Title: Positive Linear Maps on Matrix Algebras

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Abstract:

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Φ is *completely positive* when Φ is of the form

$\Phi(A) = \Sigma V_j{}^* A V_j$ for all A in M_n .

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- The main question: Are there any tractable structure theory for positive linear maps?
- Must each positive linear map (restricted to real symmetric matrices) be realized as a completely positive linear map?

The simplest counter-example of a positive linear map that does not have completely positive effect

The promised counter-example is a linear map

$\Phi : M_3 \to M_3$ such that

$$\Phi\left(\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}\right) = \begin{bmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{bmatrix} + \begin{bmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{bmatrix}$$

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- Conversely, if $f \geq 0$, does it follow that f is a sum of squares of polynomials?
- Upon homogenization, it is sufficient to consider this problem in forms (homogeneous polynomials). In the context of forms, Hilbert (1888) has solved the problem completely.

Are positive forms sums of squares?

deg. \ var.	2	3	4	5	
2	✓	✓	✓	✓	...
4	✓	✓	×	×	...
6	✓	×	×	×	...
8	✓	×	×	×	...

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- But, Hilbert's method is very complicated, with no hope of practical construction.
- In 1967, Motzkin gave a concrete example of degree-6 form of 3 variables.
- In 1973, R.M. Robinson gave a concrete example of degree-4 form of 4 variables.

Bi-quadratic forms

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- The class of such forms arise naturally in several different connections.
- In particular, each linear map $\Phi: M_n \to M_m$ determines a biquadratic form $y^*\Phi(xx^*)y$.
- Thus positive linear maps induce positive biquaratic forms while completely positive linear maps induce sums of squares.

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- This was false as I worked out the case of a positive biquadratic form

$$B(\mathbf{x}, \mathbf{y}) = (x_1^2y_1^2+x_2^2y_2^2 +x_3^2y_3^2)$$
$$-2(x_1x_2y_1y_2 +x_2x_3y_2y_3 + x_3x_1y_3y_1)$$
$$+(x_1^2y_2^2+x_2^2y_3^2+x_3^2y_1^2)$$

associated with the special positive linear maps.

In $B(\mathbf{x}, \mathbf{y}) = B(x_1, x_2, x_3; y_1, y_2, y_3)$,

letting $x_1 = X$, $x_2 = W$, $x_3 = Z$,

$y_1 = Y$, $y_2 = Z$, $y_3 = W$,

we get

$$Q(X, Y, Z, W) = W4 + X^2Y^2 + Y^2Z^2 + Z^2X^2 - 4XYZW$$

which is a positive degree 4 form but not sum of squares.

(Proof) Q is positive because

arithmetic mean $\geq$ *geometric mean.*

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- Therefore Q is not sum of squares.

