Title: Strongly correlated spin systems from the point of view of quantum information theory

Date: Jan 25, 2006 11:00 AM

URL: http://pirsa.org/06010014

Abstract: The concept of entanglement plays a central role in the field of strongly correlated quantum systems: it gives rise to fascinating phenomena such as quantum phase transitions and topological quantum order, but also represents a main obstacle to our ability to simulate such systems. We will discuss some new developments in which ideas, originating from the field of quantum information theory, led to valuable insights into the structure of entanglement in quantum spin systems and to novel powerful simulation methods

Strongly Correlated Quantum Systems from the point of view of quantum information theory

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Overview

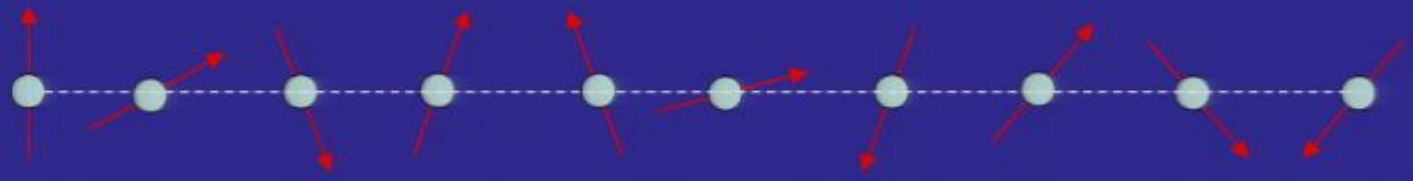
- Spin systems:
  - Properties of ground states
  - Monogamy of entanglement
- Numerical Renormalization Group methods
  - Matrix Product States
  - Projected Entangled Pair States
- Applications / Examples
  - Entanglement Length
  - Topological Quantum Order
  - One-Way Quantum Computer

Strongly Correlated quantum systems

- Recent years have seen an explosion of work on materials that lie outside of the conventional weakly-interacting solid-state paradigm:
  - Exotic materials such as high T_c superconductors and quantum magnets exhibit their remarkable properties due to strong quantum correlations
  - Experimental breakthroughs with atomic gases (optical lattices) provide a perfect playground for probing such systems
- Central question in theoretical condensed matter physics:
  - Derive effective Hamiltonians describing the physics of those exotic materials, and characterize the ground states, the different phases, order parameter, free energy, ...
  - Despite huge efforts many elementary questions are still unsolved
- MAJOR problem: people come up with theories but in most cases there is no way to check whether they are right: simulation of Hamiltonians is hard
  - Naively, the computational cost scales exponentially in the number of subsystems

Spin systems

- Typical effective Hamiltonians are described on spin systems: e.g. ground state of spin ½ Heisenberg antiferromagnet on a regular lattice (square, Kagome, …):



$$H = \sum_{\langle i,j \rangle} S_x^i S_x^j + S_y^i S_y^j + \Delta S_z^i S_z^j$$

- Statistical mechanics questions:
  - Spectrum (gapped or critical?); phase diagram
  - Correlation length
  - Order parameter (symmetry breaking?) Topological features?
  - Simulation (Monte Carlo, RG, …)

- QIT questions (about STATES)
  - Can such a system be used as a Quantum Repeater if we have local control?
  - Relation between correlations and entanglement?
  - Do generic ground states have an efficient (polynomial) description?
  - How does the geometry / symmetry affects the amount of entanglement?
  - Can this state be used as a resource for universal quantum computation?
  - How to create ground states with a quantum computer?

Ground states of spin systems

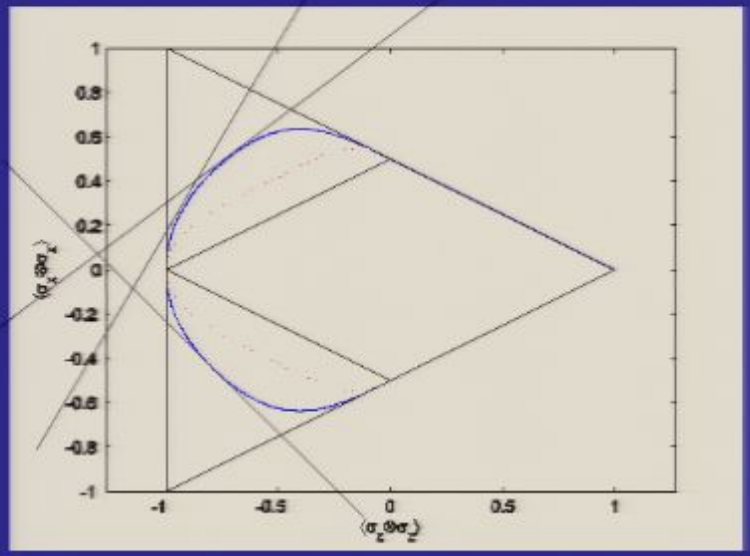
- Ground states of local spin Hamiltonians are very special:

$$H = \sum_{\langle i,j \rangle} S_x^i S_x^j + S_y^i S_y^j + \Delta S_z^i S_z^j$$

  - Translational invariance implies that energy is completely determined by n.n. reduced density operator ρ of 2 spins:

$$E = N.\mathrm{Tr}\left(\rho\, H_{ij}\right)$$

  - Finding ground state energy is equivalent to maximizing E over all possible ρ arising from states with the right symmetry
  - The extreme points of the convex set {ρ} therefore correspond to ground states: ground states are completely determined by their reduced density operators!
  - They obey the Perimeter Law
  - So there is hope for understanding / simulating ground states: it is in principle enough to reproduce the local properties; they live in a tiny part of the Hilbert space!



e.g.: $H = \sum_{\langle i,j \rangle} S_x^i S_x^j + S_y^i S_y^j + \Delta S_z^i S_z^j$

The Hamiltonian defines hyperplanes in this convex set; convex set is parameterized as $2x + \Delta z + E(\Delta) = 0$

cond-mat/0505140

- Difficulty in characterizing this convex set is due to monogamy / frustration properties of entanglement: a singlet cannot be shared
  - The extreme points of the convex set are determined by a hierarchy of semidefinite constraints; the set heavily depends on the geometry of the lattice
  - A full characterization in the case of infinite dimensions: separable states (quantum de-Finetti theorem) R. Werner, Lett. Math. Phys. 17, 359 (1989)
  - There is also a complete solution for bosonic quadratic Hamiltonians (Gaussian states) M. Wolf, FV, IC, Phys. Rev. Lett. 92, 087903 (2004)
  - this monogamy property is the main source of frustration for people doing numerical simulations

Monogamy of entanglement

- Coffman-Kundu-Wootters relation for any n-qubit state

T. Osborne, FV, quant-ph/0502176

$$\sum_{i>1} C_{1i}^2 \leq 1 - \left\langle S_n^1 \right\rangle^2$$

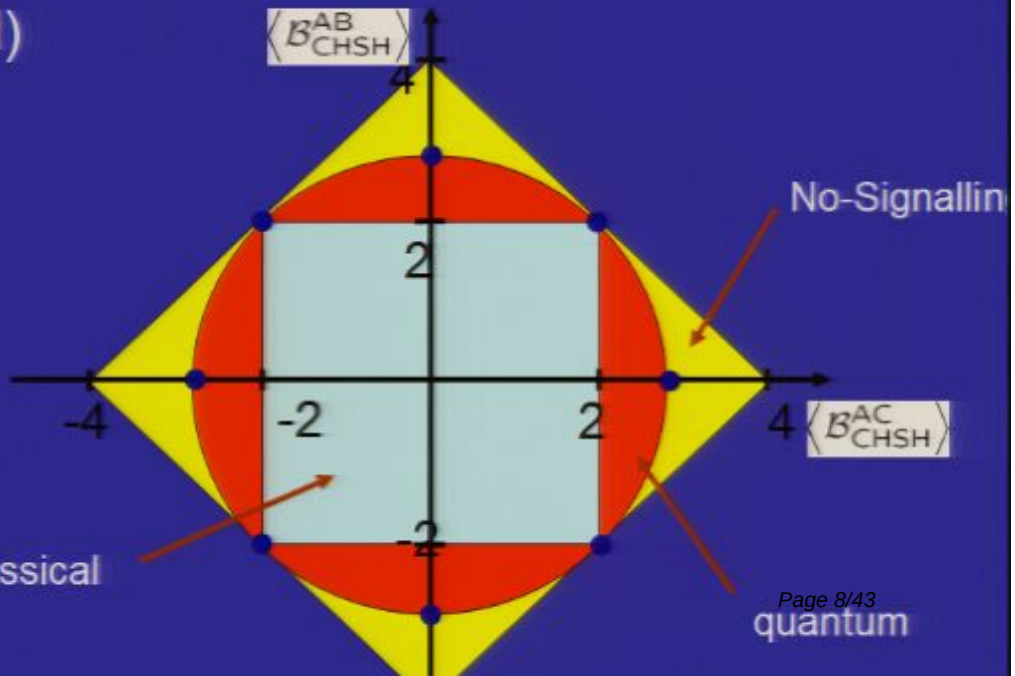
  - Consider lattice with coordination number d: $C \leq \sqrt{\frac{1-\left\langle S_n^1 \right\rangle^2}{d}}$

