Title: Entanglement-assisted invariance, ignorance, and information in quantum physics

Date: Nov 09, 2005 04:00 PM

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Abstract: I shall discuss entanglement - assisted invariance (symmetry exhibited by correlated quantum states) and describe how it can be used to understand the nature of ignorance, and, hence, the origin of probabilities in quantum physics. WHZ, Phys. Rev. Lett. 90, 120404 (2003); Rev. Mod. Phys. 75, 715 (2003); Phys. Rev. 71, 052105 (2005) (quant-ph/0405161).

ENTANGLEMENT-ASSISTED INVARIANCE, IGNORANCE, AND INFORMATION IN QUANTUM PHYSICS*

Born's Rule# from entanglement

$$|\Psi\rangle = \sum_{k=1}^{N} \psi_k |s_k\rangle \Rightarrow p_k = |\psi_k|^2$$

#Max Born, Zeitschrift fur Physik, 37, pp. 863-867 (1926).

*Wojciech H. Zurek, "Environment-assisted invariance, entanglement, and probabilities in quantum physics", PRL **90**, 120404 (2003); also Section VI D of "Decoherence…", RMP **75,** 715-765 (2003);also "Probabilities from entanglement…", quant-ph/0405161, PRA, May'05.

H. Barnum, "No-signaling-based version of Zurek's derivation of quantum probabilities", quant-ph/0312150; M. Schlosshauer & A. Fine, quant-ph…: M. Schlosshauer, Rev. Mod. Phys, Oct '04 issue…

Axioms of Quantum Theory

- The Universe consists of systems
- Pure state is a vector in the Hilbert space of a system
- Composite pure state is a vector in a tensor product of constituent Hilbert spaces
- Evolutions of isolated systems are unitary
- Observables are associated with Hermitean operators
- The only possible outcome of a measurement is an eigenstate (and the corresponding eigenvalue) of such an operator*
- **Probabilities of various outcomes are given by Born's Rule.#**

*Can be explained by decoherence and einselection, but that means relying on Born's Rule

#Need to derive without circularity!

Copy from a translation, p. 52-57 in "Quantum Theory and Measurement", John Archibald Wheeler & WHZ, eds. (Princeton U. Press, 1983)

I.2 ON THE QUANTUM MECHANICS OF COLLISIONS

[Preliminary communication]†

MAX BORN

Through the investigation of collisions it is argued that quantum mechanics in the Schrödinger form allows one to describe not only stationary states but also quantum jumps.

† This report was originally intended for *die Naturwissenschaften*, but could not be accepted there for lack of space. I hope that its publication in this journal [*Zeitschrift für Physik*] does not seem out of place [M.B.].

Originally published under the title, "Zur Quantenmechanik der Stossvorgänge," *Zeitschrift für Physik*, *37*, 863–67 (1926); reprinted in *Dokumente der Naturwissenschaft*, *1*, 48–52 (1962) and in M. Born (1963); translation into English by J.A.W. and W.H.Z., 1981.

The calculation gives this result: The scattered wave created by this perturbation has asymptotically at infinity the form:

$$\psi^1_n(x, y, z; q_k) = \sum_m \iint_{\alpha^2+\beta^2+\gamma^2=1} d\omega\, \Phi_{n,m}(\alpha, \beta, \gamma) \sin k_{n,m}(\alpha x + \beta y + \gamma z + \delta)\psi^0_m(q_k).$$

This means that the perturbation, analyzed at infinity, can be regarded as a superposition of solutions of the unperturbed problem. If one calculates the energy belonging to the wavelength $\lambda_{n,m}$ according to the de Broglie formula, one finds

where the ν^0_{nm} are the frequ…

If one translates this re…
possible. $\Phi_{n,m}(\alpha, \beta, \gamma)$ gi…
direction, to be thrown ou…
the phase change δ. Here i…
cost of the energy of the at…
collision of the second kind…

If one translates this result into terms of particles, only one interpretation is possible. $\Phi_{n,m}(\alpha, \beta, \gamma)$ gives the probability* for the electron, arriving from the z-direction, to be thrown out into the direction designated by the angles α, β, γ, with the phase change δ. Here its energy τ has increased by one quantum $h\nu^0_{nm}$ at the cost of the energy of the atom (collision of the first kind for $W^0_n < W^0_m$, $h\nu^0_{nm} < 0$; collision of the second kind $W^0_n > W^0_m$, $h\nu^0_{nm} > 0$).

Schrödinger's quantum …
the question of the effect of … description. One gets no answer to the question, "what is the state after the collision," but only to the question, "how probable is a specified outcome of the collision" (where naturally the quantum mechanical energy relation must be fulfilled).

Here the whole problem of determinism comes up. From the standpoint of our quantum mechanics there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties (like phases or the internal atomic motions) and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment—as to the impossibility of prescribing conditions for a causal evolution—is a pre-established harmony founded on the nonexistence of such conditions? I myself am inclined to give up determinism in the world of atoms. But that is a philosophical question for which physical arguments alone are not decisive.

In practical terms indeterminism is present for experimental as well as for theoretical physicists. The "yield function" Φ so much investigated by experimentalists is now also sharply defined theoretically. One can determine it from the potential energy of interaction, $V(x, y, z; q_k)$. However, the calculations required

* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity Φ_{nm}.

The calculation gives this result: The scattered wave created by ... tion has asymptotically at infinity the form:

$$\psi^{(1)}_{n}(x, y, z; q_k) = \sum_m \iint_{\alpha x + \beta y + \gamma z > 0} d\omega\, \Phi_{n,m}(\alpha, \beta, \gamma) \sin k_{n,m}(\alpha x + \beta y + \ldots$$

This means that the perturbation, analyzed at infinity, can be regar... position of solutions of the unperturbed problem. If one calcula... belonging to the wavelength $\lambda_{n,m}$ according to the de Broglie form...

$$W_{n,m} = h\nu^0_{nm} + \tau,$$

where the ν^0_{nm} are the frequencies of the unperturbed atom.

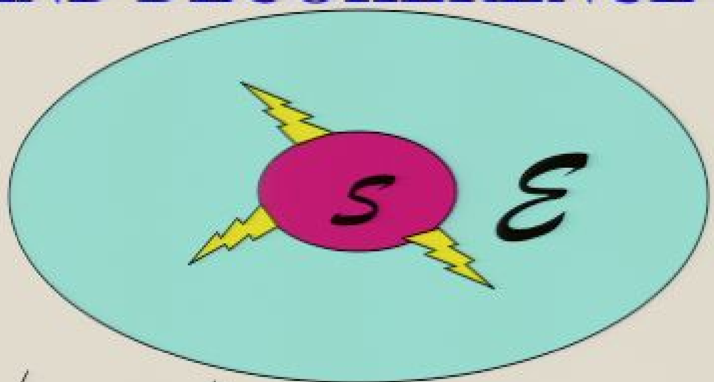
If one translates this result into terms of particles, only one interpretation is possible. $\Phi_{n,m}(\alpha, \beta, \gamma)$ gives the probability* for the electron, arriving from the z-direction, to be thrown out into the direction designated by the angles α, β, γ, with the phase change δ. Here its energy τ has increased by one quantum $h\nu^0_{nm}$ at the cost of the energy of the atom (collision of the first kind for $W^0_n < W^0_m$, $h\nu^0_{nm} < 0$; collision of the second kind $W^0_n > W^0_m$, $h\nu^0_{nm} > 0$).

Schrödinger's quantum mechanics therefore gives quite a definite answer to the question of the effect of the collision; but there is no question of any causal description. One gets no answer to the question, "what is the state after the collision," but only to the question, "how probable is a specified outcome of the collision" (where naturally the quantum mechanical energy relation must be fulfilled).

Here the whole problem of determinism comes up. From the standpoint of our quantum mechanics there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties (like phases or the internal atomic motions) and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment—as to the impossibility of prescribing conditions for a causal evolution—is a pre-established harmony founded on the nonexistence of such conditions? I myself am inclined to give up determinism in the world of atoms. But that is a philosophical question for which physical arguments alone are not decisive.

!{ * Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n,m}$.

