Title: Twistor space, amplitudes and unitarity methods

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Abstract:

Twistor space, Amplitudes and Unitarity Methods

Seminar at the Perimeter Institute

Niels Emil Jannik Bjerrum-Bohr

Includes work in collaboration with

Zvi Bern, Steven Bidder, Lance Dixon, David Dunbar, Harald Ita, Warren Perkins and K. Risager

$N = 1$ Supersymmetric One-loop Amplitudes and the Holomorphic Anomaly of Unitarity Cuts,
Phys.Lett.B606:189-201,2005, [hep-th/0410296]
S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar

Twistor space structure of the box coefficients of N = 1 one-loop amplitudes,
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JHEP 0505, 056 (2005) [hep-th/0501137].
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One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries,
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Six-point one-loop $N = 8$ supergravity NMHV amplitudes and their IR behaviour,
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N. E. J. Bjerrum-Bohr, D. C. Dunbar and H. Ita

Recursive Calculation of One-Loop QCD Integral Coefficients,
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MHV-vertices for Gravity Amplitudes,
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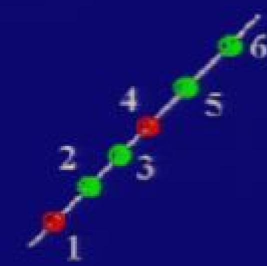
Twistor space

N=4 super Yang-Mills is dual to a string theory in twistor space? (Witten)

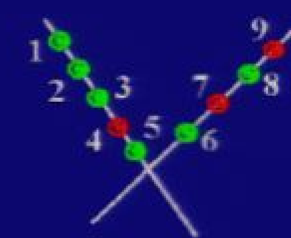
Topological String Theory with twistor target space ← ? → Perturbative N=4 super Yang-Mills

Recently huge activity and progress in the calculation of amplitudes

- New very efficient techniques for calculating amplitudes



(Witten)



Amplitudes : lines in twistor space?

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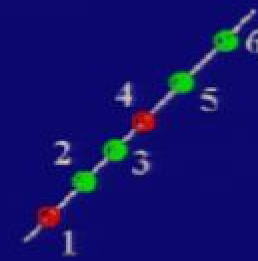
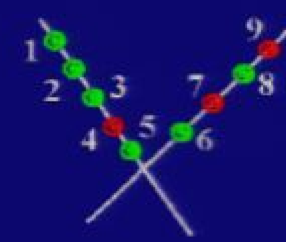


(Witten)

Key feature to rapid progress:

New inspiration from twistor space

\+ expressions for amplitudes (simple)



Amplitudes : lines in twistor space?

Outline of talk

- Yang-Mills field theories
  - Amplitudes for supersymmetric multiplets

Motivation:

The LHC collider approaching

Data from collider experiments

New physics?

- Supersymmetry?
- Higgs?
- ...

Need: Perturbative calculations!

- New insights (twistor space?)

- Gravity
  - Amplitudes in quantum gravity theories

Motivation:

Theoretical interests

- New insights?

Twistor space structure for gravity amplitudes?

Twistor Space and helicity formalism

Amplitudes

- Feynman diagrams :

  Not very ideal!

- Number of diagrams :

  Grows very rapidly with many legs!

Amplitudes :

- Momentum vectors : $(p_i \cdot p_j)$
- External polarisation tensors : $(p_i \cdot e_j)$, $(e_i \cdot e_j)$

Possible Simplifications

Specifying the external polarisation tensors:

- Spinor-helicity formalism

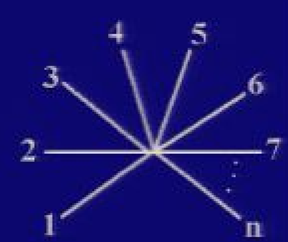
$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

- Colour ordering

Recursion

Loop amplitudes

$Tr(T_1\ T_2\ ..\ T_n)$



- Unitarity
- Supersymmetric decomposition

Amplitudes

- Feynman diagrams :
  Not very ideal!
- Number of diagrams :
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Traditional Feynman diagram expressions:
Very complicated!!

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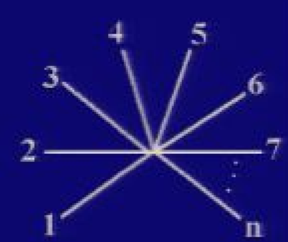

Helicity states formalism

SO(3,1) locally isomorph $SL(2) \times SL(2)$

- Spinor products :

$$\langle \lambda_1, \lambda_2 \rangle = -\epsilon^{ab} \lambda_{1a} \lambda_{2b}, \quad [\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$$

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- Polarisation : other matter types

gravitons, fermions, scalars etc

8

Scattering amplitudes in D=4

- Amplitudes can hence be expressed completely specifying
  - The external helicities

    e.g. : $A(1^+, 2^-, 3^+, 4^+, \ldots)$
  - The spinor variables

$$\lambda, \tilde{\lambda}$$

Spinor Helicity formalism

Note on notation

- We will use the notation:

$$s_{i,i+1} \equiv K_{i,i+1}^2 = (p_i + p_{i+1})^2 \quad \text{and} \quad t_{i,j} \equiv K_{i,j}^2 = (p_i + \ldots + p_j)^2$$

$$[k|K_{i,j}|l\rangle \equiv \langle k^+|\not{K}_{i,j}|l^+\rangle \equiv \langle l^-|\not{K}_{i,j}|k^-\rangle \equiv \langle l|K_{i,j}|k] \equiv \sum_{a=i}^{j} [k\,a]\,\langle a\,l\rangle$$

$$\langle k|K_{i,j}K_{m,n}|l\rangle \equiv \langle k^-|\not{K}_{i,j}\not{K}_{m,n}|l^+\rangle \equiv \sum_{a=i}^{j}\sum_{b=m}^{n} \langle k\,a\rangle\,[a\,b]\,\langle b\,l\rangle$$

$$\langle k|[q,K]|l\rangle \equiv \langle k|qK|l\rangle - \langle k|Kq|l\rangle$$

MHV-amplitudes

- Tree amplitudes : (n) same helicities vanishes

$$A^{tree}(1^+,2^+,3^+,4^+,..) = 0$$

- Tree amplitudes : (n-1) same helicities vanishes

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11

Twistor space

- Transformation of amplitudes into twistor space (Penrose)

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}} \to \mu_{\dot{a}}$$

- In metric signature (+ + - -) :

2D Fourier transform

$$\Phi(\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{a}}\tilde{\lambda}_{\dot{a}})\Phi(\tilde{\lambda})$$

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  Tree amplitudes on degenerate algebraic curves

  Degree : number of negative helicities

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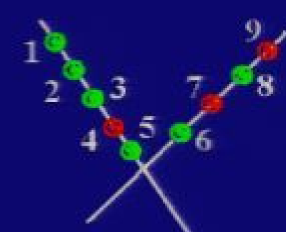

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CSW expansion of amplitudes

- In the CSW-construction : off-shell MHV-amplitudes building blocks for more complicated amplitude expressions (Cachazo, Svrcek and Witten)
- Off-shell MHV vertices:

$$V(1,2,\ldots,r^-,\ldots,s^-,\ldots,n-1,Q^+) = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{n-1} \langle \lambda_i, \lambda_{i+1} \rangle \langle \lambda_{n-1}, \lambda_Q \rangle \langle \lambda_Q, \lambda_1 \rangle}$$

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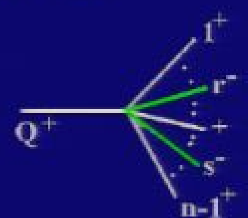

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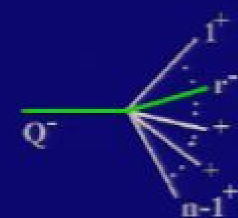

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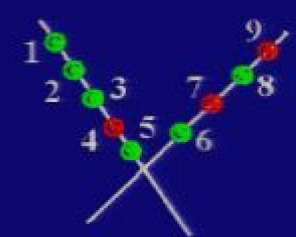

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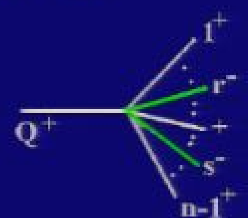

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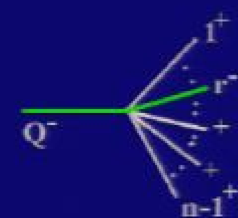
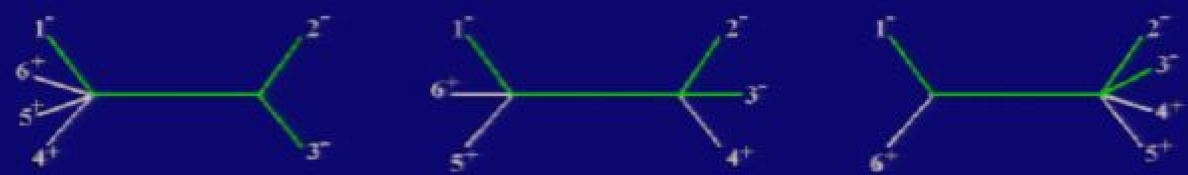


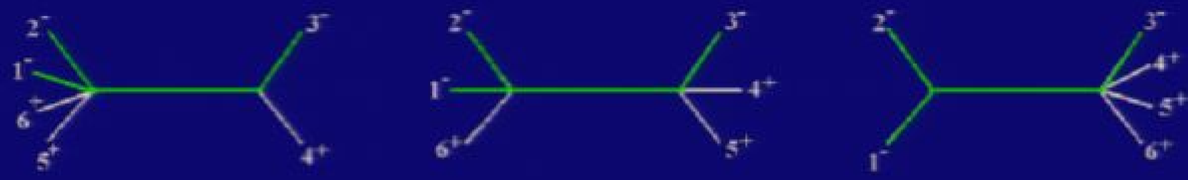
CSW expansion of amplitudes

- Example of $A_6(1^-,2^-,3^-,4^+,5^+,6^+)$

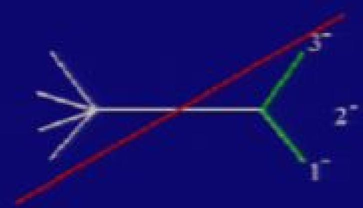
(1)(23)



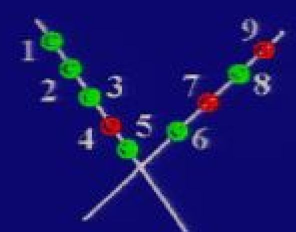
(12)(3)



(2)(13)



NMHV(3−):



15

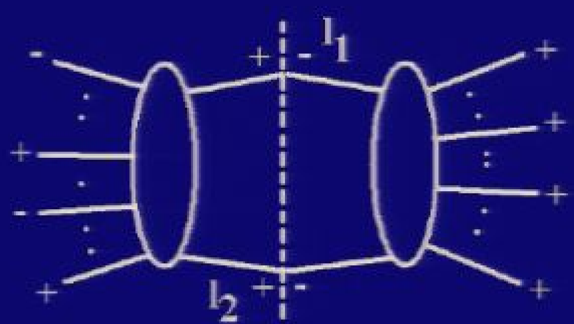
Loop amplitudes

Unitarity cuts

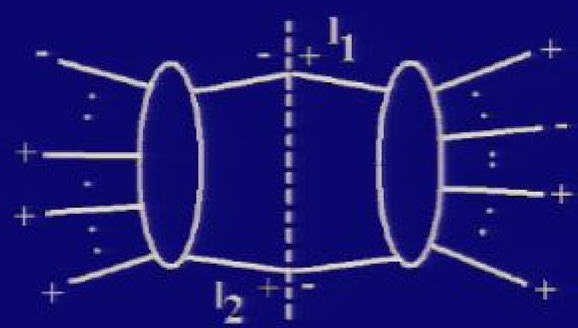
- Unitarity methods are building on the cut equation

$$C_{i,i+1,\ldots,j} = \mathrm{Im}_{(p_i+p_{i+1}+\ldots+p_j)^2>0} A^{1-\mathrm{loop}}$$

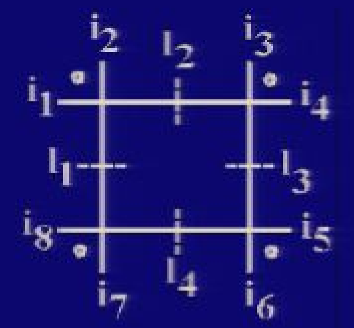
Singlet:



Non-singlet:



Quadruple cuts



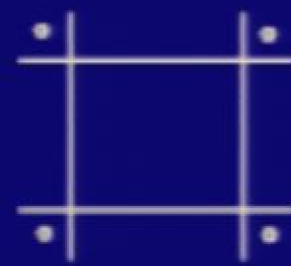
(Cachazo, Britto, Feng)

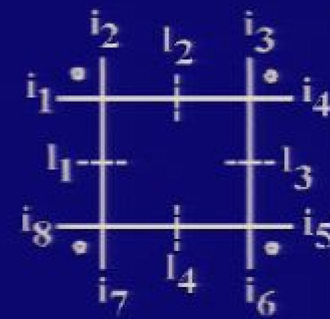
$$C_{i,\ldots,j} \equiv \frac{i}{2}\int dLIPS\Big[A^{\mathrm{tree}}(\ell_1, i, \ldots, j, \ell_2) \times A^{\mathrm{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)\Big].$$

Supersymmetric decomposition

- Super-symmetry $\Rightarrow$ simplicity
  - N=4 : scalar boxes

    (Cachazo Britto Feng;
    Bern, Dixon, Del Duca, Kosower;
    Bern, Dixon, Kosower)





(Algebra)

  - N=1 : scalar boxes,
    triangles,
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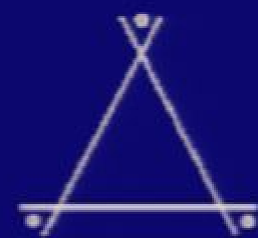
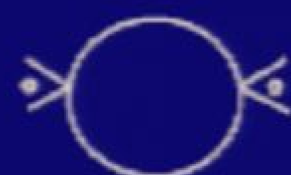


Basis



(Bern, Dixon, Dunbar, Kosower)

(but no extra rational pieces)

Supersymmetric decomposition

The three types of multiplets are:

$$A_n^{\mathcal{N}=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]}$$
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- QCD amplitudes for gluons :

□ Combine:

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- Super-symmetry $\Rightarrow$ simplicity
  - N=4 : scalar boxes

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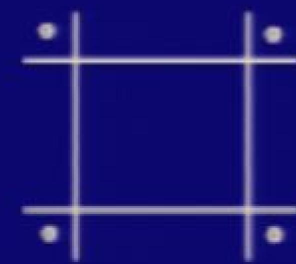

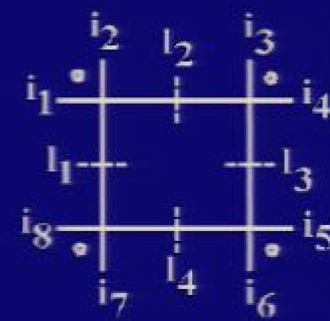

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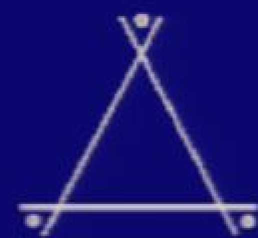

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Recursion for loops

Avoid integration?

Loop amplitudes via recursion?

Proof for the BCFW tree relations (Britto, Cachazo, Feng, Witten) +

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Inspiration :

Recursion : finite loop amplitudes (Bern, Dixon, Kosower).

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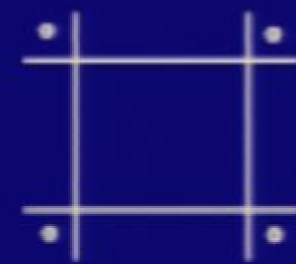

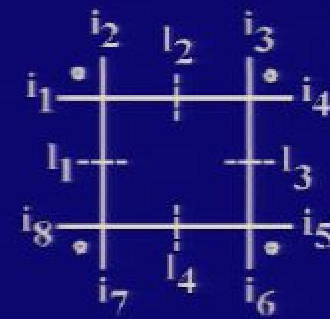

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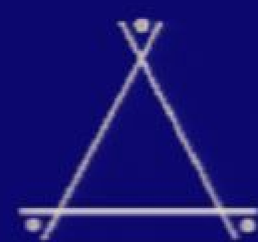

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- Feedback from progress for calculating loop amplitudes into trees.

