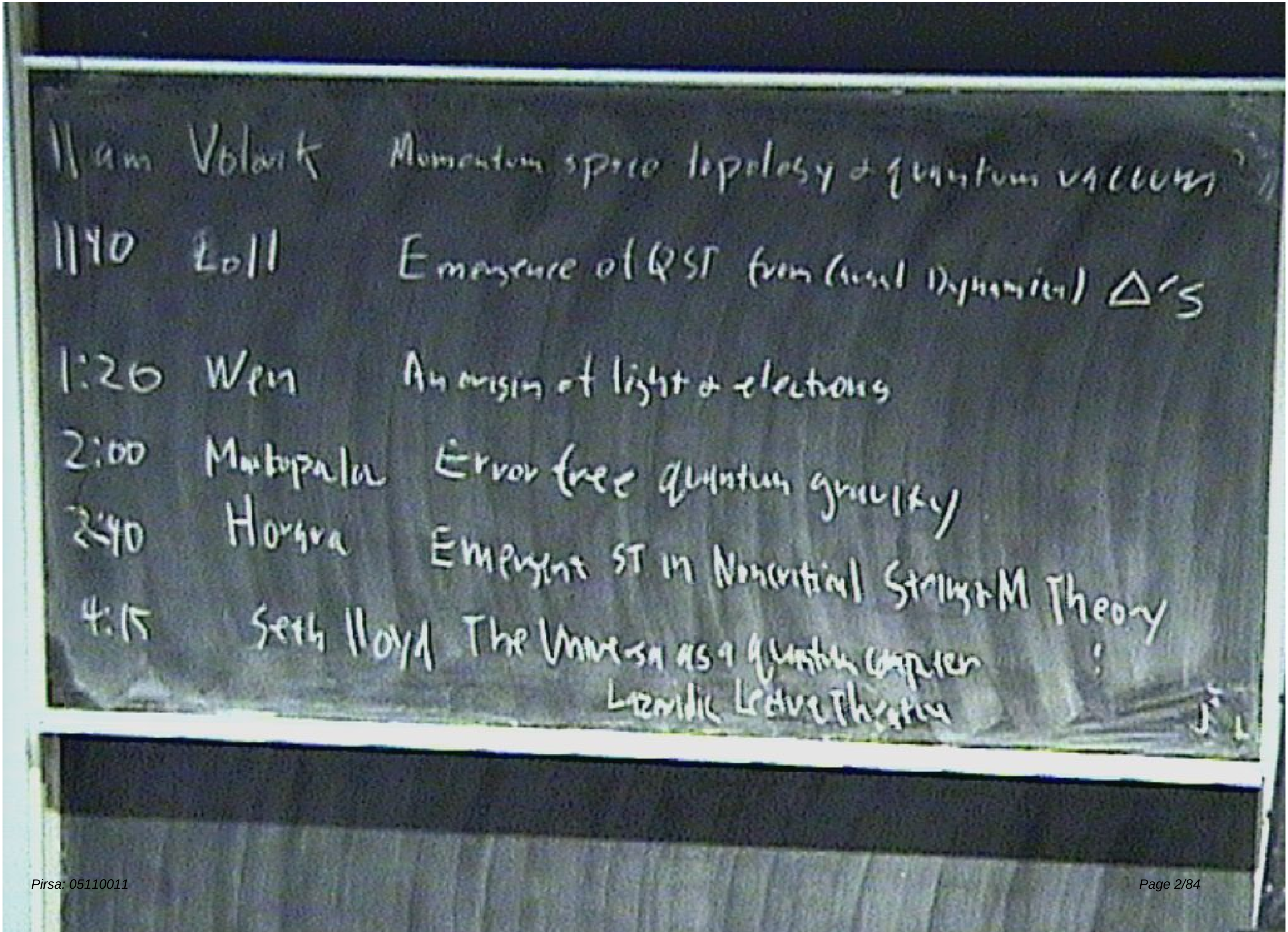
Title: Momentum-space topology and universality classes of quantum vacua

Date: Nov 18, 2005 11:00 AM

URL: http://pirsa.org/05110011

Abstract:



11 am	Volovik	Momentum space topology & quantum vacuum
1140	Loll	Emergence of QST from (causal Dynamical) △'s
1:20	Wen	An origin of light & electrons
2:00	Markopoulou	Error free quantum gravity
2:40	Horava	Emergent ST in Noncritical String/M Theory
4:15	Seth Lloyd	The Universe as a quantum computer Lazaridis Theatre

Momentum-space topology
&
universality classes of quantum vacua

Perimeter Institute
November 18, 2005

G. Volovik

Helsinki University of Technology & Landau Institute

- **p-space topology of Fermi surface & Lifshitz quantum phase transition**

- **p-space topology of Fermi points**
  chiral fermions, Standard Model, emergent gauge fields and gravity, chiral anomaly,
  quantum phase transitions at BEC-BCS crossover, CPT violation vs Higgs mechanism

- **p-space topology of Fermi lines**
  quantum phase transitions in *d*-wave superconductors and BEC-BCS crossover

- **p-space topology in fully gapped 2+1 systems**
  skyrmion in **p**-space, plateau transitions, Chern-Simons term, quantum statistics
  of **r**-space skyrmions, quantization of physical parameters, QHE and spin QHE

- **p-space topology in 1+1 systems**
  edge states, fermion zero modes, quantum 1D Ising

GUT vs anti-GUT

Big Bang

10^{19}GeV
Planck scale

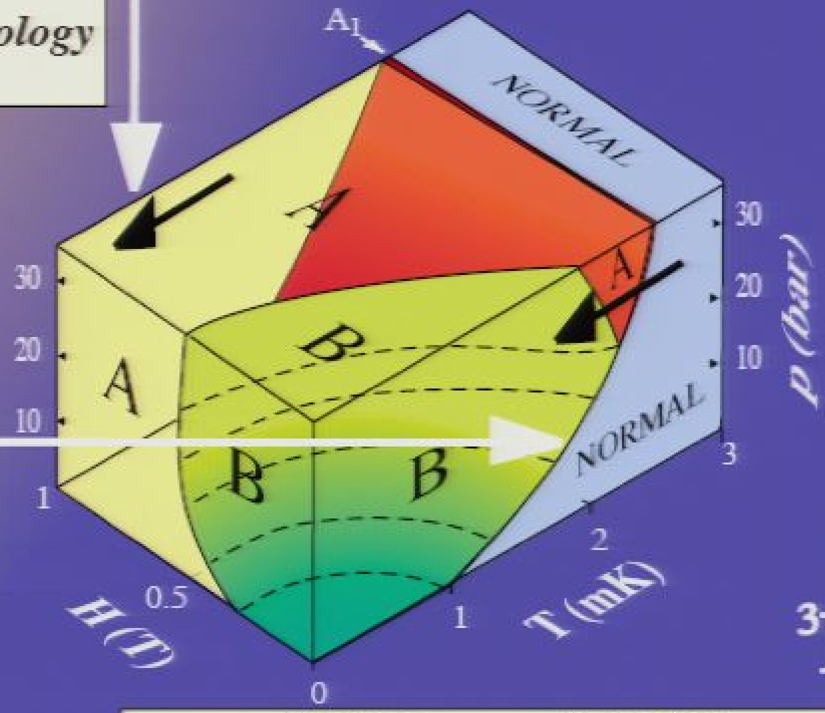
10^{15}GeV
GUT scale

10^{2}GeV
electroweak scale

anti-GUT:
chiral fermions, Lorentz invariance
gauge fields, gravity,
gauge invariance, spin ...
all gradually emerge

symmetry from topology

GUT:
symmetry breaking &
topology from symmetry



^{3}He

***GUT* in Standard Model**
symmetry breaking phase transitions

$SO(10) \rightarrow SU(3)\times SU(2)\times U(1) \rightarrow SU(3)\times U(1)$

***GUT* in superfluid ^{3}He**
symmetry breaking phase transitions

$SO(3) \times SO(3)\times U(1) \rightarrow U(1)\times U(1) \rightarrow SO(3)$

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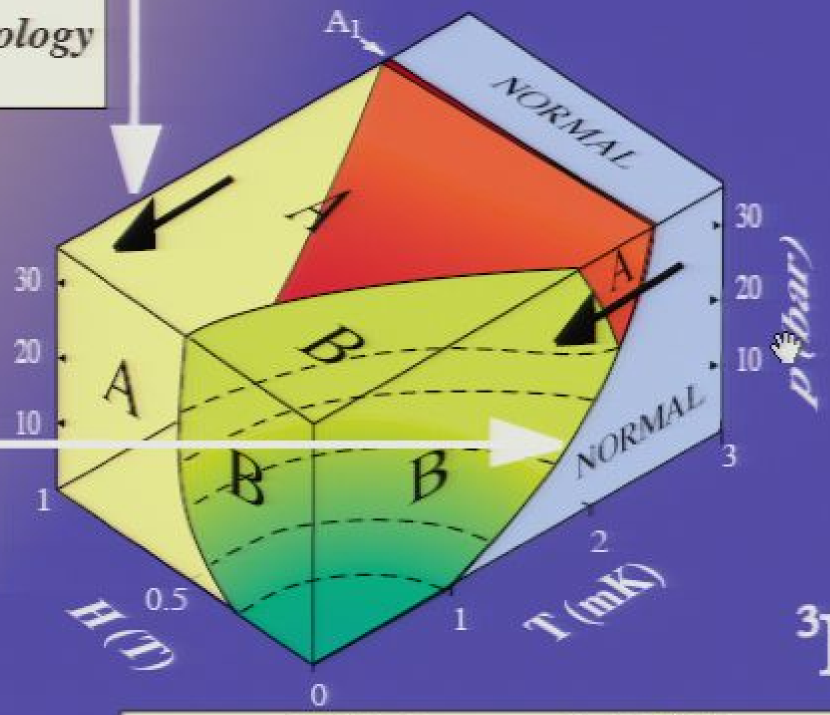

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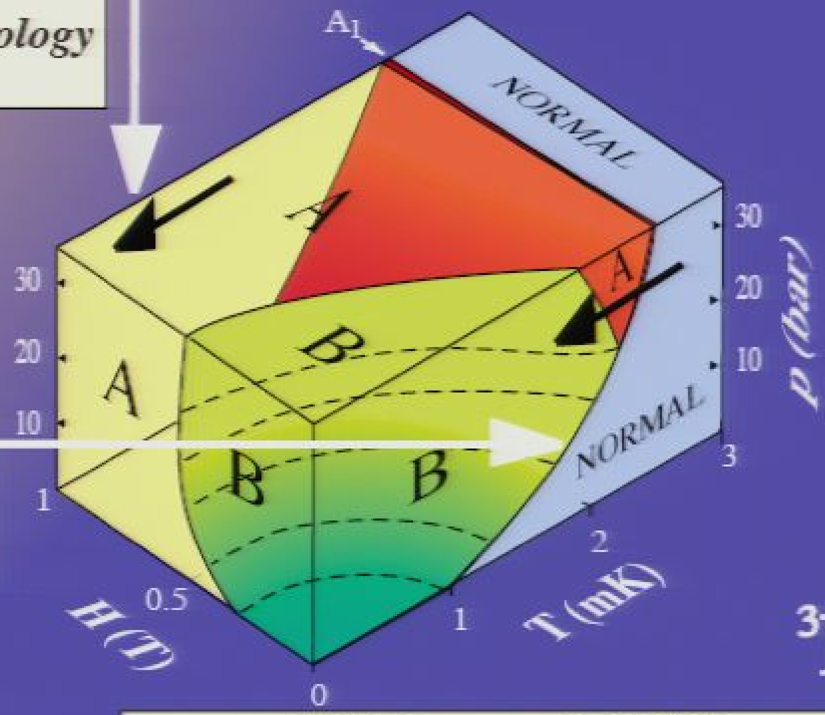

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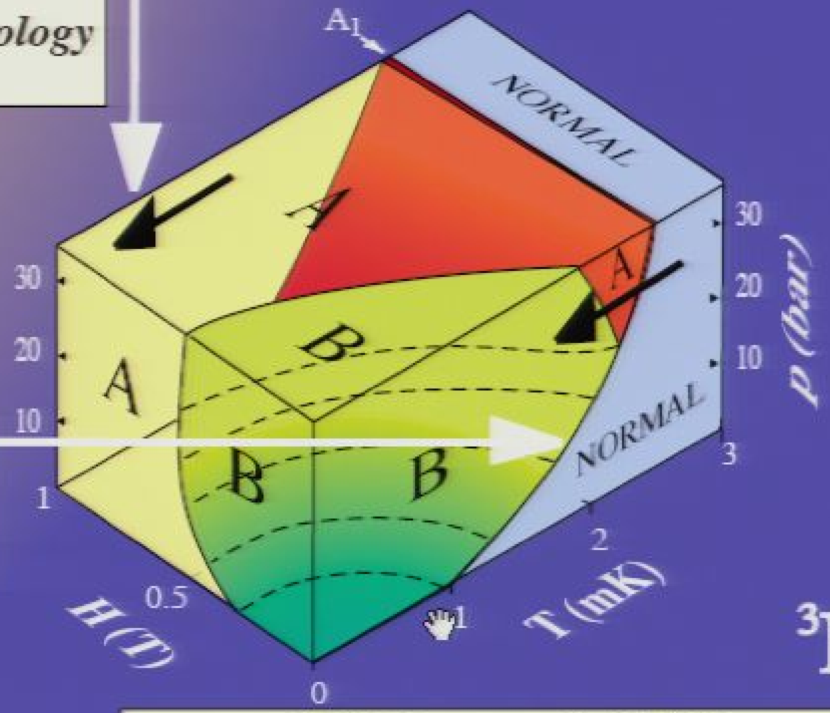

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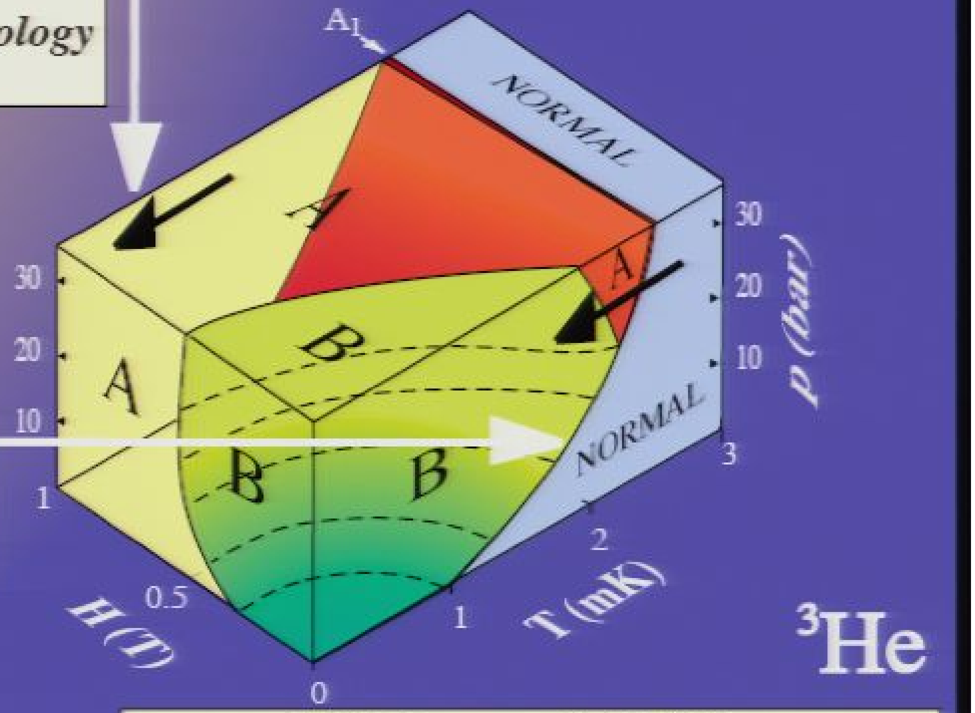

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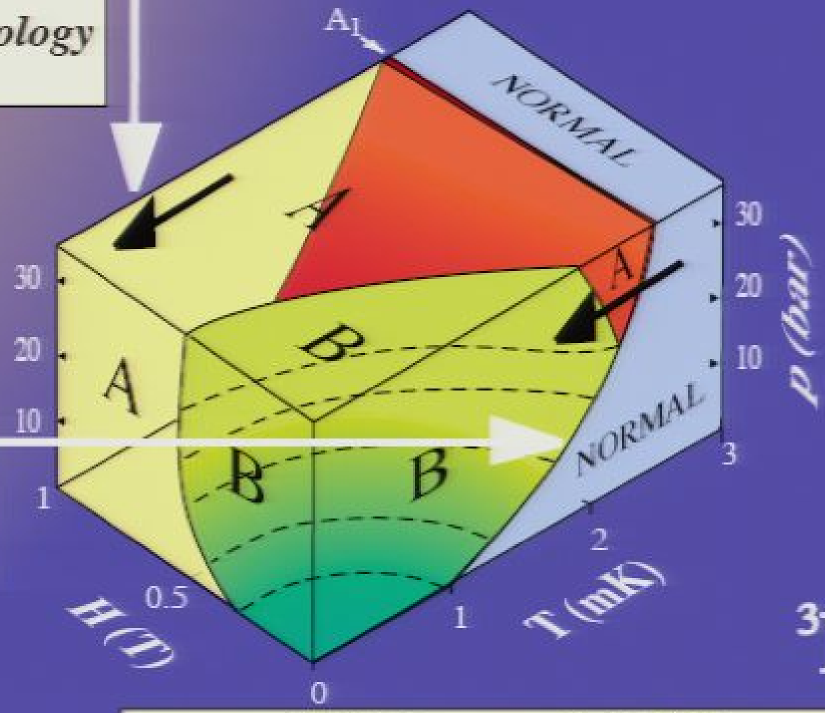

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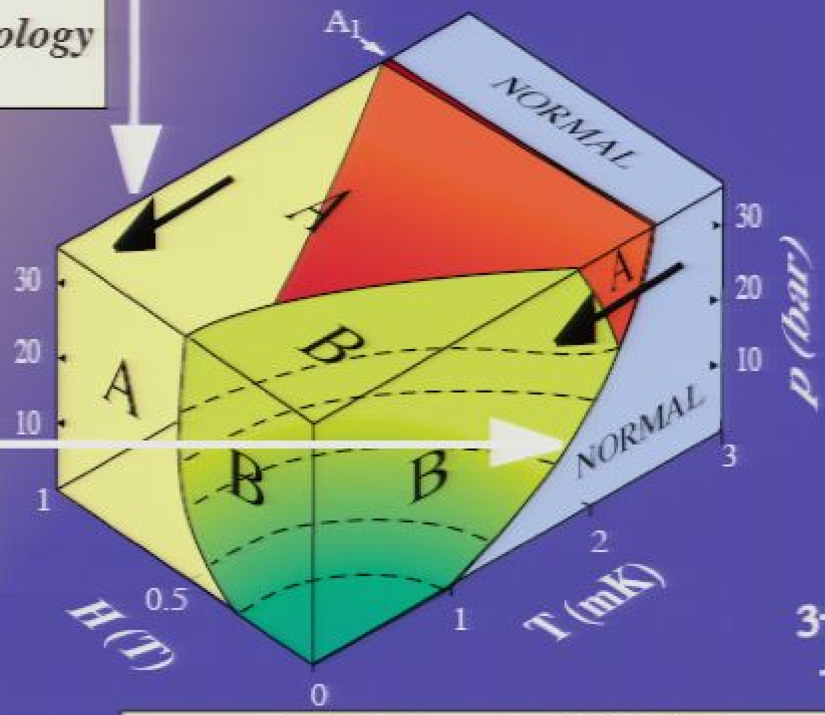

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$$T_{\text{electroweak}} = 10^{-17} E_{\text{Planck}}$$

$$T_{\text{CMBR}} = 10^{-30} E_{\text{Planck}}$$

high-energy physics is physics of ultra-low temperatures

cosmology too

physics at low T is determined by low-lying excitations

all other degrees are frozen out

low-lying excitations live near Fermi surface

or near Fermi point

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03

physics at low T is determined by low-lying excitations

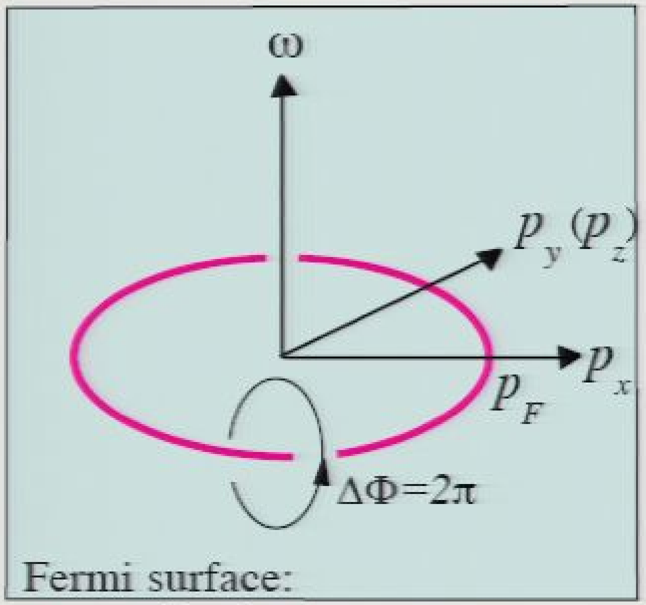
low-lying excitations live near Fermi surface

universality classes of quantum field theory

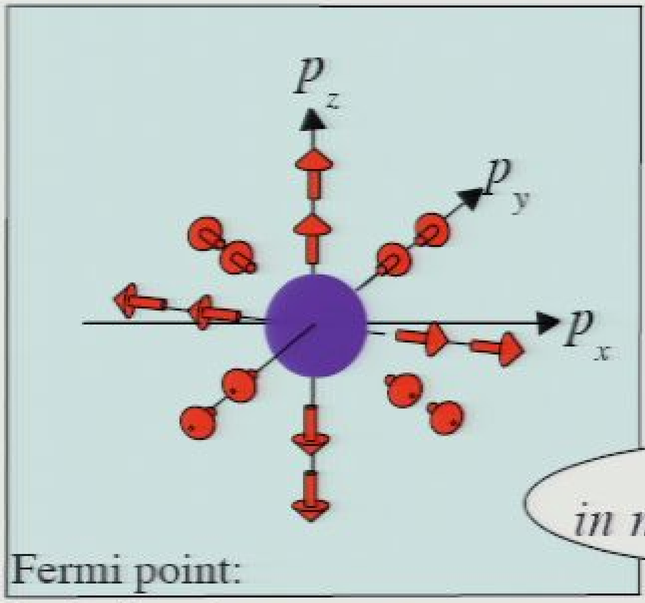
or near Fermi point

Fermi surface class

Fermi point class



Fermi surface:
vortex line in **p**-space



Fermi point:
hedgehog in **p**-space

topology in momentum space

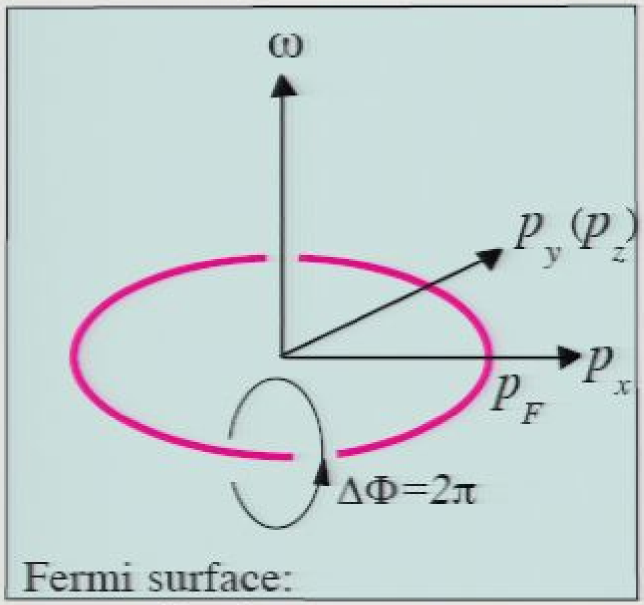
physics at low T is determined by low-lying excitations

low-lying excitations live near Fermi surface

universality classes of quantum field theory

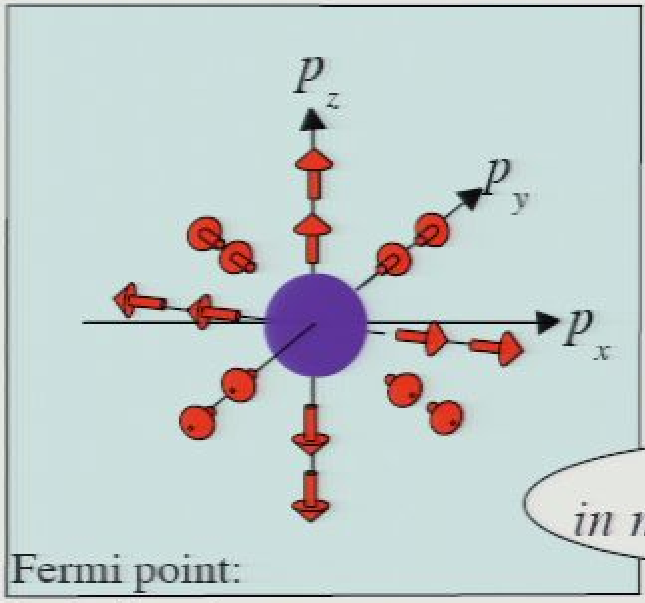
or near Fermi point

Fermi surface class



Fermi surface:
vortex line in **p**-space

Fermi point class



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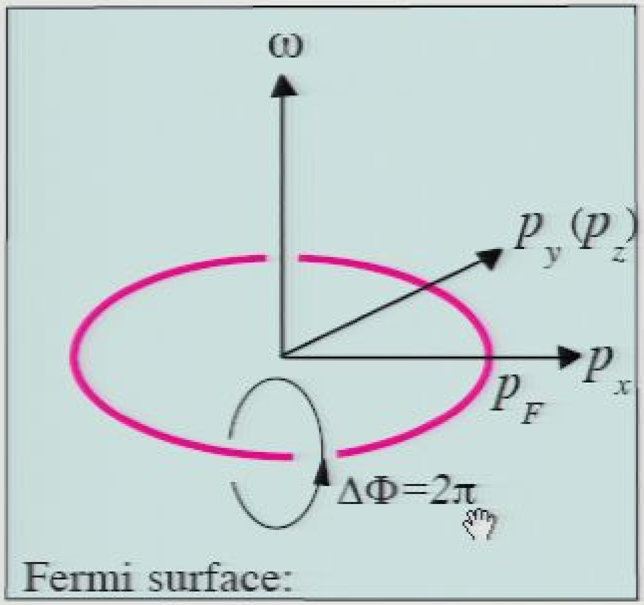
↓ Fermi surface class

