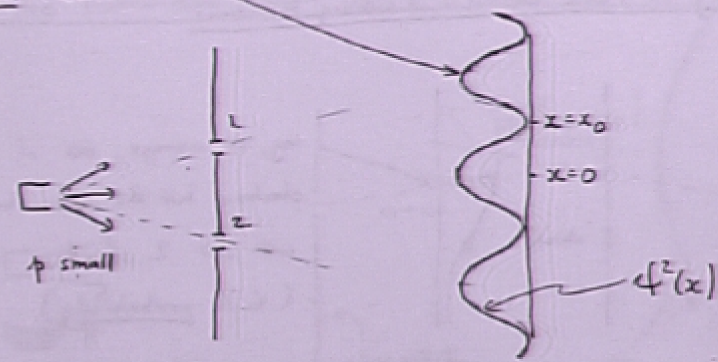
Title: ISSYP 2005 - Lecture C

Date: Aug 11, 2005 10:35 AM

URL: http://pirsa.org/05080005

Abstract:



<u>actual</u> $\psi^2(x)$ is <u>totally different</u> from expected $\psi^2(x)$!!

Most striking is the fact that :

a) With just slit 1 open , 60 electrons hit near $x=x_0$

b) " " " 2 " , 40 " " " "

c) With <u>both</u> slits open, <u>zero</u> electrons hit near $x=x_0$!?

--- shouldn't it be $60+40=100$?

-- something <u>extremely</u> strange is going on.

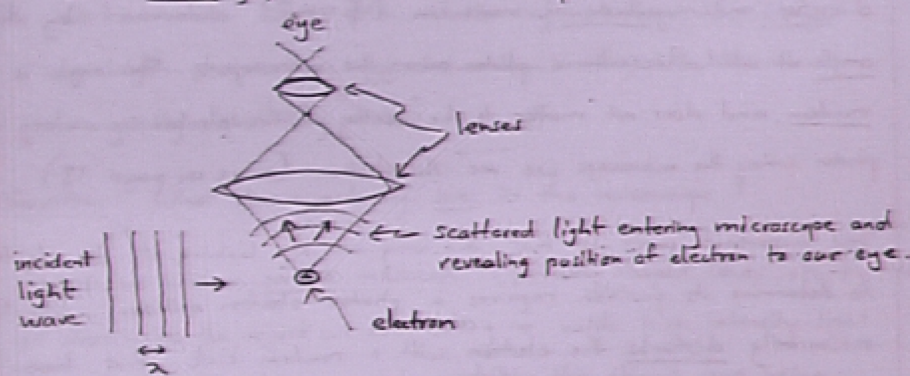
<u>NB</u>: even though electrons are definitely <u>particles</u> (wherever they happen to hit the observation screen, they always arrive as a tiny lump of "something"), careful experiments reveal that the observed $\psi^2(x)$ (<u>probability</u> electron hits at position x) is <u>identical</u> to the $\psi^2(x)$ we found for a <u>water wave</u> (or any other type of wave) <u>interfering with itself</u> !! -- provided we select the right <u>wavelength</u>, λ.

Heisenberg Uncertainty Relation (as a consequence of $\lambda p = h$)

Main idea: In making any measurement one cannot avoid disturbing the "system" one is performing the measurement on.

Example: Let's try to measure the position of an electron as accurately as we possibly can.

To measure its position, we need to look at it (involves shining light on it !)
To do it accurately, let's use a microscope:



The relation $\lambda p = h$ tells us that "every" particle phenomenon (of momentum p) is associated with waves of wavelength $\lambda = h/p$. We saw this previously for electrons of momentum p. Conversely, "every" wave phenomenon (of wavelength λ) is associated with particles of momentum $p = h/\lambda$. Thus, the incident light wave above (of wavelength λ) can be viewed as a shower of particles (called photons), each with momentum $p = h/\lambda$. The greater the intensity of the light illuminating the electron, the more particles in the shower

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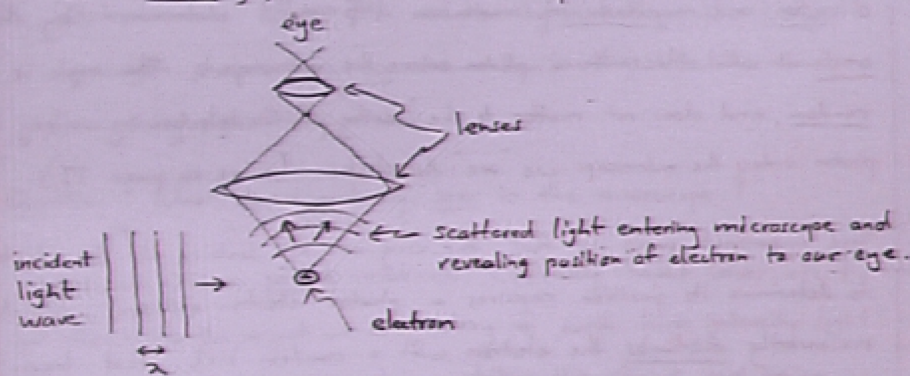

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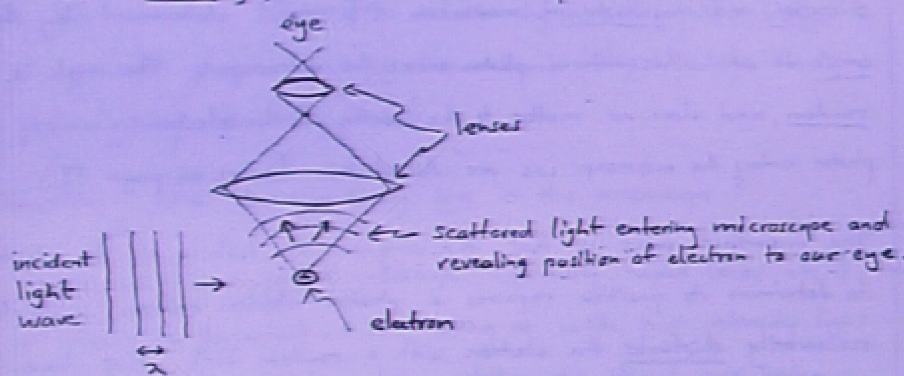

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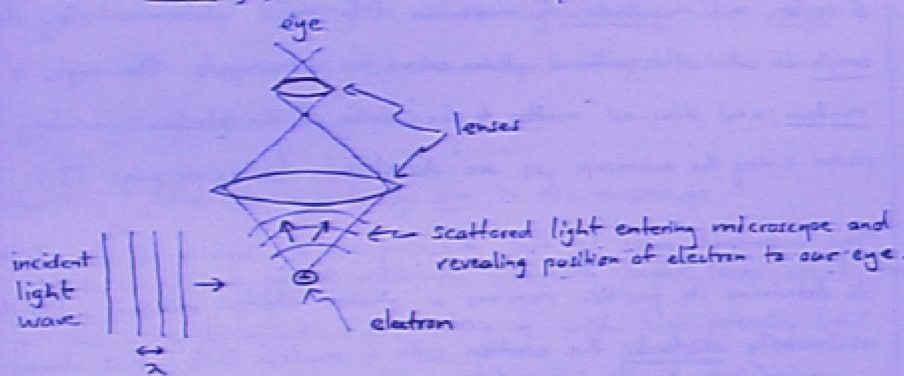

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In order to "see" the electron, the incident photon must "collide" with the electron and be scattered into the microscope. As with billiard balls, this collision causes the electron to <u>recoil</u>, in this case with a direction and magnitude of momentum ($\vec{p} = m\vec{v}$) determined by the <u>angle</u> at which the scattered photon enters the microscope. This angle is <u>random</u>, and does not matter to the "seeing" of the electron: as long as the photon enters the microscope we "see" the electron (more on page 13)

The important point is that the very act of "looking at" the electron to determine its position requires a photon-electron "collision", which necessarily <u>disturbs</u> the electron with a random "kick" that leaves the recoiling electron with a random momentum about as big as the momentum of the incident photon: h/λ.