EINSELECTION*, POINTER BASIS, AND DECOHERENCE



$$|\Phi_{\mathcal{SE}}(0)\rangle = |\psi_{\mathcal{S}}\rangle \otimes |\varepsilon_0\rangle = \left(\sum_i \alpha_i |\sigma_i\rangle\right) \otimes |\varepsilon_0\rangle \xrightarrow[\text{Entanglement}]{\text{Interaction}} \sum_i \alpha_i |\sigma_i\rangle \otimes |\varepsilon_i\rangle = |\Phi_{\mathcal{SE}}(t)\rangle$$

REDUCED DENSITY MATRIX $\rho_{\mathcal{S}}(t) = Tr_{\mathcal{E}} |\Phi_{\mathcal{SE}}(t)\rangle\langle\Phi_{\mathcal{SE}}(t)| = \sum_i |\alpha_i|^2 |\sigma_i\rangle\langle\sigma_i|$

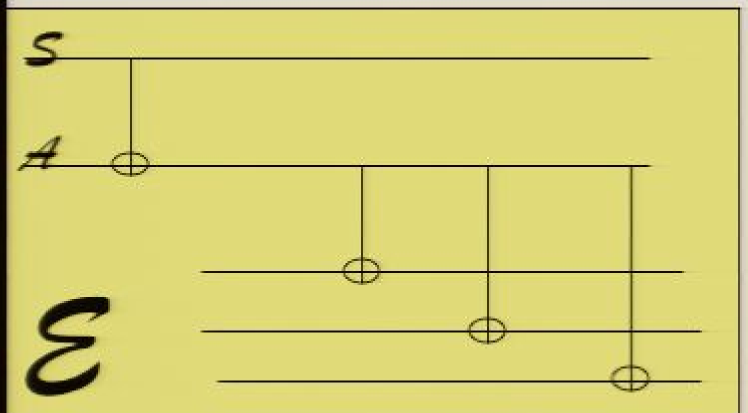
EINSELECTION* leads to POINTER STATES

(same states appear on the diagonal of $\rho_{\mathcal{S}}(t)$ for times long compared to the decoherence time)

***Environment INduced superSELECTION**

IMPLICATIONS OF DECOHERENCE AND EINSELECTION

1. MEASUREMENTS



$$\left(\sum_k \alpha_k |s_k\rangle\right)|A_0\rangle|\varepsilon_0\rangle$$

$$\Downarrow$$

$$\left(\sum_k \alpha_k |s_k\rangle|A_k\rangle\right)|\varepsilon_0\rangle$$

$$\Downarrow$$

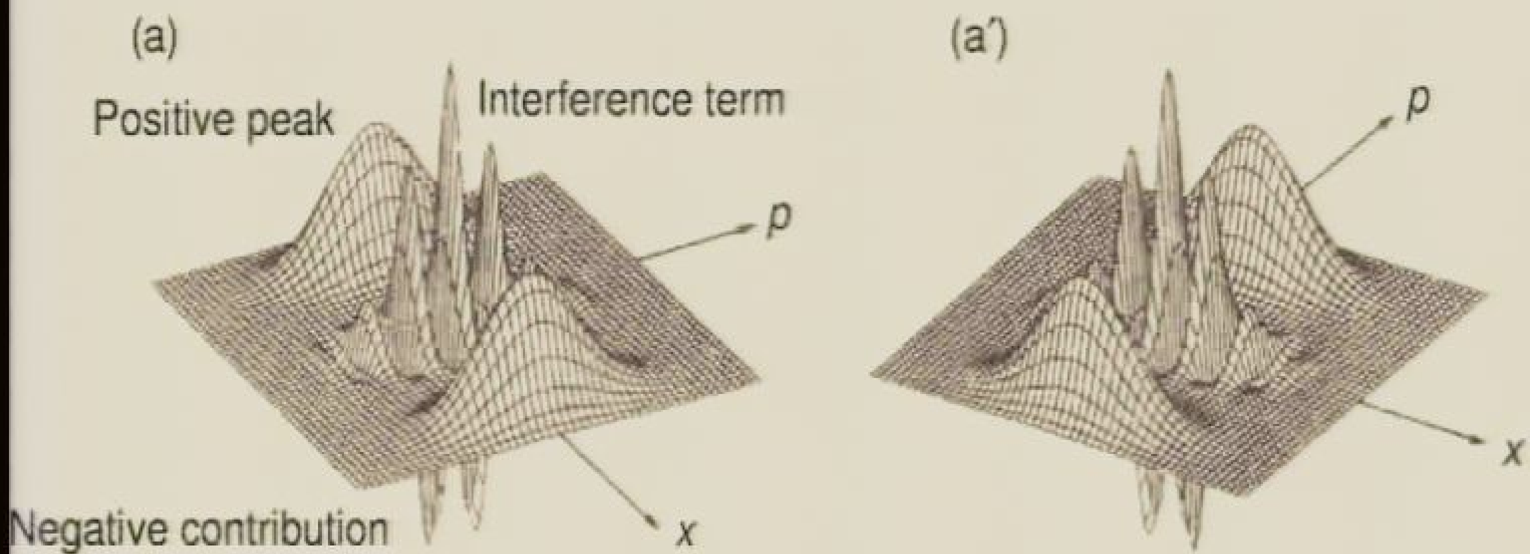
$$\sum_k \alpha_k |s_k\rangle|A_k\rangle|\varepsilon_k\rangle = |\Phi_{\mathcal{SAE}}\rangle$$

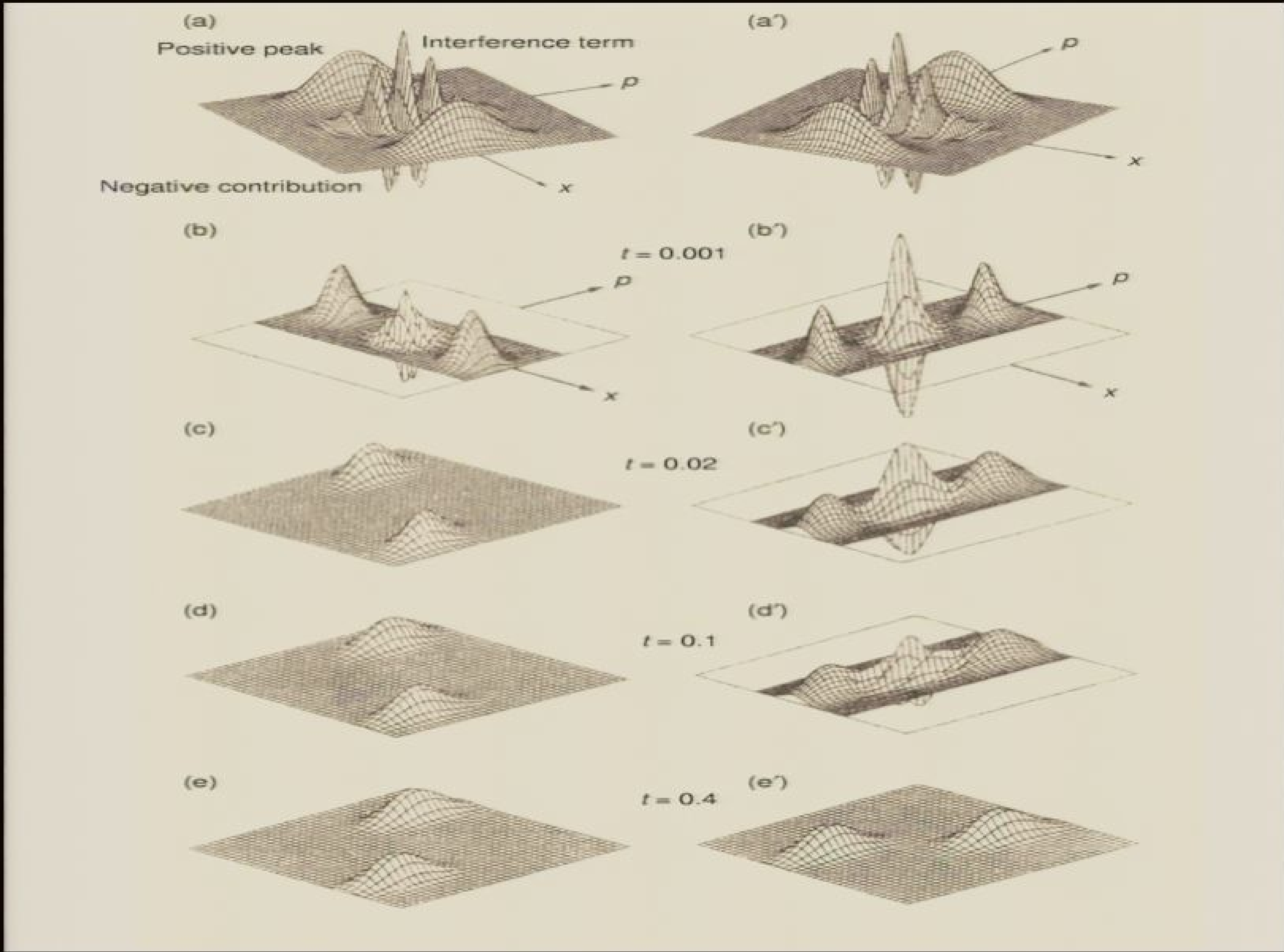
$$\rho_{\mathcal{SA}} = Tr_{\mathcal{E}}|\Phi_{\mathcal{SAE}}\rangle\langle\Phi_{\mathcal{SAE}}| \cong \sum_k |\alpha_k|^2 |s_k\rangle\langle s_k||A_k\rangle\langle A_k|$$

