Title: Noncommtuative geometry and the origin of time

Date: Sep 14, 2005 02:00 PM

URL: http://pirsa.org/05090005

Abstract: Noncommutative geometry is a more general formulation of geometry that does not require coordinates to commute. As such it unifies quantum theory and geometry and should appear in any effective theory of quantum gravity. In this general talk we present quantum groups as a microcosm of this unification in the same way that Lie groups are a microcosm of usual geometry, and give a flavour of some of the deeper insights they provide. One of them is the ability to interchange the roles of quantum theory and gravity by `arrow reversal'. Another is that noncommutative spaces typically carry a canonical 1-parameter evolution or intrinsic time created from the fundamental conflict between noncommuting coordinates and differential calculus. In physical terms one could say that quantising space typically has an anomaly for the spatial translation group and this forces the system to evolve. We give an example where we derive Schroedinger's equation in this way.

Speaker	Title	Date/Time
Toby Wiseman	n/a	2005-12-06 11:00 AM
Evelyn Fox Keller	n/a	2005-11-04 2:00 PM
Thomas Levi	n/a	2005-11-01 11:00 AM
Charles Bennett	tba	2005-10-26 2:00 PM
Sean Hartnoll	n/a	2005-10-11 11:00 AM
Jürgen Kurths	Synchronization: A universal concept in nonlinear sciences	2005-09-28 2:00 PM
Michael Martin Nieto	The Pioneer Anomaly: The data, it's meaning, and a future test	2005-09-27 2:00 PM
Mark Newman	Epidemics, Erdos numbers, and the Internet: The structure and function of networks	2005-09-21 2:00 PM
Rouzbeh Allahverdi	tba	2005-09-20 11:00 AM
Shahn Majid	Noncommtuative geometry and the origin of time	2005-09-14 2:00 PM
David Berenstein	tba	2005-09-13 2:00 PM
Sebastian Franco	tba	2005-09-13 11:00 AM
Rafael Sorkin	Is a past-finite causal order the inner basis of spacetime?	2005-09-07 2:00 PM
David Deutsch	Qubit Field Theory	2005-08-24 4:00 PM
Jens Eisert	Entanglement in many-body systems	2005-08-10 4:00 PM

Noncommutative geometry and the origin of time

1. Hopf algebra axioms (invariant under arrow reversal)

$$H \otimes H \xrightarrow{\mathrm{m}} H \qquad H \xrightarrow{\Delta} H \otimes H$$

$$\mathbb{C} \xrightarrow{1} H \qquad H \xrightarrow{S} H \qquad H \xrightarrow{\epsilon} \mathbb{C}$$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$m(S \otimes \mathrm{id})\Delta = 1\epsilon = m(\mathrm{id} \otimes S)\Delta$$

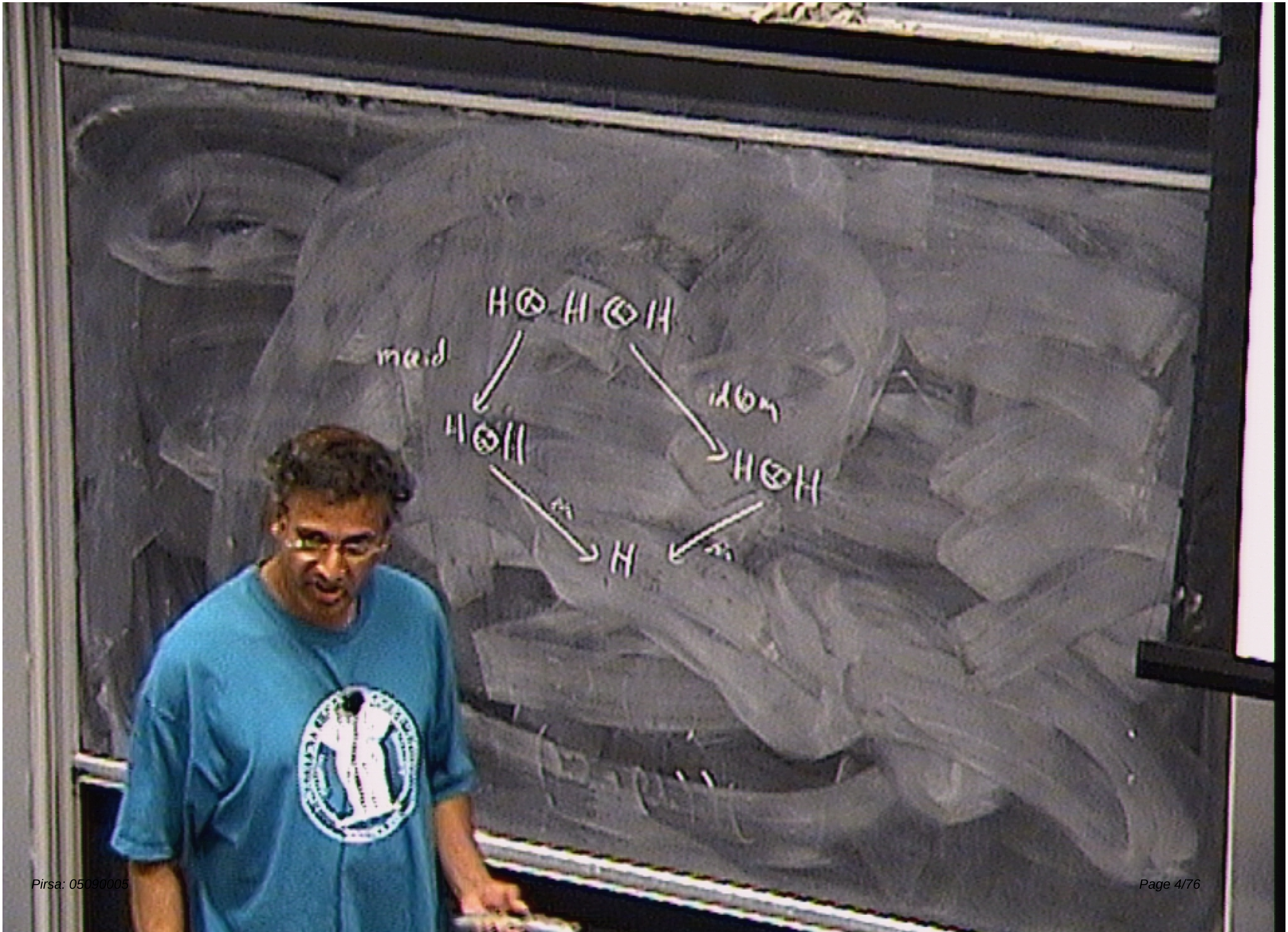
$\Rightarrow$ (dual Hopf algebra) $\quad H \leftrightarrow H^*$

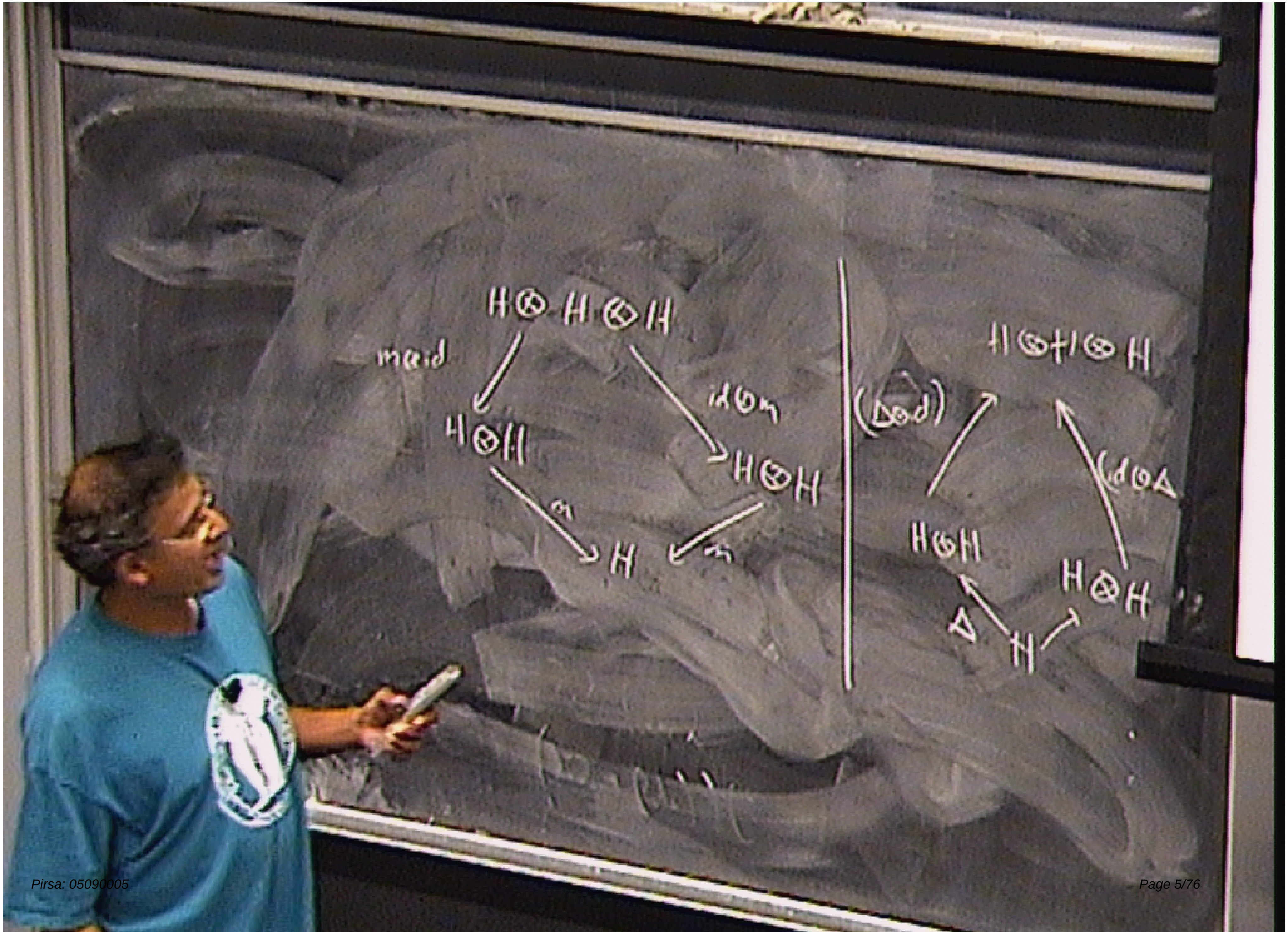
\`prove one get one free property'

$$S(ab) = S(b)S(a) \qquad \Delta S = \mathrm{flip}(S \otimes S)\Delta$$

\`for every phenomenon a cophenomenon'

action observable coaction state




H⊗H⊗H
m⊗id
id⊗m
H⊗H
H⊗H
m
m
H
(Δ⊗id)
(id⊗Δ)
H⊗H⊗H
H⊗H
H⊗H
Δ
H

Noncommutative geometry and the origin of time

1. Hopf algebra axioms (invariant under arrow reversal)

$$H \otimes H \xrightarrow{m} H \qquad\qquad H \xrightarrow{\Delta} H \otimes H$$

$$\mathbb{C} \xrightarrow{1} H \qquad H \xrightarrow{S} H \qquad H \xrightarrow{\epsilon} \mathbb{C}$$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$m(S \otimes \mathrm{id})\Delta = 1\epsilon = m(\mathrm{id} \otimes S)\Delta$$

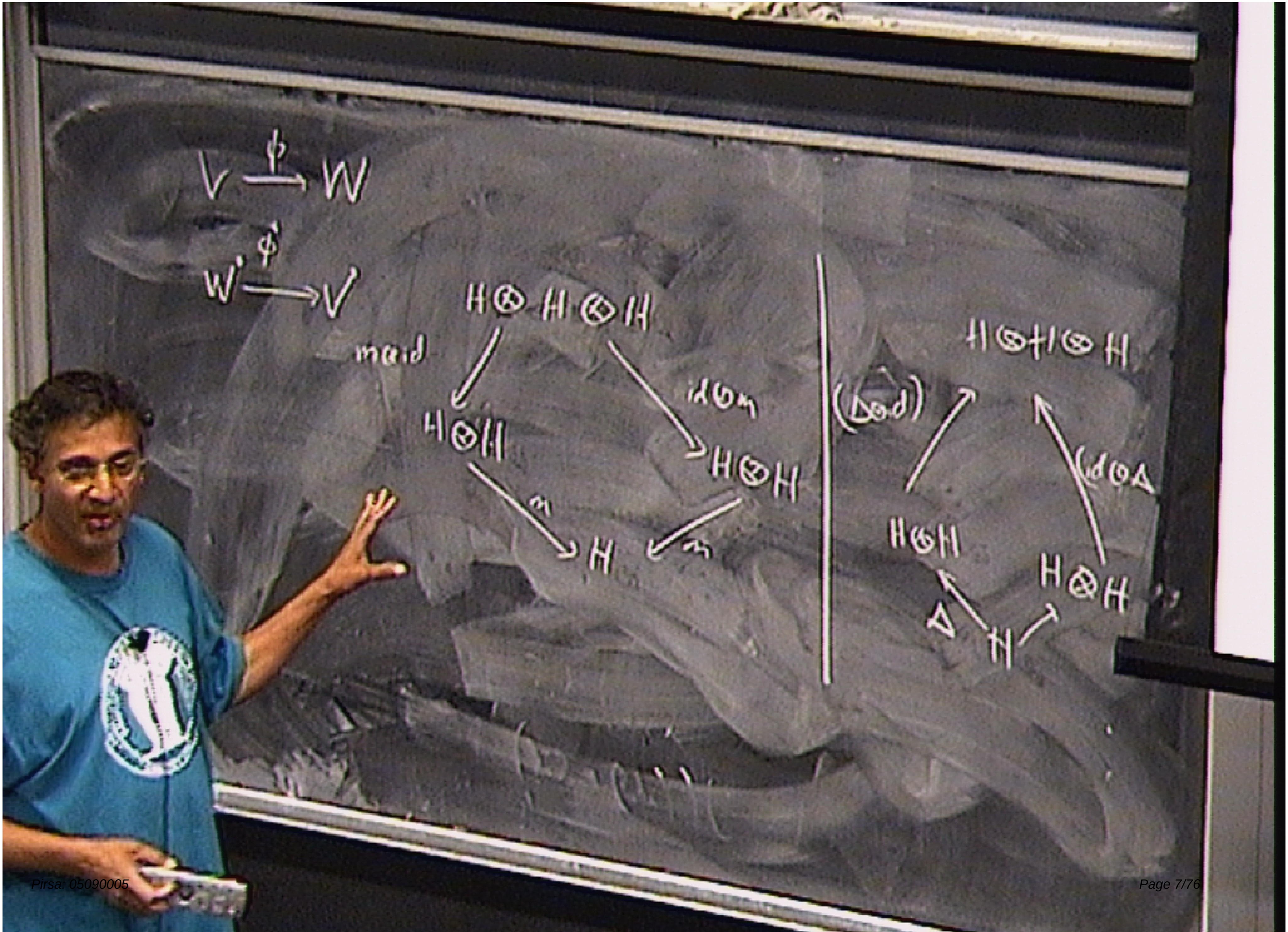
$$\Rightarrow \text{(dual Hopf algebra)} \quad H \leftrightarrow H^*$$

'prove one get one free property'

$$S(ab) = S(b)S(a) \qquad \Delta S = \mathrm{flip}(S \otimes S)\Delta$$

'for every phenomenon a cophenomenon'

action observable coaction state


V → W
φ
W* → V*
φ*
H⊗H⊗H
m⊗id
id⊗m
H⊗H
H⊗H
m
m
H
(Δ⊗id)
H⊗H⊗H
id⊗Δ
H⊗H
H⊗H
Δ
H

Noncommutative geometry and the origin of time

1. Hopf algebra axioms (invariant under arrow reversal)

$$H \otimes H \xrightarrow{\mathsf{m}} H \qquad\qquad H \xrightarrow{\Delta} H \otimes H$$

$$\mathbb{C} \xrightarrow{1} H \qquad H \xrightarrow{S} H \qquad H \xrightarrow{\epsilon} \mathbb{C}$$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$m(S \otimes \mathrm{id})\Delta = 1\epsilon = m(\mathrm{id} \otimes S)\Delta$$

$$\Rightarrow \text{(dual Hopf algebra)} \quad H \leftrightarrow H^*$$

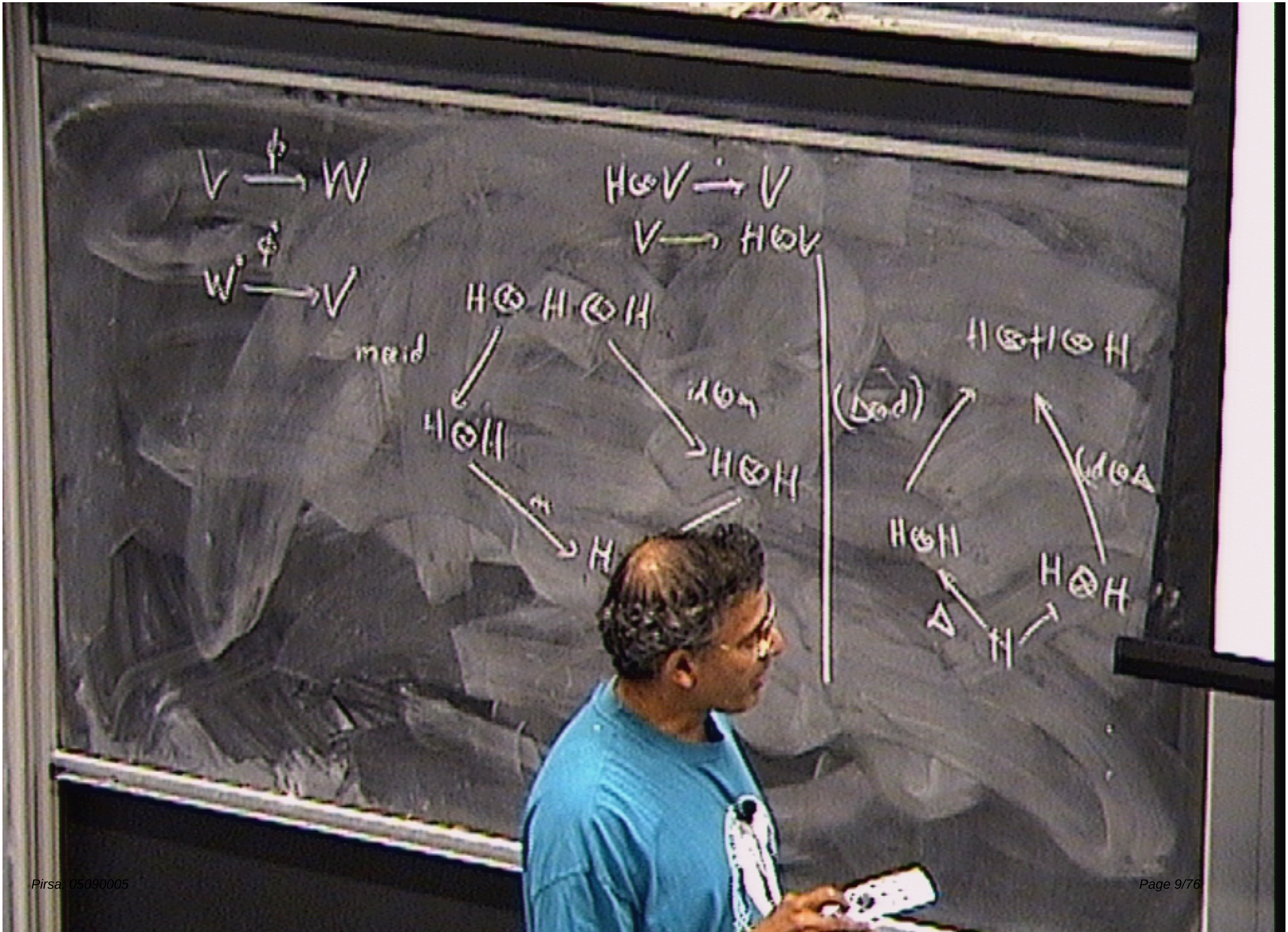
'prove one get one free property'