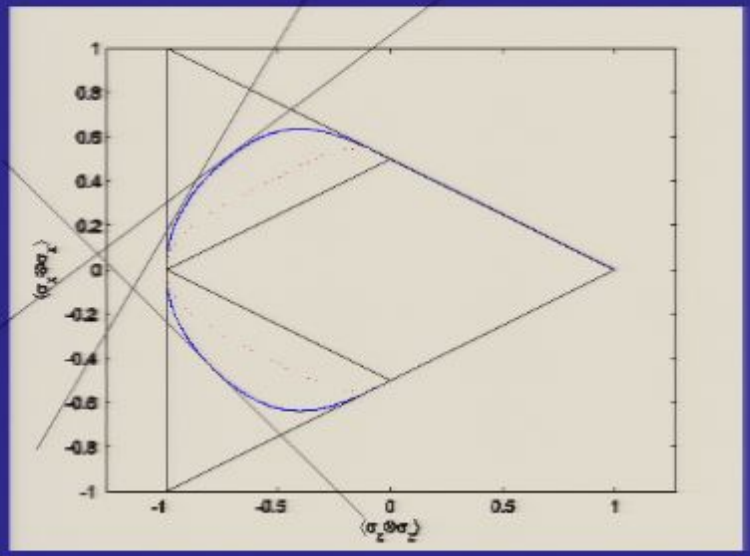
- Monogamy of CHSH-Bell correlations: (generalization of Cirel'son bound)

$$\sum_{i\geq 2} \beta_{1i}^2 \leq 8$$

B. Toner, FV (2006)

Such a monogamy relation provides basic intuition for security of quantum cryptographic protocols





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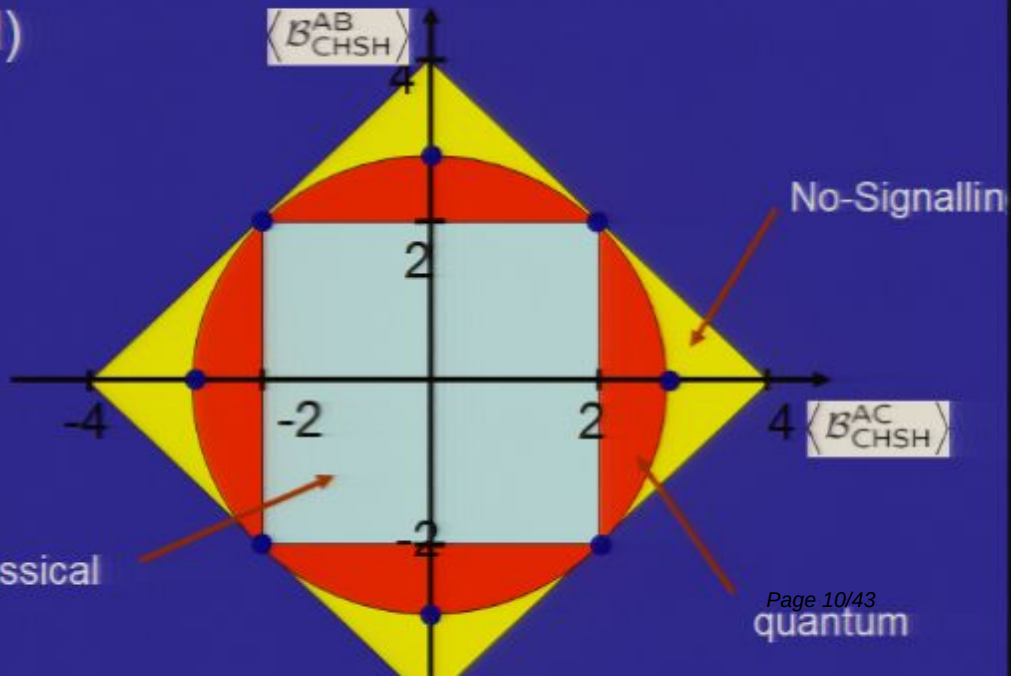
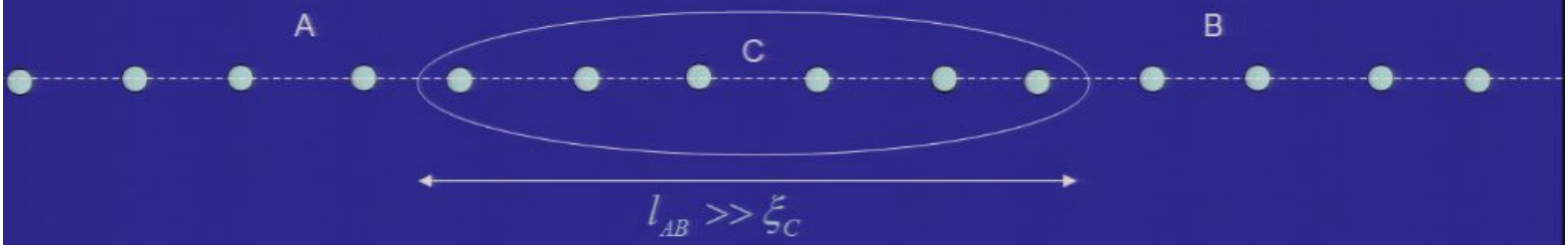


Ground states of spin Systems (II)

- Ground states of gapped local Hamiltonians have a finite correlation length:

M. Hastings '04

$$\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \approx \exp\left(-\frac{l_{AB}}{\xi_C}\right)$$



- Let's analyze this statement from the point of view of quantum information theory:
  - As all purifications of ρ_{AB} are separable, there exists a unitary in region C that disentangles the two parts
  - Blocking the spins in blocks of $\log(\xi_C)$ spins, then we can write the state as:

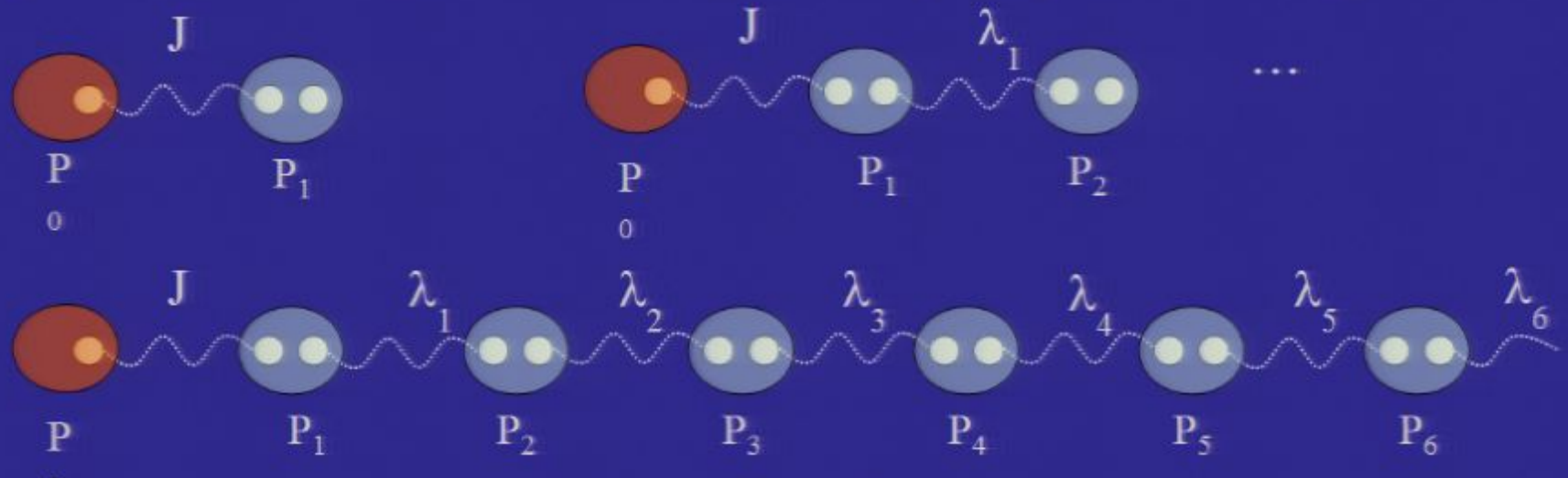
$$|\psi_{ABC}\rangle = \sum_{\alpha,\beta,i_l,i_r} U_{\alpha\beta}^{i_l i_r} |l_\alpha\rangle |i_l\rangle |i_r\rangle |r_\beta\rangle$$

  - Doing this recursively yields a Matrix Product State (MPS):

$$|\psi_{i_1 i_2 \ldots i_N}\rangle = \sum_{\alpha,\beta,i} Tr\left[A^{i_1} A^{i_2} \ldots A^{i_N}\right] |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$

MPS and Wilson's Numerical Renormalization Group

- MPS effectively appeared in the context of numerical renormalization group in the
- Main original idea of Wilson back in the '70 when studying the Kondo impurity problem: diagonalize Hamiltonian in a recursive way and always project on low-energy sector



$$|\psi_\alpha\rangle = \sum_{i_0 i_1 \cdots i_N} a^{i_0} A_1^{i_1} A_2^{i_2} \cdots A_{N-1}^{i_{N-1}} [A_N^{i_N}]_{:,\alpha} |i_0 i_1 \cdots i_N\rangle$$

- Numerical Renormalization group method can be understood of as a variational method in the space of matrix product states: construction of effective Hamiltonian in the space of MPS

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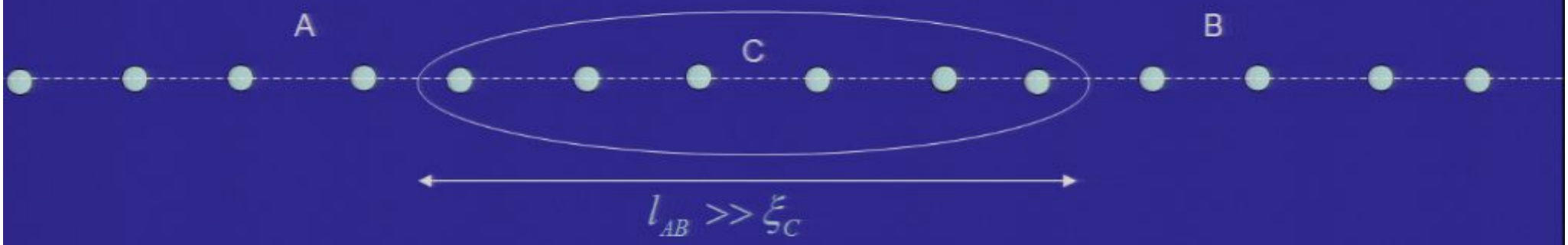