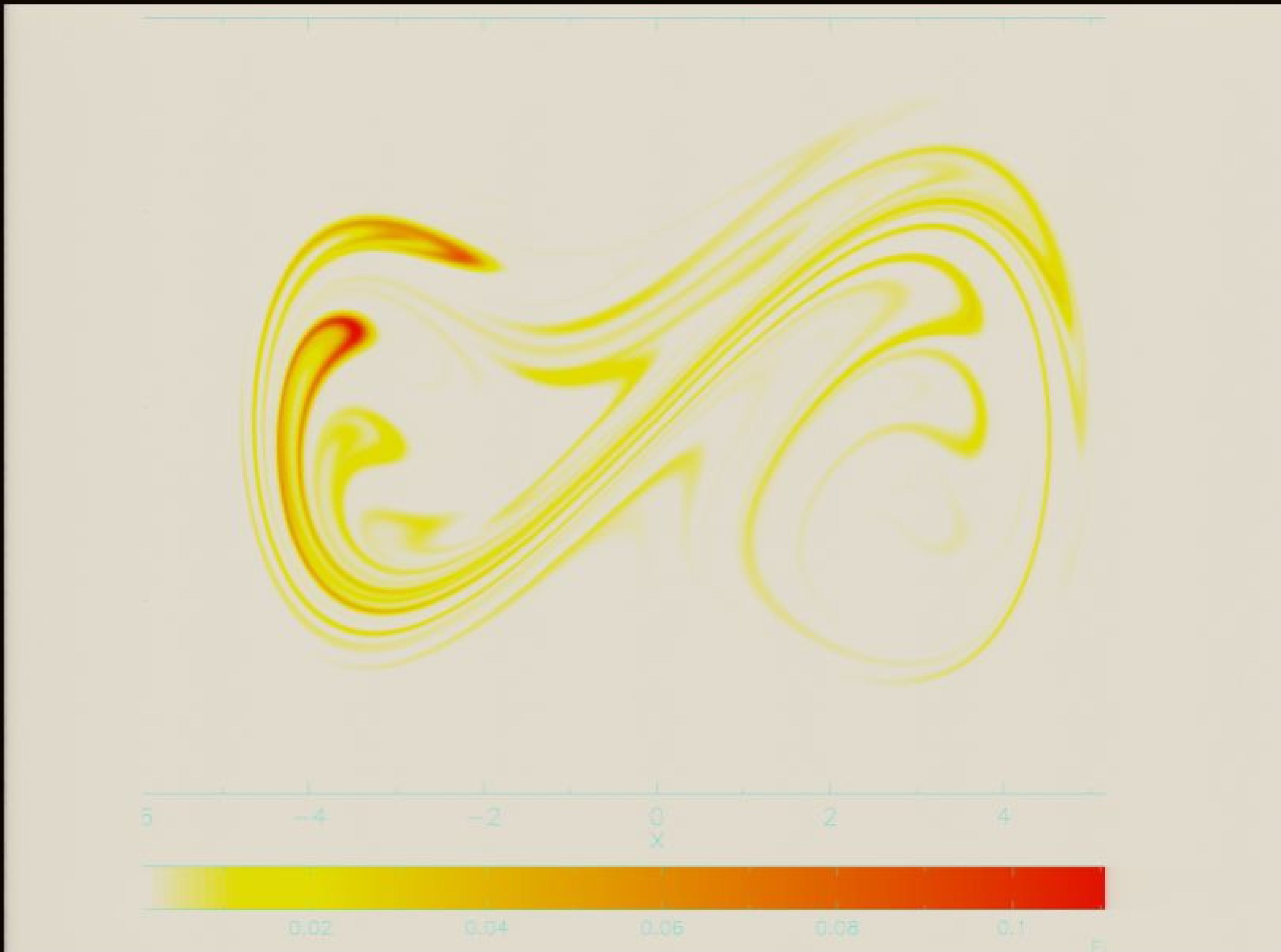
2. DYNAMICS

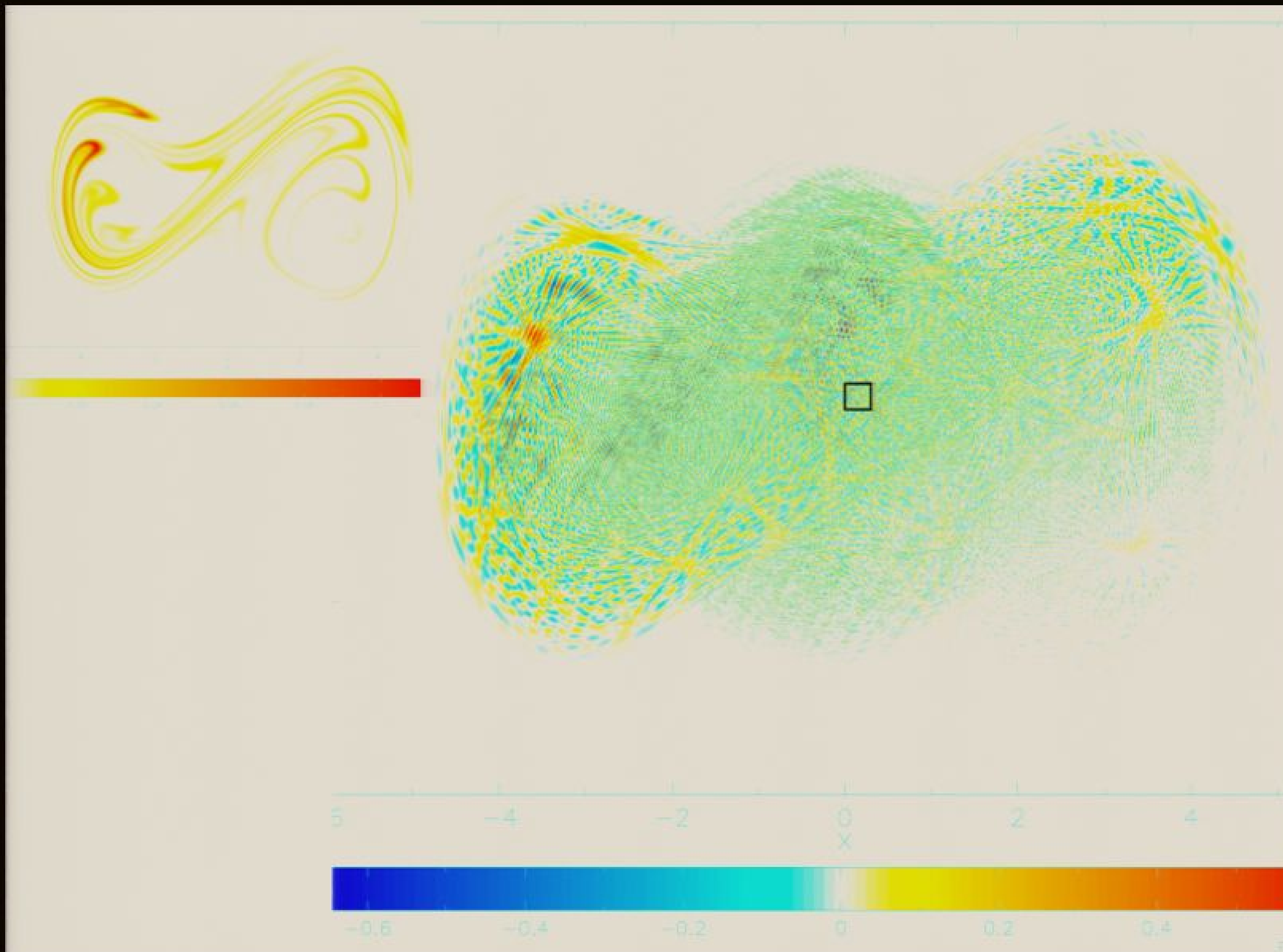
- States in Hilbert space "censored", restricted to localized quantum approximations of points
- Classical equations of motion