- IR divergent terms $\Rightarrow$ compact tree expressions

$$A_n^{\text{loop}} = \sum_i \frac{(-s_{i,i+1})^{-\epsilon}}{\epsilon^2} A_n^{\text{tree}} + O(\epsilon)$$

- Compact result for tree amplitudes (Bern, Dixon and Kosower; Roiban Spradlin and Volovich)

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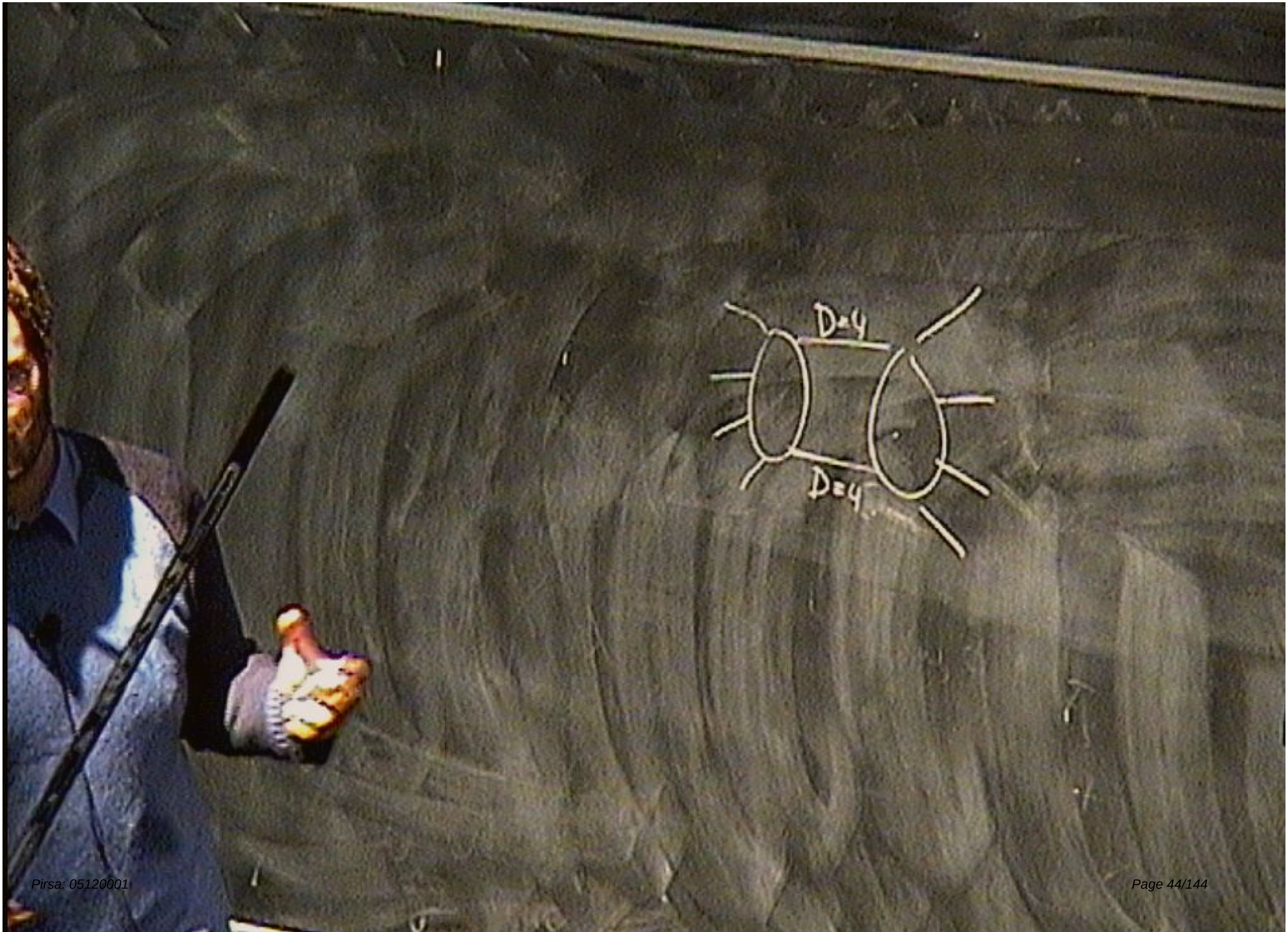
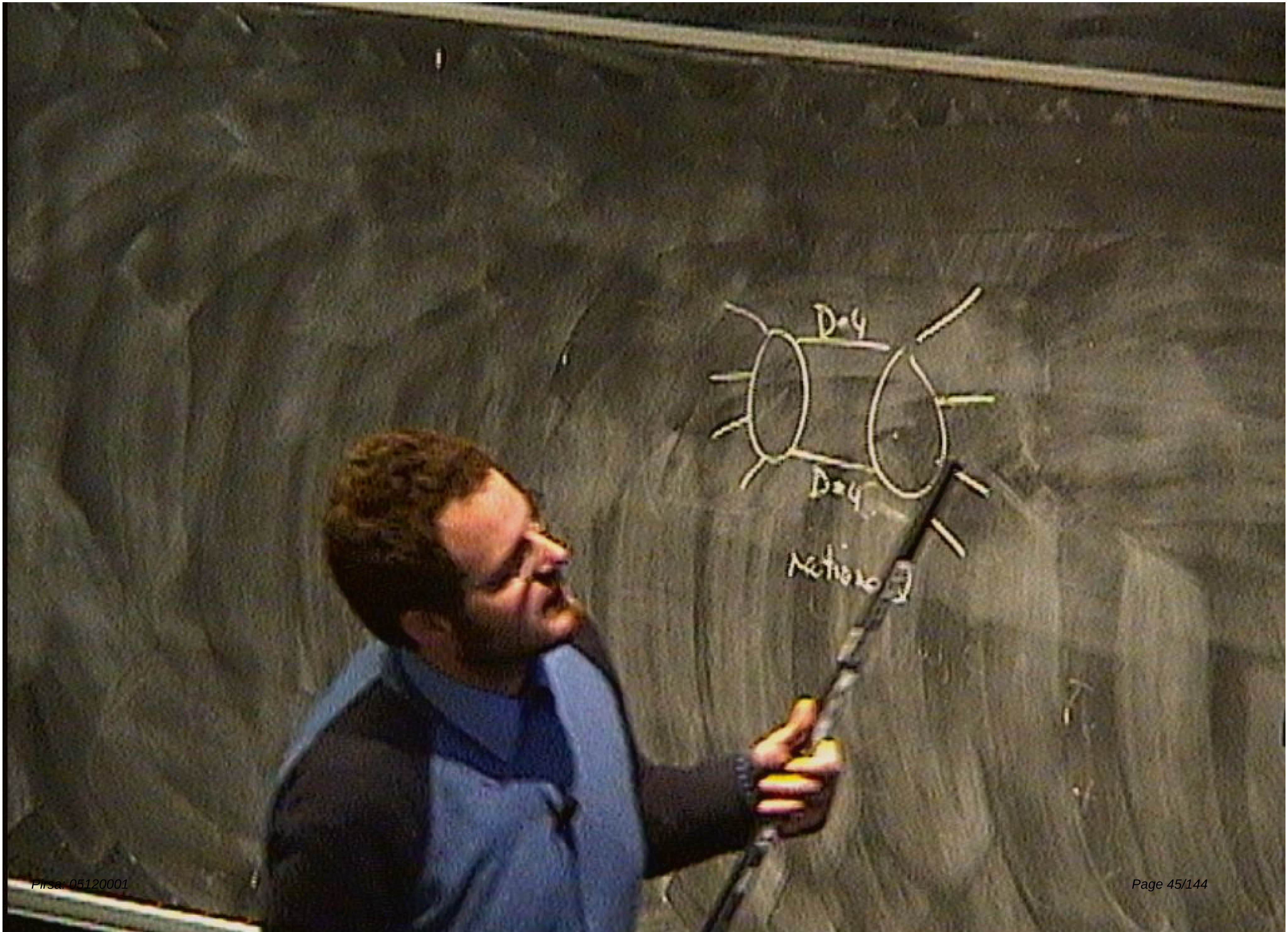

D=4
D=4


D=4
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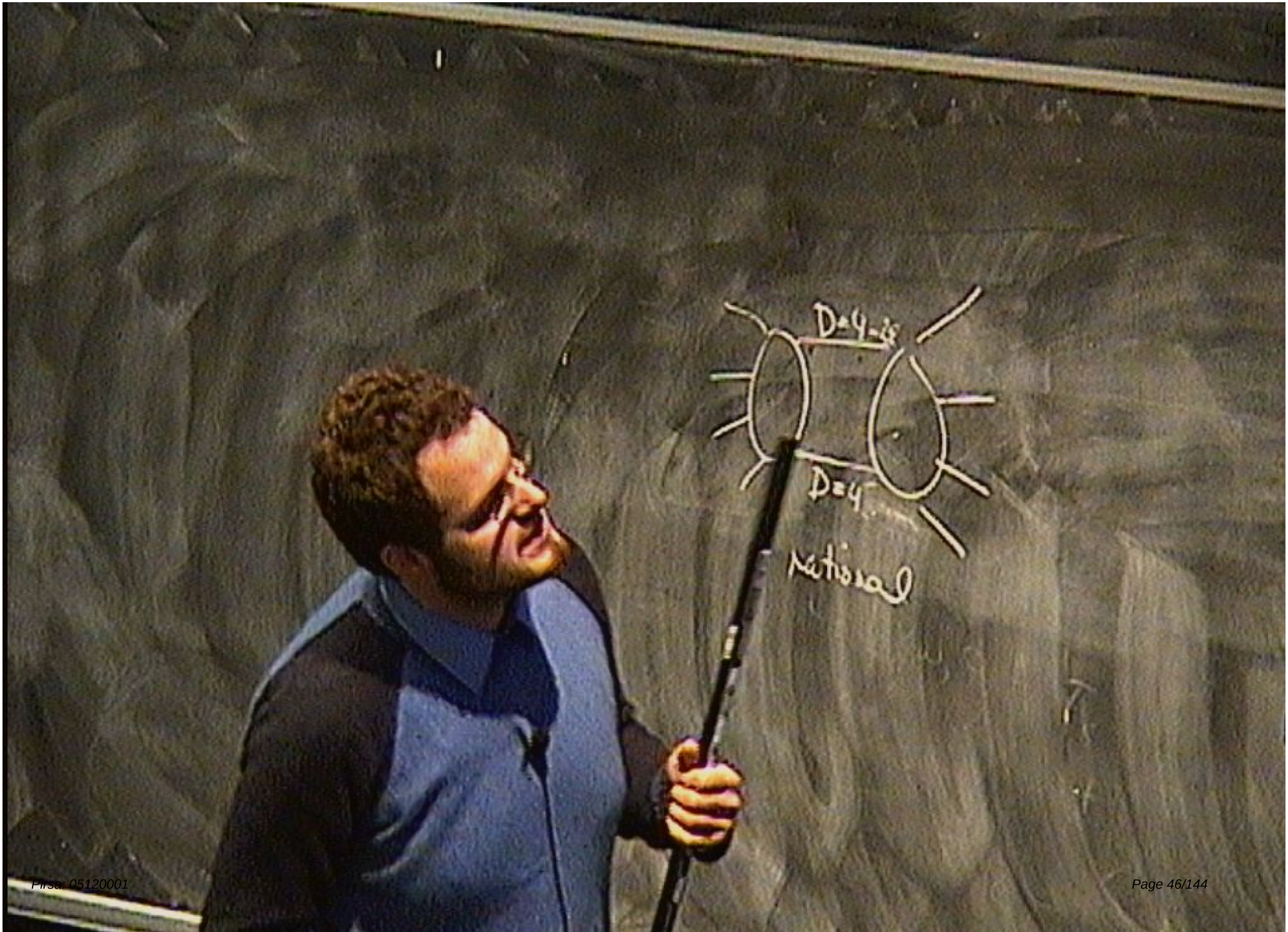
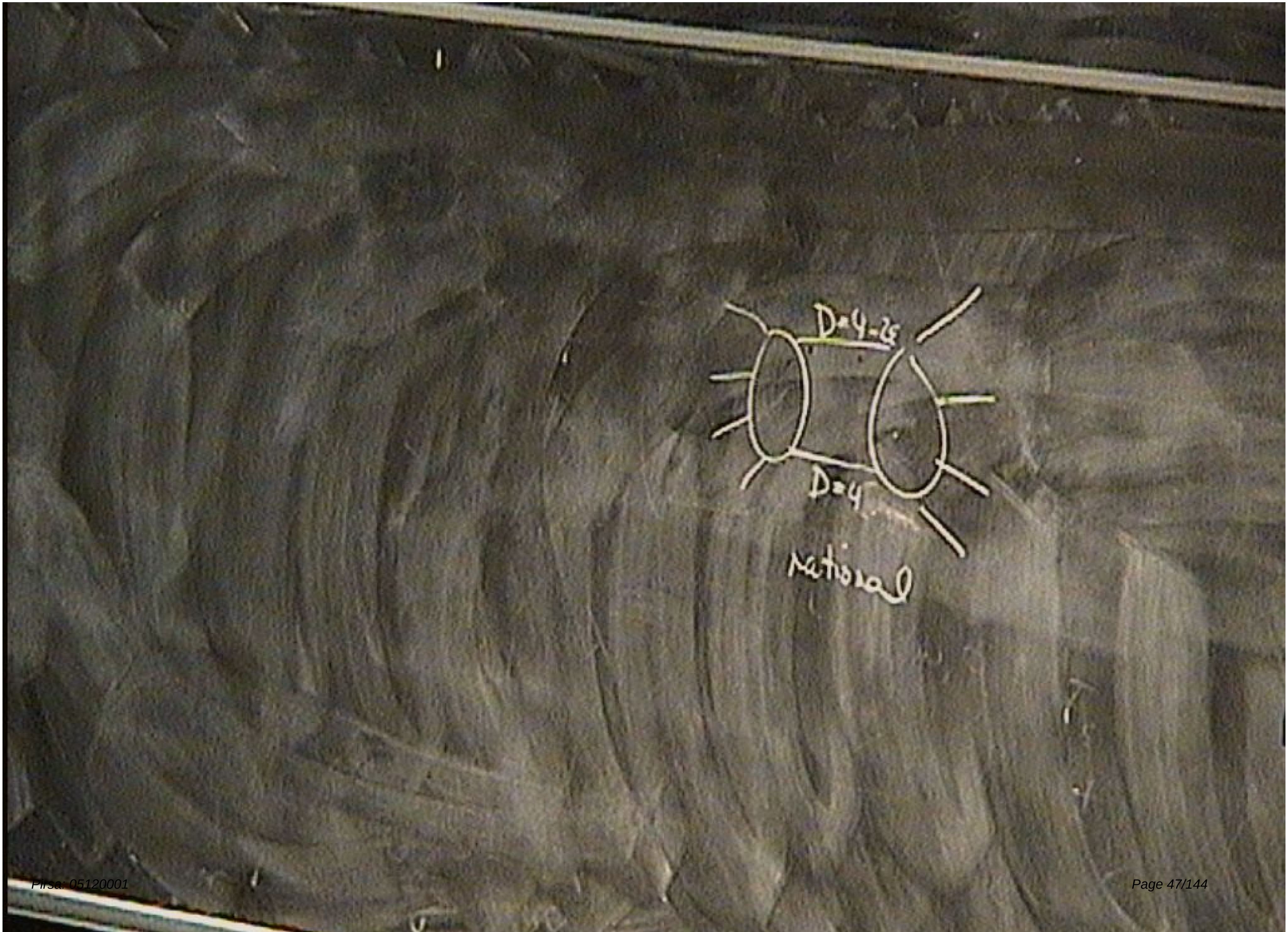

D=4
rational


D=4-2ε
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BCFW Recursion for trees

$$A(p_1, p_2, \ldots, p_n)$$

Complex momentum space!!

- Shift of the spinors :

$$\tilde\lambda_a \to \tilde\lambda_a + z\tilde\lambda_b \qquad p_a(z) = \lambda_a\tilde\lambda_a + z\lambda_a\tilde\lambda_b$$

$$\lambda_b \to \lambda_b - z\lambda_a \qquad p_b(z) = \lambda_b\tilde\lambda_b - z\lambda_a\tilde\lambda_b$$

a and b will remain on-shell even after shift

- The amplitude transforms as

$$A(p_1, p_2, \ldots, p_n) \to A(p_1, p_2, \ldots, p_a(z), \ldots, p_b(z), \ldots, p_n) \equiv A(z)$$

- We can now evaluate the contour integral over A(z)

$$\frac{1}{2\pi i}\oint \frac{dz}{z} A(z) = C_\infty = A(0) + \sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

22

BCWF Recursion for trees

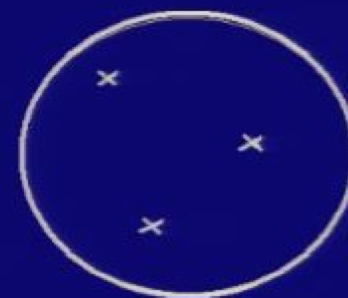
Given that:

- A(z) vanish for $z \to \infty$
- A(z) is a rational function
- A(z) has simple poles

$(C_\infty = 0)$

(Britto, Cachazo, Feng, Witten)

$$A(0) \;=\; -\sum_{\alpha} \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}$$



- Residues : Determined by factorization properties

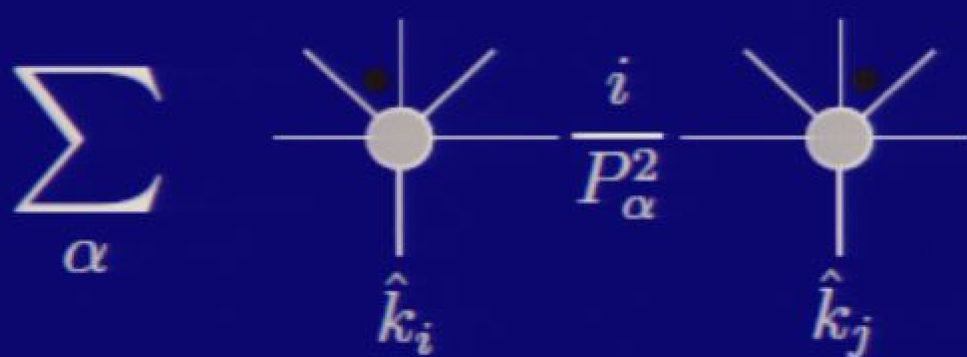
- Tree amplitude : Factorise in product of tree amplitudes

$$A \;\xrightarrow{K_{i,j}^2 \to 0}\; A(k_i, \cdots, k_j, K_{i,j}) \times \frac{i}{K_{i,j}^2} \times A(k_{j+1}, \cdots, k_{i-1}, -K_{i,j})$$

Recursion for tree amplitudes

- Tree-level : No other factorizations in complex plane

$$A(0) = \sum_{\alpha,h} A^{h}_{n-m_\alpha+1}(z_\alpha)\frac{i}{K_\alpha^2}A^{-h}_{m_\alpha+1}(z_\alpha)$$

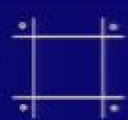
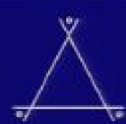


Recursion for loops

- Universal factorization properties for loop amplitudes (Bern, Chalmers)

  Structure of singularities
  - Simple poles (physical)
  - Spurious pole singularities (non-physical)

- Basis of integrals :

$A = \sum_i C_i F_i = \sum c_i$  $+ \sum t_i$  $+ \sum b_i$ 

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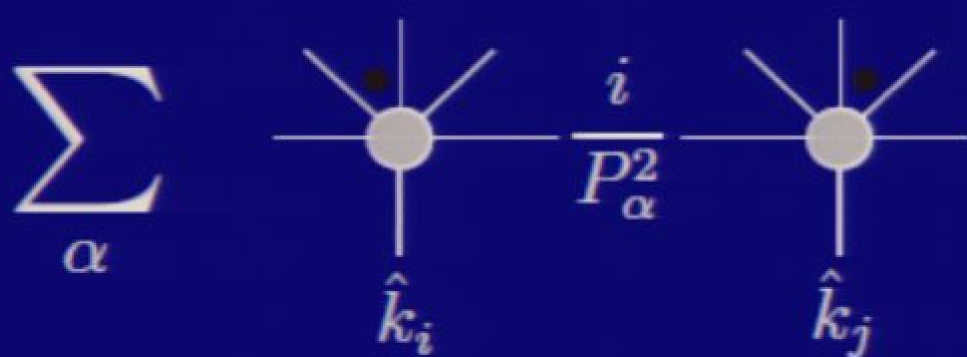

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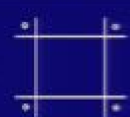
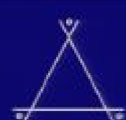
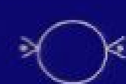
$$A = \sum_i C_i F_i = \sum c_i$$  $$+ \sum t_i$$  $$+ \sum b_i$$ 

(Bern, Dixon, Dunbar, Kosower)

- Starting expressions for recursion!

