Title: Solving pure Yang-Mills in 2+1 dimensions

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Abstract: I review our recent work on confinement in 2+1 Yang Mills theory using Karabali-Nair variables. I'll discuss our successful prediction of the glueball spectrum, including the manifestations of the QCD string.

Solving Pure Yang-Mills in 2+1 Dimensions

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based on
hep-th/0512111
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with D. Minic and
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Remarks

- the solution of the Yang-Mills theory is certainly one of the grand problems of theoretical physics
- one has always expected that, if such a solution were to be found, it would be in the large N limit
- a basic problem is in identifying the important degrees of freedom, and tractably rewriting the theory in their terms
- we should expect to see both the asymptotically free regime as well as low energy confining physics
- we should demonstrate:
  - useful variables
  - non-perturbative vacuum – the 'Master field'
  - demonstrate important observable consequences
    - *e.g., signals of confinement: area law, string tension, mass gap*
    - *in pure Yang-Mills, compute the spectrum of glueball states*

Outline

1. Introductory remarks on YM and results of low dimension toy models
2. The 'experimental' data for 2+1 pure YM
   - preview of analytic results
3. Hamiltonian formalism
   - collective field ideas and large N
   - the Karabali-Nair parameterization
4. The Vacuum Wavefunctional
5. Correlation Functions and Glueball Spectrum
6. Comments on the QCD string
7. Outlook

QCD Basics

- pure Yang-Mills theory is given by the path integral

$$Z = \int \frac{[dA^a_\mu]}{Vol\ G} e^{iS_{YM}[A]}$$

with

$$S_{YM}[A] = -\frac{1}{2g^2_{YM}} \int d^{D+1}x\ tr\ F^2_{\mu\nu}$$

gauge group $SU(N)$

$$A_\mu = A^a_\mu t^a$$

$$tr\ t^a t^b = \frac{1}{2}\delta^{ab}$$

- we will be primarily interested in D=2 here.
  - in this case, g^2_{YM} has units of mass, and we define

$$m = \frac{g^2_{YM} N}{2\pi}$$

't Hooft coupling

- this is the basic (bare) mass scale in the theory.
  - conceptually different than D=3, where the bare YM coupling is dimensionless and the physical mass scale is generated dynamically
  - nevertheless, D=2 is otherwise quite similar to D=3 (*asymptotic freedom*)
  - believed to *confine* at long distances

Phases of QCD

- Short distance:
  - free theory at arbitrarily high energies
    - *perturbative regime of free massless gluons*
- Long distance:
  - confinement of colour charges
    - *generation of a mass gap (no massless excitations in spectrum)*
    - *hope to compute the spectrum of gauge invariant states (here, "glueballs")*
  - Phenomenology:
    - *expect some effective* QCD *string picture.*

    - *this is not expected to be a "fundamental string theory" but should have features in common.*

Toy Models for Confinement

- 1+1 QCD
  - in the 1970's, 't Hooft showed that confinement can be seen directly by computing Feynman diagrams (large N)
    - *the pole of the quark propagator moves off to infinity, because of an IR divergence.*
    - *poles appear in multi-particle channels*
  - partition function of Euclidean pure YM on Riemann surface computed exactly (Witten)
    - *re-interpreted term by term as contributions of a* QCD *string theory (Gross & Taylor)*
    - *this may be related directly to (Das-Jevicki) collective field theory, and to one-matrix model*

Minahan & Polychronakos; etc.

- 2+1
  - lattice compact QED (Polyakov '75)
    - *explicit demonstration of confinement, condensation of magnetic monopoles*
  - Georgi-Glashow model (Polyakov '77)
  - pure Yang-Mills (Feynman '81)
    - *argued that theory should confine, with mass gap generated because configuration space is compact.*
    - *details incorrect.*

"dual superconductor"

- see also Seiberg-Witten; AdS/CFT

Experiment

- in 2+1 Yang-Mills, the 'experimental data' consists of a number of lattice simulations, largely by M. Teper, et al