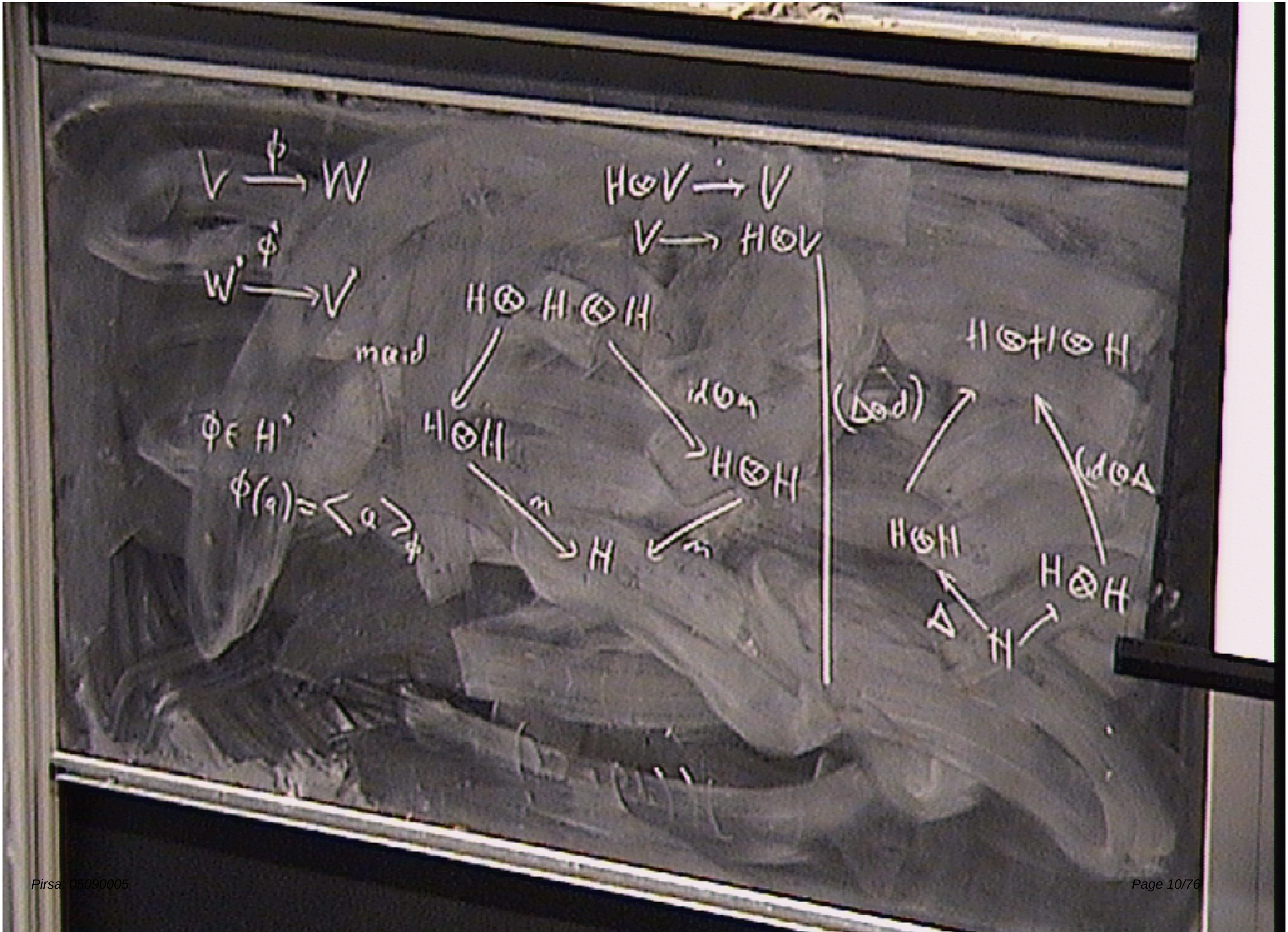
$$S(ab) = S(b)S(a) \qquad \Delta S = \mathrm{flip}(S \otimes S)\Delta$$

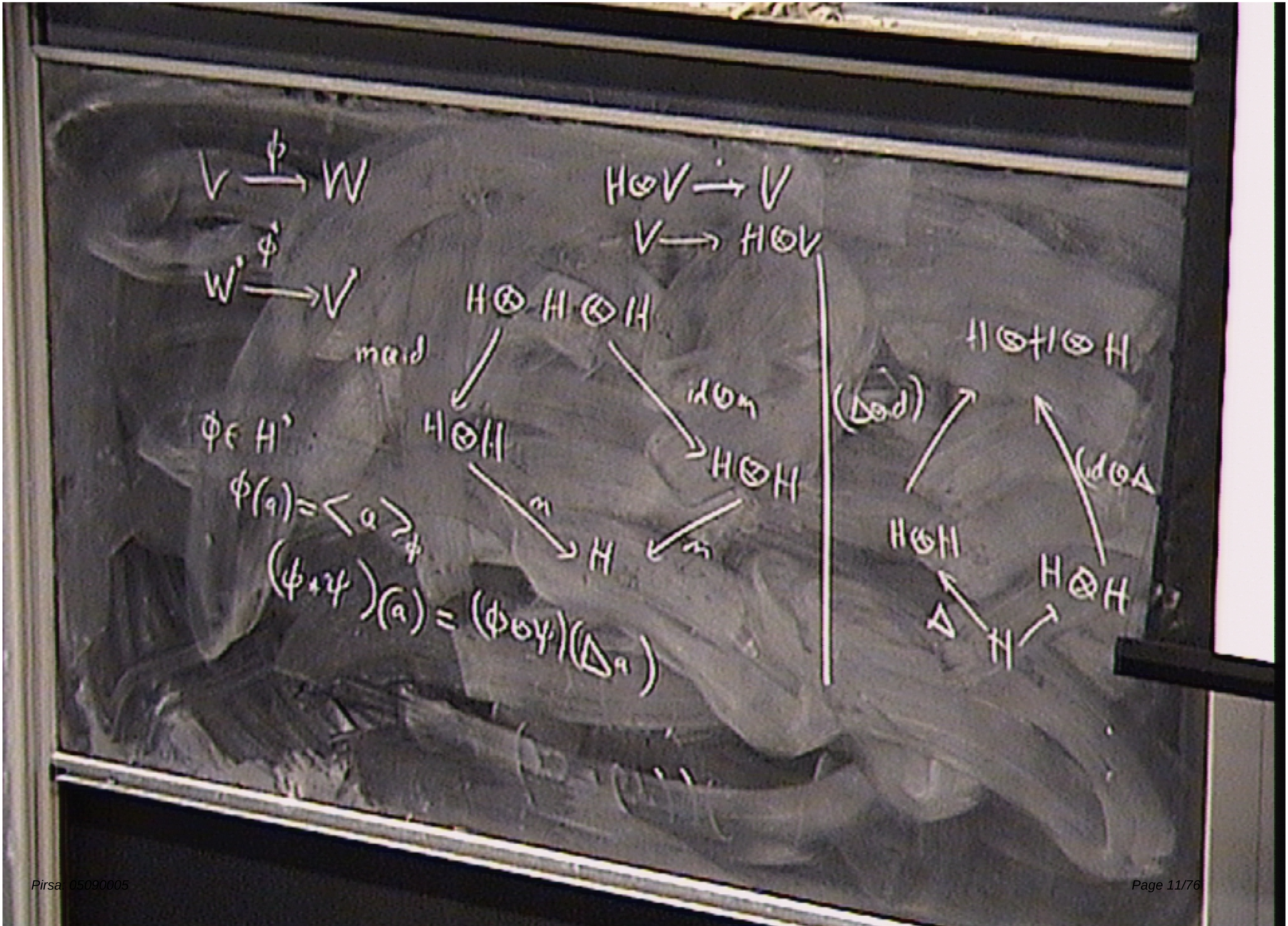
'for every phenomenon a cophenomenon'

action observable coaction state

ntegral coentropy cointegral entropy


V → W
H⊗V → V
V → H⊗V
H⊗H⊗H
m⊗id
H⊗H
id⊗m
H⊗H
(Δ⊗id)
H⊗H⊗H
id⊗Δ
H⊗H
H⊗H
Δ
H


V →φ W
W* →φ* V*
φ ∈ H*
φ(a) = ⟨a⟩φ
H⊗H⊗H
m⊗id
id⊗m
H⊗H
H⊗H
m
m
H
H⊗V → V
V → H⊗V
(Δ⊗id)
H⊗H⊗H
(id⊗Δ
H⊗H
H⊗H
Δ
H


V → W (φ)
W' → V' (φ')
φ ∈ H'
φ(a) = ⟨a⟩_φ
(φ * ψ)(a) = (φ ⊗ ψ)(Δa)
H⊗V → V
V → H⊗V
H⊗H⊗H
m ⊗ id
id ⊗ m
H⊗H
H⊗H
m
m
H
(Δ⊗id)
(id⊗Δ)
H⊗H⊗H
H⊗H
H⊗H
Δ
H

Noncommutative geometry and the origin of time

1. Hopf algebra axioms (invariant under arrow reversal)

$$H \otimes H \xrightarrow{m} H \qquad\qquad H \xrightarrow{\Delta} H \otimes H$$

$$\mathbb{C} \xrightarrow{1} H \qquad H \xrightarrow{S} H \qquad H \xrightarrow{\epsilon} \mathbb{C}$$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$m(S \otimes \mathrm{id})\Delta = 1\epsilon = m(\mathrm{id} \otimes S)\Delta$$

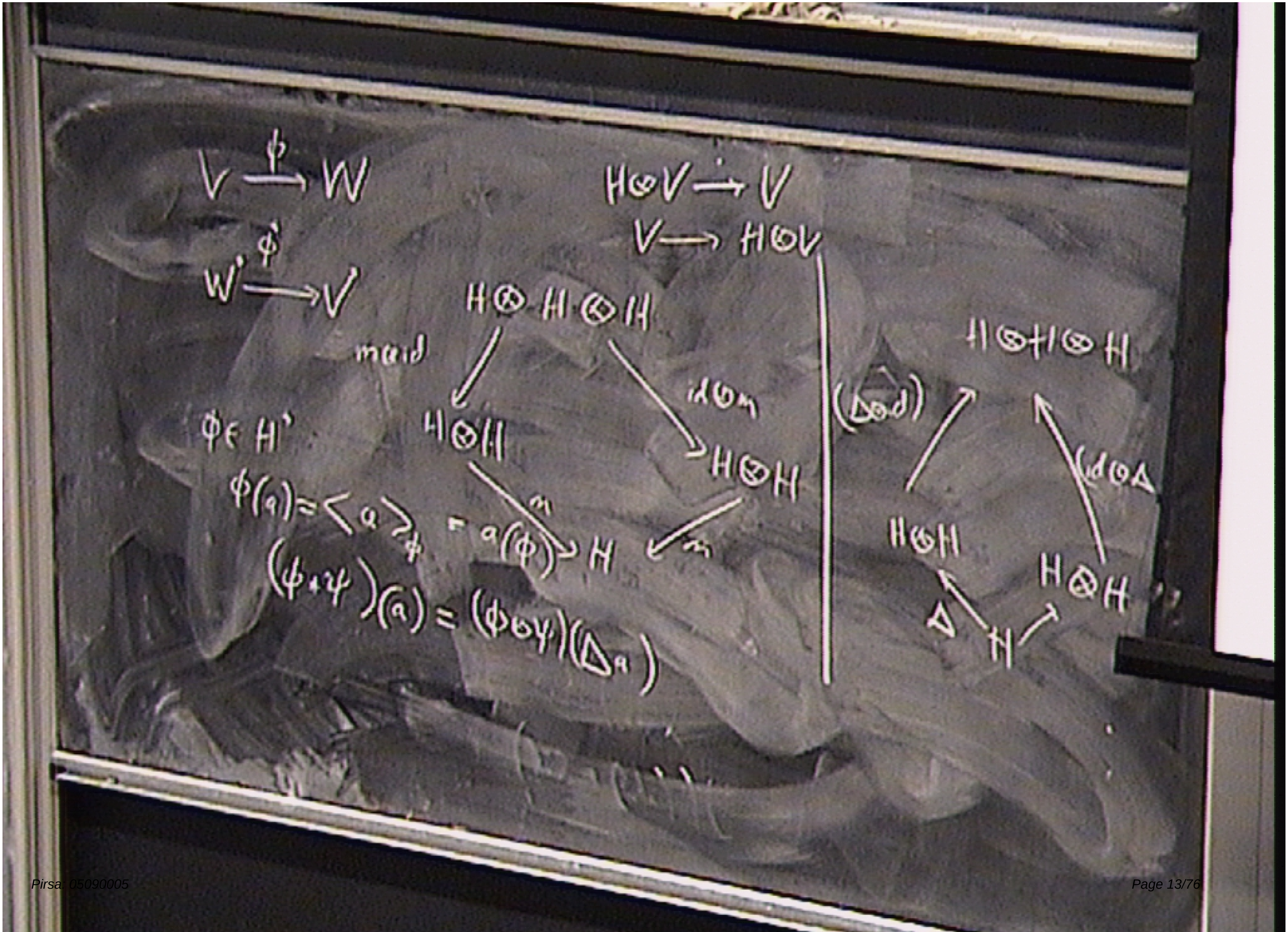
$\Rightarrow$ (dual Hopf algebra) $\quad H \leftrightarrow H^*$

`prove one get one free property'

$$S(ab) = S(b)S(a) \qquad \Delta S = \mathrm{flip}(S \otimes S)\Delta$$

`for every phenomenon a cophenomenon'

ction observable coaction state


V →φ W
W* →φ* V*
φ ∈ H*
φ(a) = ⟨a⟩φ = a(φ)
(φ∗ψ)(a) = (φ⊗ψ)(Δa)
m⊗id
id⊗m
H⊗H⊗H
H⊗H
m
H
H⊗V → V
V → H⊗V
(Δ⊗id)
(id⊗Δ)
H⊗H⊗H
Δ
H

One object, two points of view

coordinate alg

$$H = \mathbb{C}(G) = \{\delta_x | x \in G\}$$

$$\delta_x \delta_y = \delta_{x,y}\delta_x$$

$$\Delta\delta_x = \sum_{yz=x} \delta_y \otimes \delta_z$$

$$S\delta_x = \delta_{x^{-1}}$$

$$\int \delta_x = 1$$

symmetry alg

$$H = \mathbb{C}G = \{x \in G\}$$

$$x.y = xy$$

$$\Delta x = x \otimes x$$

$$Sx = x^{-1}$$

$$\int x = \delta_{x,e}$$

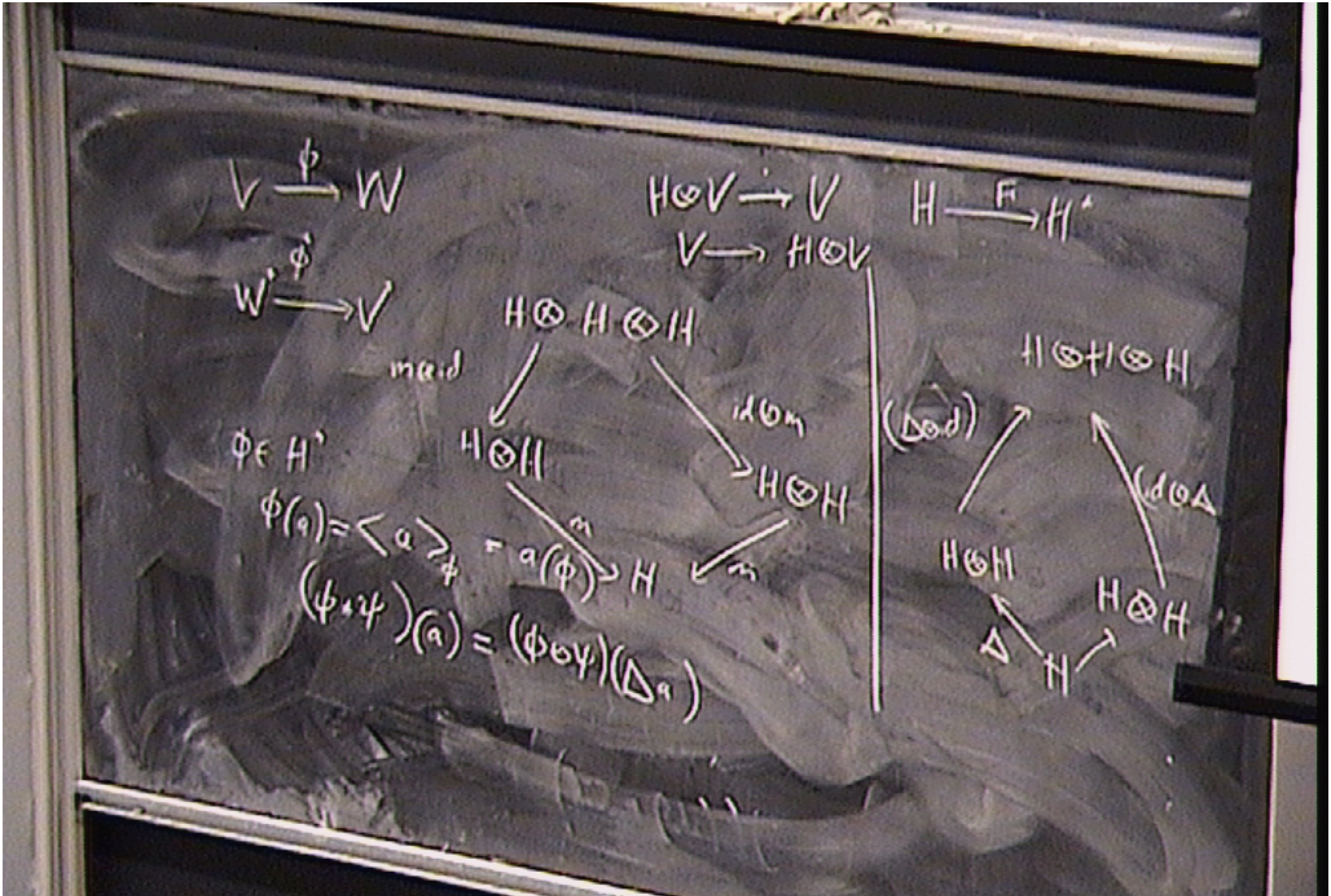
Fourier tranform

$$\mathbb{C}(G) \to \mathbb{C}G = \text{“}\mathbb{C}(\hat{G})\text{”}$$

$$\mathcal{F}(f) = (\int f \sum_x \delta_x)x = \int \mathrm{d}x f(x) x$$

curved posn | gravity | noncom mom

noncom posn | cogravity | curved mom



V —φ→ W
W* —φ*→ V*
H⊗V → V
V → H⊗V
H —F→ H*
H⊗H⊗H
m⊗id
id⊗m
H⊗H
H⊗H
m
m
H
φ∈ H*
φ(a) = ⟨a⟩φ = a(φ)
(φ∗ψ)(a) = (φ⊗ψ)(Δa)
H⊗H⊗H
(Δ⊗id)
(id⊗Δ)
H⊗H
H⊗H
Δ
H

One object, two points of view

coordinate alg

$$H = \mathbb{C}(G) = \{\delta_x | x \in G\}$$

$$\delta_x \delta_y = \delta_{x,y}\delta_x$$

$$\Delta \delta_x = \sum_{yz=x} \delta_y \otimes \delta_z$$

$$S\delta_x = \delta_{x^{-1}}$$

$$\int \delta_x = 1$$

symmetry alg

$$H = \mathbb{C}G = \{x \in G\}$$

$$x.y = xy$$

$$\Delta x = x \otimes x$$

$$Sx = x^{-1}$$

$$\int x = \delta_{x,e}$$

Fourier tranform

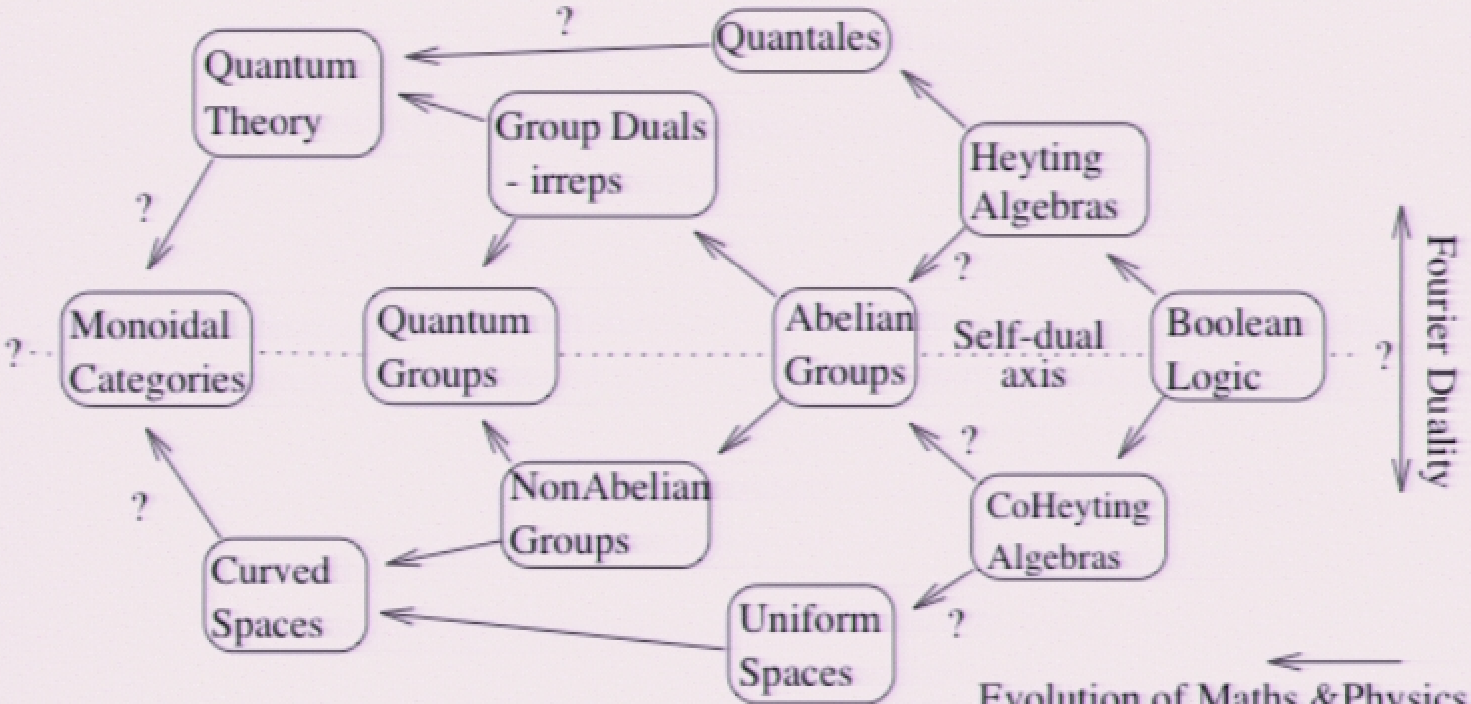
$$\mathbb{C}(G) \to \mathbb{C}G = \text{“}\mathbb{C}(\hat{G})\text{”}$$

$$\mathcal{F}(f) = (\int f \sum_x \delta_x)x = \int \mathrm{d}x f(x) x$$

curved posn | gravity | noncom mom

noncom posn | cogravity | curved mom

Category Hopf as microcosm of quantum-gravity



Theorem (SM'87)

$$0 \to \mathbb{C}(M) \rightleftarrows H \rightleftarrows U(\mathfrak{g}) \to 0 \;\Rightarrow\; H \cong \mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$$

