Title: Exact black hole degeneracies and the topological string

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Abstract:

Exact Black Hole Degeneracies and the Topological String

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D.S., Strominger and Yin, hep-th/0505094
D.S., Strominger and Yin, hep-th/0506151
D.S. and Yin, hep-th/0508174

Motivation

Our work is motivated by a suggestive relation between 4D BPS black holes and the topological string, noticed last year by Ooguri, Strominger and Vafa.

The black holes in question come from type II string theory compactified on a Calabi-Yau threefold $\mathcal{M}$. For concreteness, let us focus on IIA compactification.

Review: BH Entropy and Attractors

The Kähler and complex structure moduli of $\mathcal{M}$ become vector and hypermultiplets, respectively, in the effective theory.

Wrapping D-branes on appropriate cycles in $\mathcal{M}$, we obtain a BPS black hole in four dimensions.

At the horizon, the vector multiplet vevs are fixed by the attractor mechanism:

$$\langle X^\Lambda \rangle = f^\Lambda(p,q)$$

The hypermultiplets, however, are unconstrained.

Therefore, the black hole entropy can depend only on the (attractor-fixed) vector multiplet vevs.

Review: Topological Strings

Now let us compare with the topological string. The topological A-model on $\mathcal{M}$ computes the "prepotential"

$$F_{top}(X^\Lambda, g_{top}) = \sum_h g_{top}^{2h-2} F_h(X^\Lambda)$$

which also depends only on the vector multiplets X^Λ.

Each term in the prepotential corresponds to a coupling

$$\delta\mathcal{L} \sim \int d^4\theta W^{2h} F_h(X^\Lambda) \sim R^2 F^{2h-2} F_h(X^\Lambda) + \ldots$$

between the vector multiplets and the SUGRA multiplet.

The OSV relation vs. "conjecture"

These higher-derivative couplings modify the area law for black hole entropy (Wald; Cardoso, de Wit & Mohaupt).

All other terms in the effective Lagrangian are believed to depend on the hypermultiplet vevs. Thus, they cannot affect the black hole entropy.

Thus, we have a precise relation between the topological string and black hole entropy!

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Thus, we have a precise relation between the topological string and black hole entropy!

OSV showed that this relation takes a very simple form

$$\mathcal{F}(p,\phi) = S_{BH}(p,q) - q_\Lambda \phi^\Lambda$$

where

$$\mathcal{F}(p,\phi) \equiv 2\mathrm{Re}\, F_{top}(X^\Lambda = p^\Lambda + \frac{i\phi^\Lambda}{\pi})\,; \qquad \phi^\Lambda \equiv \frac{\partial S_{BH}(p,q)}{\partial q^\Lambda}$$

Then they conjectured an exact relation

$$|Z_{top}(p,\phi)|^2 \sim \sum_q d_{BH}(p,q) e^{-\phi\cdot q}$$

where

$$Z_{top} \sim e^{F_{top}}\,, \qquad d_{BH} \sim e^{S_{BH}}$$

The OSV conjecture is currently imprecise. Some obvious questions:

- Why is the RHS invariant under $\phi \to \phi + 2\pi i k$ but not the LHS?
- What is the nonperturbative completion of Z_{top}?
- What is the correct BH degeneracy to use?
- Are there non-trivial measure factors on either side?

Clearly, we need a better understanding of both Z_{top} and d_{BH} in order to address these questions.

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Clearly, we need a better understanding of both Z_{top} and d_{BH} in order to address these questions.

In particular, it would be nice to have exact expressions for d_{BH}.

However, exact formulas for the degeneracies of $\mathcal{N} = 2$ black holes are hard to come by.

Black holes with more supersymmetry might be easier to understand. This motivates us to study BPS black holes in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theory.

In this talk, we will derive formulas for the exact degeneracies of large BPS black holes in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theory.

We will then exhibit the OSV transform of these degeneracies and compare with the topological string.