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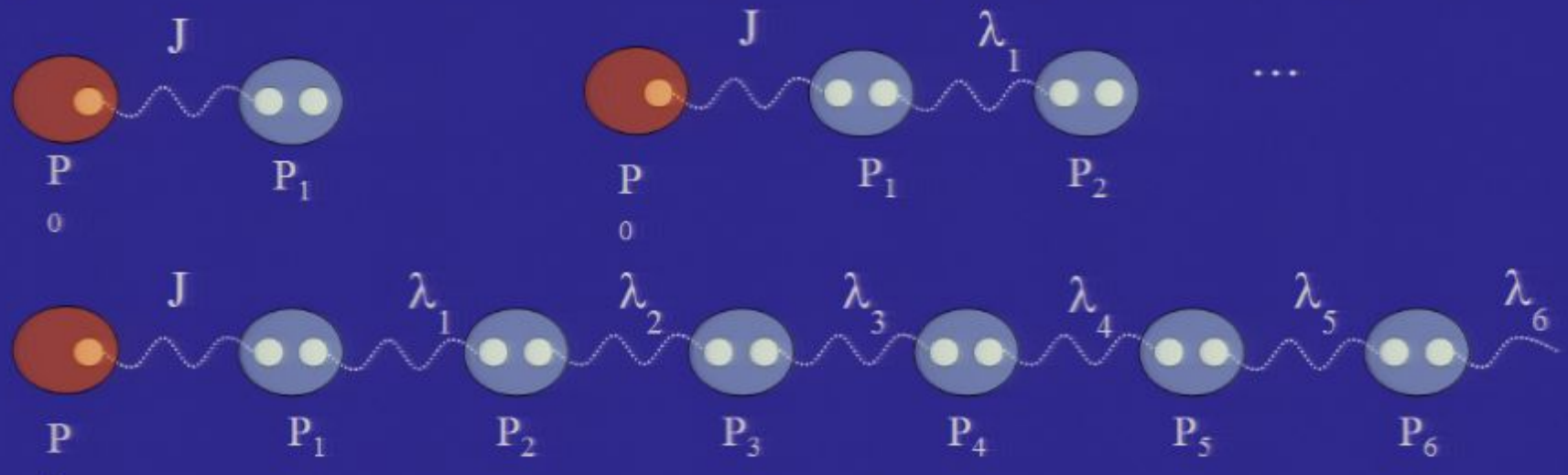

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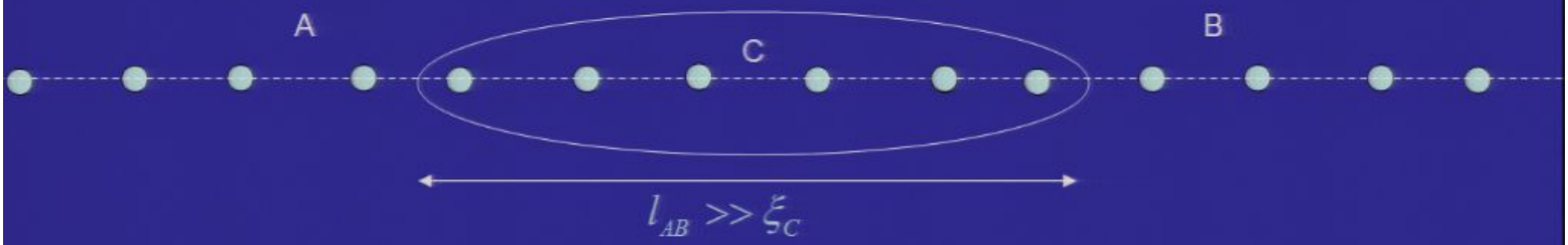

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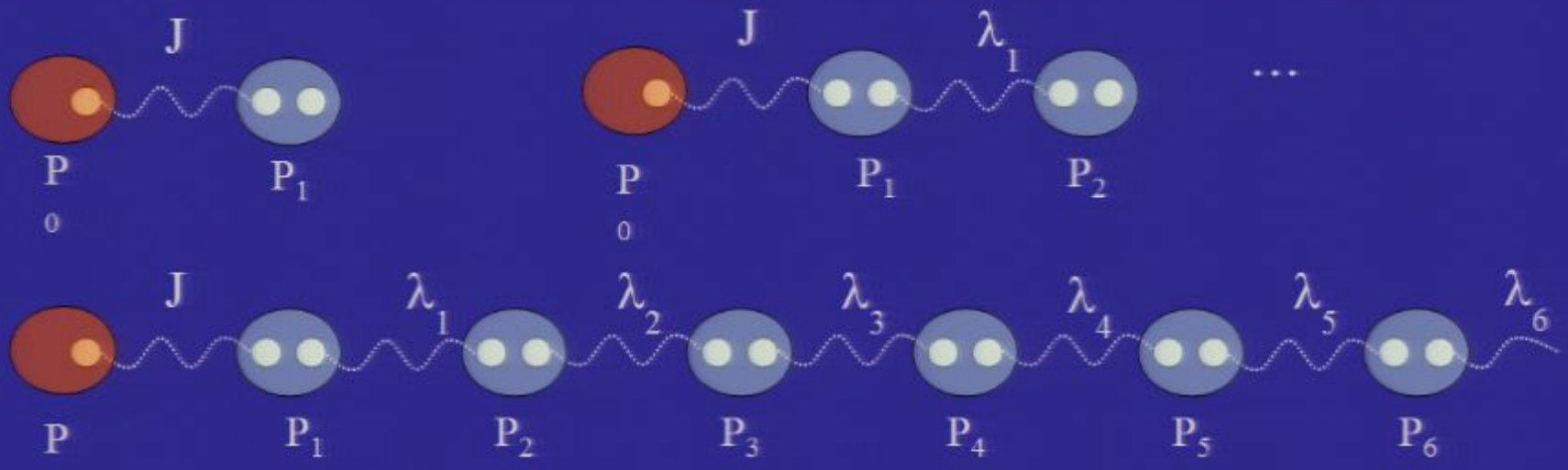

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Matrix Product States (MPS)

$$|I\rangle = \sum_{i=1}^{D} |i\rangle|i\rangle$$

$$\text{Map} : H^D \otimes H^D \rightarrow H^d$$

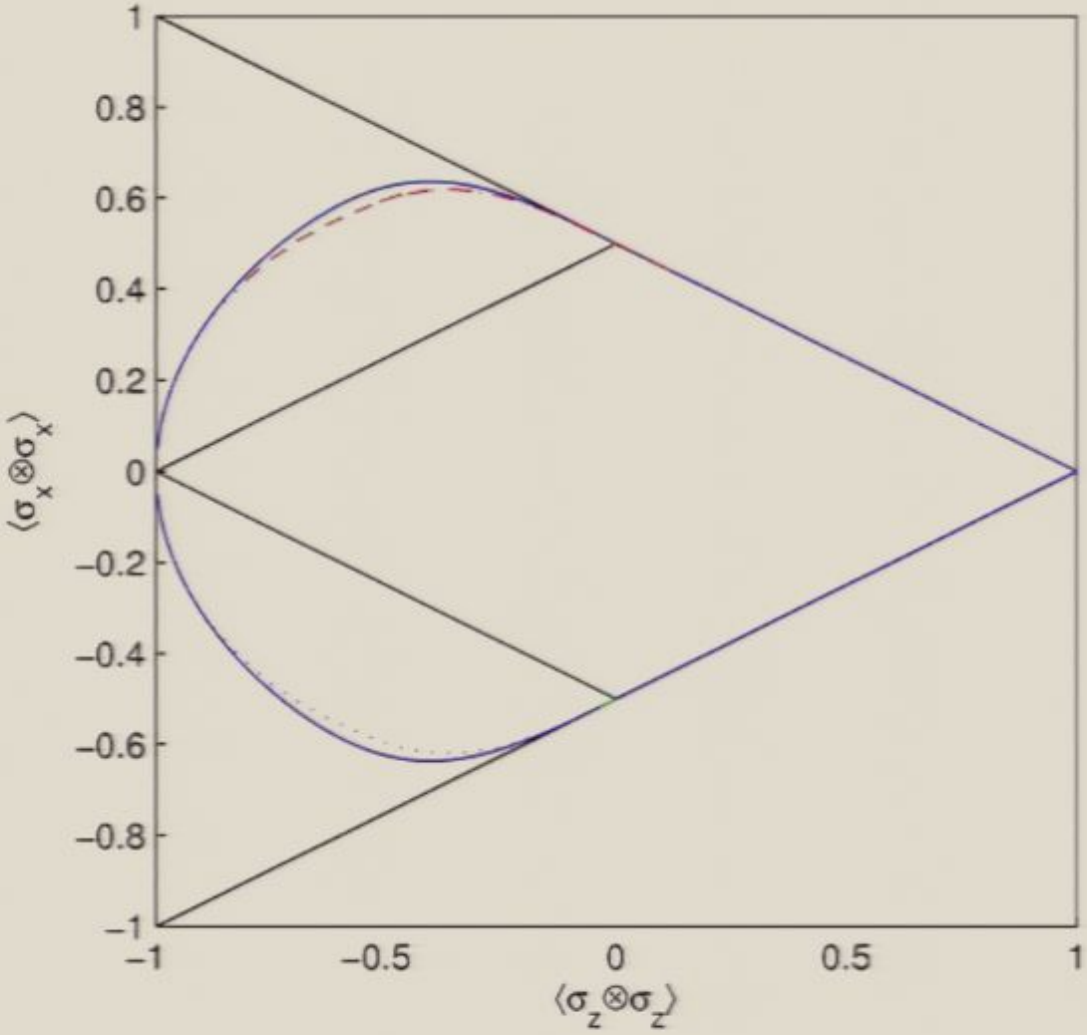
State is defined on a d^N dimensional Hilbert space

- Gives a LOCAL description of a multipartite state
- Translational invariant by construction if all maps chosen equal to each other
- Formally introduced by Fannes, Nachtergaele and Werner in '92 as generalizations of AKLT-states; proved to be ground states of gapped local quantum Hamiltonians
- The number of parameters scales linearly in N (# qubits)
- The set of all MPS is complete: *Every* state can be represented as a MPS as long as D is taken large enough
- The point is: if we consider the set of MPS with fixed D, their reduced density operators already approximate the ones obtained by all translational invariant ones very well (and hence also of all possible ground states)
- MPS have bounded Schmidt rank D (cfr. Vidal)
- Correlations can be calculated efficiently: contraction of D^2x D^2 matrices

