Under these dualities $\mathcal{Z}_{D4}$ should be invariant or perhaps transform as a modular form. This descends to the following formal equality:

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More generally: at large $\chi(P) = P^3 + c_2 \cdot P$, $N - F^2/2$ must become very large before single centered BH configurations start to exist, so these are very much suppressed in this sum. Leading contributions will come from exactly the multicentered bound states we have been considering, not deviating much from pure $D6[S_1] - \overline{D6[S_2]}$ system.

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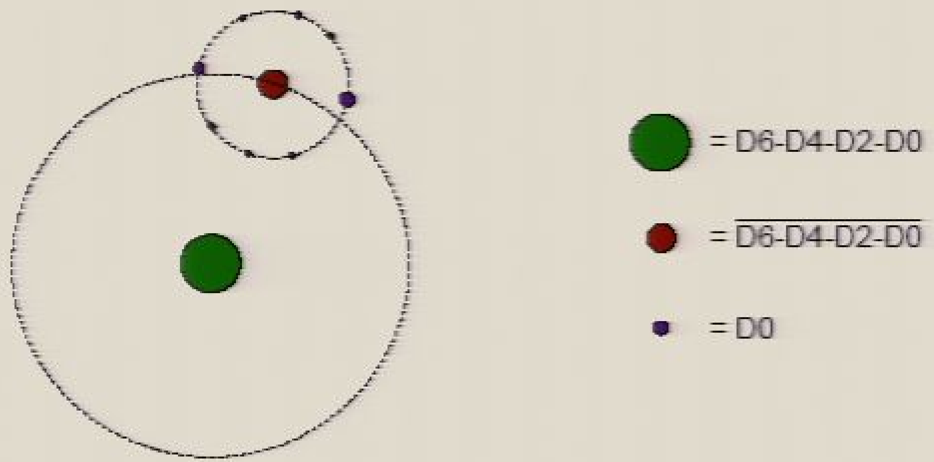

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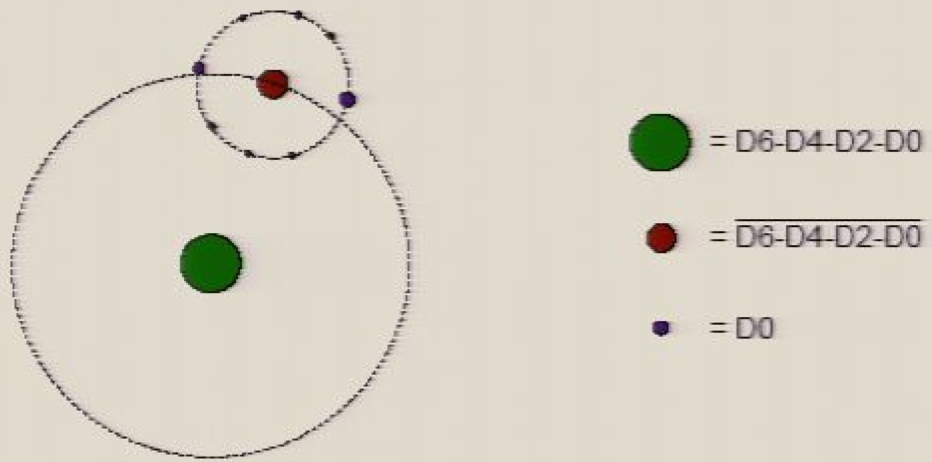

- Each D6+D4+D2+D0 can be monodromy transformed by B-shifts to D6+D2+D0, which in turn is described by ideal sheaf whose BPS states are counted by reduced Donaldson-Thomas generating function $\mathcal{Z}'_{DT} = \mathcal{Z}_{DT}/\mathcal{Z}^0_{DT}$. We factor out contributions of degree zero because these correspond to D0-branes in cloud.

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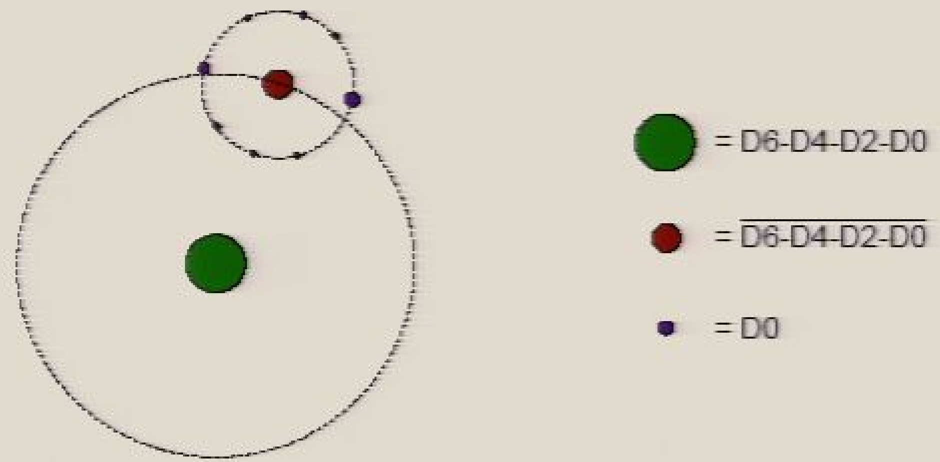