↓ Fermi point class



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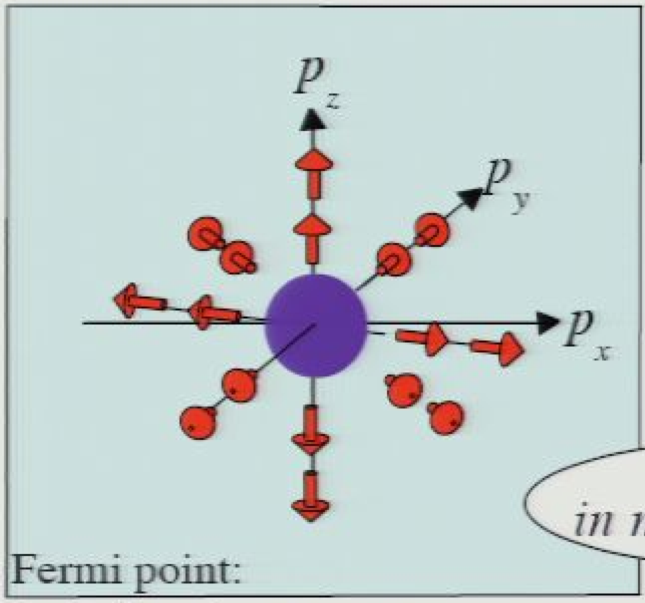

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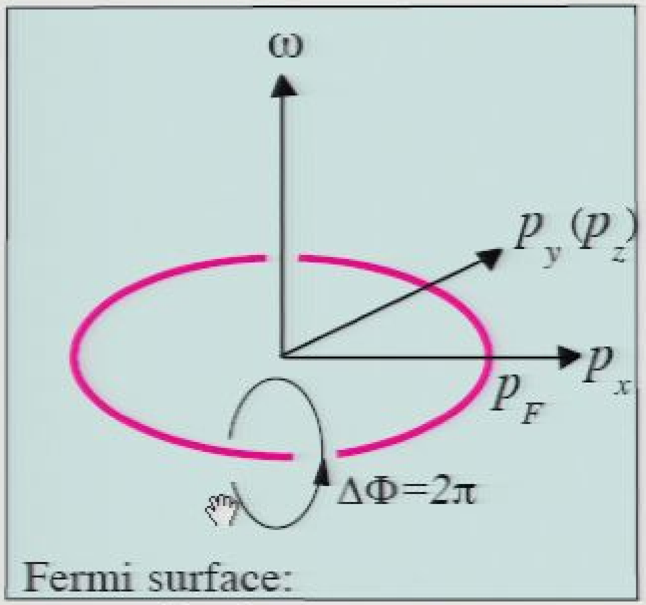
Fermi point class



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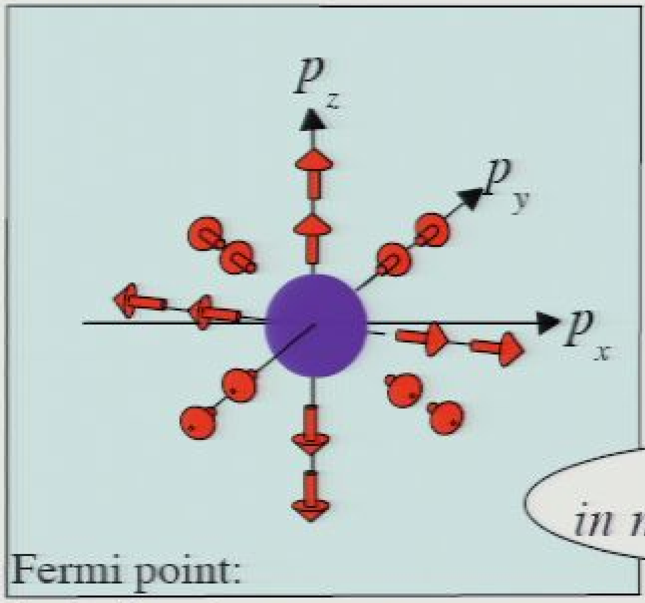

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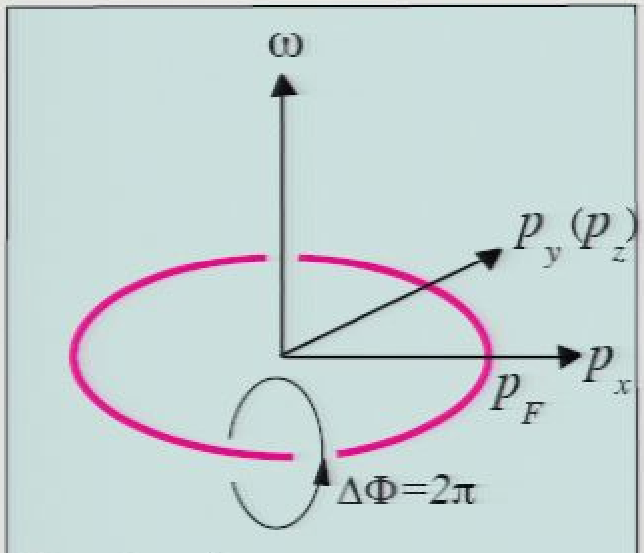

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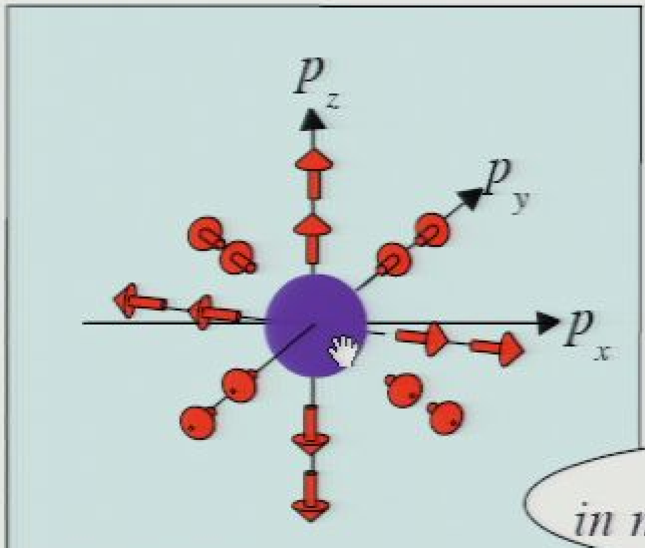


Fermi point:
hedgehog in **p**-space

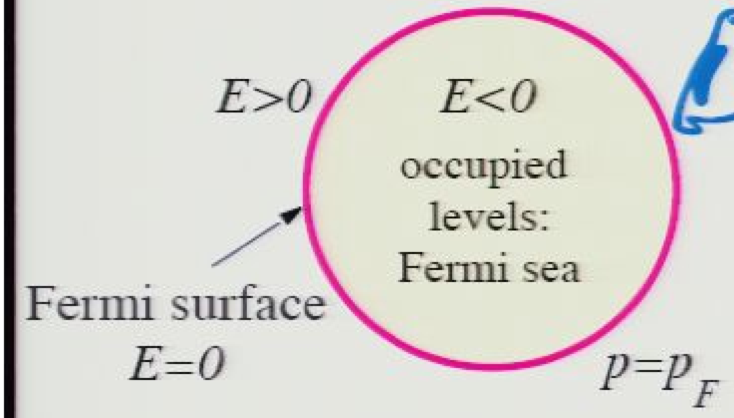
topology in momentum space

Topological stability of Fermi surface: route to Landau Fermi-liquid

$$E = \frac{p^2}{2m} - \mu = v_F (p - p_F)$$

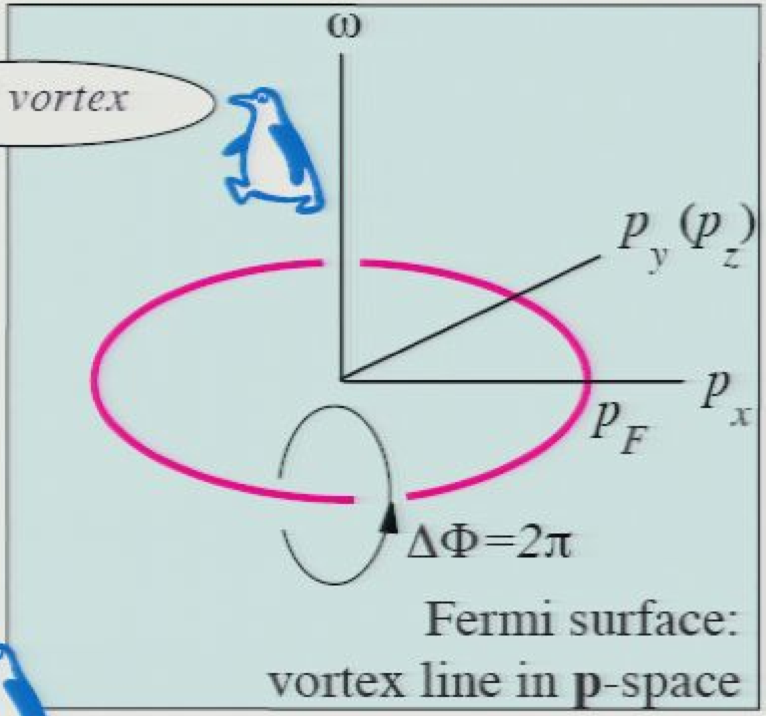
Green's function

$$G^{-1} = i\omega - v_F (p - p_F)$$



domain wall ?

no! vortex



Fermi surface is robust to interaction

then it must survive in Fermi liquid

is this the reason
why Landau theory of Fermi liquid works?

phase of Green's function

$$G(\omega, \mathbf{p}) = |G| e^{\,i\Phi}$$

has winding number N=1

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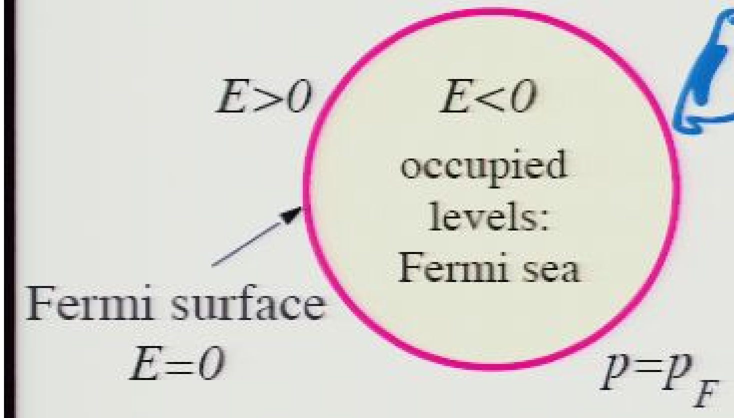

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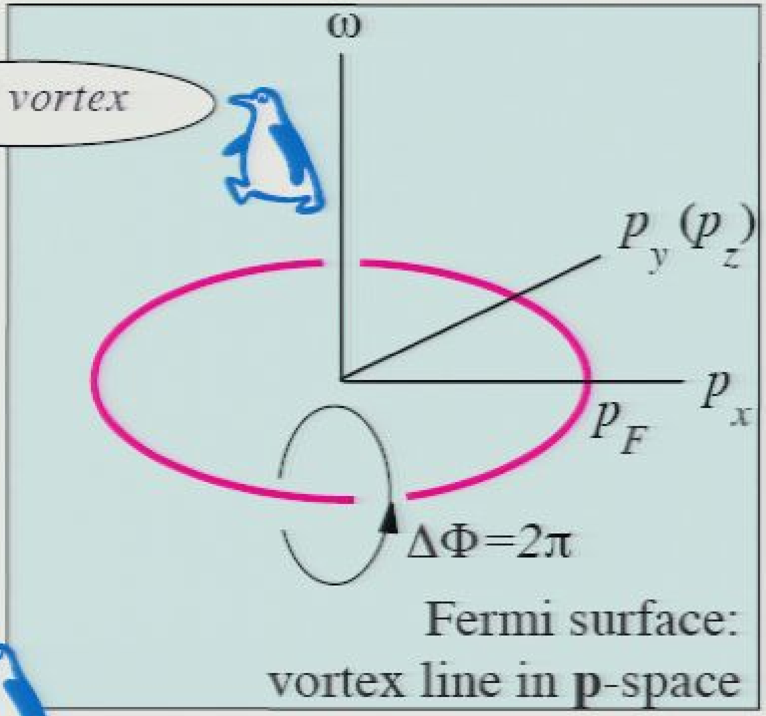

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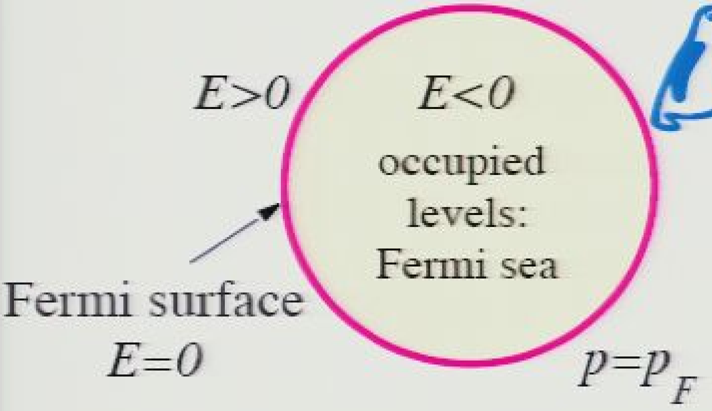

$E>0$

$E<0$

occupied
levels:
Fermi sea

Fermi surface
$E=0$

$p=p_F$

Green's function

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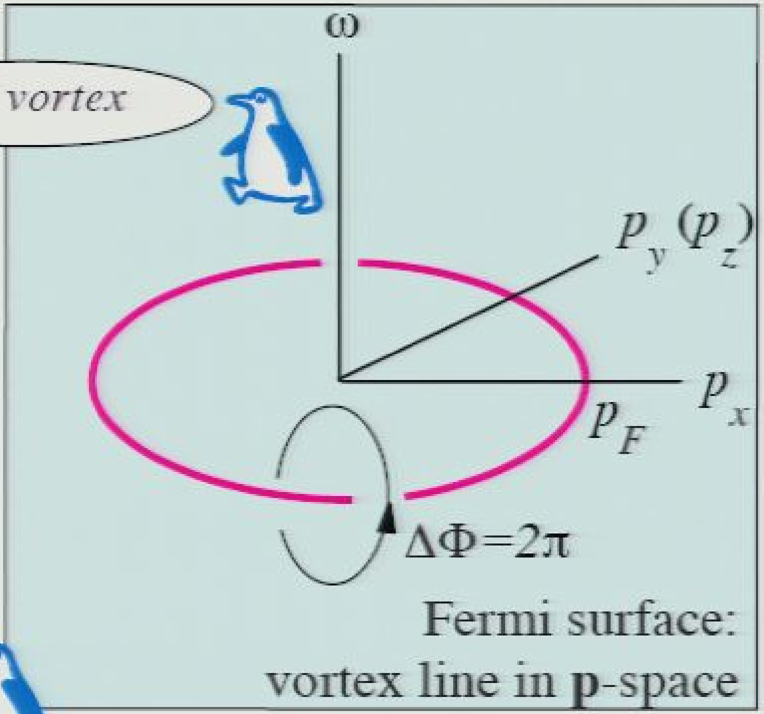

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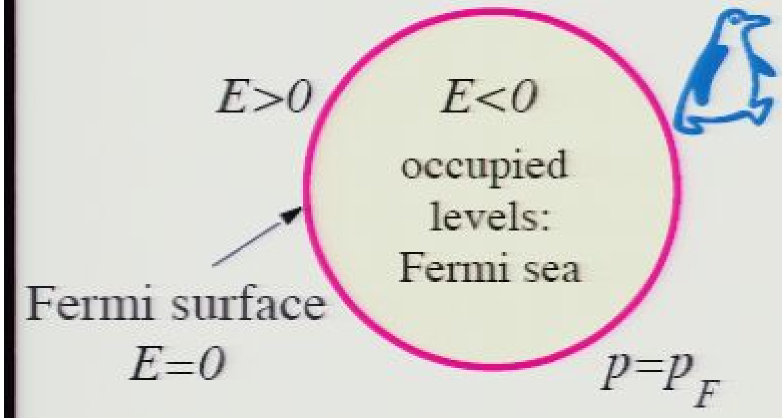

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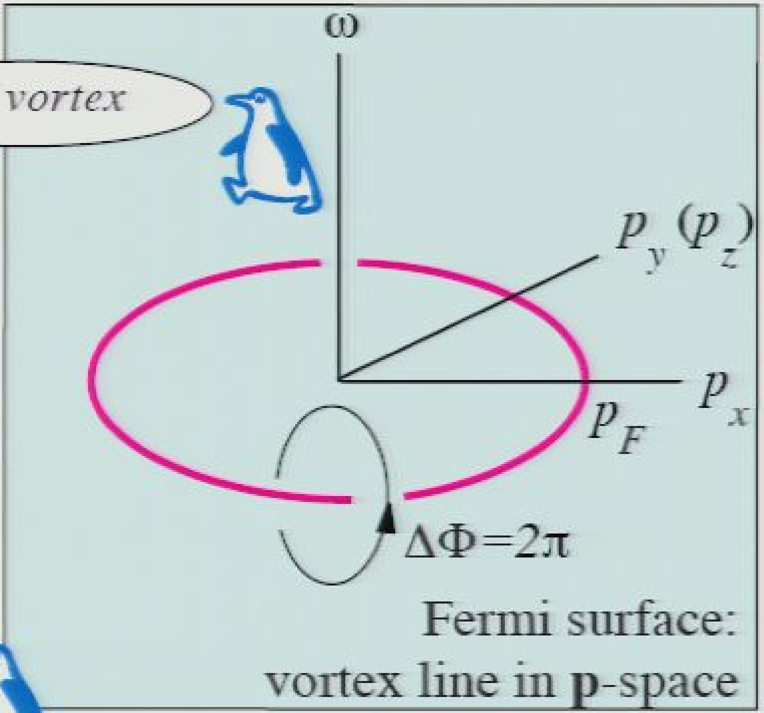

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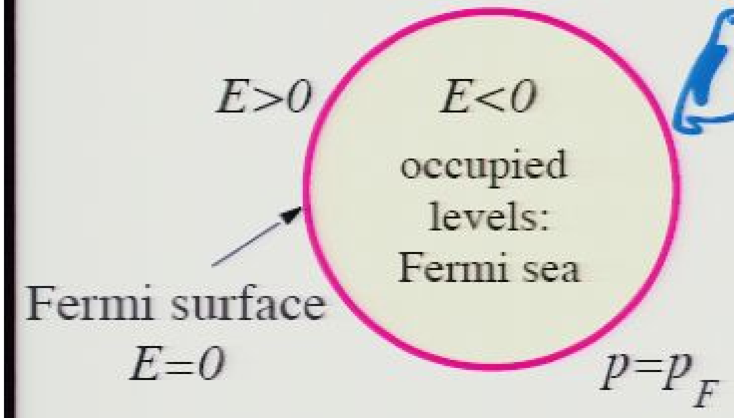

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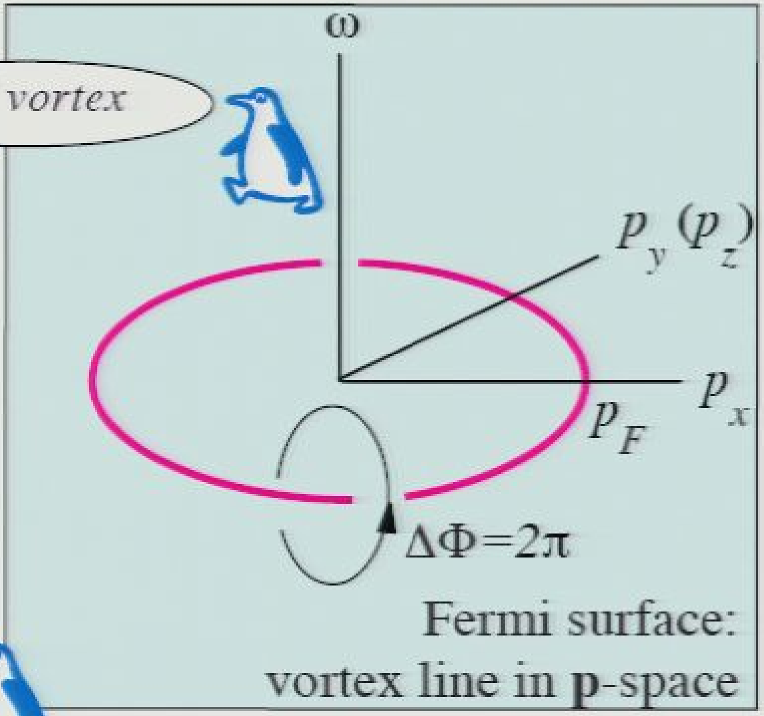