We write: $\Delta p \equiv$ <u>uncertainty</u> in momentum of recoiling electron
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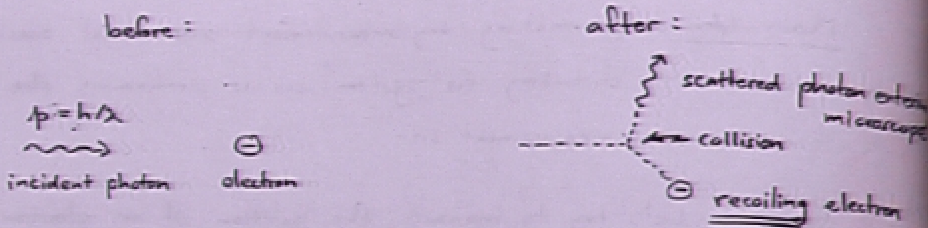

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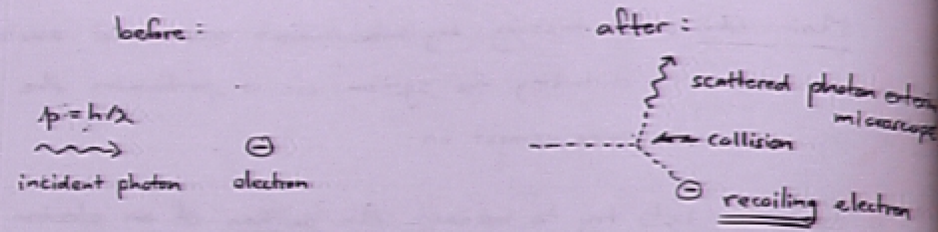

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While this disturbance can never be eliminated completely, there are two ways to minimize it :

(1) The greater the intensity of light illuminating the electron the more photons striking the electron, each imparting a random "kick", and thus the greater the uncertainty in the final electron momentum. Therefore, let us reduce the intensity of light so low that we are using only a single scattered photon to observe the electron.

Then $\Delta p \approx h/\lambda$ ← measure of minimum disturbance of electron

(2) We can make $\Delta p \approx h/\lambda$ smaller by increasing λ. Longer wavelength photons have less momentum : they strike the electron more gently, disturbing it less.

Question: What do we actually see in the microscope ?

Suppose we have a perfect microscope (perfectly shaped lenses, etc.), and we illuminate the object we are looking at with high intensity light of wavelength λ (many photons striking the object and being scattered into the microscope). What we see is a clear, magnified image of the object.

Now let us make the object smaller and smaller until its size is much less than the wavelength of light we are using. For example, let us look at an electron, which is essentially a "point particle". Will it appear as an infinitely small "dot", whose position we can accurately measure ? NO !

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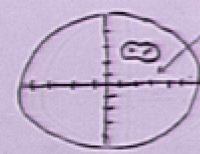
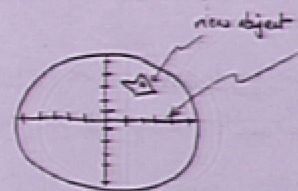
View in Microscope:

- object size $\gg \lambda$:
  ↑
  "much greater than"



measure size and location of object using calibrated grid inside eyepiece.

- now decrease the size of the object and correspondingly increase the magnification, so that the apparent size of our new object is about the same as in the previous case:



same grid lines; but because of the higher magnification, the space between tick marks indicates a smaller distance.

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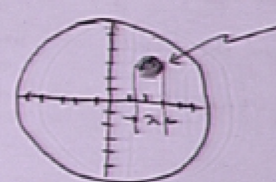

Regardless of the shape and size of the object, we now see just a fuzzy disk (called an "Airy disk"), whose diameter according to the calibrated grid is about equal to the wavelength of light being used — no details of the object are apparent.

This phenomenon has nothing to do with imperfections in the construction of the microscope — it is a fundamental consequence of the wave nature of light, in particular, the phenomenon of wave diffraction. This is called the diffraction limit of the microscope.

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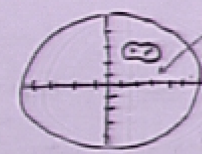

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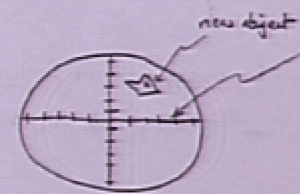

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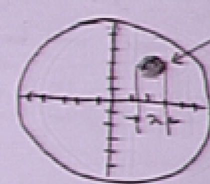

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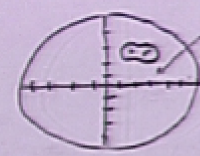

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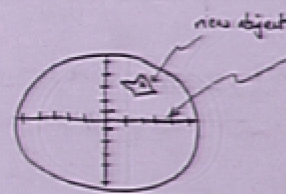

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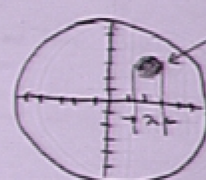

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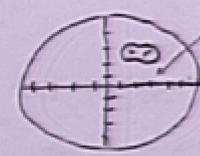

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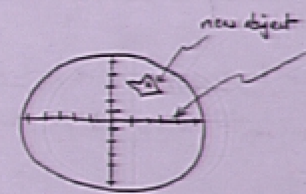

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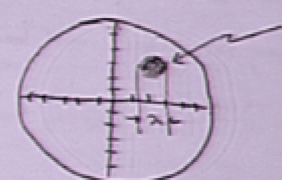

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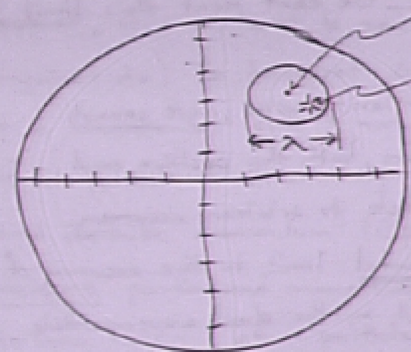
assume e^- is fixed in position

- Now, if we illuminate our electron with high intensity light, we would see an Airy disk about the size of the wavelength of light, the centre of which would give a good estimate of the position of the electron.

Problem: To disturb the electron as little as possible with our observation, we already agreed to use low intensity light — in fact, to use a single photon!