D-dimensional

$\langle\Psi|\Psi\rangle \equiv$

Convex set of reduced density operators of ground states of XXZ-chains approximated with MPS of D=1,2



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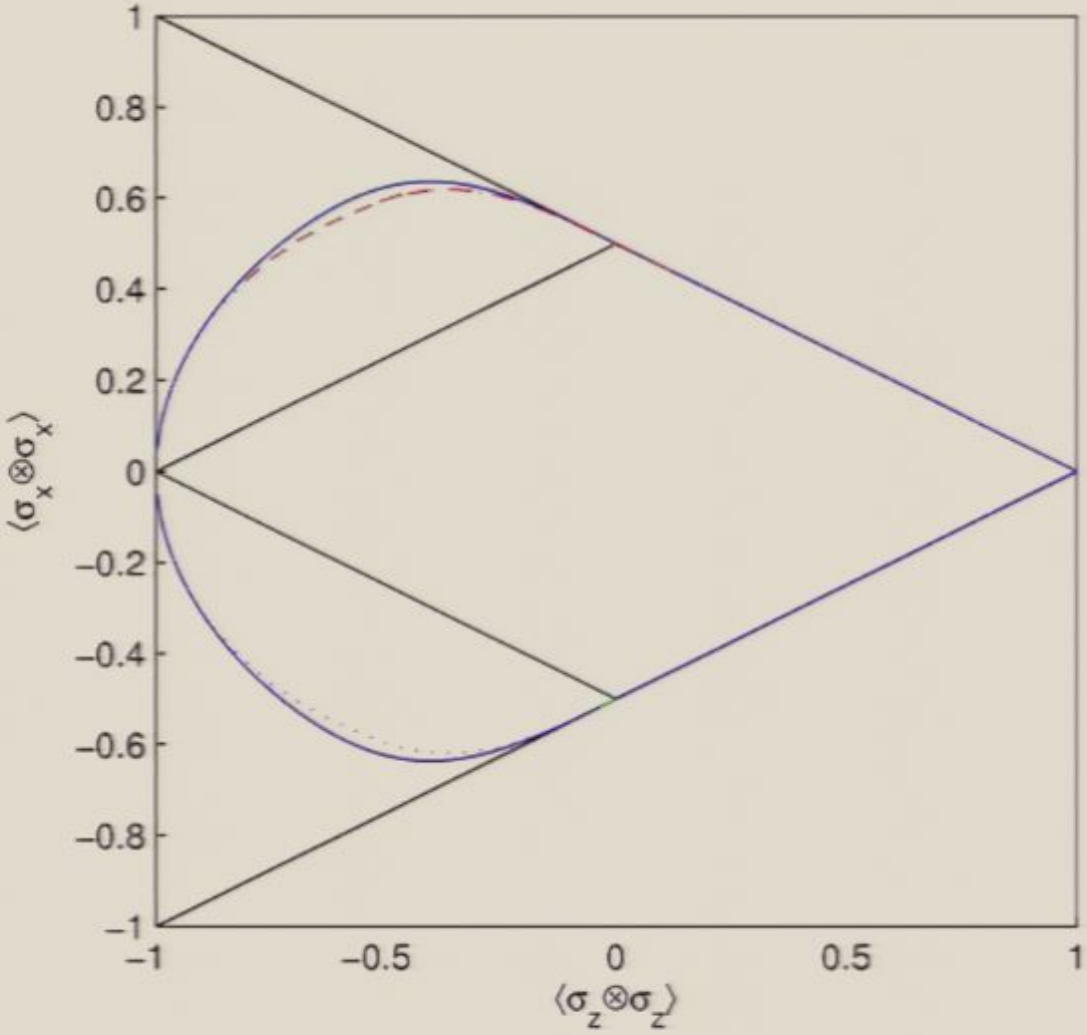


- So how good will MPS approximate ground states of 1-D systems? We want to bound $\left\| |\psi_{ex}\rangle - |\psi_D\rangle \right\|$

- We need the following result:
  - For 1-D critical systems, the Renyi entropy of a contiguous block of L spins scales as

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \ln\left(\mathrm{Tr}\, \rho^\alpha\right) = \frac{c+\bar{c}}{12}\left(1+\frac{1}{\alpha}\right)\ln(L)$$

Korepin, Cardy, Kitaev, Vidal, ..

- We want to make a statement on how D has to increase to represent the exact ground state faithfully when the number of particles N increases:
  - Choose an epsilon:

$$\text{if } D_N \leq cst\left(\frac{N^2}{(1-\alpha)\varepsilon}\right)^{\frac{\alpha}{1-\alpha}} \exp\left(S_\alpha(\rho_{\max})\right) \Rightarrow \exists\, |\psi_{\mathrm{D}}^{\mathrm{N}}\rangle : \left\| |\psi_{ex}^N\rangle - |\psi_D^N\rangle \right\| \leq \frac{\varepsilon}{N}$$

- It shows that D only has to grow as a polynomial in the number of particles to obtain a given precision, and moreover MPS will be able to represent long-range properties faithfully, even in the case of critical systems!

Variational methods using MPS

- Goal: find MPS that minimizes the energy of a 1-D spin Hamiltonian
- All expectation values and hence the energy $E = \langle \psi_{\text{vbs}} | H | \psi_{\text{vbs}} \rangle$ are multi-quadratic in the variables P_k
- Strategy for minimizing energy:
  - Fix all projectors P_i except the j^{th}
  - Both the energy and the norm $\langle \psi_{\text{vbs}} | \psi_{\text{vbs}} \rangle$ are quadratic functions of the variable P_j and hence the minimal energy by varying P_i can be obtained by a simple generalized eigenvalue problem:

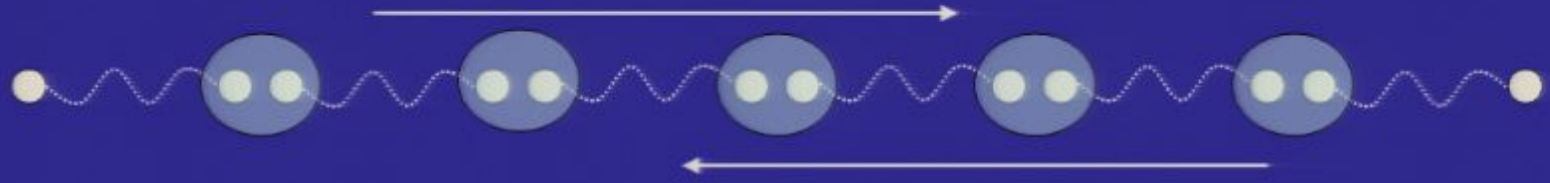
$$\min_x \frac{x^\dagger H_{eff} x}{x^\dagger N x}$$

H_{eff} and N are function of the Hamiltonian and all other projectors, and can efficiently be calculated by the transfer matrix method

  - Move on to the $(j+1)^{th}$ particle and repeat previous steps (sweep) until convergence

Phys. Rev. Lett. 93, 227205 (2004)

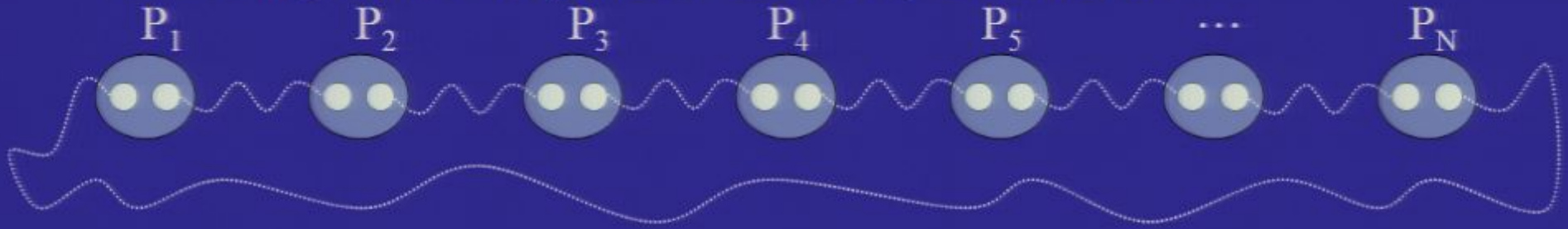
S. White's DMRG method

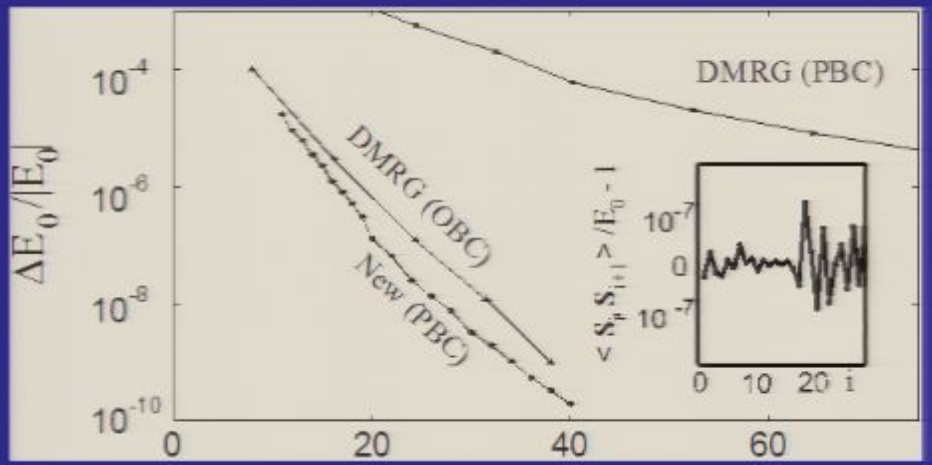


$$|\psi_j\rangle = \sum_{\alpha,\beta,i} [A_j^i]_{\alpha,\beta} |\alpha\rangle_j |i\rangle_j |\beta\rangle_j$$

$$H_{eff}^{new} = f(A_{j\pm1}^i, H_{eff}^{old}, H)$$

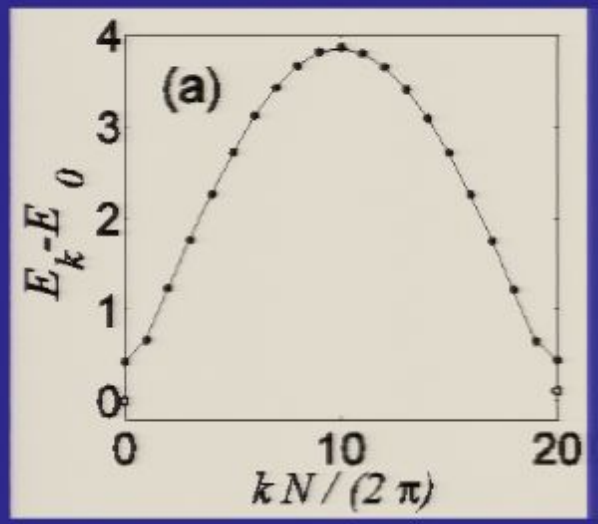
- Extending DMRG to periodic boundary conditions:

