Title: Beyond i.i.d. in quantum information theory

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Abstract: The information spectrum approach gives general formulae for optimal rates of codes in many areas of information theory. In this talk I shall relate the information spectrum approach to Shannon information theory and explore its relationship to ``entropic" properties including subadditivity, chain rules, Araki-Lieb inequlities, and monotonicity.

Beyond i.i.d. in Quantum Information

How I learned to stop worrying
and love the information spectrum

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Engineering and Physical Sciences
Research Council

UNIVERSITY OF
CAMBRIDGE

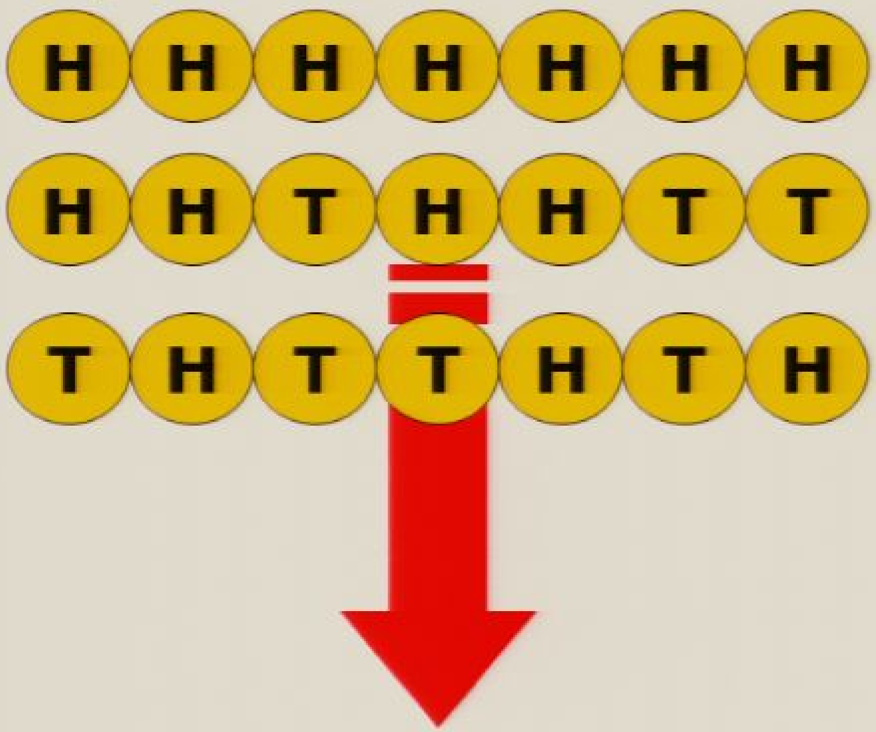
Contents

- Information, entropy and i.i.d.
- Beyond i.i.d.?
- Introduction to spectral entropy
- Source coding: an example
- Properties of spectral information
- What's next?
- The End

What is Information?

- The Action of "Informing"
- The *difference* between what you *know now* and what you *knew before*
- *"Information in communication theory relates not to what you do say, but what you could say."* – Shannon & Weaver 1949

How Much Information?



- Amount of information depends on *initial uncertainty*

$$h(x_i) = f\left(\frac{1}{p(x_i)}\right)$$

Properties of Uncertainty

- Zero for certainties $p(x_i) = 1 \Rightarrow h(x_i) = 0$
- Additive for two independent events $p(x_i, y_j) = p(x_i)p(y_j)$ $\Rightarrow h(x_i, y_j) = h(x_i) + h(y_j)$
- Logarithm function is only choice

$$h(x_i) = \log\left(\frac{1}{p(x_i)}\right) = -\log p(x_i)$$

Entropy

The *Entropy* measures uncertainty:

$$H(X) = \langle h(X) \rangle = -\sum_i p(x_i) \log_2 p(x_i)$$

Logarithm to base 2 gives *bits*

- Example: Coin flip has uncertainty of 1 bit

Conditional Entropy

- Uncertainty in X, now that you know Y
- Difference in entropies (*chain rule*):

$$H(X|Y) = H(X,Y) - H(Y)$$

- Classically this must be *positive*, you cannot be more uncertain about Y alone, than you are about X *and* Y

Mutual Information

- The *difference* between what you *didn't know before* and what you *don't know now*
- Uncertainty in X – Uncertainty in X given Y

$$I(X:Y) = H(X) + H(Y) - H(X,Y)$$

Mutual Information of X and Y

Information Theory

- Operational quantities
  - Data Compression $R = H(X)$
  - Noisy Channel Capacity $C = \max I(X:\Lambda X)$
- Assumptions
  - Asymptotically zero error probability
  - Maximize the rate (most efficient use of resources)
  - i.i.d.!