Using a single photon, what will we see?

Answer: a tiny flash of light coming from a random location somewhere inside the Airy disk:



Note: the precise location of the flash of light inside the Airy disk is related to the angle at which the scattered photon enters the microscope (see page 10).

--- in any case, both are random.

Conclusion: By using many photons, and averaging the locations of the flashes, we can determine the position of the electron (centre of Airy disk) quite accurately. But with only one photon flash, our uncertainty in the position of the electron is $\Delta x \approx \lambda$

(we cannot assume the single flash is coming from the centre of ...

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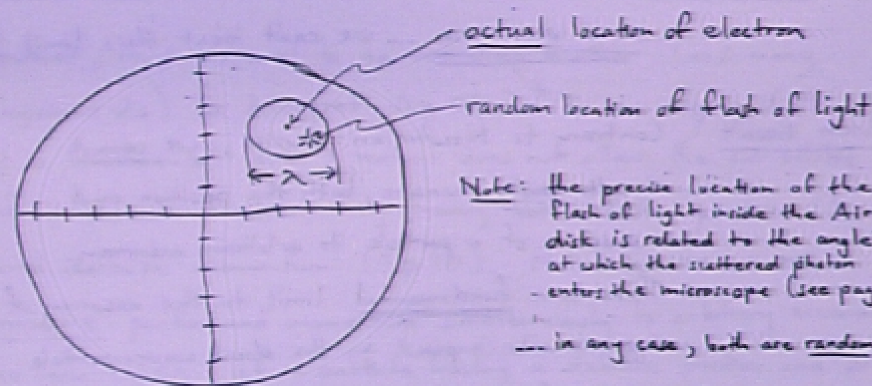

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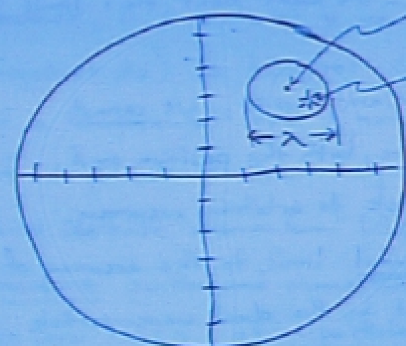

<u>actual</u> location of electron

random location of flash of light

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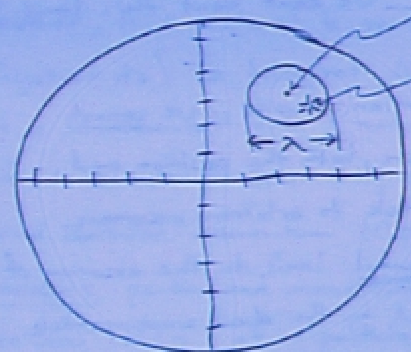

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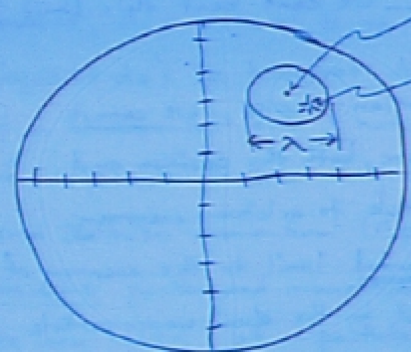

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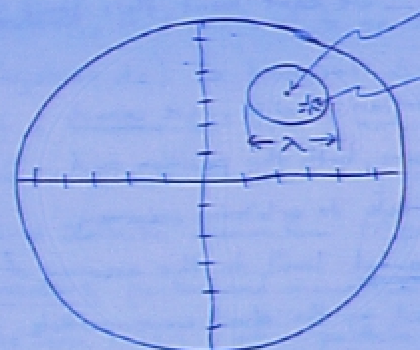

Note: the precise location of the flash of light inside the Airy disk is related to the angle at which the scattered photon enters the microscope (see page 10).

--- in any case, both are random.

Conclusion: By using many photons, and averaging the locations of the flashes, we can determine the position of the electron (centre of Airy disk) quite accurately. But with only one photon flash, our uncertainty in the position of the electron is $\Delta x \approx \lambda$

(we cannot assume the single flash is coming from the centre of ...

Summary: We agreed to use one photon to observe the position of the electron in order to minimize the disturbance of the electron's momentum. Got: $\Delta p \approx h/\lambda$.
However, using just one photon, our uncertainty in the position of the electron is $\Delta x \approx \lambda$

Multiply: $\Delta x \, \Delta p \approx \lambda \cdot \frac{h}{\lambda} = h$ (a universal constant)

using shorter wavelength light will allow us to pin down the location of the electron more accurately

but shorter wavelength photons have greater momentum (h/λ), kick the electron harder, and thus introduce greater uncertainty in the electron's momentum

— and vice versa — we can't beat this limit !!

Main Result: Contrary to Newtonian physics, we cannot simultaneously measure both the position and momentum of a particle to arbitrary accuracy. There is a fundamental limit to the accuracy of measurements imposed by the dual wave-particle aspect of all natural phenomena ($\lambda p = h$), given by the Heisenberg Uncertainty relation:

$$\boxed{\Delta x \, \Delta p \geq \frac{\hbar}{2}}$$

Mathematically, constant on right is determined to be precisely $\hbar/2$, where $\hbar = h/2\pi$

"greater than or equal to" i.e. $\frac{\hbar}{2}$ or worse

pronounced "h-bar"

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22-144 200 SHEETS

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We "derived" the relation $\Delta x \Delta p \geq \hbar/2$ by considering the "practical example" of measuring the position of an electron with the least possible disturbance of the electron's momentum. Thus, one may conclude that $\Delta x \Delta p \geq \hbar/2$ is a statement about a fundamental limit on the accuracy of measurements.

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What would this mean?

Answer: a complete breakdown of the Newtonian world view at the scale of atoms and subatomic particles!

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- a.k.a. quantum secret codes, 1984
- Certain people will pay a lot of money to keep particular things secret.
- Eg. banks, the military, computer industry



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- Eg: buying things over the internet and https:// encryption
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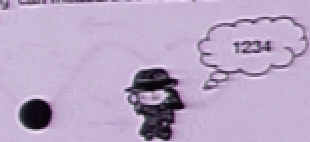
- Let us say you want to hide your PIN or PIN number for your bank account.
- Assume it is 1234
- Take an electron and, secretly, 'push it' so that it moves with a momentum of 1234 kg ms^{-1} (or 1.234 kg ms^{-1} or 1.234×10^{-25} kg ms^{-1} etc.)

p=1234 kg ms^{-1}

- Assume *Newtonian physics* is true for the moment
- Someone 'snooping' can measure both the position and the momentum and thus learn the PIN



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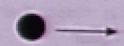

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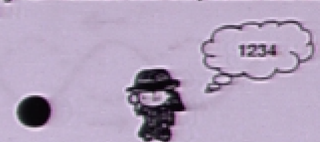
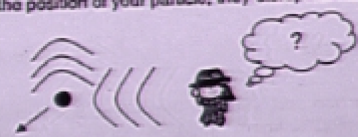


In quantum mechanics

- Consider someone trying to snoop and discover your PIN
- If they measure the position of your particle, they disrupt its momentum