Factorization of loop amplitudes

- Avoid : Logarithmic branch cuts
- Want to construct recursion relations :
    - not for full amplitudes but for coefficients of the integral functions

- Problems:
    – Complete set of poles in amplitudes not present in coefficients.
    – Spurious non-physical singularities can interfere in recursion.

Factorization of loop amplitudes

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$$A_n^{\text{one-loop}} \xrightarrow{K^2_{i,i+m-1} \to 0} \sum_{h=\pm}\Bigg[A_{m+1}^{\text{one-loop}}(\ldots, K^h_{i,i+m-1}, \ldots) \frac{i}{K^2_{i,i+m-1}} A^{\text{tree}}_{n-m+1}(\ldots, (-K_{i,i+m-1})^{-h}, \ldots)$$

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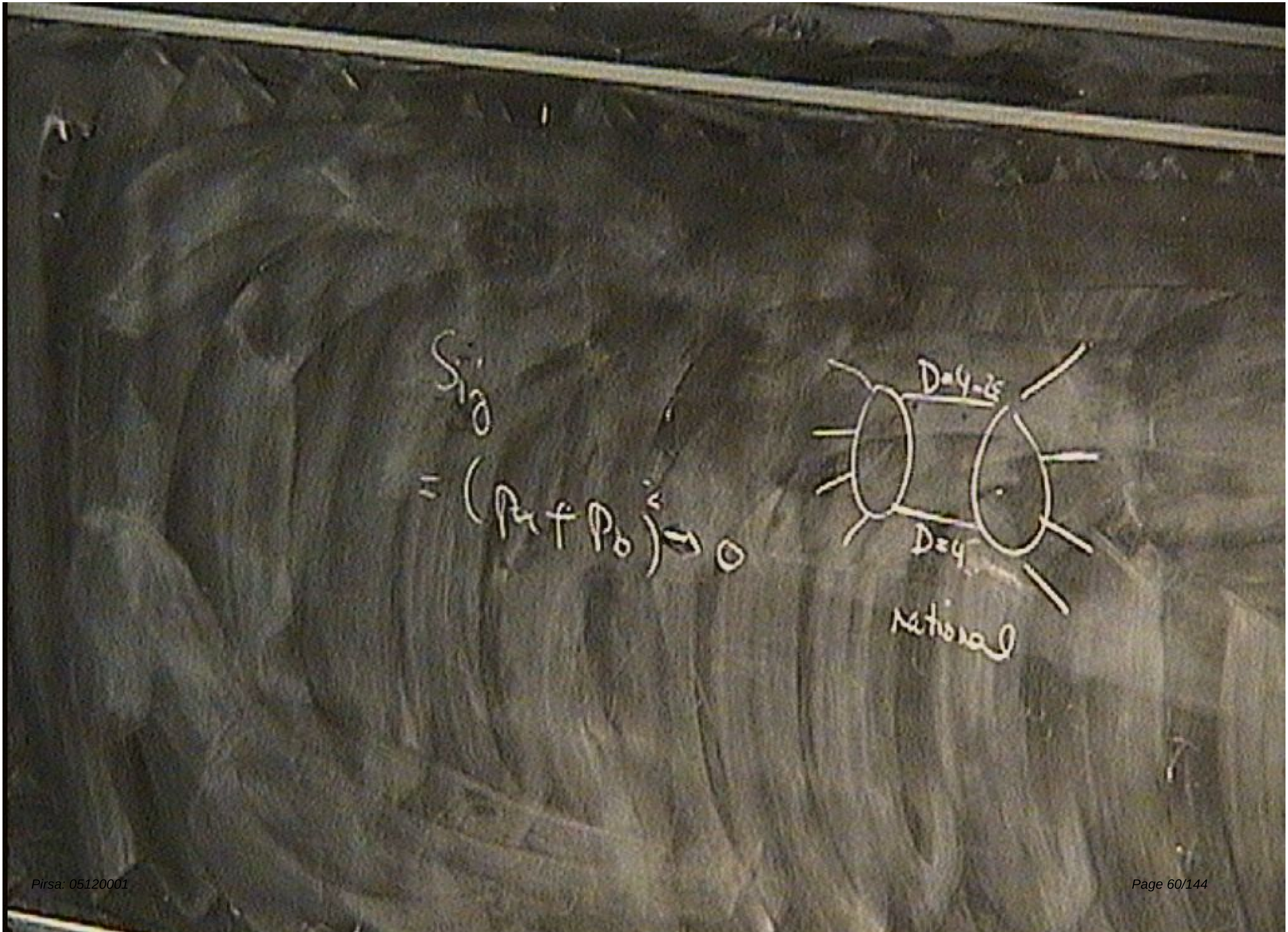

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- Now we consider the behaviour of a single term in this expansion

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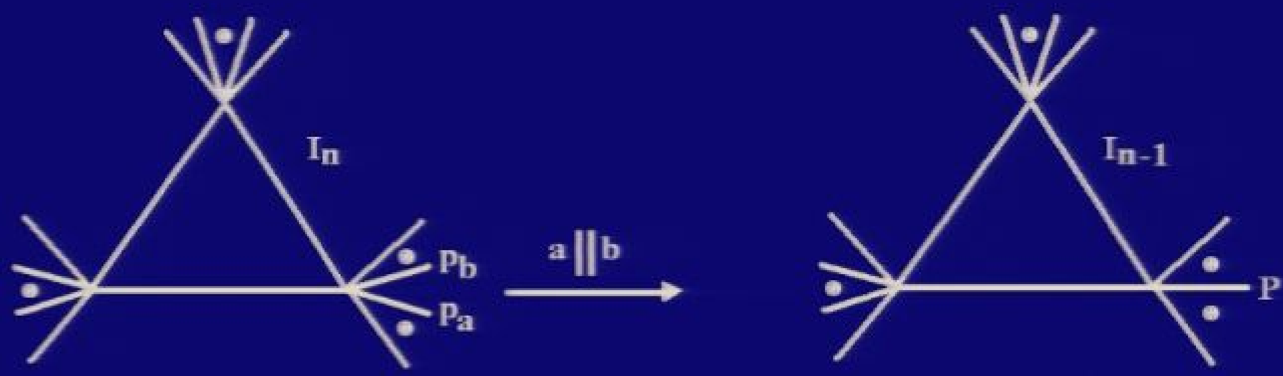

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Recursion for simple case

- Coefficients behave as if they were tree amplitudes

$$c_{i,n} \xrightarrow{K^2 \to 0} \sum_h A^h_{n-m+1} \frac{i}{K^2} c^{-h}_{i,m+1}$$

- Assuming well behaved denominators

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Recursion for simple case

- Coefficients behave as if they were tree amplitudes

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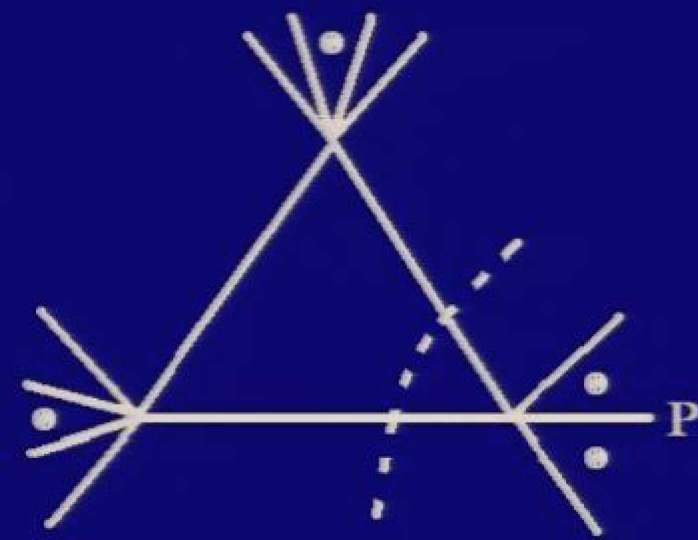
Recursion for simple case

- Sufficient conditions:
  - Shifted tree cluster well behaved as $z \to \infty$
  - All loop momenta dependent kinematic poles unmodified by shift.



Simple case

- Simple example of recursion :

  Integral coefficients for split *n*-point amplitudes

$$A(1^-, 2^-, \cdots l^-, (l+1)^+, \cdots, n^+)$$

Sufficient criteria : satisfied!

Example 5pt → 6pt

- Check that criteria for recursion is satisfied:
  - Look at cutted diagram
  - The shifted tree amplitude is

$$A^{\text{tree}}(3^-, 4^+, 5^+, \ell_s^+, \ell_s'^-; z) = i \frac{\langle 3\, \ell_s \rangle^2 \langle 3\, \ell_s' \rangle^2}{\langle 3\, 4 \rangle \left(\langle 4\, 5 \rangle + z \langle 3\, 5 \rangle \right) \langle 5\, \ell_s \rangle \langle \ell_s\, \ell_s' \rangle \langle \ell_s'\, 3 \rangle}$$

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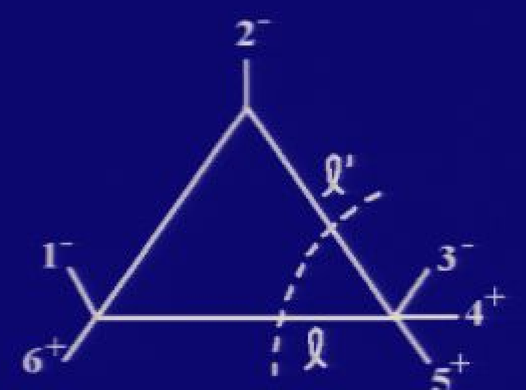

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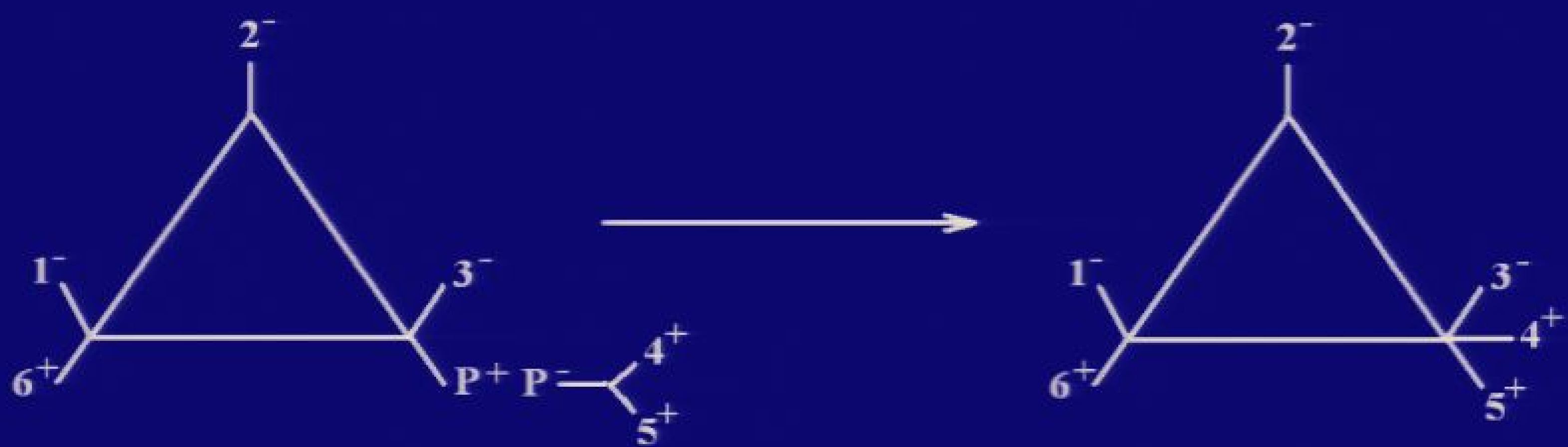
Example 5pt → 6pt

- Both criteria immediately satisfied
  - z-dependence factors out of the integrant
  - The shifted coefficient times the integral vanishes as $|z| \to \infty$

$$c_6(6^+,1^-;2^-;3^-,4^+,5^+) \xrightarrow{4\|5} \text{Split}^{\text{tree}}_{-}(4^+,5^+)\, c_5(6^+,1^-;2^-;3^-,(4+5)^+)$$

Example 5pt $\to$ 6pt

- Can consider recursion of 5pt coefficient into a 6pt coefficient :



Example 5pt → 6pt

- Integral functions:

$$\mathrm{K}_0(r) = \frac{1}{\epsilon(1-2\epsilon)}(-r)^{-\epsilon} = \Big(-\log(-r) + 2 + \frac{1}{\epsilon}\Big) + \mathcal{O}(\epsilon)$$

$$\mathrm{L}_0(r) = \frac{\log(r)}{1-r}, \qquad \mathrm{L}_2(r) = \frac{\log(r) - (r - 1/r)/2}{(1-r)^3}$$

- Five gluon (N=1) amplitude is

$$A^{\mathcal{N}=1\,\mathrm{chiral}}(1^-,2^-,3^-,4^+,5^+) = \frac{1}{2}A^{\mathrm{tree}}(1^-,2^-,3^-,4^+,5^+)\big(\mathrm{K}_0(t_{5,1}) + \mathrm{K}_0(t_{3,4})\big) + \frac{1}{2}c^{\mathcal{N}=1\,\mathrm{chiral}}(5^+,1^-;2^-;3^-,4^+)\frac{\mathrm{L}_0\big(-t_{5,1}/(-t_{5,2})\big)}{t_{5,2}}$$

Example 5pt → 6pt

- The 5pt point coefficient is

$$c^{\mathcal{N}=1\ \text{chiral}}(5^+,1^-;2^-;3^-,4^+) \equiv -i\,\frac{[4\,5]^2\ [4|[k_2,K_{5,1}]|5]}{[5\,1][1\,2][2\,3][3\,4]}$$

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$$\tilde\lambda_3 \to \tilde\lambda_3 - z\tilde\lambda_4$$
$$\lambda_4 \to \lambda_4 + z\lambda_3$$

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- Hence we can do recursion

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$$\begin{aligned} c_6 &= c(6^+, 1^-; 2^-; \hat{3}^-, \hat{K}_{45}^+)\,\frac{i}{s_{45}}\,A(\hat{4}^+, 5^+, (-\hat{K}_{45})^-) \\ &= -i\,\frac{[\hat{K}_{45}|P|6]\,[\hat{K}_{45}|\tilde{P}|6]\,\left[\hat{K}_{45}|[k_2, K_{6,2}]|6\right]}{[6\,1][1\,2][2\,\hat{3}][\hat{3}\,\hat{K}_{45}]}\,\frac{i}{s_{45}}\,\frac{(-i)\,[\hat{4}\,5]^3}{[(-\hat{K}_{45})\,\hat{4}][5\,(-\hat{K}_{45})]} \\ &= i\,\frac{\langle 3|K_{3,5}P|6]\,\langle 3|K_{3,5}\tilde{P}|6]\,\langle 3|K_{3,5}[k_2, K_{6,2}]|6]}{[2|K_{3,5}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,t_{3,5}} \end{aligned}$$

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General case and 7pt and 8pt

- The recursion : easily extendable to general cases
- 7pt: Two orders for recursion:
    - First add plus (+) leg then minus (-)
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- Different orders of recursions: Displayed as paths in helicity diagrams

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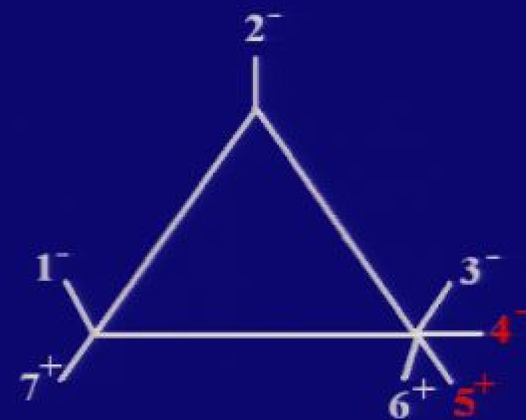

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Helicity diagrams

- The various contributions to coefficients : Organized in terms of helicity diagrams

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All paths (T_R, T_L)

$$T_{P_L,P_R} = i\,(-1)^{l+\kappa+\kappa'} \frac{\langle l|Q_R\,P\,Q_L|1\rangle\,\langle l|Q_R\,\tilde{P}\,Q_L|1\rangle\,\langle l|Q_R[k_2,K_L]Q_L|1\rangle\,\langle r,r+1\rangle}{[1\,2][2\,3]...[l-1,l]\langle l,l+1\rangle...\langle n-1,n\rangle\langle n,1\rangle}$$

$$\times \frac{\prod_{i=1}^{N}[\alpha_i-1,\alpha_i]\,\prod_{i=2}^{N}\langle\rho_i,\rho_i+1\rangle}{\langle\rho_N|K_N|\alpha_N-1]\,\prod_{i=1}^{N-1}\langle\rho_i|K_i|\alpha_i-1]\,[\alpha_i|\bar{K}_i|\rho_{i+1}+1\rangle}$$