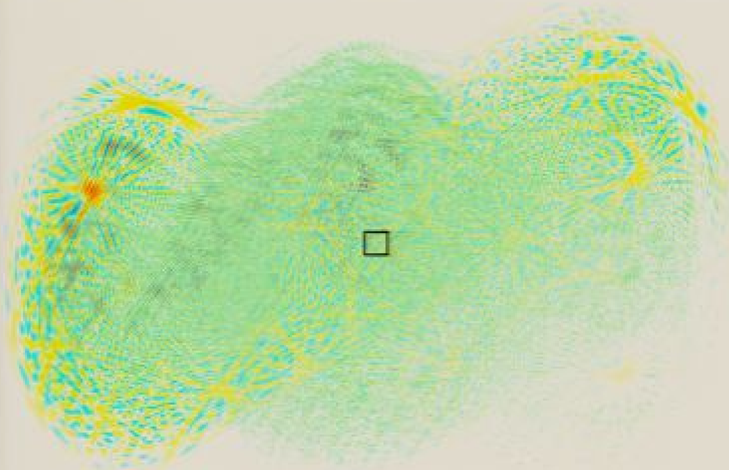
$$i\hbar \frac{d\rho_{\mathcal{SE}}}{dt} = [H, \rho_{\mathcal{SE}}] \Rightarrow \dot{W}_{\mathcal{S}}(x,p) \cong \{H_{\mathcal{S}}, W_{\mathcal{S}}\}_P$$

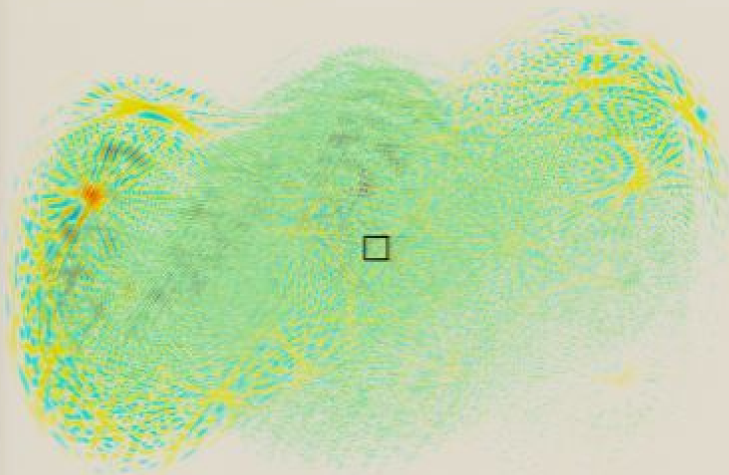

(a)
Positive peak
Interference term
p
x
Negative contribution
(a′)
p
x

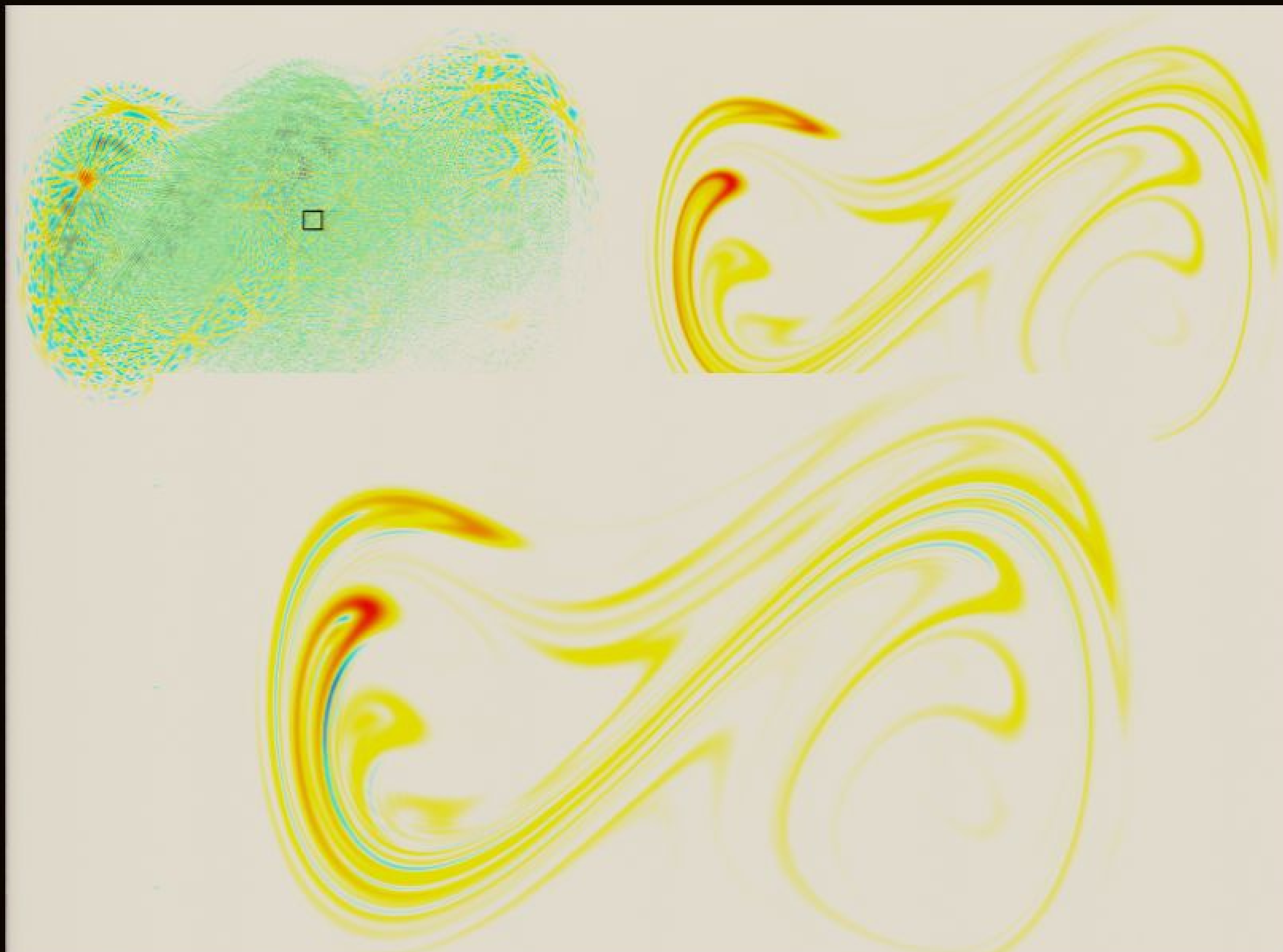

(a)
Positive peak
Interference term
p
x
Negative contribution
(a′)
p
x
(b)
t = 0.001
p
x
(b′)
p
x
(c)
t = 0.02
(c′)
(d)
t = 0.1
(d′)
(e)
t = 0.4
(e′)


-4
-2
0
2
4
x
0.02
0.04
0.06
0.08
0.1


-4
-2
0
2
4
x
-0.6
-0.4
-0.2
0
0.2
0.4







POINTER STATES FROM THE **PREDICTABILITY SIEVE**

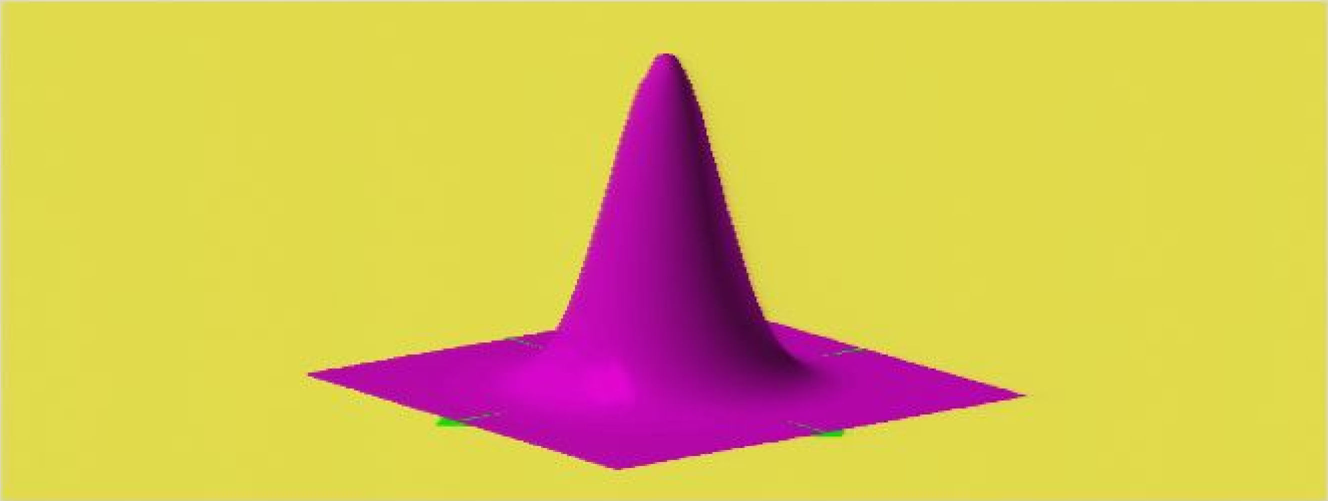
States in the Hilbert space of the open system evolve from pure into mixed under the influence of both the self-Hamiltonian and the interaction Hamiltonian. They can be sorted according to predictability (*e.g.* measured by entropy or by purity *h(r)*).

$$|\psi\rangle \Rightarrow \rho_\psi(t) \;\ldots\ldots\; h(\rho_\psi(t)) = Tr\rho_\psi^2(t)$$

$$|\varphi\rangle \Rightarrow \rho_\varphi(t) \;\ldots\ldots\; h(\rho_\varphi(t)) = Tr\rho_\varphi^2(t)$$

.

$$|\xi\rangle \Rightarrow \rho_\xi(t) \;\ldots\ldots\; h(\rho_\xi(t)) = Tr\rho_\xi^2(t)$$



DECOHERENCE AND EINSELECTION

Thesis: Quantum theory can explain emergence of the classical.