Teper:
hep-lat/9804008
Lucini & Teper:
hep-lat/0206027

	$m_G/\sqrt{\sigma}$					
state	SU(2)	SU(3)	SU(4)	SU(5)	SU(4)	SU(6)
0^{++}	4.716(21)	4.330(24)	4.239(34)	4.180(39)	4.235(25)	4.196(27)
0^{++*}	6.78(7)	6.485(55)	6.383(77)	6.22(8)	6.376(45)	6.20(7)
0^{++**}	8.07(10)	8.21(10)	8.12(13)	7.87(18)	7.93(7)	8.22(12)
0^{--}		6.464(48)	6.27(6)	6.06(11)	6.230(44)	6.097(80)
0^{--*}		8.14(8)	7.84(13)	7.85(15)	8.20(15)*	7.98(15)
2^{++}	7.81(6)	7.12(7)	7.14(8)	7.15(12)	7.17(8)	6.67(18)
2^{++*}			8.50(17)	8.56(15)	8.06(22)	8.89(20)
2^{--}		8.73(10)	8.25(21)	8.25(18)	8.49(13)	8.52(20)

Table 4: Glueball masses in units of the string tension, in the continuum limit. Reanalysis of [2] on left; new calculations on right. *from Lucini & Teper '02*

- they extract masses of some low lying states for smallish values of N, and extrapolate to large N
- (there is also info on states with other J^{PC} quantum numbers for small N in the '98 paper)

Glueball Masses: analytic results

- we have computed these masses using an analytic technique, with the following results

TABLE I: 0^{++} glueball masses in QCD_3. All masses are in units of the square root of the string tension. Results of AdS/CFT computations in the supergravity limit are also given for comparison. The percent difference between our prediction and lattice data is given in the last column.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff, %
0^{++}	4.065 ± 0.055	4.07(input)	4.10	0.8
0^{++*}	6.18 ± 0.13	7.02	5.41	12.5
0^{++**}	7.99 ± 0.22	9.92	6.72	16
0^{++***}	9.44 ± 0.38^a	12.80	7.99	15

[a] Mass of 0^{++***} state was computed on the lattice for $SU(2)$ only [9]. The number quoted here was obtained by a simple rescaling of $SU(2)$ result.

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0^{--*}	7.63 ± 0.37	9.34	7.46	2.3
0^{--**}	8.96 ± 0.65	12.37	8.77	2.2

from hep-th/0512111

- the results agree extremely well with the lattice data
  - analytic methods make use of a re-parameterization of the gauge fields within a Hamiltonian framework, pioneered by Karabali and Nair
  - we have new results for the ground-state wavefunctional and simple correlators, for large N

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2+1 YM in the Hamiltonian Formalism

- we consider 2+1 SU(N) Yang-Mills theory with Hamiltonian

$$\mathcal{H}_{YM} = \frac{1}{2}\int Tr\,\left(g_{YM}^2\Pi_i^2 + \frac{1}{g_{YM}^2}B^2\right)$$

- we choose the temporal or Hamiltonian gauge, $A_0 = 0$, leaving the gauge fields $A = (A_1 + iA_2)/2, \bar{A} = (A_1 - iA_2)/2$ dynamical

$z = x_1 - ix_2, \bar{z} = x_1 - ix_2$
$A_i = -it^a A_i^a$

- $\Pi_i \sim E_i$ is the momentum conjugate to A_i
  - $\text{quantize}: \Pi_i^a(x) \to i\frac{\delta}{\delta A_i^a(x)}$, 'position representation' : $\psi[A_i^a(x)]$
- time-independent gauge transformations preserve the gauge condition, and the gauge fields transform as a connection

$$A \mapsto gAg^{-1} - \partial g g^{-1}, \quad \bar{A} \mapsto g\bar{A}g^{-1} - \bar{\partial} g g^{-1}, \quad g(z,\bar{z}) \in SU(N)$$

- Gauss' law implies that observables and physical states are gauge invariant
- hard to deal with gauge-fixing, so we would like to perform a field redefinition to gauge-invariant variables
  - traditionally, this is taken to mean Wilson loops $W_R(C) = tr_R P e^{i\oint_C A}$