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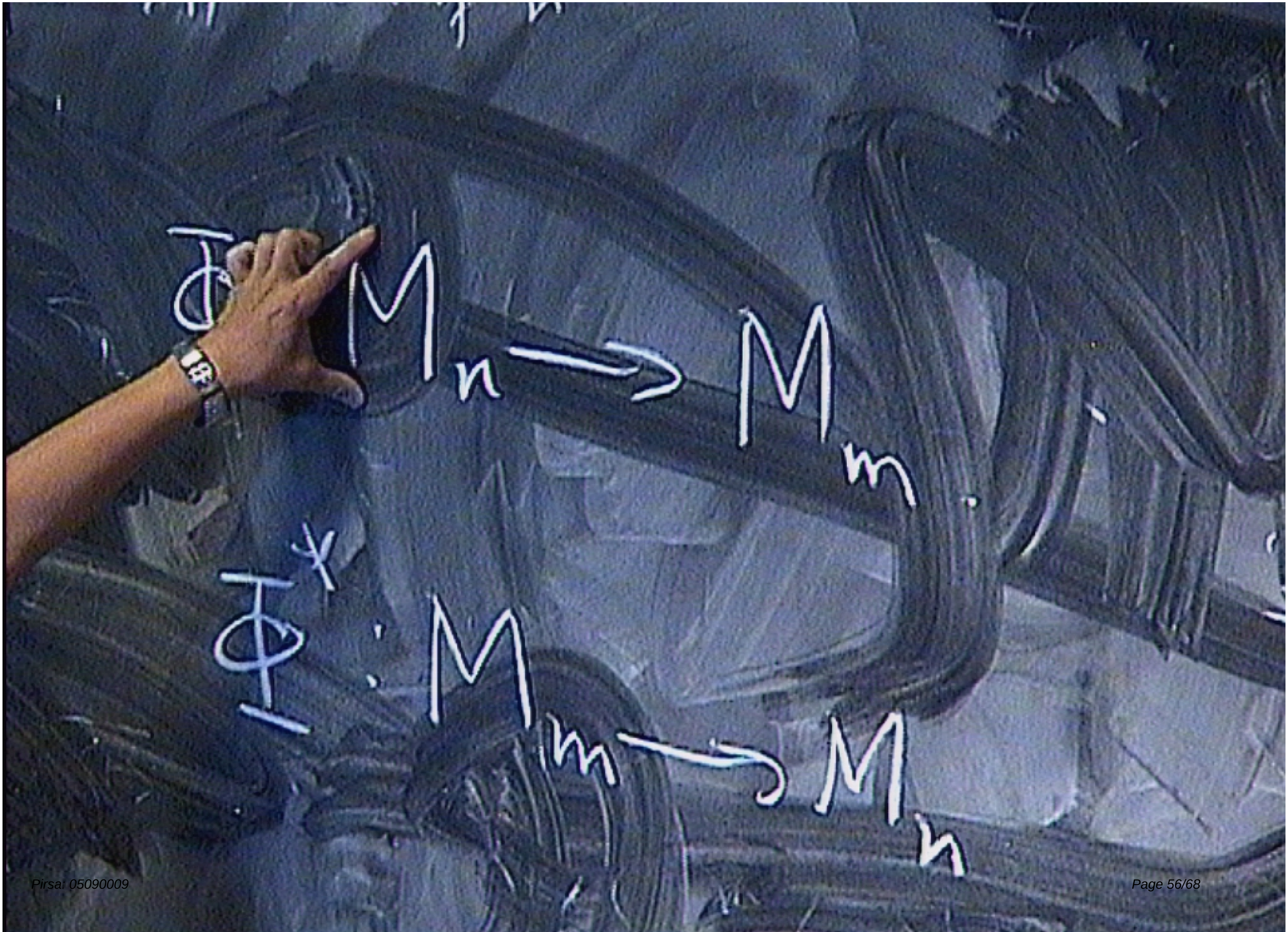
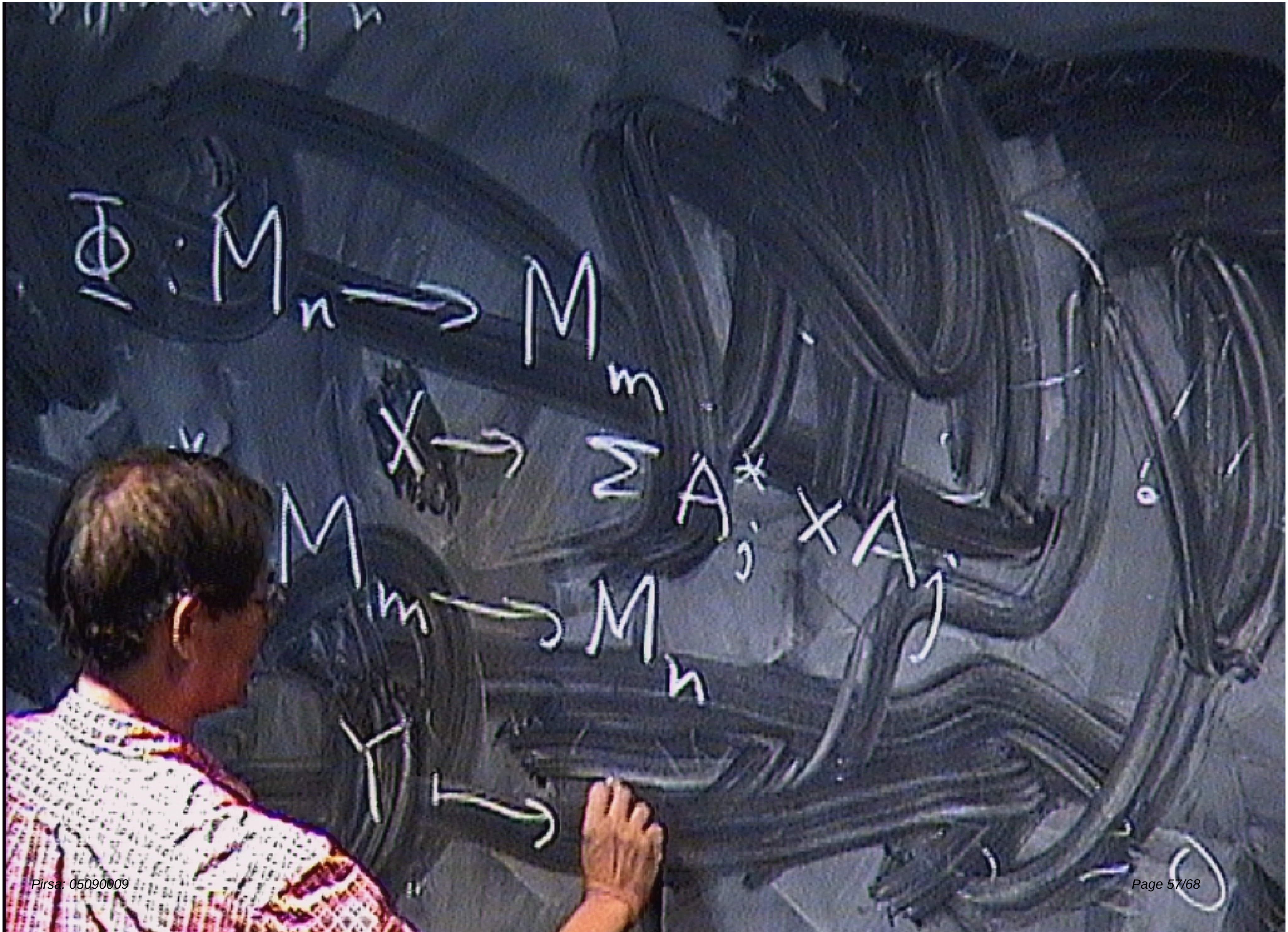
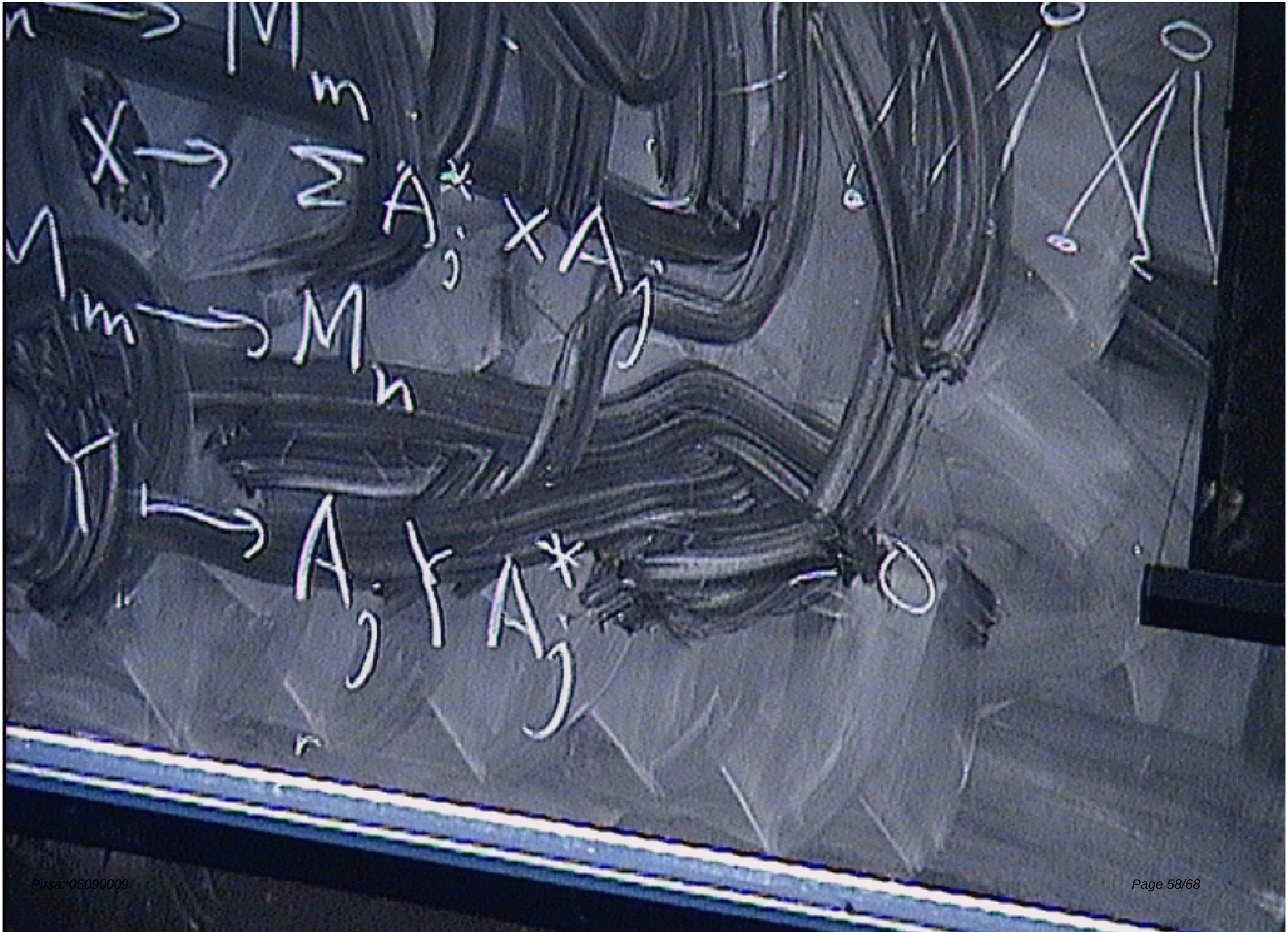
- Therefore S is not sum of squares.