Principle of superposition loses its validity in "open" systems, that is, systems interacting with their environments.

Decoherence restricts stable states (states that can persist, and, therefore, "exist") to the exceptional…

Pointer states that exist or evolve predictably in spite of the immersion of the system in the environment.

Predictability sieve can be used to 'sift' through the Hilbert space of the open system in search of these pointer states.

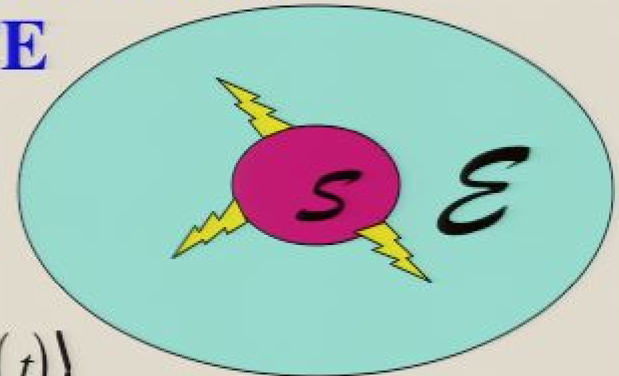
EINSELECTION (or Environment INduced superSELECTION) is the process of selection of these preferred pointer states.

For macroscopic systems, decoherence and einselection can be very effective, enforcing a ban on Schroedinger cats.

Einselection enforces an effective border that divides quantum from classical, making a point of view similar to Bohr's Copenhagen Interpretation possible, although starting from a rather different standpoint (i. e., no *ab initio* classical domain of the universe).

(Paz, Zeh, Joos, Caldeira, Leggett, Kiefer, Gell-Mann, Hartle, Unruh…)

EINSELECTION*, POINTER BASIS, AND DECOHERENCE



$$|\Phi_{S\mathcal{E}}(0)\rangle = |\psi_S\rangle \otimes |\varepsilon_0\rangle = \left(\sum_i \alpha_i |\sigma_i\rangle\right) \otimes |\varepsilon_0\rangle$$

Interaction ⟹ Entanglement

$$\sum_i \alpha_i |\sigma_i\rangle \otimes |\varepsilon_i\rangle = |\Phi_{S\mathcal{E}}(t)\rangle$$

REDUCED DENSITY MATRIX

........Depends on Born's Rule!!!

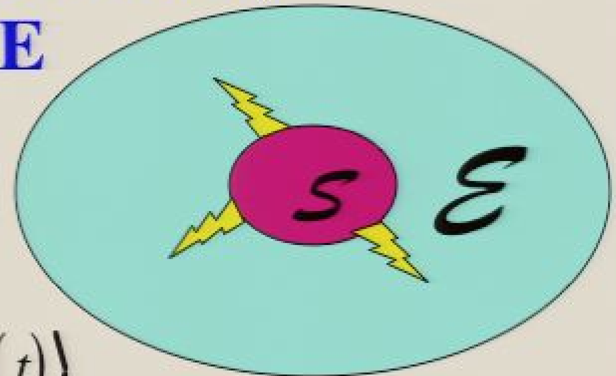
$$\rho_S(t) = Tr_{\mathcal{E}} |\Phi_{S\mathcal{E}}(t)\rangle\langle\Phi_{S\mathcal{E}}(t)| = \sum_i |\alpha_i|^2 |\sigma_i\rangle\langle\sigma_i|$$

EINSELECTION* leads to POINTER STATES

(same states appear on the diagonal of $\rho_S(t)$ for times long compared to the decoherence time)

***Environment INduced superSELECTION**

EINSELECTION*, POINTER BASIS, AND DECOHERENCE



$$\left|\Phi_{S\mathcal{E}}(0)\right\rangle = \left|\psi_S\right\rangle \otimes \left|\varepsilon_0\right\rangle = \left(\sum_i \alpha_i \left|\sigma_i\right\rangle\right) \otimes \left|\varepsilon_0\right\rangle$$

Interaction / Entanglement $\Longrightarrow$ $$\sum_i \alpha_i \left|\sigma_i\right\rangle \otimes \left|\varepsilon_i\right\rangle = \left|\Phi_{S\mathcal{E}}(t)\right\rangle$$

REDUCED DENSITY MATRIX

........*Depends on Born's Rule!!!*

$$\rho_S(t) = Tr_{\mathcal{E}} \left|\Phi_{S\mathcal{E}}(t)\right\rangle\left\langle\Phi_{S\mathcal{E}}(t)\right| = \sum \left|\alpha_i\right|^2 \left|\sigma_i\right\rangle\left\langle\sigma_i\right|$$

????????

EINSELECTION* leads to POINTER STATES

(same states appear on the diagonal of $\rho_S(t)$ for times long compared to the decoherence time)

*Environment INduced superSELECTION

BORN'S RULE

$$|\Psi\rangle = \sum_{k=1}^{N} \psi_k |s_k\rangle \Rightarrow p_k = |\psi_k|^2$$

This is how trace and reduced density matrices are justified (since introduction by Landau, 1927). Gleason (1956) proved (using elements of Bohr's Copenhagen interpretation) that Born's rule is the right measure on the subspaces of Hilbert space. Gleason's proof gives no insight into physical significance of Born's rule.

ENVARIANCE

Environment - Assisted Invariance

DEFINITION:

Consider a composite quantum object consisting of system $\mathcal{S}$ and environment $\mathcal{E}$. **When the combined state $\psi_{\mathcal{SE}}$ is transformed by:**

$$U_{\mathcal{S}} = u_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}}$$

but can be "untransformed" by acting solely on $\mathcal{E}$, that is, if there exists:

$$U_{\mathcal{E}} = \mathbf{1}_{\mathcal{S}} \otimes u_{\mathcal{E}}$$

then $\psi_{\mathcal{SE}}$ is ENVARIANT with respect to $u_{\mathcal{S}}$.

$$U_{\mathcal{E}}(U_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = U_{\mathcal{E}}|\varphi_{\mathcal{SE}}\rangle = |\psi_{\mathcal{SE}}\rangle$$

Envariance is a property of the joint state $\psi_{\mathcal{SE}}$ of two systems, $\mathcal{S}$ & $\mathcal{E}$.

ENVARIANCE

Entanglement - Assisted Invariance

DEFINITION:

Consider a composite quantum object consisting of system $\mathcal{S}$ and environment $\mathcal{E}$. **When the combined state $\psi_{\mathcal{SE}}$ is transformed by:**

$$U_{\mathcal{S}} = u_{\mathcal{S}} \otimes \mathbf{1}_{\mathcal{E}}$$

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Envariance is a property of the joint state $\psi_{\mathcal{SE}}$ of two systems, $\mathcal{S}$ & $\mathcal{E}$.

ENTANGLED STATE AS AN EXAMPLE OF ENVARIANCE: LEMMA1

Schmidt decomposition:

$$|\psi_{\mathcal{S}\mathcal{E}}\rangle = \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle$$

Above Schmidt states $|s_k\rangle$, $|\varepsilon_k\rangle$ are orthonormal and α_k complex.

Lemma 1: Unitary transformations with Schmidt eigenstates:

$$u_{\mathcal{S}}(s_k) = \sum_{k=1} \exp(i\phi_k)|s_k\rangle\langle s_k|$$

leave $\psi_{\mathcal{S}\mathcal{E}}$ envariant.