Source Coding

Binary alphabet {0,1} with probabilities 0.75 and 0.25 respectively, for *n=4*

$$x \in \{0000\} \mapsto p(x) = 0.75^4 \approx 0.316$$

$$x \in \{0001, 0010, 0100, 1000\} \mapsto p(x) = 0.75^3 \cdot 0.25 \approx 0.105$$

$$x \in \{0011, 0101, 1001, 0110, 1010, 1100\} \mapsto p(x) = 0.75^2 \cdot 0.25^2 \approx 0.035$$

$$x \in \{0111, 1011, 1101, 1110\} \mapsto p(x) = 0.75 \cdot 0.25^3 \approx 0.012$$

$$x \in \{1111\} \mapsto p(x) = 0.25^4 \approx 0.004$$

Source Coding II

Total probabilities for numbers of 0's and 1's give *typical sequences*

$$T_0 = \{0000\} \mapsto p(T_0) = 0.75^4 \approx 0.316$$

$$T_1 = \{0001, 0010, 0100, 1000\} \mapsto p(T_1) = 4 \cdot 0.75^3 \cdot 0.25 \approx 0.422$$

$$T_2 = \{0011, 0101, 1001, 0110, 1010, 1100\} \mapsto p(T_2) = 6 \cdot 0.75^2 \cdot 0.25^2 \approx 0.211$$

$$T_3 = \{0111, 1011, 1101, 1110\} \mapsto p(T_3) = 4 \cdot 0.75 \cdot 0.25^3 \approx 0.047$$

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Source Coding III

Define a typical set

$$\mathrm{T}_n = \left\{ x_i : H(X) - \varepsilon \le -\tfrac{1}{n} \log p(x_i) \le H(X) + \varepsilon \right\}$$
$$= \left\{ x_i : e^{-n(H(X)+\varepsilon)} \le p(x_i) \le e^{-n(H(X)-\varepsilon)} \right\}$$

then

$$\lim_{n\to\infty} p(\mathrm{T}_n) = 1$$
$$|\mathrm{T}_n| \le e^{n(H(X)+\varepsilon)}$$

$$1 = \sum_i p(x_i)$$

$$\geq \sum_{x_i \in \mathbb{T}_n} p(x_i)$$

$$1 = \sum_i p(x_i)$$

$$\geq \sum_{x_i \in H_n} p(x_i)$$

$$\geq \sum_{x_i \in H_n} e^{-n(H(x)+\varepsilon)}$$

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i.i.d.

independent and

identically

distributed

Is this a *reasonable* assumption?

i.i.d.

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Is this a *reasonable* assumption?

Who Cares?

- Me
- Applications - Absolute physical bounds require minimal assumptions
- Theory - Deeper insight

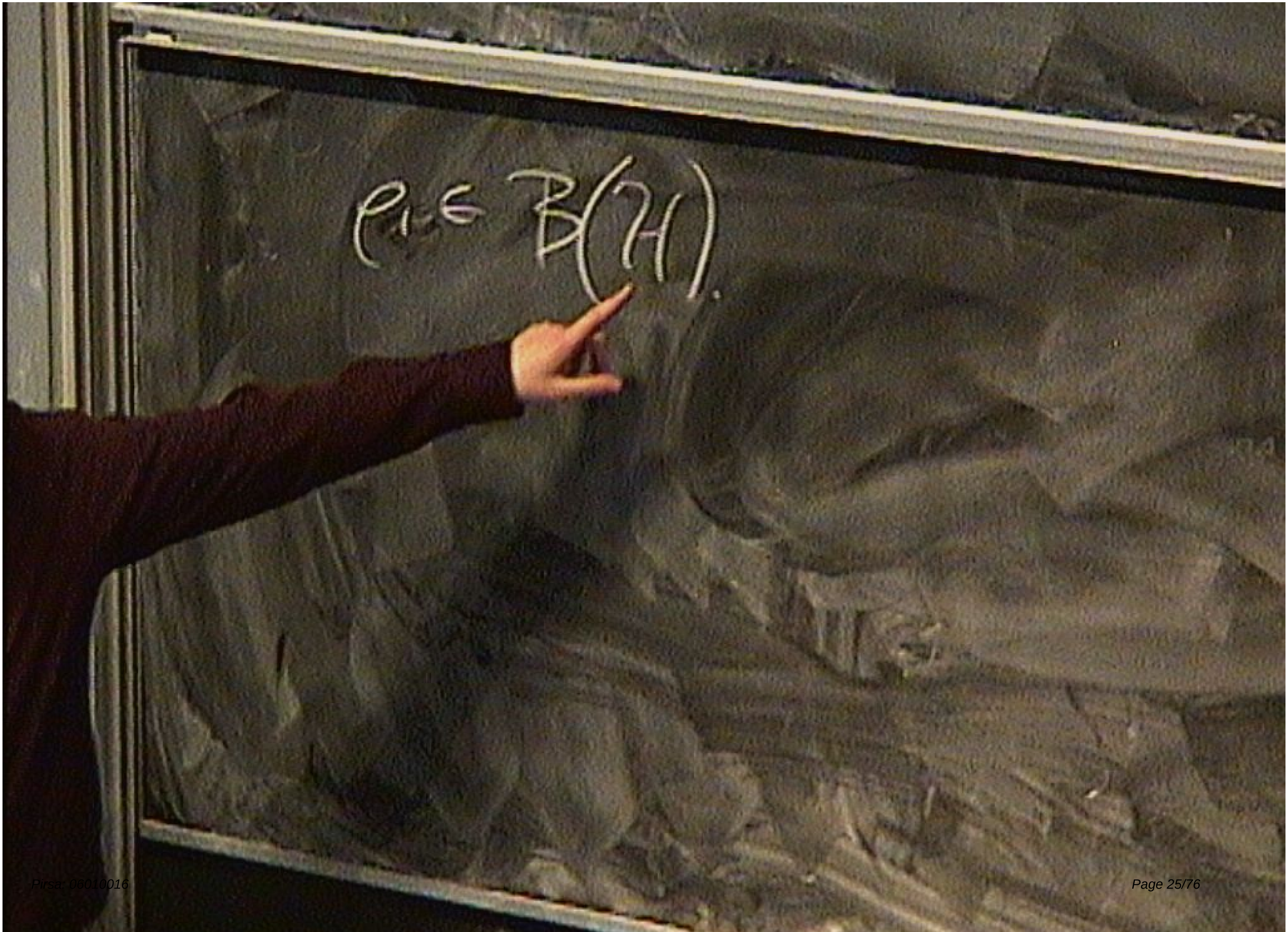
Non-i.i.d.