Let $L(M_n, M_m) = \{$all linear maps: $M_n \to M_m\}$. There is a natural linear isomorphism between $L(M_n, M_m)$ and $M_n \otimes M_m \simeq M_{nm}$, assigning each linear map $\Phi: M_n \to M_m$ to a big matrix $[\Phi(E_{jk})]_{j,k=1}^{n} \in M_n(M_m)$. Moreover, M_p is identifiable with $\{$linear functionals on $M_p\}$ since each $A \in M_p$ induces a linear functional ρ_A by $\rho_A(X) = \text{trace}(AX)$. Henceforth, we get a chart showing natural correspondences among different classes.

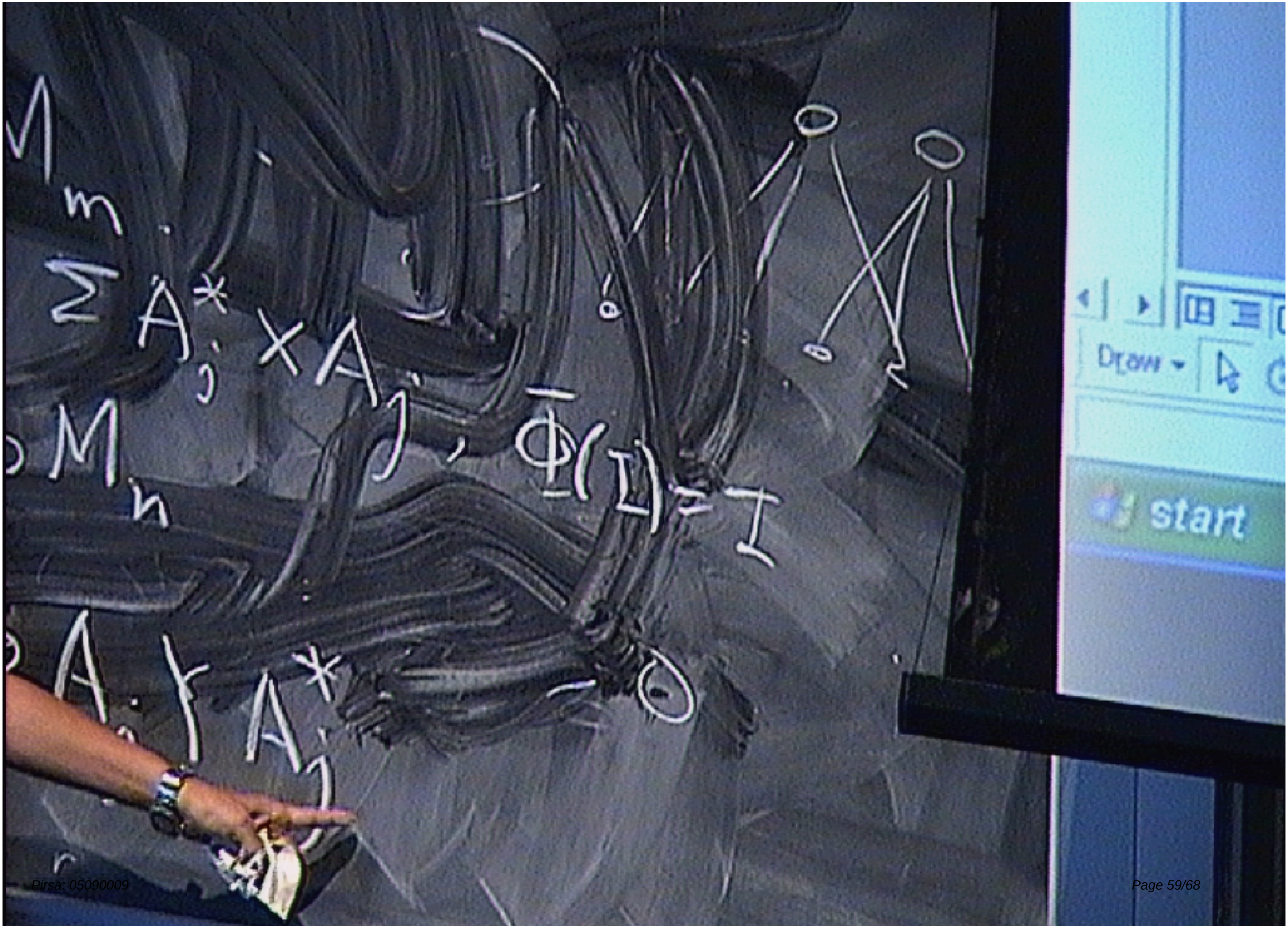
$L(M_n, M_m)$	$\longleftrightarrow$	$M_n \otimes M_m$	$\longleftrightarrow$	linear functionals on $M_n \otimes M_m$
$\uparrow$		$\uparrow$		$\uparrow$
hermitian-preserving linear maps	$\longleftrightarrow$	$(M_n \otimes M_m)^h$	$\longleftrightarrow$	linear functionals assuming real values on $(M_n \otimes M_m)^h$
$\uparrow$		$\uparrow$		$\uparrow$
positive linear maps	$\longleftrightarrow$	?	$\longleftrightarrow$	linear functionals assuming positive values on $M_n^+ \otimes M_m^+$
$\uparrow$		$\uparrow$		$\uparrow$
completely positive linear maps	$\longleftrightarrow$	$(M_n \otimes M_m)^+$	$\longleftrightarrow$	linear functionals assuming positive values on $(M_n \otimes M_m)^+$