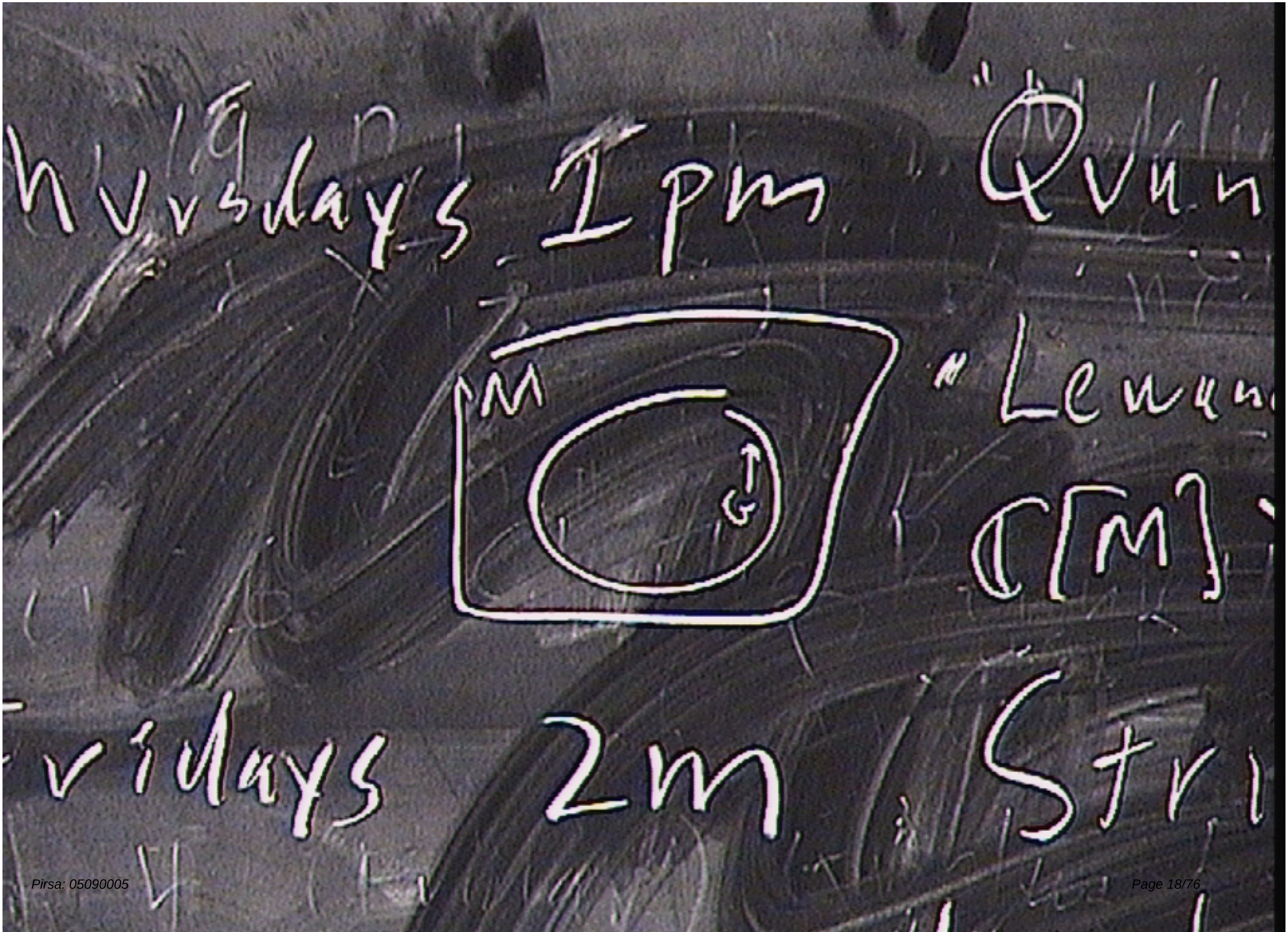
posn alg obs mom

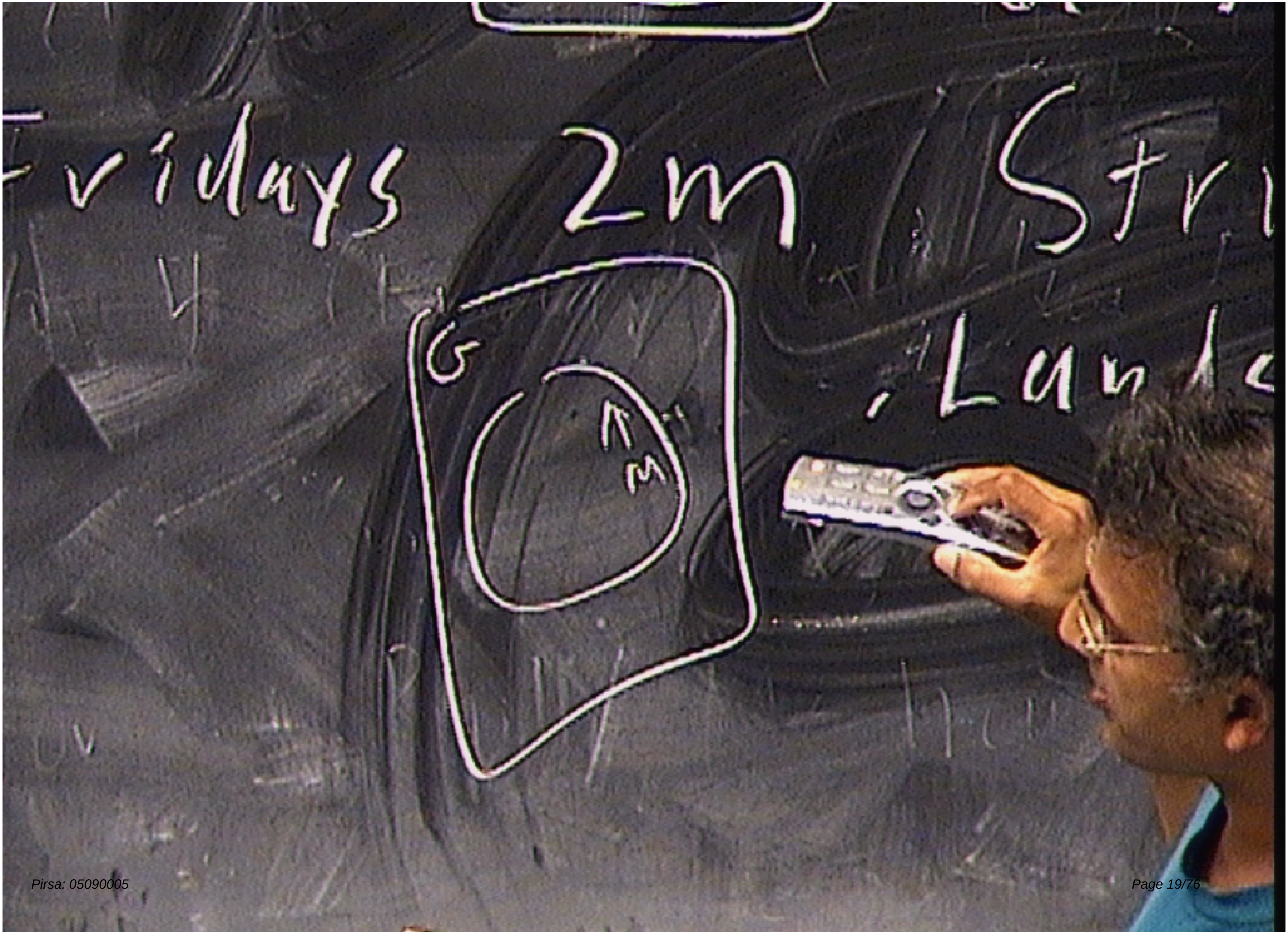
$$H^* \cong U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G)$$

$$\Leftrightarrow \; (G, M)$$ matched pair eqns `toy einstein eq'

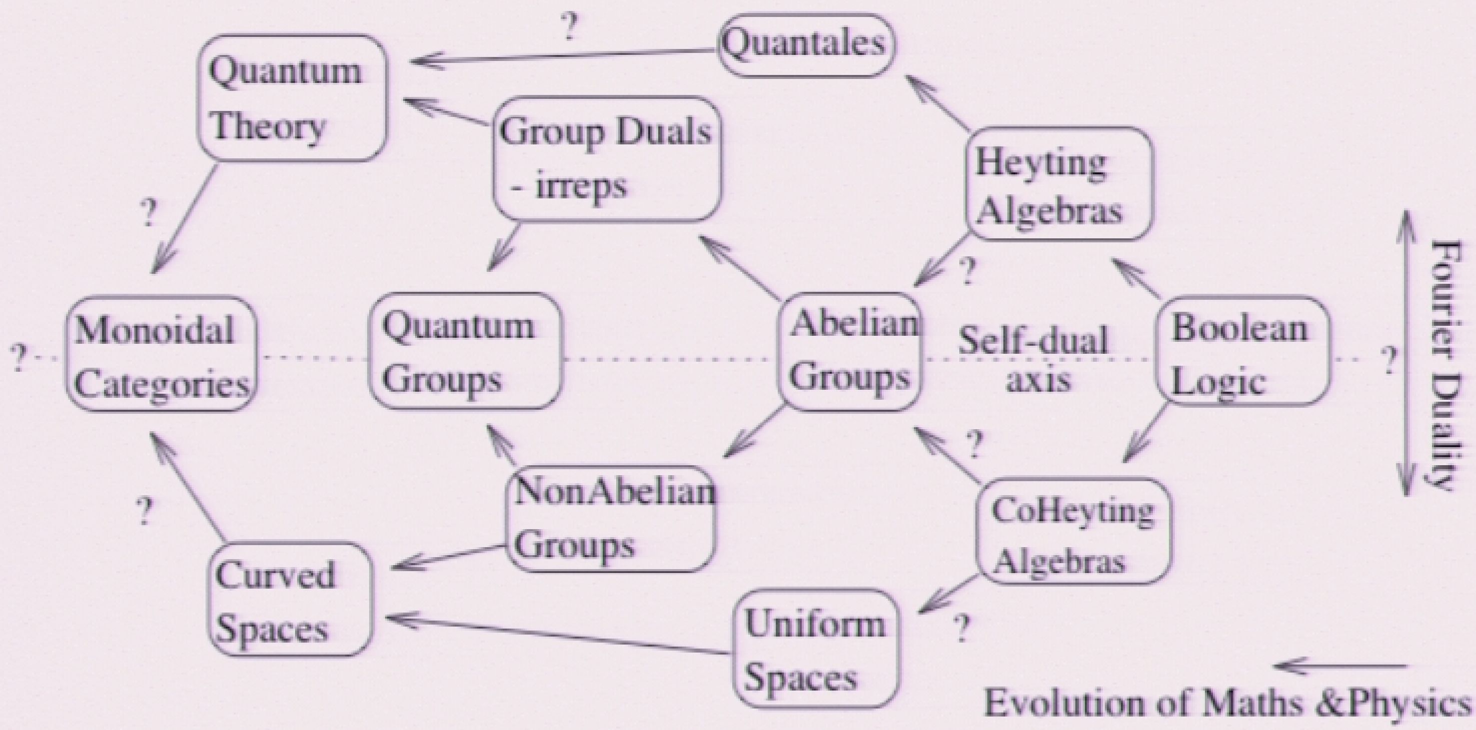
e.g. $$G = M = \mathbb{R} \;\Rightarrow\; [p, x] = \imath\hbar(1 - e^{-\gamma x})$$

$$\Delta x = x \otimes 1 + 1 \otimes x \quad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p$$


hursdays 1pm
"Quan
M
"Lemn
C[M]
ridays 2m
Stri


2m
G
M

Category Hopf as microcosm of quantum-gravity



Theorem (SM'87)

$$0 \to \mathbb{C}(M) \rightleftarrows H \rightleftarrows U(\mathfrak{g}) \to 0 \;\Rightarrow\; H \cong \mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$$

posn alg obs mom

$$H^* \cong U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G)$$

$$\Leftrightarrow \; (G, M)$$ matched pair eqns `toy einstein eq'

e.g. $G = M = \mathbb{R} \;\Rightarrow\; [p, x] = \imath\hbar(1 - e^{-\gamma x})$

$\Delta x = x \otimes 1 + 1 \otimes x \qquad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p$

ummary of Hopf paradigm

lg of obs is Hopf $\Leftrightarrow$ $\phi(x) = x(\phi)$ obs-state duality

nding one of self-dual type = QM+toy gravity (both modified in the process)

The next self-dual paradigm? $\mathrm{Mon}_* = \{F : \mathcal{C} \to \mathcal{V}\}$ `category of monoidal functors'

$(F, \mathcal{C}, \mathcal{V})^\circ = F^\circ : \mathcal{C}^\circ \to \mathcal{V}$ `dual'(SM 1989)

.g. (Fredenhagen) cov QFT is a monoidal functor

F:{Globally hyp manifolds} $\longrightarrow$ {C*-algebras}

QUANTUM-GRAV=finding self-dual object in Mon??

(modify both sides of cov QFT)

. Noncommutative geometry of an algebra A

quantum groups approach: these should be `differentiable')

Differential Structure

Ω^1 a((db)c)=(a(db))c (A-A bimodule)

$\mathrm{d} : A \to \Omega^1$ d(ab)=(da)b+a(db) (Leibniz rule)

$adb\} = \Omega^1$ $\ker \mathrm{d} = \mathbb{C}.1$ (connectedness)

Hopf case - ask it to be translation invariant

$\Rightarrow$ $\Omega = \oplus\Omega^n, \ \mathrm{d}^2 = 0$ ... Hodge * ... metric ... bundles ...

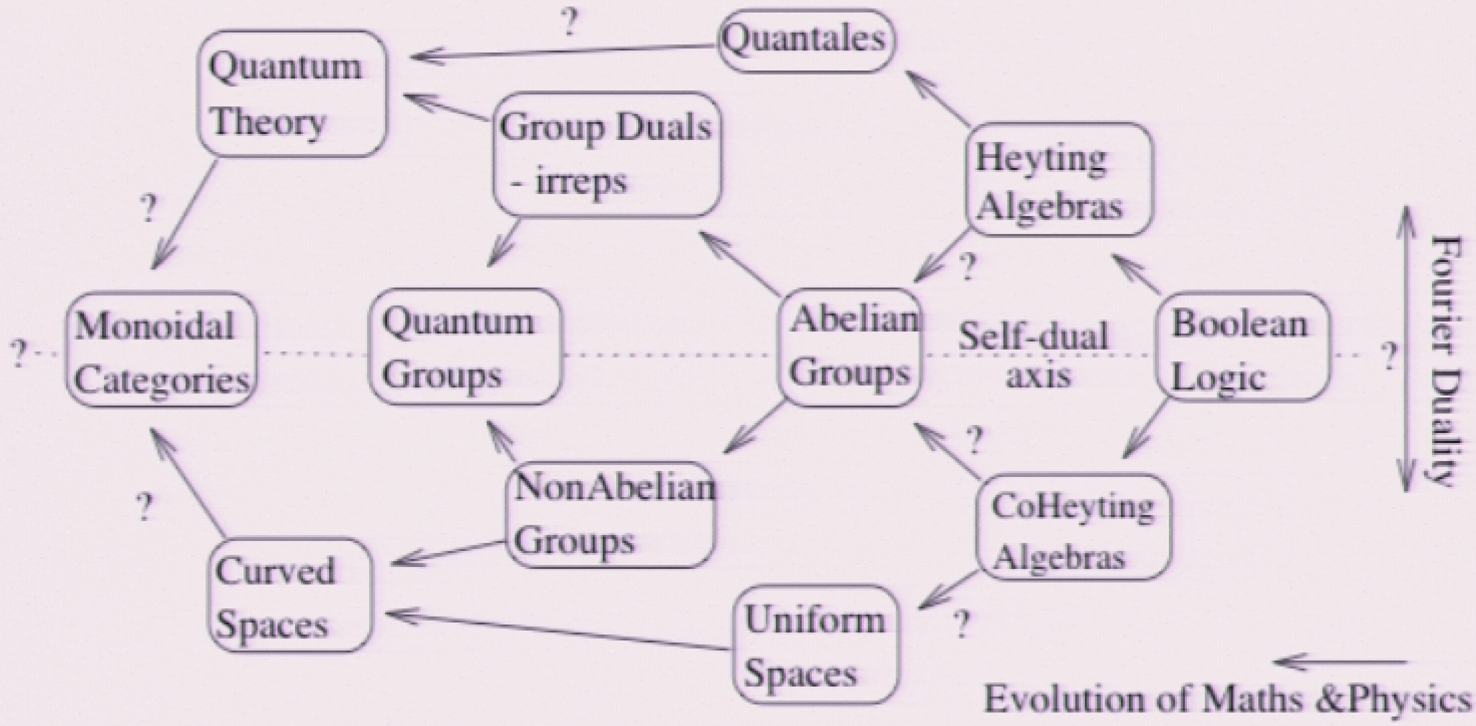
heorem: for noncom deformation $\Omega^1(A_\lambda)$

$$\exists\theta \in \Omega^1 \text{such that } [\theta, a] = \lambda \mathrm{d}a$$

ut need not have any classical analogue

$\mathrm{d}x_i \to \mathrm{d}x_i$ $? \to \theta$

Category Hopf as microcosm of quantum-gravity



Theorem (SM'87)

$$0 \to \mathbb{C}(M) \rightleftarrows H \rightleftarrows U(\mathfrak{g}) \to 0 \;\Rightarrow\; H \cong \mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$$

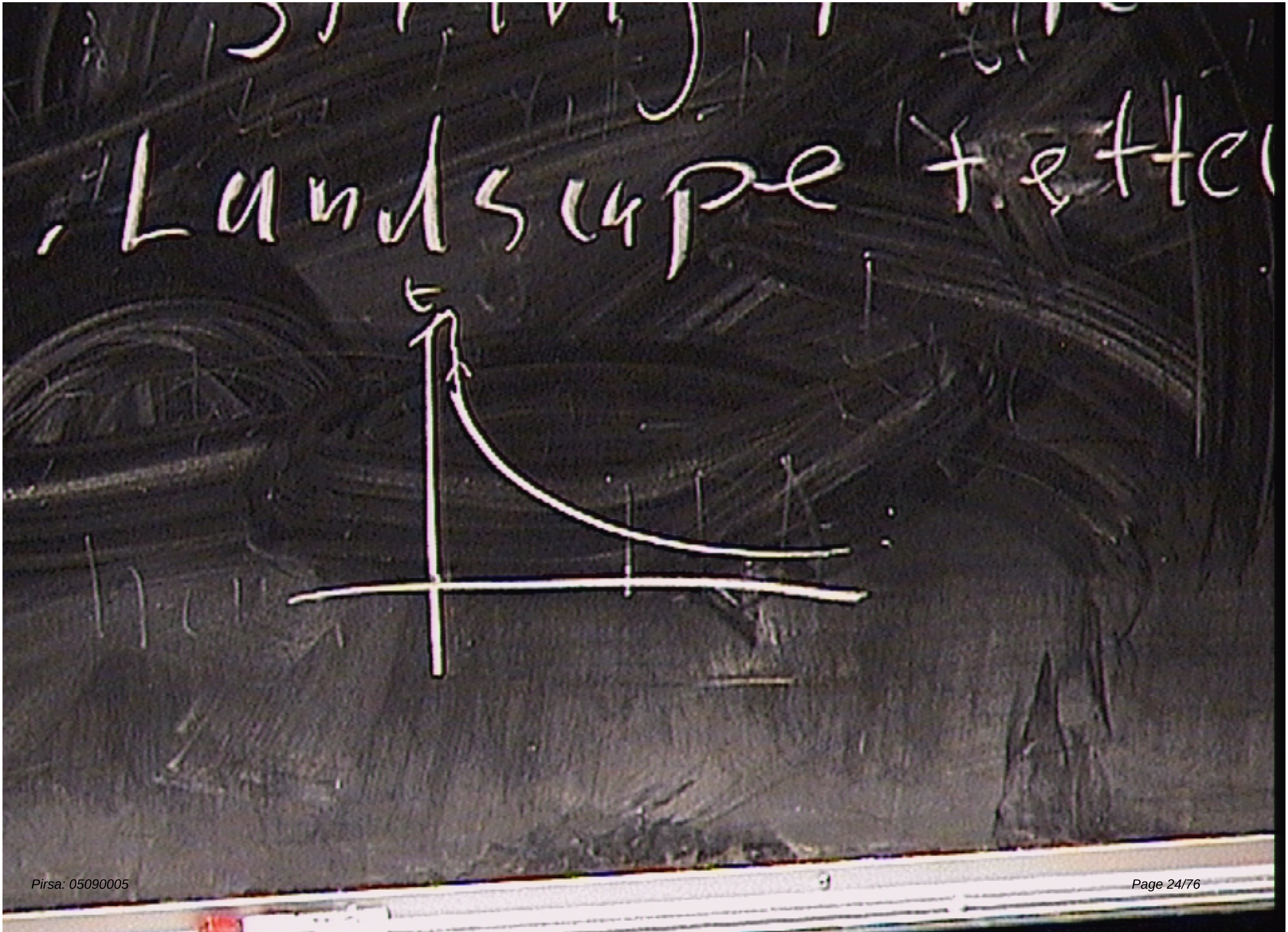
posn alg obs mom

$$H^* \cong U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G)$$

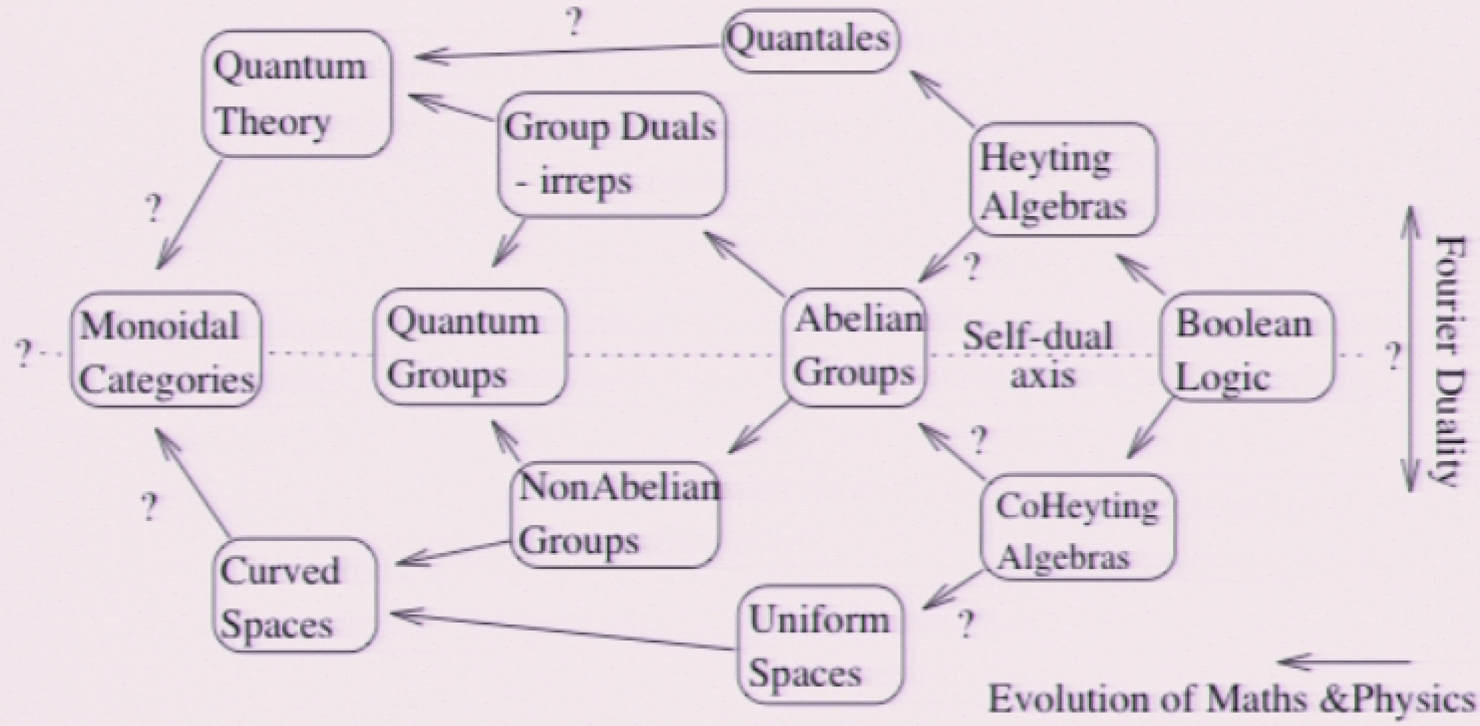
$$\Leftrightarrow \; (G, M)$$ matched pair eqns `toy einstein eq'

e.g. $G = M = \mathbb{R} \;\Rightarrow\; [p, x] = \imath\hbar(1 - e^{-\gamma x})$