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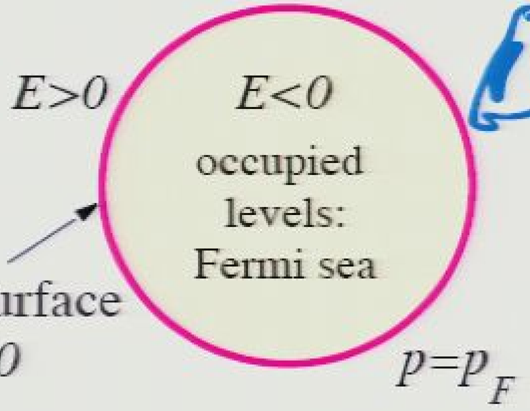

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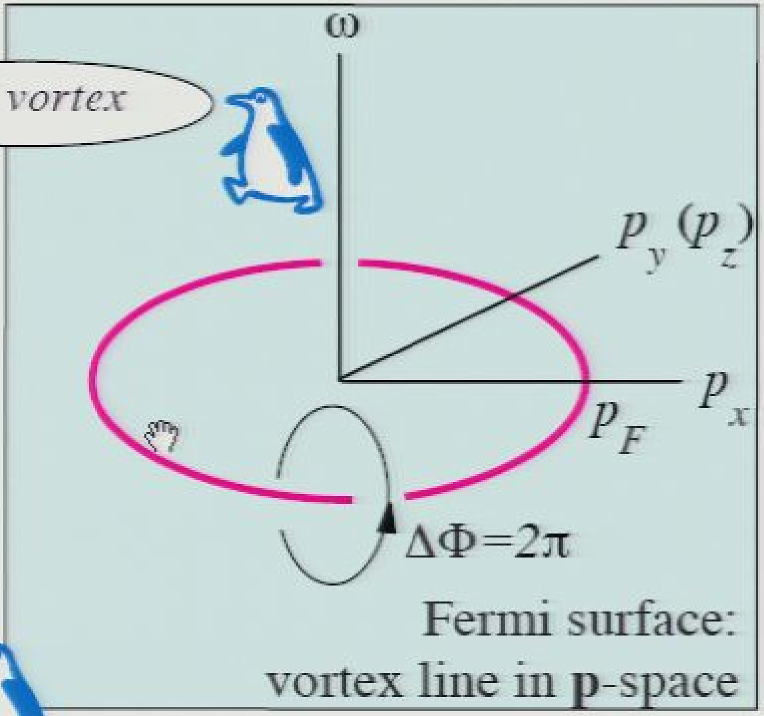

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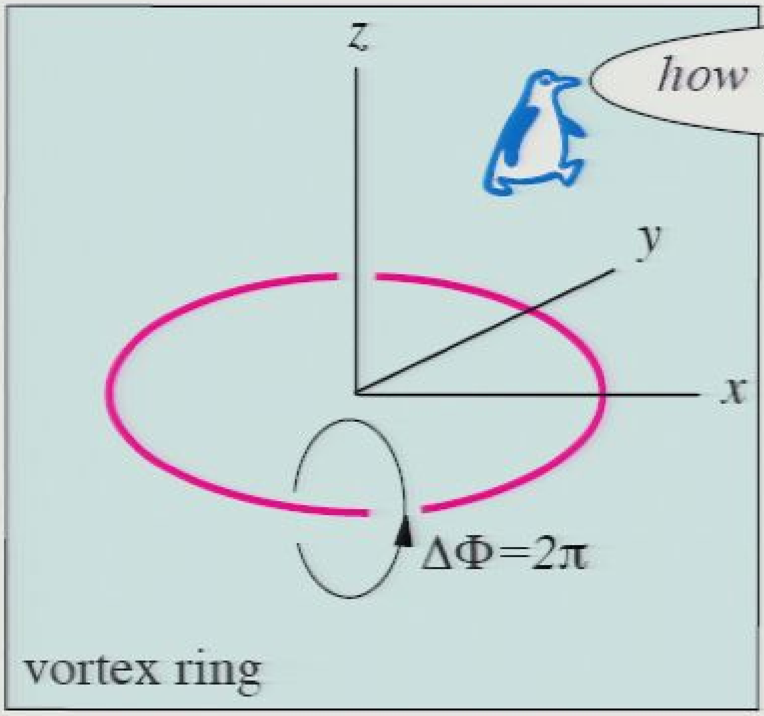
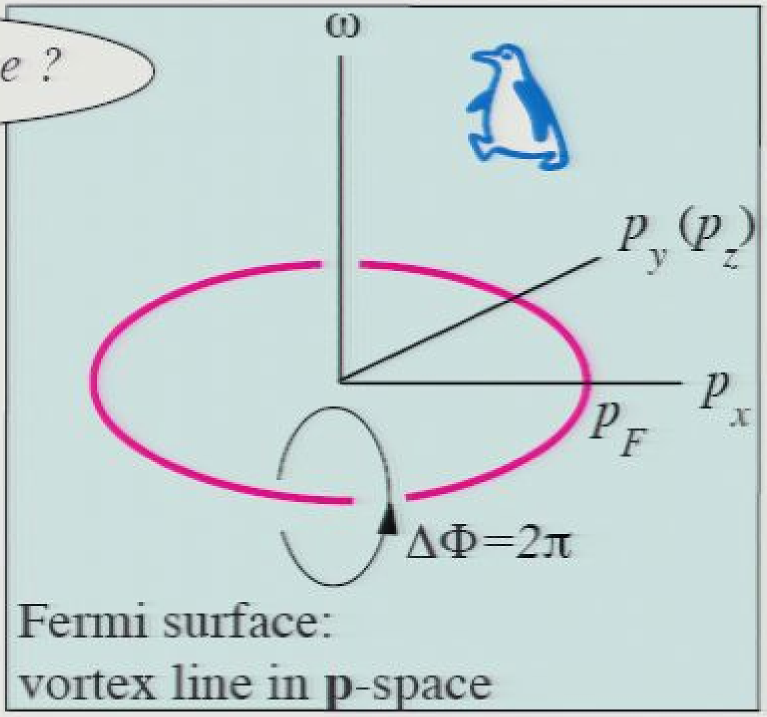
quantized vortex vs Fermi surface

Topology in **r**-space



how is it in ***p****-space ?*

Topology in **p**-space



$$\Psi(\mathbf{r})=|\Psi|\, e^{\,i\Phi}$$

order parameter

$$G(\omega,\mathbf{p})=|G|\, e^{\,i\Phi}$$

Green's function

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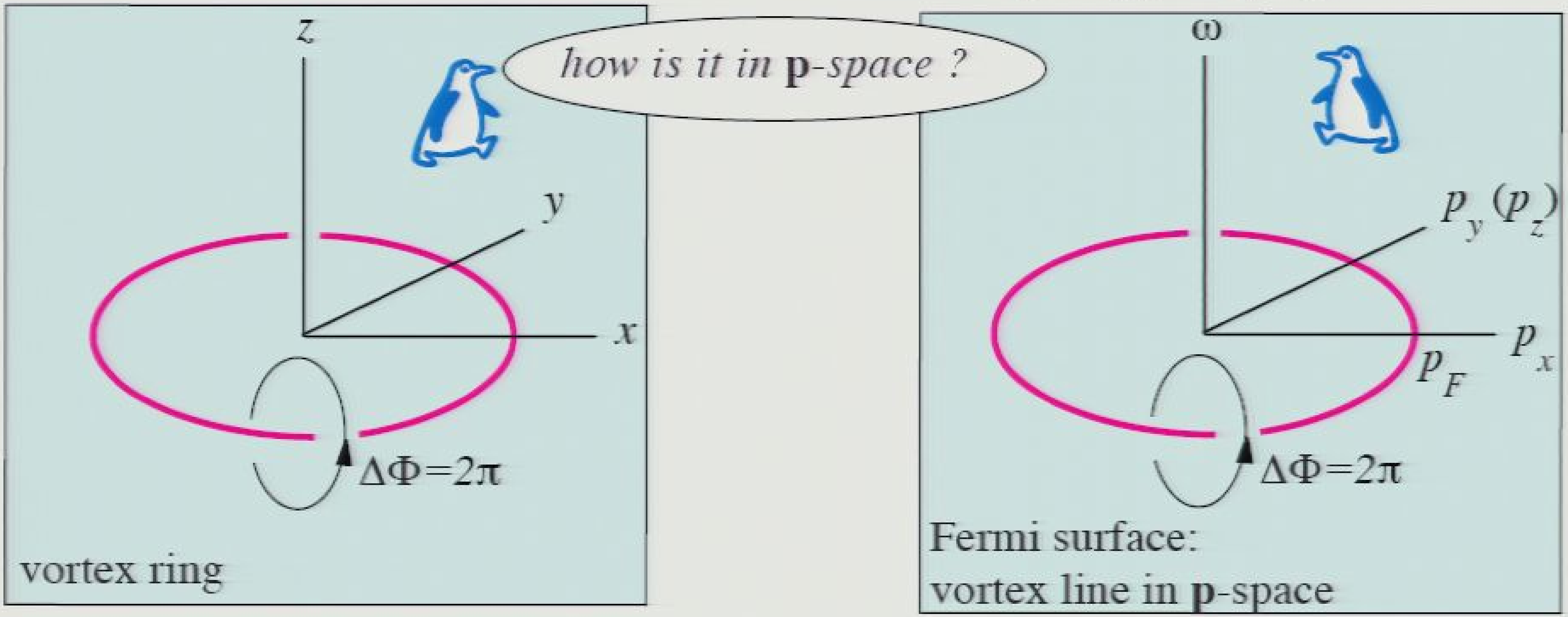
quantized vortex vs Fermi surface

Topology in **r**-space

Topology in **p**-space



vortex ring

Fermi surface:
vortex line in **p**-space

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order parameter

$$G(\omega,\mathbf{p})=|G|\,e^{\,i\Phi}$$

Green's function

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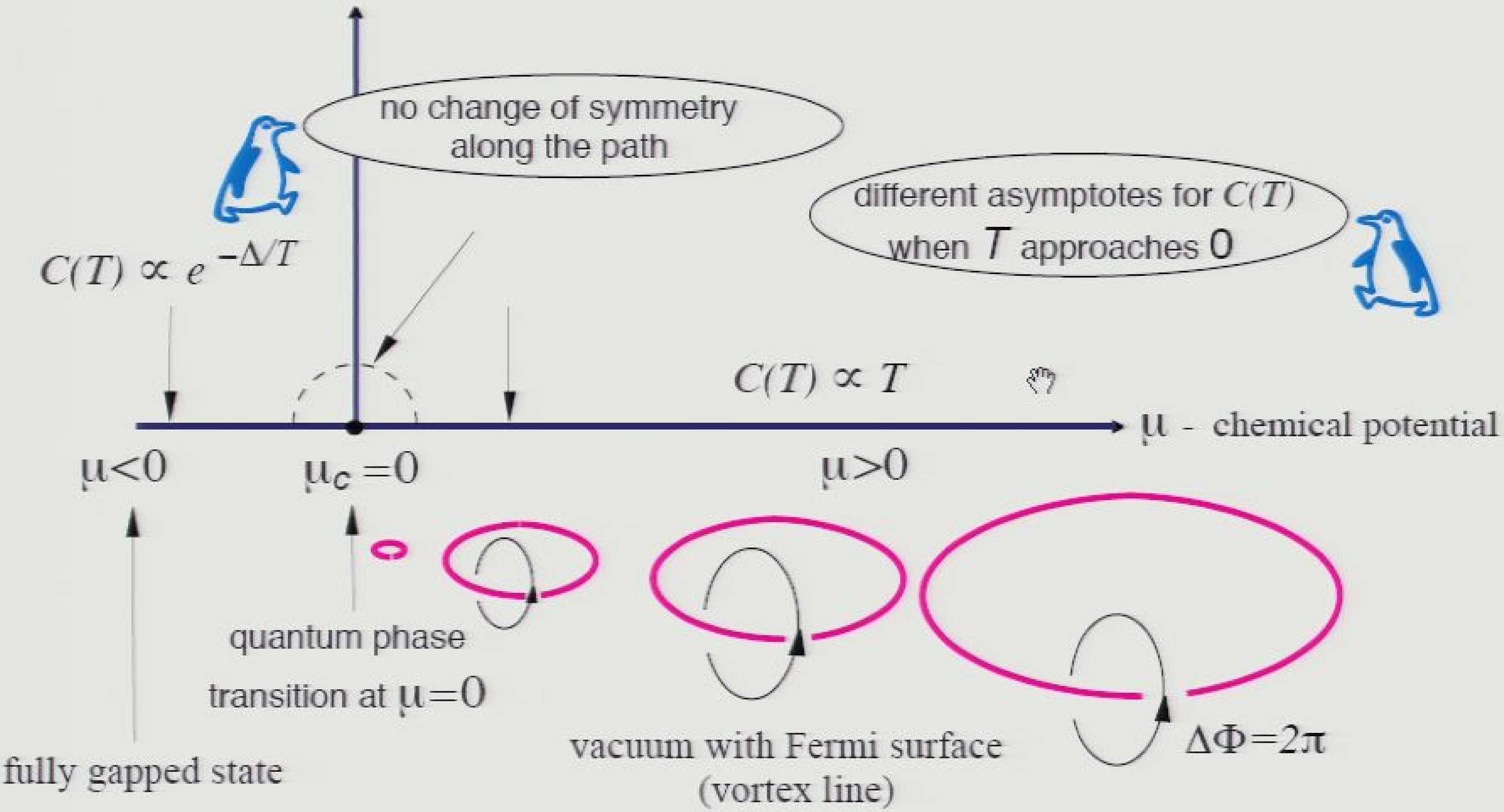
Lifshitz transition: vortex loop shrinks

$$E = \frac{p^2}{2m} - \mu$$



T (temperature)

no change of symmetry along the path

different asymptotes for $C(T)$ when T approaches 0

$C(T) \propto e^{-\Delta/T}$

$C(T) \propto T$

μ - chemical potential

$\mu<0$

$\mu_c=0$

$\mu>0$

quantum phase transition at $\mu=0$

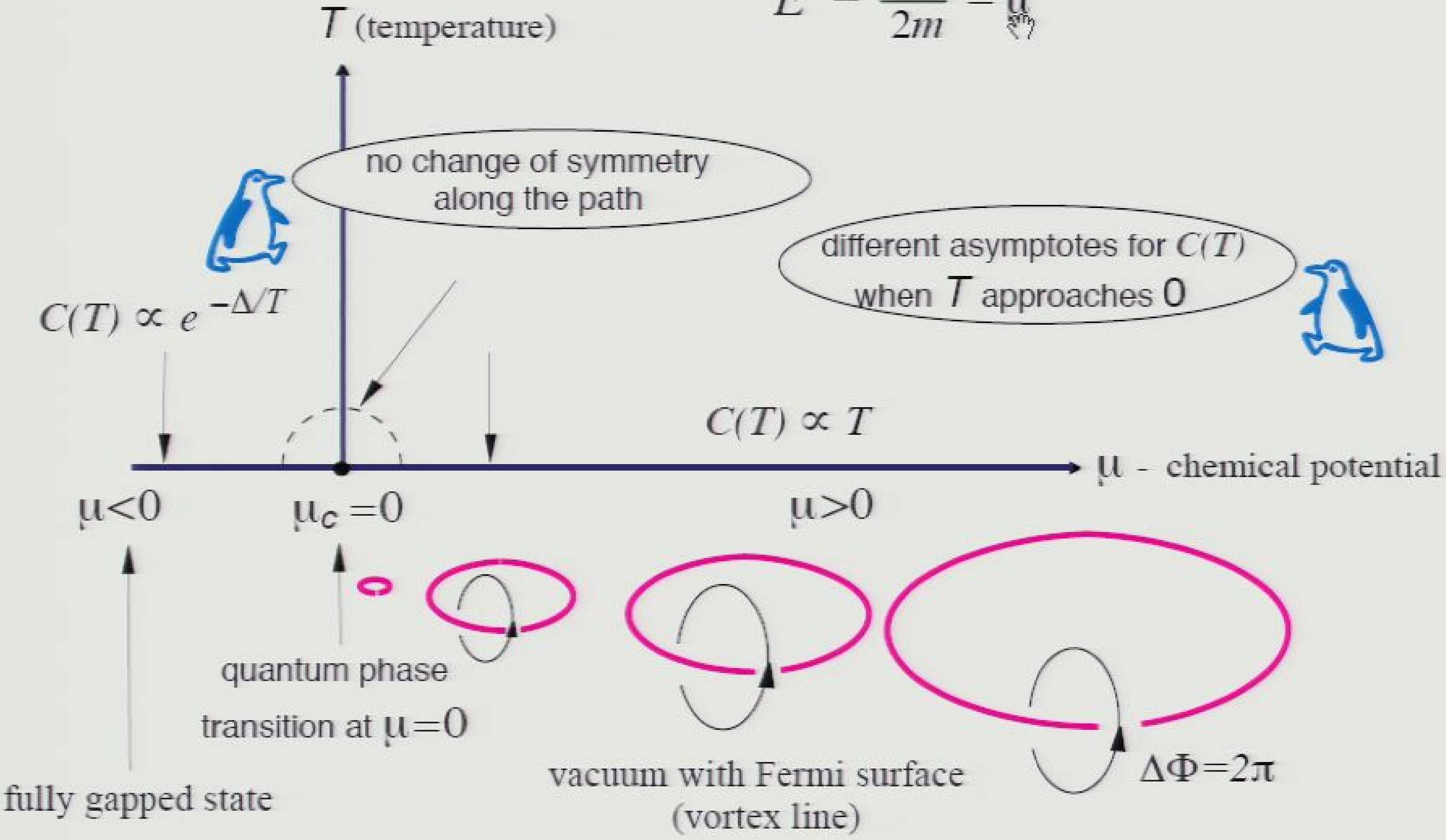
vacuum with Fermi surface (vortex line)

$\Delta\Phi=2\pi$

fully gapped state

Lifshitz transition: vortex loop shrinks

$$E = \frac{p^2}{2m} - \mu$$



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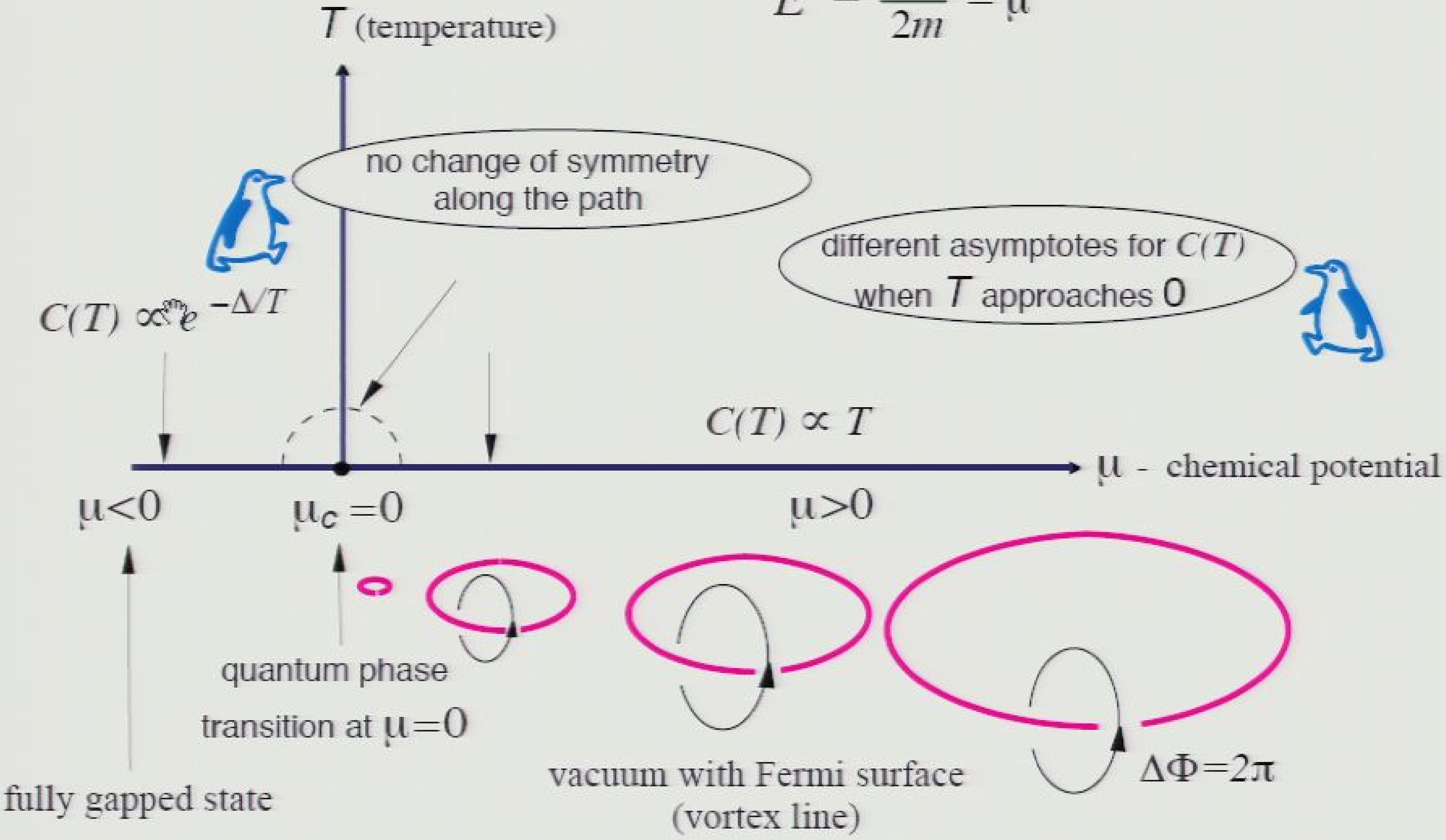