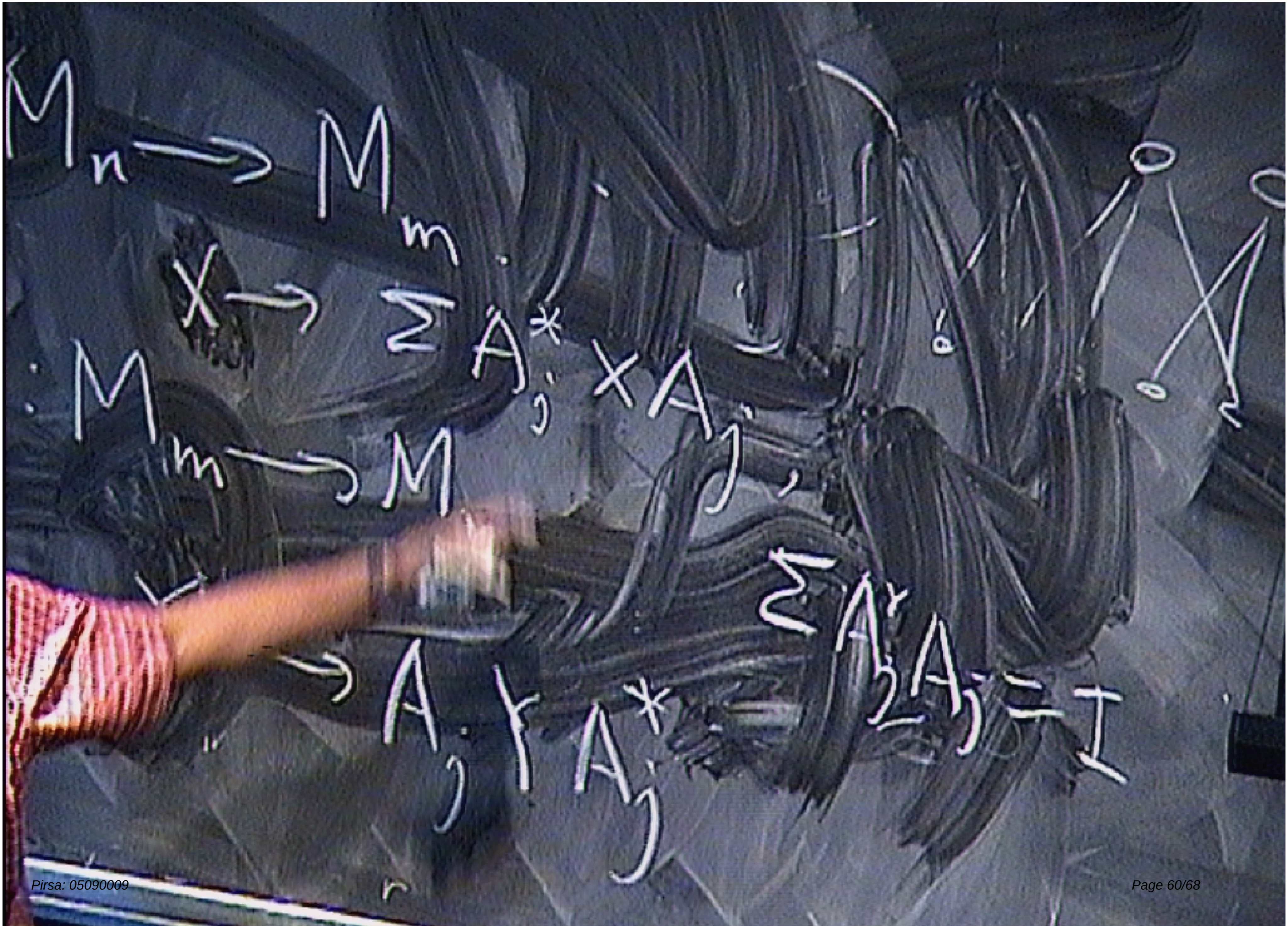

$\Phi: M_n \to M_m$
$\Phi^*: M_m \to M_n$

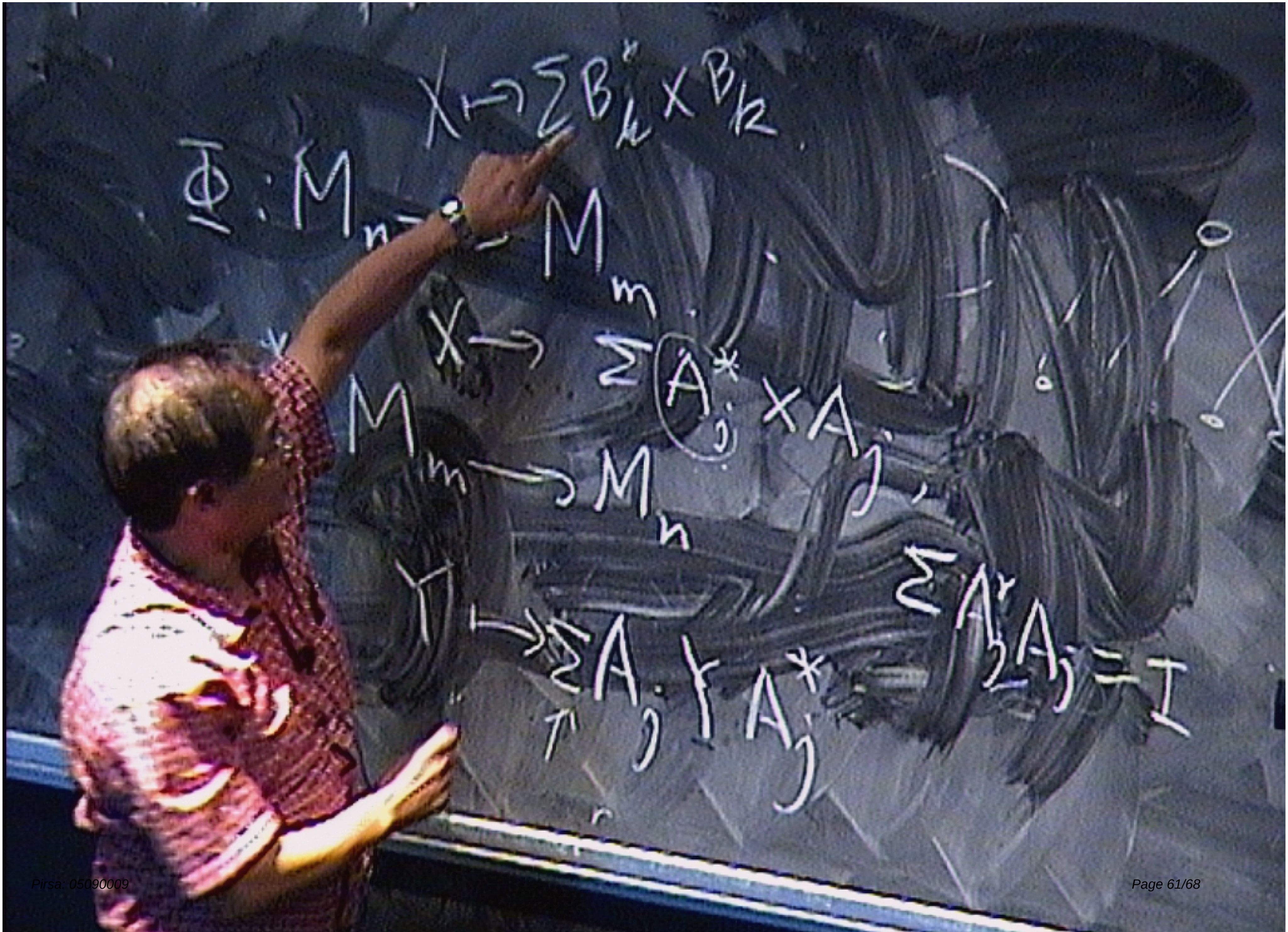

Φ: M_n ⟶ M_m
X ⟶ Σ A_j^* X A_j
M_m ⟶ M_n
Y ⟼



M_m → M_n
X ↦ Σ A_j* X A_j
M_n → M_m
Y ↦ A_j Y A_j*


Draw
start


M_n → M_m
X → Σ A_j* X A_j
M_m → M_n
→ A_j Y A_j*
Σ A_j* A_j = I


$X \mapsto \sum B_k^* X B_k$
$\Phi : M_n \to M_m$
$X \mapsto \sum A_j^* X A_j$