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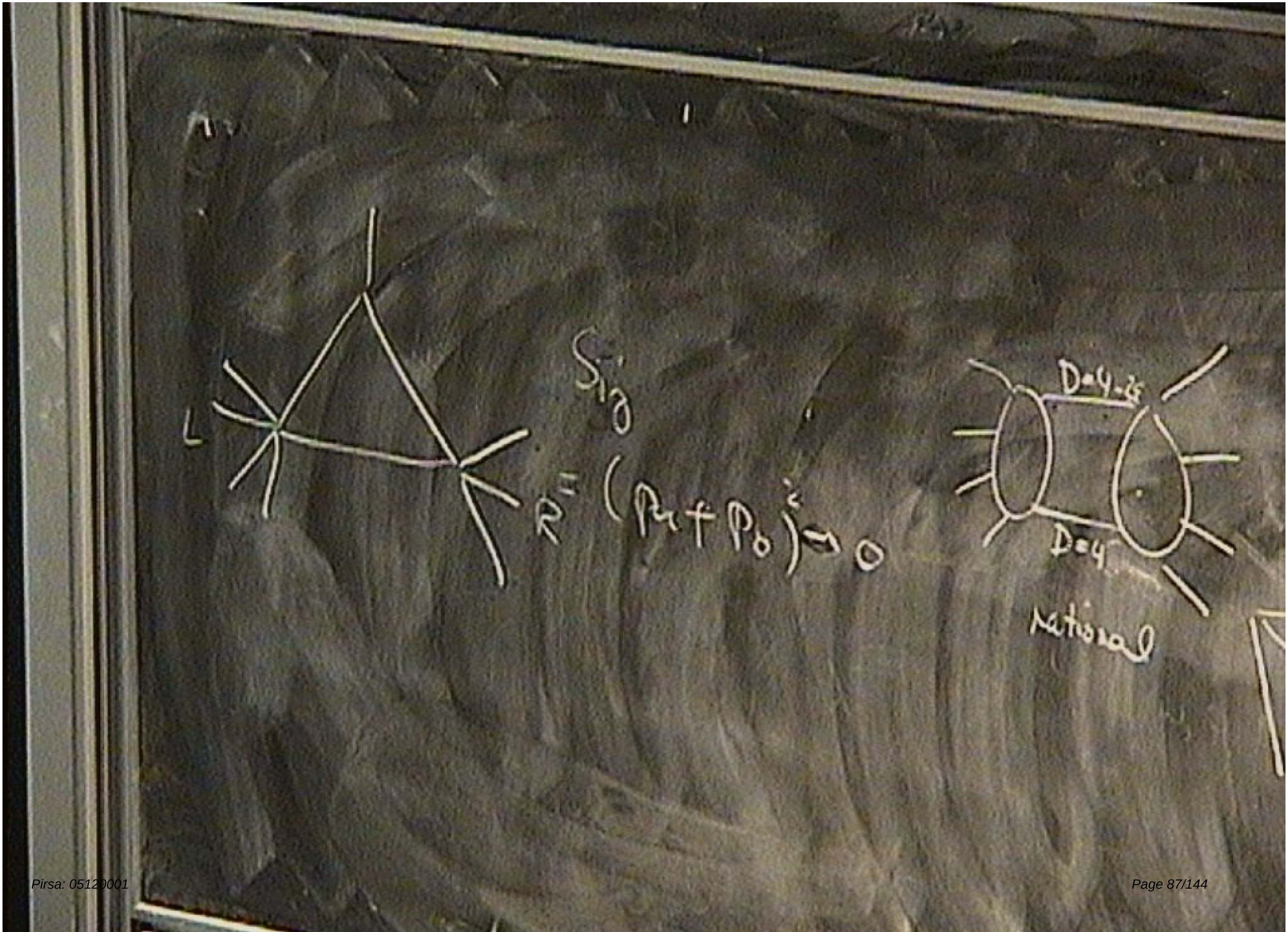
$$\times \frac{1}{K_N^2\,K_{N'}^2\,\prod_{i=1}^{N-1}\bar{K}_i^2\,K_i^2\;\prod_{j=1}^{N'-1}{K'}_j^2{\bar{K}'}_j{}^2}$$


L
D=4
rational

All paths (T_R,T_L)

- Where

$$K_i = K_{\alpha_i,\rho_i} \qquad K'_j = K_{\beta_j,\sigma_j}$$
$$\bar{K}_i = K_{\alpha_i,\rho_{i+1}} \qquad \bar{K}'_j = K_{\beta_j,\sigma_{j+1}}$$
$$K_R = K_{\alpha_\kappa,\rho_1} \qquad K'_L = K_{\beta_{\kappa'},\sigma_1}$$
$$Q_R = K_N \bar{K}_{N-1} K_{N-1} \ldots \bar{K}_1 K_1 \qquad Q_L = K'_1 \bar{K}'_1 \ldots K'_{N'-1} \bar{K}'_{N'-1} K'_{N'}$$
$$P = k_m K_R\,,1 \qquad \tilde{P} = K_R k_m\,,1$$

NMHV result

- Reproduction of result for NMHV case split-helicity amplitudes N=1

(Bidder, NEJBB, Dunbar, Perkins)

$$A_n^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) = \frac{A^{\text{tree}}}{2}\left(\mathrm{K}_0(s_{n1})+\mathrm{K}_0(s_{34})\right) - \frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_0[t_{3,r}/t_{2,r}]}{t_{2,r}}$$

$$-\frac{i}{2}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_0[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}} - \frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_0[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}}$$

NMHV result

- New result for NMHV case split-helicity amplitudes $A^{[0]}$

$$A_n^{[0]}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) = \frac{1}{3}A_n^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+)$$
$$-\frac{i}{3}\sum_{r=4}^{n-1}\hat{d}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{2,r}]}{t_{2,r}^3}-\frac{i}{3}\sum_{r=4}^{n-2}\hat{g}_{n,r}\frac{\mathrm{L}_2[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}^3}-\frac{i}{3}\sum_{r=4}^{n-2}\hat{h}_{n,r}\frac{\mathrm{L}_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}^3}$$
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(Bidder, NEJBB, Dixon, Dunbar)

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$$\hat{d}_{n,r} = \frac{\langle 3|K_{3,r}k_2|1\rangle\,\langle 3|k_2K_{2,r}|1\rangle\,\langle 3|K_{3,r}\left[k_2, K_{2,r}\right]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r+1\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle}$$

$$\hat{g}_{n,r} = \sum_{j=1}^{r-3}\frac{\langle 3|K_{3,j+3}K_{2,j+3}P|1\rangle\,\langle 3|K_{3,j+3}K_{2,j+3}\tilde{P}|1\rangle\,\langle 3|K_{3,j+3}K_{2,j+3}\left[k_{r+1}, K_{2,r}\right]|1\rangle\,\langle j+3\,j+4\rangle}{[2|K_{2,j+3}|j+3\rangle[2|K_{2,j+3}|j+4\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\ldots\langle n\,1\rangle\,t_{3,j+3}\,t_{2,j+3}}$$

$$\hat{h}_{n,r} = \hat{g}_{n,n-r+2}\Big|_{(123..n)\to(321n..4)}$$

New results

Split helicity amplitude contributions :

Pieces of QCD amplitudes

N=4 amplitudes : Calculated (Bern, Dixon, Del Duca, Kosover; Bern, Dixon, Kosower)

Split-helicity : trees (Britto,Feng,Roiban,Spradlin,Volovich)

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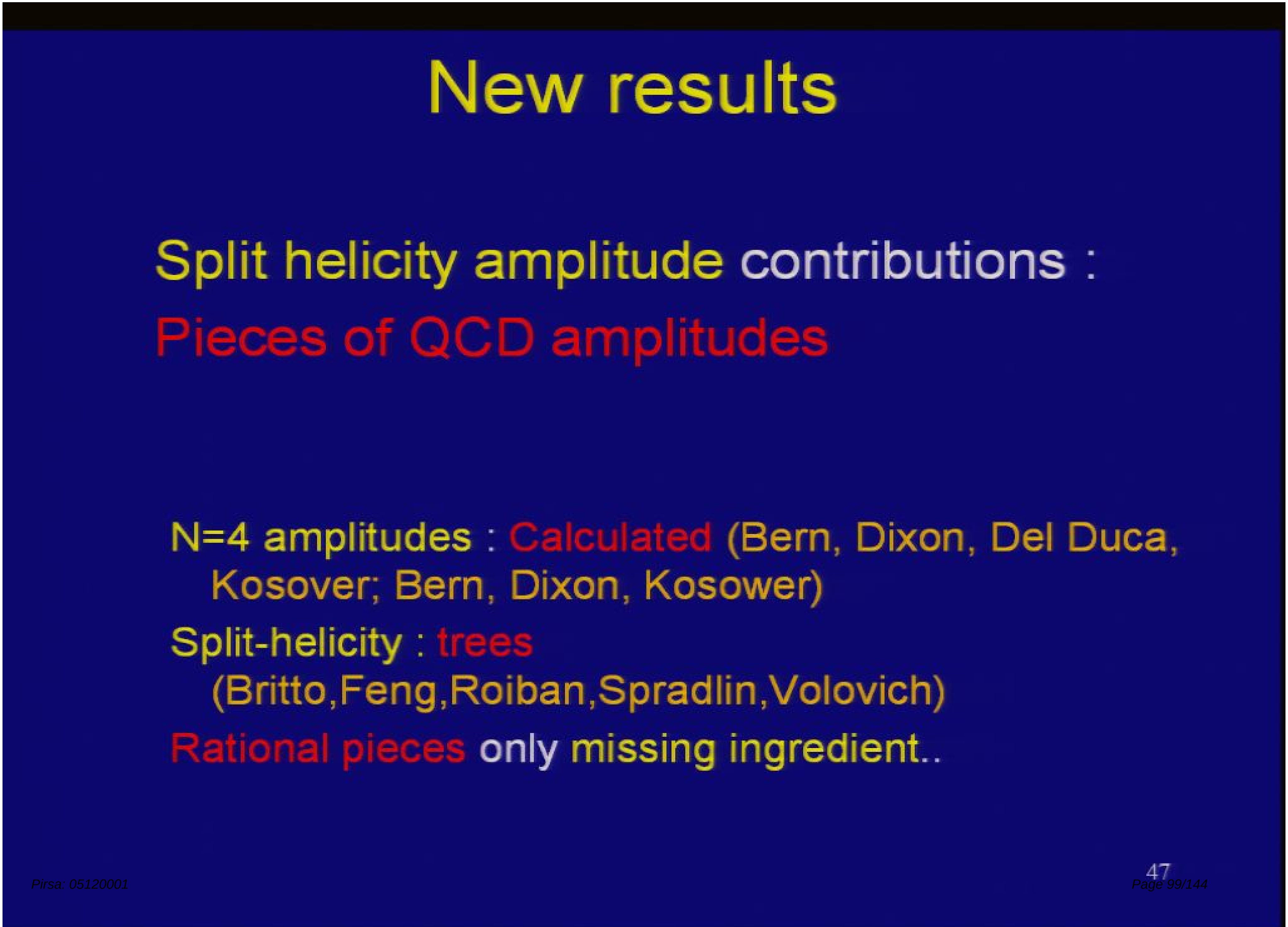
48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Gravity

Gravity Amplitudes

- Traditional methods :

  Expand Einstein-Hilbert Lagrangian

  $$\int d^D\, 1/\kappa^2 g^{1/2} R$$

  order by order in κ

  $$g_{\mu\nu} \to \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$$

$$1/\kappa^2\, R(g) \to h\partial^2 h + \kappa\, h^2\partial^2 h + \kappa^2 h^3\partial^2 h + ..$$

Gravity Amplitudes

KLT relationship (Kawai, Lewellen and Tye)

→ Gravity loop amplitudes (Bern, NEJBB and Dunbar)

The KLT relationship relates open and closed strings

$$\mathcal{M}_n^{(\text{closed string})} \sim \sum_{\Pi,\tilde{\Pi}} e^{i\pi\Phi(\Pi,\tilde{\Pi})} \mathcal{A}_n^{\text{left (open string)}}(\Pi) \times \mathcal{A}_n^{\text{right (open string)}}(\tilde{\Pi}),$$

$$M_3^{\text{tree}}(1,2,3) = -\, i A_3^{\text{tree}}(1,2,3) A_3^{\text{tree}}(1,2,3)$$

$$M_4^{\text{tree}}(1,2,3,4) = -\, i s_{12} A_4^{\text{tree}}(1,2,3,4) A_4^{\text{tree}}(1,2,4,3)$$

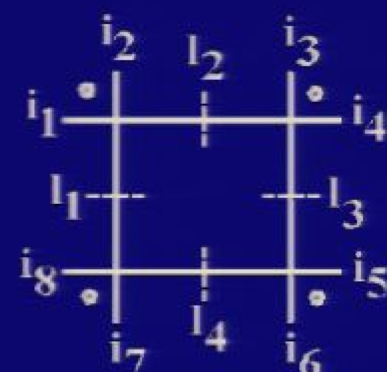
$$M_5^{\text{tree}}(1,2,3,4,5) = i s_{12} s_{34}\, A_5^{\text{tree}}(1,2,3,4,5) A_5^{\text{tree}}(2,1,4,3,5) + i s_{13} s_{24}\, A_5^{\text{tree}}(1,3,2,4,5) A_5^{\text{tree}}(3,1,4,2,5).$$

Quadruple cuts in complex momenta

- Observation : Quadruple cuts of N = 4 box coefficients

$\Rightarrow$ Coefficients of box functions by algebra

(Britto, Cachazo and Feng)



Solving the on-shell conditions

$$\ell_1^2 = 0, \quad \ell_2^2 = 0, \quad \ell_3^2 = 0, \quad \ell_4^2 = 0$$

$$\hat{c} = \frac{1}{2}\sum_{\mathcal{S}}\Big(A^{\text{tree}}(\ell_1, i_1, \ldots, i_2, \ell_2) \times A^{\text{tree}}(\ell_2, i_3, \ldots, i_4, \ell_3) \times A^{\text{tree}}(\ell_3, i_5, \ldots, i_6, \ell_4) \times A^{\text{tree}}(\ell_4, i_7, \ldots, i_8, \ell_1)\Big)$$

51

Supergravity amplitudes

$$\hat{c}_{N=8}^{(2^-3^-)(4^+5^+)6^+1^-} = \frac{1}{2} M^{\text{tree}}(2^-, 3^-, -\ell_1{}^+, \ell_3{}^+) \times M^{\text{tree}}(4^+, 5^+, -\ell_3{}^-, \ell_5{}^-)$$

$$\times M^{\text{tree}}(6^+, -\ell_5{}^+, \ell_6{}^-) \times M^{\text{tree}}(1^-, -\ell_6^+, \ell_1^-)$$

$$= \frac{1}{2} s_{23} A^{\text{tree}}(2^-, 3^-, -\ell_1{}^+, \ell_3{}^+) A^{\text{tree}}(3^-, 2^-, -\ell_1{}^+, \ell_3{}^+)$$

$$\times s_{45} A^{\text{tree}}(4^+, 5^+, -\ell_3{}^-, \ell_5{}^-) A^{\text{tree}}(5^+, 4^+, -\ell_3{}^-, \ell_5{}^-)$$

$$\times A^{\text{tree}}(6^+, -\ell_5{}^+, \ell_6{}^-) A^{\text{tree}}(6^+, -\ell_5{}^+, \ell_6{}^-)$$

$$\times A^{\text{tree}}(1^-, -\ell_6{}^+, \ell_1{}^-) A^{\text{tree}}(1^-, -\ell_6{}^+, \ell_1{}^-)$$

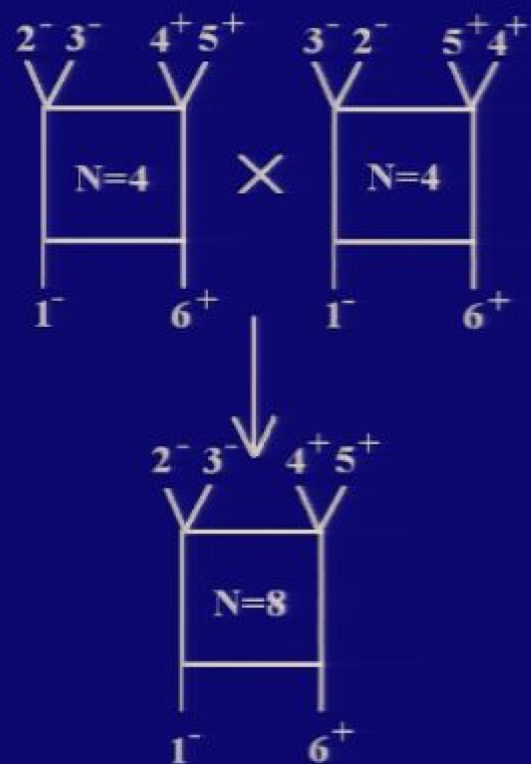
$$= \frac{1}{2} s_{23} s_{45} \times \Big(A^{\text{tree}}(2^-, 3^-, -\ell_1{}^+, \ell_3{}^+) \times A^{\text{tree}}(4^+, 5^+, -\ell_3{}^-, \ell_5{}^-)$$

$$\times A^{\text{tree}}(6^+, -\ell_5{}^+, \ell_6{}^-) \times A^{\text{tree}}(1^-, -\ell_6{}^+, \ell_1{}^-) \Big)$$

$$\times \Big(A^{\text{tree}}(3^-, 2^-, -\ell_1{}^+, \ell_3{}^+) \times A^{\text{tree}}(5^+, 4^+, -\ell_3{}^-, \ell_5{}^-) \times$$

$$A^{\text{tree}}(6^+, -\ell_5{}^+, \ell_6{}^-) \times A^{\text{tree}}(1^-, -\ell_6{}^+, \ell_1{}^-) \Big)$$

$$= 2 s_{23} s_{45} \times \hat{c}_S^{(2^-3^-)(4^+5^+)6^+1^-} \times \hat{c}_S^{(3^-2^-)(5^+4^+)6^+1^-}$$



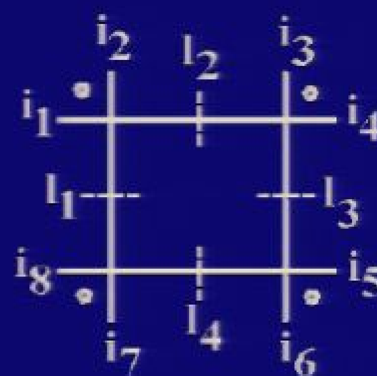
Quadruple cuts in complex momenta

- Observation : Quadruple cuts of N = 4 box coefficients

$\Rightarrow$ Coefficients of box functions by algebra

(Britto, Cachazo and Feng)

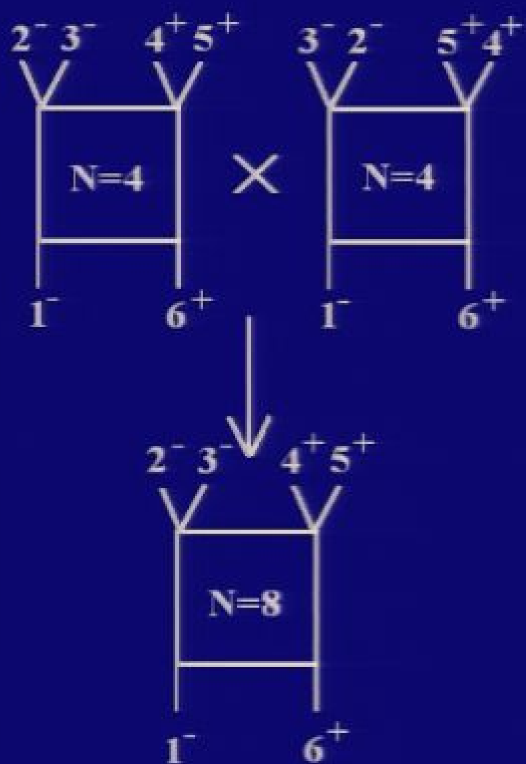
Solving the on-shell conditions



$$\ell_1^2 = 0, \quad \ell_2^2 = 0, \quad \ell_3^2 = 0, \quad \ell_4^2 = 0$$

$$\hat{c} = \frac{1}{2}\sum_{\mathcal{S}}\Big(A^{\text{tree}}(\ell_1, i_1, \ldots, i_2, \ell_2) \times A^{\text{tree}}(\ell_2, i_3, \ldots, i_4, \ell_3) \times A^{\text{tree}}(\ell_3, i_5, \ldots, i_6, \ell_4) \times A^{\text{tree}}(\ell_4, i_7, \ldots, i_8, \ell_1)\Big)$$