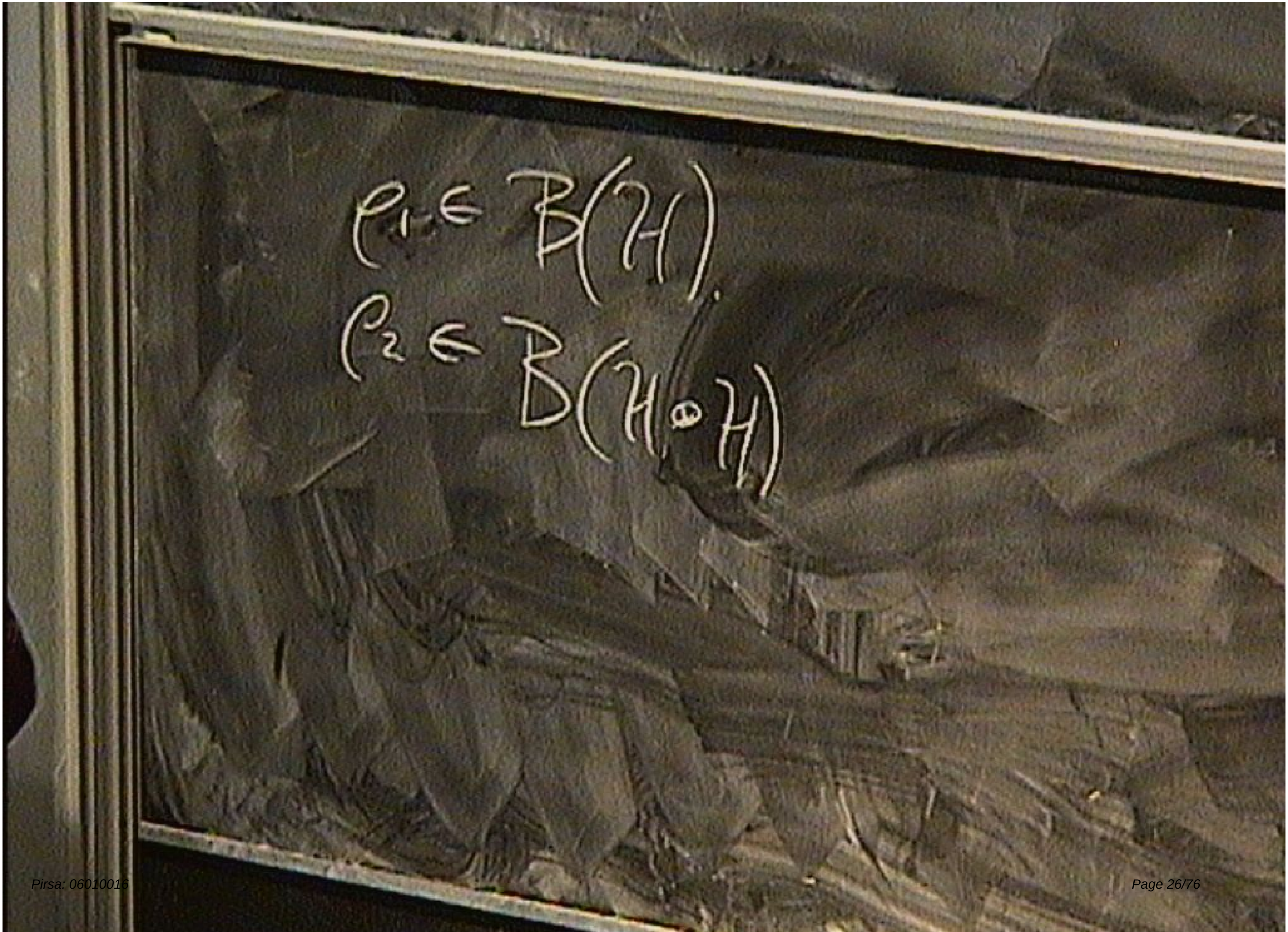
- Treat sequences of distributions (states)