$M_m \to M_n$
$Y \mapsto \sum A_j Y A_j^*$
$\sum A_j^* A_j = I$

$X \mapsto \sum B_k^* X B_k$

$\Phi : M_n \to M_m$

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$\Phi^* : M_m \to M_n$

$Y \mapsto \sum A_j Y A_j^*$

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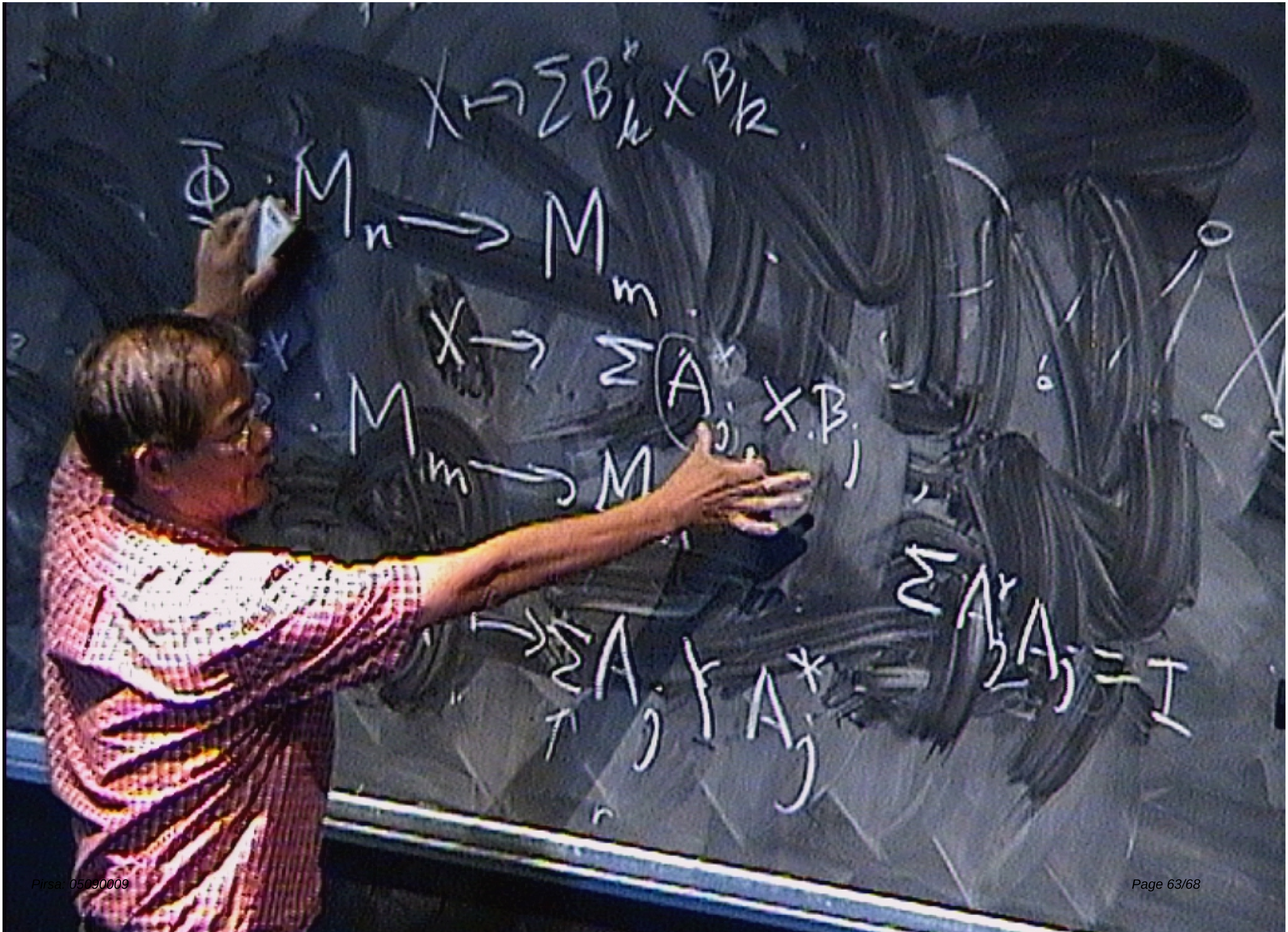
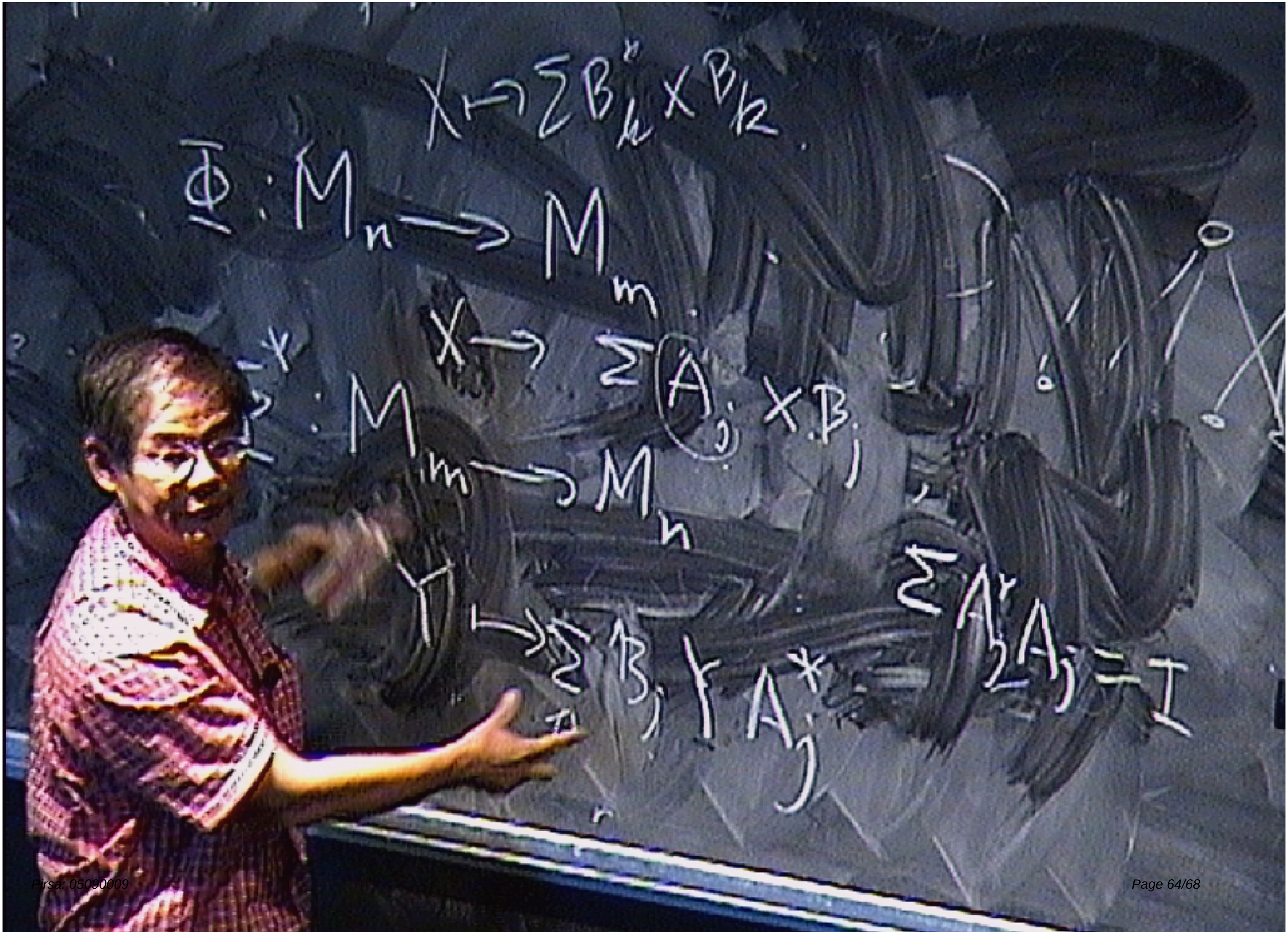