Supergravity amplitudes



$$
\begin{aligned}
\hat{c}_{N=8}^{(2^-3^-)(4^+5^+)6^+1^-} &= \frac{1}{2} M^{\text{tree}}(2^-,3^-,-\ell_1{}^+,\ell_3{}^+) \times M^{\text{tree}}(4^+,5^+,-\ell_3{}^-,\ell_5{}^-) \\
&\qquad \times M^{\text{tree}}(6^+,-\ell_5{}^+,\ell_6{}^-) \times M^{\text{tree}}(1^-,-\ell_6^+,\ell_1^-) \\
&= \frac{1}{2} s_{23} A^{\text{tree}}(2^-,3^-,-\ell_1{}^+,\ell_3{}^+) A^{\text{tree}}(3^-,2^-,-\ell_1{}^+,\ell_3{}^+) \\
&\qquad \times s_{45} A^{\text{tree}}(4^+,5^+,-\ell_3{}^-,\ell_5{}^-) A^{\text{tree}}(5^+,4^+,-\ell_3{}^-,\ell_5{}^-) \\
&\quad \times A^{\text{tree}}(6^+,-\ell_5{}^+,\ell_6{}^-) A^{\text{tree}}(6^+,-\ell_5{}^+,\ell_6{}^-) \\
&\qquad \times A^{\text{tree}}(1^-,-\ell_6{}^+,\ell_1{}^-) A^{\text{tree}}(1^-,-\ell_6{}^+,\ell_1{}^-) \\
&= \frac{1}{2} s_{23} s_{45} \times \Big(A^{\text{tree}}(2^-,3^-,-\ell_1{}^+,\ell_3{}^+) \times A^{\text{tree}}(4^+,5^+,-\ell_3{}^-,\ell_5{}^-) \\
&\qquad \times A^{\text{tree}}(6^+,-\ell_5{}^+,\ell_6{}^-) \times A^{\text{tree}}(1^-,-\ell_6{}^+,\ell_1{}^-) \Big) \\
&\quad \times \Big(A^{\text{tree}}(3^-,2^-,-\ell_1{}^+,\ell_3{}^+) \times A^{\text{tree}}(5^+,4^+,-\ell_3{}^-,\ell_5{}^-) \times \\
&\qquad A^{\text{tree}}(6^+,-\ell_5{}^+,\ell_6{}^-) \times A^{\text{tree}}(1^-,-\ell_6{}^+,\ell_1{}^-) \Big) \\
&= 2 s_{23} s_{45} \times \hat{c}_S{}^{(2^-3^-)(4^+5^+)6^+1^-} \times \hat{c}_S{}^{(3^-2^-)(5^+4^+)6^+1^-}
\end{aligned}
$$

Supergravity amplitudes

$$\hat{c}_{N=8}^{(ab)c(de)f} = 0$$

$$\hat{c}_{N=8}^{(ab)(cd)ef} = 2s_{ab}s_{cd} \times \left(\sum_{i=NS,S} \hat{c}_i^{(ab)(cd)ef} \times \hat{c}_i^{(ba)(dc)ef} \right)$$

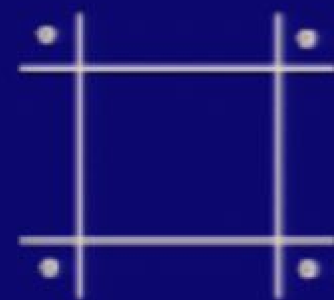
$$\hat{c}_{N=8}^{(abc)def} = 2s_{ab}s_{c\ell_c} \sum_{i=NS,S} \left(\hat{c}^{(abc)def} \hat{c}_i^{(bac)def} + \hat{c}_i^{(abc)def} \hat{c}_i^{(bca)def} + \hat{c}_i^{(abc)def} \hat{c}_i^{(cba)def} \right)$$

$$+ 2s_{ac}s_{b\ell_c} \sum_{i=NS,S} \left(\hat{c}_i^{(acb)def} \hat{c}_i^{(cab)def} + \hat{c}_i^{(acb)def} \hat{c}_i^{(cba)def} + \hat{c}_i^{(acb)def} \hat{c}_i^{(bca)def} \right)$$

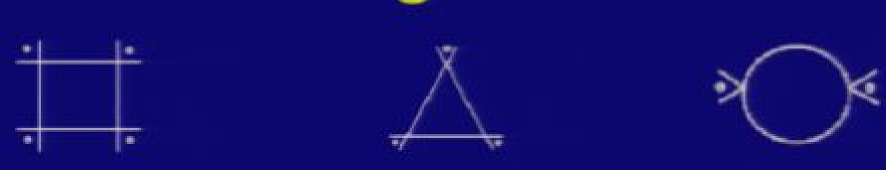
Sample expressions for other box-coefficients at seven, eight and n-pt have also been presented.

Supergravity amplitudes

- Assumption : Only integral functions = box-functions



Full amplitude : boxes + triangles + bubbles?



Supergravity amplitudes

- Demonstrated :

  (Bern, Dunbar, Dixon, Perelstein and Rozowsky)

  One-loop MHV N = 8 supergravity amplitudes

  $\Rightarrow$ much better power behaviour than expected.

  - Cut-constructibility : Demonstrated for MHV amplitudes
    - For 6pt : Direct calculation
    - All n : Factorisation properties

- IR behaviour : NMHV six-point (NEJBB, Dunbar and Ita)
- Conjecture supported!

  $\Rightarrow$ Only box functions appear

  N = 8 supergravity :

  Same one-loop power counting as N = 4 super-Yang-Mills?

Twistor space properties

- Twistor-space properties N=8 Supergravity:
  - $\Rightarrow$ More complicated!

$$A_n^{\text{tree MHV}}(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i,\mu_i) \quad \text{N=4}$$

$$\sim \int d^4x \prod_{i=1}^{n} \delta^2(\mu_{i\dot a} + x_{a\dot a}\lambda_i^a) A_n^{\text{tree MHV}}(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i)$$

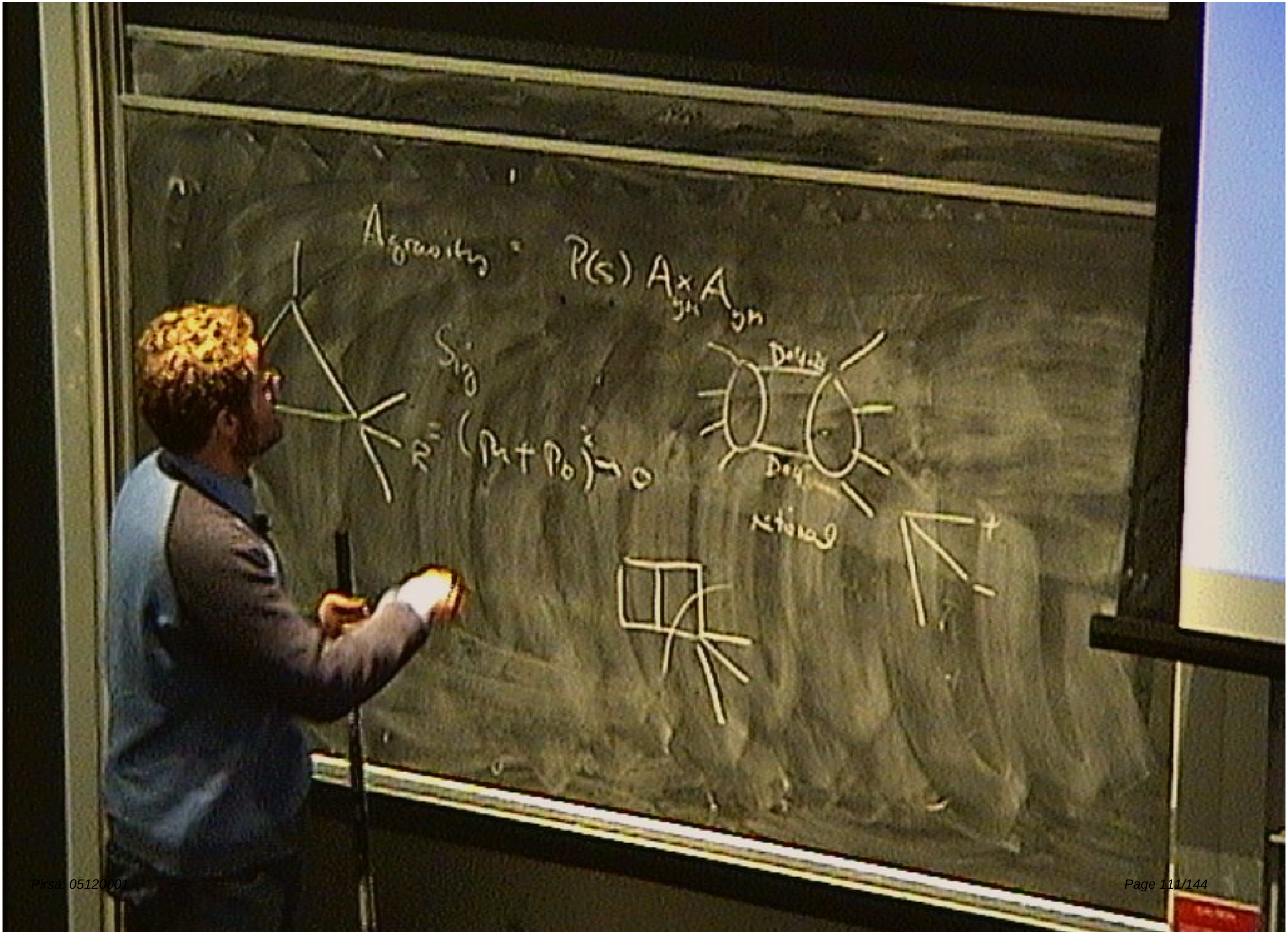
$\Uparrow$

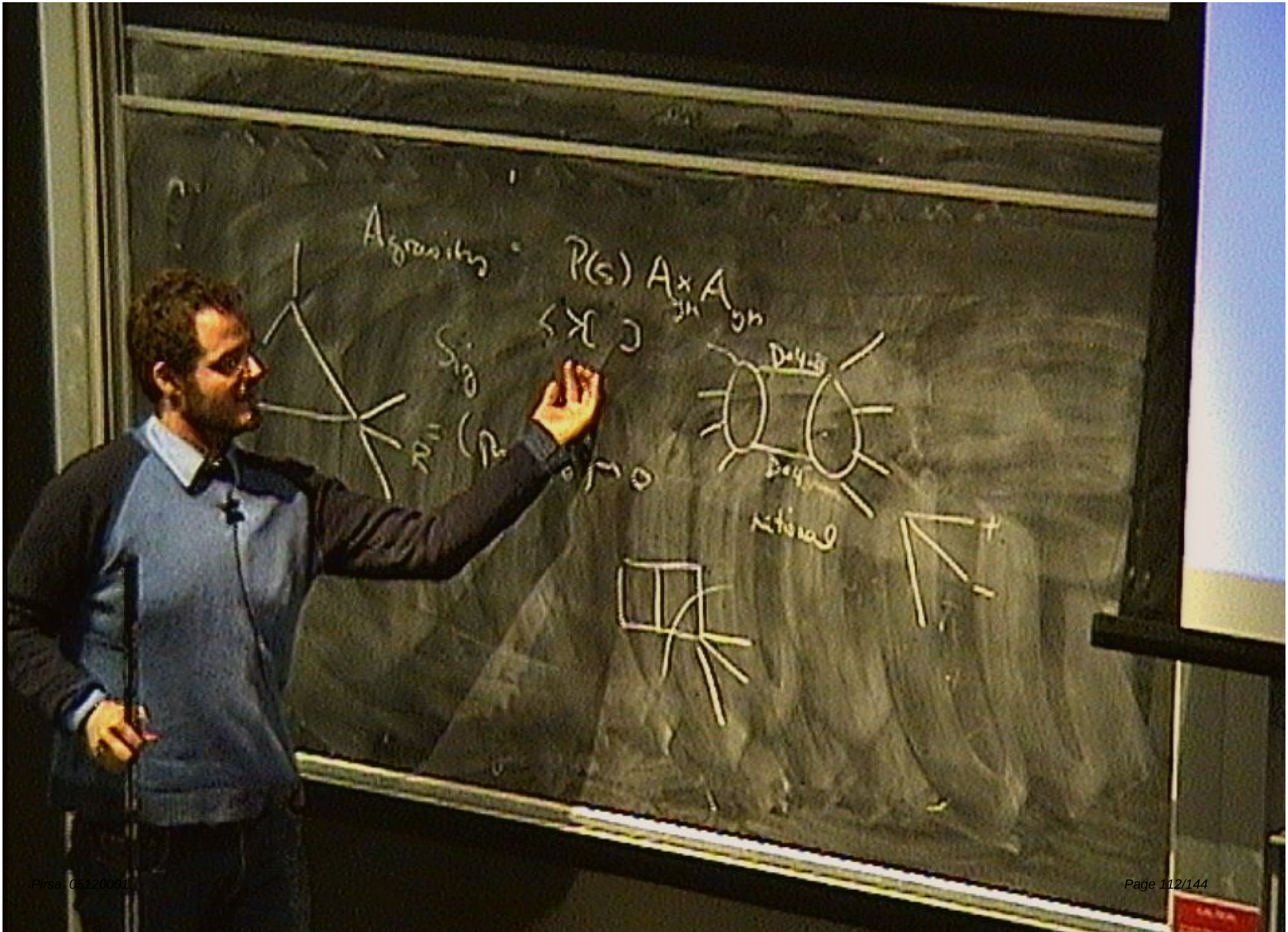
δ-functions

Derivatives of δ-functions

$$M_n^{\text{tree MHV}}(1^+,2^+,\ldots,p^-,\ldots,q^-,\ldots,n^+)(\lambda_i,\mu_i) \quad \text{N=8} \quad \Downarrow$$

$$\sim \int d^4x P\Big(-i\frac{\partial}{\partial \mu_{i\dot a}}\Big) \prod_{i=1}^{n} \delta^2(\mu_{i\dot a} + x_{a\dot a}\lambda_i^a)$$


Agravity = P(s) A_YM A_YM



Twistor space properties

- Twistor-space properties N=8 Supergravity:
  - ⇒ More complicated!

$$A_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i, \mu_i) \quad \text{N=4}$$

$$\sim \int d^4x \prod_{i=1}^{n} \delta^2(\mu_{i\dot a} + x_{a\dot a}\lambda_i^a) A_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i)$$

⇑

δ-functions

Derivatives of δ-functions

$$M_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i, \mu_i) \quad \text{N=8} \quad \Downarrow$$

$$\sim \int d^4x P\Big(-i\frac{\partial}{\partial \mu_{i\dot a}}\Big) \prod_{i=1}^{n} \delta^2(\mu_{i\dot a} + x_{a\dot a}\lambda_i^a)$$

Twistor space properties

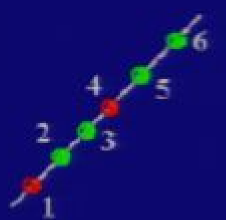
- For gravity : Guaranteed that

$$F^P M_n^{\text{tree MHV}}(1.....n) = 0 \qquad \text{for} \quad P > 2(n-3)$$

- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

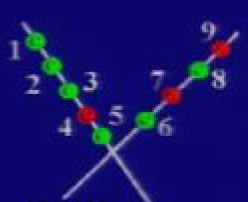
$$K^2 M_5^{\text{tree MHV}} = 0 \quad K^2 M_5^{\text{tree googly}} = 0$$

- Tree amplitudes :



$$F^4_{ijk} M_6^{\text{tree MHV}} = 0$$
$$F^5_{ijk} M_7^{\text{tree MHV}} = 0$$
$$F^6_{ijk} M_8^{\text{tree MHV}} = 0$$

and

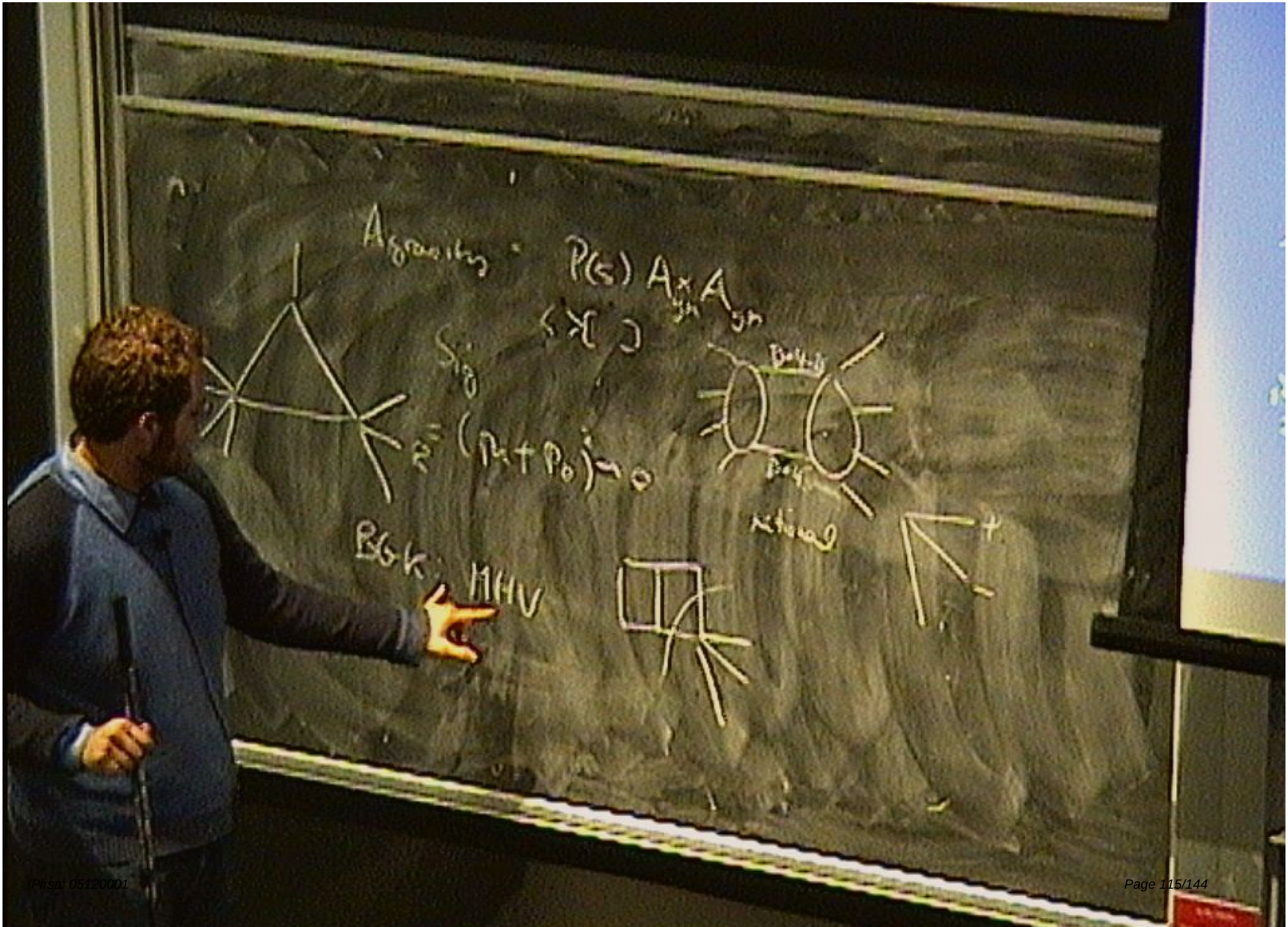


$$K^3_{ijkl} M_6^{\text{tree }(---+++)} = 0$$
$$K^4_{ijkl} M_7^{\text{tree }(---++++)} = 0$$

Checked by using computer algebra.