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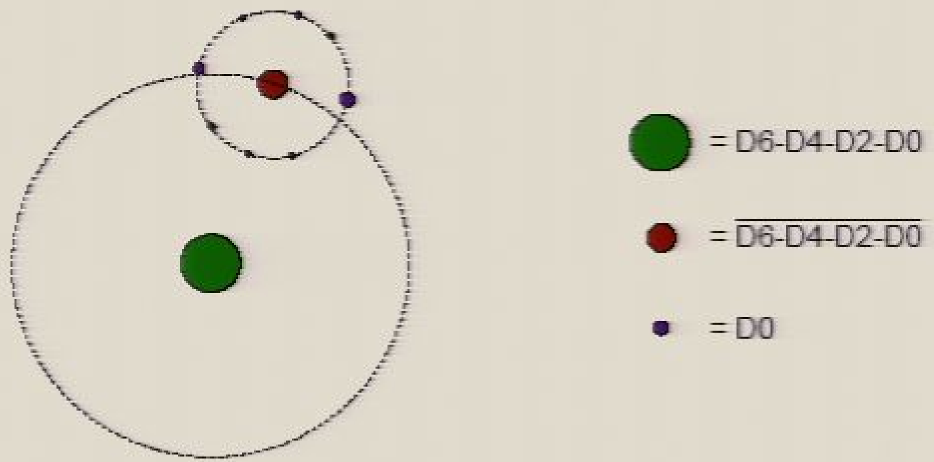

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- D0-brane cloud gives contribution given by coefficients of Mac Mahon $M(q)^{-\chi(X)}$ (degree zero DT invariants).
- Landau degeneracy from $D6 - \overline{D6}$ equals $\chi(\mathcal{M}_P) = P^3/6 + c_2 \cdot P/12$.

Computing $\mathcal{Z}_{osv}$

Thus, at large $\chi(P) = P^3 + c_2 \cdot P$, after some work:

$$\mathcal{Z}_{osv} \approx \chi(\mathcal{M}_P)(\phi^0/2)^{1-b_1} M(e^{4\pi/\phi^0})^{-\chi(X)} e^{-\frac{\pi}{6\phi^0}(P^3+c_2\cdot P)}$$
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in exact agreement with (refined) OSV conjecture!

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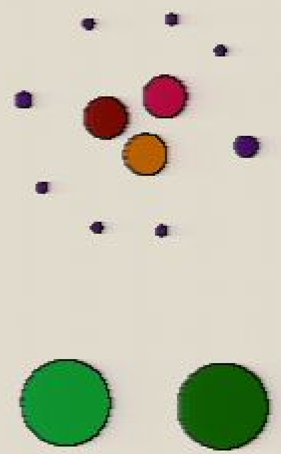

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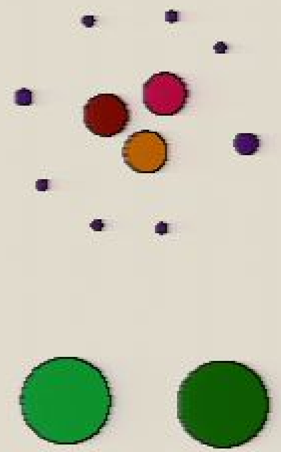

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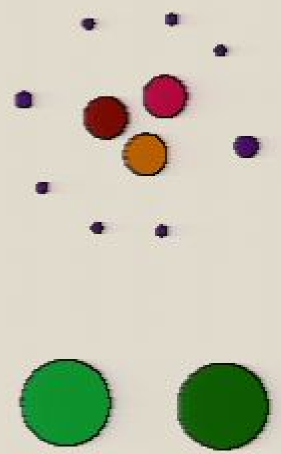

- More centers [DGOV]
- More complicated Landau degeneracies or non-factorization Landau - internal states.