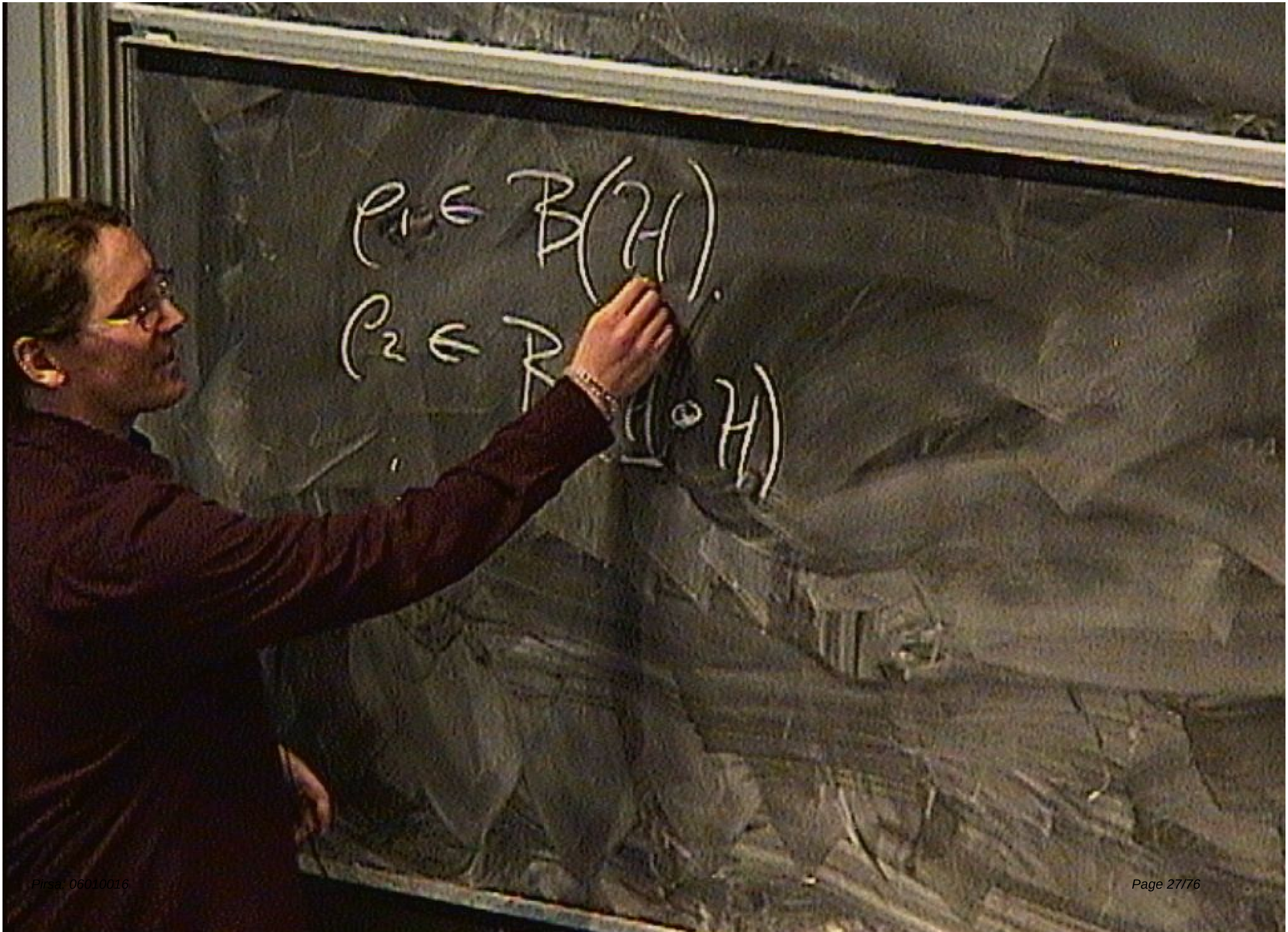
$$\rho = \{\rho_n\}_{n=1}^{\infty}$$

- i.i.d. a subset of all sequences

$$\rho = \{\rho^{\otimes n}\}_{n=1}^{\infty}$$




ρ1 ∈ B(H)