$\Delta x = x \otimes 1 + 1 \otimes x \qquad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p$


Landscape

Category Hopf as microcosm of quantum-gravity



Theorem (SM'87)

$$0 \to \mathbb{C}(M) \rightleftarrows H \rightleftarrows U(\mathfrak{g}) \to 0 \;\Rightarrow\; H \cong \mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$$

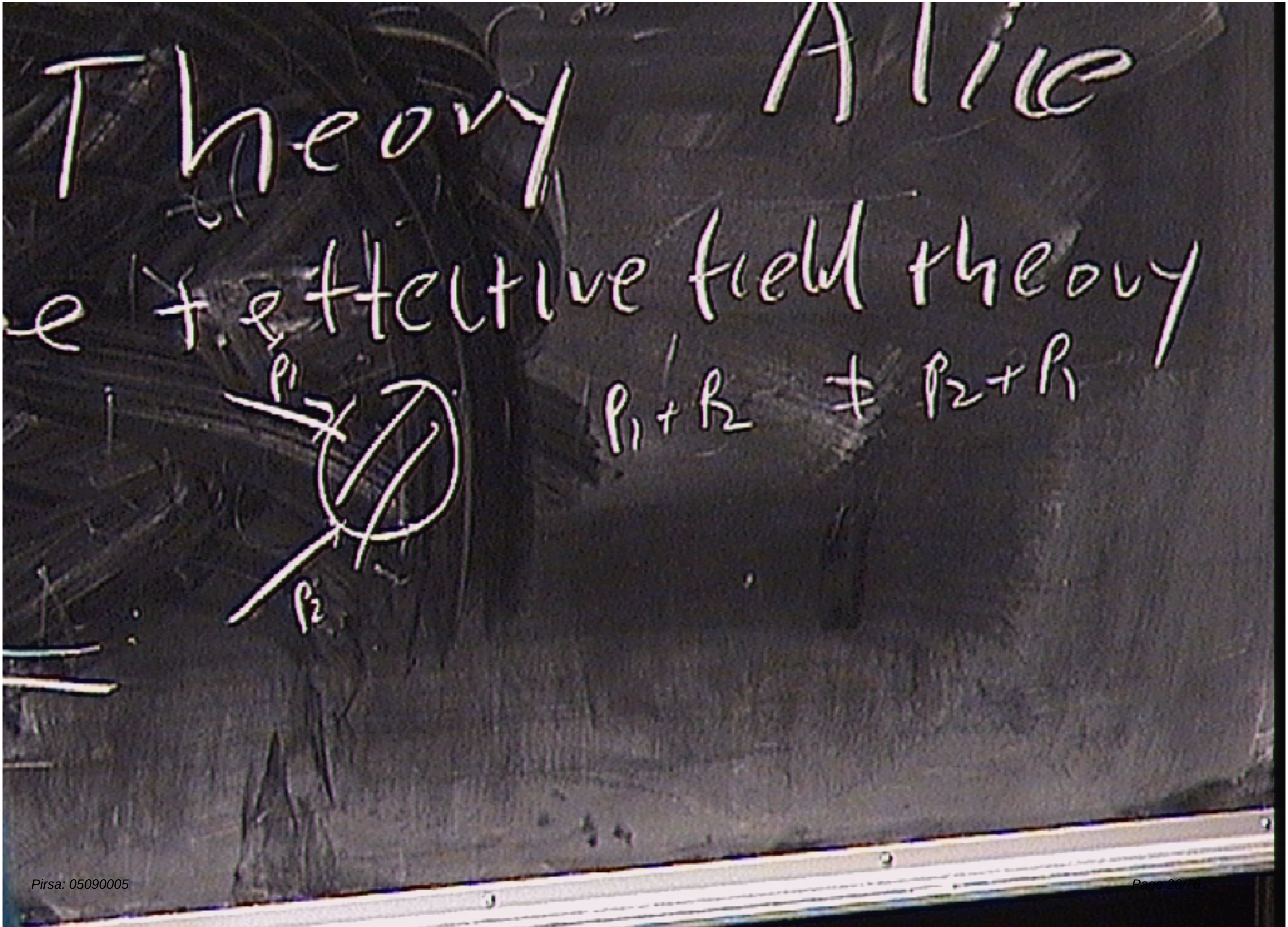
posn   alg obs   mom

$$H^* \cong U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G)$$

$$\Leftrightarrow \; (G, M)$$ matched pair eqns `toy einstein eq'

e.g. $G = M = \mathbb{R} \;\Rightarrow\; [p, x] = \imath\hbar(1 - e^{-\gamma x})$

$\Delta x = x \otimes 1 + 1 \otimes x \qquad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p$


Theory Alice
effective field theory
p_1
p_2
$p_1+p_2 \neq p_2+p_1$

ummary of Hopf paradigm

lg of obs is Hopf $\Leftrightarrow$ $\phi(x) = x(\phi)$ obs-state duality

nding one of self-dual type = QM+toy gravity (both modified in the process)

The next self-dual paradigm? $\mathrm{Mon}_* = \{F : \mathcal{C} \to \mathcal{V}\}$ `category of monoidal functors'

$(F, \mathcal{C}, \mathcal{V})^\circ = F^\circ : \mathcal{C}^\circ \to \mathcal{V}$ `dual'(SM 1989)

.g. (Fredenhagen) cov QFT is a monoidal functor

F:{Globally hyp manifolds} $\longrightarrow$ {C*-algebras}

QUANTUM-GRAV=finding self-dual object in Mon??
(modify both sides of cov QFT)

One object, two points of view

coordinate alg

$$H = \mathbb{C}(G) = \{\delta_x | x \in G\}$$

$$\delta_x \delta_y = \delta_{x,y} \delta_x$$

$$\Delta \delta_x = \sum_{yz=x} \delta_y \otimes \delta_z$$

$$S\delta_x = \delta_{x^{-1}}$$

$$\int \delta_x = 1$$

symmetry alg

$$H = \mathbb{C}G = \{x \in G\}$$

$$x.y = xy$$

$$\Delta x = x \otimes x$$

$$Sx = x^{-1}$$

$$\int x = \delta_{x,e}$$

Fourier tranform

$$\mathbb{C}(G) \to \mathbb{C}G = \text{“}\mathbb{C}(\hat{G})\text{”}$$

$$\mathcal{F}(f) = (\int f \sum_x \delta_x)x = \int \mathrm{d}x f(x) x$$

curved posn gravity noncom mom

noncom posn cogravity curved mom

. Noncommutative geometry of an algebra A

quantum groups approach: these should be `differentiable')

Differential Structure

Ω^1 a((db)c)=(a(db))c (A-A bimodule)

$\mathrm{d} : A \to \Omega^1$ d(ab)=(da)b+a(db) (Leibniz rule)

$\{a\mathrm{d}b\} = \Omega^1 \qquad \ker \mathrm{d} = \mathbb{C}.1$ (connectedness)

Hopf case - ask it to be translation invariant

$\Rightarrow \quad \Omega = \oplus\Omega^n,\ \mathrm{d}^2 = 0$ ... Hodge * ... metric ... bundles ...

heorem: for noncom deformation $\Omega^1(A_\lambda)$

$$\exists\theta \in \Omega^1 \text{such that } [\theta, a] = \lambda \mathrm{d}a$$

ut need not have any classical analogue

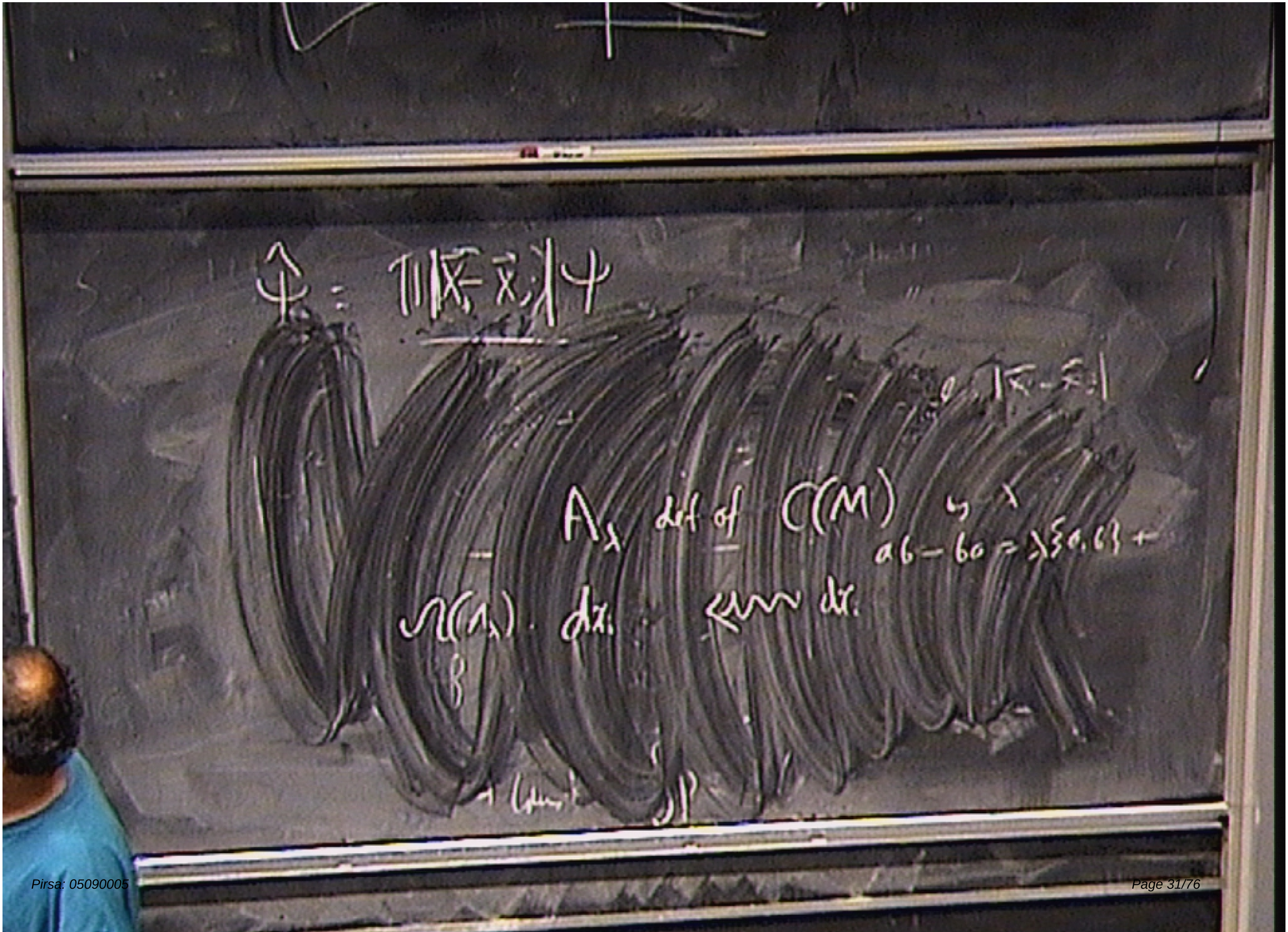
$$\mathrm{d}x_i \to \mathrm{d}x_i \qquad ? \to \theta$$

$|x_i - x_j| \varphi$

$|\bar{x}_i - \bar{x}_j|$

A_λ def of $C(M)$ by λ

$ab - ba = \lambda\{a, b\} + \dots$


A_λ def of C(M)
ab − ba = λ{a,b} +

'artial derivatives on A def. by $\mathrm{d}a = (\partial^i a)\mathrm{d}x_i + (\partial^0 a)\theta$

$h = \lim_{\lambda\to 0} \partial^0/\lambda$ is induced class. hamiltonian

Summary

Axioms of an a translation inv. diffl calculus on most known quantum groups => an extra cotangent `time' direction

(`classical translation invariance of differentiation is anomalous on quantisation')

;emiclassical analysis:

$[a, \mathrm{d}b] = \lambda\hat{\nabla}_a(\mathrm{d}b) + O(\lambda^2)$ $\hat{\nabla}$ `Poisson preconnection'

Jacobi $\leftrightarrow$ $R_{\hat{\nabla}}$

Can prove non-existence of flat transl-invariant $\hat{\nabla}$

. Noncommutative geometry of an algebra A

quantum groups approach: these should be `differentiable')

Differential Structure

Ω^1 a((db)c)=(a(db))c (A-A bimodule)

$\mathrm{d}: A \to \Omega^1$ d(ab)=(da)b+a(db) (Leibniz rule)

$\{a\mathrm{d}b\} = \Omega^1$ $\ker \mathrm{d} = \mathbb{C}.1$ (connectedness)

Hopf case - ask it to be translation invariant

$\Rightarrow \quad \Omega = \oplus\Omega^n, \ \mathrm{d}^2 = 0$ ... Hodge * ... metric ... bundles ...

heorem: for noncom deformation $\Omega^1(A_\lambda)$

$$\exists\theta \in \Omega^1 \text{such that } [\theta, a] = \lambda \mathrm{d}a$$

ut need not have any classical analogue

$$\mathrm{d}x_i \to \mathrm{d}x_i \qquad ? \to \theta$$

artial derivatives on A def. by $\mathrm{d}a = (\partial^i a)\mathrm{d}x_i + (\partial^0 a)\theta$

$h = \lim_{\lambda \to 0} \partial^0/\lambda$ is induced class. hamiltonian

Summary

Axioms of an a translation inv. diffl calculus on most known quantum groups => an extra cotangent `time' direction

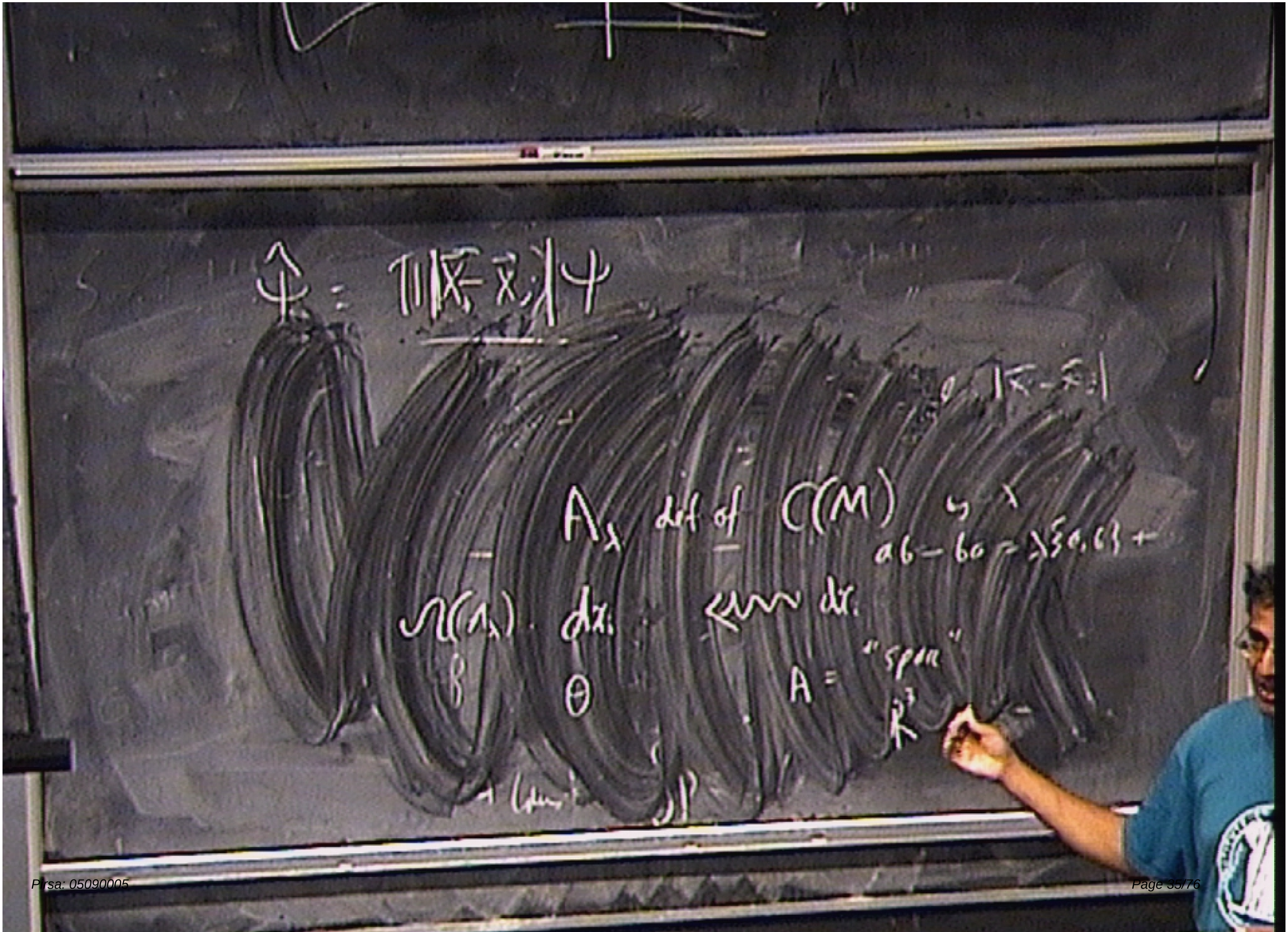
(`classical translation invariance of differentiation is anomalous on quantisation')

emiclassical analysis:

$[a, \mathrm{d}b] = \lambda \hat{\nabla}_a(\mathrm{d}b) + O(\lambda^2)$ $\hat{\nabla}$ `Poisson preconnection'

Jacobi $\leftrightarrow$ $R_{\hat{\nabla}}$

Can prove non-existence of flat transl-invariant $\hat{\nabla}$


A_λ def of C(M)
ab − ba = λ{a,b} +
"span"
A =

Example (spin space model)   2+1 QG (Schroers, Batista-SM, Freidel)

$$\mathbb{R}^3_\lambda = U(su_2): \ [x_i, x_j] = \imath 2\lambda \epsilon_{ijk} x_k$$

`coadjoint quantization'

$$\cong \oplus_j M_{2j+1}(\mathbb{C}), \ M_{2j+1} = \mathbb{R}^3_\lambda / \ x^2 = 4\lambda^2 j(j+1)$$

`fuzzy sphere'

