Title: Two exponential separations in communication complexity through bounded-error quantum state indistinguishability

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Abstract: We consider the problem of bounded-error quantum state identification: given one of two known states, what is the optimal probability with which we can identify the given state, subject to our guess being correct with high probability (but we are permitted to output "don't know" instead of a guess). We prove a direct product theorem for this problem. Our proof is based on semidefinite programming duality and the technique may be of wider interest. Using this result, we present two new exponential separations in the simultaneous message passing model of communication complexity. Both are shown in the strongest possible sense: -- we describe a relation that can be computed with O(log n) classical bits of communication in the presence of shared randomness, but needs n^(1/3) communication if the parties don't share randomness, even if communication is quantum; -- we describe a relation that can be computed with O(log n) classical bits of communication in the presence of shared entanglement, but needs (almost) n^(1/3) communication if the parties share randomness but no entanglement, even if communication is quantum.

Two Exponential Separations in Communication Complexity Through Quantum States Indistinguishability

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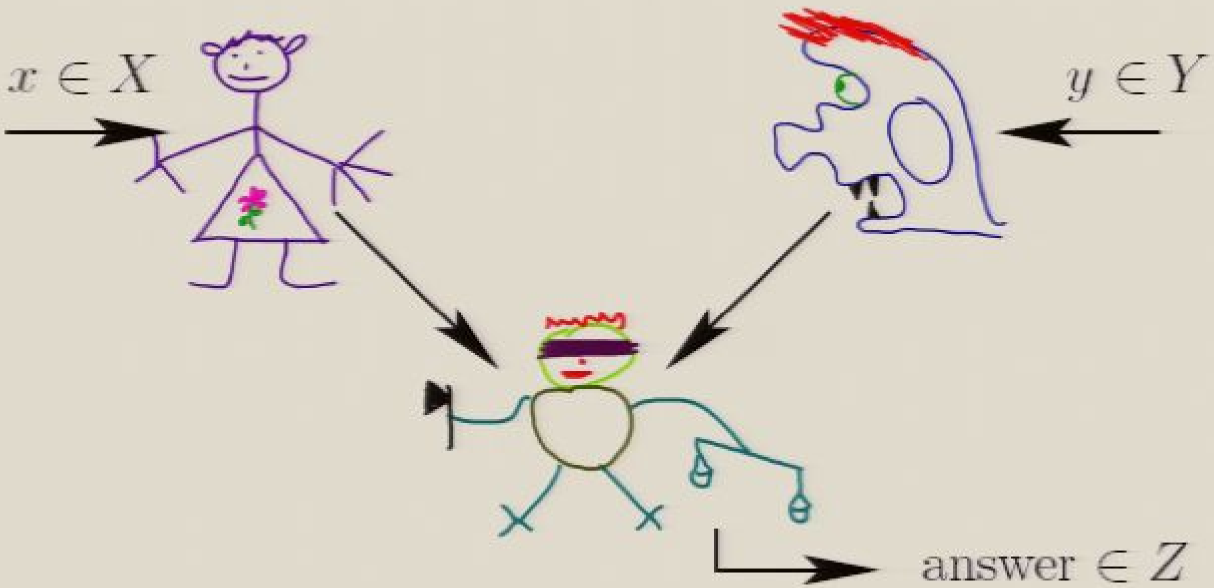
University of Calgary

Joint work with:

Julia Kempe, Oded Regev, Ronald de Wolf

Communication Complexity: the SMP Model



$P \subseteq X \times Y \times Z$

Is $(x, y, z) \in P$?

Simultaneous Message Passing:

- Alice receives x and sends a message to the referee;
- (at the same time) Bob receives y and sends a message to the referee;
- the referee reads the messages and produces an answer.