ρ2 ∈ B(H ⊗ H)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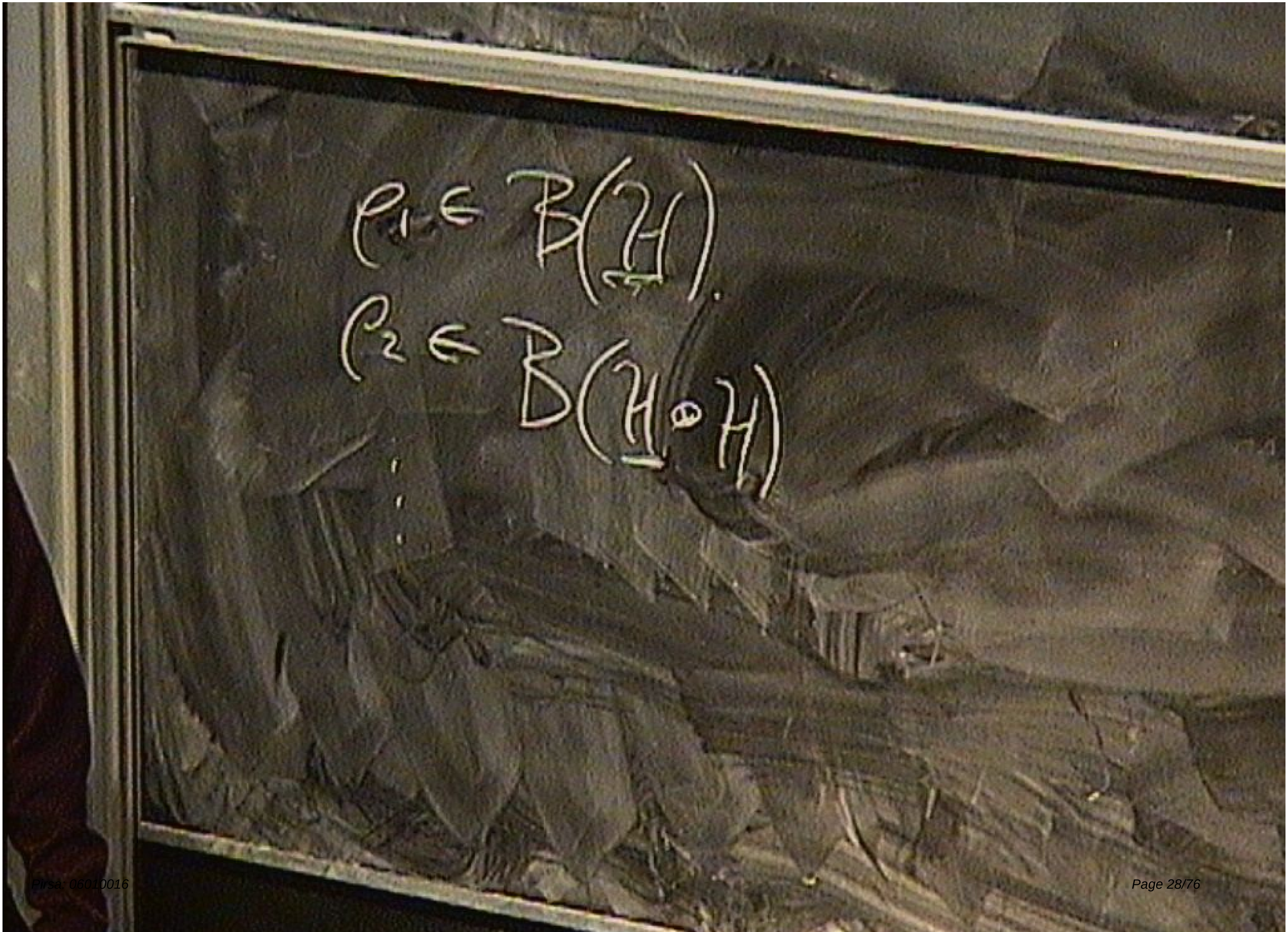
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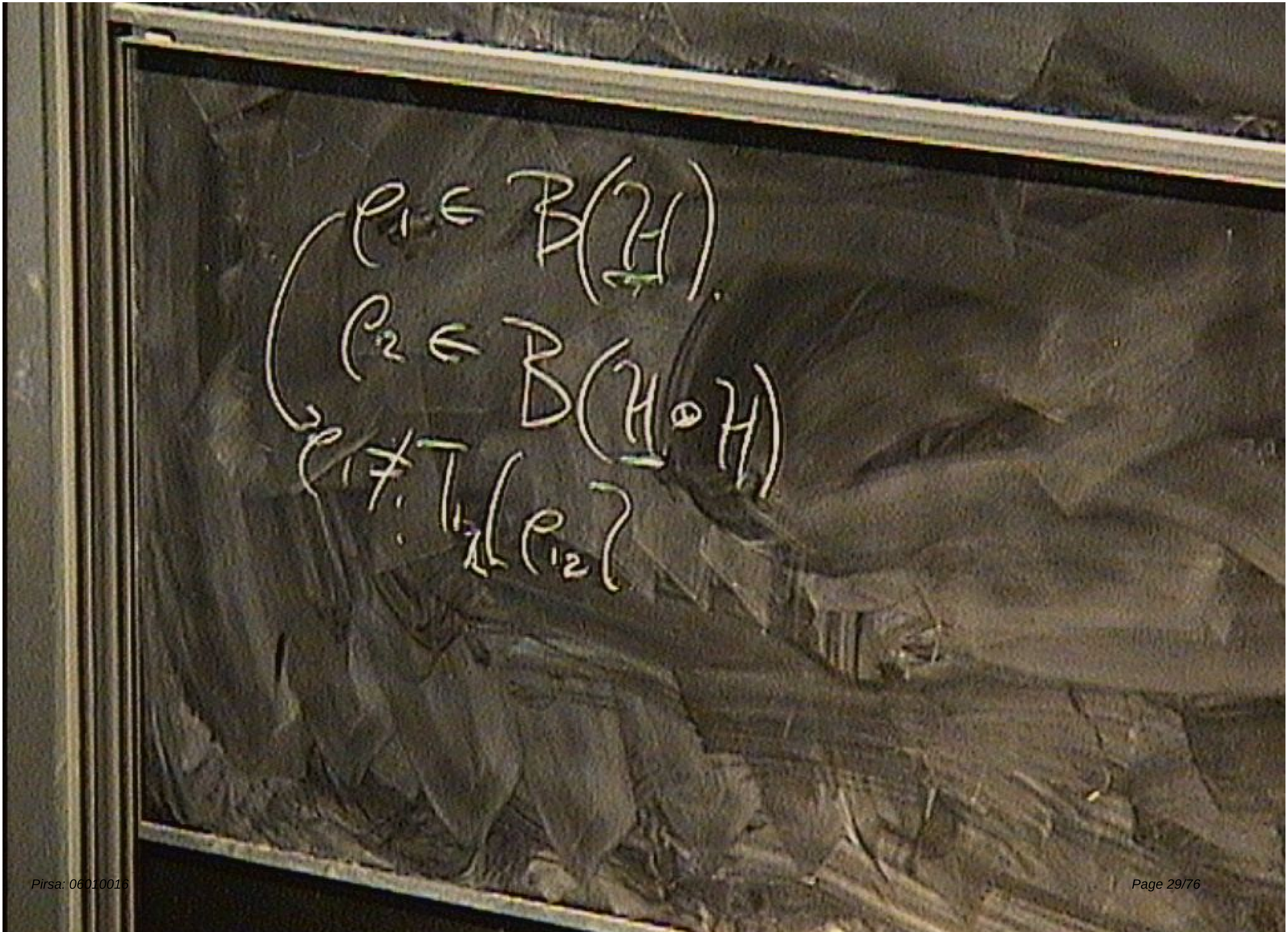