Proof:

$$u_{\mathcal{S}}(s_k)|\psi_{\mathcal{S}\mathcal{E}}\rangle = \sum_{k=1} \alpha_k \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle, \quad u_{\mathcal{E}}(\varepsilon_k) = \sum_{k=1} \exp\{i(-\phi_k + 2\pi l_k)\}|\varepsilon_k\rangle\langle\varepsilon_k|$$

$$u_{\mathcal{E}}(\varepsilon_k)\{u_{\mathcal{S}}(s_k)|\psi_{\mathcal{S}\mathcal{E}}\rangle\} = \sum_{k=1} \alpha_k \exp\{i(\phi_k - \phi_k + 2\pi l_k)|s_k\rangle|\varepsilon_k\rangle = \sum_{k=1} \alpha_k \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle = |\psi_{\mathcal{S}\mathcal{E}}\rangle$$

ENVARIANCE -- SOME PROPERTIES

$$U_{\mathcal{E}}(U_{\mathcal{S}}|\psi_{\mathcal{SE}}\rangle) = U_{\mathcal{E}}|\varphi_{\mathcal{SE}}\rangle = \exp(i\phi)|\psi_{\mathcal{SE}}\rangle$$

- Envariant $\psi_{\mathcal{SE}}$ is an eigenstate of two unitary transformations with a unit (or unimodular) eigenvalue.
- Envariance can be defined for density matrices of $\mathcal{SE}$, but this will not be necessary, as one can instead purify the state of $\mathcal{SE}$ in the usual way, by introducing $\mathcal{E}'$, so the density matrix of $\mathcal{SE}$ is given by: $\rho_{\mathcal{SE}} = Tr_{\mathcal{E}'}|\Psi_{\mathcal{SEE}'}\rangle\langle\Psi_{\mathcal{SEE}'}|$
- A product of envariant transformations of $\psi_{\mathcal{SE}}$ is an envariant transformation of $\psi_{\mathcal{SE}}$.
- All envariant transformations have Schmidt eigenstates.

For additional discussion, see WHZ, quant-ph/0211037, PRL, 90, 120404 (2003); *Decoherence, einselection, and the quantum origin of the classical* RMP, 75, 715 (2003); and especially *Probabilities from entanglement…*, quant-ph/0405161, PRA May '05

ENTANGLED STATE AS AN EXAMPLE OF ENVARIANCE: LEMMA 2

Schmidt decomposition:

$$|\psi_{\mathcal{SE}}\rangle = \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle$$

Schmidt states $|s_k\rangle$, $|\varepsilon_k\rangle$ are orthonormal and α_k complex.

Lemma 2: ALL envariant transformations have Schmidt eigenstates, that is, have the form of:

$$u_{\mathcal{S}}(s_k) = \sum_{k=1} \exp(i\phi_k)|s_k\rangle\langle s_k|$$

Proof (by contradiction): Consider a unitary transformation that does not have Schmidt eigenstates. It will change the Schmidt states of the system in $\psi_{\mathcal{SE}}$. But a change of Schmidt states cannot be undone by any transformation acting only on the environment. QED.

Remark: When absolute values of some of the coefficients are equal, Schmidt basis is not unique in the corresponding subspace…

PHASE ENVARIANCE THEOREM

Fact 1: Unitary transformations must act on the system to alter its state (if they act only somewhere else, system is not effected).

Fact 2: The state of the system is all that is necessary/available to predict measurement outcomes (including their probabilities).

Fact 3: A state of the composite system is all that is needed/available to determine the state of the system.

Moreover, "entanglement happens":

$$|\psi_{\mathcal{S}\mathcal{E}}\rangle \propto \sum_{k=1}^{N} \alpha_k |s_k\rangle|\varepsilon_k\rangle$$

THEOREM 1: Probabilities of the system $\mathcal{S}$ alone can depend only on the absolute values of Schmidt coefficients $|\alpha_k|$, and not on their phases.

Proof: Phases of α_k can be changed by acting on $\mathcal{S}$ alone. But the state of the whole can be restored by acting only on $\mathcal{E}$. So the change of the phases of Schmidt coefficients could not have affected $\mathcal{S}$! QED.

$\therefore$ By phase envariance, $\{|\alpha_k|, |s_k\rangle\}$ must provide a complete local description of the system alone. NOTE: Same info as density matrix!!!

Envariance of entangled states: the case of equal coefficients

$$|\psi_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

In this case ANY orthonormal basis is Schmidt. In particular, in the Hilbert subspace spanned by any two $\{|s_k\rangle, |s_l\rangle\}$ one can define a Hadamard basis;

$$|\pm\rangle = (|s_k\rangle \pm |s_l\rangle)/\sqrt{2}$$

This can be used to generate 'new kind' of envariant transformations:

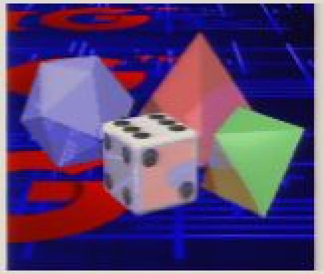
A ***SWAP:*** $u_{\mathcal{S}}(k \leftrightarrow l) = \exp(i\varphi_{kl})|s_k\rangle\langle s_l| + h.c.$

Can be 'undone' by the ***COUNTERSWAP:***

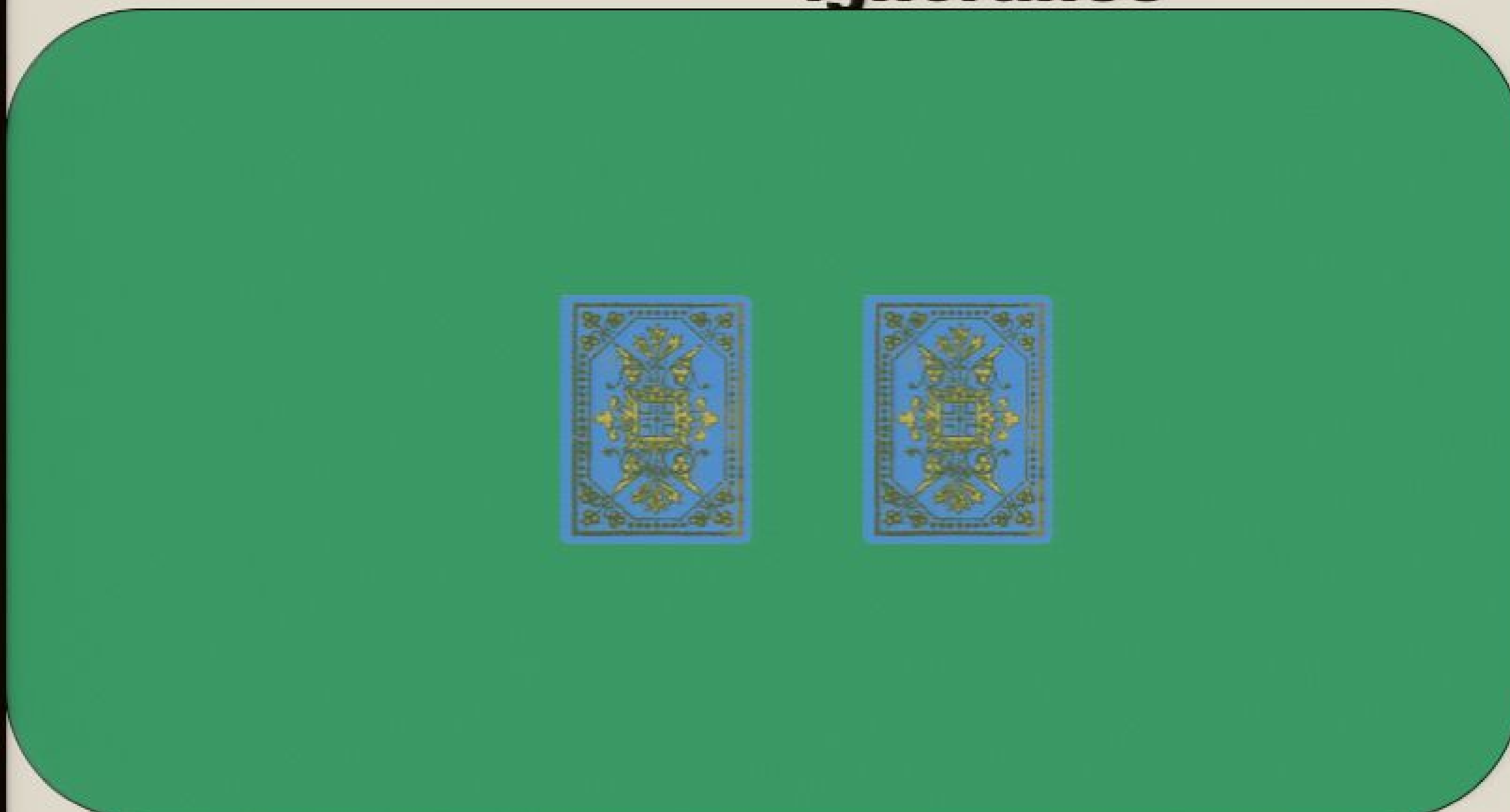
$$u_{\mathcal{E}}(k \leftrightarrow l) = \exp\{i(-\varphi_{kl} - \varphi_k + \varphi_l)\}|\varepsilon_l\rangle\langle\varepsilon_k| + h.c.$$