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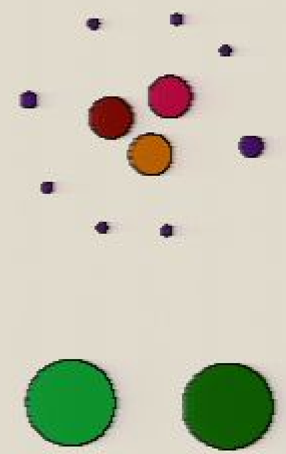

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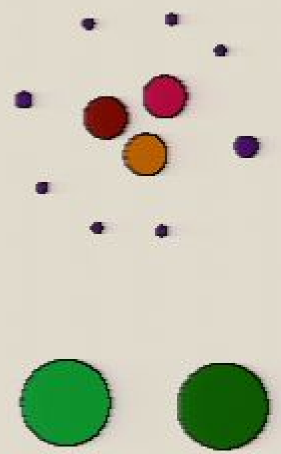

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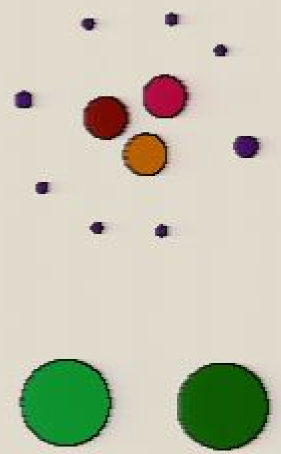

- More centers [DGOV]
- More complicated Landau degeneracies or non-factorization Landau - internal states.
- More D6-branes
- More precision
- But: suspect asymptotically exact in limit $P^3/\phi^0 \to \infty$, $P/\phi^0 \gg 1$.
- ...

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Dominant contributions

So we have

$$\mathcal{Z}_{osv} = (\phi^0/2)^w e^{\frac{\pi}{2\phi^0}\Phi^2} \sum_{F,N} \Omega(F,N)\, e^{-\frac{4\pi}{\phi^0}(-N+\frac{F^2}{2}+\frac{\chi(P)}{24})+\frac{2\pi i}{\phi^0}\Phi\cdot F}$$

We take as usual $\mathrm{Re}\,\phi^0 < 0$.

Comparison to OSV conjecture

Up to prefactor refinement (which was not specified in conjecture), matches exactly in $\phi^0 \to 0$ approximation:

$$\mathcal{Z}_{OSV} \sim e^{F_{top}(\lambda, t)} \, e^{\overline{F_{top}(\lambda, t)}}$$

with substitutions

$$\lambda \to \frac{4\pi i}{\phi^0}, \quad t^A \to \frac{-ip^A + \phi^A}{\phi^0}.$$

OSV partition function for D4

Define

$$\mathcal{Z}_{osv}(\phi^0, \Phi^A) = \sum_{q_0, q_A} \Omega(q_0, q_2)\, e^{-\pi\phi^0 q_0 - \pi\Phi^A q_A}$$

where $\Omega(q_0, q_A)$ is suitable index of BPS states of a D4 wrapped on very ample divisor $P = p^A D_A$ with D2 charges $q_A = D_A \cdot F$ and D0-charge $q_0 = -N + F^2/2 + \chi(P)/24$.

Susy condition [MMMS]:

$$F^{2,0} = 0$$

$\rightsquigarrow$ puts constraints on divisor embedding in X. [MGM,S et al]

Because in general $H^2(P) \gg H^2(X)$, there are many different (F, N) giving same (q_0, q_A). Each sector gives moduli space $\mathcal{M}_{P,F,N}$ of divisors deformations + D0-positions, and

$$\Omega(q_0, q_A) = \sum_{F,N \Leftrightarrow q_0, q_A} \chi(\mathcal{M}_{P,F,N})$$

Evaluation $\mathcal{Z}_{osv}$ at small ϕ^0

In continuum approximation for sum over F ($\Leftrightarrow$ large $|q_0|$ approx. $\Leftrightarrow$ small $|\phi^0|$ approx.): $\mathcal{Z}_{osv}$ can be evaluated as Gaussian boson-fermion integral with Q-symmetry, giving:

$$\mathcal{Z}_{osv} \approx \hat{\chi}(\mathcal{M}_0) \left(\frac{\phi^0}{2}\right)^{1-b_1} \exp\left(-\frac{\pi}{6\phi^0}(P^3 + c_2 \cdot P) + \frac{\pi}{2\phi^0}\Phi^2\right).$$

where "differential geometric Euler characteristic"

$$\hat{\chi}(\mathcal{M}_0) \equiv \frac{1}{\pi^n} \int_{\mathcal{M}} \det R,$$

with R curvature form of natural $H^{2,0}$ metric on $\mathcal{M}_0$.

singular $\Rightarrow$ not at all obvious that

$$\hat{\chi} = \chi_{top} = (\frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P)/|\mathrm{Aut}|,$$

but comparison to independent results [Shih-Yin] indicate it is!

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We take as usual $\mathrm{Re}\,\phi^0 < 0$.

The leading contribution comes from $N = 0, F = 0$ because $N \geq 0, F^2 \leq 0$ on susy configurations [There is actually one "bad" positive susy F^2 mode, but this disappears in regularized version.]

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