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$$

lassify calculi by repns of su(2) - we use pauli $\Longrightarrow$

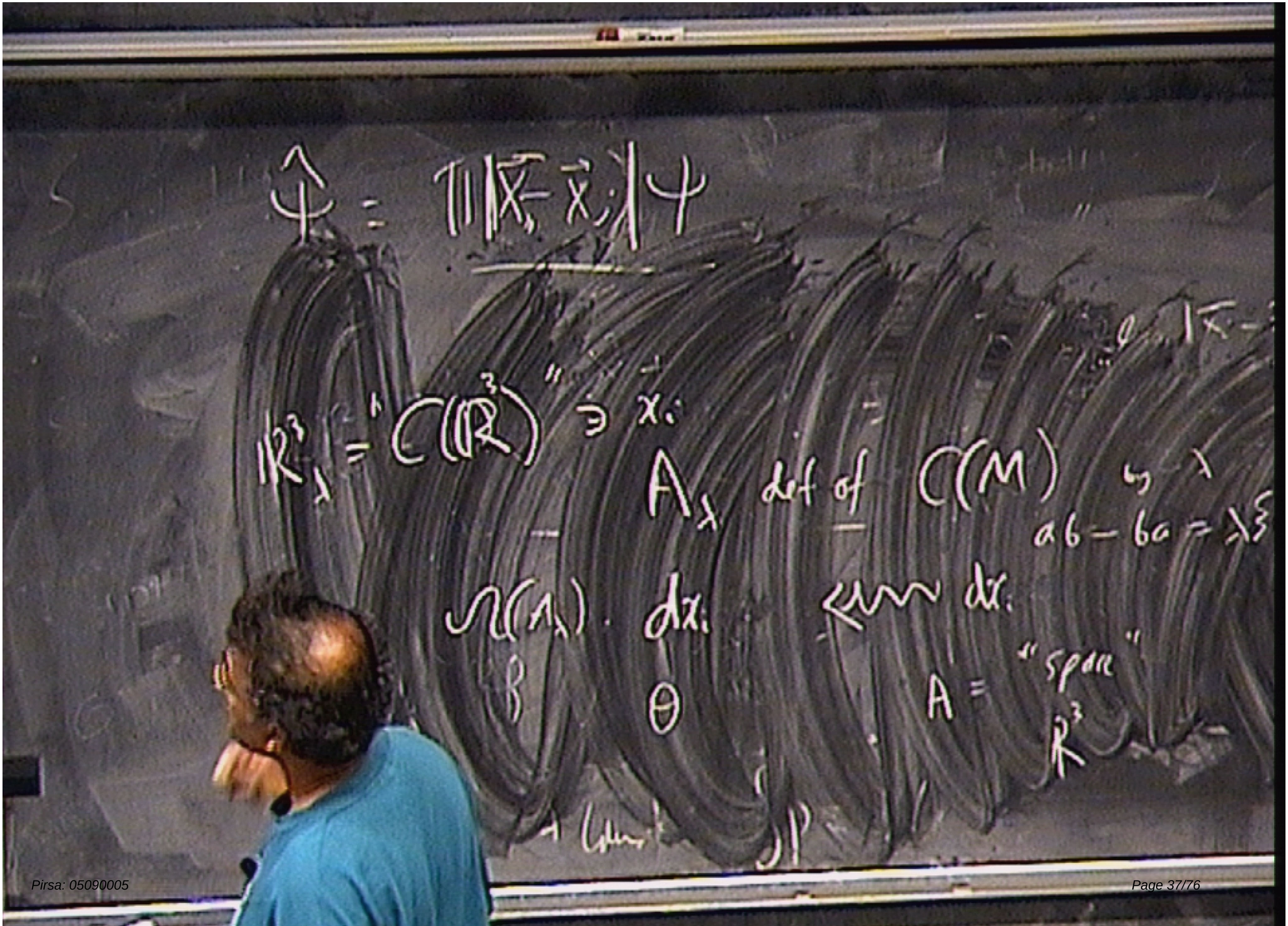
$$\Omega^1(\mathbb{R}^3_\lambda) = \mathbb{R}^3_\lambda . M_2(\mathbb{C}), \quad \mathrm{d}x_i = \lambda \sigma_i, \ \theta \propto \mathrm{id}$$

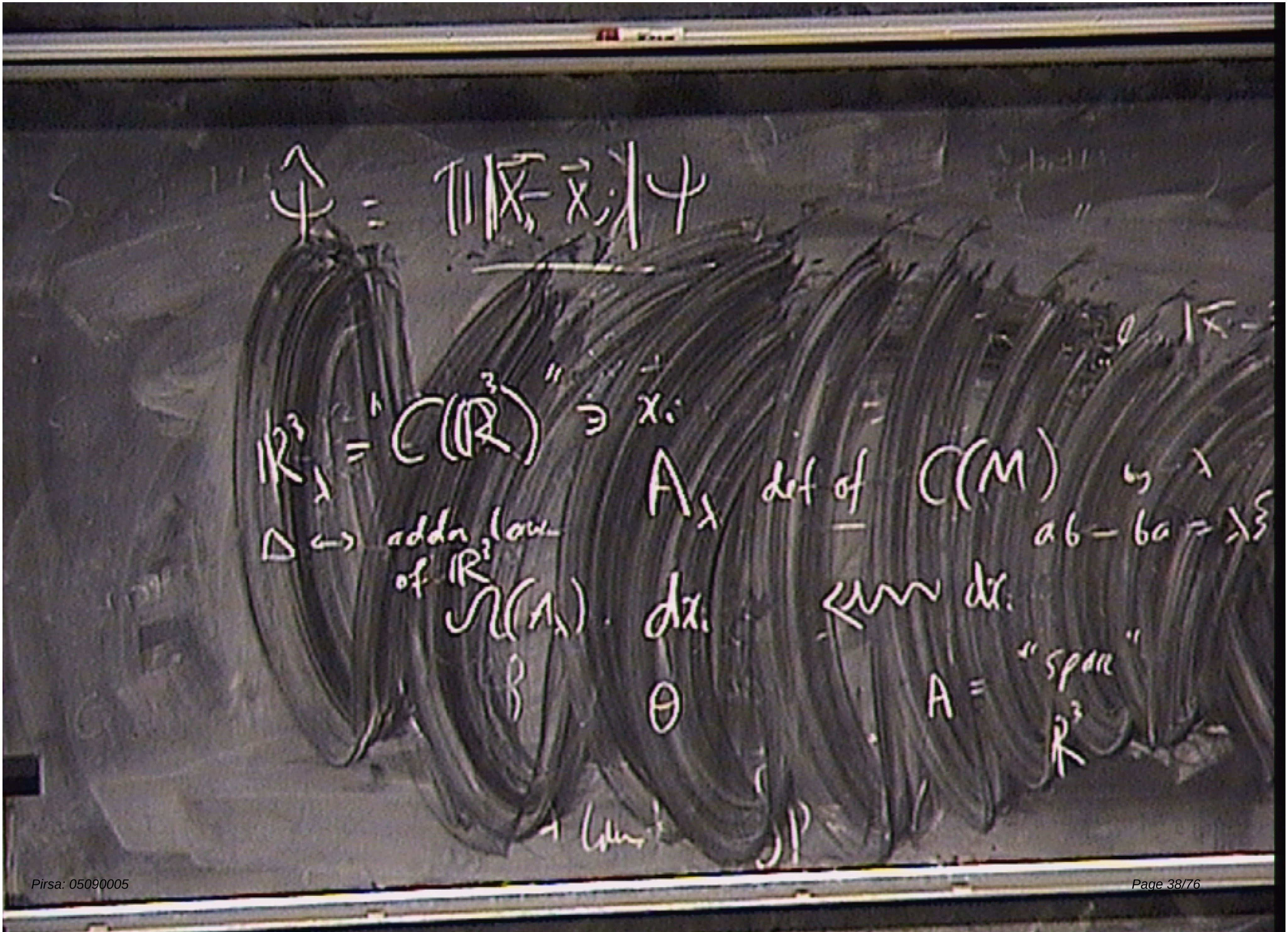
$$(\mathrm{d}x_i)x_j - x_j \mathrm{d}x_i = \imath \lambda \epsilon_{ij}{}^k \mathrm{d}x_k + \imath \mu \delta_{ij} \theta.$$

$$x_i \theta - \theta x_i = \imath \frac{\lambda^2}{\mu} \mathrm{d}x_i$$

$\mu \neq 0$ is induced free param for normln of θ

ts use Hopf Fourier trans to find $\partial^i, \partial^0 : \mathbb{R}^3_\lambda \to \mathbb{R}^3_\lambda$


A_λ def of C(M) by λ
ab − ba = λξ
A = "space"
R^3



$\hat{\Psi} = \Pi |\vec{X}_i - \vec{x}_i| \Psi$
$\mathbb{R}^3_\lambda = $ "$C(\mathbb{R}^3)$" $\ni x_i$
A_λ def of $C(M)$ by λ
$ab - ba = \lambda \xi$
$\Delta \leftrightarrow$ add a flow of $\mathbb{R}^3$
$\Omega(A_\lambda)$ dx_i
β θ
$A =$ "space" $\mathbb{R}^3$

Example (spin space model)

2+1 QG (Schroers, Batista-SM, Freidel)

$$\mathbb{R}^3_\lambda = U(su_2):\ [x_i, x_j] = \imath 2\lambda\epsilon_{ijk}x_k$$

`coadjoint quantization'

$$\cong \oplus_j M_{2j+1}(\mathbb{C}),\ M_{2j+1} = \mathbb{R}^3_\lambda/\ x^2 = 4\lambda^2 j(j+1)$$

`fuzzy sphere'

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$$

lassify calculi by repns of su(2) - we use pauli $\Longrightarrow$

$$\Omega^1(\mathbb{R}^3_\lambda) = \mathbb{R}^3_\lambda.M_2(\mathbb{C}), \quad \mathrm{d}x_i = \lambda\sigma_i,\ \theta \propto \mathrm{id}$$

$$(\mathrm{d}x_i)x_j - x_j\mathrm{d}x_i = \imath\lambda\epsilon_{ij}{}^k\mathrm{d}x_k + \imath\mu\delta_{ij}\theta.$$

$$x_i\theta - \theta x_i = \imath\frac{\lambda^2}{\mu}\mathrm{d}x_i$$

$\mu \neq 0$ is induced free param for normln of θ

ts use Hopf Fourier trans to find $\partial^i, \partial^0 : \mathbb{R}^3_\lambda \to \mathbb{R}^3_\lambda$

$: \mathbb{C}[SU_2] \to \mathbb{R}^3_\lambda \quad \mathcal{F}(f) = \int_{SU_2} \mathrm{d}u f(u) u \approx \int_{\mathbb{R}^3} \mathrm{d}^3 p \, J(\vec{p}) f(\vec{p}) e^{\imath \vec{p}\cdot x}$

curved mom coord alg

p^i parametrize waves, are local coordinates on mom space

iff-ops become multiplic. i.e. acting on plane waves:

$$\partial^i = \imath \frac{p^i}{\lambda|\vec{p}|}\sin(\lambda|\vec{p}|) \qquad \partial^0 = \frac{\imath\mu}{\lambda^2}(\cos(\lambda|p|) - 1)$$

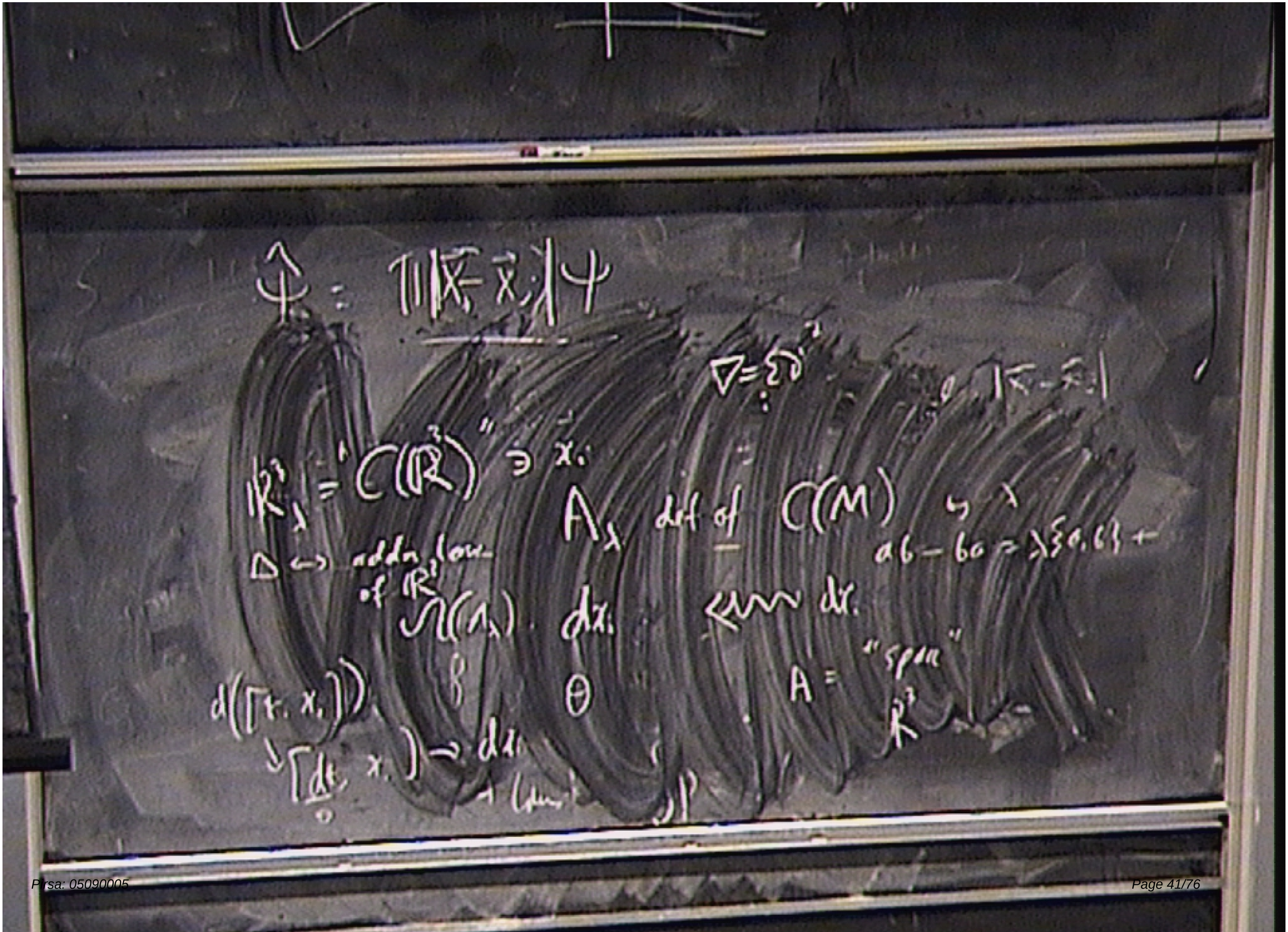
$$= \imath\vec{p}\cdot x \implies (\mathrm{d}X).X = X\mathrm{d}X - \imath\nu\theta, \quad \theta X = X\theta - \imath\nu' \mathrm{d}X \Rightarrow \mathrm{d}X^n$$

$$\partial^0 = \frac{\imath\mu}{\lambda^2}\left(\sqrt{1+\lambda^2\nabla^2} - 1\right) = \imath\frac{\mu}{2}\nabla^2\psi + O(\lambda^2)$$

Induced evol is Schroedingers eqn as $\lambda \to 0$

Now lets adjoin t, $\quad [t, x_i] = 0 \qquad \mathrm{d}t = \theta$

$$\implies [t, \mathrm{d}x_i] = \imath\frac{\lambda^2}{\mu}\mathrm{d}x_i, \; [t,\theta] = \imath\frac{\lambda^2}{\mu}\theta \;\Rightarrow\; f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_\mu = \mathrm{d}x_\mu f(t)$$


def of
"span"

$$\Rightarrow \quad \mathrm{d}f(t) = (\partial^t f(t))\mathrm{d}t; \quad \partial^t f(t) \equiv \frac{f(t) - f(t - \imath\frac{\lambda^2}{\mu})}{\imath\frac{\lambda^2}{\mu}}$$

$$\begin{aligned}
(a(x)f(t)) &= (\partial^i a \mathrm{d}x_i)f + (\partial^0 a \mathrm{d}t)f + a\partial^t f \mathrm{d}t \\
&= (\partial^i a)f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_i + \left((\partial^0 a)f(t - \imath\frac{\lambda^2}{\mu}) + a\partial^t f\right)\mathrm{d}t \\
&\equiv \tilde{\partial}^i(af)\mathrm{d}x_i + \tilde{\partial}^0(af)\mathrm{d}t.
\end{aligned}$$

n this extended space-time the NCSE becomes

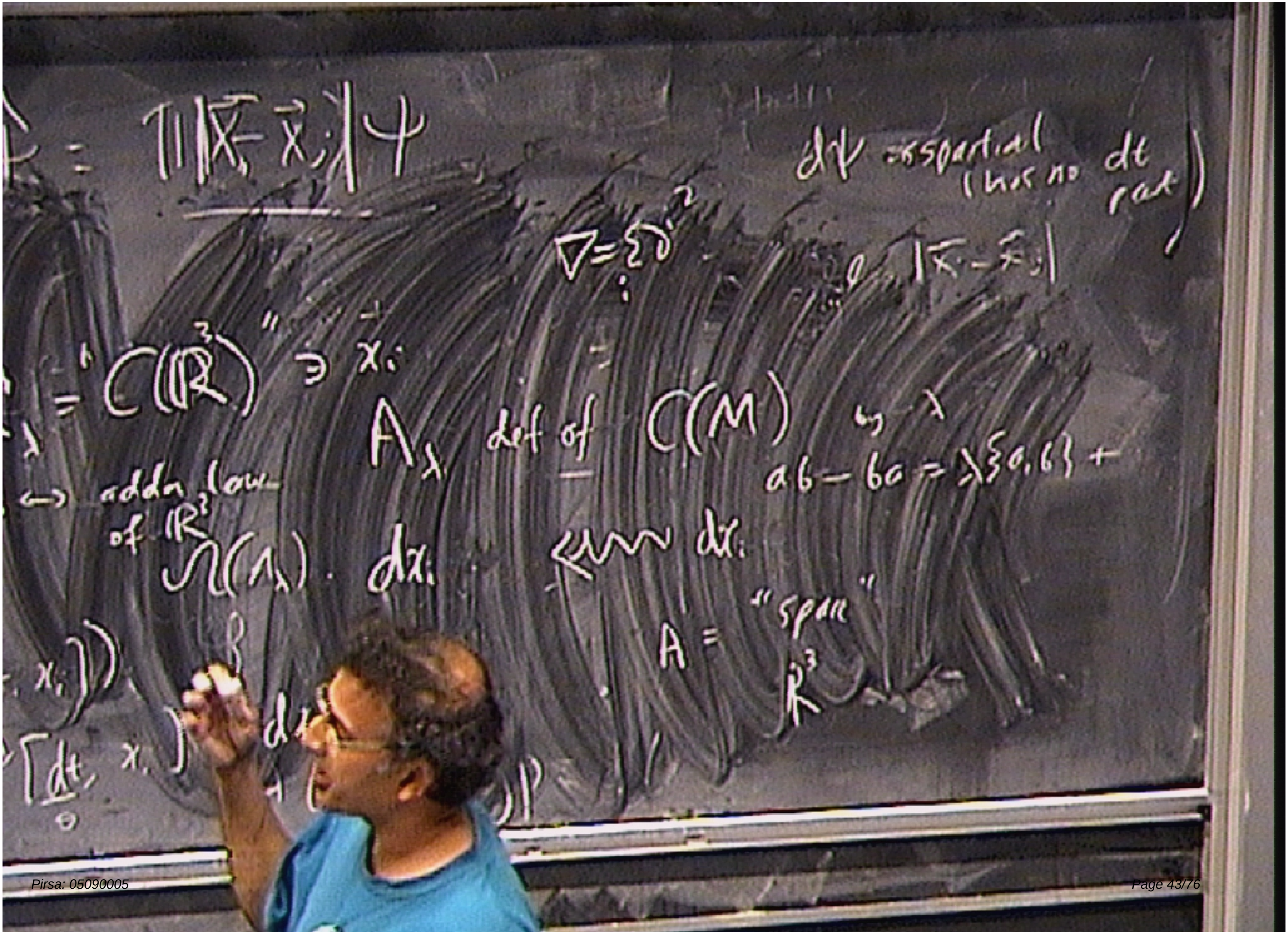
$$\tilde{\partial}^0\psi(x,t) = 0 \;\Leftrightarrow\; \frac{\imath\mu}{\lambda^2}\left(\psi(x,t) - \psi(x, t - \imath\frac{\lambda^2}{\mu})\right) = \partial^0\psi(x, t - \imath\frac{\lambda^2}{\mu})$$

$$\Leftrightarrow \quad \psi(x, t + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1 + \lambda^2\nabla^2}\right)\psi(x,t)$$

Vhich is solved by

$$\psi_{\vec{p}}(x,t) = e^{\imath\vec{p}\cdot x + \imath p^0 t}, \quad e^{-\frac{\lambda^2}{\mu}p^0} = \cos(\lambda|\vec{p}|), \quad |p| < \frac{\pi}{2\lambda}$$

dispersion reln Planckian cutoff


dψ is spatial
(has no dt
part)
def of
C(M)
by λ
ab − ba = λ{a,b} +
"span"

$$\Rightarrow \quad \mathrm{d}f(t) = (\partial^t f(t))\mathrm{d}t; \quad \partial^t f(t) \equiv \frac{f(t) - f(t - \imath\frac{\lambda^2}{\mu})}{\imath\frac{\lambda^2}{\mu}}$$

$$\begin{aligned} (a(x)f(t)) &= (\partial^i a \mathrm{d}x_i)f + (\partial^0 a \mathrm{d}t)f + a\partial^t f \mathrm{d}t \\ &= (\partial^i a)f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_i + \left((\partial^0 a)f(t - \imath\frac{\lambda^2}{\mu}) + a\partial^t f\right)\mathrm{d}t \\ &\equiv \tilde{\partial}^i(af)\mathrm{d}x_i + \tilde{\partial}^0(af)\mathrm{d}t. \end{aligned}$$

n this extended space-time the NCSE becomes

$$\tilde{\partial}^0\psi(x,t) = 0 \;\Leftrightarrow\; \frac{\imath\mu}{\lambda^2}\left(\psi(x,t) - \psi(x, t - \imath\frac{\lambda^2}{\mu})\right) = \partial^0\psi(x, t - \imath\frac{\lambda^2}{\mu})$$

$$\Leftrightarrow \quad \psi(x, t + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1 + \lambda^2\nabla^2}\right)\psi(x,t)$$