A variables do not create physical excitations

Gauge Invariant Formalism

- would like to transform to gauge invariant variables $\{\Phi\}$
- path integral would transform $\rightarrow \int [d\Phi] \frac{1}{\det \frac{d\Phi}{dA}} e^{iS}$
  - the Jacobian is typically hard to compute
- a natural choice is to take variables to be Wilson loops
  - expectation value is order parameter for confinement $\langle W_R(C) \rangle \sim e^{-\sigma A + \ldots}$
  - Wilson loops are a complete set of operators but are over-complete and constrained
    - *at large N, they become independent, due to factorization* $\langle \Phi\Phi \ldots \rangle \rightarrow \langle \Phi \rangle \langle \Phi \rangle \ldots$
- equation of motion $\leftrightarrow$ loop equation (Makeenko & Migdal)
  - hard to proceed
- can compute (formally!!) in Hamiltonian formalism (Sakita '80; Jevicki & Sakita '81)
  - Hamiltonian has "collective field form"
  - formally, if one knew the Jacobian, one could do a saddle point approximation, and compute
    - *validity is equivalent to large N*
  - this is essentially what we will do, in a more convenient parameterization

Karabali-Nair Parameterization

- it is possible to parameterize the gauge fields as

$$A = -\partial M M^{-1}, \qquad \bar{A} = M^{-\dagger} \bar{\partial} M^{\dagger}$$

where M is complex, invertible, unimodular

A traceless $\leftrightarrow \det M = 1$
$M \in SL(N, \mathbb{C})$

- M transforms linearly under gauge transformations

$$M \mapsto gM$$

- gauge invariant variables may be written simply

$$H = M^{\dagger} M$$

- note that these are *local fields*. Roughly, M may be thought of as analogous to an open Wilson line, and H a closed loop
- the Wilson loop evaluates to

$$\Phi(C) = TrPe^{i \oint_C (A dz + \bar{A} d\bar{z})} = TrPe^{-i \oint_C dz \; \partial H H^{-1}}$$

- dependence on C is an artifact; one can use the local H variables instead.
  - although Wilson loop retains its usefulness as an order parameter for confinement

Holomorphic Invariance

- one might wonder if the parameterization is well-defined
  - does H capture all of the physics? Is the parameterization one-to-one?
- in fact, there is a new *holomorphic invariance* acting on M on the right, which is not seen by the original gauge fields

$$M(z,\bar{z}) \mapsto M(z,\bar{z})h^\dagger(\bar{z}) \qquad M^\dagger(z,\bar{z}) \mapsto h(z)M^\dagger(z,\bar{z})$$

$$H(z,\bar{z}) \mapsto h(z)H(z,\bar{z})h^\dagger(\bar{z})$$

- the appearance of this can be seen by attempting to invert the defining relations

$$M(z,\bar{z}) = \left(1 - \int d^2w\ G(z,w)A(w,\bar{w}) + \dots\right)\bar{V}(\bar{z}) \qquad \partial_z G(x,y) = \delta^{(2)}(x-y)$$

- so one must ensure that all results are holomorphic invariant
- one could simply fix the gauge $\bar{V} = 1$, and then *enforce holomorphic invariance on physical states;* in general, all physical formulae must be holomorphic invariant

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The Jacobian

- now, a change of variables is not too remarkable, classically. However, in this particular case, the path integral Jacobian of the transformation can be worked out -- in fact it is given in terms of the level $-2c_A$ hermitian Wess-Zumino-Witten model

$$d\mu[C] = \sigma \, d\mu[H] e^{2c_A S_{WZW}[H]}$$

$$d\mu[H] \leftrightarrow ds_H^2 = \int Tr\,(\delta H H^{-1})^2$$

$$S_{WZW}[H] = -\frac{1}{2\pi}\int d^2z\, Tr\, H^{-1}\partial H H^{-1}\bar{\partial} H + \frac{i}{12\pi}\int d^3x \epsilon^{\mu\nu\lambda} Tr\, H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\lambda H$$

Polyakov & Weigmann

- this is both gauge and holomorphic invariant
- thus the inner product on states can be written in the position representation as an overlap integral of gauge and holomorphic invariant wave functionals with non-trivial measure

$$\langle 1|2\rangle = \int d\mu[H] e^{2c_A S_{WZW}[H]} \Psi_1^* \Psi_2$$

- this non-trivial measure has important consequences -- e.g., $\Psi = 1$ is normalizable!
    - in fact, this is an approximation to the ground-state wavefunctional