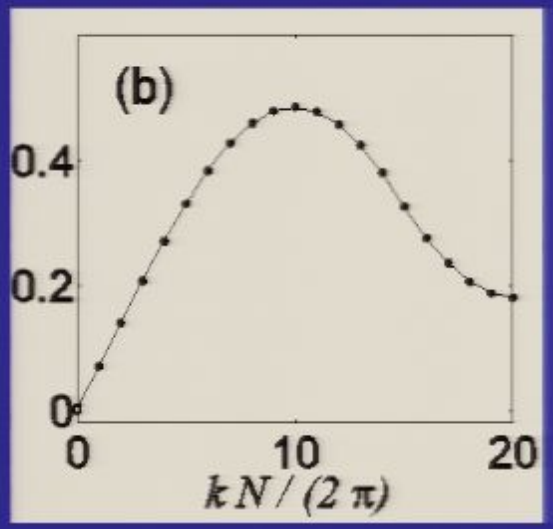


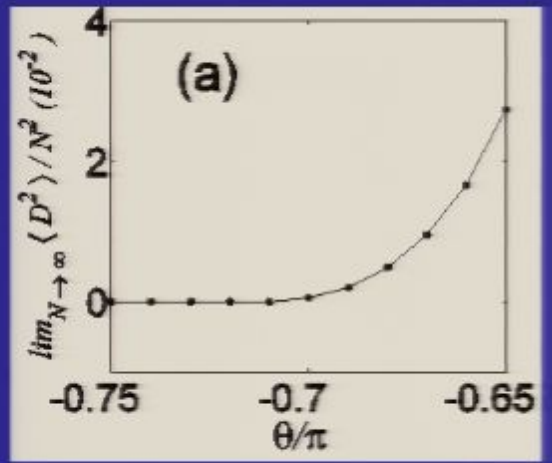


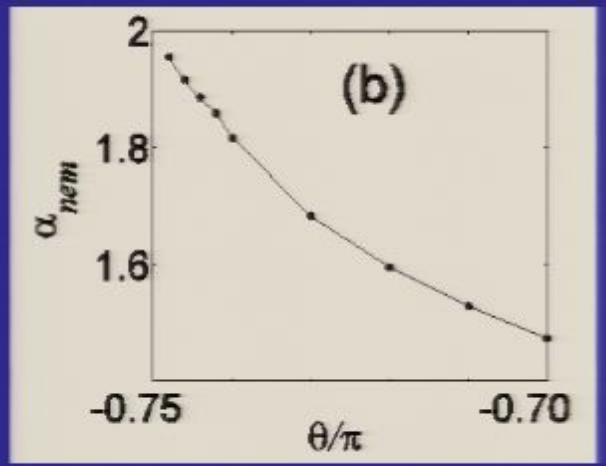
Phys. Rev. Lett. 93, 227205 (2004)

- We can also enforce MPS to have a definite momentum quantum number by taking linear combinations of them
- Example: study of nematic phase in bilinear-biquadratic S=1 Heisenberg spin chain



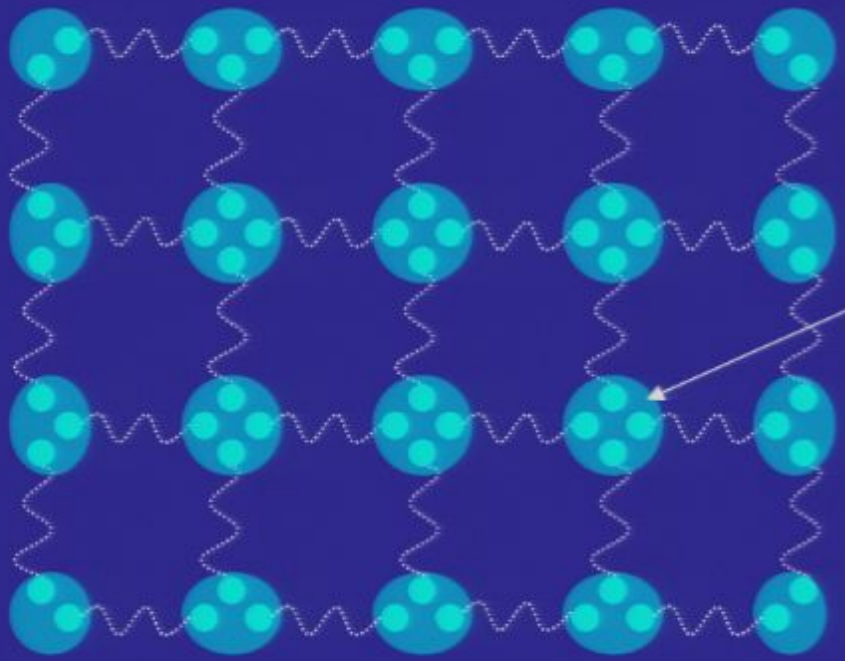






Generalizations of MPS to higher dimensions

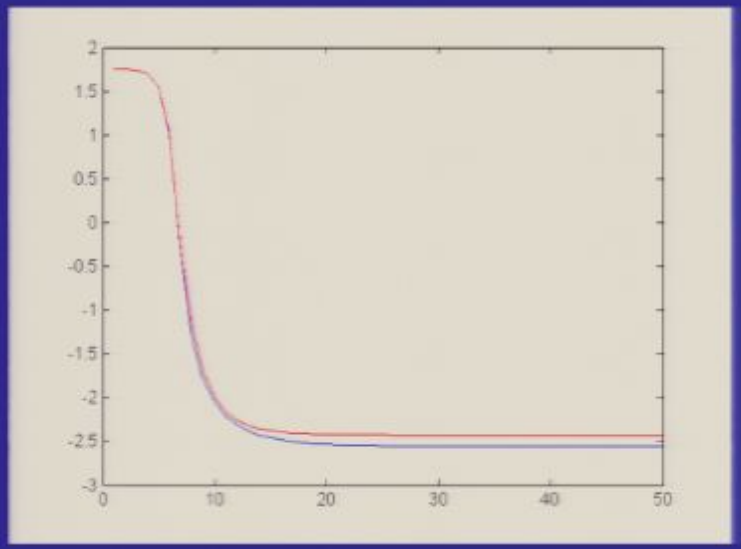
- The MPS picture can be generalized to any geometry: Projected Entangled Pair States (PEPS)



- Properties: Perimeter Law automatically fulfilled; local properties can be approximated very well ; guaranteed to be ground states of local Hamiltonians; again: every state can be written as a PEPS

cond-mat/0407066

- How to calculate correlation functions?
  - Assume for simplicity translational invariance on an infinite 2-D lattice
  - Instead of contracting matrices, we have to contract tensors:



  - Because of translational invariance, it is enough to calculate largest eigenvector of "transfer matrix" which can be extremely well approximated by using the 1-D MPS methods described before
  - We also have a method for contracting tensors in an variational approximate way when not translational invariant : this allows to extend renormalization group method sketched before to higher dimensions!

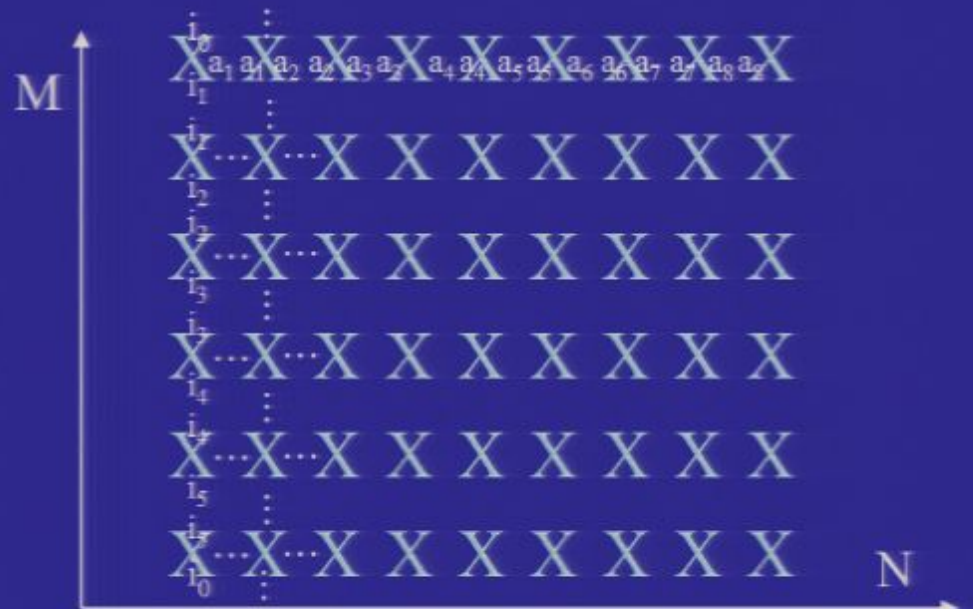
- Similarly: calculation of free energy and classical partition functions

$$F(\beta) = \mathrm{Tr}\left[\exp\left(-\beta \mathrm{H}\right)\right]$$

$$\approx \mathrm{Tr}\left[\left\{\exp\left(-\frac{\beta \mathrm{H}_1}{\mathrm{M}}\right)\exp\left(-\frac{\beta \mathrm{H}_2}{\mathrm{M}}\right)\cdots\exp\left(-\frac{\beta \mathrm{H}_q}{\mathrm{M}}\right)\right\}^{M}\right]$$

$$H = \sum_{\alpha=1}^{q} H_\alpha \quad ; \quad \mathrm{H}_\alpha = \sum_{i=1}^{N} X^{\alpha}_{i,i+1} \quad ; \quad \left[X^{\alpha}_{i,i+1}, X^{\alpha}_{i+1,i+2}\right] = 0$$