X ↦ Σ B_k^* X B_k
Φ : M_n → M_m
X → Σ A_j X B_j
M_m → M
Σ A_j^* A_j = I



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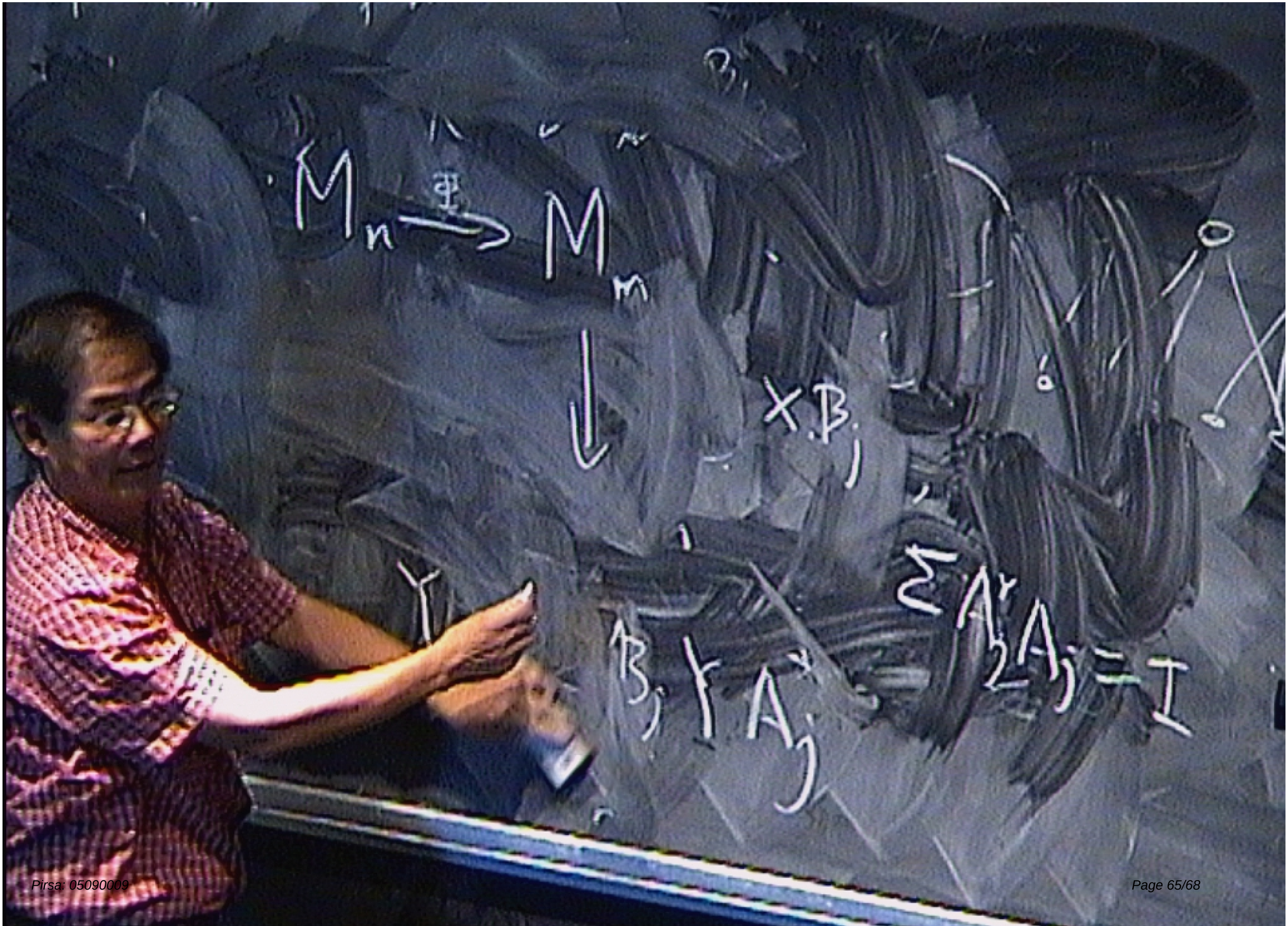
$\sum A_j^* A_j = I$