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$C(T) \propto T$

μ - chemical potential

$\mu<0$

$\mu_c=0$

$\mu>0$

quantum phase transition at $\mu=0$

fully gapped state

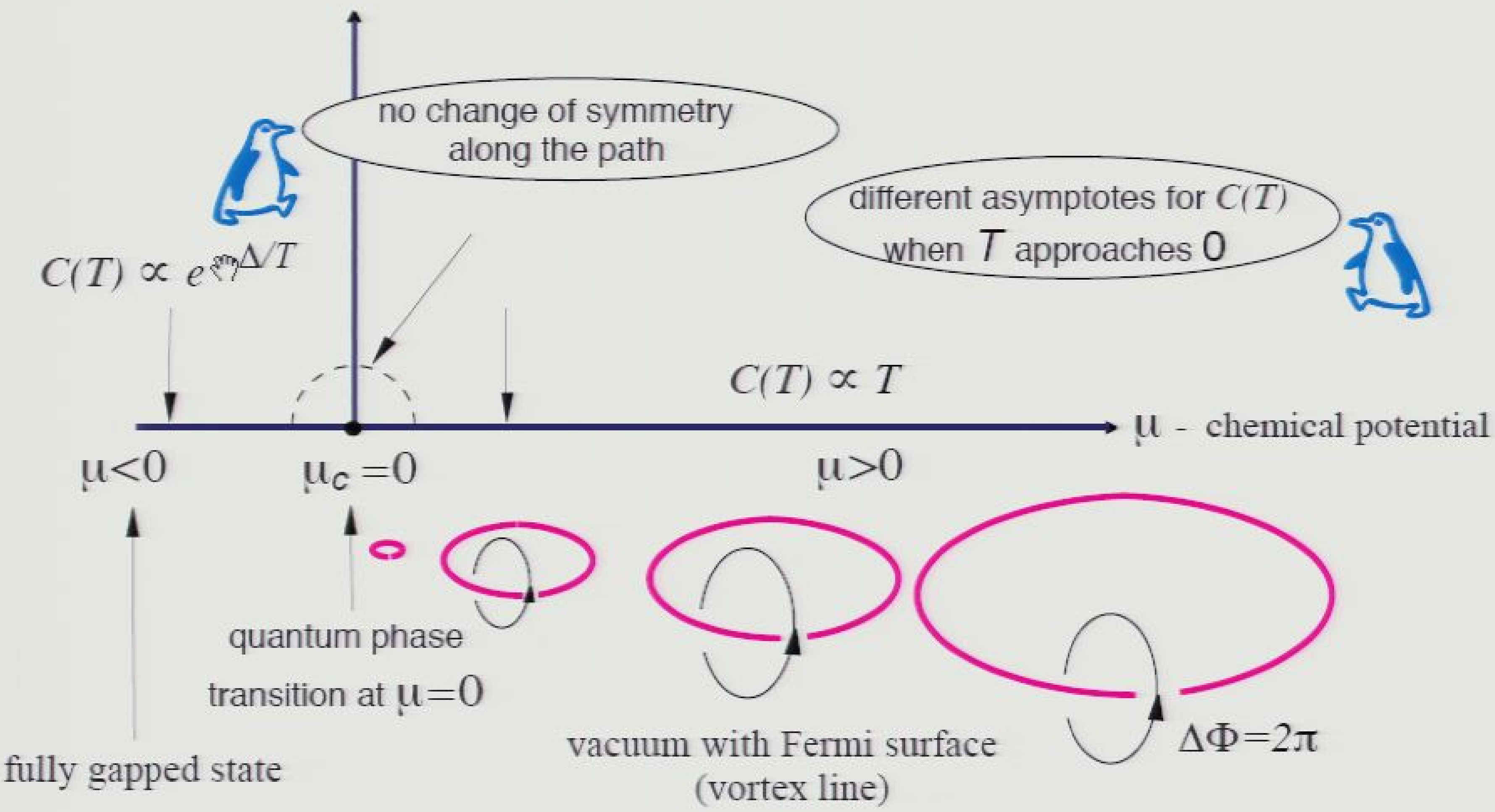
vacuum with Fermi surface (vortex line)

$\Delta\Phi=2\pi$

Lifshitz transition: vortex loop shrinks

$$E = \frac{p^2}{2m} - \mu$$

T (temperature)



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quantum phase transitions induced by p-space topology

transitions between **ground states** of the **same symmetry**,
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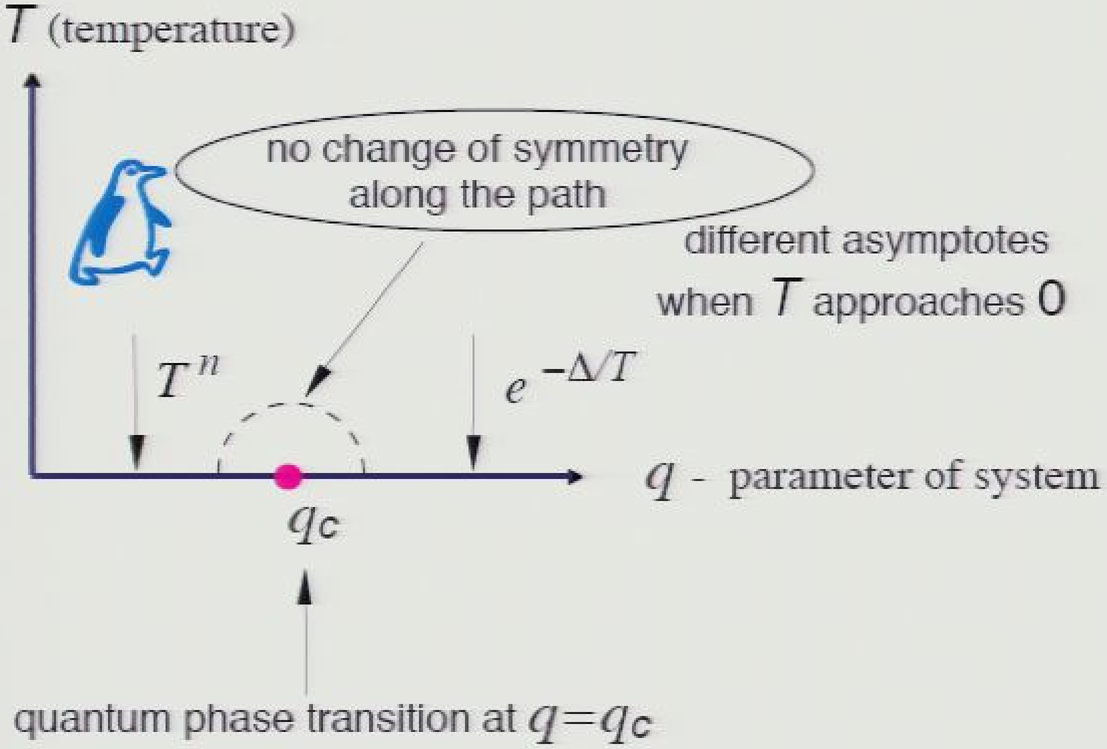

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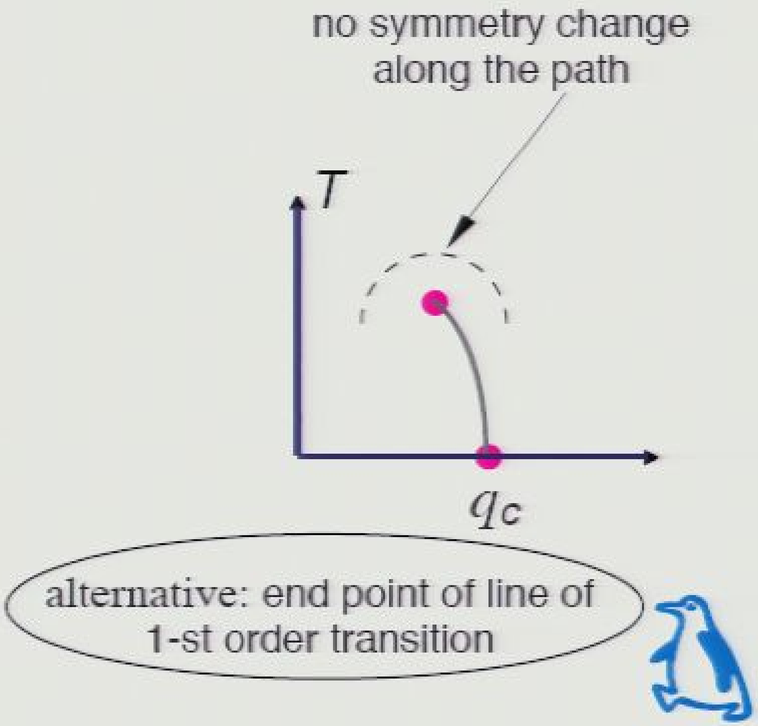
quantum phase transition at $q=q_c$

Lifshitz transition, plateau transition, ...



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alternative: end point of line of 1-st order transition

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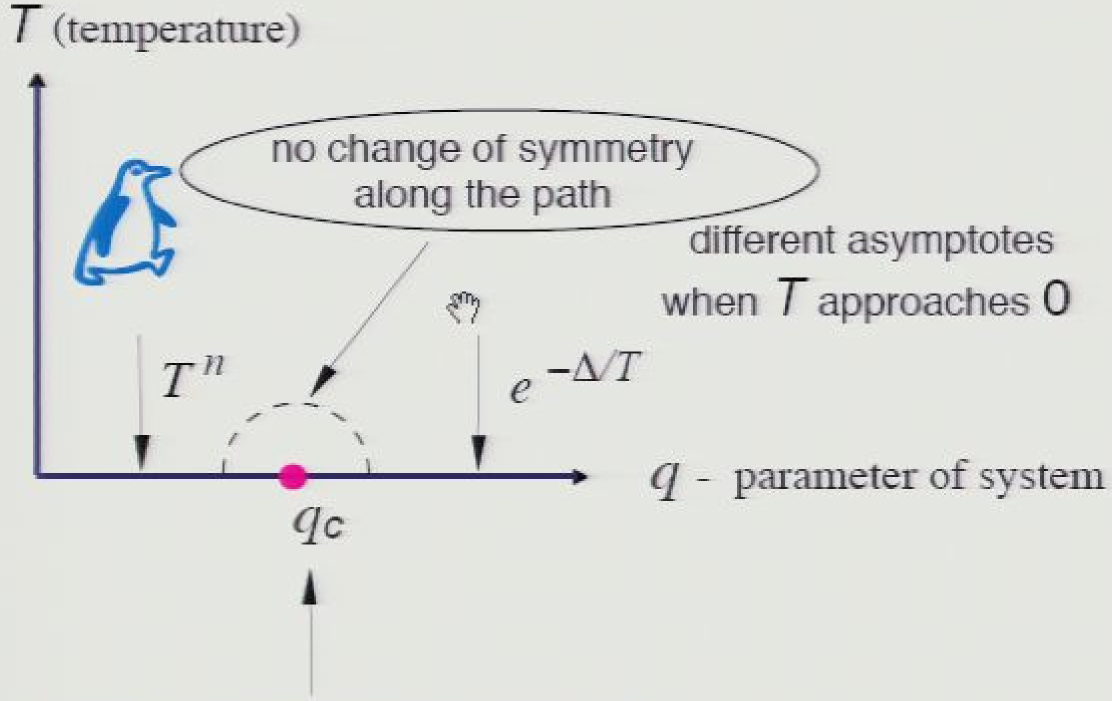

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different asymptotes when T approaches 0

T^n

$e^{-\Delta/T}$

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q_c

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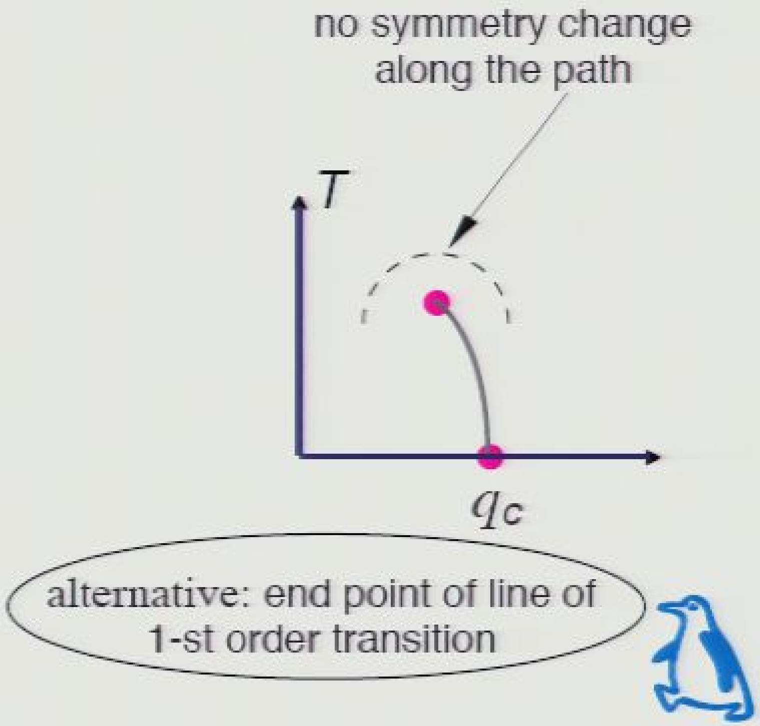

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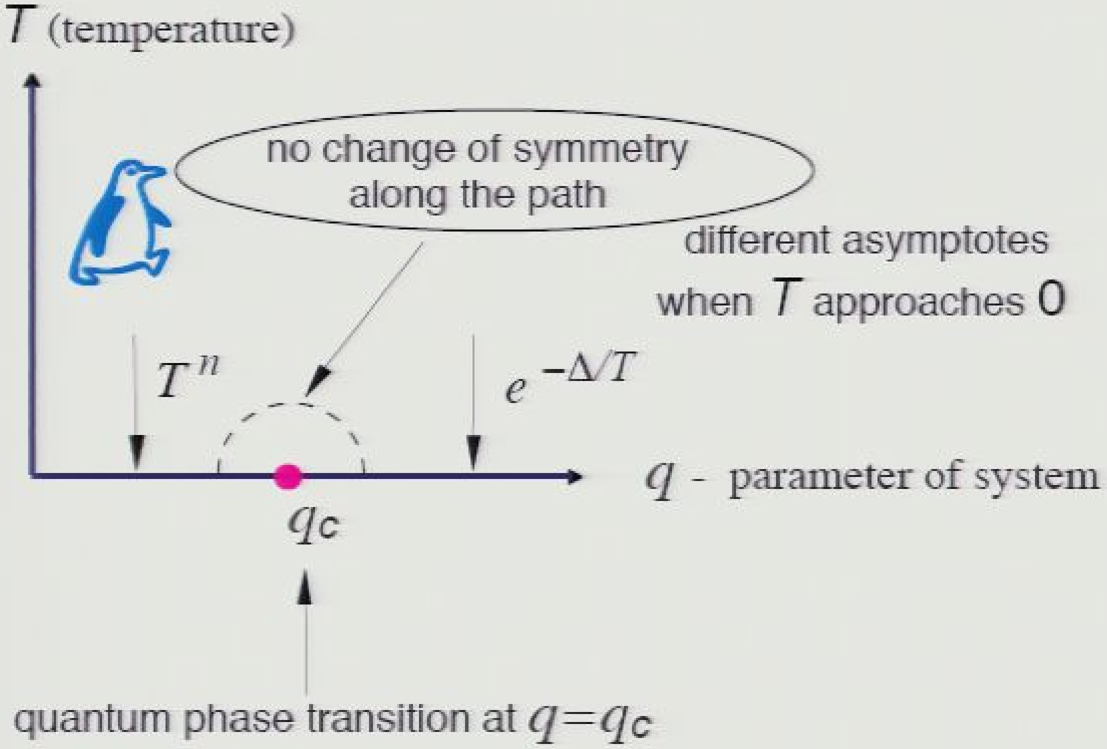

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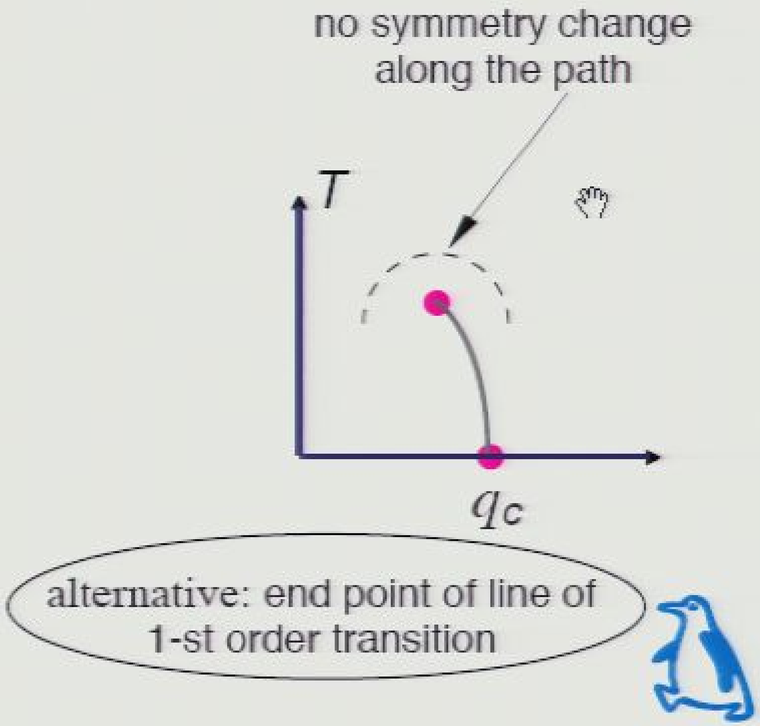

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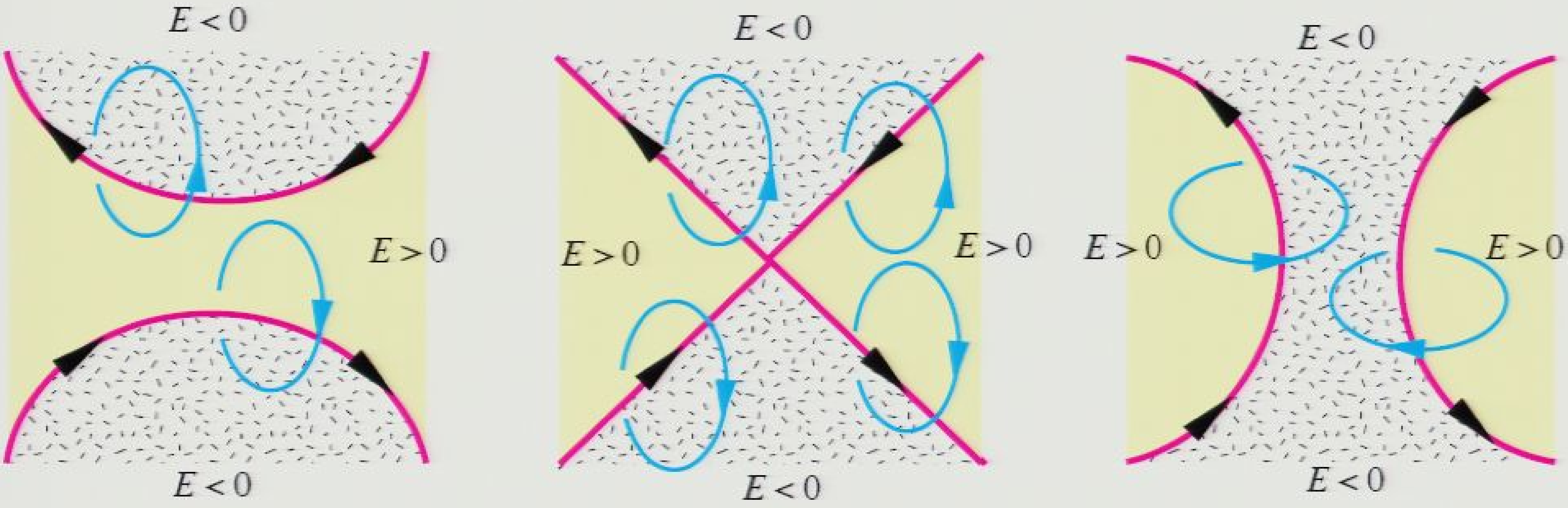
Lifshitz transition as reconnection of vortex lines in p-space

$$E(\mathbf{p}) = p_x^2 - p_y^2 - \mu$$



Fermi surface
before Lifshitz transition

$\mu < 0$

Fermi surface
at Lifshitz point
of quantum phase transition

$\mu = 0$

Fermi surface
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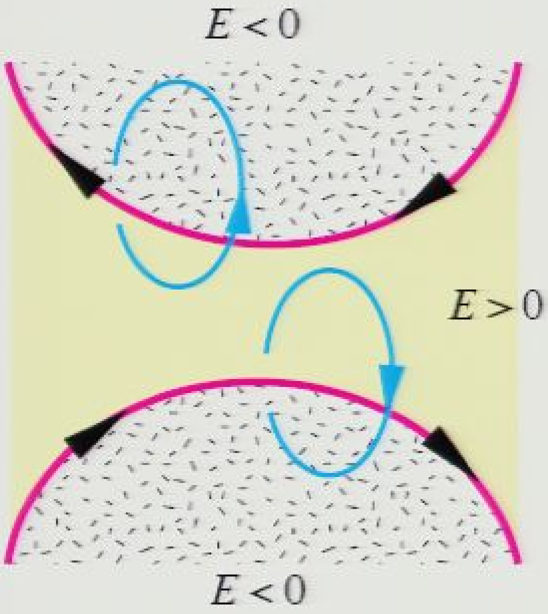

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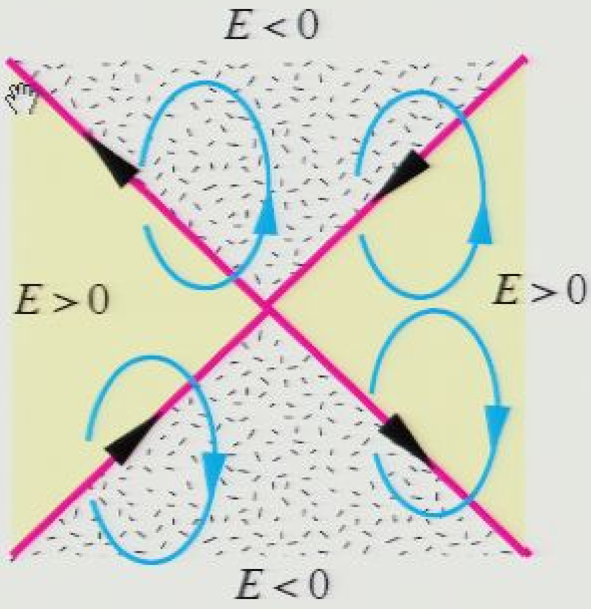

Fermi surface
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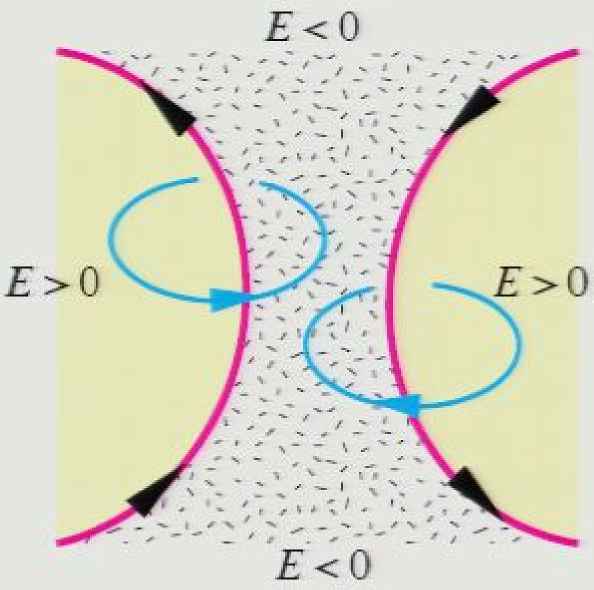