From this pattern we postulate the general behaviour,

$$F_{ijk}^{n-2} M_n^{\text{tree MHV}} = 0$$
$$K_{ijkl}^{n-3} M_n^{\text{tree NMHV}} = 0$$


MHV

Twistor space properties

- For gravity : Guaranteed that

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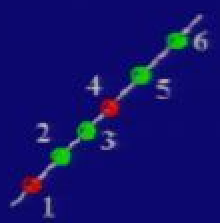
- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

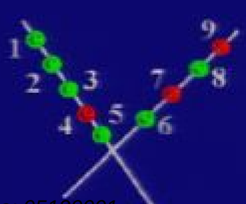
$$K^2 M_5^{\text{tree MHV}} = 0 \qquad K^2 M_5^{\text{tree googly}} = 0$$

- Tree amplitudes :



$$F_{ijk}^4 M_6^{\text{tree MHV}} = 0$$
$$F_{ijk}^5 M_7^{\text{tree MHV}} = 0$$
$$F_{ijk}^6 M_8^{\text{tree MHV}} = 0$$

and

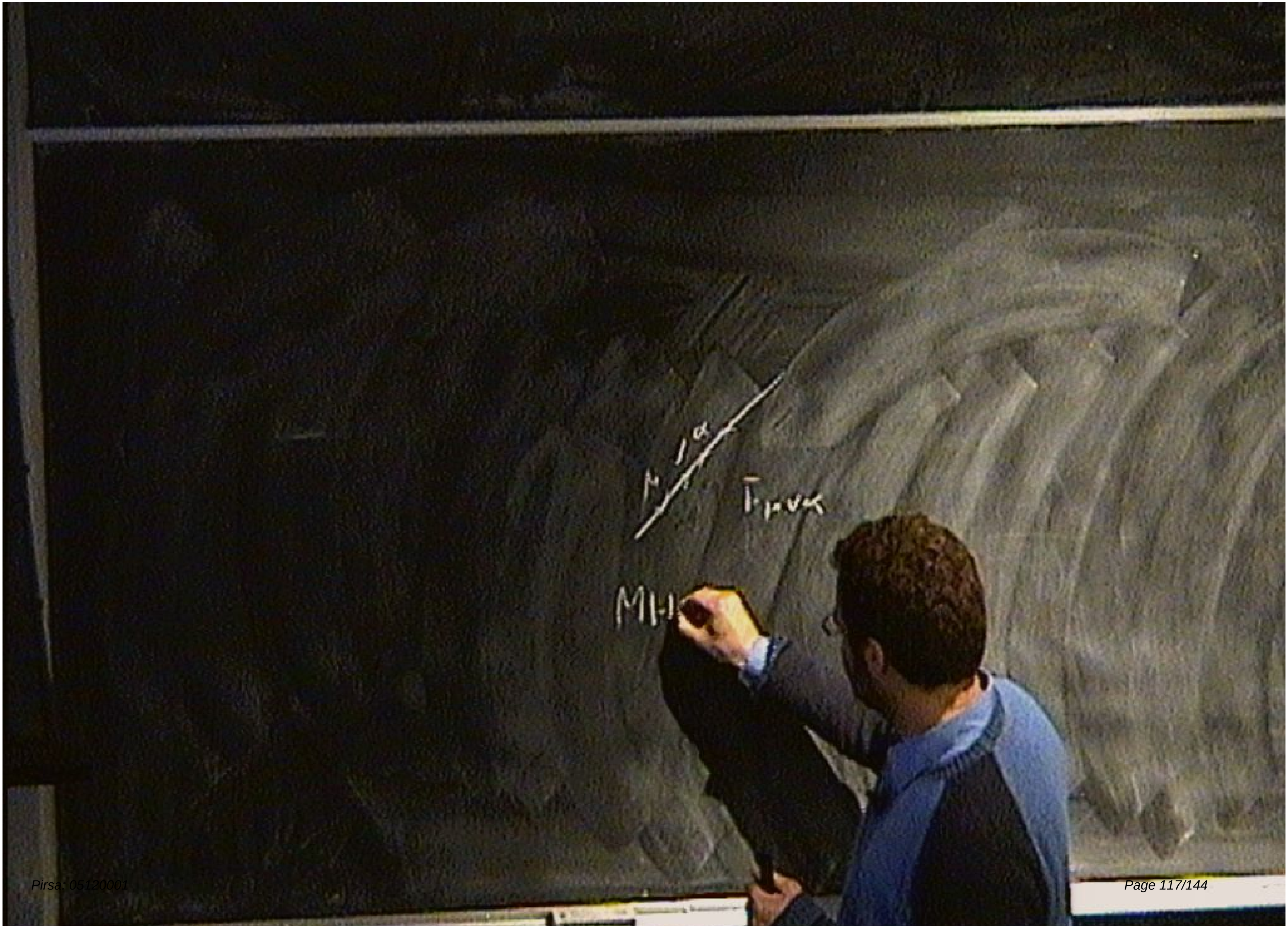


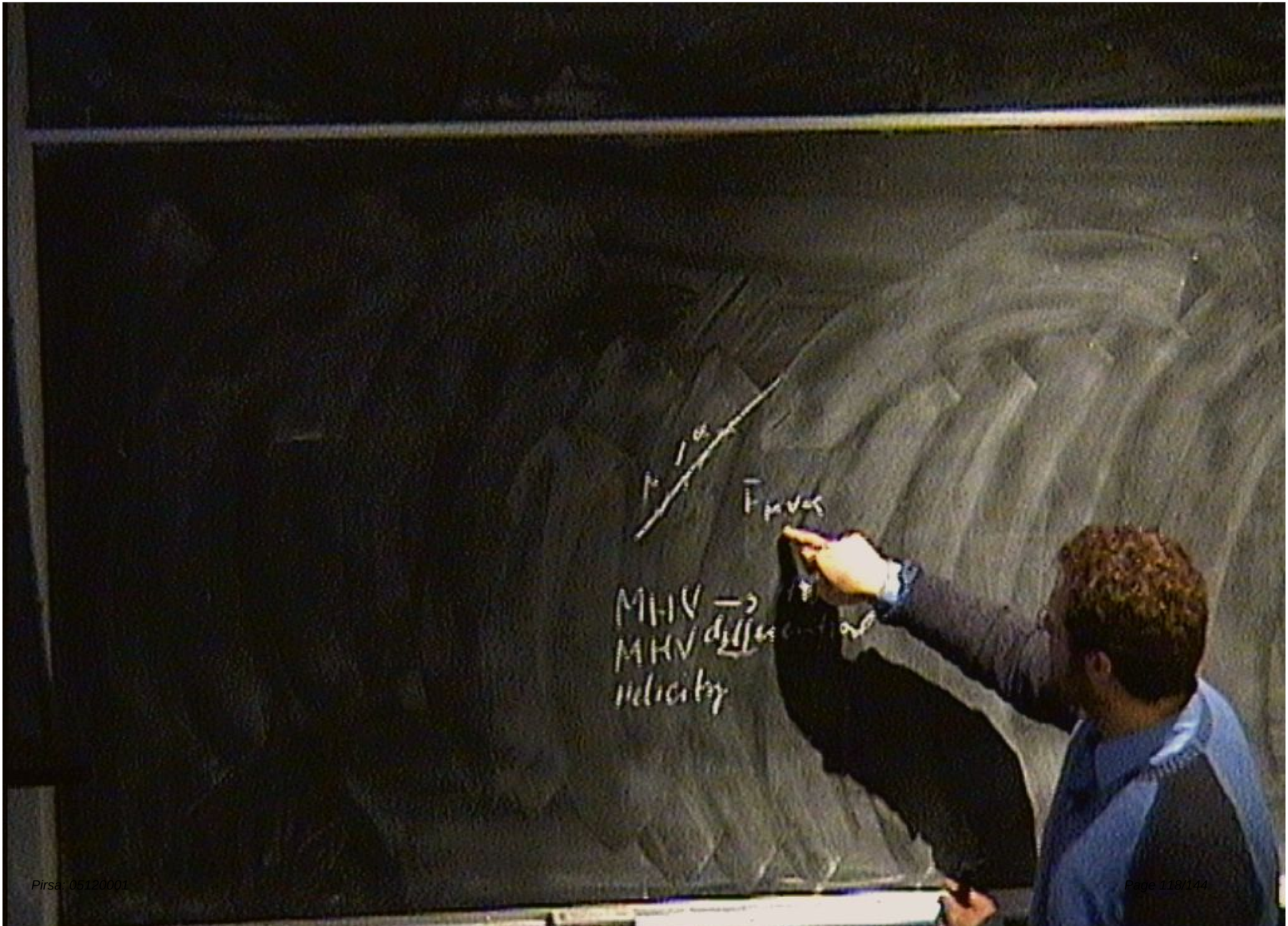
$$K_{ijkl}^3 M_6^{\text{tree }(---+++)} = 0$$
$$K_{ijkl}^4 M_7^{\text{tree }(---++++)} = 0$$

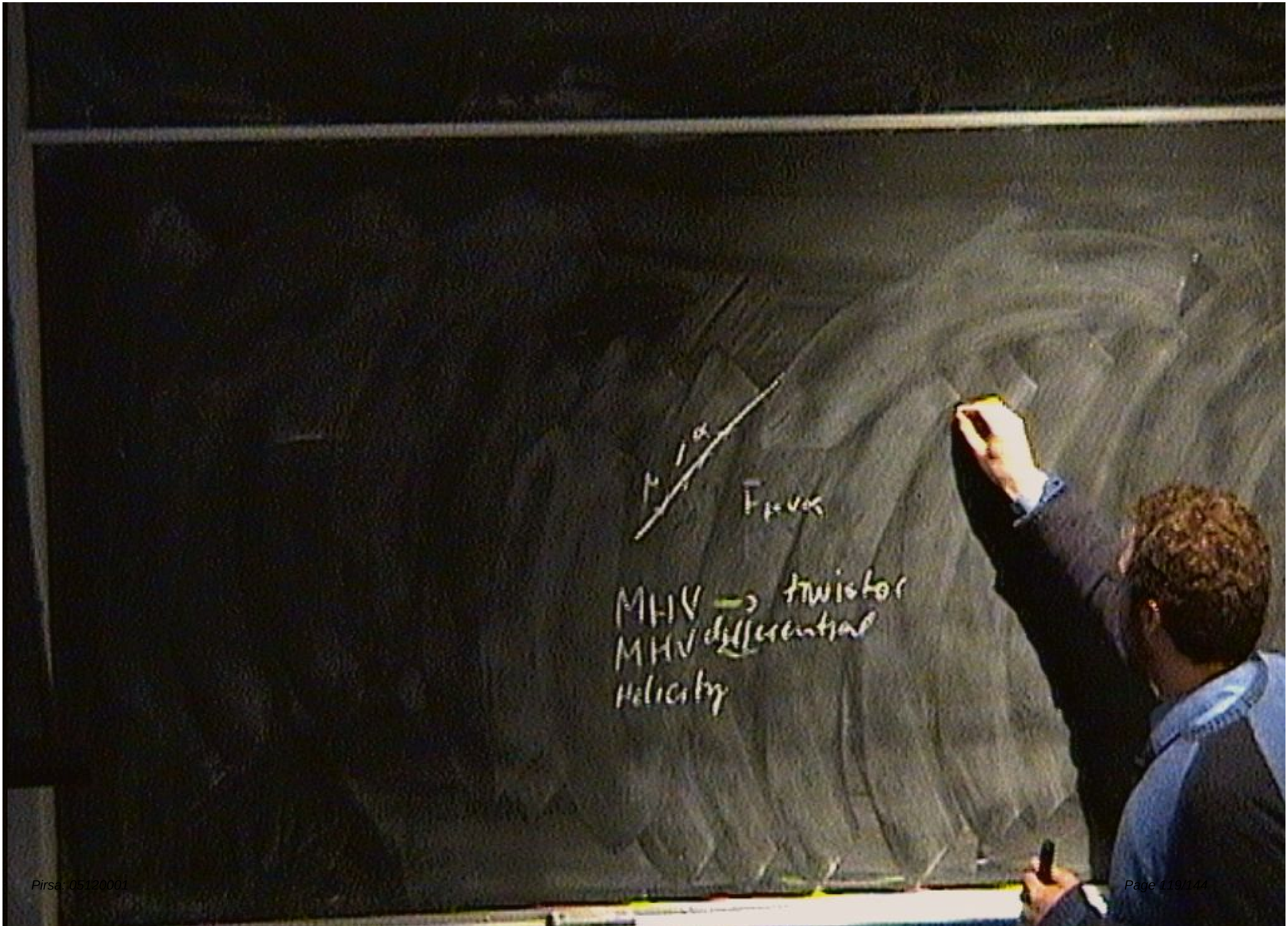
Checked by using computer algebra.

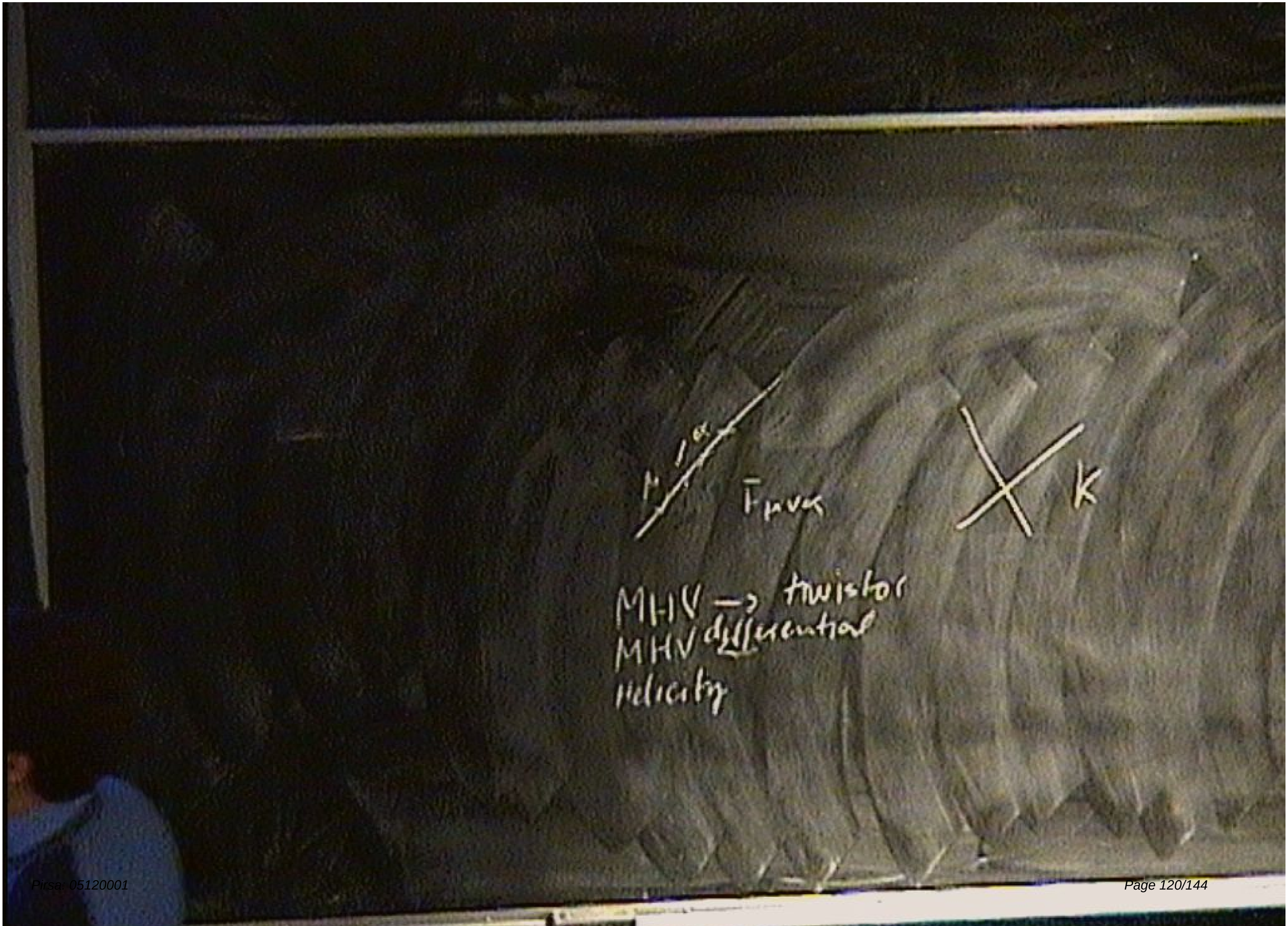
From this pattern we postulate the general behaviour,

$$F_{ijk}^{n-2} M_n^{\text{tree MHV}} = 0$$
$$K_{ijkl}^{n-3} M_n^{\text{tree NMHV}} = 0$$




MHV →
MHV


MHV → twistor
MHV differential
helicity


MHV → twistor
MHV differential
helicity
K

Twistor space properties

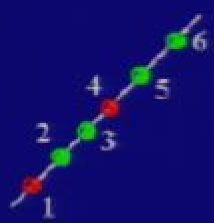
- For gravity : Guaranteed that

$$F^P M_n^{\text{tree MHV}}(1.....n) = 0 \qquad \text{for} \quad P > 2(n-3)$$

- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

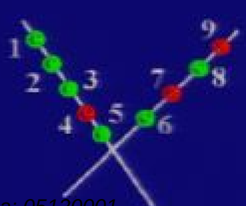
$$K^2 M_5^{\text{tree MHV}} = 0 \quad K^2 M_5^{\text{tree googly}} = 0$$

- Tree amplitudes :



$$F^4_{ijk} M_6^{\text{tree MHV}} = 0$$
$$F^5_{ijk} M_7^{\text{tree MHV}} = 0$$
$$F^6_{ijk} M_8^{\text{tree MHV}} = 0$$

and



$$K^3_{ijkl} M_6^{\text{tree }(---+++)} = 0$$
$$K^4_{ijkl} M_7^{\text{tree }(---++++)} = 0$$

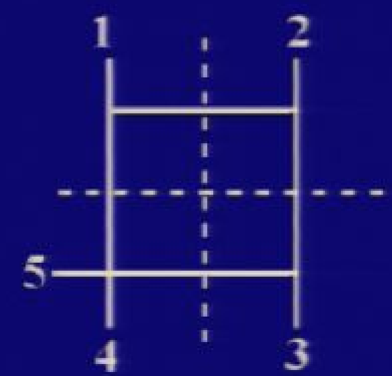
Checked by using computer algebra.