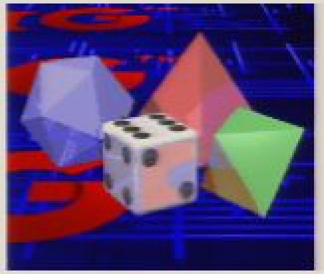
LEMMA 3: **Swaps of states are envariant when their Schmidt coefficients have the same absolute value.**

Symmetries can reflect ignorance

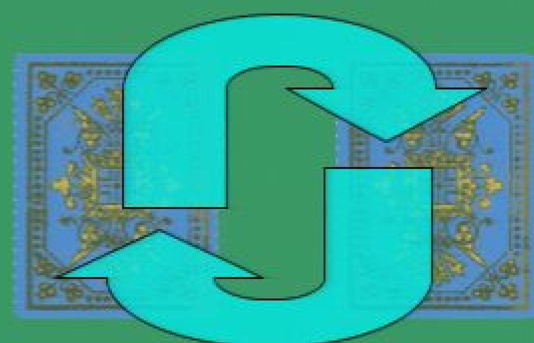


Symmetries can reflect ignorance

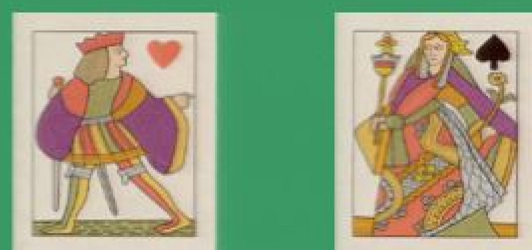


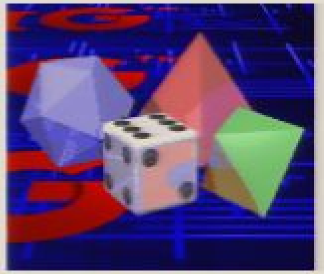


Symmetries can reflect ignorance



Symmetries can reflect ignorance



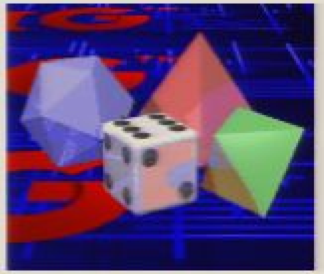


Symmetries can reflect ignorance

Probabilities from envariance

(Environment-assisted iNVARIANCE)

$$|0_S\rangle|0_E\rangle + |1_S\rangle|1_E\rangle \xrightarrow{\text{swap in } S} |1_S\rangle|0_E\rangle + |0_S\rangle|1_E\rangle \xrightarrow{\text{swap in } E} |1_S\rangle|1_E\rangle + |0_S\rangle|0_E$$



Symmetries can reflect ignorance

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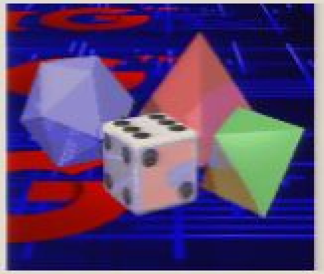

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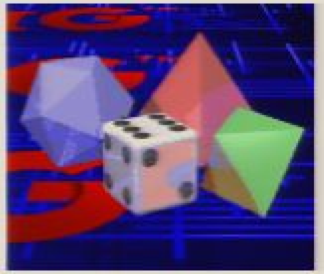

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$p = |\psi|^2$ follows!

Envariance of entangled states: the case of equal coefficients

$$|\psi_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

In this case ANY orthonormal basis is Schmidt. In particular, in the Hilbert subspace spanned by any two $\{|s_k\rangle, |s_l\rangle\}$ one can define a Hadamard basis;

$$|\pm\rangle = (|s_k\rangle \pm |s_l\rangle)/\sqrt{2}$$

This can be used to generate 'new kind' of envariant transformations:

A ***SWAP:*** $u_{\mathcal{S}}(k \leftrightarrow l) = \exp(i\varphi_{kl})|s_k\rangle\langle s_l| + h.c.$

Can be 'undone' by the ***COUNTERSWAP:***

$$u_{\mathcal{E}}(k \leftrightarrow l) = \exp\{i(-\varphi_{kl} - \varphi_k + \varphi_l)\}|\varepsilon_l\rangle\langle\varepsilon_k| + h.c.$$

LEMMA 3: Swaps of states are envariant when their Schmidt coefficients have the same absolute value.

Probabilities of envariantly swappable states

$$|\psi_{\mathcal{SE}}\rangle \propto \sum_{k=1}^{N} \exp(i\phi_k)|s_k\rangle|\varepsilon_k\rangle$$

By the Phase Envariance Theorem the set of pairs $|\alpha_k|,\ |s_k\rangle$ provides a complete description of $\mathcal{S}$. But all $|\alpha_k|$ are equal.

With additional assumption about probabilities, can prove

THEOREM 2: Probabilities of envariantly swappable states are equal.

(a) "pedantic assumption"; when states get swapped, so do probabilitites;

(b) when the state of the system does not change under any unitary in a part of its Hilbert space, probabilities of any set of basis states are equal.

(c) When there is one-to-one correlation between $|s_k\rangle, |\varepsilon_k\rangle$ (Barnum '03).

Therefore, by normalization: $$p_k = \frac{1}{N} \quad \forall_k$$

Moreover: $$p_{k_1 \vee k_2 \vee \dots \vee k_m} = \frac{m}{N}$$

Special case with unequal coefficients

Consider system $\mathcal{S}$ with two states $\{|0\rangle, |2\rangle\}$

The environment $\mathcal{E}$ has three states $\{|0\rangle, |1\rangle, |2\rangle\}$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

$$|\psi_{\mathcal{SE}}\rangle = \sqrt{\frac{2}{3}}|0\rangle|+\rangle + \sqrt{\frac{1}{3}}|2\rangle|2\rangle$$

An auxilliary environment $\mathcal{E}'$ interacts with $\mathcal{E}$ so that:

$$|\psi_{\mathcal{SE}}\rangle|\mathcal{E}'_0\rangle = \left(\sqrt{\frac{2}{3}}|0\rangle|+\rangle + \sqrt{\frac{1}{3}}|2\rangle|2\rangle\right)|0\rangle \Rightarrow \sqrt{\frac{2}{3}}|0\rangle(|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2} + \sqrt{\frac{1}{3}}|2\rangle|2\rangle|2\rangle =$$

$$= (|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |2\rangle|2\rangle|2\rangle)/\sqrt{3}$$

States $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|2\rangle|2\rangle$ have equal coefficients. Therefore, Each of them has probability of 1/3. Consequently:

p(0) = p(0,0)+p(0,1) = 2/3, and p(2) = 1/3.

..... BORN's RULE!!!

$\circlearrowleft SL(2,Z)$

$$|k\rangle|0\rangle \rightarrow |k\rangle|k\rangle$$

Special case with unequal coefficients

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..... **BORN's RULE!!!**

each fixed p

$$\sim \sqrt{2}\,|0\rangle|\varepsilon_0\rangle + \sqrt{1}\,|2\rangle|2\rangle$$

$$\sim \sqrt{2}\, |0\rangle|\varepsilon_0\rangle + \sqrt{1}\, |2\rangle|2\rangle$$

$$\mathrm{Tr}\,\rho_{S\varepsilon}^2 = 1 \qquad \mathrm{Tr}\left(|\psi_{S\varepsilon}\rangle\langle\psi_{S\varepsilon}| \right)^2 =$$

Probabilities from Envariance

The case of commensurate probabilities: $|\psi_{\mathcal{SE}}\rangle = \sum_{k=1}^{N} \underbrace{\sqrt{m_k/M}}_{\alpha_k} |s_k\rangle|\varepsilon_k\rangle$

Attach the auxiliary environment $\mathcal{E}'$:

$$|\psi_{\mathcal{SE}}\rangle|e_0'\rangle = \left(\sum_{k=1}^{N} \underbrace{\sqrt{m_k/M}}_{\alpha_k} |s_k\rangle \left(\underbrace{\sum_{j_k=1}^{m_k} \frac{1}{\sqrt{m_k}} |e_{j_k}\rangle}_{|\varepsilon_k\rangle} \right) \right) |e_0'\rangle \Rightarrow$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{M}} \sum_{j=1}^{M} |s_{k(j)}\rangle|e_j\rangle|e_j'\rangle}$$

THEOREM 3: The case with commensurate probabilities can be reduced to the case with equal probabilities. **BORN's RULE follows:**

$$p_j = \frac{1}{M}, \qquad \boxed{p_k =} \sum_{j_k=1}^{m_k} p_{j_k} = \frac{m_k}{M} \boxed{= |\alpha_k|^2}$$

General incommensurate case follows by continuity. QED.