– can be done by contracting tensors by "variational dimensional reduction"



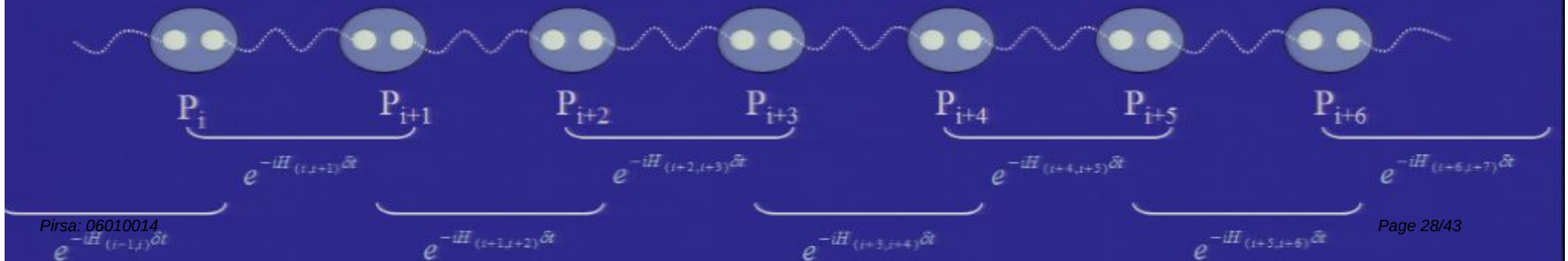
Variational dimensional reduction of MPS

- Given a D-dimensional $|\psi_D\rangle$ MPS parameterized by the DxD matrices A^i, find $|\chi_{D'}\rangle$ parameterized by D'xD'matrices B^i (D'< D) such as to minimize

$$\||\psi\rangle - |\chi\rangle\|^2 = \mathrm{Tr}\left[\left(\sum_i B_1^i \otimes B_1^i\right)\left(\sum_i B_2^i \otimes B_2^i\right)\cdots\left(\sum_i B_N^i \otimes B_N^i\right)\right]$$
$$-2\mathrm{Tr}\left[\left(\sum_i B_1^i \otimes A_1^i\right)\left(\sum_i B_2^i \otimes A_2^i\right)\cdots\left(\sum_i B_N^i \otimes A_N^i\right)\right] + cst$$

  - This can again be minimized variationally by iteratively solving linear equations

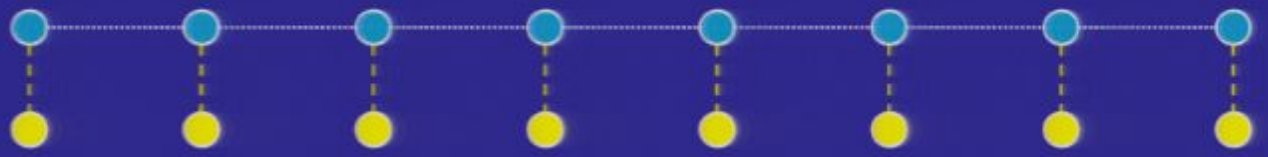
- This allows to treat real/imaginary time evolution of spin systems in a variational way by Trotter decomposition (cfr. methods of Vidal for time evolution)



Using ancilla's to simulate finite-T

- Imaginary time evolution on a purification of the maximally mixed state:

$$e^{-\beta H} = e^{-\frac{\beta H}{2}} I e^{-\frac{\beta H}{2}}$$



$$|\Psi\rangle = \sum_{s_1,\ldots,s_N} \sum_{a_1,\ldots,a_N} \mathrm{Tr}\left(\prod_{k=1}^{N} A_k^{s_k,a_k} \right) |s_1 a_1, \ldots, s_N a_N\rangle.$$

Phys. Rev. Lett. 93, 207204 (2004)

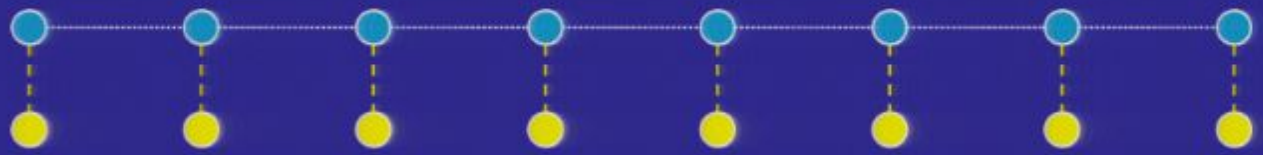
- How to simulate thermal states on a quantum computer?

Exploiting quantum parallelism to simulate random quantum systems

B. Paredes, FV, IC, Phys. Rev. Lett. 95, 140501 (2005)

- Randomness can be encoded in ancilla's:

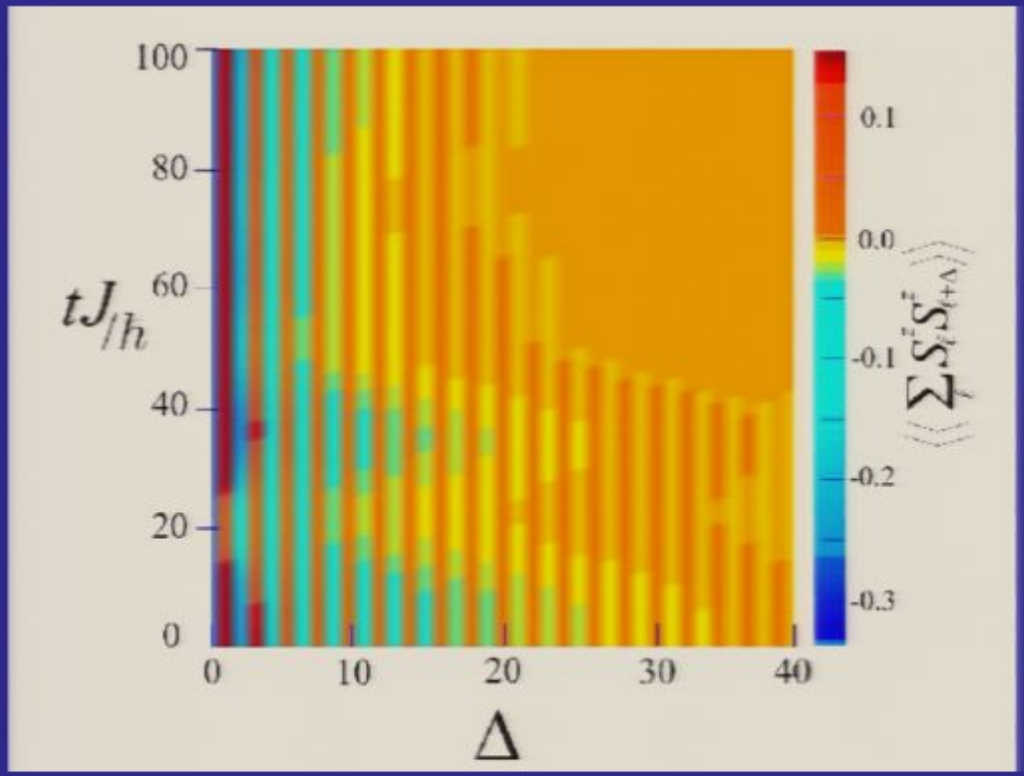
$$H = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} + \sum_{i=1}^{N} (-1)^{r(i)} S_i^z$$



$$H_{simulation} = I_{anc} \otimes \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1} + \sum_{i=1}^{N} S_{i,anc}^z \otimes S_i^z$$

  - By initializing the ancilla in the $|+,+,...+\rangle$ state, all possible (exponentially many) realizations of the Hamiltonian are ran in parallel
  - Ensemble average obtained by tracing out ancilla's
  - Finding ground states: adiabatic evolution
- Can easily be generalized to include random interactions

- We can of course also use numerical MPS ansatz
  - E.g. real time evolution of the ground state of Heisenberg antiferromagnet in the presence of random magnetic field in z-direction

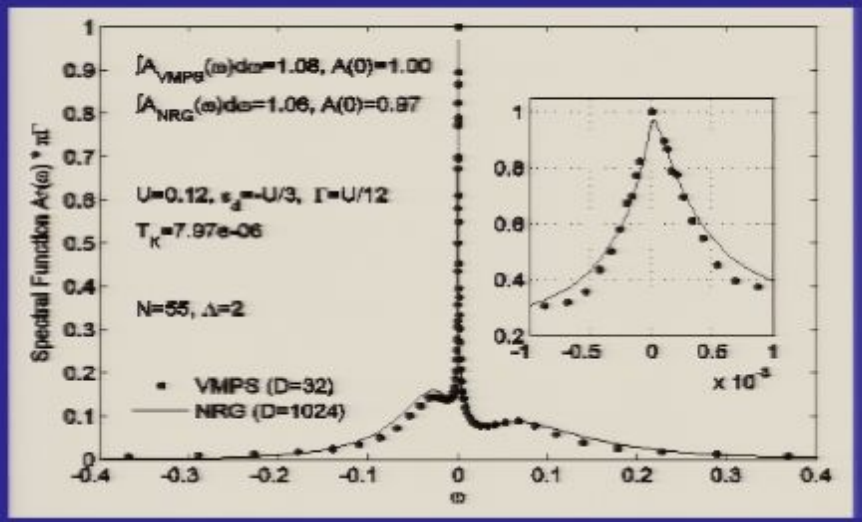


Green's functions

- Goal: calculate $\langle \Psi | a \frac{1}{H - \omega + i\eta} a^* | \Psi \rangle$