Vhich is solved by

$$\psi_{\vec{p}}(x,t) = e^{\imath\vec{p}\cdot x + \imath p^0 t}, \quad e^{-\frac{\lambda^2}{\mu}p^0} = \cos(\lambda|\vec{p}|), \quad |p| < \frac{\pi}{2\lambda}$$

dispersion reln Planckian cutoff

$\hat{\ } = \prod |\vec{x}_i - \vec{x}_j| \psi$ $\qquad \mu = \frac{1}{m}$ $\qquad d\psi$ is spatial (has no dt part)

$\nabla = \sum_i \partial_i^2$ $\qquad |\vec{x}_i - \vec{x}_j|$

$= C(\mathbb{R}^3)$ " $\ni \vec{x}_i$

A_λ def of $C(M)$ by λ

$ab - ba = \lambda\{a, b\} +$

$\leftrightarrow$ add a flow of $\mathbb{R}^3$

$\Omega(A_\lambda)$ $\quad dx_i$ $\quad \longleftarrow$ dx_i

$A =$ "span" $\mathbb{R}^3$

x_i)) $\quad \beta \quad \theta$

$[dt, x_i] = dx_i$

$\Rightarrow$ wave speed $\quad |\frac{\partial p^0}{\partial \vec{p}}| = \frac{\mu}{\lambda}|\tan(\lambda|\vec{p}|)|$ (cf `bicross-product model')

How to detect? Exist minimal uncertainty states s.t.

$$\langle j,\phi,\psi|\vec{x}|j,\phi,\psi\rangle = r(\sin\phi\,\cos\psi, \sin\phi\,\sin\psi, \cos\phi)$$

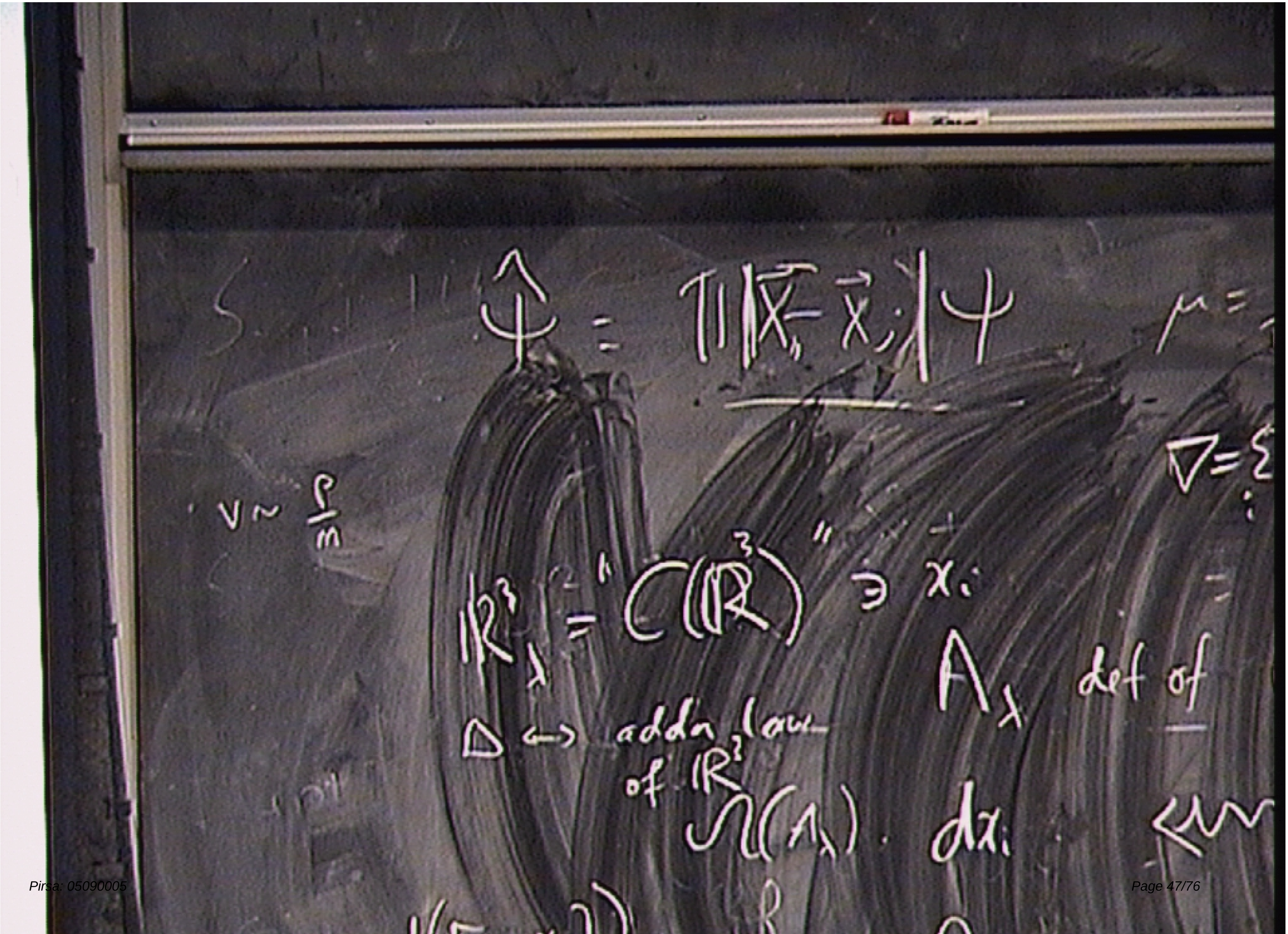
$$|j,\phi,\psi\rangle = \sum_{k=0}^{2j} 2^{-j}\sqrt{\binom{2j}{k}}(1+\cos\phi)^{\frac{2j-k}{2}}(1-\cos\phi)^{\frac{k}{2}}e^{\imath k\psi}|j,j-k\rangle$$

$\Rightarrow \quad r = 2j\lambda$

$$|\langle j,\phi,\psi|j,\phi',\psi'\rangle|^2 = \left(\frac{1}{2}(1+\cos(\mathrm{angle}(\phi,\psi|\phi',\psi')))\right)^{2j}$$

(cf Penrose spin network) In such states we see fuzzy waves:

$$\langle \tfrac{1}{2},\phi,\psi|e^{\imath\vec{p}\cdot x}|\tfrac{1}{2},\phi,\psi\rangle = \cos\lambda|\vec{p}| + \imath\frac{\vec{p}\cdot\langle x\rangle}{\lambda|\vec{p}|}\sin\lambda|\vec{p}|$$



v ~ p/m
Δ ⟷ add'n law
of ℝ³
def of

$$\Rightarrow \; \mathrm{d}f(t) = (\partial^t f(t))\mathrm{d}t; \quad \partial^t f(t) \equiv \frac{f(t) - f(t - \imath\frac{\lambda^2}{\mu})}{\imath\frac{\lambda^2}{\mu}}$$

$$\begin{aligned} (a(x)f(t)) &= (\partial^i a \mathrm{d}x_i)f + (\partial^0 a \mathrm{d}t)f + a\partial^t f \mathrm{d}t \\ &= (\partial^i a)f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_i + \left((\partial^0 a)f(t - \imath\frac{\lambda^2}{\mu}) + a\partial^t f\right)\mathrm{d}t \\ &\equiv \tilde{\partial}^i(af)\mathrm{d}x_i + \tilde{\partial}^0(af)\mathrm{d}t. \end{aligned}$$

n this extended space-time the NCSE becomes

$$\tilde{\partial}^0\psi(x,t) = 0 \;\Leftrightarrow\; \frac{\imath\mu}{\lambda^2}\left(\psi(x,t) - \psi(x, t - \imath\frac{\lambda^2}{\mu})\right) = \partial^0\psi(x, t - \imath\frac{\lambda^2}{\mu})$$

$$\Leftrightarrow \quad \psi(x, t + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1 + \lambda^2\nabla^2}\right)\psi(x,t)$$

Vhich is solved by

$$\psi_{\vec{p}}(x,t) = e^{\imath\vec{p}\cdot x + \imath p^0 t}, \quad e^{-\frac{\lambda^2}{\mu}p^0} = \cos(\lambda|\vec{p}|), \quad |p| < \frac{\pi}{2\lambda}$$

dispersion reln Planckian cutoff

$\Rightarrow$ wave speed $|\frac{\partial p^0}{\partial \vec{p}}| = \frac{\mu}{\lambda}|\tan(\lambda|\vec{p}|)|$ (cf `bicross-product model')

How to detect? Exist minimal uncertainty states s.t.

$$\langle j,\phi,\psi|\vec{x}|j,\phi,\psi\rangle = r(\sin\phi\,\cos\psi, \sin\phi\,\sin\psi, \cos\phi)$$

$$j,\phi,\psi\rangle = \sum_{k=0}^{2j} 2^{-j}\sqrt{\binom{2j}{k}}(1+\cos\phi)^{\frac{2j-k}{2}}(1-\cos\phi)^{\frac{k}{2}}e^{\imath k\psi}|j,j-k\rangle$$

$\Rightarrow \quad r = 2j\lambda$

$$\langle j,\phi,\psi|j,\phi',\psi'\rangle|^2 = \left(\frac{1}{2}(1+\cos(\mathrm{angle}(\phi,\psi|\phi',\psi')))\right)^{2j}$$

cf Penrose spin network) In such states we see fuzzy waves:

$$\langle \tfrac{1}{2},\phi,\psi|e^{\imath\vec{p}\cdot x}|\tfrac{1}{2},\phi,\psi\rangle = \cos\lambda|\vec{p}| + \imath\frac{\vec{p}\cdot\langle x\rangle}{\lambda|\vec{p}|}\sin\lambda|\vec{p}|$$

$$\Rightarrow \ \mathrm{d}f(t) = (\partial^t f(t))\mathrm{d}t; \quad \partial^t f(t) \equiv \frac{f(t) - f(t - \imath\frac{\lambda^2}{\mu})}{\imath\frac{\lambda^2}{\mu}}$$

$$\begin{aligned} (a(x)f(t)) &= (\partial^i a \mathrm{d}x_i)f + (\partial^0 a \mathrm{d}t)f + a\partial^t f \mathrm{d}t \\ &= (\partial^i a)f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_i + \left((\partial^0 a)f(t - \imath\frac{\lambda^2}{\mu}) + a\partial^t f\right)\mathrm{d}t \\ &\equiv \tilde{\partial}^i(af)\mathrm{d}x_i + \tilde{\partial}^0(af)\mathrm{d}t. \end{aligned}$$

n this extended space-time the NCSE becomes

$$\tilde{\partial}^0\psi(x,t) = 0 \Leftrightarrow \frac{\imath\mu}{\lambda^2}\left(\psi(x,t) - \psi(x, t - \imath\frac{\lambda^2}{\mu})\right) = \partial^0\psi(x, t - \imath\frac{\lambda^2}{\mu})$$

$$\Leftrightarrow \quad \psi(x, t + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1 + \lambda^2\nabla^2}\right)\psi(x,t)$$

Vhich is solved by

$$\psi_{\vec{p}}(x,t) = e^{\imath\vec{p}\cdot x + \imath p^0 t}, \quad e^{-\frac{\lambda^2}{\mu}p^0} = \cos(\lambda|\vec{p}|), \quad |p| < \frac{\pi}{2\lambda}$$

dispersion reln Planckian cutoff

$\Rightarrow$ wave speed $|\frac{\partial p^0}{\partial \vec{p}}| = \frac{\mu}{\lambda}|\tan(\lambda|\vec{p}|)|$ (cf `bicross-product model')

How to detect? Exist minimal uncertainty states s.t.

$$\langle j,\phi,\psi|\vec{x}|j,\phi,\psi\rangle = r(\sin\phi\,\cos\psi, \sin\phi\,\sin\psi, \cos\phi)$$

$$j,\phi,\psi\rangle = \sum_{k=0}^{2j} 2^{-j}\sqrt{\binom{2j}{k}}(1+\cos\phi)^{\frac{2j-k}{2}}(1-\cos\phi)^{\frac{k}{2}}e^{\imath k\psi}|j,j-k\rangle$$

$\Rightarrow \quad r = 2j\lambda$

$$\langle j,\phi,\psi|j,\phi',\psi'\rangle|^2 = \left(\frac{1}{2}(1+\cos(\text{angle}(\phi,\psi|\phi',\psi')))\right)^{2j}$$

cf Penrose spin network) In such states we see fuzzy waves:

$$\langle \tfrac{1}{2},\phi,\psi|e^{\imath\vec{p}\cdot x}|\tfrac{1}{2},\phi,\psi\rangle = \cos\lambda|\vec{p}| + \imath\frac{\vec{p}\cdot\langle x\rangle}{\lambda|\vec{p}|}\sin\lambda|\vec{p}|$$

4 Outlook

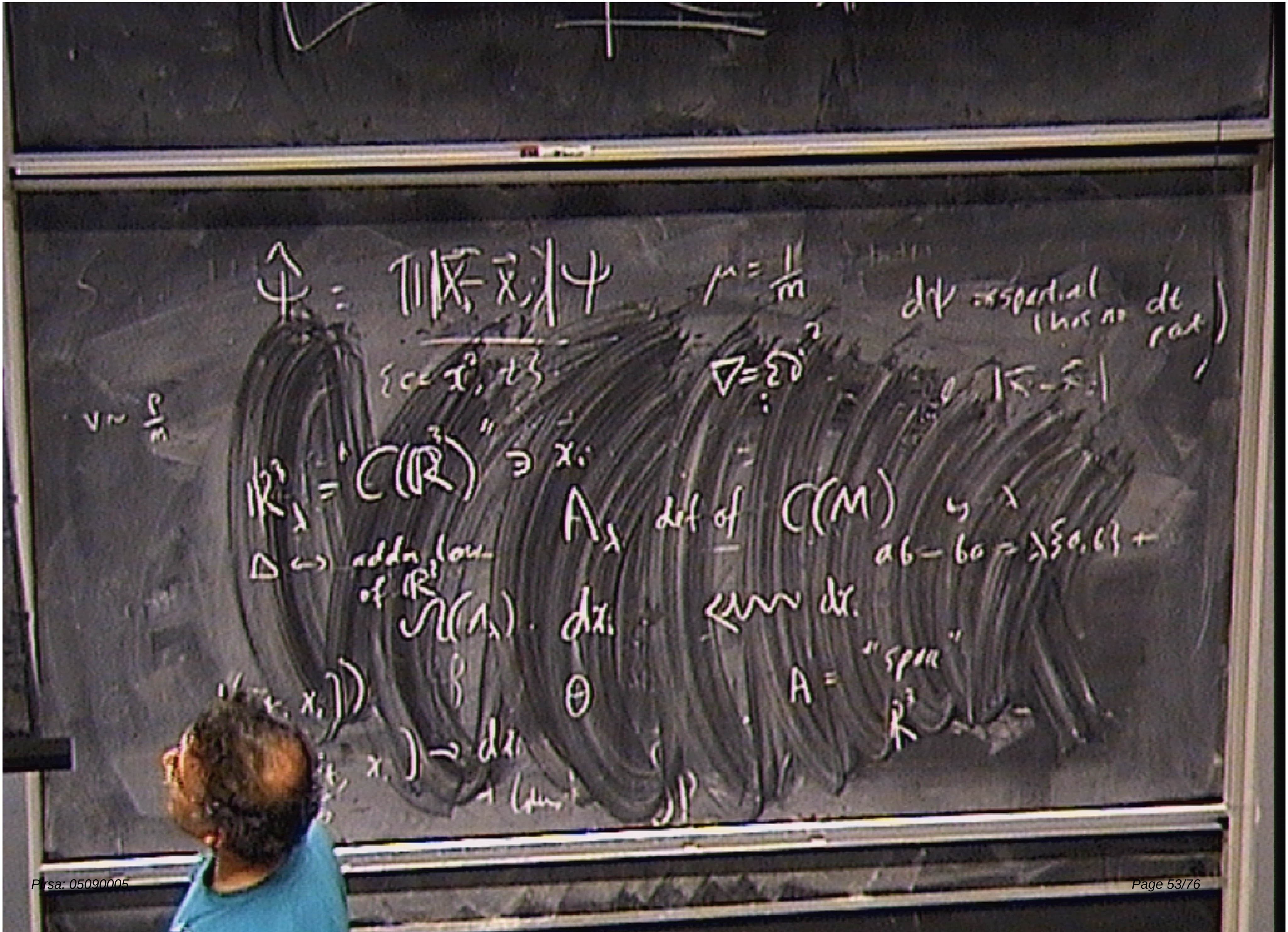
NC space-time models

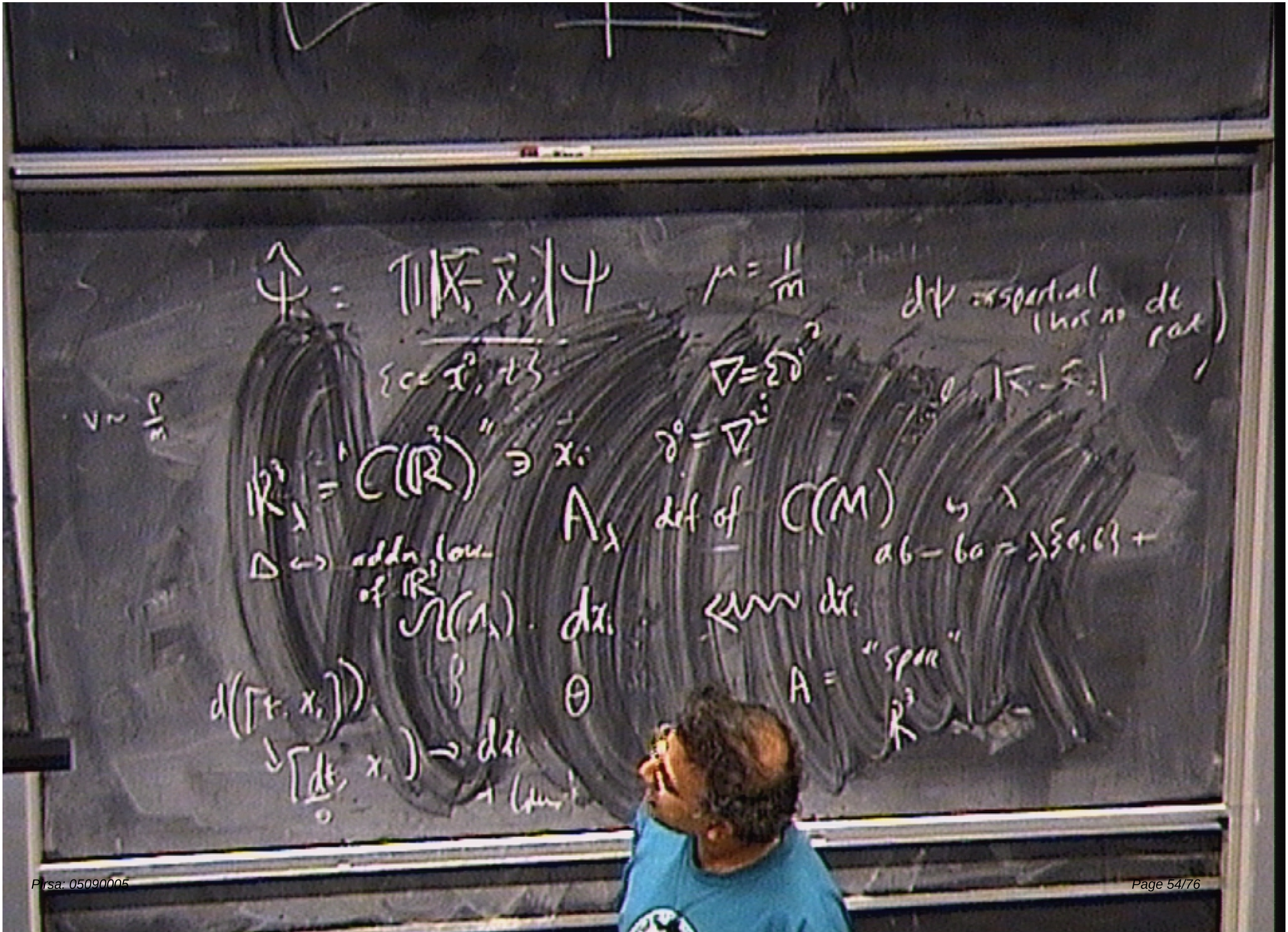
- solve hydrogen atom, electromag.(exist polar coords)
- $U_q(su_2) = \mathbb{C}_\lambda[B_+]$ curved version - cosmol const
- $\mathbb{C}_q[SU_2]$ time generation $\Rightarrow$ q-Friedman model
- fermions, field theory incl. bicrossprod (kappa) model

NCG in general

- $\Omega^1(\mathbb{C}(\mathrm{Set})) \leftrightarrow$ Graphs $\mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\delta_x \mathrm{d}\delta_y$

  QG = $\sum_{\mathrm{Graphs}} \int \mathrm{d}A\mathrm{d}E\; e^{S(A,E)}$
- qua. random walks, qua. information, qua. measurement
- extend obs-state duality, Einstein's eqn as self-duality
- general cogravity - mom. dependent $\hbar$?


def of C(M)
A =


def of C(M) by λ
ab − ba = λ{a,b} +
A = “span”

4 Outlook

NC space-time models

- solve hydrogen atom, electromag.(exist polar coords)
- $U_q(su_2) = \mathbb{C}_\lambda[B_+]$ curved version - cosmol const
- $\mathbb{C}_q[SU_2]$ time generation $\Rightarrow$ q-Friedman model
- fermions, field theory incl. bicrossprod (kappa) model

NCG in general

- $\Omega^1(\mathbb{C}(\mathrm{Set})) \leftrightarrow$ Graphs $\mathrm{d}f = \sum_{x \to y} (f(y) - f(x))\delta_x \mathrm{d}\delta_y$

  QG = $\sum_{\mathrm{Graphs}} \int \mathrm{d}A \mathrm{d}E \; e^{S(A,E)}$
- qua. random walks, qua. information, qua. measurement
- extend obs-state duality, Einstein's eqn as self-duality
- general cogravity - mom. dependent $\hbar$?