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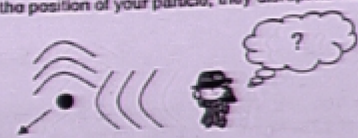

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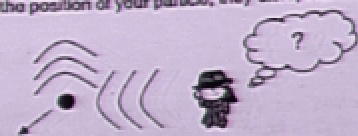

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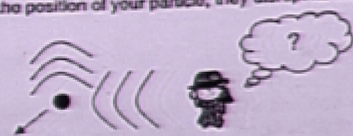

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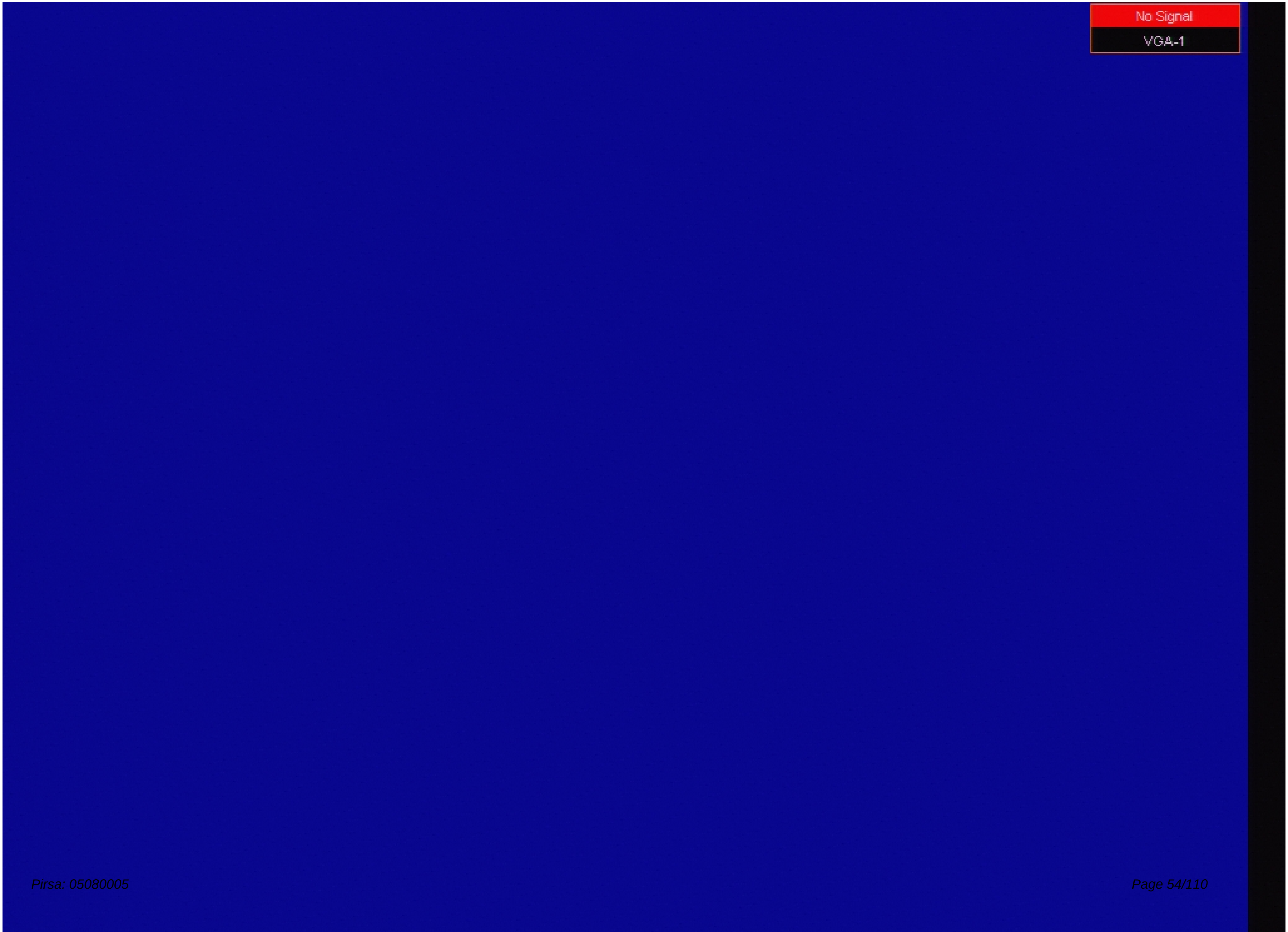
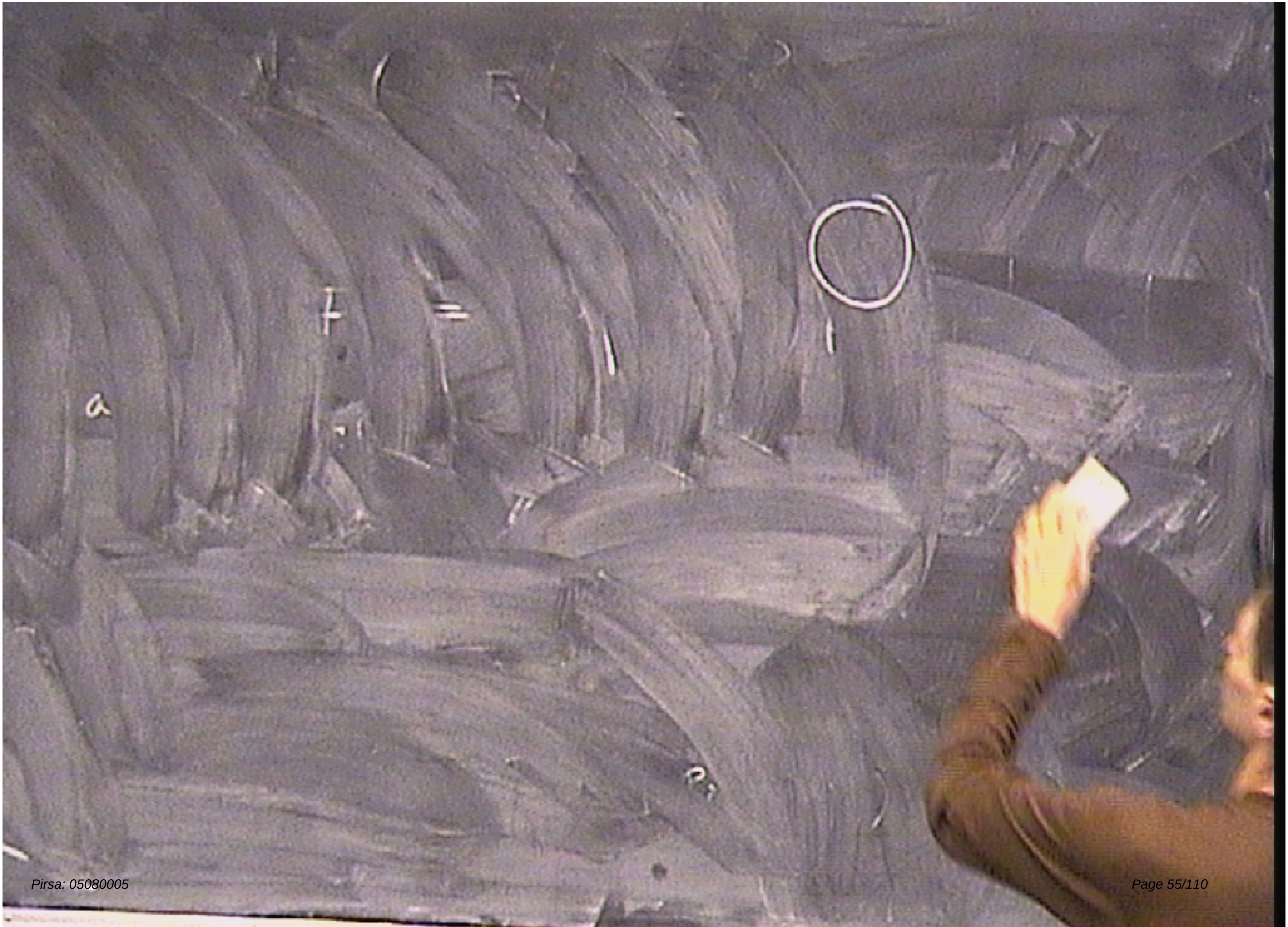
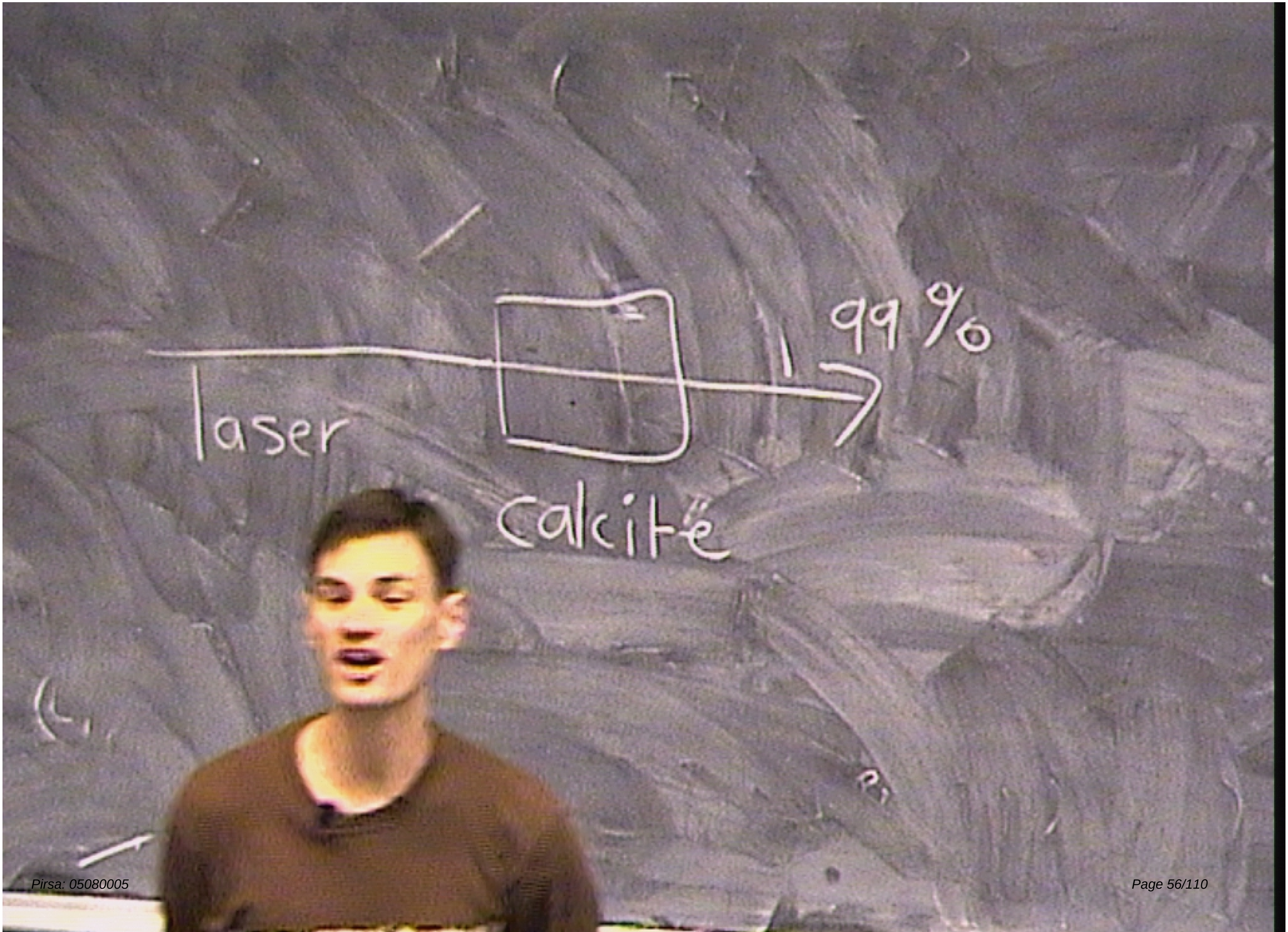