M_n Φ M_m
×B_j
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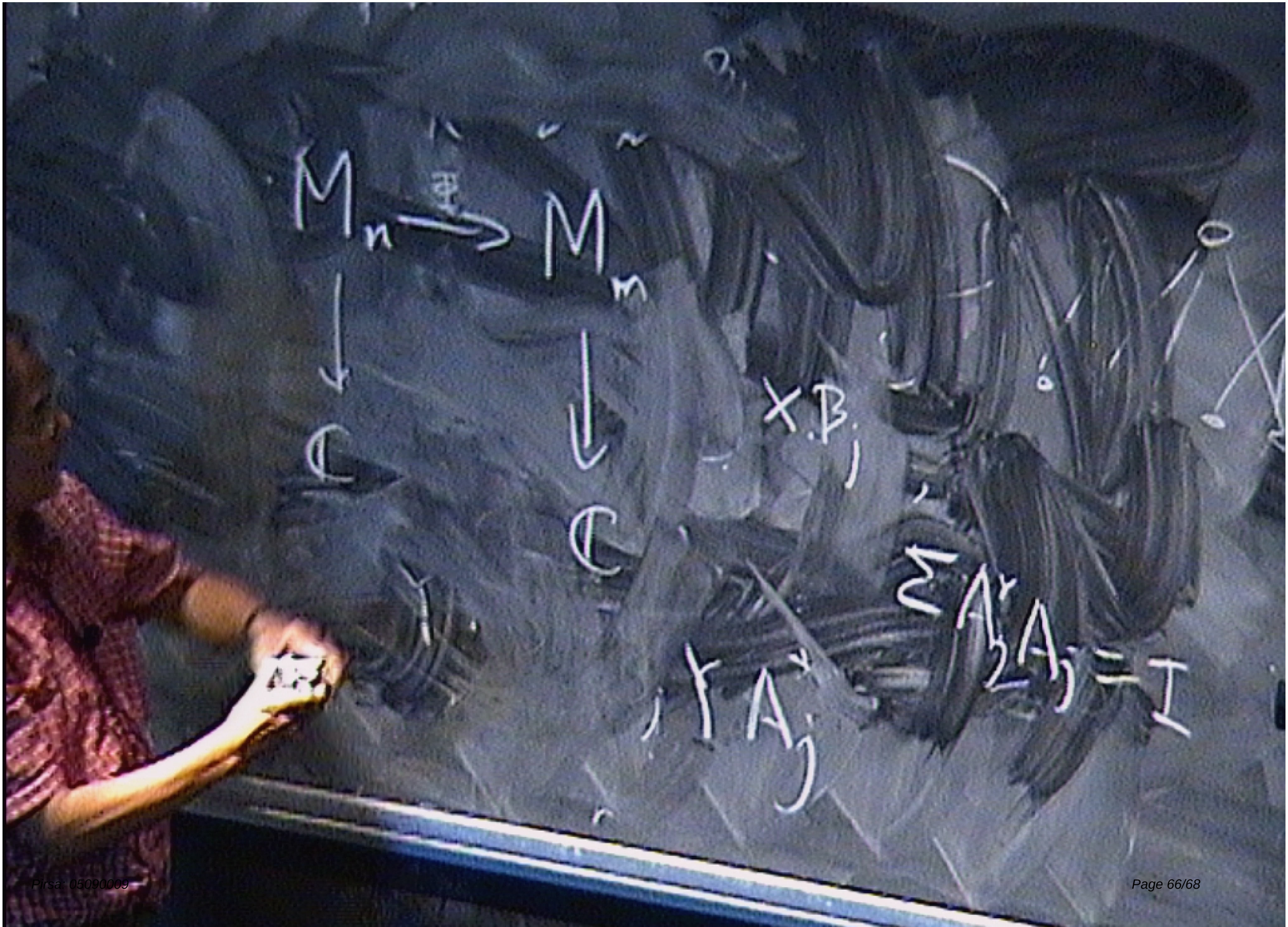
Σ A_j A_j = I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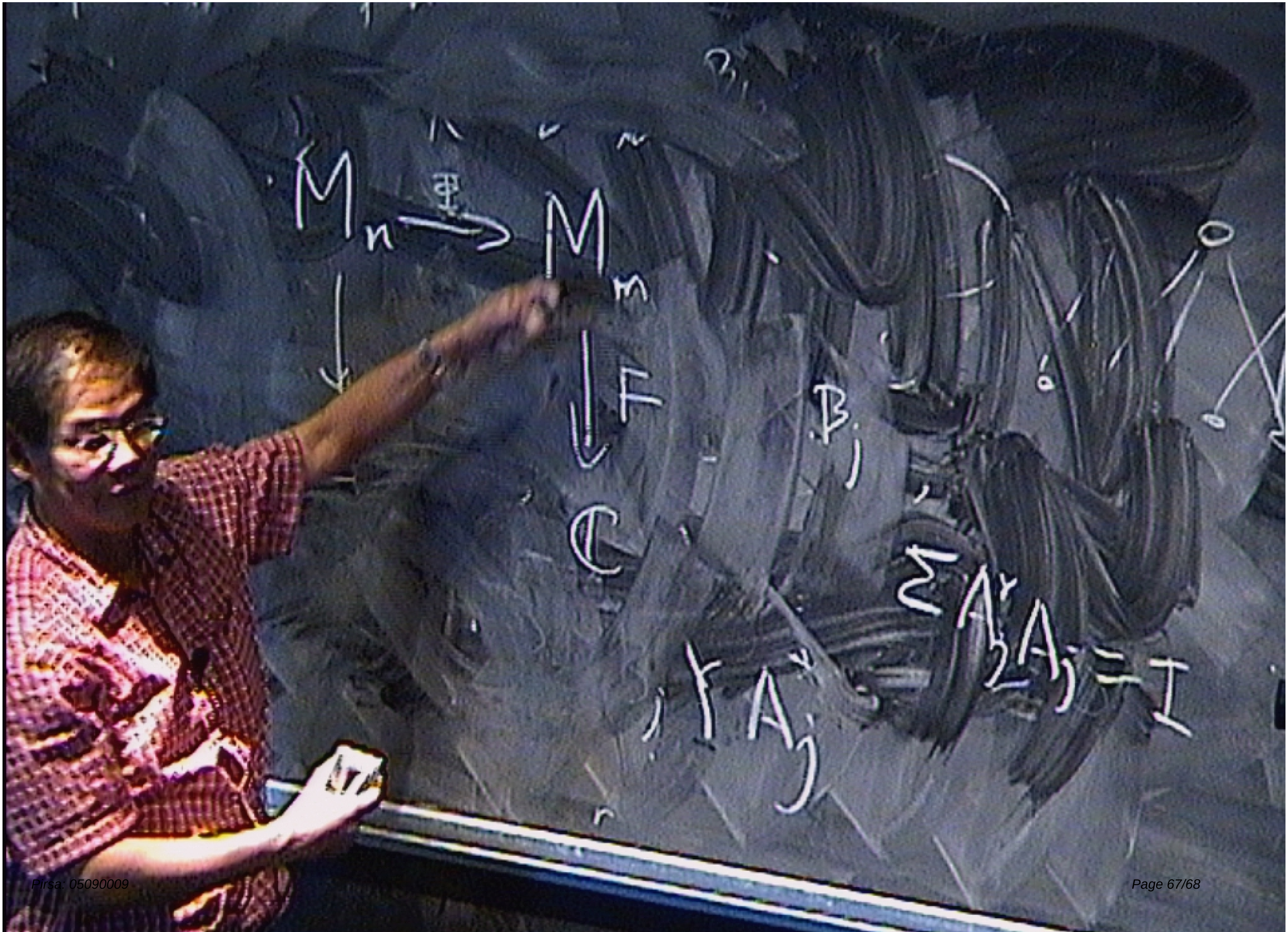