The Hamiltonian

- it is natural to introduce the 'current'

$$J = \frac{c_A}{\pi} \partial H H^{-1}$$

J is a connection for holomorphic invariance:

$$J \mapsto hJh^{-1} + \tfrac{\pi}{c_A}\partial h h^{-1}$$

- the YM Hamiltonian can then be rewritten in terms of J

$$\mathcal{H}_{KN}[J] = m\left(\int_x J^a(x)\frac{\delta}{\delta J^a(x)} + \int_{x,y}\Omega_{ab}(x,y)\frac{\delta}{\delta J^a(x)}\frac{\delta}{\delta J^b(y)}\right) + \frac{\pi}{mc_A}\int_x \bar{\partial}J^a\bar{\partial}J^a$$

(recall *m* is the 't Hooft coupling)

Karabali & Nair

- this has the collective field form and

$$\Omega_{ab}(x,y) = \frac{c_A}{\pi^2}\frac{\delta_{ab}}{(x-y)^2} - \frac{i}{\pi}\frac{f_{abc}J^c(x)}{(x-y)}$$

- the derivation of the Hamiltonian has involved a careful gauge-invariant regularization
  - this is true of all computations that we will discuss, but the details will be suppressed

Wavefunctionals

- a wavefunctional in position representation may be regarded as a functional of H, or as a functional of J
  - specifically, note that $\bar{\partial} J$ and $D = \partial - \frac{\pi}{c_A} J$ transform homogeneously under holomorphic transformations

$$\bar{\partial} J \mapsto h(z) \bar{\partial} J h^{-1}(z)$$

    - *thus, these are the building blocks for holomorphic invariant functionals*
  - in fact, we will find that, at large N, $\bar{\partial} J$ plays a very special role, essentially a *string oscillator*
  - note also that J satisfies a `reality condition' (analogous to hermiticity of *H*)

$$\bar{\partial} J = [D, \bar{J}] \qquad \bar{J} = \frac{c_A}{\pi} \bar{\partial} H H^{-1}$$

- more precisely, paying attention to spacetime quantum numbers, we can build invariants (with $J^{PC} = 0^{++}$) as traces of products of $\bar{\partial} J$ and $\Delta = \bar{\partial} D + D \bar{\partial}$
- consider the vacuum wavefunctional Ψ_0
  - this will satisfy the functional Schrödinger equation

$$\mathcal{H}_{KN} \Psi_0 = E_0 \Psi_0$$

$\mathbf{J^{PC}}$

- Spin J: $SO(2) \subset SO(2,1)$
  - thus spin is just a charge

J = +1	$\bar{\partial}$
J = −1	$J, \quad D$
J = 0	$\bar{\partial}J, \quad D\bar{\partial} + \bar{\partial}D$

- Parity: $x_1 \to x_1, \quad x_2 \to -x_2$ and charge conjugation

$$
\begin{aligned}
P: \quad & z \to \bar{z} \\
& A_{i\bar{j}} \to \bar{A}_{i\bar{j}} \\
& M \to M^{-\dagger} \\
& H \to H^{-1} \\
& \bar{\partial}J \to -H^{-1}\bar{\partial}JH \\
& \Delta \to +H^{-1}\Delta H
\end{aligned}
$$

$$
\begin{aligned}
C: \quad & z \to z \\
& A_{i\bar{j}} \to -A_{j\bar{i}} \quad \bar{A}_{i\bar{j}} \to -\bar{A}_{j\bar{i}} \\
& M_{i\bar{\alpha}} \to (M^{-1})_{\alpha\bar{i}} \quad M^{\dagger}_{\alpha\bar{i}} \to (M^{-\dagger})_{i\bar{\alpha}} \\
& H_{\alpha\bar{\beta}} \to (H^{-1})_{\beta\bar{\alpha}} \\
& J_{\alpha\bar{\beta}} \to -J_{\beta\bar{\alpha}} \\
& ([D,\phi])_{\alpha\bar{\beta}} \mapsto +([D,\phi^C])_{\beta\bar{\alpha}}
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& ([D, \phi])_{\alpha\bar{\beta}} \mapsto +([D, \phi^C])_{\beta\bar{\alpha}}
\end{aligned}
$$