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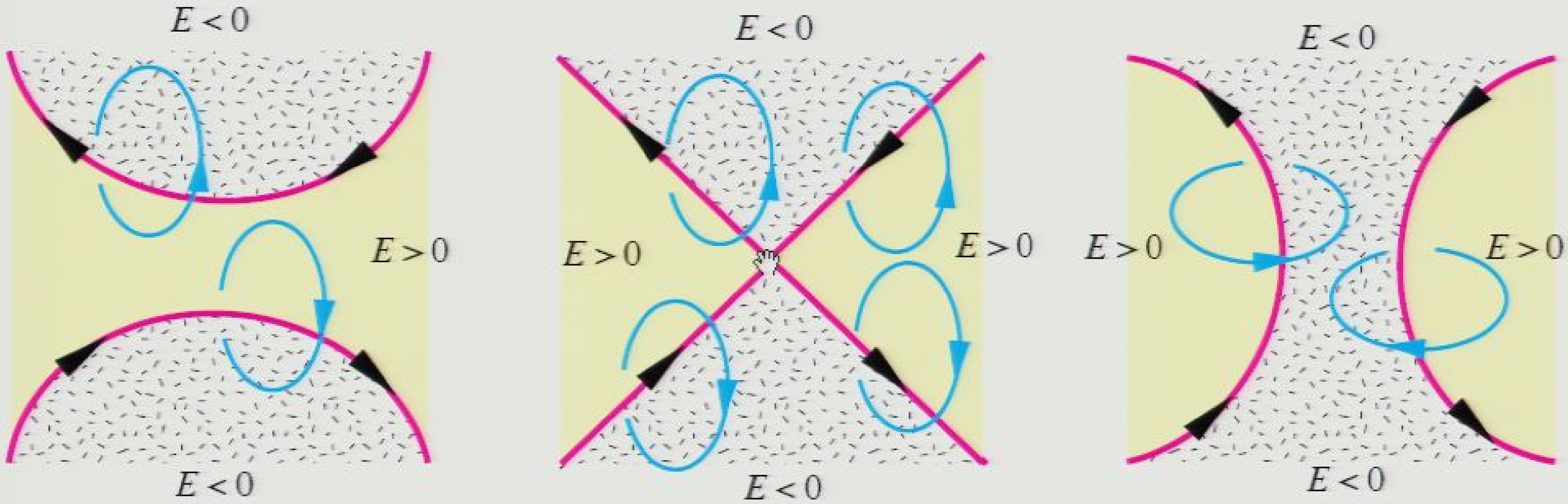

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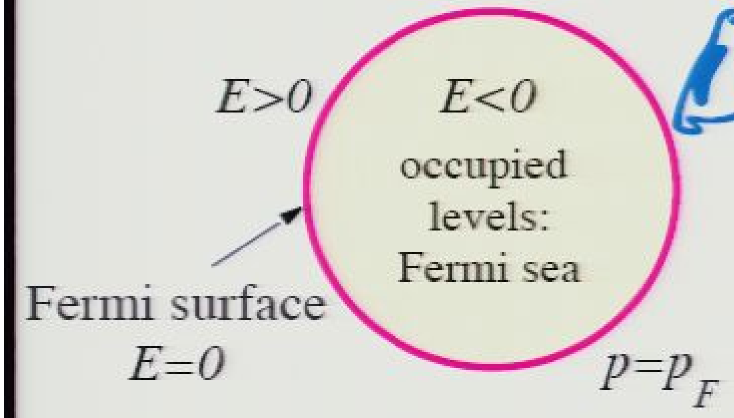
Topological stability of Fermi surface: route to Landau Fermi-liquid

$$E = \frac{p^2}{2m} - \mu = v_F\,(p-p_F)$$

Green's function

$$G^{-1} = i\omega - v_F\,(p-p_F)$$



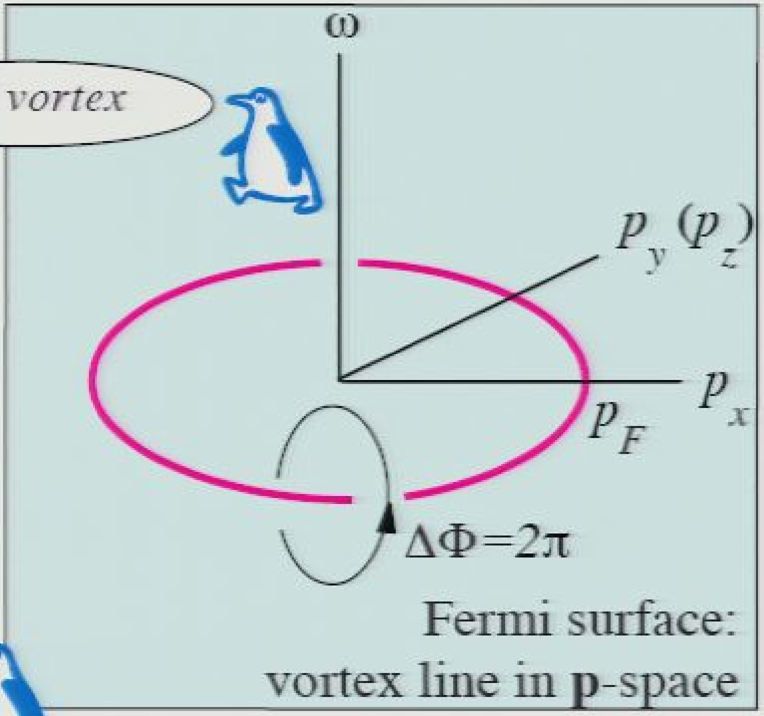
domain wall ?

no! vortex

Fermi surface is robust to interaction

then it must survive in Fermi liquid

is this the reason
why Landau theory of Fermi liquid works?



Fermi surface:
vortex line in **p**-space

phase of Green's function

$$G(\omega,\mathbf{p}) = |G|\,e^{\,i\Phi}$$

has winding number N=1

Lifshitz transition: vortex loop shrinks

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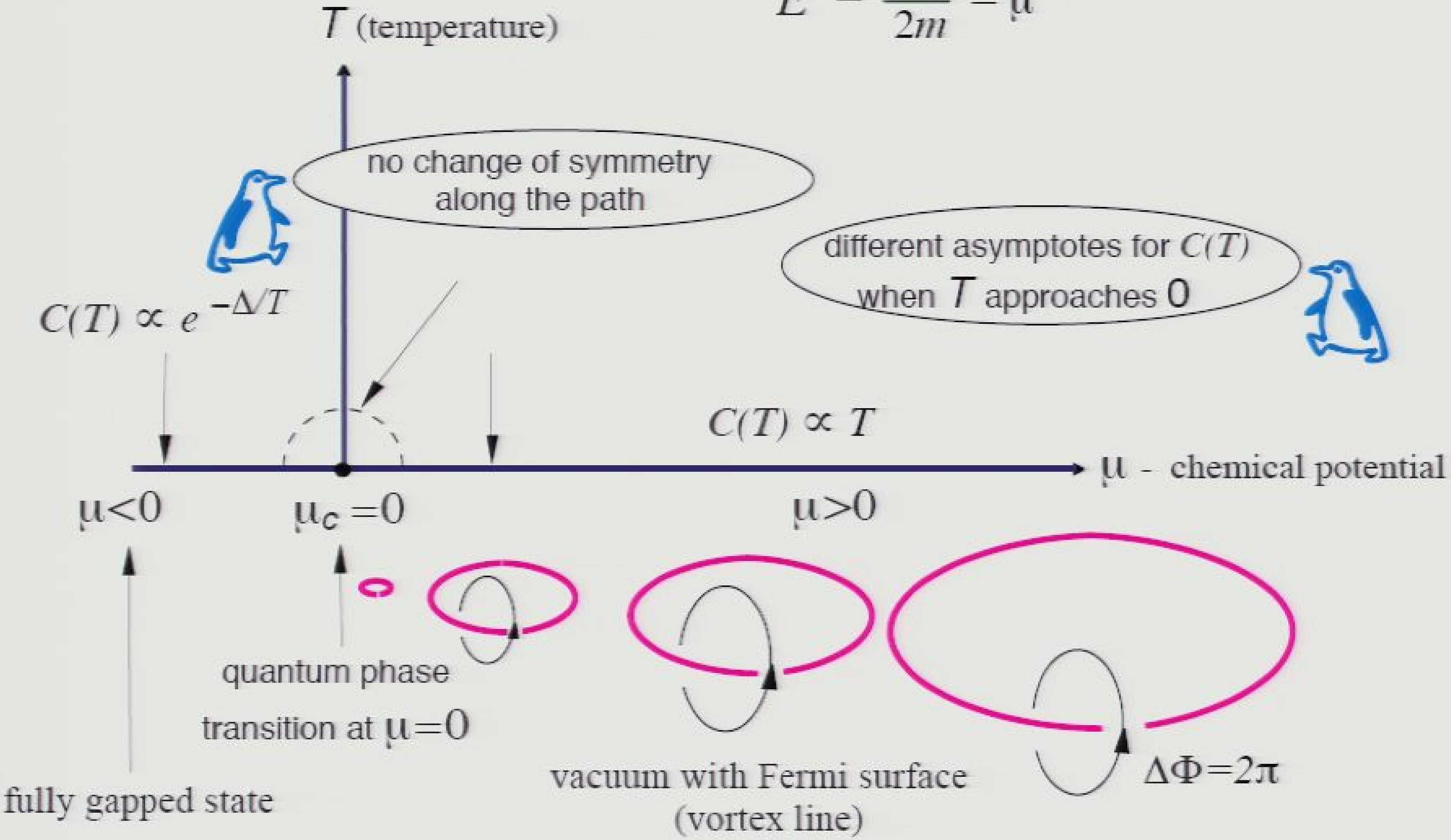

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$l = 4$

Lifshitz transition: vortex loop shrinks

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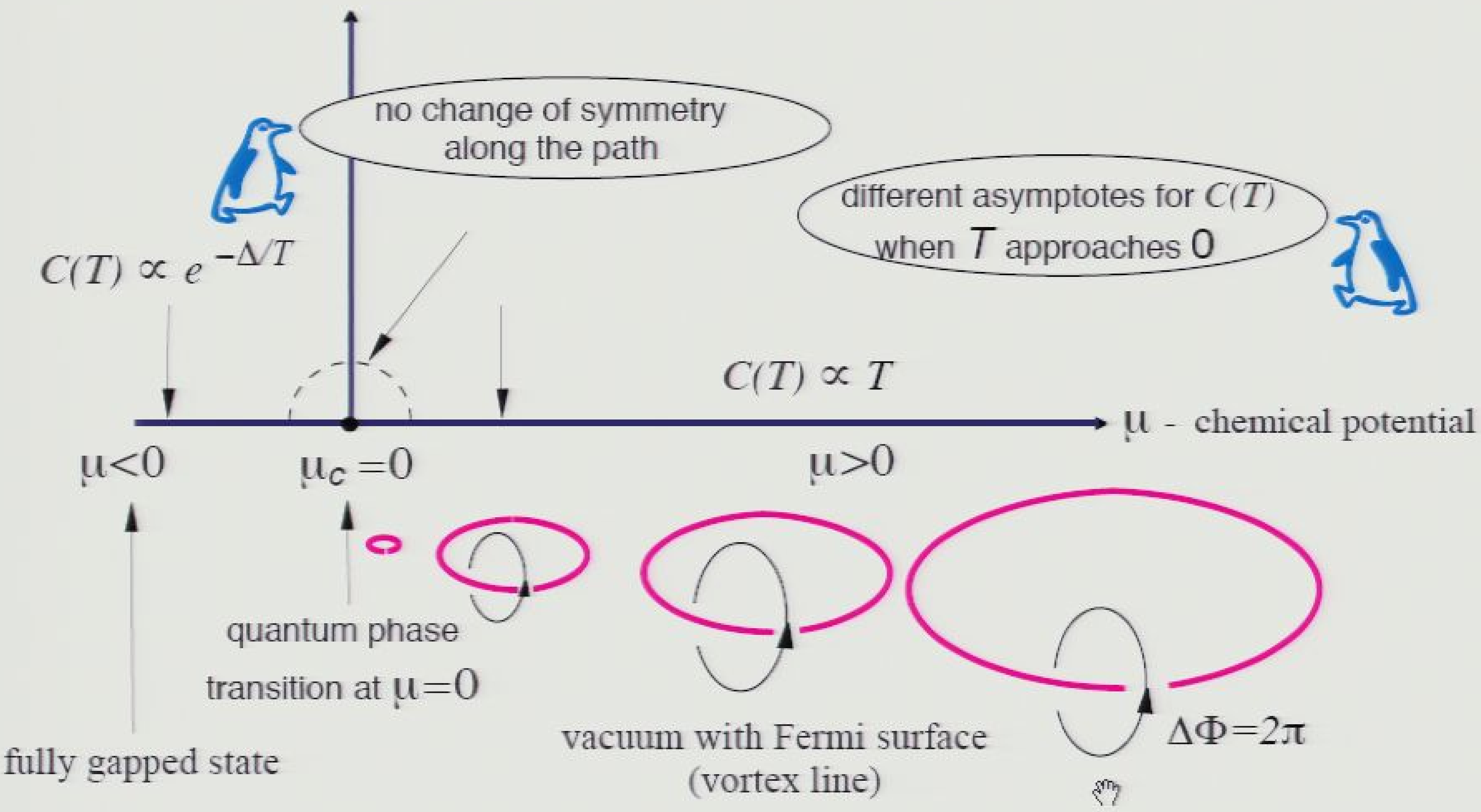

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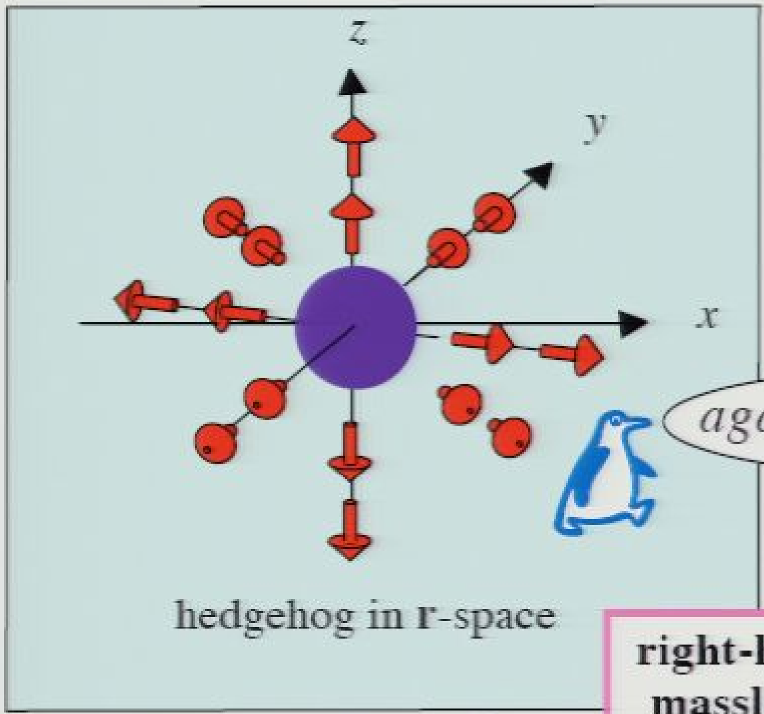
right-handed electron vs magnetic hedgehog

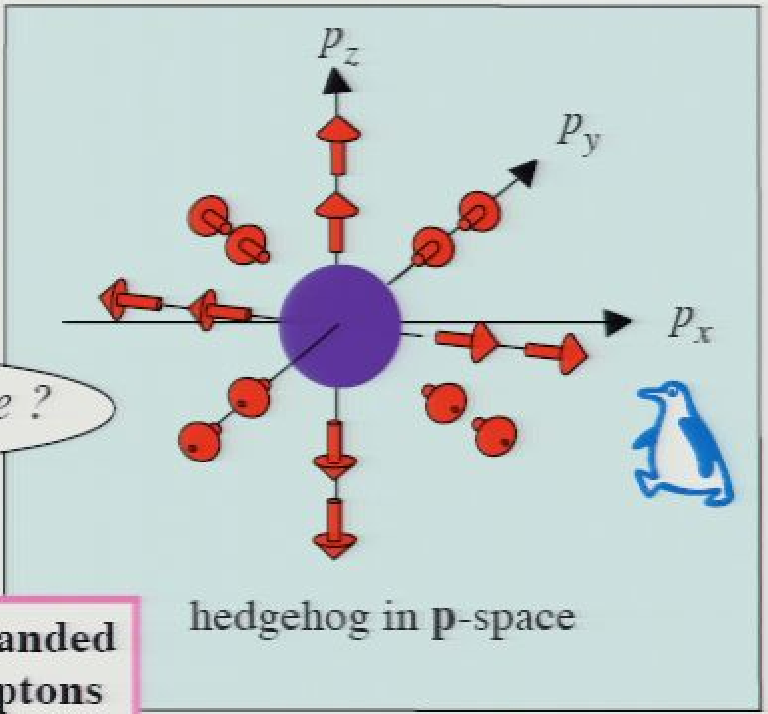
Topology in **r**-space



hedgehog in **r**-space

$$\mathbf{M}(\mathbf{r}) = |\mathbf{M}|\hat{\mathbf{r}}$$

Topology in **p**-space



hedgehog in **p**-space

$$\sigma(\mathbf{p}) = \hat{\mathbf{p}}$$

again no difference ?

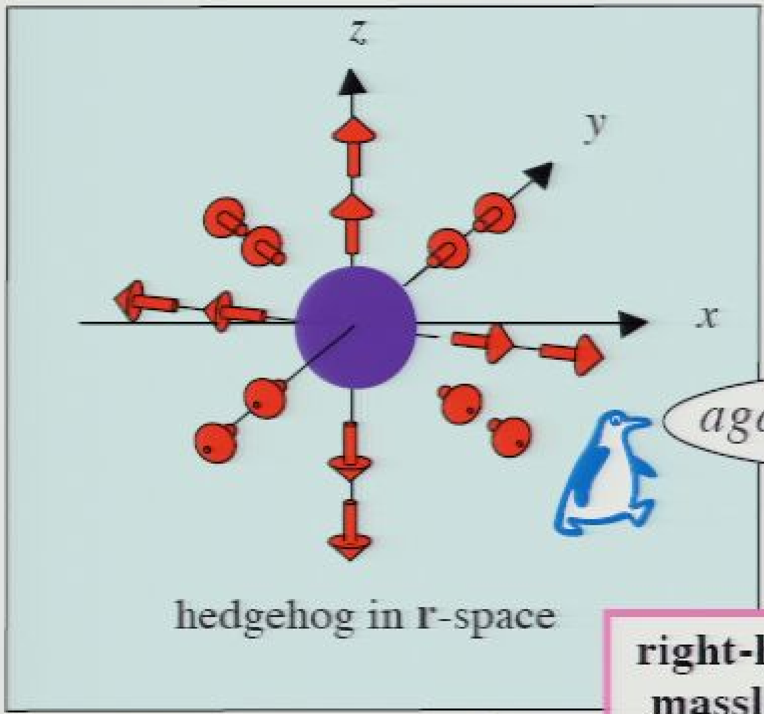
right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

right-handed electron =
hedgehog in **k**-space with spines = spins

$$H = + c\, \sigma \cdot \mathbf{p}$$

right-handed electron vs magnetic hedgehog

Topology in **r**-space



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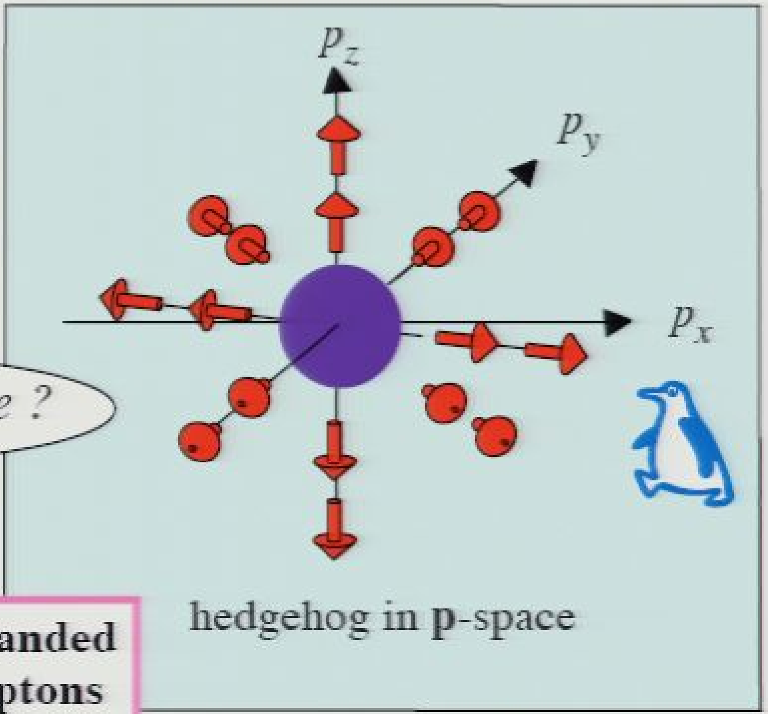

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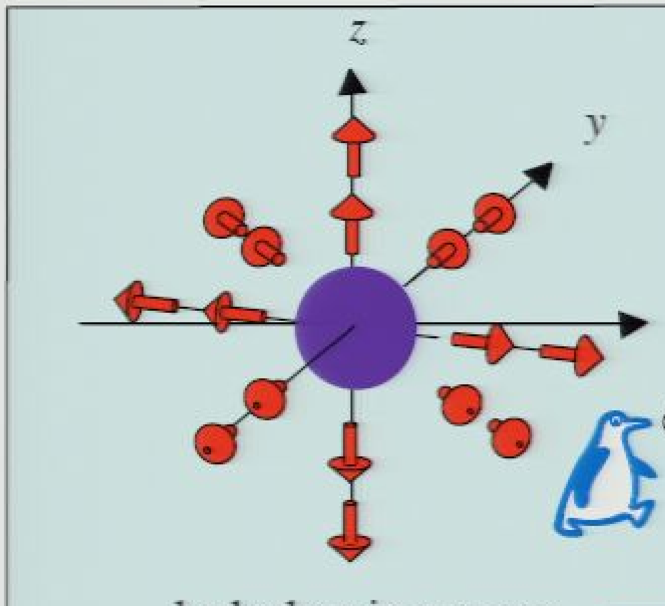

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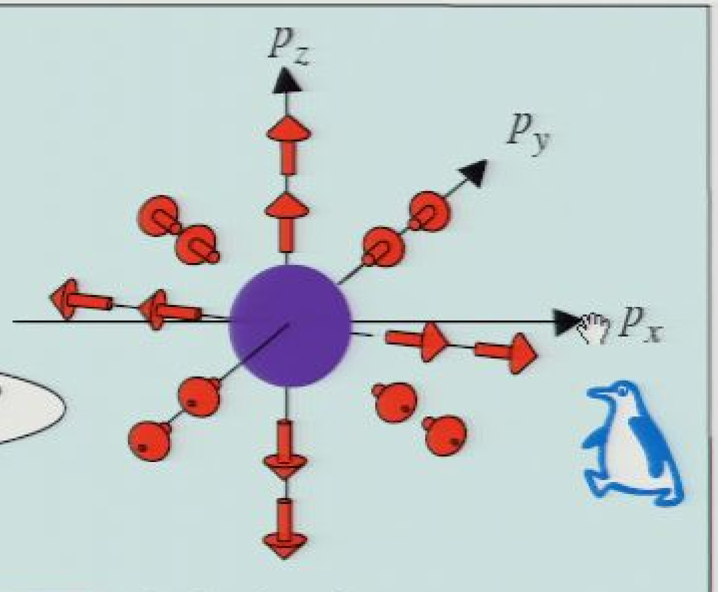