$\prod |\vec{x}_j - \vec{x}_i| \psi$ $\mu = \frac{1}{m}$

$d\psi$ is spatial (has no dt part)

$\{c = x^2, t\}$

$\nabla = \sum \partial_i^2$

$\partial^0 = \nabla^2$

$C(\mathbb{R}^3) \ni x_i$

A_λ def of $C(M)$ by λ

$ab - ba = \lambda\{a,b\} + \ldots$

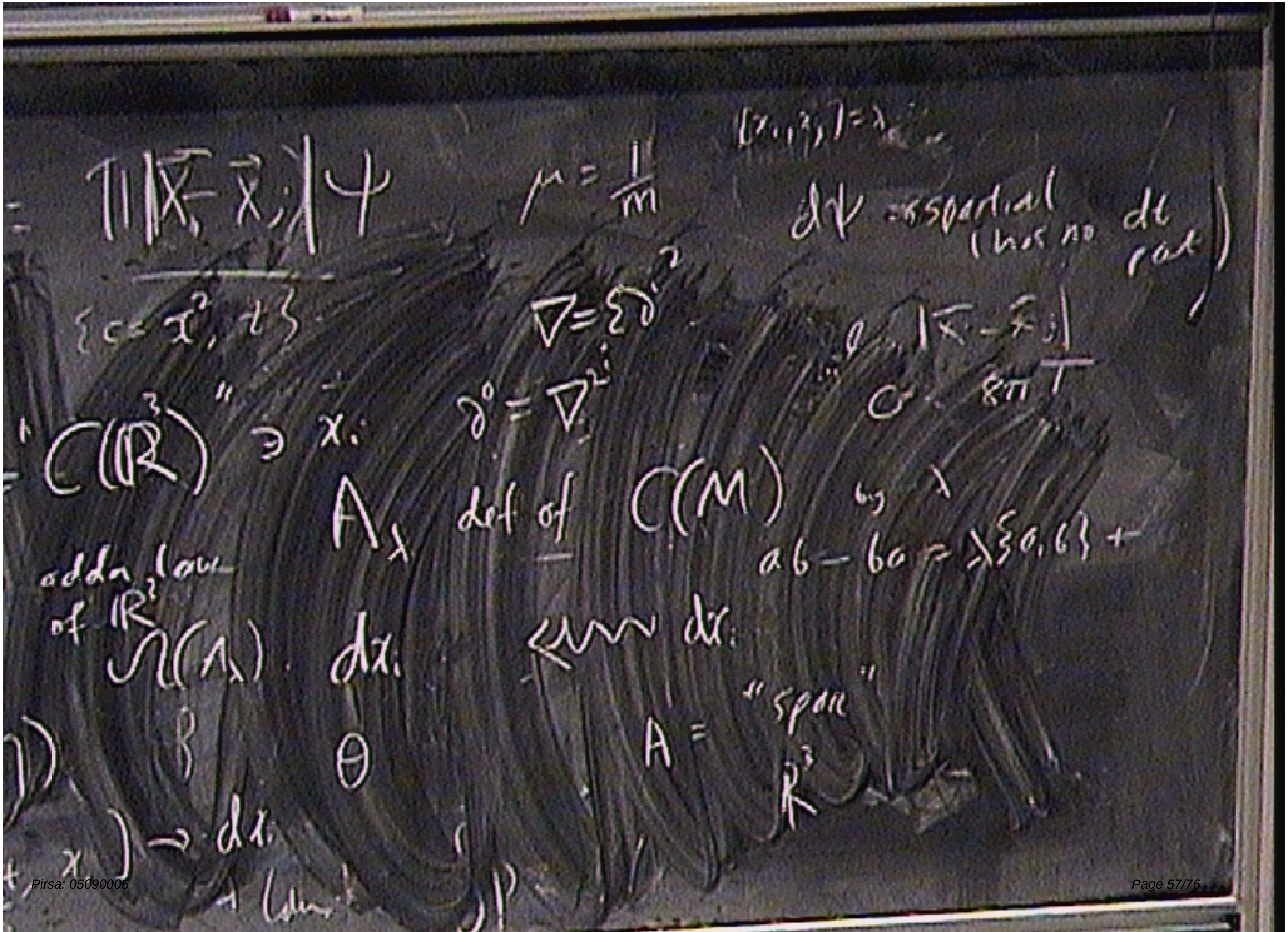
addn law of $\mathbb{R}^3$

$\Omega(A_\lambda)$ dx_i $\longleftrightarrow$ dx_i

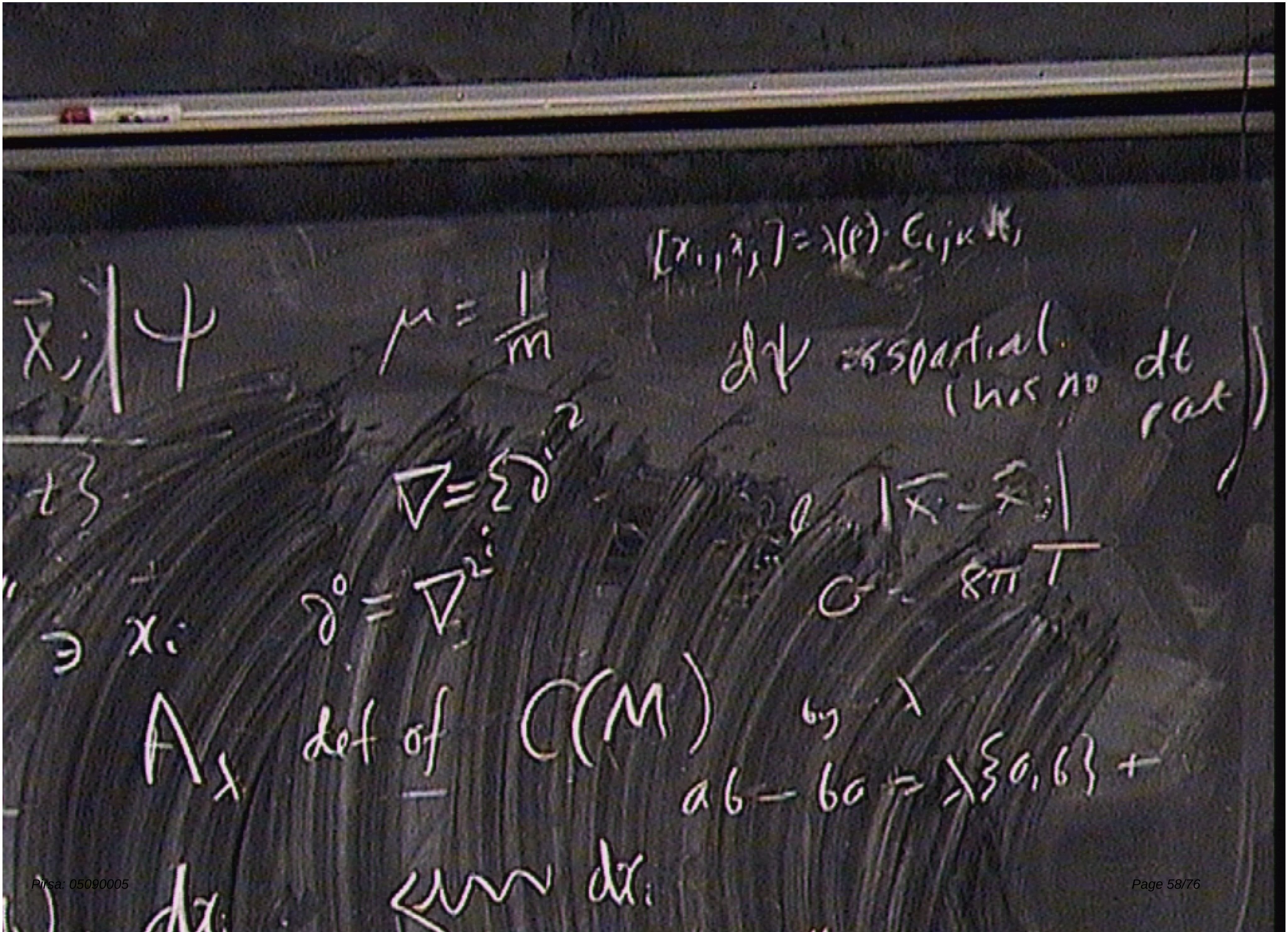
"space"

β θ $A = \mathbb{R}^3$

$) \to dx_i$



μ = 1/m
dψ is spatial
(has no dt
part)
C(M)
def of
by λ
ab − ba = λ{a, b} +
add'n law
of R³
A =
"span"


μ = 1/m
spatial
(has no dt
part)
A_λ def of C(M) by λ
ab − ba = λ{a, b} +

4 Outlook

NC space-time models

- solve hydrogen atom, electromag.(exist polar coords)
- $U_q(su_2) = \mathbb{C}_\lambda[B_+]$ curved version - cosmol const
- $\mathbb{C}_q[SU_2]$ time generation $\Rightarrow$ q-Friedman model
- fermions, field theory incl. bicrossprod (kappa) model

NCG in general

- $\Omega^1(\mathbb{C}(\mathrm{Set})) \leftrightarrow$ Graphs $\mathrm{d}f = \sum_{x \to y} (f(y) - f(x))\delta_x \mathrm{d}\delta_y$

  QG = $\sum_{\mathrm{Graphs}} \int \mathrm{d}A\mathrm{d}E \, e^{S(A,E)}$
- qua. random walks, qua. information, qua. measurement
- extend obs-state duality, Einstein's eqn as self-duality
- general cogravity - mom. dependent $\hbar$?

Noncommutative geometry and the origin of time

1. Hopf algebra axioms (invariant under arrow reversal)

$$H \otimes H \xrightarrow{\mathsf{m}} H \qquad\qquad H \xrightarrow{\Delta} H \otimes H$$

$$\mathbb{C} \xrightarrow{1} H \qquad H \xrightarrow{S} H \qquad H \xrightarrow{\epsilon} \mathbb{C}$$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$m(S \otimes \mathrm{id})\Delta = 1\epsilon = m(\mathrm{id} \otimes S)\Delta$$

$$\Rightarrow \text{(dual Hopf algebra)} \quad H \leftrightarrow H^*$$

\`prove one get one free property'

$$S(ab) = S(b)S(a) \qquad \Delta S = \mathrm{flip}(S \otimes S)\Delta$$

\`for every phenomenon a cophenomenon'

action observable coaction state

ntegral coentropy cointegral entropy

One object, two points of view

coordinate alg

$$H = \mathbb{C}(G) = \{\delta_x | x \in G\}$$

$$\delta_x \delta_y = \delta_{x,y}\delta_x$$

$$\Delta\delta_x = \sum_{yz=x} \delta_y \otimes \delta_z$$

$$S\delta_x = \delta_{x^{-1}}$$

$$\int \delta_x = 1$$

symmetry alg

$$H = \mathbb{C}G = \{x \in G\}$$

$$x.y = xy$$

$$\Delta x = x \otimes x$$

$$Sx = x^{-1}$$

$$\int x = \delta_{x,e}$$

Fourier tranform

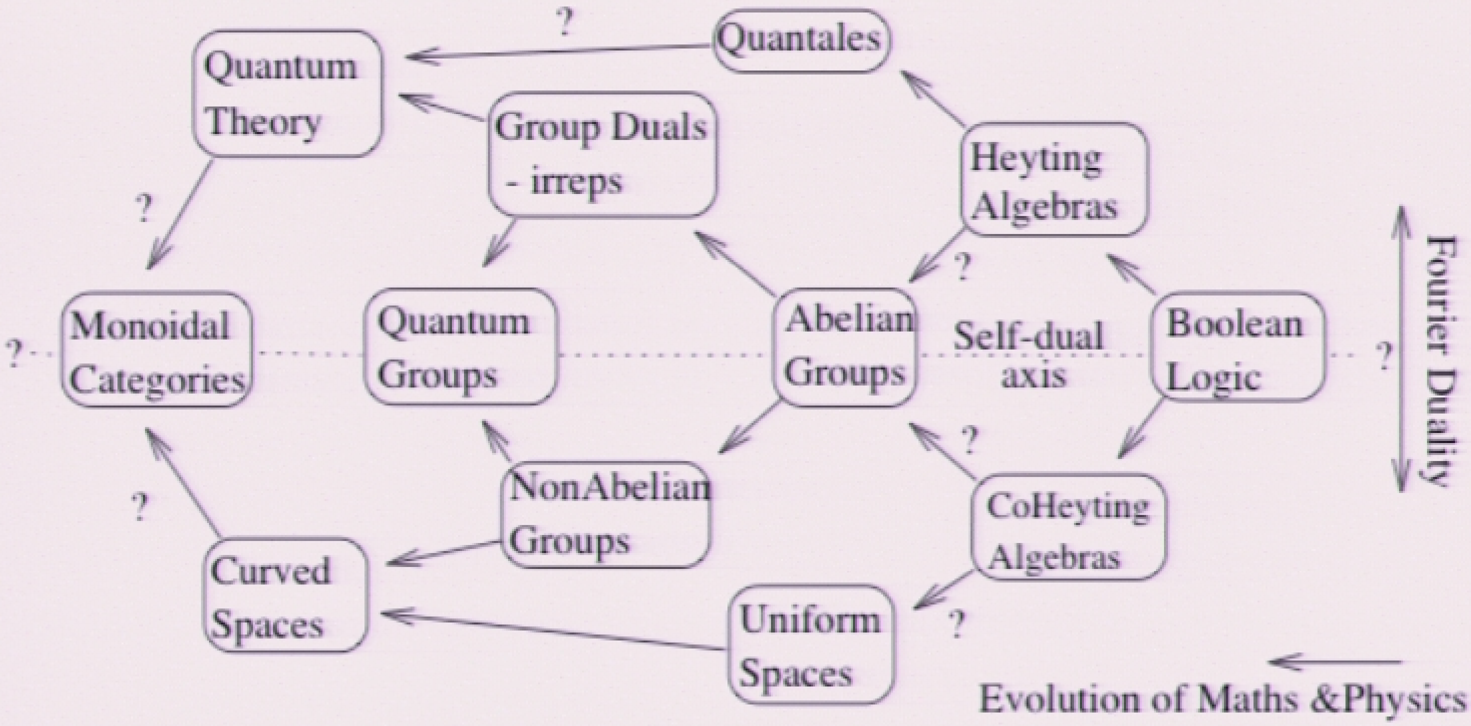
$$\mathbb{C}(G) \to \mathbb{C}G = \text{“}\mathbb{C}(\hat{G})\text{”}$$

$$\mathcal{F}(f) = (\int f \sum_x \delta_x)x = \int \mathrm{d}x f(x) x$$

curved posn — gravity — noncom mom

noncom posn — cogravity — curved mom

Category Hopf as microcosm of quantum-gravity



Theorem (SM'87)

$$0 \to \mathbb{C}(M) \rightleftarrows H \rightleftarrows U(\mathfrak{g}) \to 0 \;\Rightarrow\; H \cong \mathbb{C}(M) \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$$

posn alg obs mom

$$H^* \cong U(\mathfrak{m}) \triangleright\!\!\blacktriangleleft \mathbb{C}(G)$$

$$\Leftrightarrow\; (G, M)$$ matched pair eqns `toy einstein eq'

e.g. $$G = M = \mathbb{R} \;\Rightarrow\; [p, x] = \imath\hbar(1 - e^{-\gamma x})$$

$$\Delta x = x \otimes 1 + 1 \otimes x \qquad \Delta p = p \otimes e^{-\gamma x} + 1 \otimes p$$

. Noncommutative geometry of an algebra A

quantum groups approach: these should be `differentiable')

Differential Structure

Ω^1 a((db)c)=(a(db))c (A-A bimodule)

$\mathrm{l} : A \to \Omega^1$ d(ab)=(da)b+a(db) (Leibniz rule)

$a\mathrm{d}b\} = \Omega^1 \quad \ker \mathrm{d} = \mathbb{C}.1$ (connectedness)

Iopf case - ask it to be translation invariant

$\Rightarrow \quad \Omega = \oplus \Omega^n, \ \mathrm{d}^2 = 0$ ... Hodge * ... metric ... bundles ...

heorem: for noncom deformation $\Omega^1(A_\lambda)$

$$\exists \theta \in \Omega^1 \text{such that } [\theta, a] = \lambda \mathrm{d}a$$

ut need not have any classical analogue

$\mathrm{d}x_i \to \mathrm{d}x_i \qquad ? \to \theta$

$: \mathbb{C}[SU_2] \to \mathbb{R}^3_\lambda \quad \mathcal{F}(f) = \int_{SU_2} \mathrm{d}u f(u) u \approx \int_{\mathbb{R}^3} \mathrm{d}^3p\, J(\vec{p}) f(\vec{p}) e^{\imath \vec{p}\cdot x}$

curved mom coord alg

p^i parametrize waves, are local coordinates on mom space

iff-ops become multiplic. i.e. acting on plane waves:

$$\partial^i = \imath \frac{p^i}{\lambda|\vec{p}|}\sin(\lambda|\vec{p}|) \qquad \partial^0 = \frac{\imath\mu}{\lambda^2}(\cos(\lambda|p|) - 1)$$

$= \imath\vec{p}\cdot x \implies (\mathrm{d}X).X = X\mathrm{d}X - \imath\nu\theta, \quad \theta X = X\theta - \imath\nu'\mathrm{d}X \Rightarrow \mathrm{d}X^n$

$$\partial^0 = \frac{\imath\mu}{\lambda^2}\left(\sqrt{1+\lambda^2\nabla^2} - 1\right) = \imath\frac{\mu}{2}\nabla^2\psi + O(\lambda^2)$$

Induced evol is Schroedingers eqn as $\lambda \to 0$

Now lets adjoin t, $\quad [t, x_i] = 0 \quad \mathrm{d}t = \theta$

$\Rightarrow [t, \mathrm{d}x_i] = \imath\frac{\lambda^2}{\mu}\mathrm{d}x_i,\ [t,\theta] = \imath\frac{\lambda^2}{\mu}\theta \ \Rightarrow\ f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_\mu = \mathrm{d}x_\mu f(t)$

$$\Rightarrow \; \mathrm{d}f(t) = (\partial^t f(t))\mathrm{d}t; \quad \partial^t f(t) \equiv \frac{f(t) - f(t - \imath\frac{\lambda^2}{\mu})}{\imath\frac{\lambda^2}{\mu}}$$

$$\begin{aligned}
(a(x)f(t)) &= (\partial^i a \mathrm{d}x_i)f + (\partial^0 a \mathrm{d}t)f + a\partial^t f \mathrm{d}t \\
&= (\partial^i a)f(t - \imath\frac{\lambda^2}{\mu})\mathrm{d}x_i + \left((\partial^0 a)f(t - \imath\frac{\lambda^2}{\mu}) + a\partial^t f\right)\mathrm{d}t \\
&\equiv \tilde{\partial}^i(af)\mathrm{d}x_i + \tilde{\partial}^0(af)\mathrm{d}t.
\end{aligned}$$

n this extended space-time the NCSE becomes

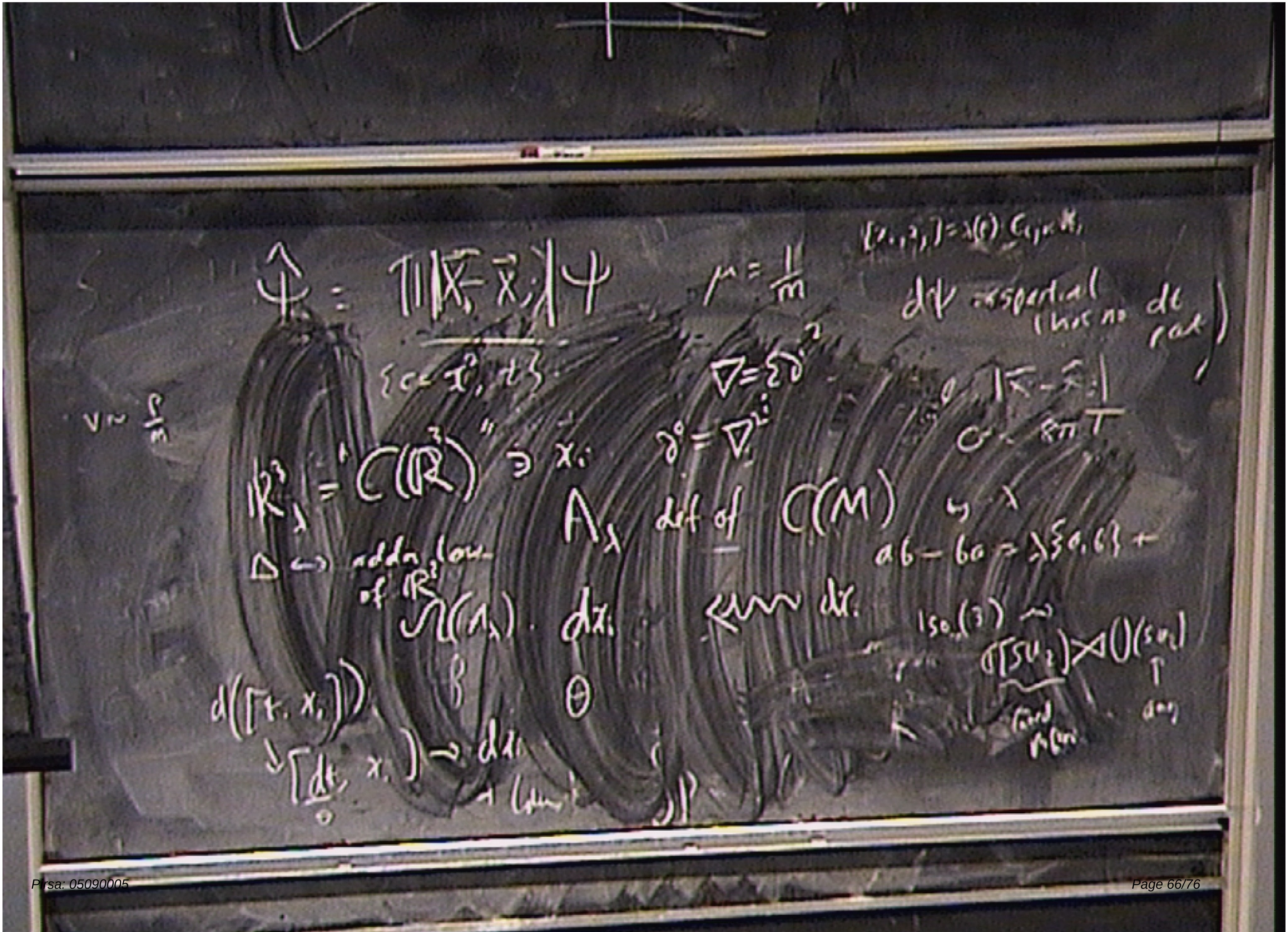
$$\tilde{\partial}^0\psi(x,t) = 0 \;\Leftrightarrow\; \frac{\imath\mu}{\lambda^2}\left(\psi(x,t) - \psi(x, t - \imath\frac{\lambda^2}{\mu})\right) = \partial^0\psi(x, t - \imath\frac{\lambda^2}{\mu})$$

$$\Leftrightarrow \quad \psi(x, t + \imath\frac{\lambda^2}{\mu}) = \left(\sqrt{1 + \lambda^2\nabla^2}\right)\psi(x,t)$$