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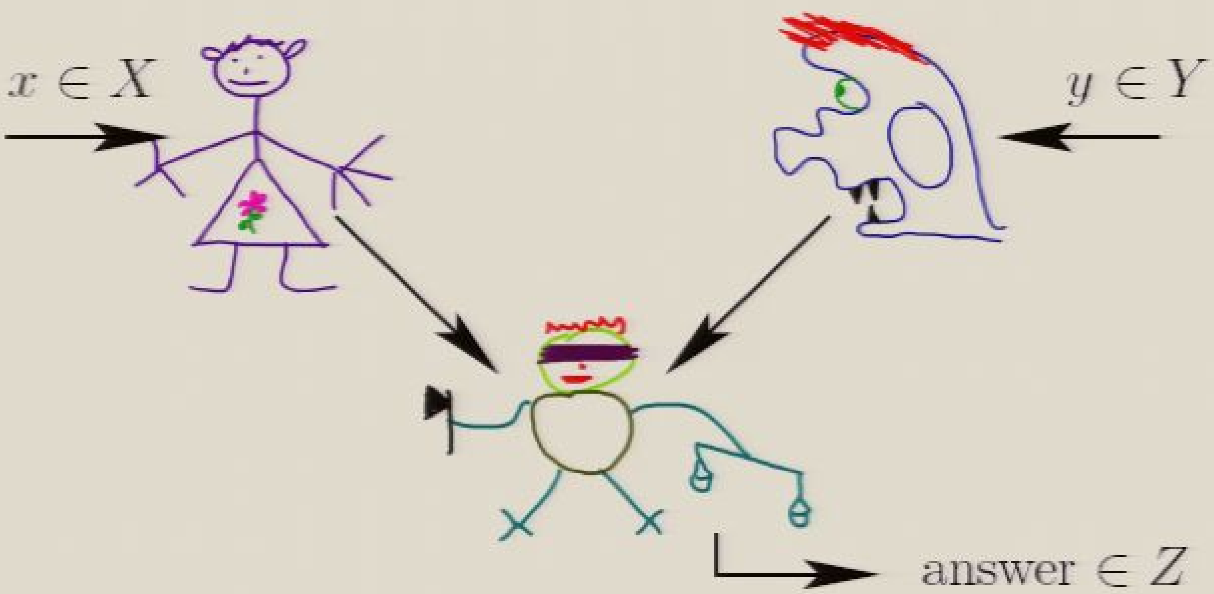

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Models: $R^{||}$, $R^{||,pub}$, $Q^{||}$, $Q^{||,ent}$

- **Communication with shared randomness:** Alice and Bob share a sequence of random bits (fair coin flips).
- **Quantum communication:** Alice and Bob send quantum messages, the referee performs a POVM measurement in order to produce the final output.
- **Communication with shared entanglement:** Alice and Bob share pairs of entangled qubits (w.l.o.g., EPR pairs).

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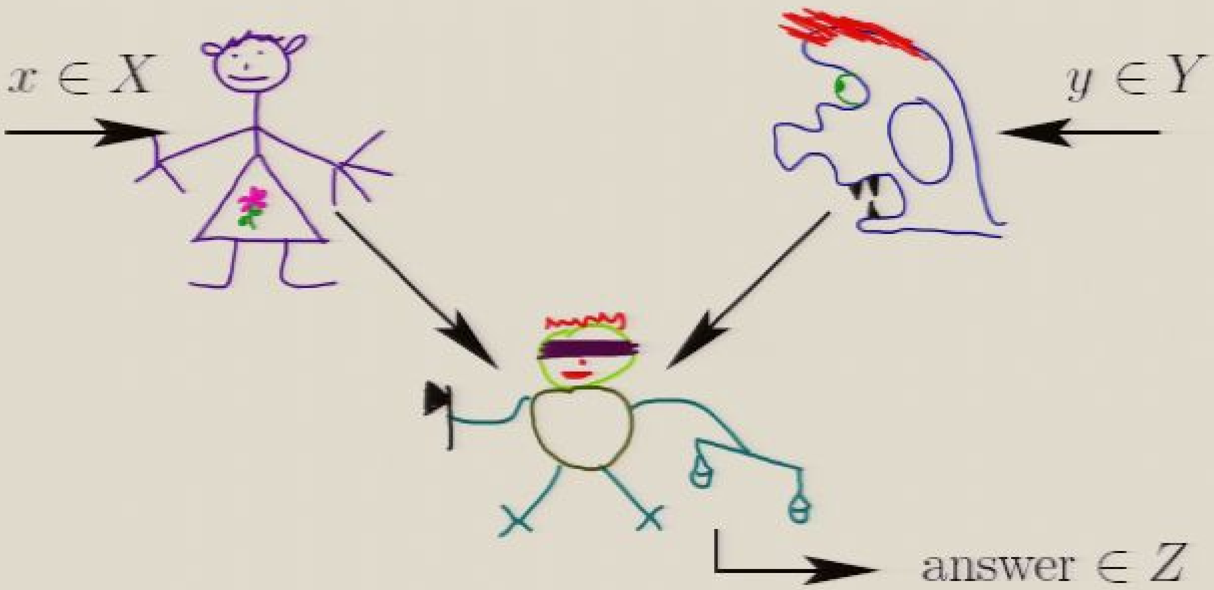