No Signal
VGA-1

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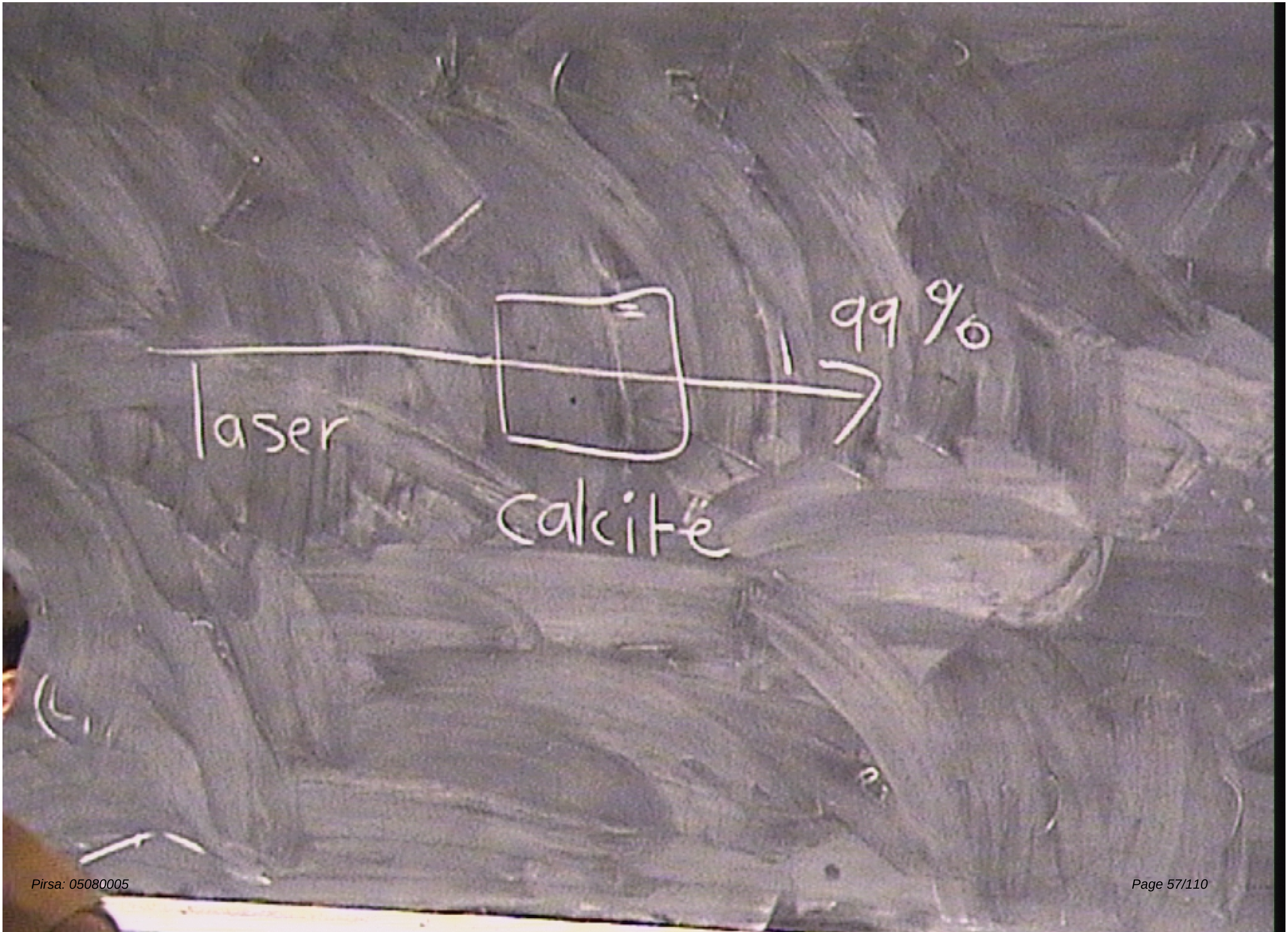
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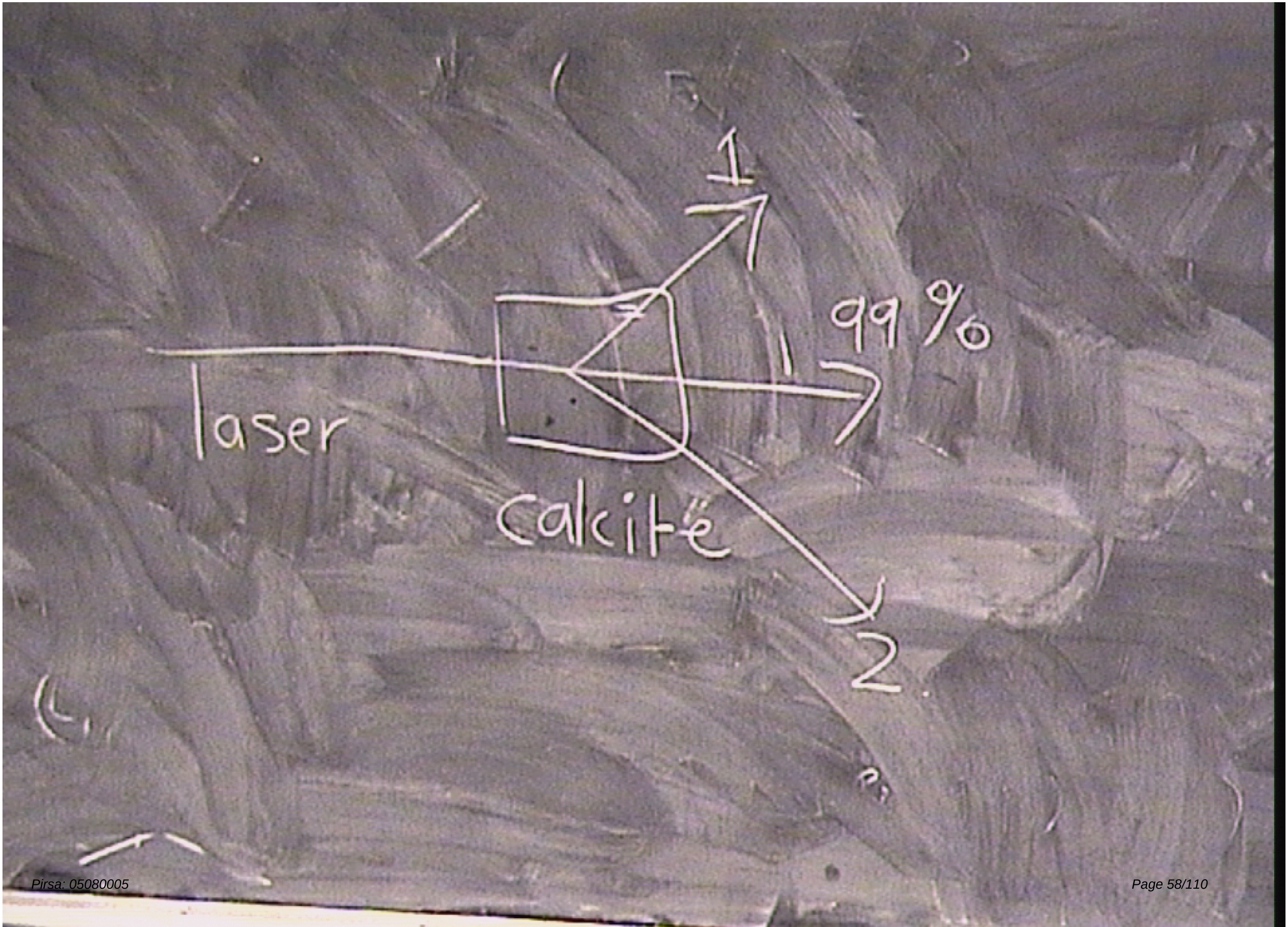