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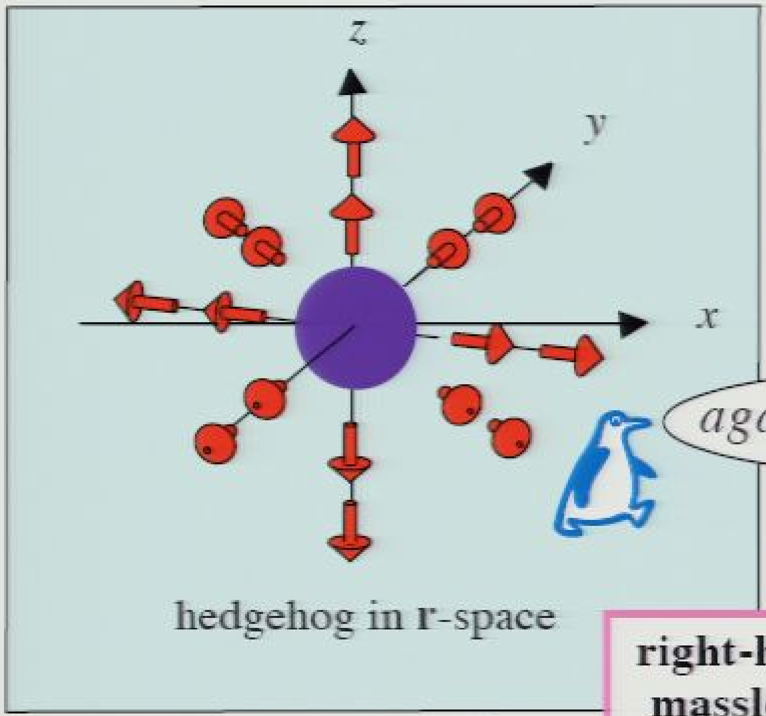

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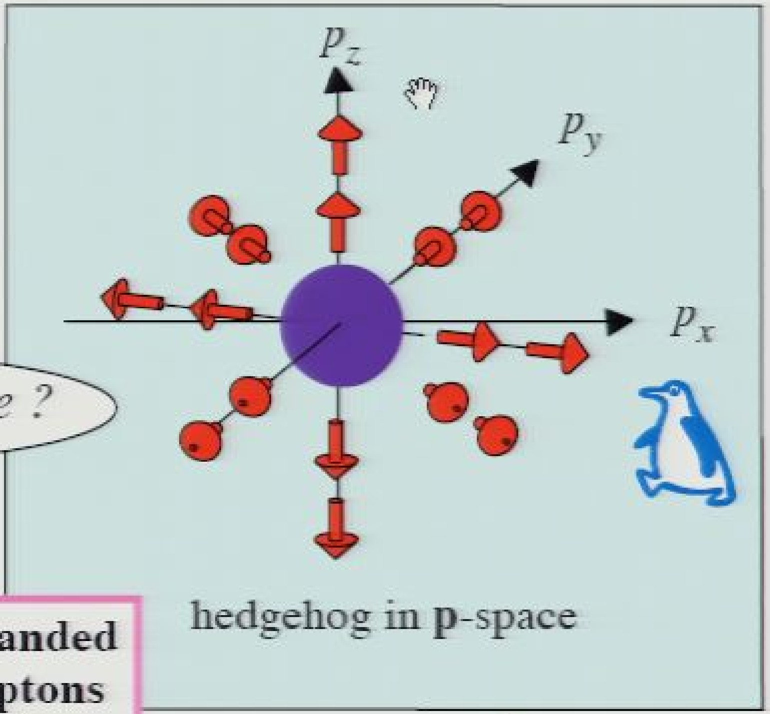

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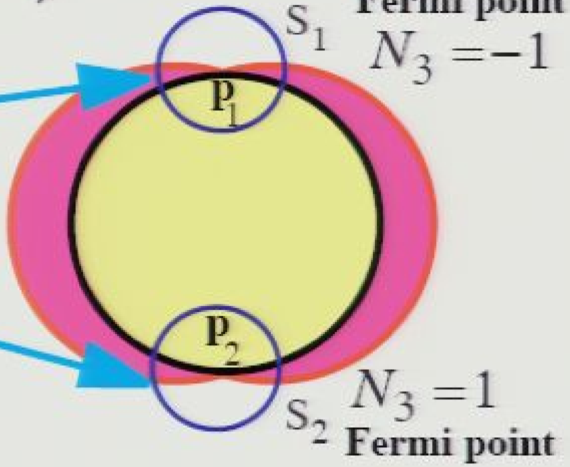
$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot \left(\partial_{p_i} \hat{\mathbf{g}} \times \partial_{p_j} \hat{\mathbf{g}}\right)$$

over 2D surface S in 3D momentum space



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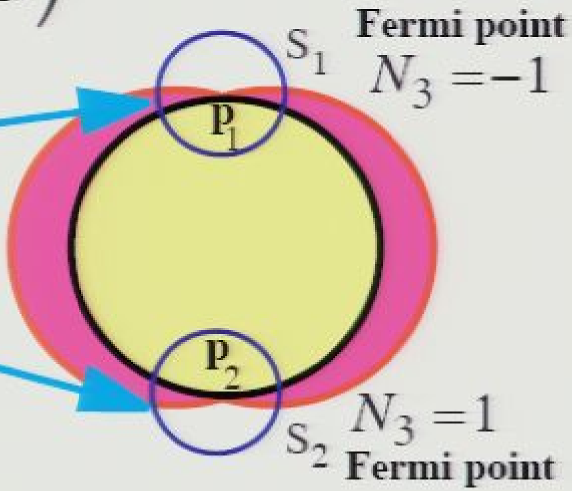

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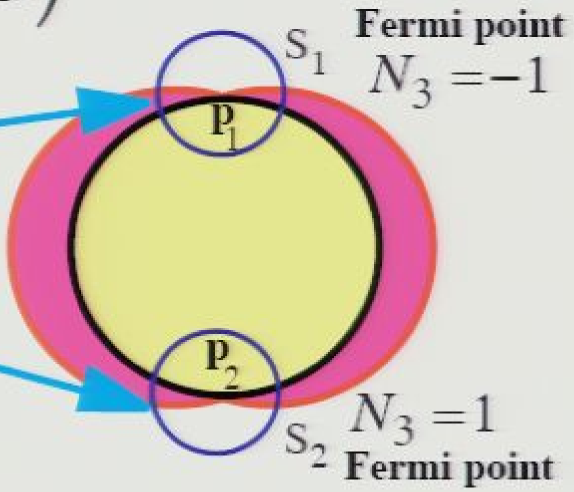

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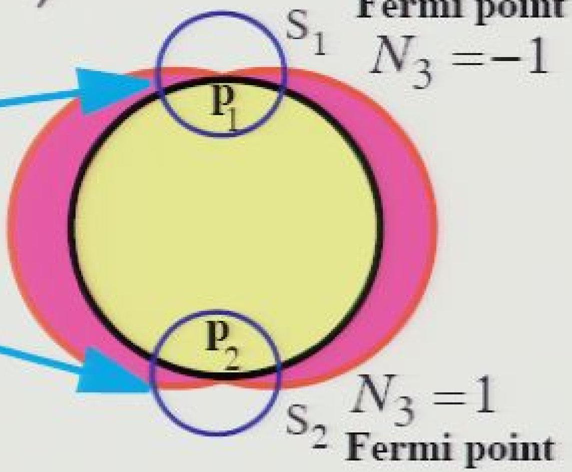

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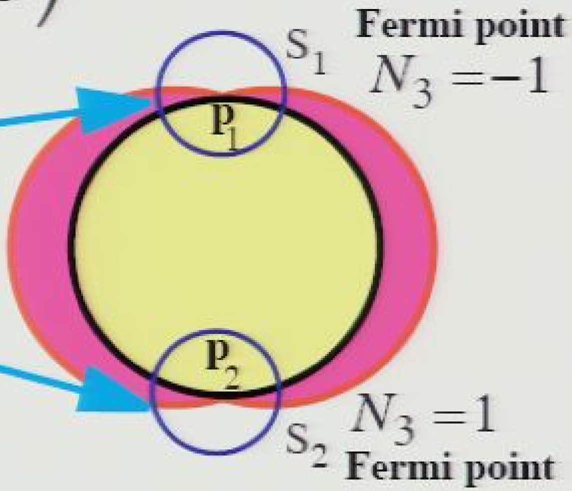

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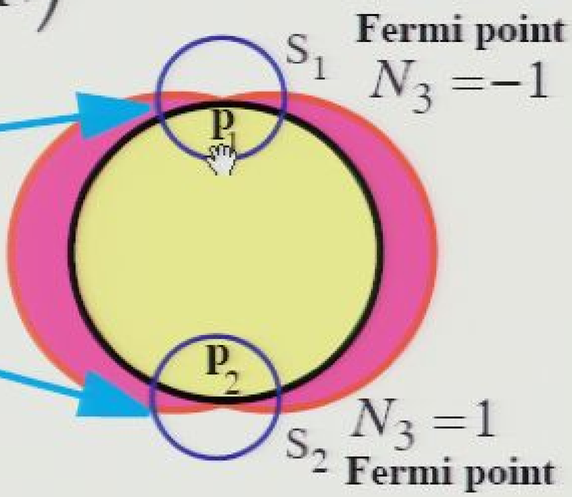

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over 2D surface S
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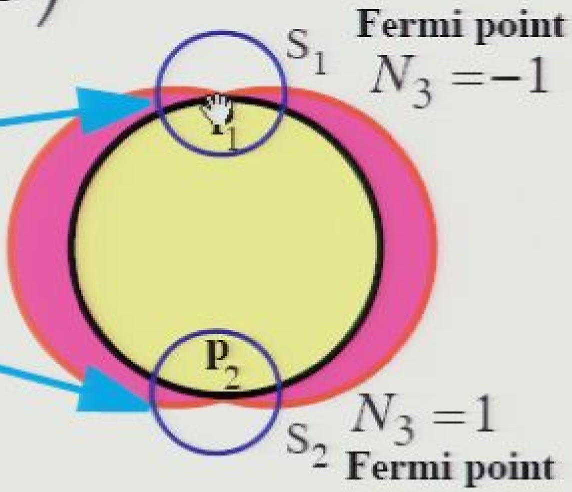

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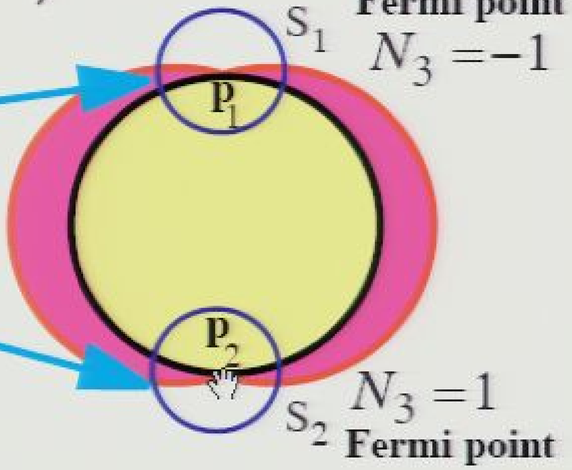

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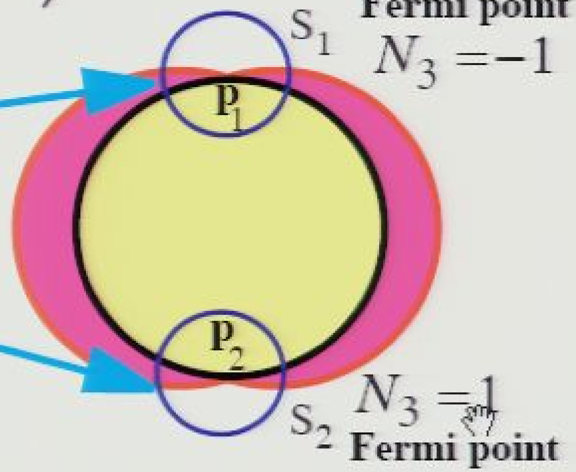

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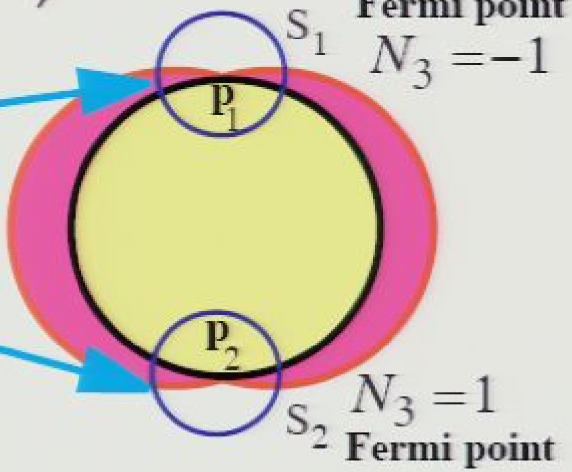

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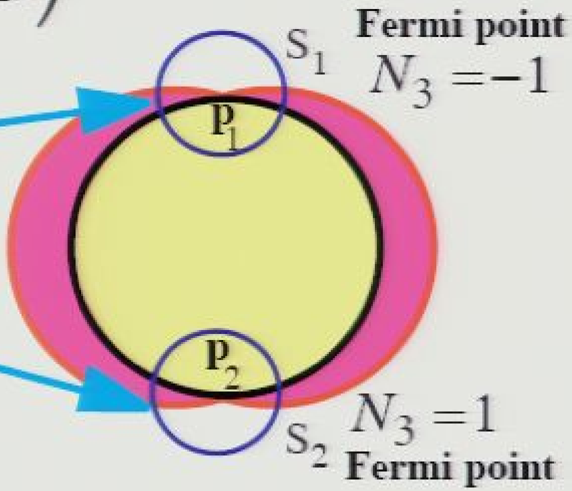

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effective
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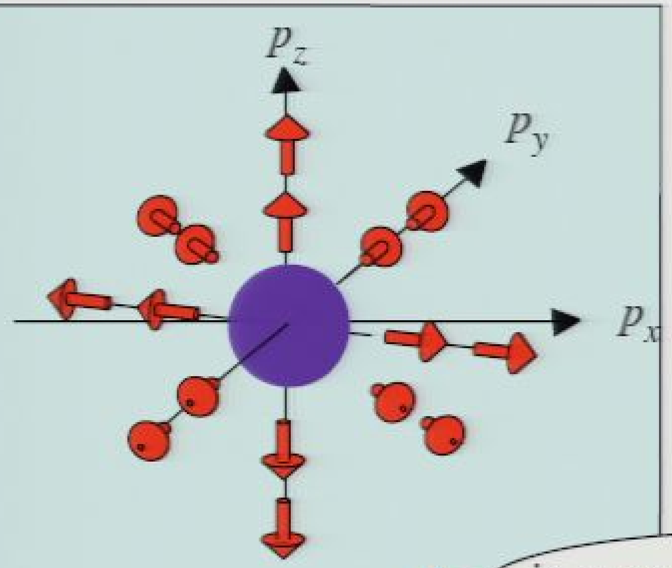

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quill

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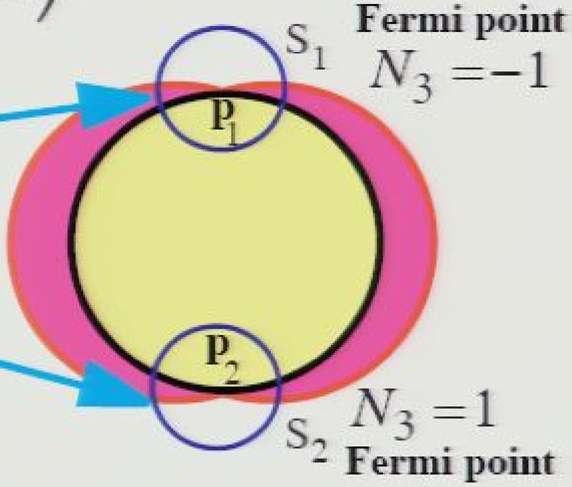

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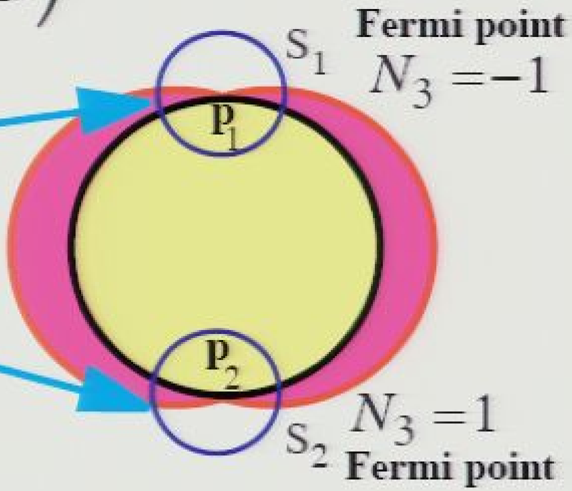

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A_μ → effective electromagnetic field

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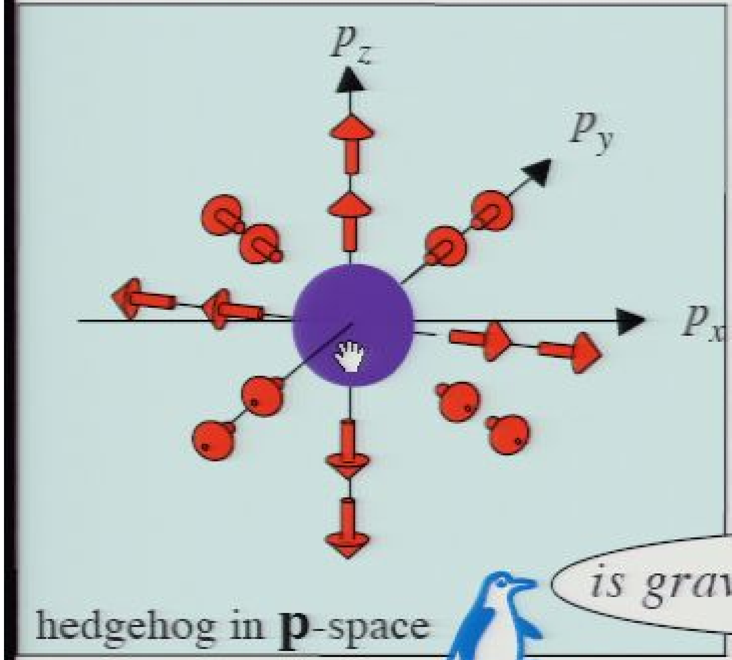

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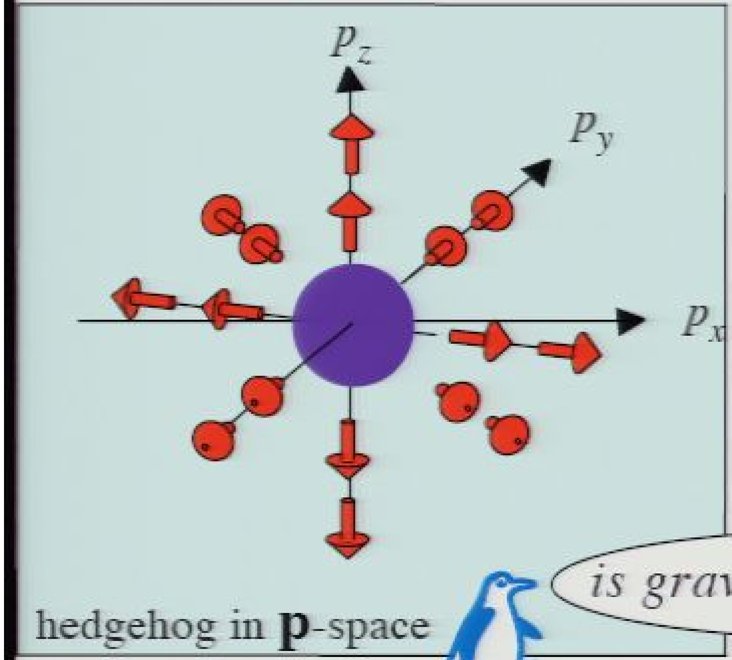

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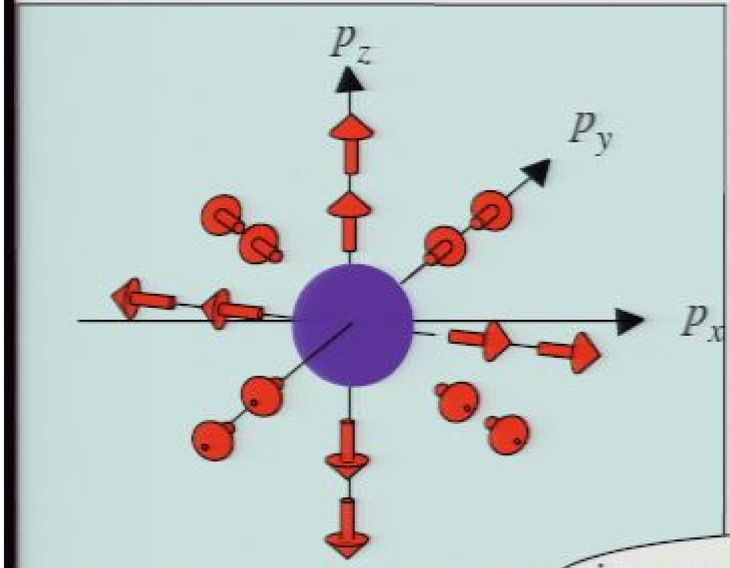

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S

$\bar{p} =$

$1 = h$

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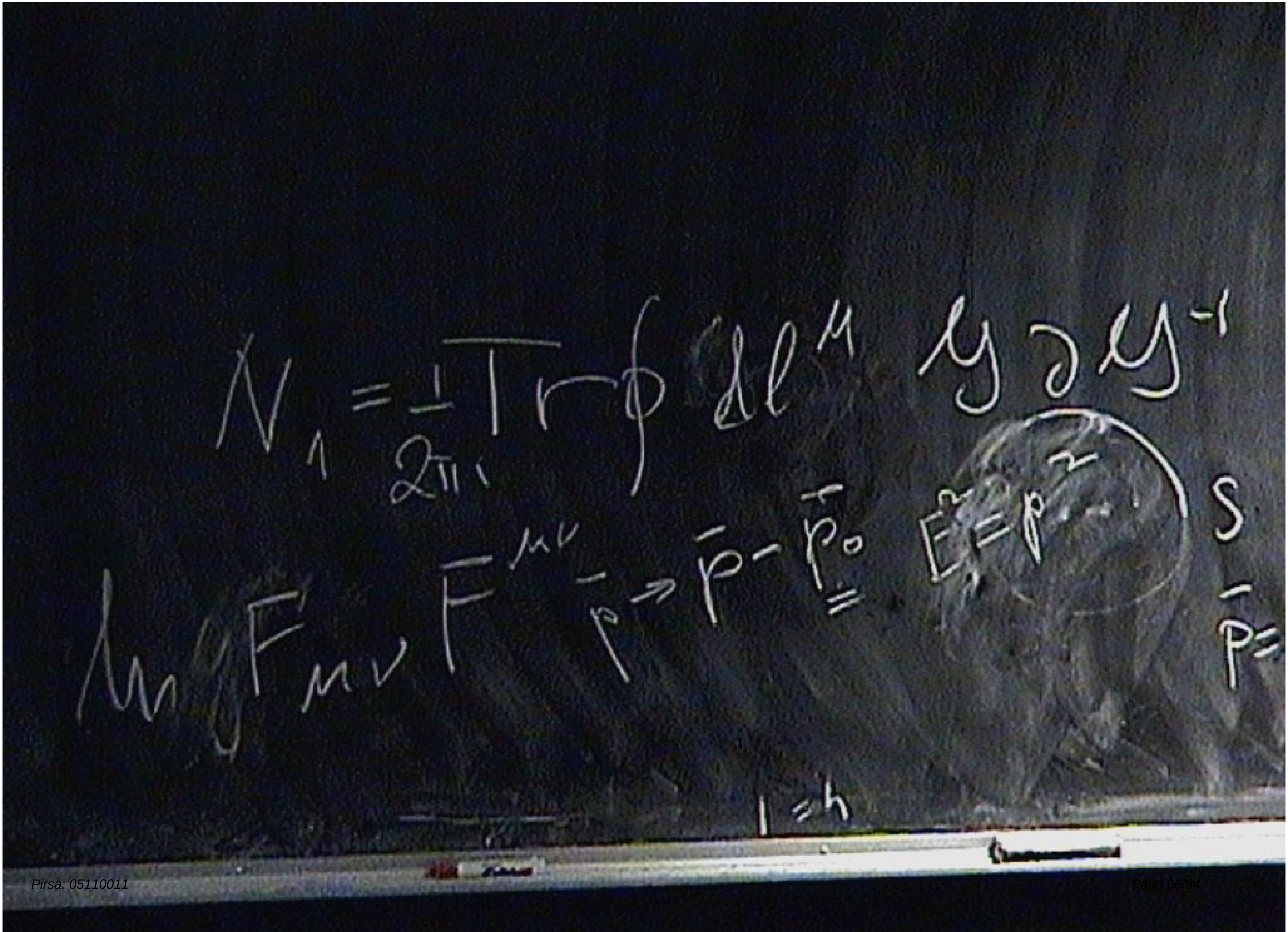