Vacuum Wave-functional

- if the KN Hamiltonian contained just the kinetic part, then $\Psi = 1$ would be a suitable *normalizable* solution (because of the non-trivial measure)
    - note the potential term vanishes in the limit of large g_{YM}^2
- more generally, the potential term will make a contribution
    - we will take as ansatz

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int tr\ \bar{\partial}J K(L)\bar{\partial}J + \ldots\right). \qquad L = \Delta/m^2$$

    - this is explicitly gauge and holomorphic invariant
    - this may be regarded as a WKB approximation
    - can also be regarded as a saddle point approximation, from the point of view of collective field theory
        - *its validity is controlled by the 1/N expansion*
    - we solve the Schrödinger equation order by order in $\bar{\partial}J$
    - note that this Gaussian part of the vacuum wavefunctional contains a (non-trivial) kernel K, which will be determined by the Schrödinger equation
        - *K contains information about the spectrum of the theory at large N*

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$$(\bar{\partial} J)) \sim [D, \bar{\partial}]$$

Vacuum Wave-functional

- if the KN Hamiltonian contained just the kinetic part, then $\Psi = 1$ would be a suitable *normalizable* solution (because of the non-trivial measure)
  - note the potential term vanishes in the limit of large g^2_{YM}
- more generally, the potential term will make a contribution
  - we will take as ansatz

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int tr\ \bar{\partial}JK(L)\bar{\partial}J + \dots\right). \qquad L = \Delta/m^2$$

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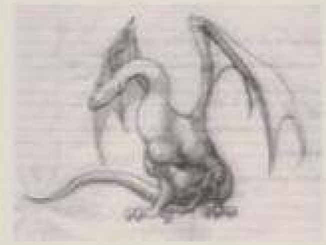
Schrödinger



- the Schrödinger equation takes the form

$$\mathcal{H}_{KN}\Psi_0 = \left[\ldots + \frac{\pi}{mc_A}\int tr\ \bar{\partial}J(\mathcal{R})\bar{\partial}J + \ldots\right]\Psi_0$$

(divergent) vacuum energy

- by careful computation (regularization required!) we find

$$\boxed{\mathcal{R} = -K(L) - \frac{L}{2}\frac{d}{dL}[K(L)] + LK(L)^2 + 1 = 0}$$

"Riccati diff. eq."

- this is a formal expression, obtained by regarding K as a power series in L, and computing term by term
- the boxed equation is a differential equation for K, which can be solved formally -- in fact, by a series of redefinitions, it can be cast as a Bessel eq.
    - this should be solved subject to a physical boundary condition
        - *at small L, we should have* $K(L) \to 1$ *(confining regime)*
        - *will also obtain correct large L behaviour (asymptotic freedom)*

Vacuum Wavefunctional

- the solution with the correct asymptotics is

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int tr\ \bar{\partial}JK(L)\bar{\partial}J + \dots\right).$$

$p \to 0,\ \ K \to 1$

$p \to \infty,\ \ K \to 2m/p$

$$K(L) = \frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})}$$

- the small L limit contains information about the string tension
  - indeed, because $\bar{\partial}J$ is similar to the Yang-Mills magnetic field B, and the computation of the expectation value of a spatial Wilson loop may be regarded as a computation in 2-dimensional Yang-Mills
  - one finds (correctly) $$\sqrt{\sigma} \simeq \frac{g_{YM}^2 N}{\sqrt{8\pi}}$$ $$\langle\Phi\rangle \sim \exp(-\sigma A)$$
- in the large L limit, the wavefunctional goes over to a form consistent with free gluons, with coupling g_{YM}^2

Correlation Functions

- we would like now to use this result to compute correlation functions of products of invariant operators $\langle \mathcal{O}_{-J,P,C}(\vec{x},t)\mathcal{O}_{J,P,C}(\vec{y},t)\rangle$
- at large distance, we will find contributions of single particle poles of the correct quantum numbers

$$\langle \mathcal{O}_{-J,P,C}(\vec{x},t)\mathcal{O}_{J,P,C}(\vec{y},t)\rangle \sim \frac{\#}{|x-y|}\sum_j e^{-m_j|x-y|}$$