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Outline

- Motivation
- 4D-5D connection
- Black holes in $\mathcal{N} = 4$ string theory
- Black holes in $\mathcal{N} = 8$ string theory
- Comparison with the topological string

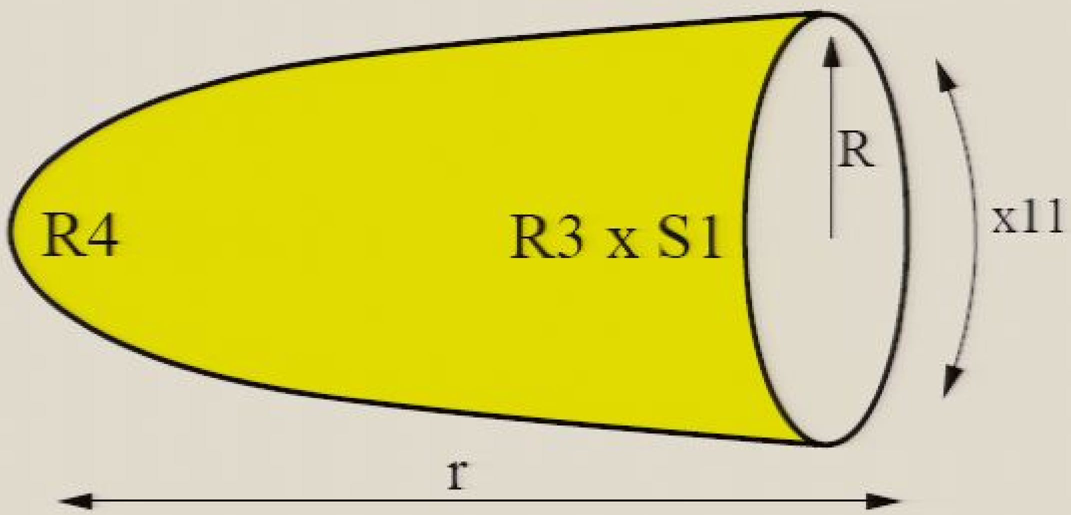
4D-5D Connection

Consider type IIA string theory compactified on a CY3 $\mathcal{M}$ with

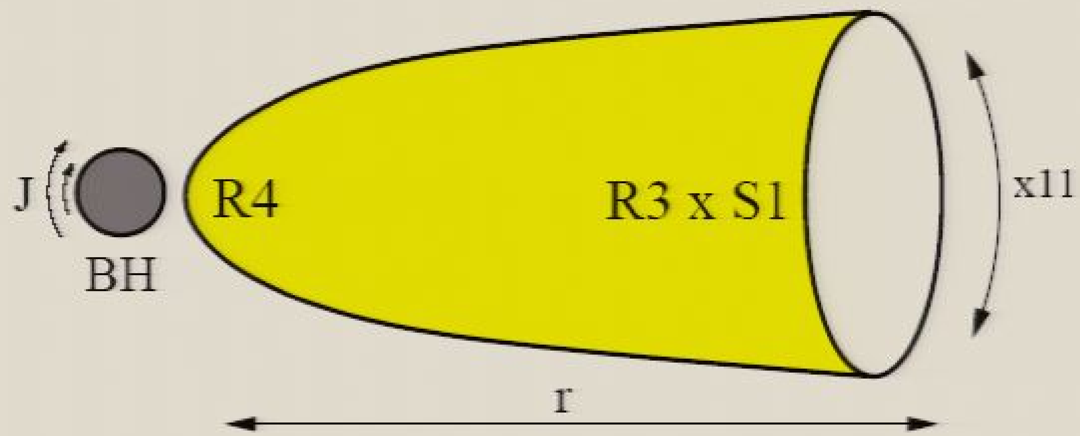
- $p^0 = 1$ D6 branes wrapped on $\mathcal{M}$
- q_A D2 branes wrapped on $\alpha^A \subset \mathcal{M}$
- q_0 D0 branes

Now lift this to M-theory.

The D6 brane lifts to Taub-NUT space:



Taub-NUT space looks like a cigar, with a tip at $r = 0$ and an asymptotic radius $R \sim g_{10}^{2/3}$.



The D2 branes lift to M2 branes sitting at the tip of Taub-NUT.

The D0 branes lift to angular momentum localized at the tip of Taub-NUT.

$$J_L = \frac{1}{2} q_0$$

So we have a spinning black hole at the tip of Taub-NUT.