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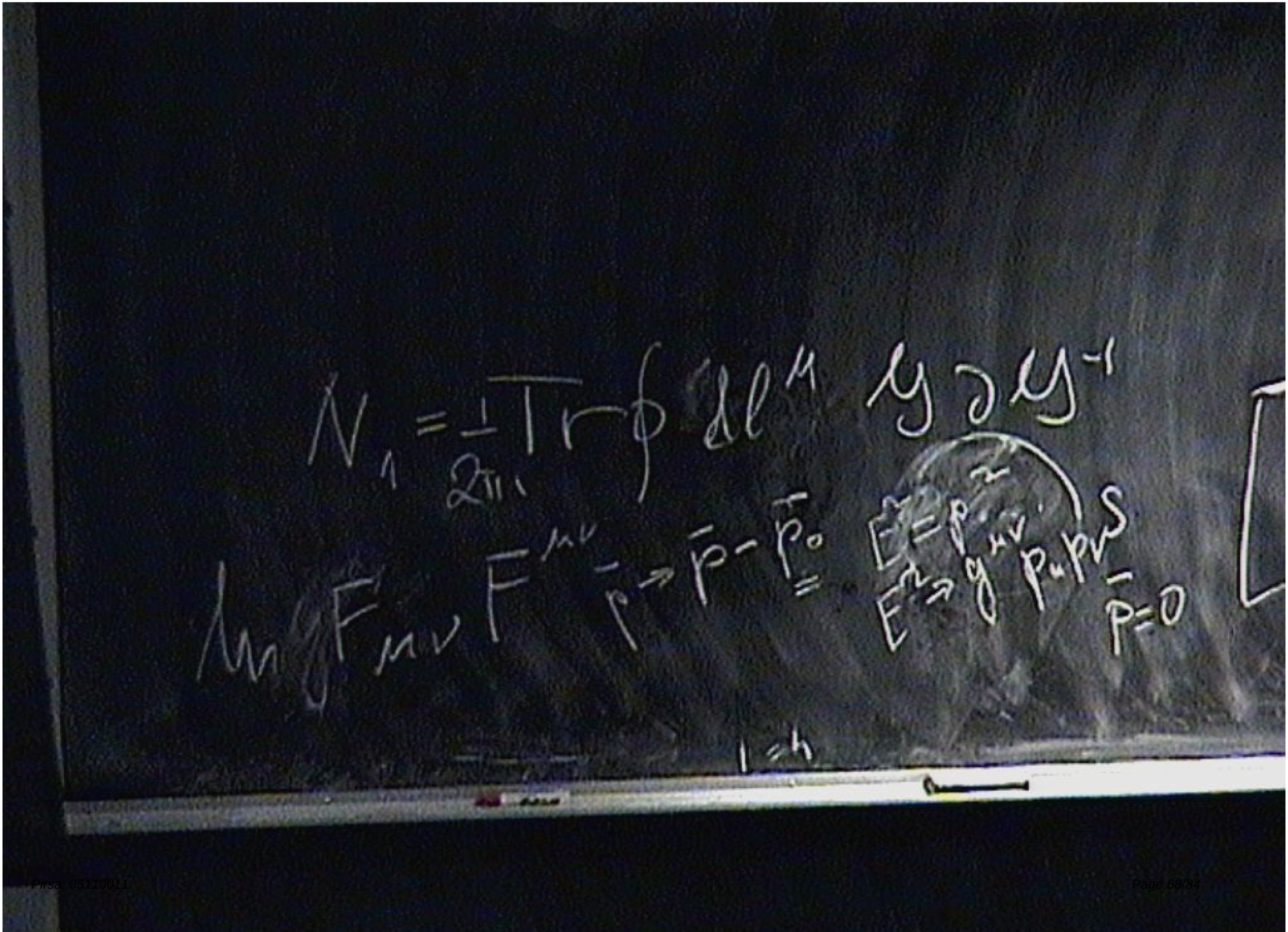

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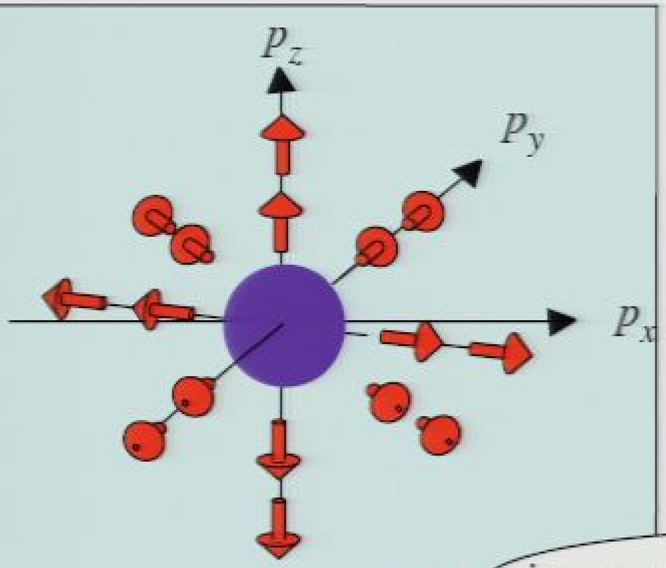

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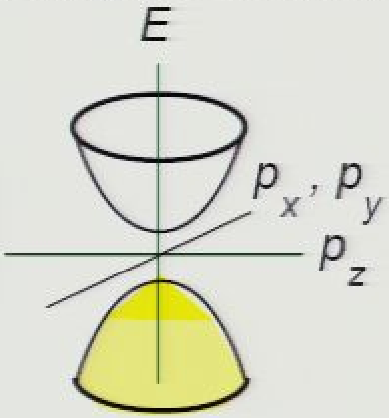
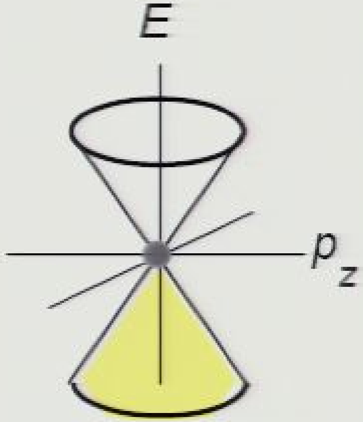


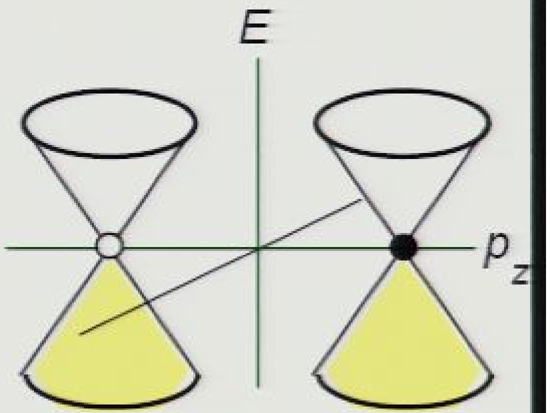
fully gapped spectrum in BEC regime



marginal Fermi point

$N_3 = 0$

at quantum phase transition



two Fermi points in BCS regime

- ● $N_3 = +1$
- ● $N_3 = 0$
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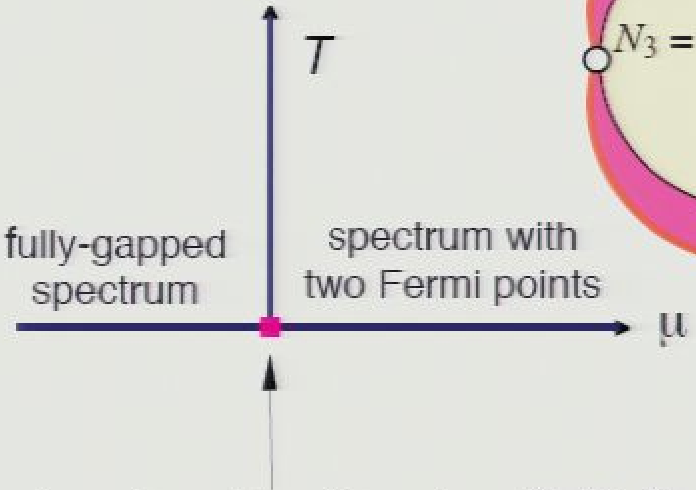
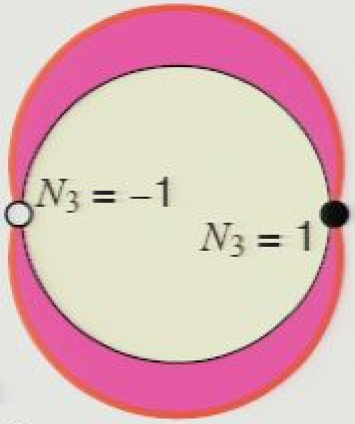
can such transition occur in quantum vacuum?



quantum phase transition at $\mu=0$, $T=0$



Many-body Schrödinger quantum mechanics for N atoms

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i + \sum_{i=1}^{N}\sum_{j=i+1}^{N} V(\mathbf{r}_i - \mathbf{r}_j)$$

Universality Classes of vacua from Theory of Everything

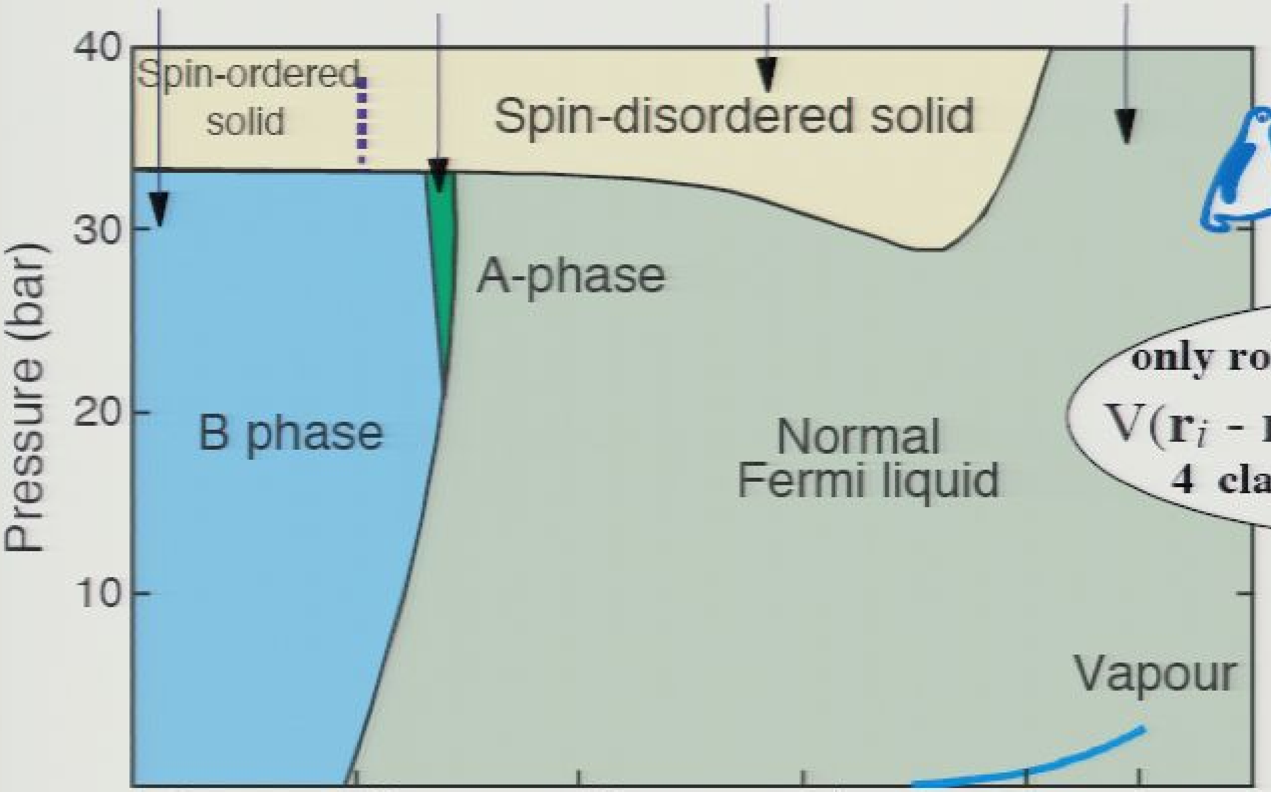
(in system of ^{3}He atoms)

Fully gapped vacuum

Fermi point class

Crystalline vacuum

Fermi surface class



background independent effective theory

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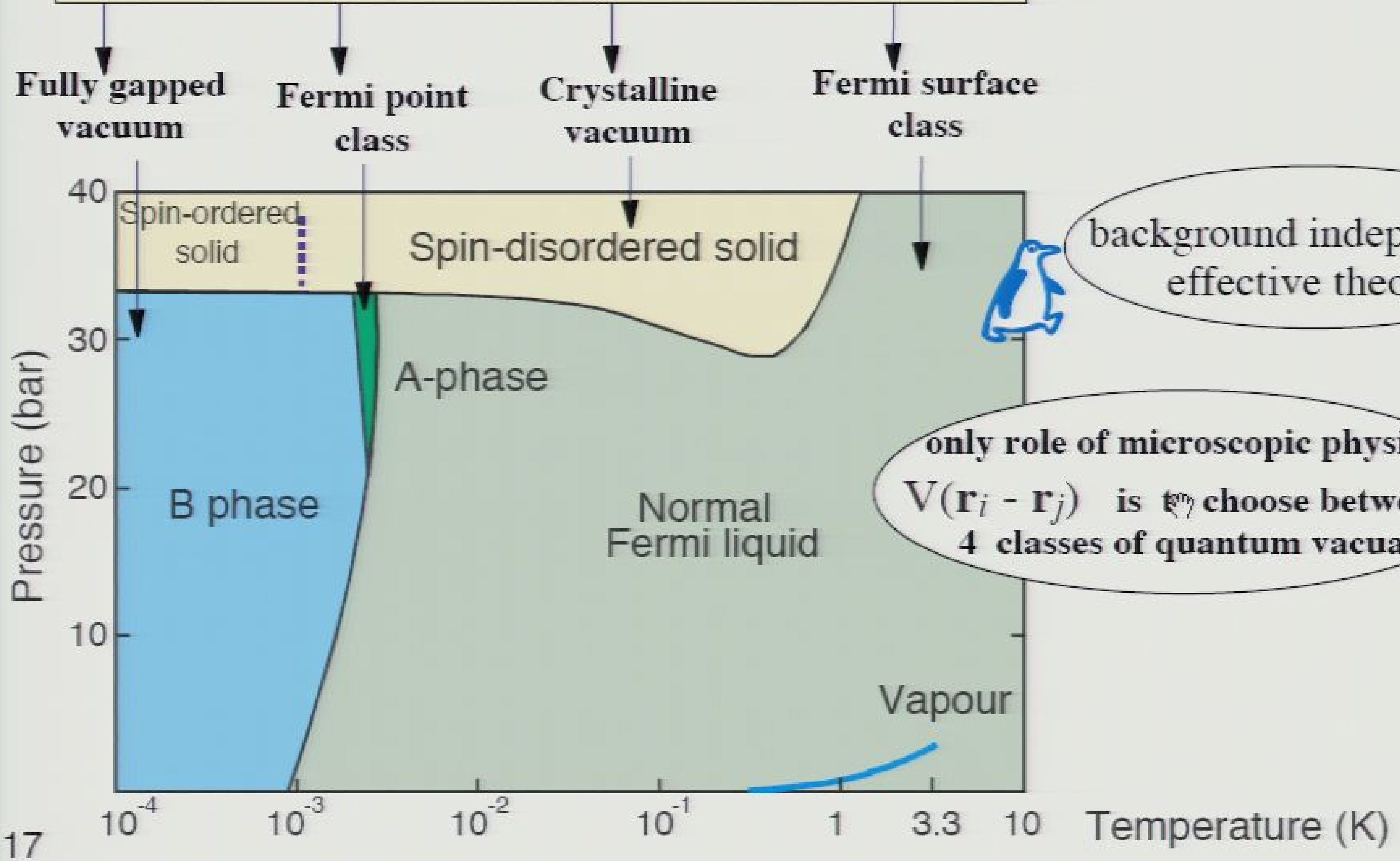

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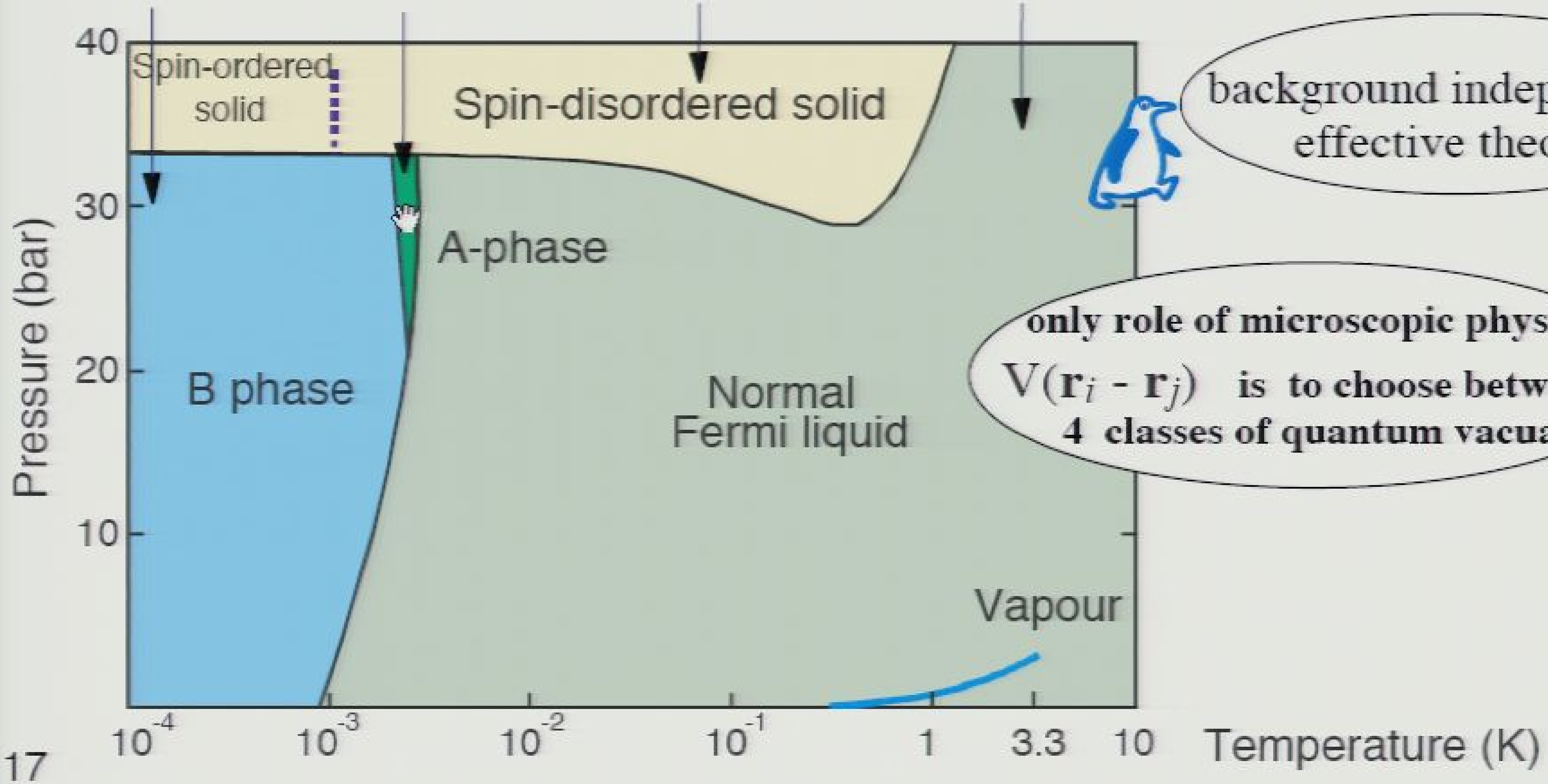

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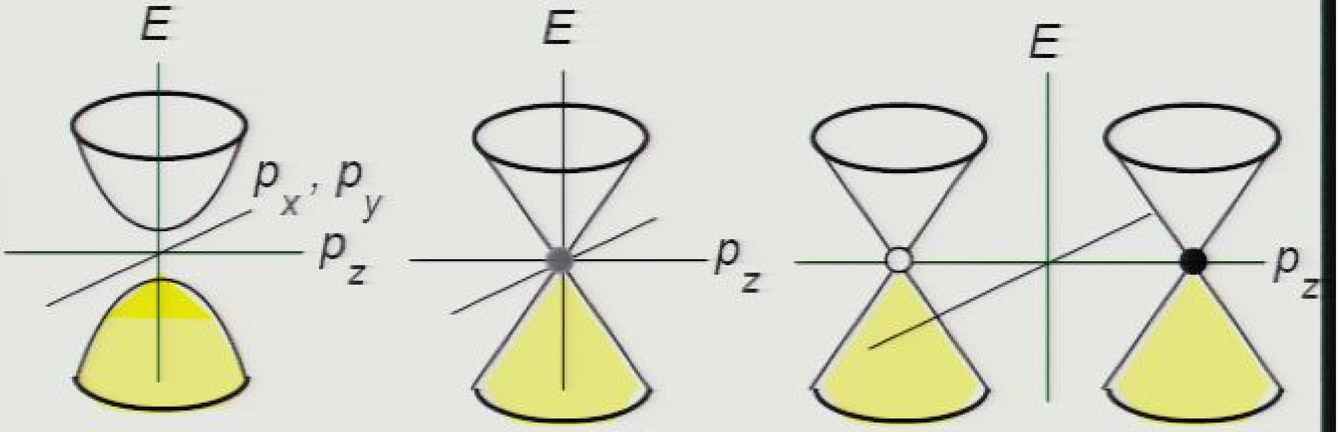
BEC-BCS quantum phase transition between *p*-wave states

$$H=\begin{pmatrix} \frac{p^2}{2m}-\mu & c(p_x+ip_y) \\ c(p_x-ip_y) & \mu-\frac{p^2}{2m} \end{pmatrix}$$



● N_3= +1

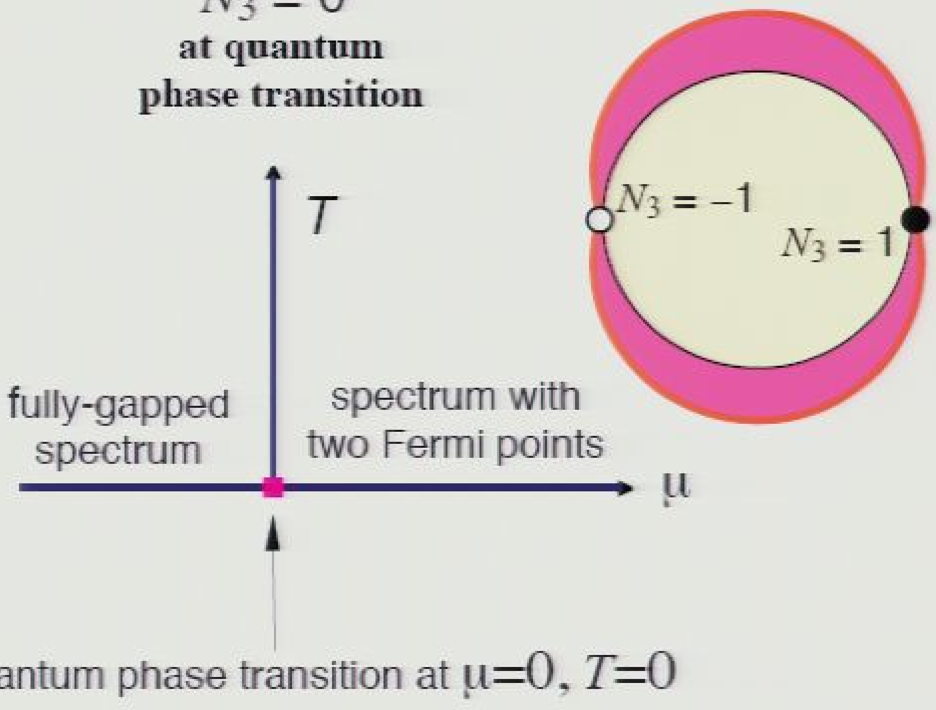
● N_3= 0

○ N_3= -1

fully gapped spectrum in BEC regime

marginal Fermi point
$N_3 = 0$
at quantum phase transition

two Fermi points in BCS regime



fully-gapped spectrum

spectrum with two Fermi points

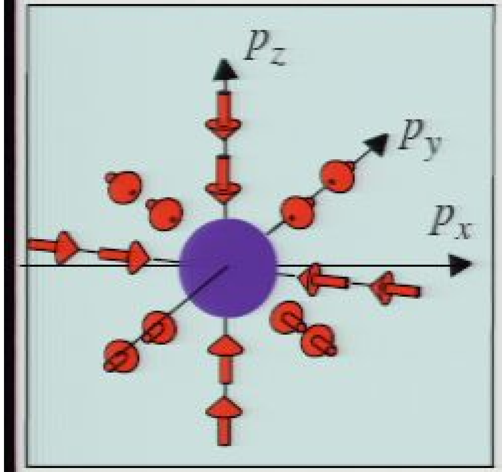
quantum phase transition at $\mu=0$, $T=0$

can such transition occur in quantum vacuum?