From this pattern we postulate the general behaviour,

$$F^{n-2}_{ijk} M_n^{\text{tree MHV}} = 0$$
$$K^{n-3}_{ijkl} M_n^{\text{tree NMHV}} = 0$$

Twistor space properties of gravity loop amplitudes

- Unitarity : loop behaviour from trees
  - Cuts of the MHV box



  - Consider the cut C123, where the gravity tree amplitude is $M_{tree}(l_5, 1, 2, 3, l_3)$.
  - This tree is annihilated by $F^3(123)$
    - Hence $F^3(123)c_{N=8}(45)123 = 0$
    - Similarly $F^3(145)c_{N=8}(45)123 = F^3(345)c_{N=8}(45)123 = 0$.
    - Remaining choices of F_{ijk} : consider more generalised cuts, e.g., C(4512) and hence $F^4(124)c_{N=8}(45)123 = 0$.

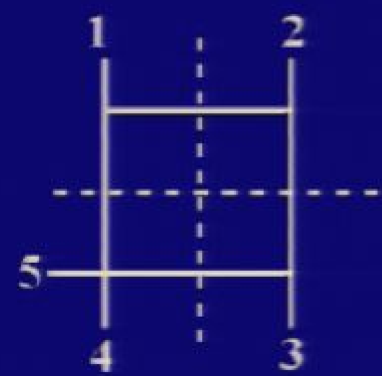
- Summarising:

$$F^3_{123}{c_{N=8}}^{(45)123} = F^3_{145}{c_{N=8}}^{(45)123} : \quad F^3_{345}{c_{N=8}}^{(45)123} = 0,$$
$$F^4_{ijk}{c_{N=8}}^{(45)123} = 0 \ \forall\ i, j, k\,.$$

58

Twistor space properties of gravity loop amplitudes

- Unitarity : loop behaviour from trees
  - Cuts of the MHV box



  - Consider the cut C123, where the gravity tree amplitude is $M_{tree}(l_5, 1, 2, 3, l_3)$.
  - This tree is annihilated by $F^3(123)$
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    - Remaining choices of F_{ijk} : consider more generalised cuts, e.g., C(4512) and hence $F^4(124)c_{N=8}(45)123 = 0$.

- Summarising:

$$F^3_{123} c_{N=8}{}^{(45)123} = F^3_{145} c_{N=8}{}^{(45)123} = F^3_{345} c_{N=8}{}^{(45)123} = 0,$$

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58

Twistor space properties

- For gravity : Guaranteed that

$$F^P M_n^{\text{tree MHV}}(1.....n) = 0 \qquad \text{for} \quad P > 2(n-3)$$

- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

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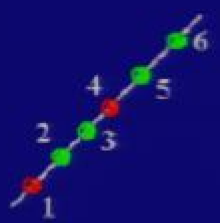
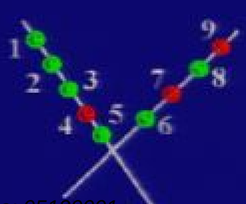


$$F_{ijk}^4 M_6^{\text{tree MHV}} = 0$$
$$F_{ijk}^5 M_7^{\text{tree MHV}} = 0$$
$$F_{ijk}^6 M_8^{\text{tree MHV}} = 0$$

Checked by using computer algebra.

From this pattern we postulate the general behaviour,

and



$$K_{ijkl}^3 M_6^{\text{tree }(---+++)} = 0$$
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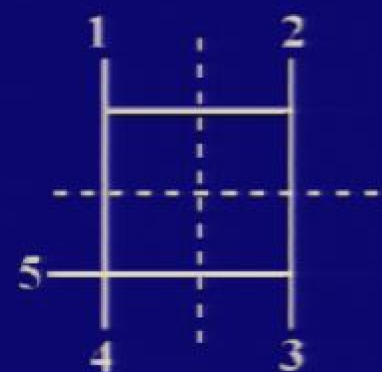
$$F_{ijk}^{n-2} M_n^{\text{tree MHV}} = 0$$
$$K_{ijkl}^{n-3} M_n^{\text{tree NMHV}} = 0$$

Twistor space properties of gravity loop amplitudes

- Unitarity : loop behaviour from trees
  - Cuts of the MHV box



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    - Remaining choices of F_{ijk} : consider more generalised cuts, e.g., C(4512) and hence $F^4(124)c_{N=8}(45)123 = 0$.

- Summarising:

$$F^3_{123} c_{N=8}{}^{(45)123} = F^3_{145} c_{N=8}{}^{(45)123} : \quad F^3_{345} c_{N=8}{}^{(45)123} = 0,$$

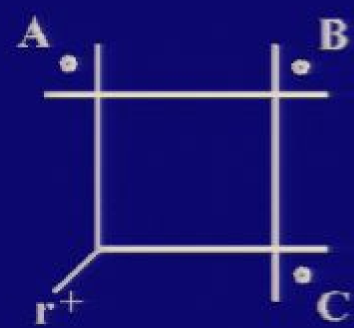
$$F^4_{ijk} c_{N=8}{}^{(45)123} = 0 \ \forall \ i, j, k\,.$$

Twistor space properties of gravity loop amplitudes

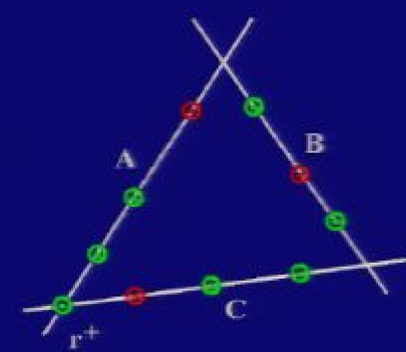
- Inspecting the general n-point case, we can now predict

$$F_{ijk}^{n-1} c_{N=8}{}^{n\text{-point}} = 0 \qquad \forall i, j, k$$

- Similarly we can deduce that (consistent with the YM picture)



$$K^{n-2} c_{N=8}{}^{n\text{-point}} = 0\,,$$



Topology :

As N=4 super-Yang-Mills

$\Rightarrow$ Points lie on three intersecting lines (Bern, Dixon and Kosower.)

$$F^{r}_{ijk} c_{N=8} = 0,\ \{ijk\} \in A \quad F^{r'}_{ijk} c_{N=8} = 0,\ \{ijk\} \in B$$

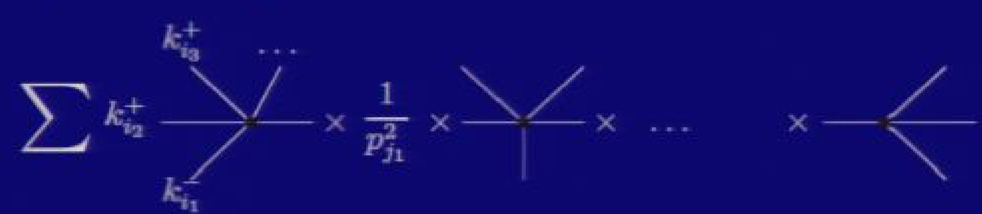
$$F^{r''}_{ijk} c_{N=8} = 0,\ \{ijk\} \in C \quad F^{r'''}_{ijk} c_{N=8} = 0,\ \{ijk\} \in D$$

CSW type expansion for gravity tree amplitudes

- BCFW for gravity (Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrtec)
- CSW expansion : gravity?
- Shift (Risager)

$$\hat{\bar{\lambda}}_{m_1} = \bar{\lambda}_{m_1} + z\langle m_2\, m_3\rangle \bar{\eta}\,,$$
$$\hat{\bar{\lambda}}_{m_2} = \bar{\lambda}_{m_2} + z\langle m_3\, m_1\rangle \bar{\eta}\,,$$
$$\hat{\bar{\lambda}}_{m_3} = \bar{\lambda}_{m_3} + z\langle m_1\, m_2\rangle \bar{\eta}\,,$$

Shift : Correct factorisation



Reproduce CSW for Yang-Mills (Risager)

CSW type expansion for gravity tree amplitudes

- Negative legs shifted in the following way

$$\hat{k}_{m_i}(z) \;=\; \lambda_{m_i}\left(\bar{\lambda}_{m_i} \;+\; z\langle m_{i-1}\, m_{i+1}\rangle\bar{\eta}\right)$$

- Analytic continuation of the amplitude into the complex plane.

$$\frac{1}{2\pi i}\oint\frac{dz}{z}M_n(z) \;=\; C_\infty \;=\; M_n(0)+\sum_\alpha \mathrm{Res}_{z=z_\alpha}\frac{M_n(z)}{z}$$

- If $M_n(z)$, 1) rational, 2) simple poles at points z, and 3) C_∞ vanishes (justified assumption) :

  $\Rightarrow M_n(0)$ = sum of residues,

$$M_n(0) \;=\; -\sum_\alpha \mathrm{Res}_{z=z_\alpha}\frac{M_n(z)}{z}$$

61

CSW type expansion for gravity tree amplitudes

- All poles : Factorise as,

$$M^{\rm MHV}(m_{i_1}^-,\cdots,P^-)\times\frac{i}{P^2}\times M^{\rm MHV}((-P)^+,m_{i_2}^-,m_{i_3}^-,\cdots)$$

- $\hat{P}^2(z)$ vanishes linearly in z :

$$\hat{P}^2 \;=\; P^2+z_\alpha\langle m_{i_2}m_{i_3}\rangle[\eta|P|m_{i_1}\rangle=0$$

- Spinor products : not z dependent (normal CSW)

$$\langle i\,\hat{P}\rangle \;=\; \frac{\langle i\,\hat{P}\rangle[\hat{P}\,\eta]}{[\hat{P}\,\eta]} \;=\; \frac{\langle i|P|\eta]}{\omega}$$

CSW type expansion for gravity tree amplitudes

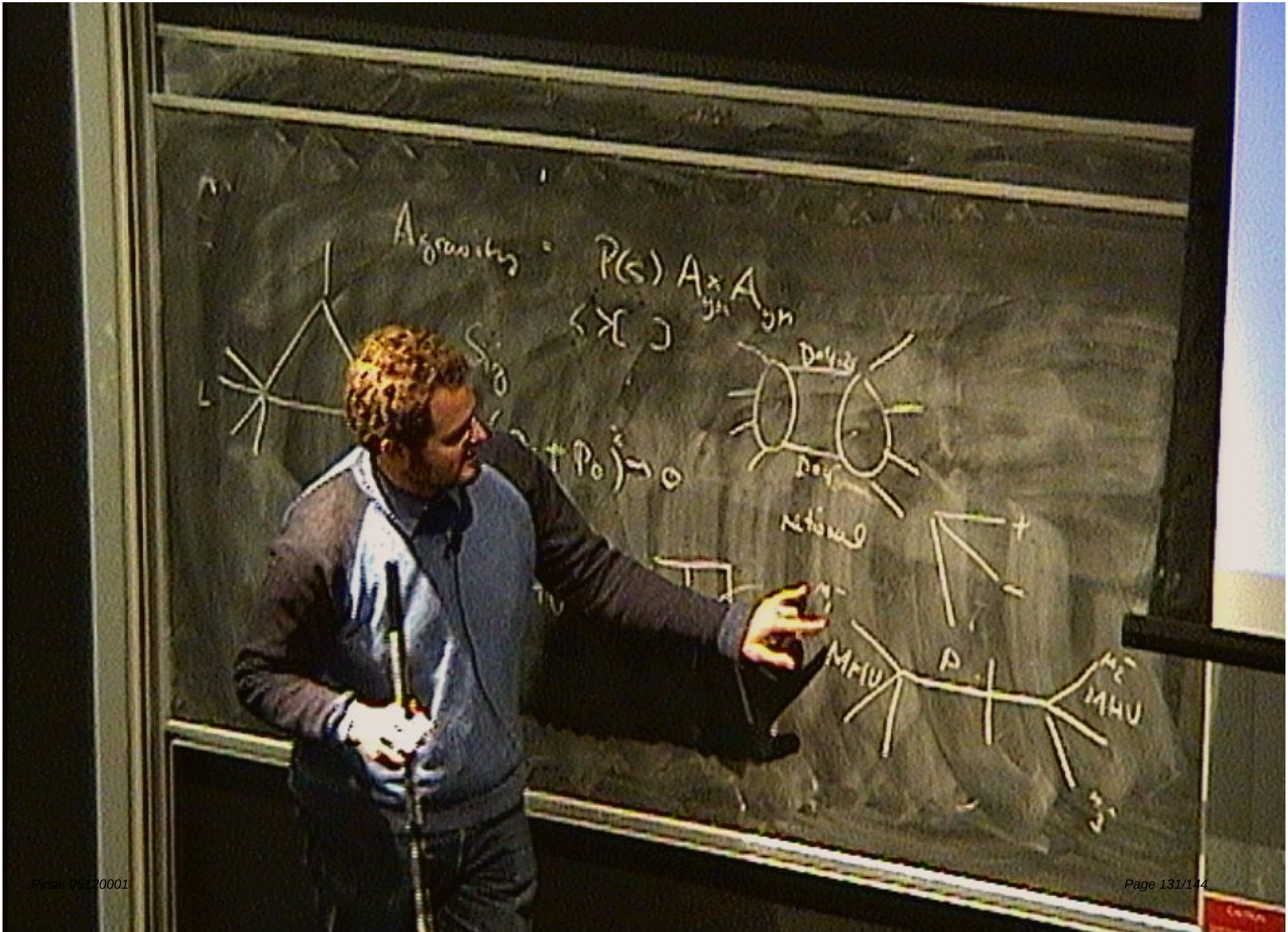
- For gravity : Substitutions

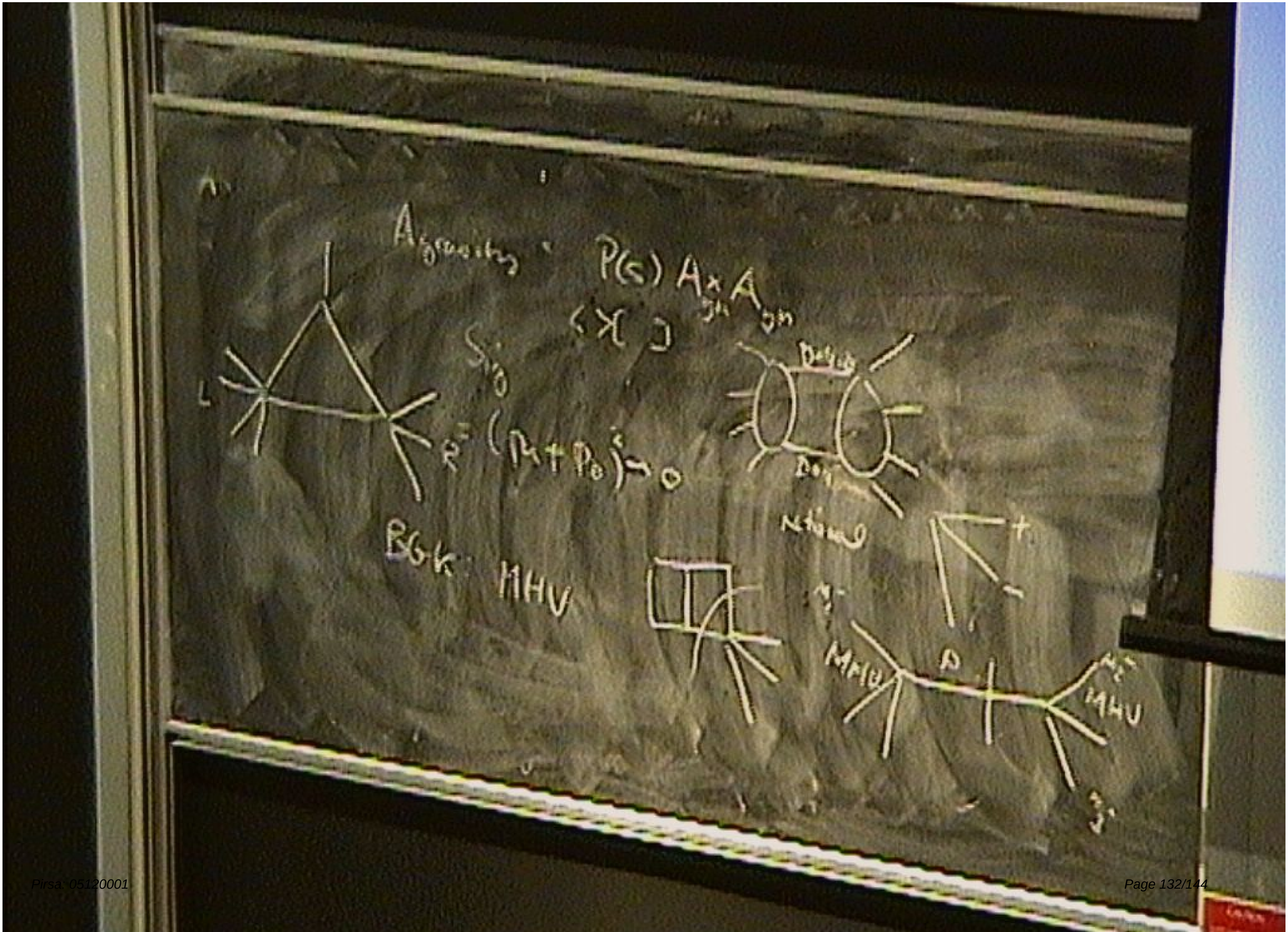
$$
\begin{aligned}
[l^+\,\hat{P}] &= \frac{[l^+\,\hat{P}]\langle \hat{P}\,\alpha\rangle}{\langle \hat{P}\,\alpha\rangle} = \frac{\omega[l^+|\hat{P}|\alpha\rangle}{[\eta|P|\alpha\rangle} = \frac{\omega[l^+|P|m_{i_1}\rangle}{[\eta|P|m_{i_1}\rangle}\,, \\
[\hat{m}_{i_2}\,\hat{m}_{i_3}] &= [m_{i_2}\,m_{i_3}] + z_\alpha[\eta|P_{m_{i_2}m_{i_3}}|m_{i_1}\rangle\,, \Leftarrow \text{non-locality} \\
[\hat{m}_{i_1}\,l^+] &= [m_{i_1}\,l^+] + z_\alpha[\eta\,l^+]\,\langle m_{i_2}\,m_{i_3}\rangle\,,
\end{aligned}
$$