Changing the Taub-NUT radius R interpolates between 5D and 4D black holes.

Since the microscopic degeneracy – appropriately defined – cannot depend on the continuous parameter R, the 4D and the 5D degeneracies must be equal (Gaiotto, Strominger and Yin):

$$d_4(p^0 = 1, q_0, q_A) = d_5(J_L = \frac{1}{2}q_0, q_A)$$

In some cases, an 5D degeneracies are known exactly. This allows us to derive the exact 4D degeneracies.

Check: classical entropy

The classical entropy of 4D D6-D2-D0 black holes with $p^0 = 1$ is given by the formula (Shmakova)

$$S_{4D} = 2\pi\sqrt{Q^3 - \frac{1}{4}q_0^2}$$

$$\left(Q^3 = (D_{ABC}y^A y^B y^C)^2,\ q_A = 3D_{ABC}y^A y^B\right)$$

Agrees with entropy of 5D spinning black hole (Kallosh, Rajaraman, Wong)

$$S_{5D} = 2\pi\sqrt{Q^3 - J_L^2}$$

confirming the identification of $J_L = q_0/2$.

String theory on $K3 \times T^2$

Type II string theory compactified on $K3 \times T^2$ preserves $\mathcal{N} = 4$ supersymmetry in four dimensions.

The U-duality group is $SL(2,\mathbb{Z}) \times SO(22,6;\mathbb{Z})$.

The electric and magnetic charge vectors $\vec{q}_e$, $\vec{q}_m$ transform in the **28** of $SO(22,6)$. We can form three $SO(22,6)$ invariants out of these charges,

$$\Omega = \begin{pmatrix} q_e^2 & q_e \cdot q_m \\ q_e \cdot q_m & q_m^2 \end{pmatrix}$$

Duality transformations

The simplest possibility consistent with U-duality is that the 4D degeneracies depend only on q_e^2, q_m^2 and $q_e \cdot q_m$,

$$d_4 = d_4\big(\Omega(p,q)\big)$$

The goal is to use the 4D-5D connection to derive an exact formula for d_4. To proceed further, we must dualize the 5D system to one whose degeneracies are known.

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To be specific, consider the following D6-D2-D0 configuration:

- A single $K3 \times T^2$ wrapped D6 brane
- q_A D2 branes wrapping 2-cycles $\alpha^A \subset K3$
- $\widetilde{q}$ D2 branes wrapping T^2
- q_0 D0 branes

The $SO(22,6)$ invariants reduce to:

$$\frac{1}{2}q_e^2 = \frac{1}{2}C^{AB}q_A q_B, \qquad q_e \cdot q_m = q_0 \qquad \frac{1}{2}q_m^2 = \widetilde{q}$$

M-theory on $K3 \times T^2$

q_A $K3$-wrapped M2 branes

$\widetilde{q}$ T^2-wrapped M2 branes

$q_0/2$ units of angular momentum

→ shrink $S^1 \subset T^2$ →

IIA on $K3 \times S^1$

q_A $K3$-wrapped D2 branes

$\widetilde{q}$ F-strings winding S^1

$q_0/2$ units of angular momentum

↓ T-duality ↓

IIB on $K3 \times S^1$

q_A D3 branes

$\widetilde{q}$ units of momentum on S^1

$q_0/2$ units of angular momentum

← U-duality ←

IIB on $K3 \times S^1$

1 D5 brane, N D1 branes

$\widetilde{q}$ units of momentum on S^1

$q_0/2$ units of angular momentum

The D1-D5 charges are related to the 4D charges by:

$$(L_0, N, J_L) = (\widetilde{q}, \frac{1}{2}C^{AB}q_A q_B + 1, \frac{1}{2}q_0) = (\frac{1}{2}q_m^2, \frac{1}{2}q_e^2 + 1, \frac{1}{2}q_e \cdot q_m)$$

We can form a 5D generating function

$$Z_{5D} = \sum_{L_0, N, J_L} d_5(L_0, N, J_L) e^{2\pi i(\rho L_0 + \sigma N + 2\nu J_L)}$$

By the 4D-5D correspondence:

$$\begin{aligned} d_4(\Omega) &= d_5(L_0 = \frac{1}{2}q_m^2,\ N = \frac{1}{2}q_e^2 + 1,\ J_L = \frac{1}{2}q_e \cdot q_m) \\ Z_{4D} &= e^{-2\pi i\sigma} Z_{5D} \end{aligned}$$

It remains to compute Z_{5D}. It has two parts,

$$Z_{5D} = Z_{D1D5} \times Z_{D5}$$

Exact 5D degeneracies

The microstates of the spinning D1-D5 system were first studied by BMPV, extending the work of Strominger & Vafa.