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Communication Cost

- A communication protocol is a description of the behavior of Alice, Bob and the referee.
- For a relation $P \subset X \times Y \times Z$, its communication cost (in a given model) is the minimum cost of a protocol which produces a good answer with probability at least $2/3$, for every possible $x \in X$ and $y \in Y$.

Our Results

- We state and prove a Quantum State Indistinguishability Lemma.
- We exhibit a relation P_1, using the Lemma we prove that $R^{\|,pub}(P_1) \in O(\log n)$ but $Q^{\|}(P_1) \in \Omega\left(n^{1/3}\right)$.
- We exhibit a relation P_2, using the Lemma we prove that $R^{\|,pub}(P_2) \in O(\log n)$ but $Q^{\|}(P_2) \in \Omega\left((n/\log n)^{1/3}\right)$.

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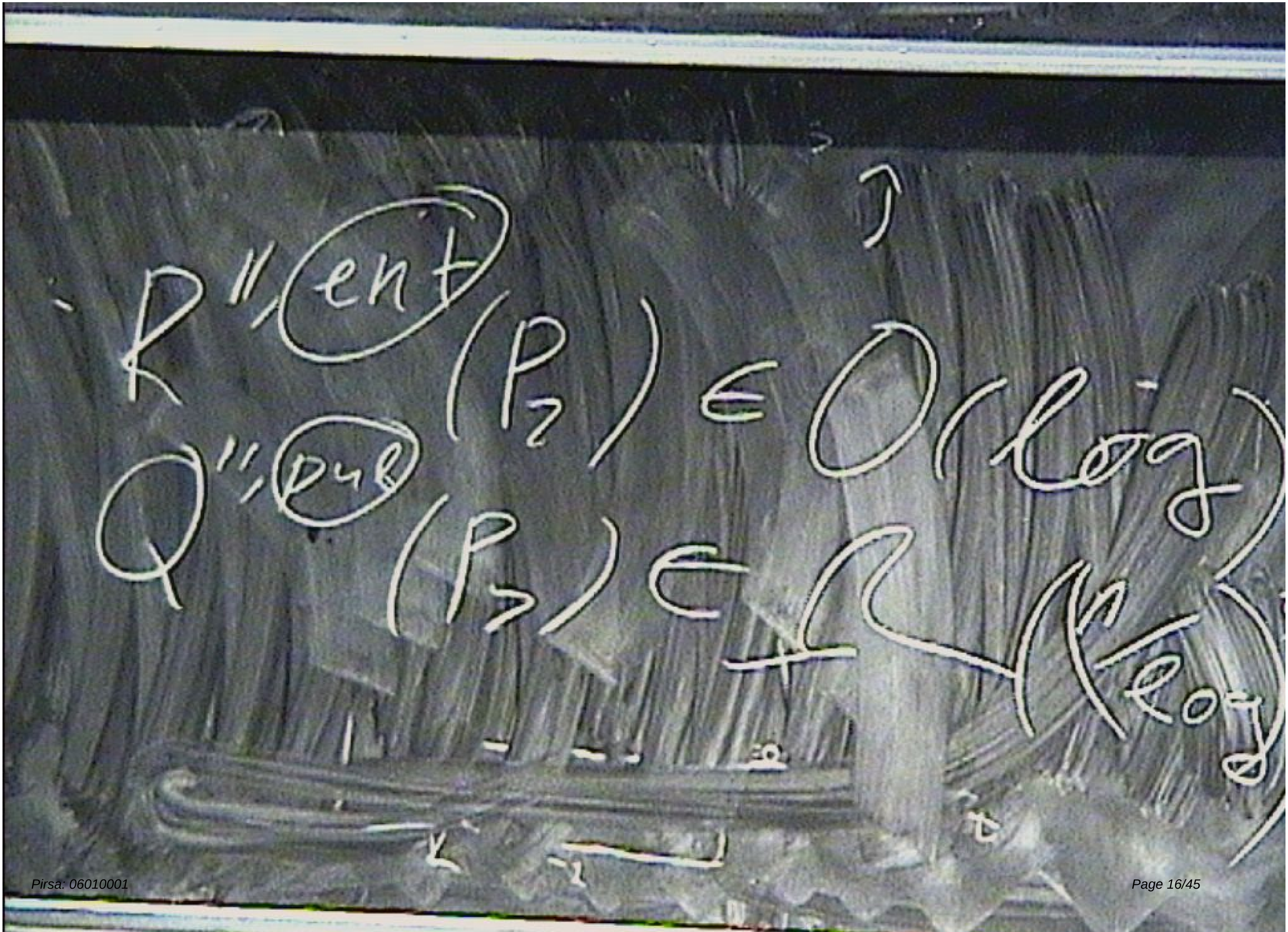
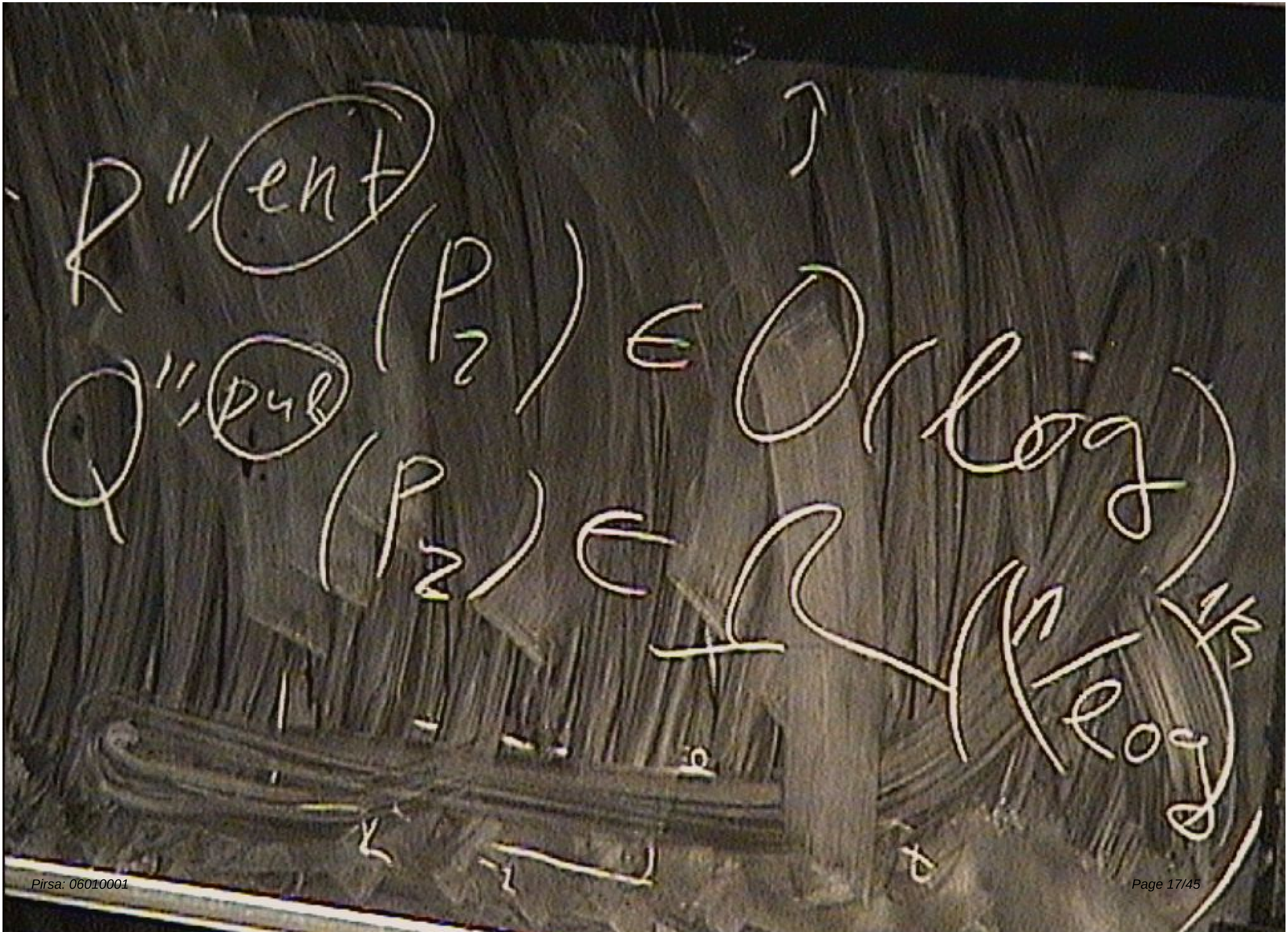
$$Q^{\parallel, pub}(P_2) \in \Omega\left(\frac{n}{\log}\right.$$