Chiral fermions in Standard Model

Family #1 of quarks and leptons

left particles



hedgehog with spines (spins) inward (N_3= −1)

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

$SU(2)_L$

0 ν_L -1/2	-1 e_L -1/2

$$H = -c\,\sigma\cdot\mathbf{p}$$

$$N_3 = -1$$

quarks

$SU(3)_C$

leptons

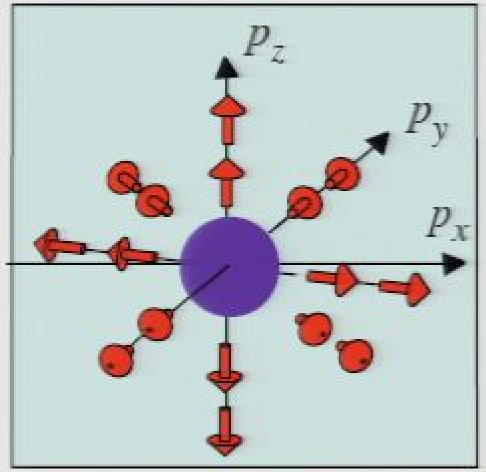
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

0 ν_R 0	-1 e_R -1

$$H = +c\,\sigma\cdot\mathbf{p}$$

$$N_3 = +1$$

right particles



hedgehog with spines (spins) outward (N_3= +1)

$$N_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr}\int dS^{\gamma}\, \mathbf{G}^{\mu}\, \mathbf{G}^{-1}\, \mathbf{G}^{\nu}\, \mathbf{G}^{-1}\mathbf{G}^{\lambda}\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

Chiral fermions in Standard Model

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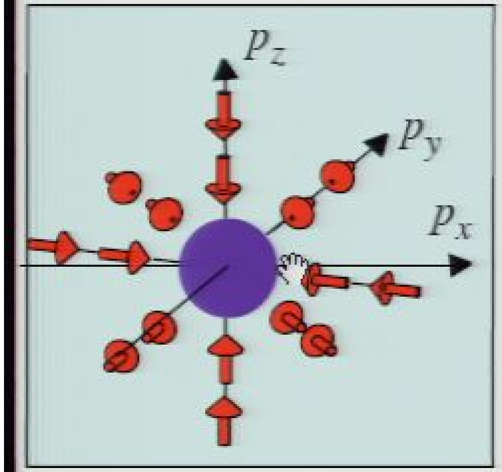
left particles



hedgehog with spines (spins) inward ($N_3 = -1$)

quarks

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

$SU(3)_C$

$SU(2)_L$

leptons

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$$H = -c\,\sigma \cdot \mathbf{p}$$

$$N_3 = -1$$

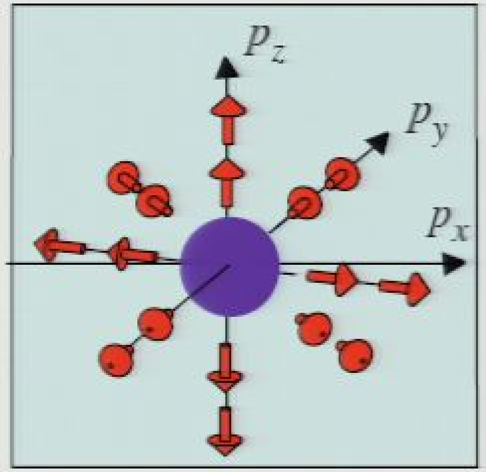
right particles

+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

0 ν_R 0	-1 e_R -1

$$H = +c\,\sigma \cdot \mathbf{p}$$

$$N_3 = +1$$



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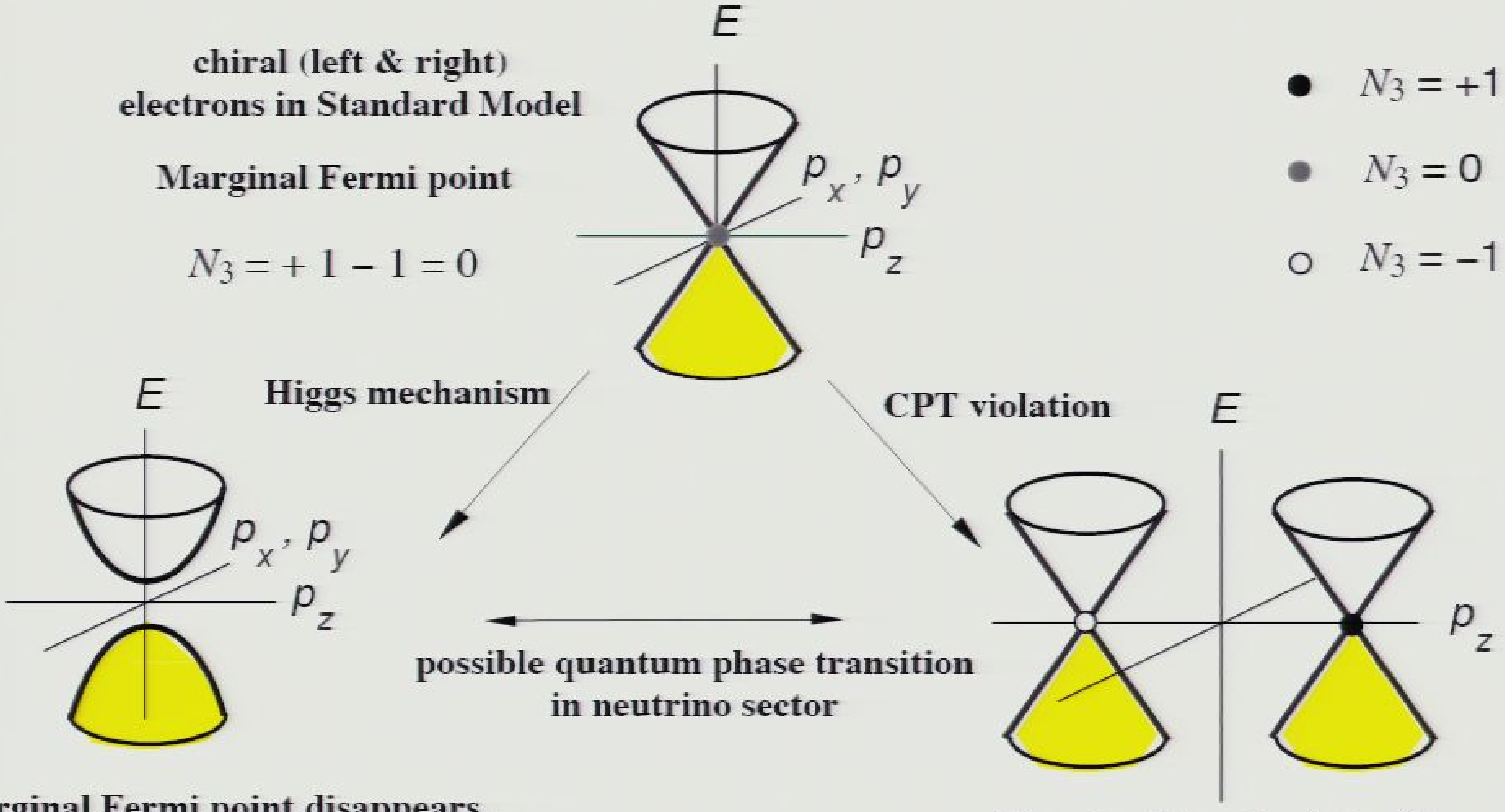
over 3D surface S in 4D momentum space

Two topological scenaria in Standard Model



chiral (left & right)
electrons in Standard Model

Marginal Fermi point

$N_3 = +1 - 1 = 0$

● $N_3 = +1$

● $N_3 = 0$

○ $N_3 = -1$

Higgs mechanism

CPT violation

possible quantum phase transition
in neutrino sector

Marginal Fermi point disappears,
massive Dirac electron is formed

Marginal Fermi point splits
into two topologically protected
Fermi points with $N_3 = +1$ and $N_3 = -1$

Two topological scenaria in Standard Model

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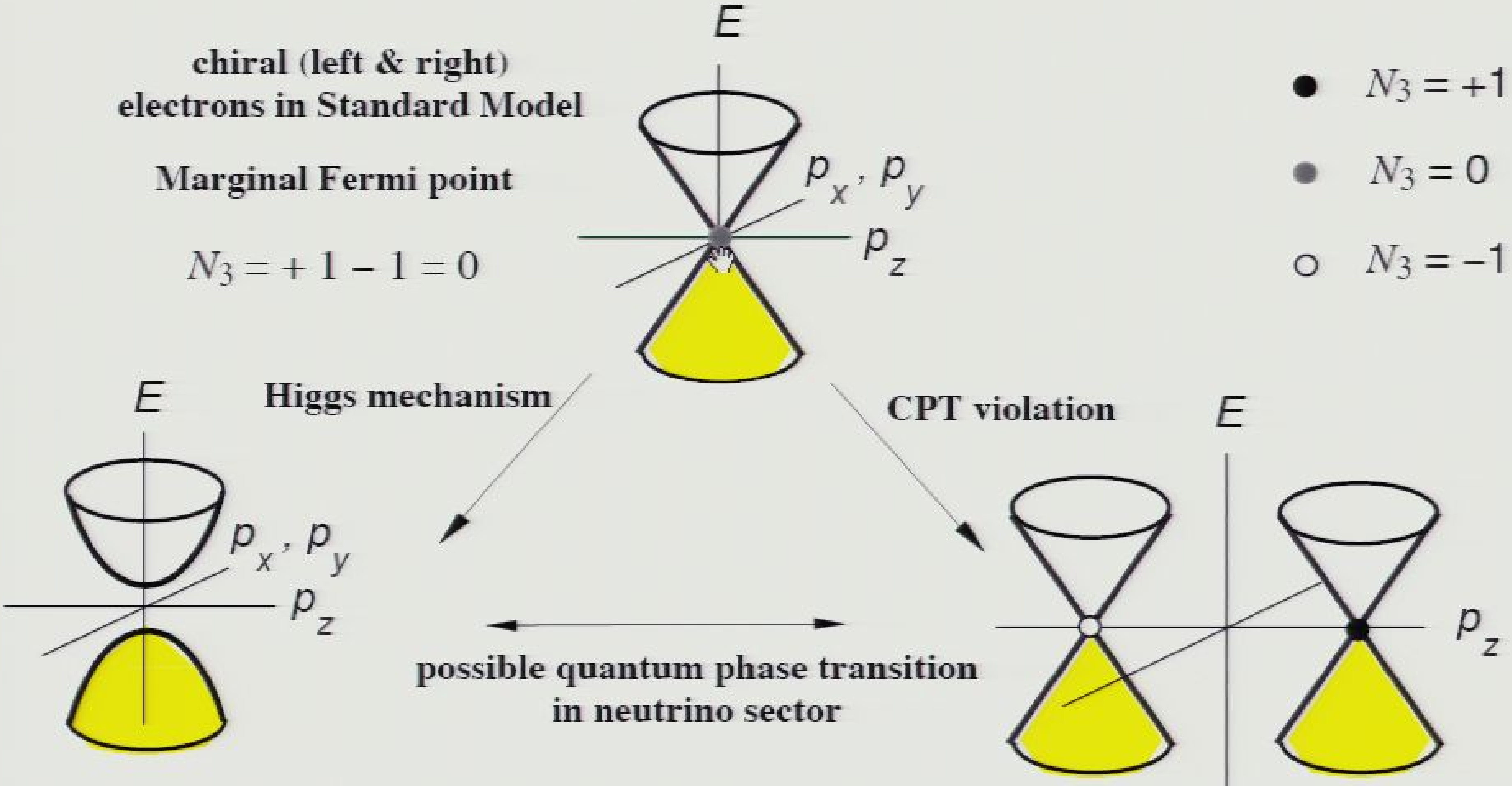

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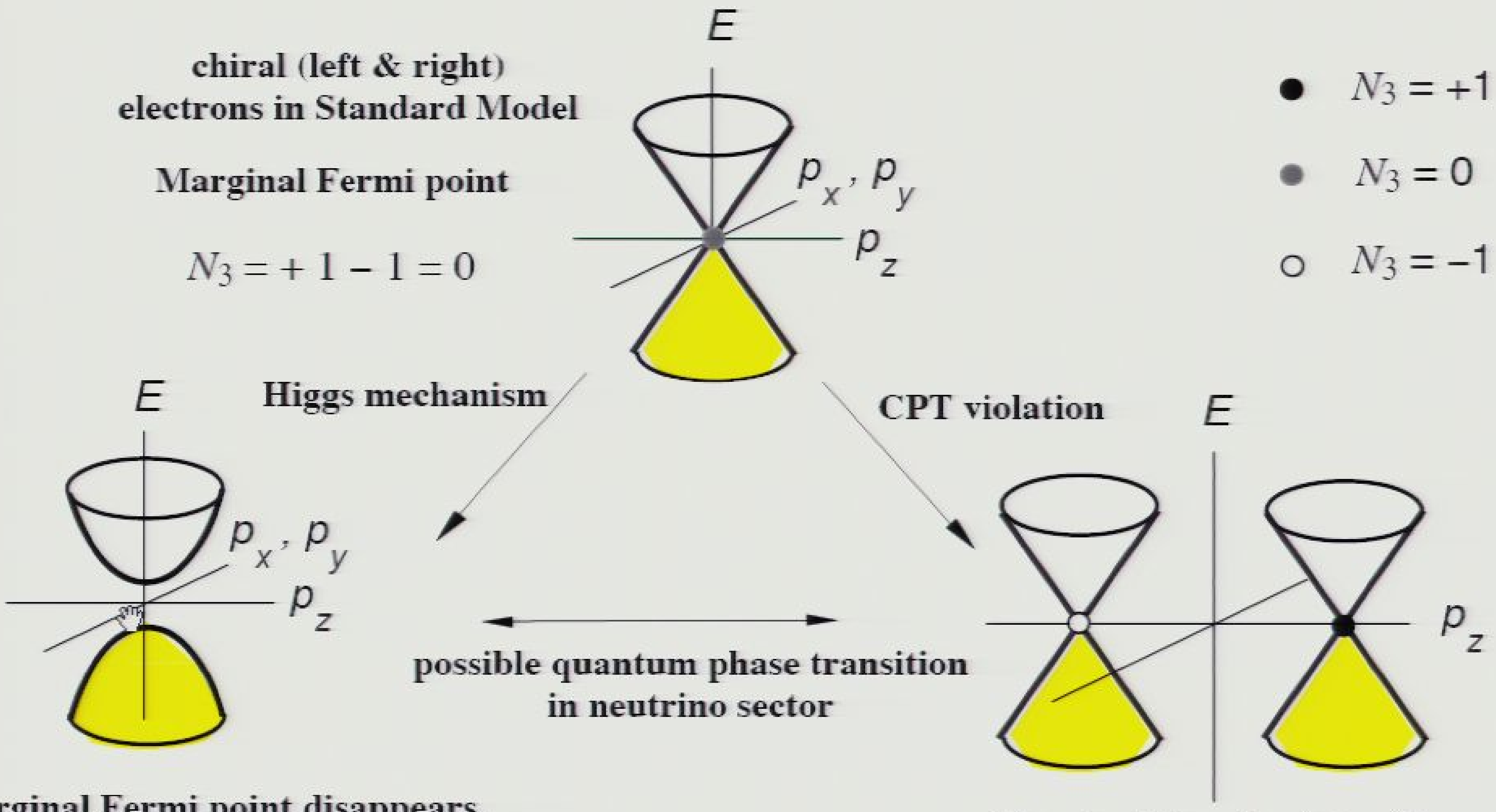

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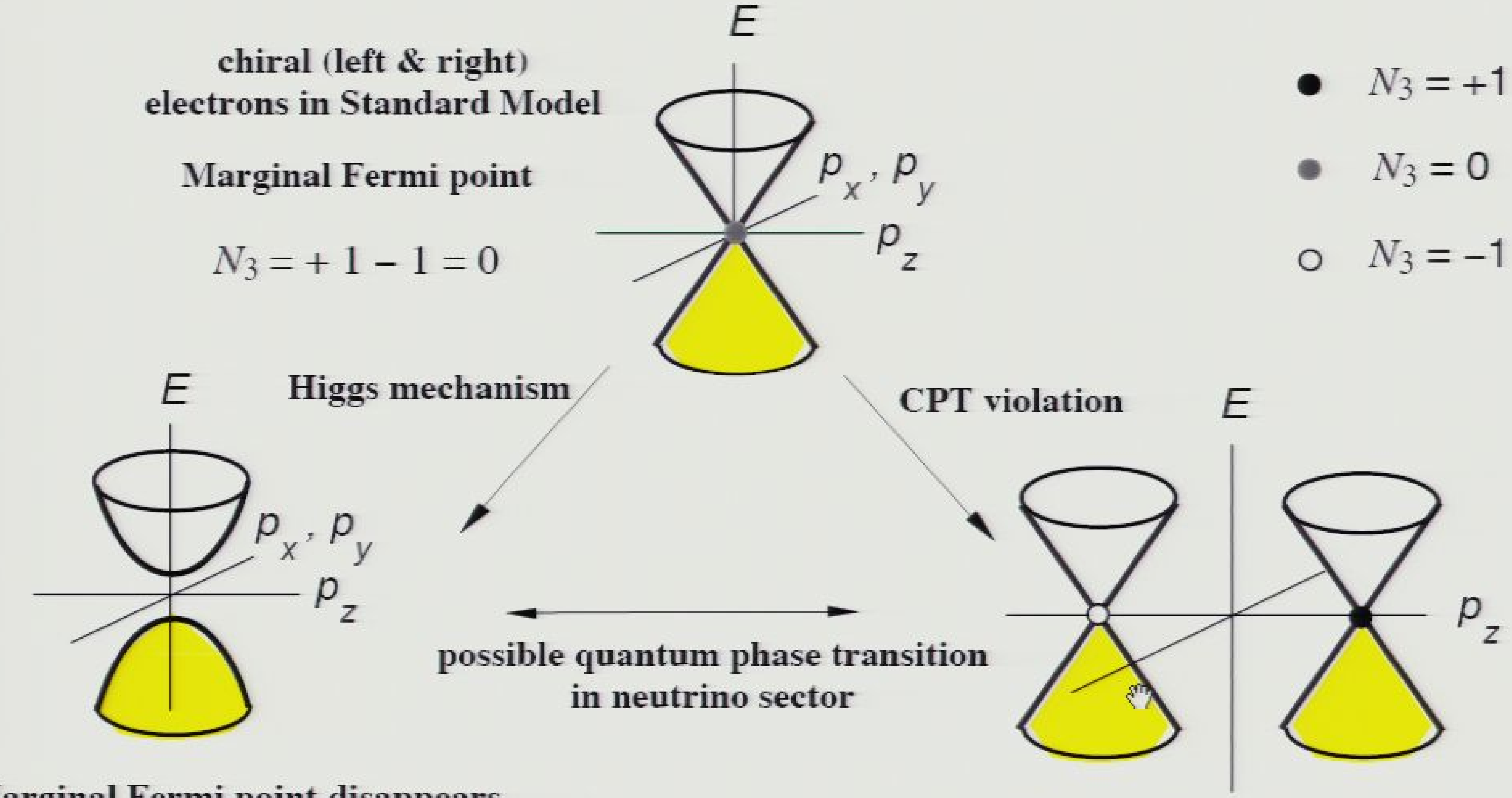

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Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

quantum statistics of topological objects

spectrum of edge states & fermion zero modes on walls & quantum vortices

chiral anomaly & vortex dynamics
etc.

$1+1=0$ $\mathbb{Z}_2$

$dl^\mu \quad g\,\partial g^{-1}$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

$$S_{CS} = \frac{1}{16\pi} N_{IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_{\mu}^{\ I} F_{\nu\lambda}^{\ \ J}$$

spin current $J_{x}^{\ z} = \delta S_{CS} / \delta A_{x}^{\ z} = \frac{1}{4\pi} (\gamma N_{ss}\, dH^{z}/dy + N_{se}\, E_{y})$

2D singlet superconductor: $N_{ss} = N/4$

quantized spin Hall conductivity:

$$\sigma_{xy}^{\ s} = \frac{\gamma N}{16\pi}$$

s-wave: $N = 0$

$p_x + ip_y$: $N = 2$

$d_{xx\text{-}yy} + id_{xy}$: $N = 4$

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