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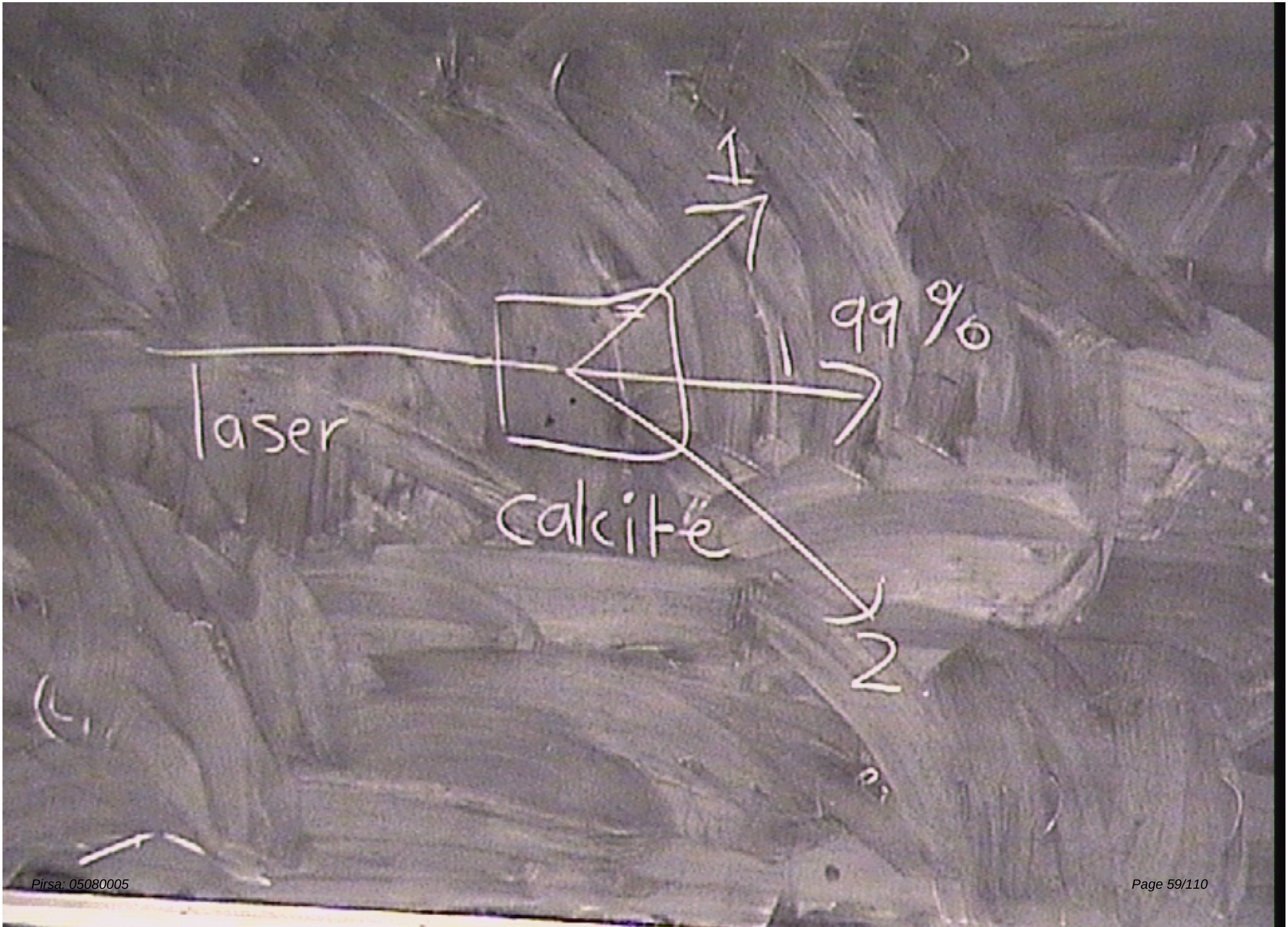
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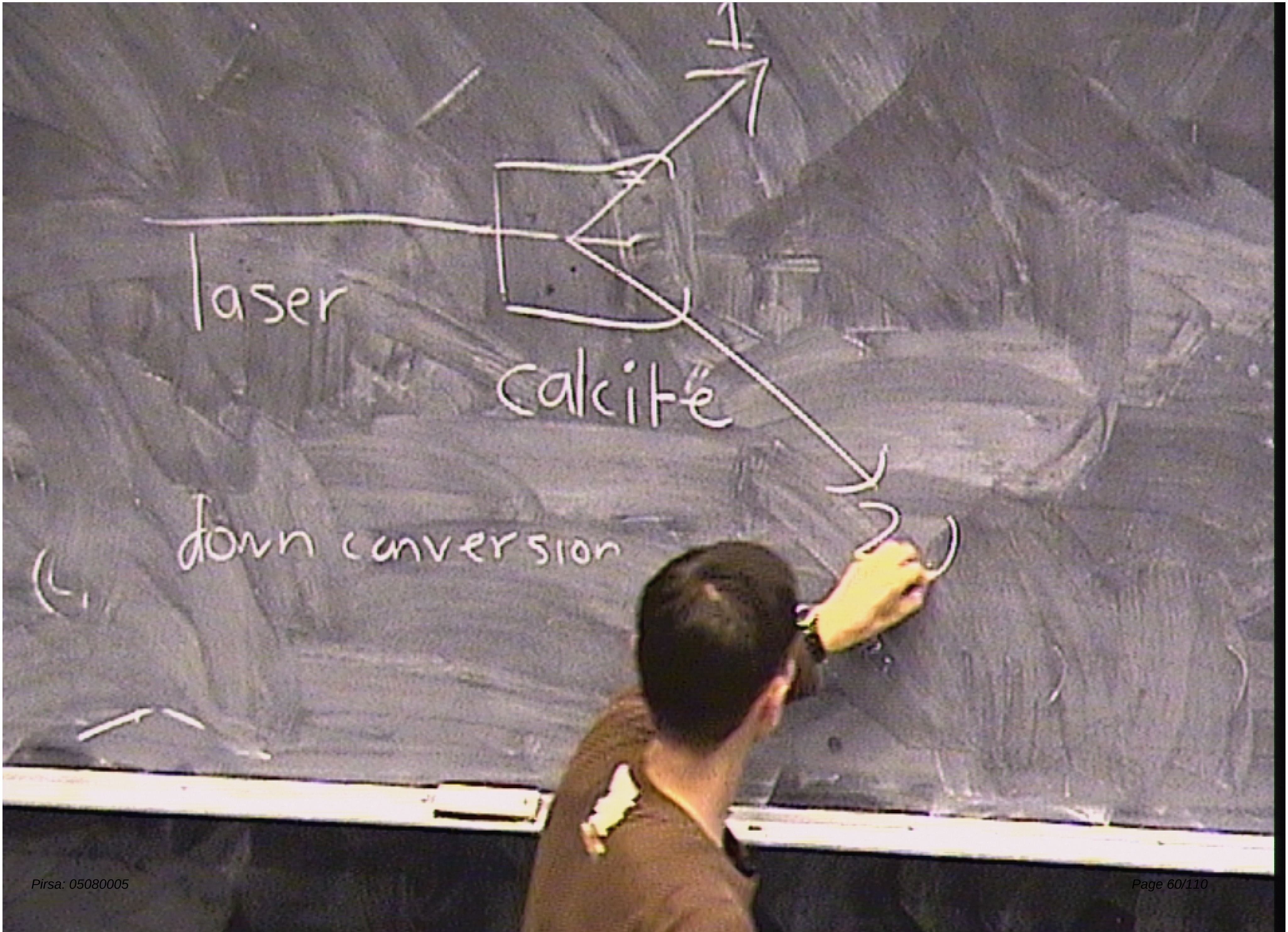