$\rho_1 \in B(\mathcal{H})$
$\rho_2 \in B(\mathcal{H} \otimes \mathcal{H})$
$\rho_1 \neq Tr_2(\rho_{12})$

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- Treat sequences of distributions (states)

$$\rho = \{\rho_n\}_{n=1}^{\infty}$$

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Spectral Projections

For an operator with spectral decomposition

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

The non-negative spectral projection is

$$\{A \geq 0\} \equiv \sum_{\lambda_i \geq 0} |i\rangle\langle i|$$

Define

$$\{A \geq B\} \equiv \{A - B \geq 0\}$$

Spectral Entropy

There are *two* spectral entropies:

$$\overline{S}(\rho) = \inf\left\{\gamma : \lim_{n\to\infty} Tr\left[\left\{\rho_n \geq e^{-n\gamma} I_n\right\}\left(\rho_n - e^{-n\gamma} I_n\right)\right] = 1\right\}$$

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Source Coding

For a source $\rho = \{\rho_n\}_{n=1}^{\infty}$

the optimal rate of compression is

$$R = \overline{S}(\rho)$$

Proof: coding I

Given error $\varepsilon_n = 1 - Tr\left[P_n \rho_n\right]$

$$\leq 1 - Tr\left[P_n\left(\rho_n - e^{-n\gamma} I_n\right)\right]$$

choosing $P_n = \left\{\rho_n \geq e^{-n\gamma} I_n\right\}$

$$\gamma > \bar{S}(\rho)$$

implies $\lim_{n\to\infty} \varepsilon_n = 0$

Proof: coding II

The rate for the code is

$$R = \overline{\lim_{n\to\infty}} \frac{1}{n} \log Tr\left[\left\{\rho_n \geq e^{-n\gamma} I_n\right\}\right]$$

and

$$Tr\left[\left\{\rho_n \geq e^{-n\gamma} I_n\right\}\right] \leq e^{n\gamma} Tr\left[\left\{\rho_n \geq e^{-n\gamma} I_n\right\}\rho_n\right]$$
$$\leq e^{n\gamma}$$

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For $0 \le A \le I$

$$Tr[A(\rho - \omega)] \le Tr[\{\rho \ge \omega\}(\rho - \omega)]$$

Proof: $$Tr[A(\rho - \omega)] = Tr\left[AU^{*}(\Pi - \Omega)U\right]$$
$$= Tr\left[\tilde{A}\Pi\right] - Tr\left[\tilde{A}\Omega\right]$$
$$\le Tr[\Pi]$$

Proof: converse

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Strong Converse

- What does the second spectral entropy correspond to?
- Strong converse for data compression

$$R < \underline{S}(\rho) \Rightarrow \varepsilon \underset{n\to\infty}{\longrightarrow} 1$$

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Relation to i.d.d.

Inequalities for entropy rate

$$\underline{S}(\rho) \leq \underline{\lim}_{n\to\infty} \frac{1}{n} S(\rho_n) \leq \overline{\lim}_{n\to\infty} \frac{1}{n} S(\rho_n) \leq \overline{S}(\rho)$$

for an i.i.d. source $\rho_n = \omega^{\otimes n}$

gives $\underline{S}(\rho) \leq S(\omega) \leq \overline{S}(\rho)$

Spectral Information Rates

Starting with the *spectral divergence rates*

$$\overline{D}(\rho|\omega) = \inf\left\{\gamma : \lim_{n\to\infty} Tr\left[\left\{\rho_n \geq e^{n\gamma}\omega_n\right\}\left(\rho_n - e^{n\gamma}\omega_n\right)\right] = 0\right\}$$

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Spectral Information Rates II

Define other rates in terms of divergences

$$\overline{S}(\rho) = -\underline{D}(\rho | I) \qquad \underline{S}(\rho) = -\overline{D}(\rho | I)$$

$$\overline{S}(A|B) = -\underline{D}\left(\rho^{AB} \middle| I^{A} \otimes \rho^{B}\right) \quad \underline{S}(A|B) = -\overline{D}\left(\rho^{AB} \middle| I^{A} \otimes \rho^{B}\right)$$

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Properties of divergences

$$\underline{D}(\rho|\omega) \leq \overline{D}(\rho|\omega)$$

and for states $\omega = \{\omega_n\}_{n=1}^{\infty}$ $\quad 0 \leq \underline{D}(\rho|\omega)$