They are described by the worldvolume CFT on the N D1 branes. This is a non-linear sigma model with target space

$$\mathcal{M} = Hilb^N(K3) \approx Sym^N(K3)$$

The degeneracies of this CFT are known exactly.

The elliptic genus of $\mathcal{M}$ provides a generating function for the exact degeneracies of this CFT:

$$\begin{aligned}\chi_N(\rho,\nu) &= \mathrm{Tr}(-1)^{2J_L+2J_R} e^{2\pi i(\rho L_0+2\nu J_L)} \\ &= \sum_{L_0,J_L} d_5'(L_0,N,J_L)e^{2\pi i(\rho L_0+2\nu J_L)}\end{aligned}$$

The full partition function of the D1D5 system is then

$$Z_{D1D5}(\rho,\sigma,\nu) \equiv \sum_N e^{2\pi i\sigma N}\chi_N(\rho,\nu)$$

The generating function $Z_{D1D5}(\rho, \sigma, \nu)$ was studied by DMVV. They derived a product formula for it:

$$Z_{D1D5} = \prod_{k \geq 0, l > 0, m \in \mathbb{Z}} (1 - e^{2\pi i(k\rho + l\sigma + m\nu)})^{-c(4kl - m^2)}$$

where $c(4kl - m^2)$ are the elliptic genus coefficients for a single $K3$:

$$\chi_1(\rho, \nu) = \chi(K3) = \sum_{h \geq 0, m \in \mathbb{Z}} c(4h - m^2) e^{2\pi i(h\rho + m\nu)}$$

We also need the contribution Z_{D5} from the single D5 brane on $K3 \times S^1$. This was calculated by AGNT:

$$\begin{aligned} Z_{D5}(\rho,\nu) &= e^{-2\pi i\rho}(e^{\pi i\nu} - e^{-\pi i\nu})^{-2}\prod_{k\geq 1}(1-e^{2\pi i(k\rho+\nu)})^{-2} \\ &\qquad \times(1-e^{2\pi i(k\rho-\nu)})^{-2}(1-e^{2\pi ik\rho})^{-20} \\ &= e^{-2\pi i(\rho+\nu)}\prod_{\substack{k>0,\, l=0,\, m\in\mathbb{Z} \\ k=l=0, m<0}}(1-e^{2\pi i(k\rho+l\sigma+m\nu)})^{-c(4kl-m^2)} \end{aligned}$$

Here we have used the fact that $c(0) = 20$, $c(-1) = 2$ and $c(n) = 0$ for $n < -1$.

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Exact 4D degeneracy

These two product formulas combine nicely to give

$$Z_{5D} = Z_{D1D5} Z_{D5} = e^{-2\pi i(\rho+\nu)} \prod_{(k,l,m)>0} \left(1 - e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{-c(4kl-m^2)}$$

where $(k, l, m) > 0$ means $k,\ l \geq 0$, $m \in \mathbb{Z}$, and if $k = l = 0$, only $m < 0$.

Therefore, the 4D partition function is:

$$Z_{4D} = e^{-2\pi i(\rho+\sigma+\nu)} \prod_{(k,l,m)>0} \left(1 - e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{-c(4kl-m^2)}$$

- Our answer exactly reproduces an old conjecture of Dijkgraaf, Verlinde and Verlinde!

- $\Phi_{10} \equiv Z_{4D}^{-1}$ is the unique weight 10 automorphic form of $Sp(2,\mathbb{Z})$.