R''ent(P2) ∈ O(log)
Q''pure(P2) ∈ Ω(log)


$R^{\text{"ent"}}(P_2) \in O(\log)$
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$R^{\|,pub}$ vs. $Q^{\|}$

- **It was known before that** there exists a relation K efficiently solvable in $Q^{\|}$ but not in $R^{\|,pub}$ (due to Bar-Yossef, Jayram and Kerenidis).
- **We show that** there exists a relation P_1 efficiently solvable in $R^{\|,pub}$ but not in $Q^{\|}$ (in fact, our protocol for P_1 in $R^{\|,pub}$ is 0-error).
- Therefore, $R^{\|,pub}$ and $Q^{\|}$ are incomparable.
- Yao has shown that any protocol from $R^{\|,pub}$ can be simulated in $Q^{\|}$ by some exponentially longer protocol.

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Our Relation P_1

Input: (Alice) $x \in \{0,1\}^n$, (Bob) $y, s \in \{0,1\}^n$ with $|s| = n/2$;
Output: Any (i, x_i, y_i) s.t. $s_i = 1$.

0-error Protocol for P_1 in $R^{\|,pub}$

For a randomly chosen $i \in \{1, \dots, n\}$:

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By repeating the protocol 2 times in parallel, the error can be reduced to 1/4.

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0-error Protocol for P_1 in $R^{||,pub}$

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P_1 Is Hard for $Q^{\|}$

Using the Indistinguishability Lemma we show that

$$Q^{\|}(P_1) \in \Omega\left(n^{1/3}\right).$$

$R^{\|,ent}$ vs. $Q^{\|,pub}$

- **We show that** there exists a relation P_2 efficiently solvable in $R^{\|,ent}$ but not in $Q^{\|,pub}$.
- In fact, our protocol for P_2 in $R^{\|,ent}$ is exact.

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Indistinguishability Lemma

Our separations of communication models will be based on the following Indistinguishability Lemma:

Lemma

Suppose that the success probability of unrestricted distinguishing of the quantum states σ_1 and σ_2 is at most $1/2 + a$ and the answering probability for constant-error distinguishing of the states ρ_1 and ρ_2 is at most b.
Then there exists a constant ε such that the answering probability for ε-error distinguishing of the family

$$\{\sigma_1 \otimes \rho_1,\ \sigma_1 \otimes \rho_2,\ \sigma_2 \otimes \rho_1,\ \sigma_2 \otimes \rho_2\}$$

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A Counterintuitive Example

Suppose that instead of **distinguishing** the elements of $\{\sigma_i \otimes \rho_j \mid i, j \in \{1, 2\}\}$ we want **to identify i $\oplus$ j**...

- Let $|\alpha_1\rangle = |\beta_1\rangle = |0\rangle$, $|\alpha_2\rangle = |\beta_2\rangle = \sqrt{1-\delta^2}\,|0\rangle + \delta\,|1\rangle$.
- Then the success probability of unrestricted distinguishing of $|\alpha_1\rangle\langle\alpha_1|$ from $|\alpha_2\rangle\langle\alpha_2|$ is $1/2 + \Theta(\delta^2)$ and the answering probability for constant-error distinguishing of $|\beta_1\rangle\langle\beta_1|$ from $|\beta_2\rangle\langle\beta_2|$ is $\Theta(\delta^2)$.
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