For sequences of CPTP maps

$$\overline{D}(T(\rho)|T(\omega)) \leq \overline{D}(\rho|\omega)$$

$$\underline{D}(T(\rho)|T(\omega)) \leq \underline{D}(\rho|\omega)$$

Properties of Spectral Information

Entropies

- Chain rules
- Strong subadditivity
- Subadditivity
- Araki-Lieb inequality

Spectral entropies

- Chain rule *inequalities*
- Strong subadditivity
- Subadditivity
- Araki-Lieb inequality

Properties (explicitly)

$$0 \le \underline{S}(\rho) \le \overline{S}(\rho) \le \log d$$

$$-\underline{S}(A) \le \overline{S}(A \mid BC) \le \overline{S}(A \mid B) \le \overline{S}(A)$$

$$\underline{S}(AB) - \overline{S}(B) \le \underline{S}(A \mid B) \le \min\left[\overline{S}(AB) - \overline{S}(B), \underline{S}(AB) - \underline{S}(B)\right]$$

$$\max\left[\overline{S}(AB) - \overline{S}(B), \underline{S}(AB) - \underline{S}(B)\right] \le \overline{S}(A \mid B) \le \overline{S}(AB) - \underline{S}(B)$$

$$\overline{S}(AB) \le \overline{S}(A) + \overline{S}(B)$$

$$\overline{S}(AB) \ge \max\left[\left|\overline{S}(A) - \overline{S}(B)\right|, \left|\underline{S}(A) - \underline{S}(B)\right|\right]$$

Yet More Properties

1. For unital CPTP sequences $\bar{S}(T(\rho)) \geq \bar{S}(\rho)$
2. For ρ_n^{AB} pure $\bar{S}(A) = \bar{S}(B)$
3. For ρ_n^{AB} "classical" states