- As shown by DVV,

$$d_4(\Omega(p,q)) = \oint d\rho d\sigma d\nu \, e^{-\pi i (q_m^2 \rho + q_e^2 \sigma + 2(q_e \cdot q_m)\nu)} Z_{4D}(\rho,\sigma,\nu)$$

has the correct asymptotics

$$d_4 \sim e^{S_{cl}} = e^{\pi \sqrt{q_e^2 q_m^2 - (q_e \cdot q_m)^2}}$$

and is manifestly U-duality invariant.

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String theory on T^6

Type II string theory compactified on T^6 preserves $\mathcal{N} = 8$ supersymmetry in four dimensions.

The U-duality group is E_7. There is a unique quartic charge invariant: the "Cremmer-Julia invariant."

In terms of the $\mathcal{N} = 4$ charges, it is

$$J = q_e^2 q_m^2 - (q_e \cdot q_m)^2$$

The simplest possibility consistent with U-duality is that the 4D degeneracies depend only on J.

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We consider the same D6-D2-D0 brane configuration as before. The same chain of dualities yields a D1-D5 system on $T^4 \times S^1$.

The charges are:

$$(L_0, N, J_L) = \left(\widetilde{q}, \frac{1}{2}C^{AB}q_A q_B, \frac{1}{2}q_0\right) = \left(\frac{1}{2}q_m^2, \frac{1}{2}q_e^2, \frac{1}{2}q_e \cdot q_m\right)$$

Using the 4D-5D correspondence,

$$d_4 = d_5\left(L_0 = \frac{1}{2}q_m^2,\ N = \frac{1}{2}q_e^2,\ J_L = \frac{1}{2}q_e \cdot q_m\right)$$

However, the elliptic genus of $Sym^N(T^4)$ vanishes because of the extra supersymmetry.

So we are forced to consider instead the modified elliptic genus,

$$\begin{aligned}\widehat{\chi}_N(\rho,\nu) &= \mathrm{Tr}(2J_R)^2(-1)^{2J_L+2J_R}e^{2\pi i(\rho L_0+2\nu J_L)} \\ &= \sum_{L_0,J_L} d_5(L_0,N,J_L)e^{2\pi i(\rho L_0+2\nu J_L)}\end{aligned}$$

The generating function for the 5D degeneracies is then

$$Z_{5D}=Z_{D1D5}=\sum_N e^{2\pi i N\sigma}\widehat{\chi}_N(\rho,\nu)$$

This was computed by Maldacena, Moore & Strominger:

$$Z_{5D} = \sum_{s,k,n,\ell} s e^{2\pi i s(k\rho + n\sigma + \ell\nu)} \widehat{c}(4nk - \ell^2)$$

where the coefficients are defined by the Fourier expansion of $\widehat{\chi}_1$

$$-\eta(\rho)^{-6}\theta_1(\nu|\rho)^2 = \sum_{k,\ell} \widehat{c}(4k - \ell^2) e^{2\pi i(k\rho + \ell\nu)}$$

Notice that when (k, n, ℓ) are coprime, there is only one term that contributes to the sum Z_{5D}.

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Notice that when (k, n, ℓ) are coprime, there is only one term that contributes to the sum Z_{5D}.

Therefore,

$$\begin{aligned}\widehat{c}(4nk-\ell^2)\big|_{(k,n,\ell)coprime} &= d_5\left(L_0=n, N=k, J_L=\frac{1}{2}\ell\right)\\ &= d_4\left(q_m^2=2n,\, q_e^2=2k,\, q_e\cdot q_m=\ell\right)\end{aligned}$$

The combination $4nk-\ell^2$ is precisely the Cremmer-Julia invariant $J=q_e^2q_m^2-(q_e\cdot q_m)^2$!