ENVARIANCE* -- SUMMARY

1. New symmetry - ENVARIANCE - of joint states of quantum systems. It is related to causality.

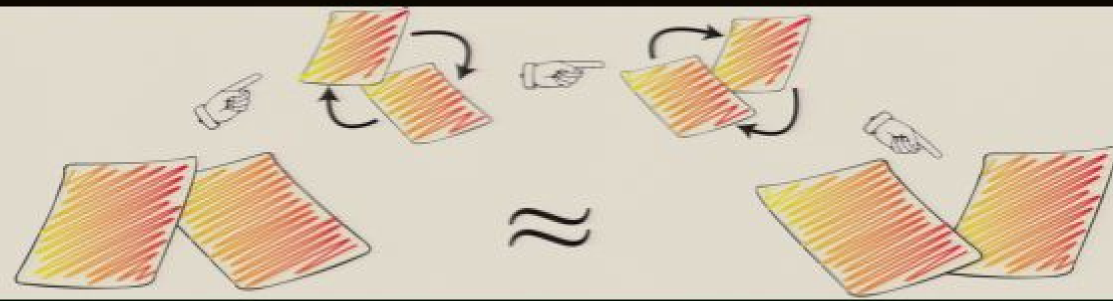
2. In quantum physics perfect knowledge of the whole may imply complete ignorance of a part.

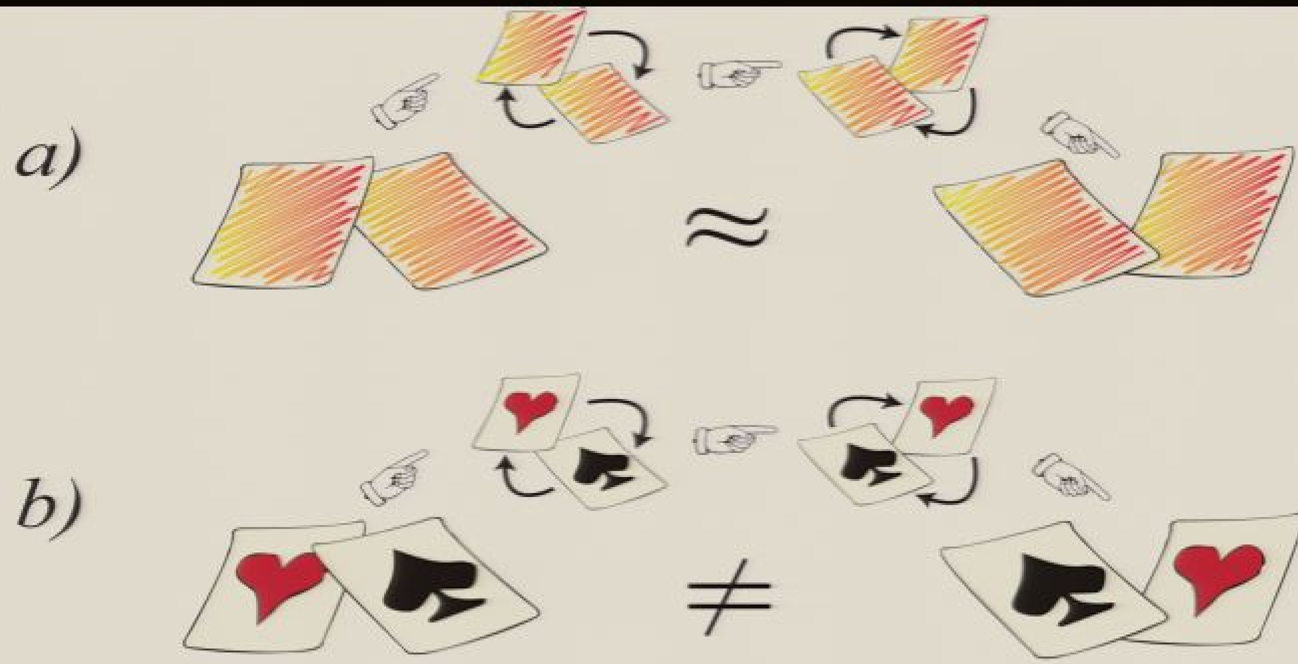
3. **BORN's RULE** is a consequence of envariance.

4. Relative frequency interpretation of probabilities naturally follows.

5. Envariance supplies a new foundation for environment - induced superselection, decoherence, quantum statistical physics, etc., by justifying the form and interpretation of reduced density matrices.

*WHZ, PRL, **90**, 120404; RMP **75**, 715 (2003); quant-ph/0405161, PRA May '05 a)

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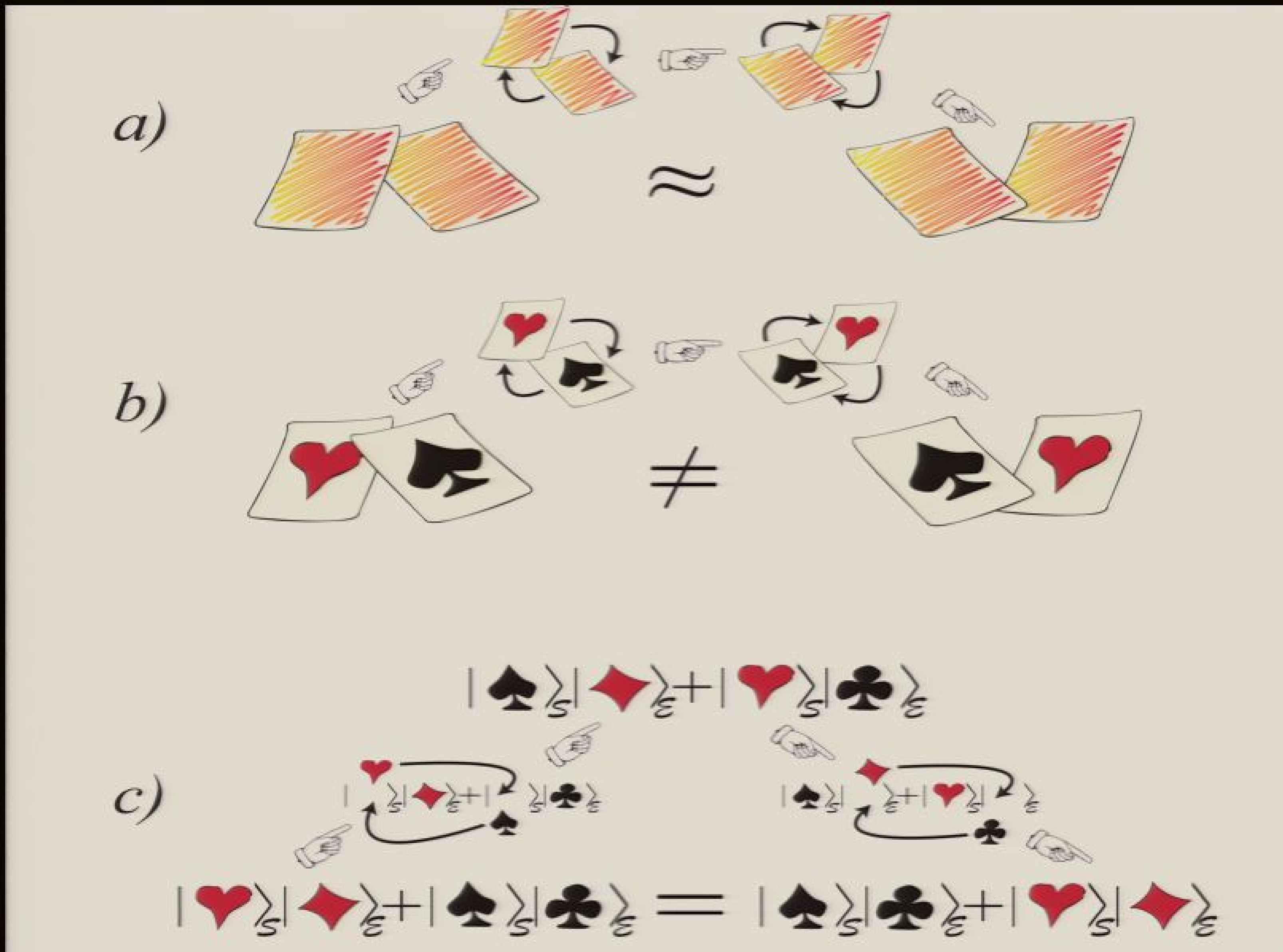
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