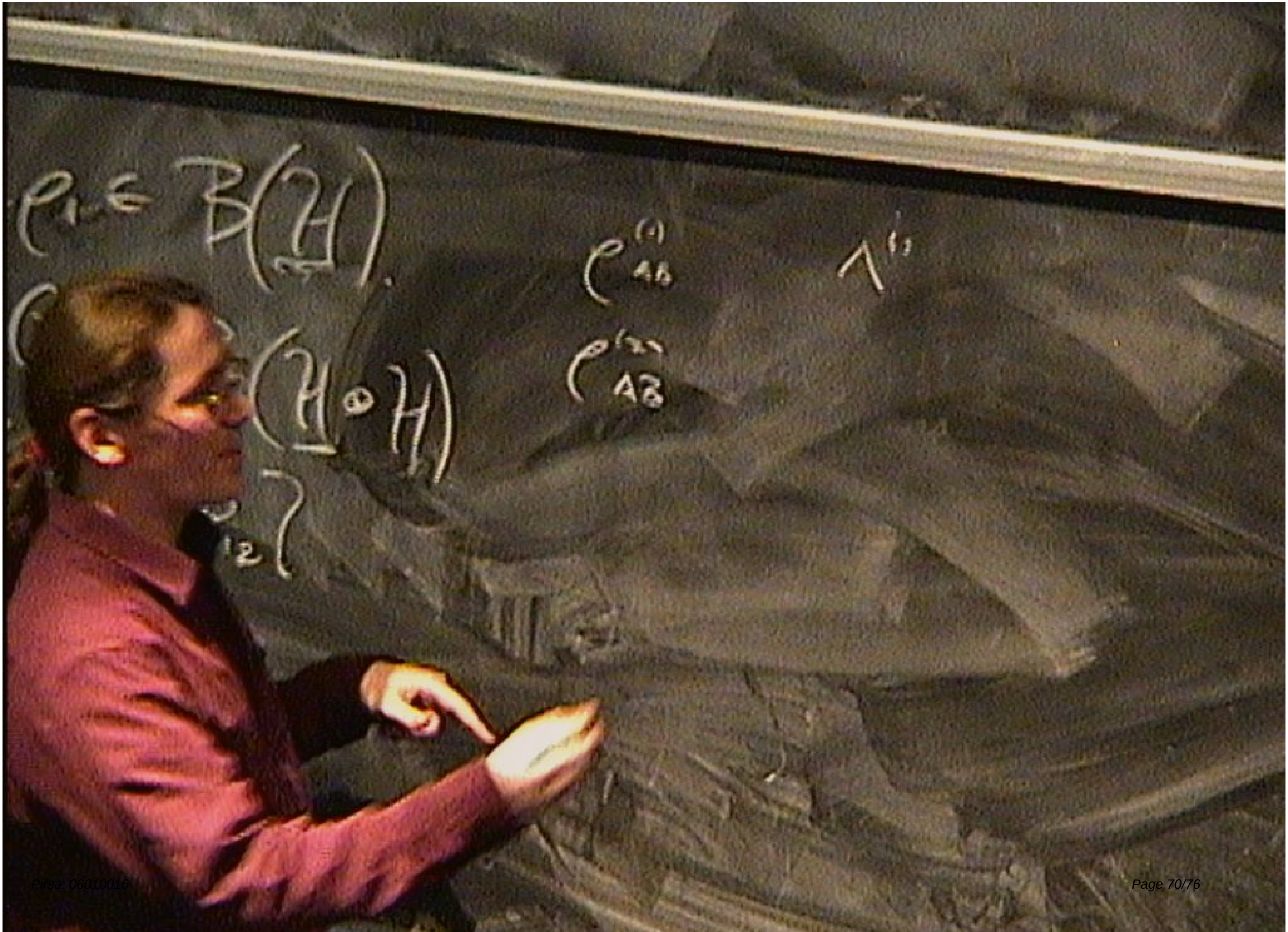
$$\underline{S}(A|B) \geq 0$$

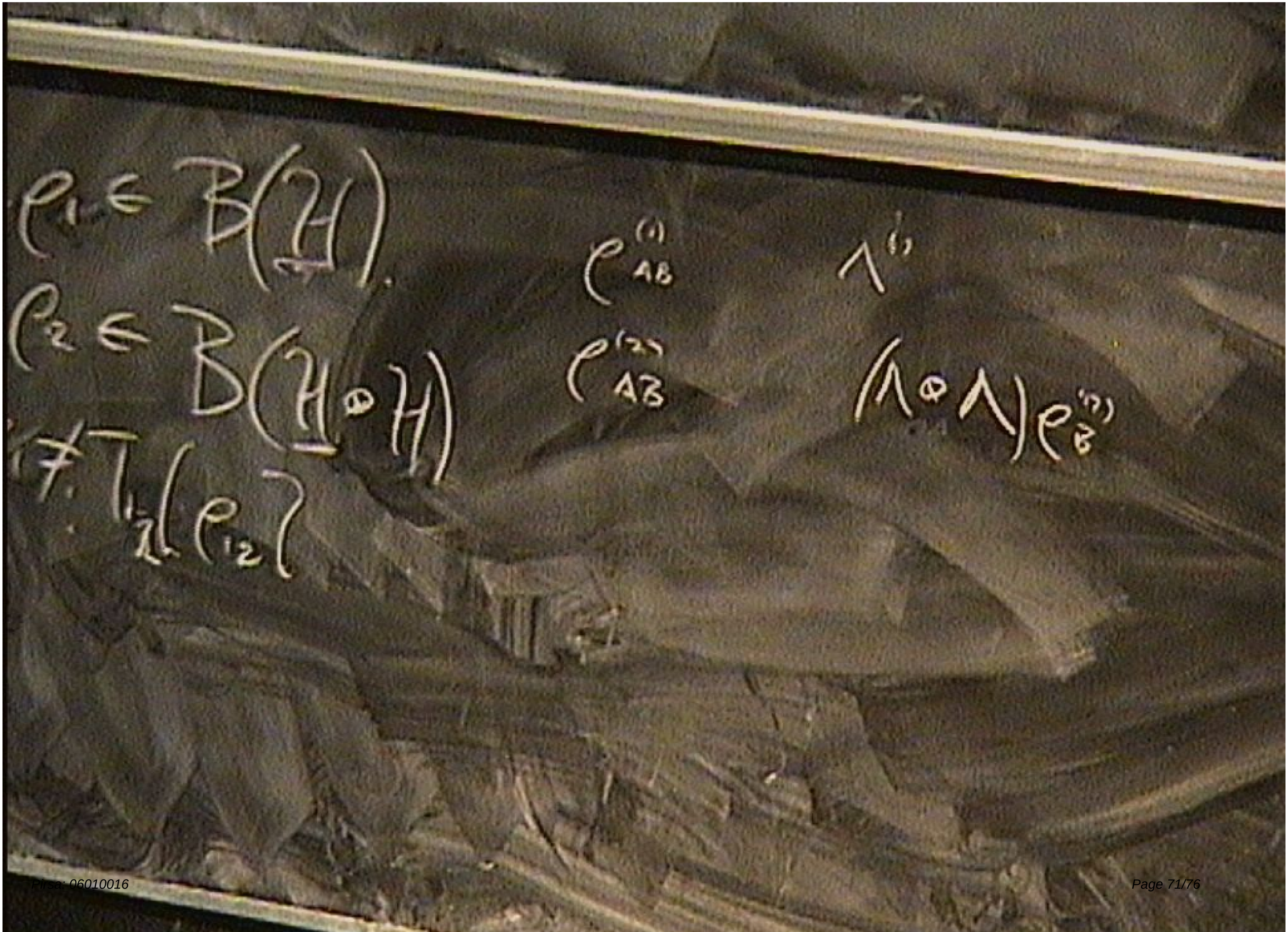
$$\bar{S}(AB) \geq \max\left[\bar{S}(A), \bar{S}(B)\right]$$

Operational Quantities

Known operational quantities:

1. Source coding $R = \overline{S}(\rho)$
2. Classical-quantum capacity $C = \max_{\rho^{AB} \in \Sigma} \underline{S}(A : \Lambda B)$
3. Quantum capacity(?) $Q \le \max_{\rho^{RQ}} \left[-\overline{S}(R | \Lambda Q) \right]$




ρ1 ∈ B(H).
ρ2 ∈ B(H ⊗ H)
∄ Tr2(ρ12)?
ρAB(1)
ρAB(2)
Λ(1)
(Λ ⊗ Λ)ρB(2)

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What's next?

- Operational quantities
  - Quantum capacities
  - Distributed compression (Slepian-Wolf, state merging)
  - Additional resources (entanglement, feedback, classical side channels)
- Classification of sources & channels
- Rate distortion theory

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THE END

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