Clearly, we can get every value of J by an appropriate choice of coprime charges. Therefore we conclude that

$$d_4 = d_4(J) = \widehat{c}(J)$$

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A short calculation starting from

$$-\eta(\rho)^{-6}\theta_1(\nu|\rho)^2 = \sum_{k,\ell} d_4(4k-\ell^2)e^{2\pi i(k\rho+\ell\nu)}$$

yields the generating function for the 4D degeneracies:

$$Z_{4D} = \sum_J d_4(J)e^{2\pi i\rho J} = \eta(4\rho)^{-6}\sum_{m\in\mathbb{Z}} e^{2\pi i\rho m^2} = \frac{\theta_3(2\rho)}{\eta(4\rho)^6}$$

Summary

We have derived the exact degeneracies of dyonic black holes in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ string theory. They are

$$d_4(\Omega(p,q)) = \oint \frac{e^{-\pi i(q_m^2\rho + q_e^2\sigma + 2(q_e\cdot q_m)\nu)}}{\Phi_{10}(\rho,\sigma,\nu)}$$

and

$$d_4(J(p,q)) = \oint e^{-2\pi i\rho J(p,q)} \frac{\theta_3(2\rho)}{\eta(4\rho)^6}$$

respectively.

The first result precisely reproduces an old conjecture of DVV. The second result is entirely new.

OSV Transforms

Finally, let us briefly discuss the OSV transforms of d_4 and their topological string intepretation:

$$Z_{\mathcal{N}=4}(p,\phi) = \sum_q d_4\big(\Omega(p,q)\big)e^{-\phi\cdot q}$$

$$Z_{\mathcal{N}=8}(p,\phi) = \sum_q d_4\big(J(p,q)\big)e^{-\phi\cdot q}$$

For technical reasons, we will restrict to the case $p^0 = 0$.

The result of the OSV transform is:

$$Z_{\mathcal{N}=4} = \sum_{\phi \to \phi + 2\pi i k} \left(\frac{e^{\frac{D_{ABC}(\pi^2 p^A p^B p^C - 3\phi^A \phi^B p^C)}{\phi^0}}}{\eta(\frac{\pi p^1 - i\phi^1}{i\phi^0})^{24} \eta(\frac{\pi p^1 + i\phi^1}{i\phi^0})^{24}} \right) \times$$

$$\left[\frac{D_{ABC} p^A p^B p^C}{(\phi^0)^3} - \frac{24 p^1}{\pi i (\phi^0)^3} \frac{\eta'(\frac{\pi p^1 - i\phi^1}{i\phi^0})}{\eta(\frac{\pi p^1 - i\phi^1}{i\phi^0})} - \frac{24 p^1}{\pi i (\phi^0)^3} \frac{\eta'(\frac{\pi p^1 + i\phi^1}{i\phi^0})}{\eta(\frac{\pi p^1 + i\phi^1}{i\phi^0})} \right] + \ldots$$

and

$$Z_{\mathcal{N}=8} = \sum_{\phi \to \phi + 2\pi i k} \left(e^{\frac{D_{ABC}(\pi^2 p^A p^B p^C - 3\phi^A \phi^B p^C)}{\phi^0}} \right) \left(\frac{D_{ABC} p^A p^B p^C}{(\phi^0)^{11}} \right) + \ldots$$

Does this mess have an interpretation in terms of the topological string?

The topological string on T^6 and $K3 \times T^2$

The topological string on $\mathcal{M} = T^6$, $K3 \times T^2$ is only nontrivial at tree-level and one-loop

$$Z_{top} = e^{F_{top}}, \quad F_{top} = \begin{cases} \frac{(2\pi i)^3 D_{ABC} t^A t^B t^C}{g_{top}^2} - 24 \log \eta(t^1) & (K3 \times T^2) \\ \frac{(2\pi i)^3 D_{ABC} t^A t^B t^C}{g_{top}^2} & (T^6) \end{cases}$$

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But many questions remain...

Open questions

- Can we think of the extra factors as measure factors for the topological string "wavefunction"? How do they fit with the holomorphic anomaly equations? (Verlinde)

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- Understand better the nonperturbative corrections. Do they contain information about baby universes and black hole fragmentation?