- to find particle states of given spacetime quantum numbers, we consider operators of a suitable form

$$e.g.,\ \mathcal{O}_{0++} = tr : \bar{\partial}J\bar{\partial}J :$$

- the correlation function is written in position space representation as

$$\int d\mu[H]e^{2c_A S_{WZW}[H]}\Psi_0^*\mathcal{O}(x)\mathcal{O}(y)\Psi_0 = \int d\mu[\bar{\partial}J]\Psi_0^*\mathcal{O}(x)\mathcal{O}(y)\Psi_0$$

- in the second half of this equation, we have changed variables from H to J
  - since the vacuum wavefunctional is Gaussian, $\bar{\partial}J$ acts as essentially a free field
  - furthermore in the large N limit, we can regard K(L) as a function of $\partial\bar{\partial}/m^2$ and correlation functions may be computed by Wick contractions with kernel K^{-1}

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0^{++} Glueballs

- thus we find $\langle tr\ \bar{\partial}J\bar{\partial}J(x)\ tr\ \bar{\partial}J\bar{\partial}J(y)\rangle \simeq K^{-2}(|x-y|)$
- this is expressed in terms of the Fourier transform
- using a product form of the Bessel function $J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty}(1 - \frac{z^2}{\gamma_{\nu,n}^2})$

  we find
$$K^{-1}(\vec{k}) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{M_n^2}{M_n^2+\vec{k}^2} \qquad M_n \equiv \gamma_{2,n}m/2$$
- Fourier transforming, we find a result which at long distance behaves as
$$K^{-1}(|x-y|) = -\frac{1}{4\sqrt{2\pi|x-y|}}\sum_{n=1}^{\infty}(M_n)^{3/2}e^{-M_n|x-y|}$$
- thus, we find the remarkable formula
$$\langle tr\ \bar{\partial}J\bar{\partial}J(x)\ tr\ \bar{\partial}J\bar{\partial}J(y)\rangle \simeq \sum_{m,n}\frac{\#}{|x-y|}e^{-(M_n+M_m)|x-y|}$$
  - with masses determined by the zeros of Bessel function
$$m_{m,n} = (\gamma_{2,m}+\gamma_{2,n})\frac{m}{2} = (\gamma_{2,m}+\gamma_{2,n})\frac{\sqrt{\sigma}}{\sqrt{2\pi}}$$

$\gamma_{2,1} = 5.14$
$\gamma_{2,2} = 8.42$
$\gamma_{2,3} = 11.62$

Comparison to Lattice

- using this result, we tabulate states

TABLE I: 0^{++} glueball masses in QCD_3. All masses are in units of the square root of the string tension. Results of AdS/CFT computations in the supergravity limit are also given for comparison. The percent difference between our prediction and lattice data is given in the last column.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff. %
0^{++}	4.065 ± 0.055	4.07(input)	4.10	0.8
0^{++*}	6.18 ± 0.13	7.02	5.41	12.5
0^{++**}	7.99 ± 0.22	9.92	6.72	16
0^{++***}	9.44 ± 0.38^{a}	12.80	7.99	15

[a] Mass of 0^{++***} state was computed on the lattice for $SU(2)$ only [9]. The number quoted here was obtained by a simple rescaling of $SU(2)$ result.

TABLE II: 0^{--} glueball masses in QCD_3. All masses are in units of the square root of the string tension. Results of ADS/CFT computations in the supergravity limit are also given for comparison. The percent difference between our prediction and lattice data is given in the last column.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff.%
0^{--}	5.91 ± 0.25	6.10	6.15	4
0^{--*}	7.63 ± 0.37	9.34	7.46	2.3
0^{--**}	8.96 ± 0.65	12.37	8.77	2.2