Vhich is solved by

$$\psi_{\vec{p}}(x,t) = e^{\imath\vec{p}\cdot x + \imath p^0 t}, \quad e^{-\frac{\lambda^2}{\mu}p^0} = \cos(\lambda|\vec{p}|), \quad |p| < \frac{\pi}{2\lambda}$$

dispersion reln Planckian cutoff


v ~ ρ/m
μ = 1/m
∇ = Σ∂²
A_λ def of C(M)
ab − ba = λ{a,b} +
Θ

Appendix

icrossproduct models $[t, x_i] = \imath\lambda x_i, \quad [x_i, x_j] = 0$

d case (SM 1987) has poinc. $\mathbb{C}[B_+]\blacktriangleright\!\!\triangleleft U(su_2)$

$$[M_i, p_j] = \frac{\imath}{2}\epsilon_{ij}{}^3\left(\frac{1-e^{-2\lambda p_3}}{\lambda} - \lambda\vec{p}^2\right) + \imath\lambda\epsilon_i{}^{k3}p_j p_k \qquad i,j = 1,2\ (t = x_3)$$

$$\Delta M_i = M_i \otimes e^{-\lambda p_3} + \lambda M_3 \otimes p_i + 1 \otimes M_i \qquad \Delta p_i = p_i \otimes e^{-\lambda p_3} + 1 \otimes p_i$$

Vave op $\frac{2}{\lambda^2}(\cosh(\lambda p_3) - 1) + \vec{p}^2 e^{\lambda p_3}$ on $e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p_3 x_3}$

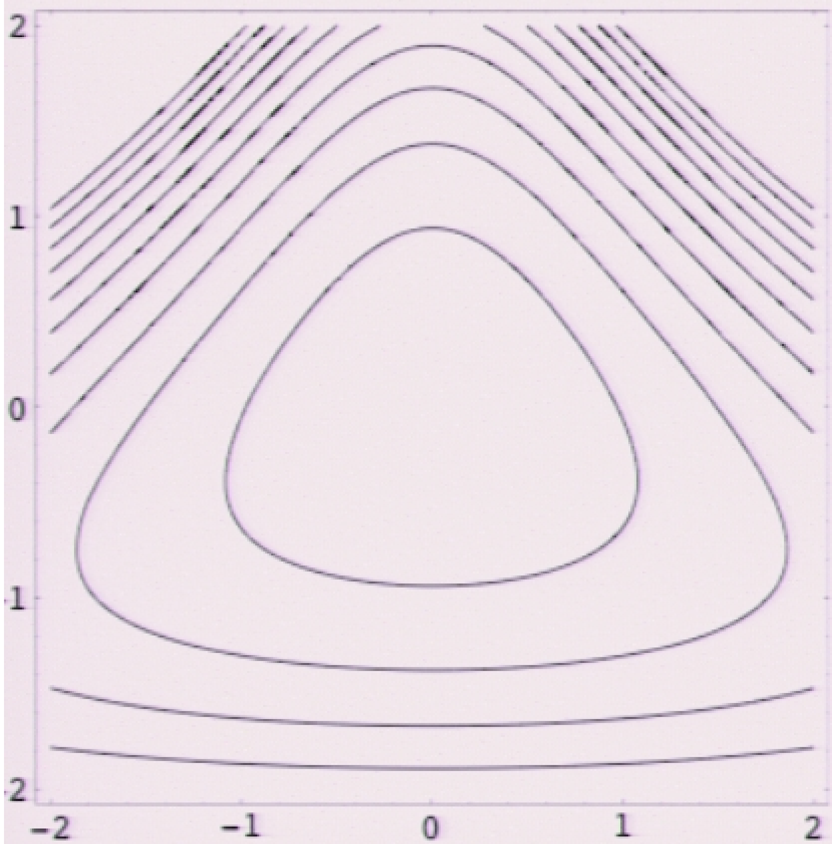
+1 case (Ruegg-SM 1994) has poinc. $U(so_{1,3})\triangleright\!\!\blacktriangleleft\mathbb{C}[B_+]$

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j \qquad i.j = 1,2,3$$

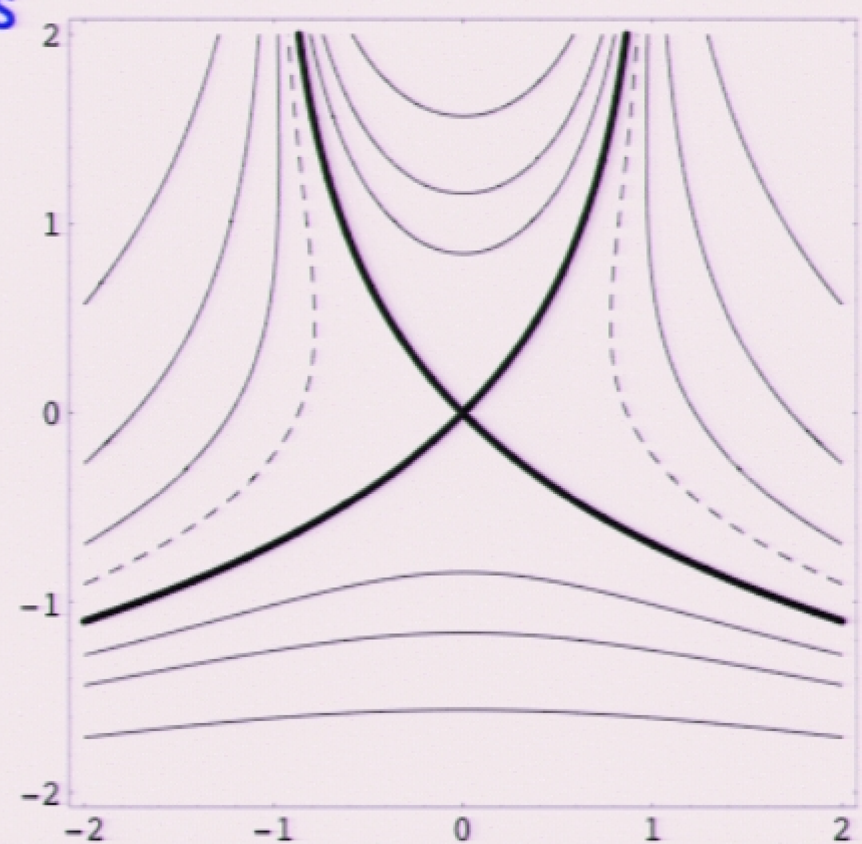
$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda\epsilon_{ijk}p^j \otimes M_k \qquad \Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

Vave op $-\frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1) + \vec{p}^2 e^{\lambda p^0}$ on $e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p^0 t}$

Orbits



Euclidean (def spheres)



Mink. (def hyperboloids)

m=0: $|\vec{p}| = \frac{1}{\lambda}(1 - e^{-\lambda p^0}) < 1/\lambda.$

$$|\frac{\partial p^0}{\partial \vec{p}}| = e^{\lambda p^0}$$

(or interpret orbits in posn space, Mackey ıantisation, Mink case is \`fake black hole')

cf. non rel. spin model

Appendix

icrossproduct models $\quad [t, x_i] = \imath\lambda x_i, \quad [x_i, x_j] = 0$

d case (SM 1987) has poinc. $\mathbb{C}[B_+]\blacktriangleright\!\!\triangleleft U(su_2)$

$$[M_i, p_j] = \frac{\imath}{2}\epsilon_{ij}{}^3\left(\frac{1-e^{-2\lambda p_3}}{\lambda} - \lambda\vec{p}^2\right) + \imath\lambda\epsilon_i{}^{k3}p_jp_k \qquad i,j = 1,2 \ (t = x_3)$$

$$\Delta M_i = M_i \otimes e^{-\lambda p_3} + \lambda M_3 \otimes p_i + 1 \otimes M_i \qquad \Delta p_i = p_i \otimes e^{-\lambda p_3} + 1 \otimes p_i$$

Vave op $\quad \frac{2}{\lambda^2}(\cosh(\lambda p_3) - 1) + \vec{p}^2 e^{\lambda p_3}$ on $\quad e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p_3 x_3}$

+1 case (Ruegg-SM 1994) has poinc. $U(so_{1,3})\triangleright\!\!\blacktriangleleft\mathbb{C}[B_+]$

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j \qquad i.j = 1,2,3$$

$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda\epsilon_{ijk}p^j \otimes M_k \qquad \Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

Vave op $\quad -\frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1) + \vec{p}^2 e^{\lambda p^0}$ on $\quad e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p^0 t}$

4 Outlook

NC space-time models

- solve hydrogen atom, electromag.(exist polar coords)
- $U_q(su_2) = \mathbb{C}_\lambda[B_+]$ curved version - cosmol const
- $\mathbb{C}_q[SU_2]$ time generation $\Rightarrow$ q-Friedman model
- fermions, field theory incl. bicrossprod (kappa) model

NCG in general

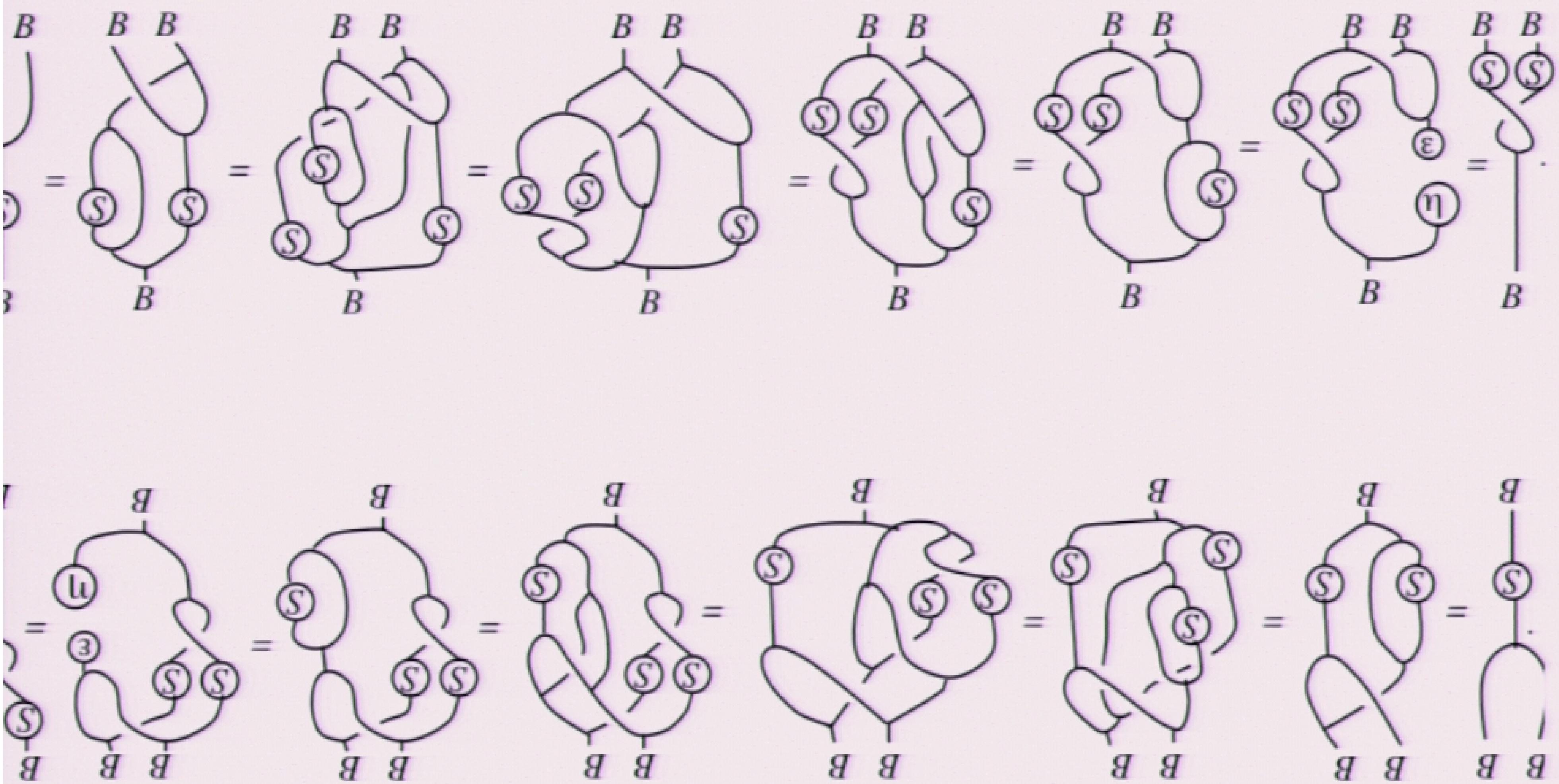
- $\Omega^1(\mathbb{C}(\mathrm{Set})) \leftrightarrow$ Graphs $\mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\delta_x \mathrm{d}\delta_y$

  QG = $\sum_{\mathrm{Graphs}} \int \mathrm{d}A\mathrm{d}E\; e^{S(A,E)}$

- qua. random walks, qua. information, qua. measurement
- extend obs-state duality, Einstein's eqn as self-duality
- general cogravity - mom. dependent $\hbar$?

Appendix

input-output symmetry (braided Hopf algebras)



Appendix

icrossproduct models $\quad [t, x_i] = \imath\lambda x_i, \quad [x_i, x_j] = 0$

d case (SM 1987) has poinc. $\mathbb{C}[B_+]\blacktriangleright\!\!\triangleleft U(su_2)$

$$[M_i, p_j] = \frac{\imath}{2}\epsilon_{ij}{}^3\left(\frac{1-e^{-2\lambda p_3}}{\lambda} - \lambda\vec{p}^2\right) + \imath\lambda\epsilon_i{}^{k3}p_j p_k \qquad i,j = 1,2 \ (t = x_3)$$

$$\Delta M_i = M_i \otimes e^{-\lambda p_3} + \lambda M_3 \otimes p_i + 1 \otimes M_i \qquad \Delta p_i = p_i \otimes e^{-\lambda p_3} + 1 \otimes p_i$$

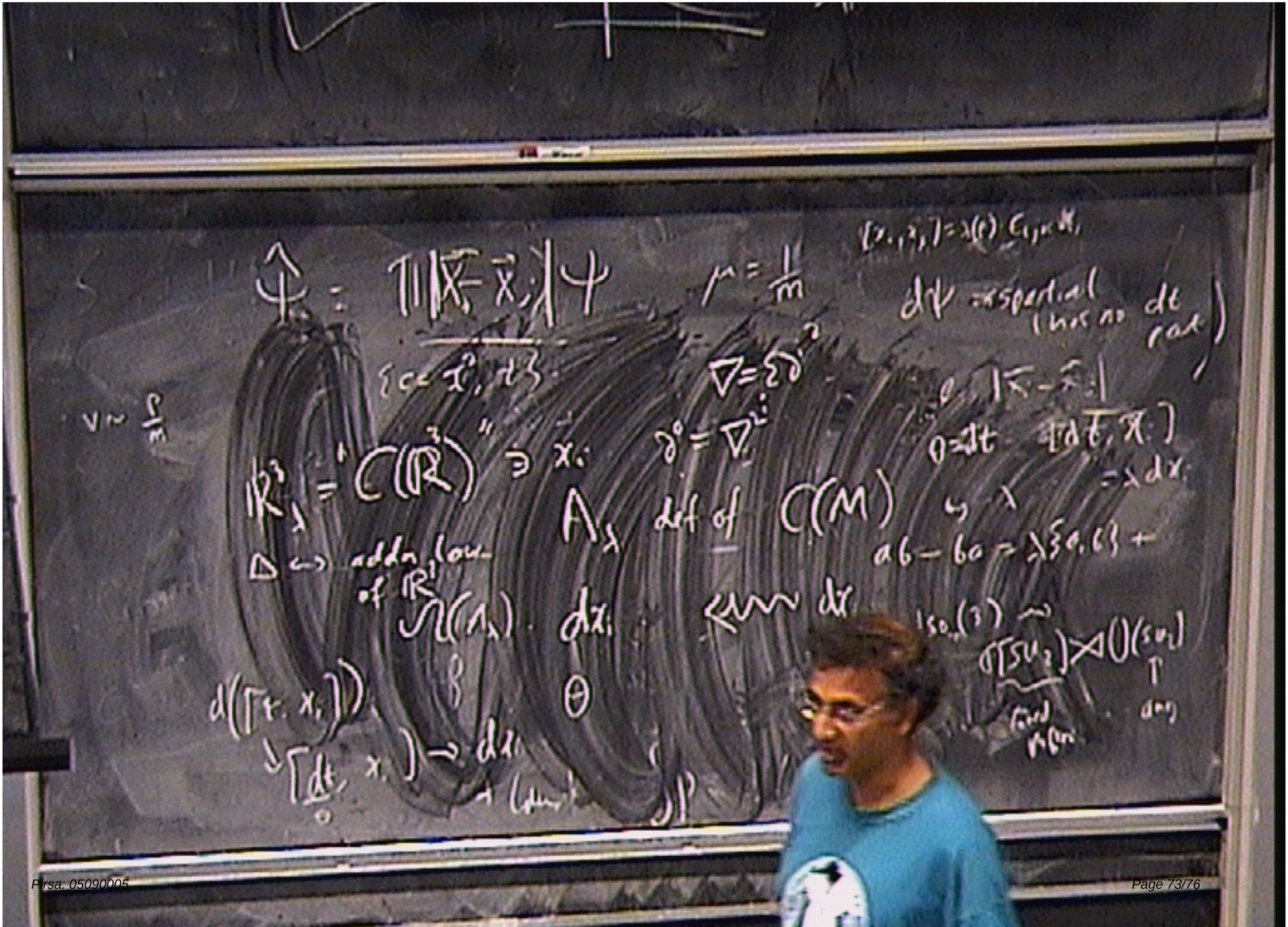
Vave op $\quad \frac{2}{\lambda^2}(\cosh(\lambda p_3) - 1) + \vec{p}^2 e^{\lambda p_3}$ on $\quad e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p_3 x_3}$

+1 case (Ruegg-SM 1994) has poinc. $U(so_{1,3})\triangleright\!\!\blacktriangleleft\mathbb{C}[B_+]$

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j \qquad i.j = 1,2,3$$

$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda\epsilon_{ijk}p^j \otimes M_k \qquad \Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

Vave op $\quad -\frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1) + \vec{p}^2 e^{\lambda p^0}$ on $\quad e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p^0 t}$



Appendix

icrossproduct models $[t, x_i] = \imath\lambda x_i, \quad [x_i, x_j] = 0$

d case (SM 1987) has poinc. $\mathbb{C}[B_+]\blacktriangleright\!\!\triangleleft U(su_2)$

$$[M_i, p_j] = \frac{\imath}{2}\epsilon_{ij}{}^3\left(\frac{1-e^{-2\lambda p_3}}{\lambda} - \lambda\vec{p}^2\right) + \imath\lambda\epsilon_i{}^{k3}p_j p_k \qquad i,j = 1,2\ (t = x_3)$$

$$\Delta M_i = M_i \otimes e^{-\lambda p_3} + \lambda M_3 \otimes p_i + 1 \otimes M_i \qquad \Delta p_i = p_i \otimes e^{-\lambda p_3} + 1 \otimes p_i$$

Vave op $\frac{2}{\lambda^2}(\cosh(\lambda p_3) - 1) + \vec{p}^2 e^{\lambda p_3}$ on $e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p_3 x_3}$

+1 case (Ruegg-SM 1994) has poinc. $U(so_{1,3})\triangleright\!\!\blacktriangleleft\mathbb{C}[B_+]$

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j \qquad i.j = 1,2,3$$

$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda\epsilon_{ijk}p^j \otimes M_k \qquad \Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

Vave op $-\frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1) + \vec{p}^2 e^{\lambda p^0}$ on $e^{\imath\vec{p}\cdot\vec{x}}e^{\imath p^0 t}$

Appendix

$a \in A$ copositive observable means

$$(\phi^* \otimes \phi)\Delta a \geq 0, \quad \forall \phi \in A^* \;\; \epsilon(a) = 1, \quad a^* = Sa$$

E.g. $A = \mathbb{C}[x]$ a has pos f.trans, $a(0) = 1, \quad \bar{a}(x) = a(-x)$

$\{e^{\imath px}\}$ are the pure copositives

dual of a random walk is `creation process'

$$|0\rangle, a_1|0\rangle, a_2a_1|0\rangle, a_3a_2a_1|0\rangle \ldots$$

Coentropy = entropy as a state on A^*; increases

Quantum random walk has step $A \to A \otimes A \to A$

$$a \to (\phi \otimes \mathrm{id})\Delta a = (\phi \otimes \mathrm{id})W(1 \otimes a)W^* = \partial_\phi a$$

(in heisenberg pic.) relates to noncom diffls

Appendix

input-output symmetry (braided Hopf algebras)