(see also Hallberg, Kuhner, White, Jeckelmann)

  - First calculate $|\Psi\rangle$
  - Variationally find $|\chi\rangle$ which minimizes $\left\| \frac{1}{H - \omega + i\eta} a^* |\Psi\rangle - |\chi\rangle \right\|_{(H-\omega)^2+\eta^2}$ This amounts to solving $\min_{|\chi\rangle} \langle \chi | (H-\omega)^2 | \chi \rangle - \langle \psi | a^* | \chi \rangle$ which can be done efficiently
  - Calculate $\langle \Psi | a | \chi \rangle$
  - e.g: applied to asymetric SIAM



- The same techniques can be used to minimize $\min_{|\chi\rangle} \frac{\langle \chi | (H-\omega)^* (H-\omega) | \chi \rangle}{\langle \chi | \chi \rangle}$ which allows to calculate gaps of non-hermitean Hamiltonians

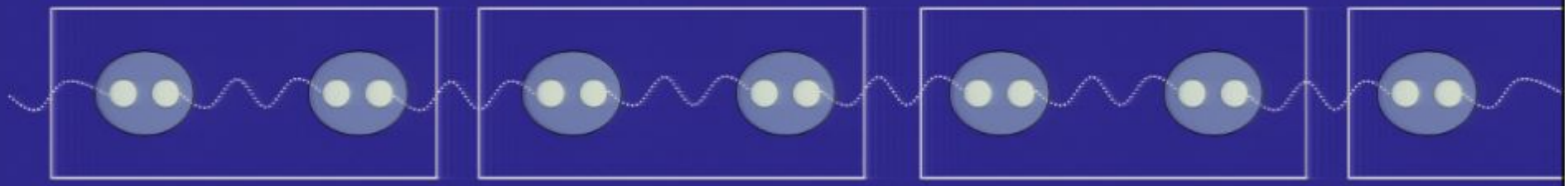
MPS for experimentalists

- Class of D-dim. MPS gives a complete characterization of all N-particle states that can be created by sequential generation through coupling to a D-level ancillary system (Markov chain)
  - Photonic qubits generated by a cavity QED source
  - Quantum dot coupled to a microcavity
  - Interaction of ions with phonons in ion trap

- 1-to-1 correspondence between maps and unitaries (isometries) occurring in cavity
  - Constructive: MPS-structure automatically yields description of how to generate states
- Example for D=2: GHZ-, cluster-, W- states (for any N)

Renormalization group transformations on states

- Goal: coarse-graining of PEPS-ground states by isometries

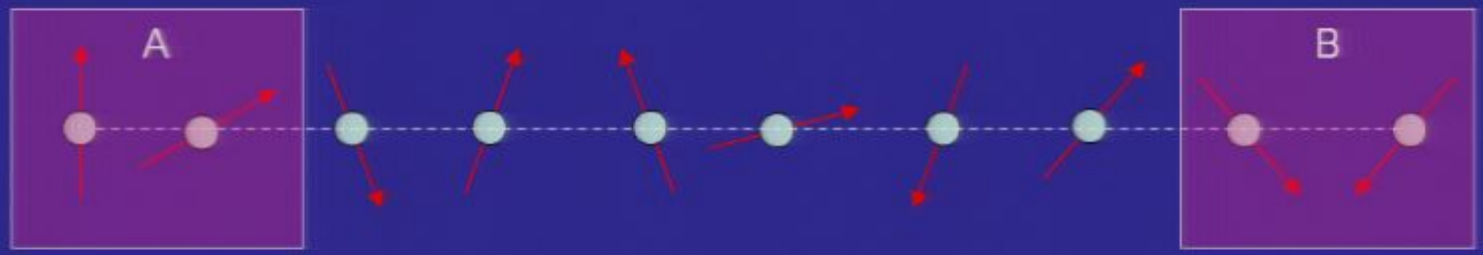


- This can be done exactly, and leads exponentially fast to a fixed point; the fixed points are scale-invariant and an alternative way of creating MPS locally
  - The fixed point of the generic case consists of the virtual subsystems becoming real, and where the ME-states are replaced with states with some entropy determined by the eigenvectors of the transfer matrix

  - A complete classification of fixed points in case of qubit bonds has been made; special cases correspond to GHZ, W, cluster and some other exotic states in QIT

Phys. Rev. Lett. 94, 140601 (2005)

Entanglement Length / Localizable Entanglement



Phys. Rev. Lett. 92, 027901 (2004); 92, 087201 (2004)

- Localizable Entanglement: how much entanglement can be *localized* between qubits separated by A, B by doing local operations assisted by classical communication?
- Gives rise to the notion of entanglement length: $\xi_E \geq \xi_C$
  - all bipartite correlations can be converted into singlets by LOCC
- Spin chain as a perfect quantum repeater: entanglement swapping
  - OK if P in MPS is spanned by set of maximally entangled states

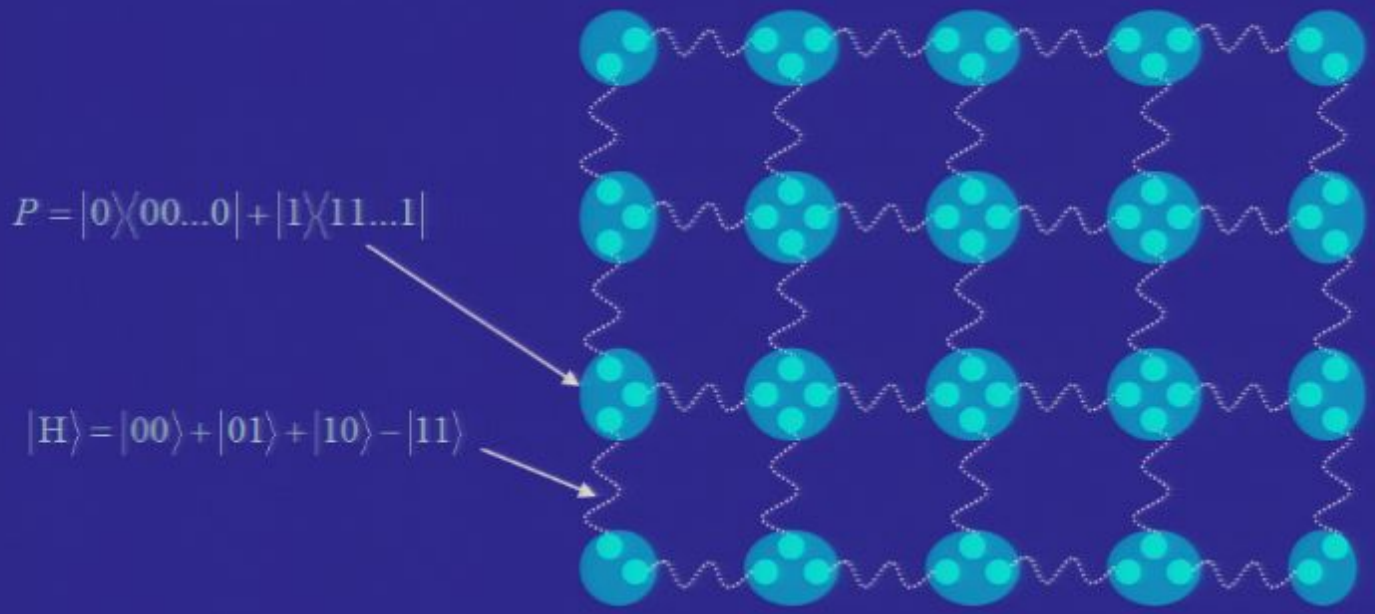
- LE can be calculated exactly in asymptotic limit of many copies:

$$\min\left(S(\varrho_A), S(\varrho_B)\right)$$

J. Smolin, F.V. A. Winter, quant-ph/0505038

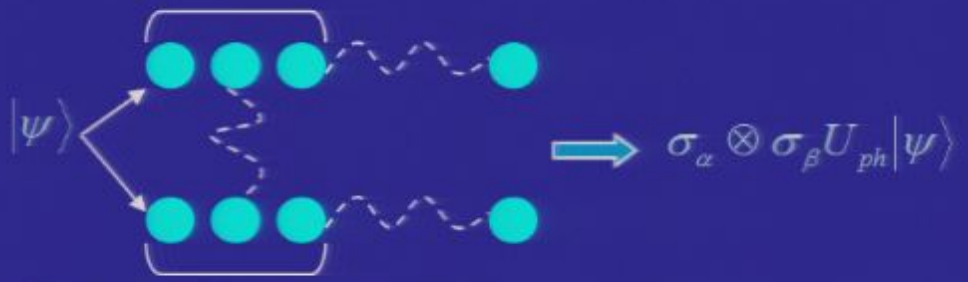
One-Way Quantum Computing: virtual qubits in action

- One-way QC is equivalent to teleportation based quantum computation (cfr. Gottesman and Chuang, '99) on "virtual" qubits



$(\langle 000| + \langle 111|)\ \sigma_x^i \otimes \sigma_x^j \otimes \sigma_z^k$

= "Postselected" teleportation-based computation



$(\langle 000| + \langle 111|)\ \sigma_x^i \otimes \sigma_x^j \otimes \sigma_z^k$

PEPS with topological quantum order

- Virtue of QIT approach: physics is about quantum states (as opposed to Hamiltonians); MPS/PEPS: guaranteed to be ground states of local Hamiltonians
  - Topological order can be defined on level of quantum states: a state exhibits topological order iff there exists another orthogonal one that is locally indistinguishable from it and no local matrix elements connect both

- Toric code state of A. Kitaev: $|\Psi\rangle \propto P_1 P_2 \cdots P_N |+\rangle|+\rangle\cdots|+\rangle$
  - Jamiolkowski: applying a nonlocal map can be implemented by taking an auxiliary entangled state and doing joint local projections on the original state and the auxiliary one; this yields a PEPS
  - Can be done for every stabilizer state which has local stabilizers (e.g. cluster)

- Resonating Valence Bond States: superposition of all possible n.n. dimer coverings is a PEPS with D=3

$$P = |0\rangle\left[\langle 0222| + \langle 2022| + \langle 2202| + \langle 2220|\right] + |1\rangle\left[\langle 1222| + \langle 2122| + \langle 2212| + \langle 2221|\right]$$

Lieb-Robinson bound

- Lieb-Robinson bound for Hamiltonian evolution states that any local spin system exhibits a finite group velocity (also critical ones):

$$\left\| \left[A(t), B(0) \right]_- \right\| \leq c \|A\| \|B\| \exp\left(-c(x - vt) \right)$$

  - Assume that we have a number of qubits on a lattice with some geometry and any quantum circuit with only gates between n.n. :

    Lieb-Robinson implies that correlations between two separated particles can only be created by a circuit whose depth is linear in their distance if one starts from a separable state (note that this statement cannot be derived from no-signaling!)