from hep-th/0512111

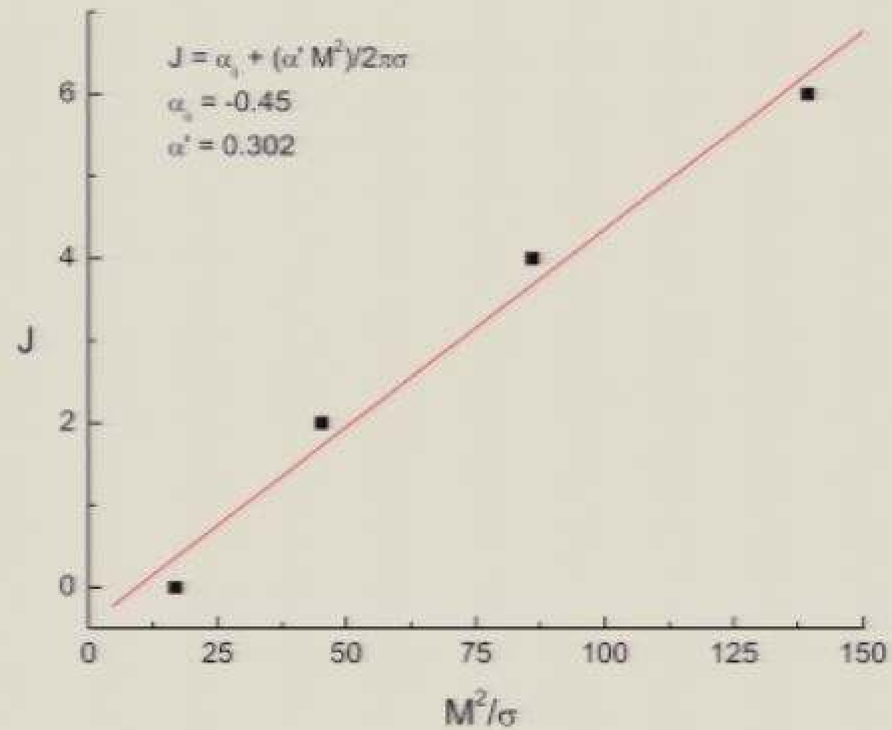
- the lowest lying 0^{++} state agrees very well with the lattice result
  - other 0^{++} states are within 10-15% of lattice

Carlsson & McKellar

    - *however, it has been suggested in the past that the masses of these states should have larger error bars*
- results for 0^{--} states come from correlation function of $tr\ \bar{\partial}J\bar{\partial}J\bar{\partial}J$ and agree with lattice within a few percent

Comments on Regge Trajectories

- preliminary work on higher spin states is encouraging
- lattice data is sparse, except for low N
- states organize into a series of straightish trajectories
  - a representative is shown here
  - in any case, a more careful analysis is required



Comments on the QCD String

- the Bessel function is essentially sinusoidal, and so its zeros are evenly spaced (better for large *n*)

$\gamma_{2,n} \sim \pi(n + 1/4)$

- thus, the predicted spectrum has approximate degeneracies

$$e.g.,\ M_1 + M_5 \simeq M_2 + M_4 \simeq M_3 + M_3$$

and the spectrum is organized into bands concentrated around a given level (which are well separated)

- at each level, one finds more and more spin states
- preliminary counting suggests that there is an approximate (in the sense that degeneracies are not exact) Hagedorn spectrum of states
    - degeneracies are more precise at high levels
- we believe this is a basic manifestation of the QCD string
    - ∂J essentially plays the role of a string oscillator
    - the departure from exact degeneracies at low levels is a sign that this is not a fundamental string (a result which is certainly expected, as the theory retains information about the asymptotically free regime)

see Leigh, Minic, Nowling, Yelnikov
hep-th/0602xxx

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Outlook

- further work
    - would like to more carefully sort out predicted states, especially higher spins
    - finite N effects? (widths?, etc.)
- more lattice simulations are required!
- 2+1 QCD
    - we believe that we can extend these results to QCD with fundamental fermions
        - *it is possible to include fermions into the KN formalism*
        - *would like to demonstrate confinement and compute meson spectrum (!)*
- 3+1 Yang-Mills
    - it's not clear that this can be handled rigorously by an extension of this formalism
        - *however, it's certainly worth a try!*
        - *preliminary numerical estimates, based on 'scaling up' the 2+1 ideas, seem to agree with 3+1 lattice results with 10% or so*