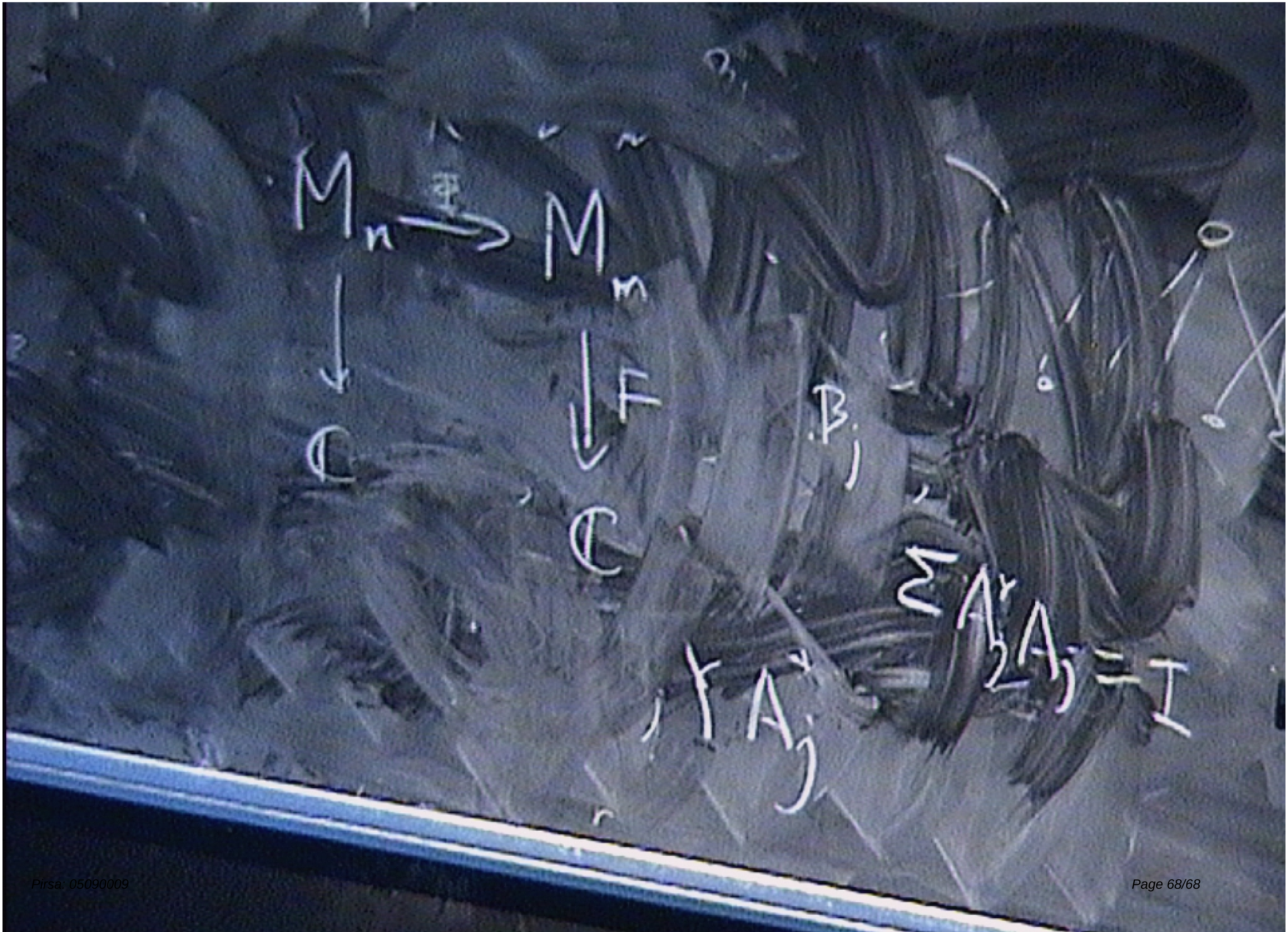
M_n
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M_m
F
ℂ
ℂ
B_j
Σ A_j* A_j = I
{A_j}