  - Similarly : depth of quantum circuit has to be at least linear in the size of the torus to create a state with topological quantum order

M. Hastings and FV, 2006

Critical PEPS

- PEPS with D=2 and power law decay of correlations: consider classical Ising model on a square lattice and temperature T.
  - It is obvious how to create quantum state with same correlations as classical one:

$$|\Psi\rangle \propto \sum_{\sigma_1\sigma_2\cdots\sigma_N} \exp\left[-\frac{\beta}{2}\sum_{\langle i,j\rangle}\sigma_i\sigma_j\right]|\sigma_1\rangle|\sigma_2\rangle\cdots|\sigma_N\rangle$$

$$\propto \exp\left[-\frac{\beta}{2}\sum_{\langle i,j\rangle}S_i^z S_j^z\right]|+\rangle|+\rangle\cdots|+\rangle$$

  - This is again a PEPS with D=2; So PEPS with very small D can exhibit power law decay of correlations: emergent phenomena
    - Area law for critical 2-D systems: no logarithmic correction (as in 1 dim.)!

- This procedure is completely general: all classical phase transitions driven by temperature have always an equivalent desription in terms of a zero-T quantum phase transition described by a PEPS and associated local Hamiltonian in the same dimension

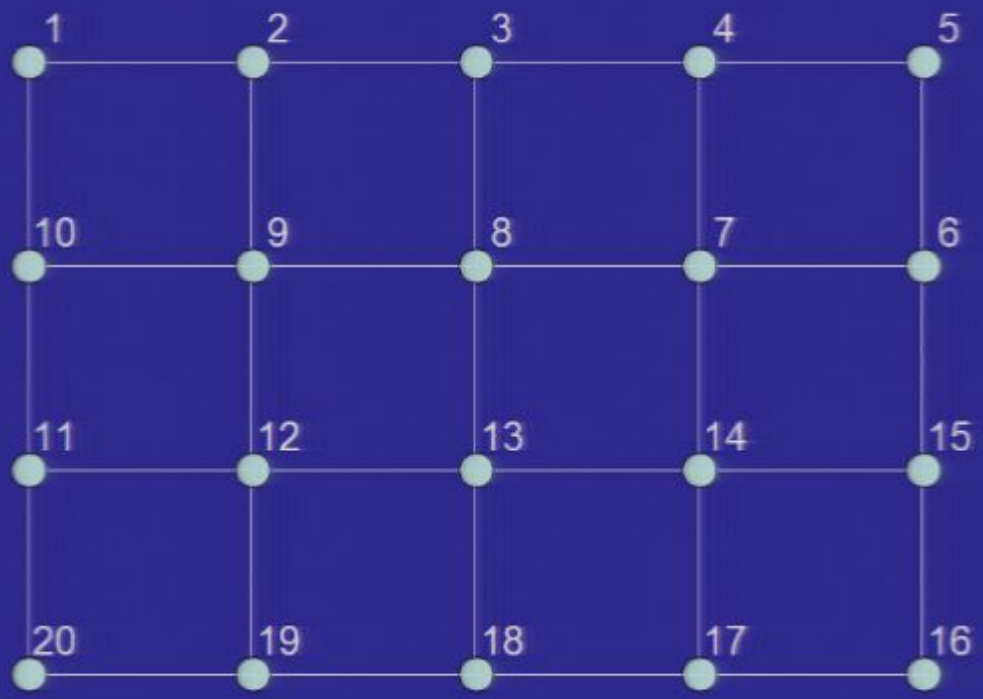
- Corollary: the problem of contracting a calculating network of tensors as in the PEPS is NP-hard (because calculating the partition function of a random 2-D classical Ising spin glass is NP-hard: Barahona 1982) ; So an oracle that could calculate expectation values of PEPS would give you the power to solve NP-hard problems; what about QMA?

2-D fermionic systems

- Fundamental question: are fermions fundamentally different from bosons/spins or can local fermionic Hamiltonians be understood as effective Hamiltonians describing low energy sector of local bosonic theories?
- Hilbert space associated to fermions is Fock space, which is obtained via second quantization:

$$|\Psi\rangle = \sum_{i_1 i_2 \ldots} c_{i_1 i_2 \ldots} \left(a_1^*\right)^{i_1} \left(a_2^*\right)^{i_2} \ldots |\Omega\rangle$$

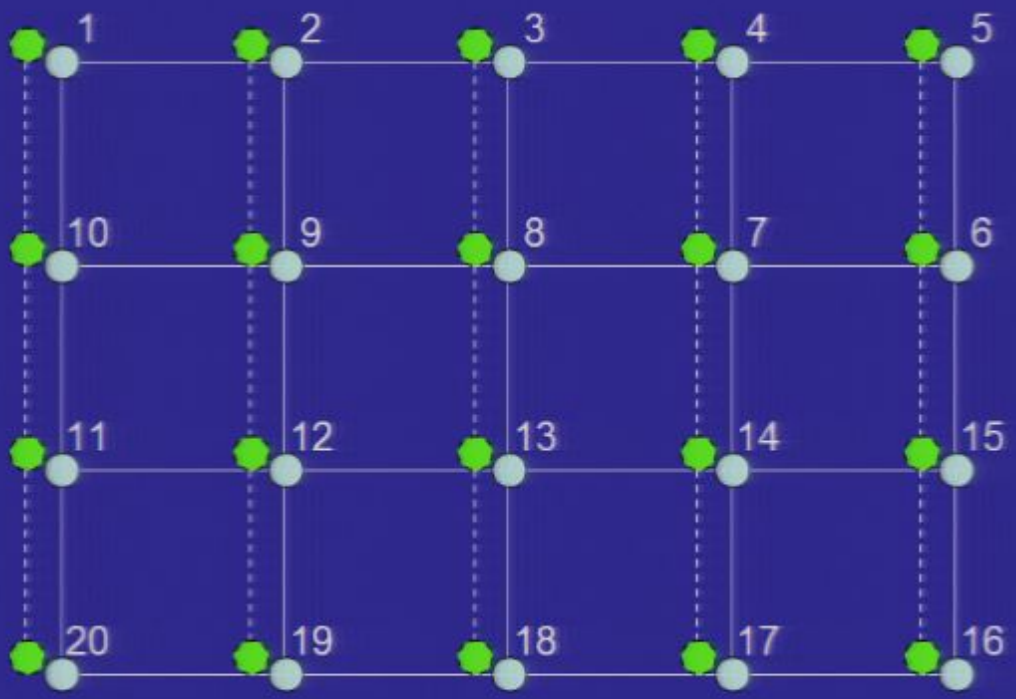
- What we want to approximate is $c_{i_1 i_2 \ldots}$
- Effective Hamiltonian for this tensor is obtained by doing the Jordan-Wigner transformation on the original one (note the ordering of the fermions in second quantization)
- Consider hopping terms in 2-D: J-W induces long-range correlations
- Solution: use auxiliary Majorana fermions to turn this Hamiltonian into a local Hamiltonian of spins (cfr. Kitaev)
- Similar but different trick applies to any geometry/dimension and multi-channel impurity problems
- Important from numerical point of view!



J. Stat. Mech. (2005) P09012

Vertical hopping terms become nonlocal by JW-transformation:

$$a_1^\dagger a_{10} + a_{10}^\dagger a_1 \xrightarrow{JW} \sigma_1^x \left(\bigotimes_{k=2}^{9} \sigma_k^z \right) \sigma_{10}^x + \sigma_1^y \left(\bigotimes_{k=2}^{9} \sigma_k^z \right) \sigma_{10}^y$$



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Solution: add ancillary chains of free fermions b_i constructed as follows: define Majorana fermions $c_i = b_i + b_i^\dagger$, $d_i = i(b_i - b_i^\dagger)$ and free Hamiltonian

$$H_{anc} = -\sum_{\langle k,l \rangle} i c_k d_l$$

As all terms $ic_k d_l$ are constants of motion (i.e. +1) and commute with each other, we can change the original vertical hopping terms $a_1^\dagger a_{10} + a_{10}^\dagger a_1 \longrightarrow \left(a_1^\dagger a_{10} + a_{10}^\dagger a_1 \right) ic_1 d_{10}$ without changing the physics of the Hamiltonian. Renumbering everything makes everything local after the JW

Conclusion

- Formalism of quantum information theory provides unique perspective on strongly correlated quantum systems
  - MPS/PEPS picture describes low-energy sector of local Hamiltonians
  - Allows to reformulate and extend numerical renormalization group methods
  - Also applies to local Markov processes (e.g. hopping, traffic)
  - Similar ideas should apply to quantum field theories / lattice gauge theories
- Key element: Frustration / monogamy property of entanglement (cfr. quantum cryptography)
- New algorithms allow to go where nobody has gone before, such as simulating 2-D Fermi-Hubbard model, crucial in the field of condensed matter theory