MHV amplitudes on the pole $\Rightarrow$ MHV vertices

- CSW type expansion for gravity

$$
\hat{\bar{\lambda}}_{m_1} = \bar{\lambda}_{m_1} + z\,\langle m_2\,m_3\rangle\,\bar{\eta} = \bar{\lambda}_{m_1} - \frac{P^2\bar{\eta}}{[\eta|P|m_1\rangle} \Leftarrow \text{Contact term!}
$$




BGK
MHV

CSW type expansion for gravity tree amplitudes

- For gravity : Substitutions

$$[l^+\,\hat{P}] = \frac{[l^+\,\hat{P}]\langle\hat{P}\,\alpha\rangle}{\langle\hat{P}\,\alpha\rangle} = \frac{\omega[l^+|\hat{P}|\alpha\rangle}{[\eta|P|\alpha\rangle} = \frac{\omega[l^+|P|m_{i_1}\rangle}{[\eta|P|m_{i_1}\rangle},$$

$$[\hat{m}_{i_2}\,\hat{m}_{i_3}] = [m_{i_2}\,m_{i_3}] + z_\alpha[\eta|P_{m_{i_2}m_{i_3}}|m_{i_1}\rangle\,, \Leftarrow \text{non-locality}$$

$$[\hat{m}_{i_1}\,l^+] = [m_{i_1}\,l^+] + z_\alpha[\eta\,l^+]\,\langle m_{i_2}\,m_{i_3}\rangle\,,$$

MHV amplitudes on the pole $\Rightarrow$ MHV vertices

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CSW type expansion for gravity tree amplitudes

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$$[\hat{m}_{i_1}\,l^+] = [m_{i_1}\,l^+] + z_\alpha[\eta\,l^+]\,\langle m_{i_2}\,m_{i_3}\rangle\,,$$

MHV amplitudes on the pole $\Rightarrow$ MHV vertices

- CSW type expansion for gravity

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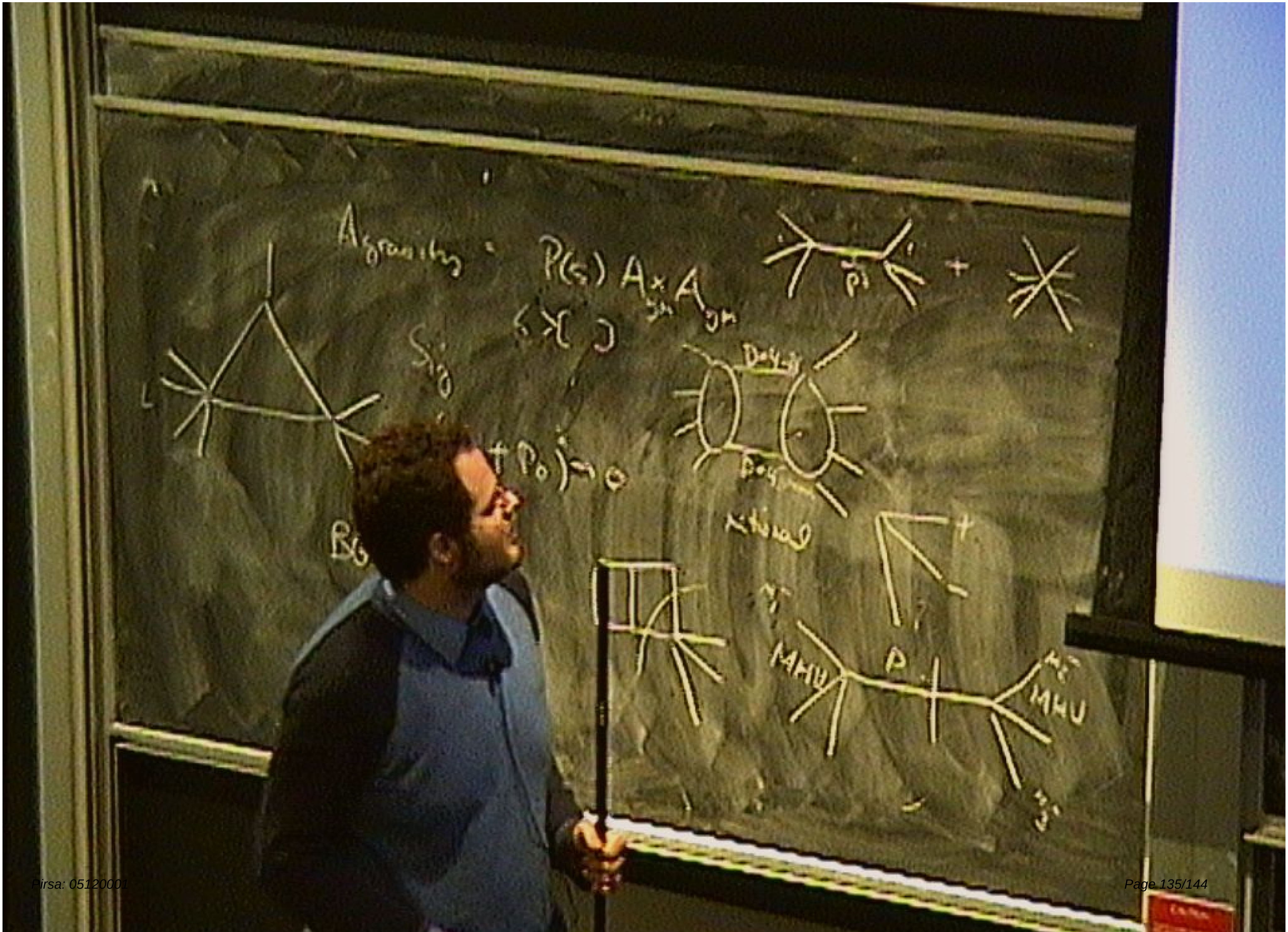
63


MHV
MHV

CSW type expansion for gravity tree amplitudes

- For gravity : Substitutions

$$[l^+ \hat{P}] = \frac{[l^+ \hat{P}]\langle \hat{P}\,\alpha\rangle}{\langle \hat{P}\,\alpha\rangle} = \frac{\omega[l^+|\hat{P}|\alpha\rangle}{[\eta|P|\alpha\rangle} = \frac{\omega[l^+|P|m_{i_1}\rangle}{[\eta|P|m_{i_1}\rangle},$$

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$$[\hat{m}_{i_1}\,l^+] = [m_{i_1}\,l^+] + z_\alpha[\eta\,l^+]\,\langle m_{i_2}\,m_{i_3}\rangle,$$

MHV amplitudes on the pole $\Rightarrow$ MHV vertices

- CSW type expansion for gravity

$$\hat{\bar{\lambda}}_{m_1} = \bar{\lambda}_{m_1} + z\,\langle m_2\,m_3\rangle\,\bar{\eta} = \bar{\lambda}_{m_1} - \frac{P^2\bar{\eta}}{[\eta|P|m_1\rangle} \Leftarrow \text{Contact term!}$$

Conclusion

- Amplitudes in QCD :

  Methods building on previous results are preferable
  - Unitarity cuts
  - Recursive techniques are examples of this

- Amplitudes via recursion $\Rightarrow$ most efficient option

  Via recursion for loop amplitudes :

  Integrations avoided!
  - Recursion for (some) integral coefficients : Possible!
  - Sufficient criteria for valid recursions (not nessesary)

Conclusion

- Apply to large classes of coefficients
  - Illustrated recursion with explicit results for *n* gluon scattering (NMHV amplitudes with split helicity)

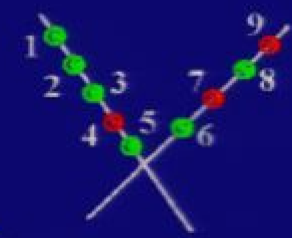
- Possible to extend : more generic shifts?

  (Not clear what such a shift should be Bena, Bern, NEJBB, Dunbar, Ita, Mastrolia)
  - All coefficients for loops in this way?

Additional results will be needed for use in colliders. Still long way to go.

Discussion and summary

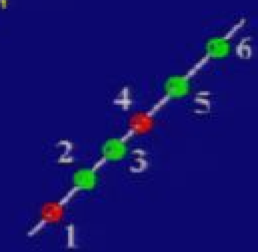
- It occurs to be very interesting that amplitudes have so simple structures as is found.



- Consequence of twistor space?
  - Simplicity due to twistor space? (same structure gravity, Yang-Mills?)

    Picture (simplicity) is very valuable!

    Inspired ⇒ exciting period in perturbative physics.



  - Gravity twistor string conjecture perhaps possible?…twistor structure present/CSW expansion possible.

Twistor space properties

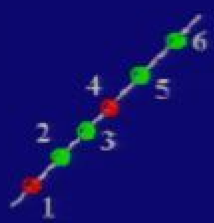
- For gravity : Guaranteed that

$$F^P M_n^{\text{tree MHV}}(1.....n) = 0 \qquad \text{for} \quad P > 2(n-3)$$

- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

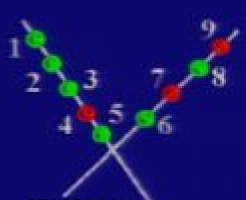
$$K^2 M_5{}^{\text{tree MHV}} = 0 \quad K^2 M_5^{\text{tree googly}} = 0$$

- Tree amplitudes :



$$F^4_{ijk} M_6^{\text{tree MHV}} = 0$$
$$F^5_{ijk} M_7^{\text{tree MHV}} = 0$$
$$F^6_{ijk} M_8^{\text{tree MHV}} = 0$$

and



$$K^3_{ijkl} M_6^{\text{tree }(---+++)} = 0$$
$$K^4_{ijkl} M_7^{\text{tree }(---++++)} = 0$$

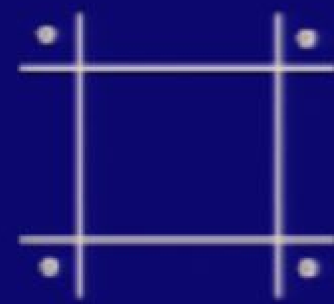
Checked by using computer algebra.

From this pattern we postulate the general behaviour,

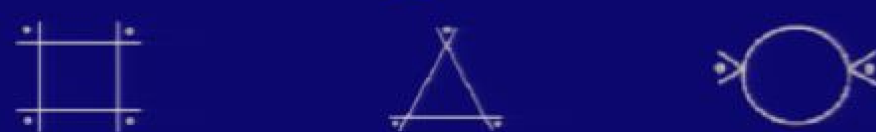
$$F^{n-2}_{ijk} M_n^{\text{tree MHV}} = 0$$
$$K^{n-3}_{ijkl} M_n^{\text{tree NMHV}} = 0$$

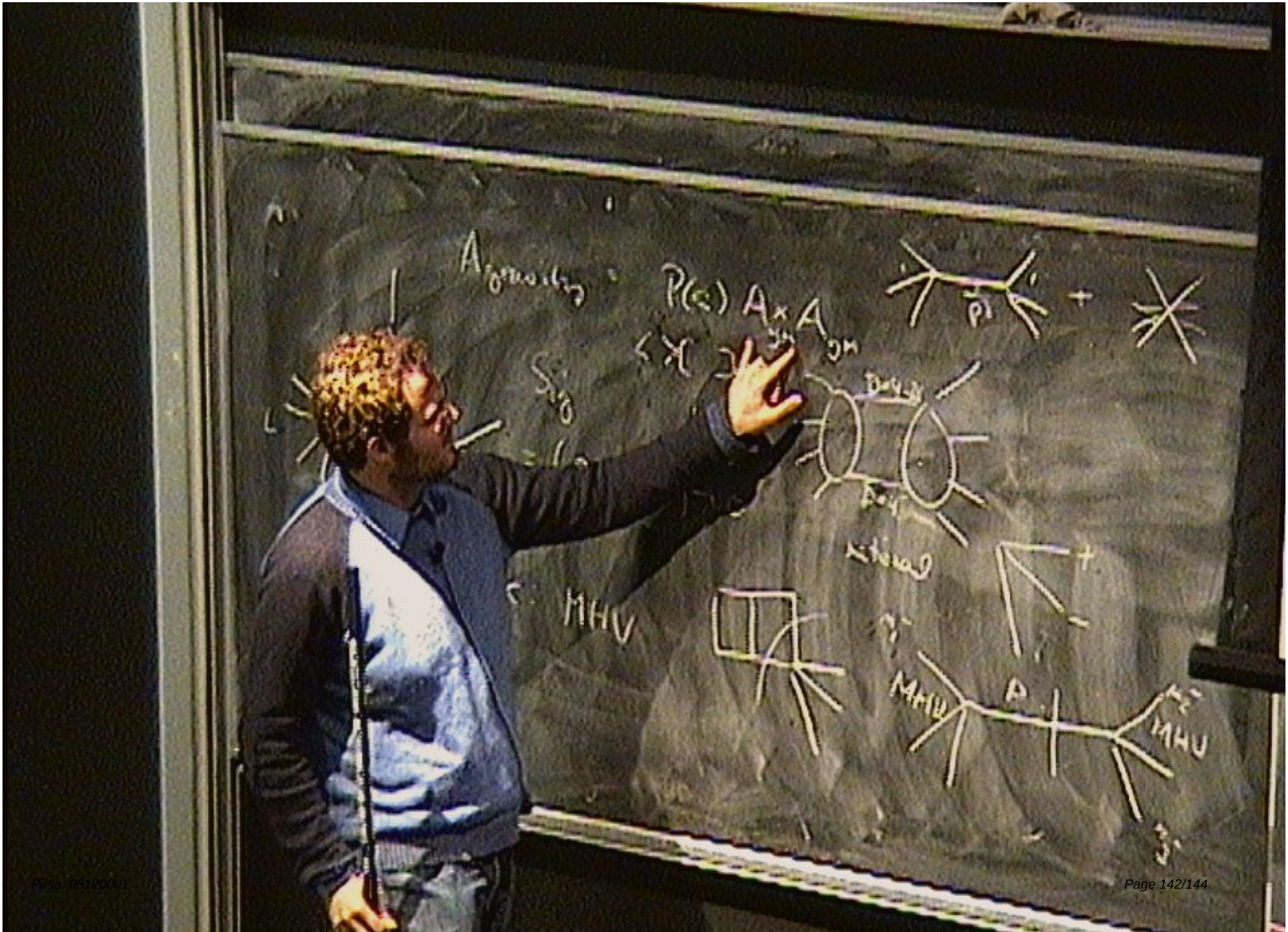
Supergravity amplitudes

- Assumption : Only integral functions = box-functions



Full amplitude : boxes + triangles + bubbles?



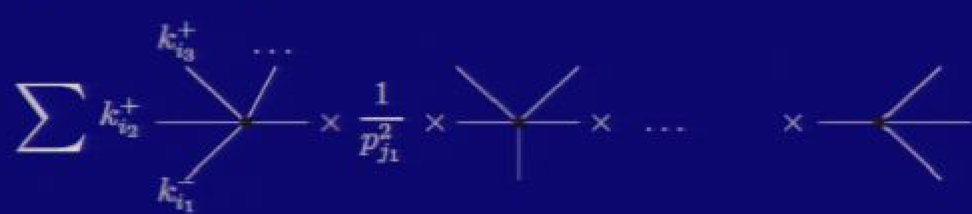

MHV

CSW type expansion for gravity tree amplitudes

- BCFW for gravity (Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrtec)
- CSW expansion : gravity?
- Shift (Risager)

$$\hat{\bar{\lambda}}_{m_1} = \bar{\lambda}_{m_1} + z\langle m_2\, m_3\rangle \bar{\eta}\,,$$
$$\hat{\bar{\lambda}}_{m_2} = \bar{\lambda}_{m_2} + z\langle m_3\, m_1\rangle \bar{\eta}\,,$$
$$\hat{\bar{\lambda}}_{m_3} = \bar{\lambda}_{m_3} + z\langle m_1\, m_2\rangle \bar{\eta}\,,$$

Shift : Correct factorisation



Reproduce CSW for Yang-Mills (Risager)

CSW type expansion for gravity tree amplitudes

- Negative legs shifted in the following way

$$\hat{k}_{m_i}(z) \;=\; \lambda_{m_i}\left(\bar{\lambda}_{m_i} \;+\; z\langle m_{i-1}\, m_{i+1}\rangle\bar{\eta}\right)$$

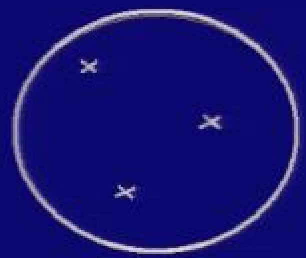
- Analytic continuation of the amplitude into the complex plane.

$$\frac{1}{2\pi i}\oint\frac{dz}{z}M_n(z) \;=\; C_\infty \;=\; M_n(0)+\sum_\alpha \mathrm{Res}_{z=z_\alpha}\frac{M_n(z)}{z}$$

- If $M_n(z)$, 1) rational, 2) simple poles at points z, and 3) C_∞ vanishes (justified assumption) :

$\Rightarrow M_n(0)$ = sum of residues,

$$M_n(0) \;=\; -\sum_\alpha \mathrm{Res}_{z=z_\alpha}\frac{M_n(z)}{z}$$



61