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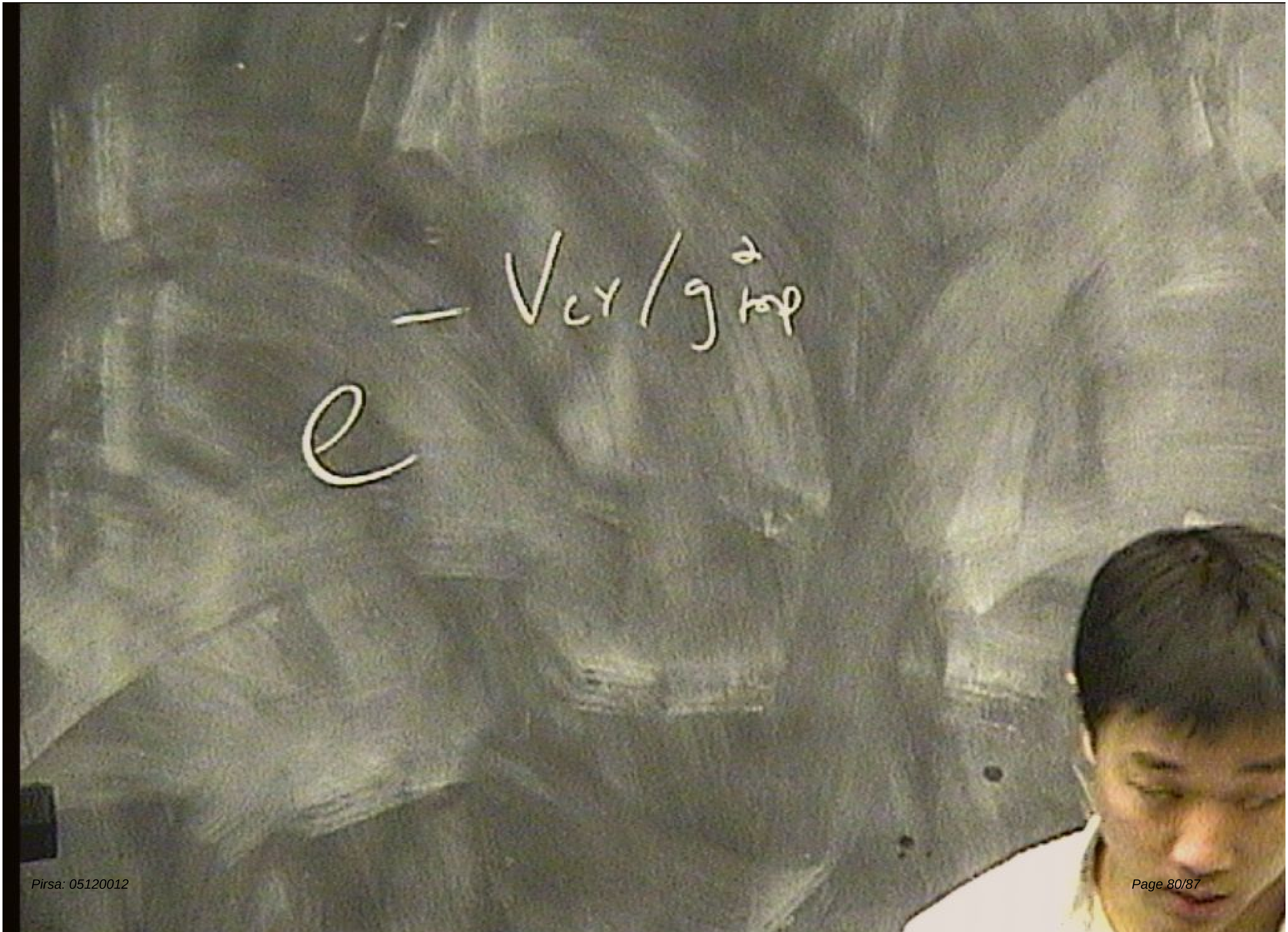
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The elliptic genus of $\mathcal{M}$ provides a generating function for the exact degeneracies of this CFT:

$$\begin{aligned}\chi_N(\rho,\nu) &= \mathrm{Tr}(-1)^{2J_L+2J_R}e^{2\pi i(\rho L_0+2\nu J_L)} \\ &= \sum_{L_0,J_L} d_5'(L_0,N,J_L)e^{2\pi i(\rho L_0+2\nu J_L)}\end{aligned}$$

The full partition function of the D1D5 system is then

$$Z_{D1D5}(\rho,\sigma,\nu) \equiv \sum_N e^{2\pi i\sigma N}\chi_N(\rho,\nu)$$