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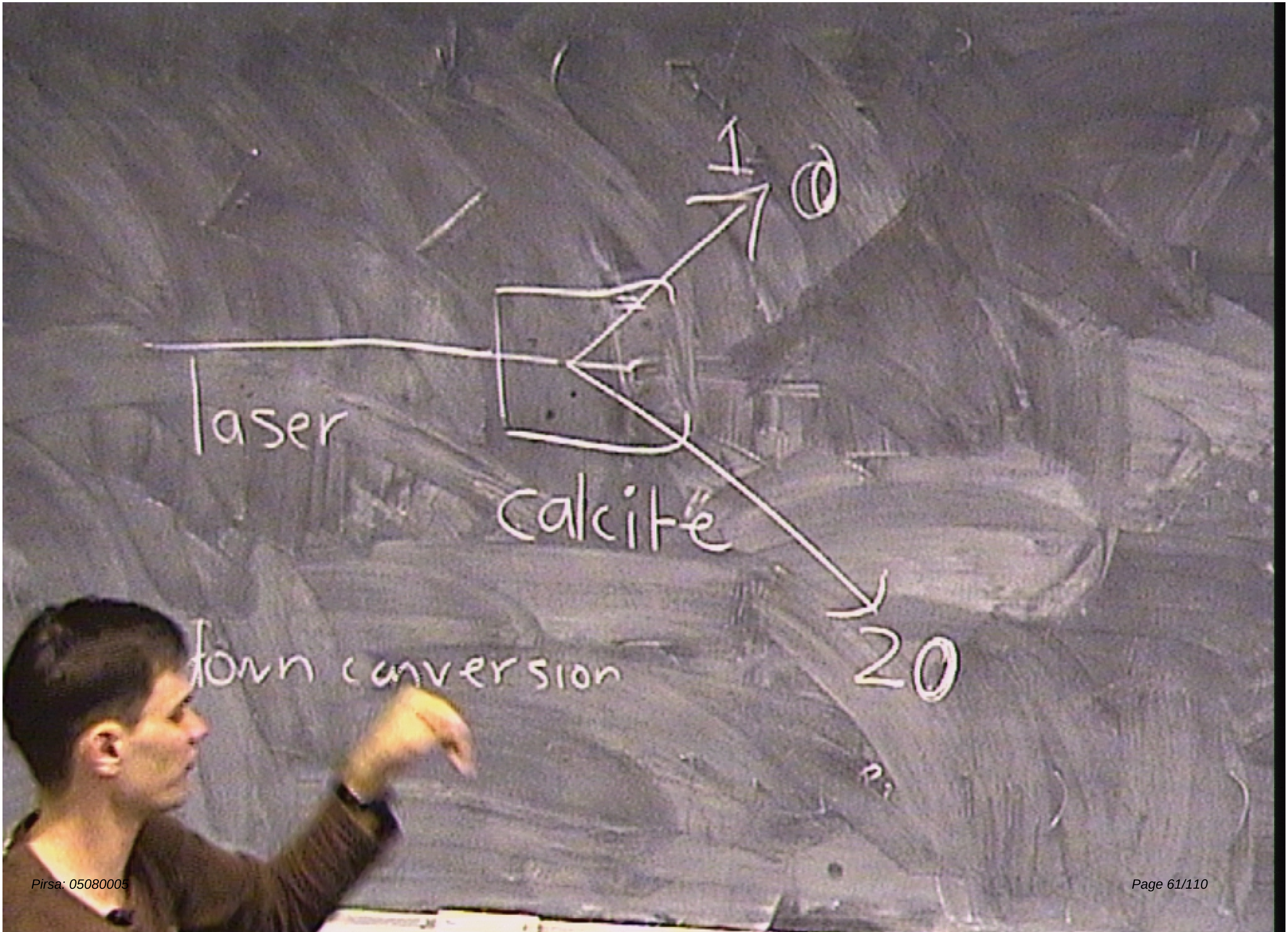
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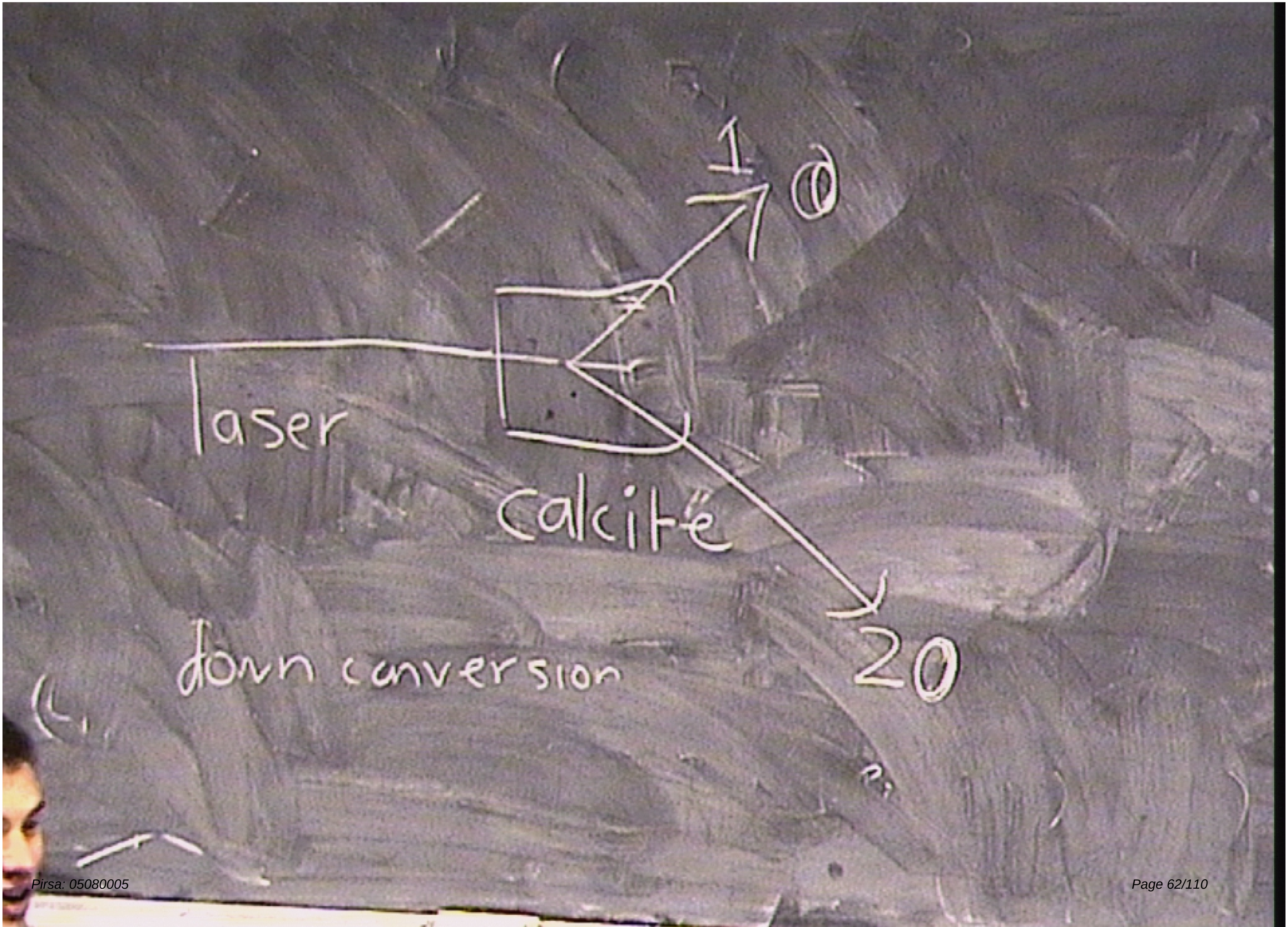
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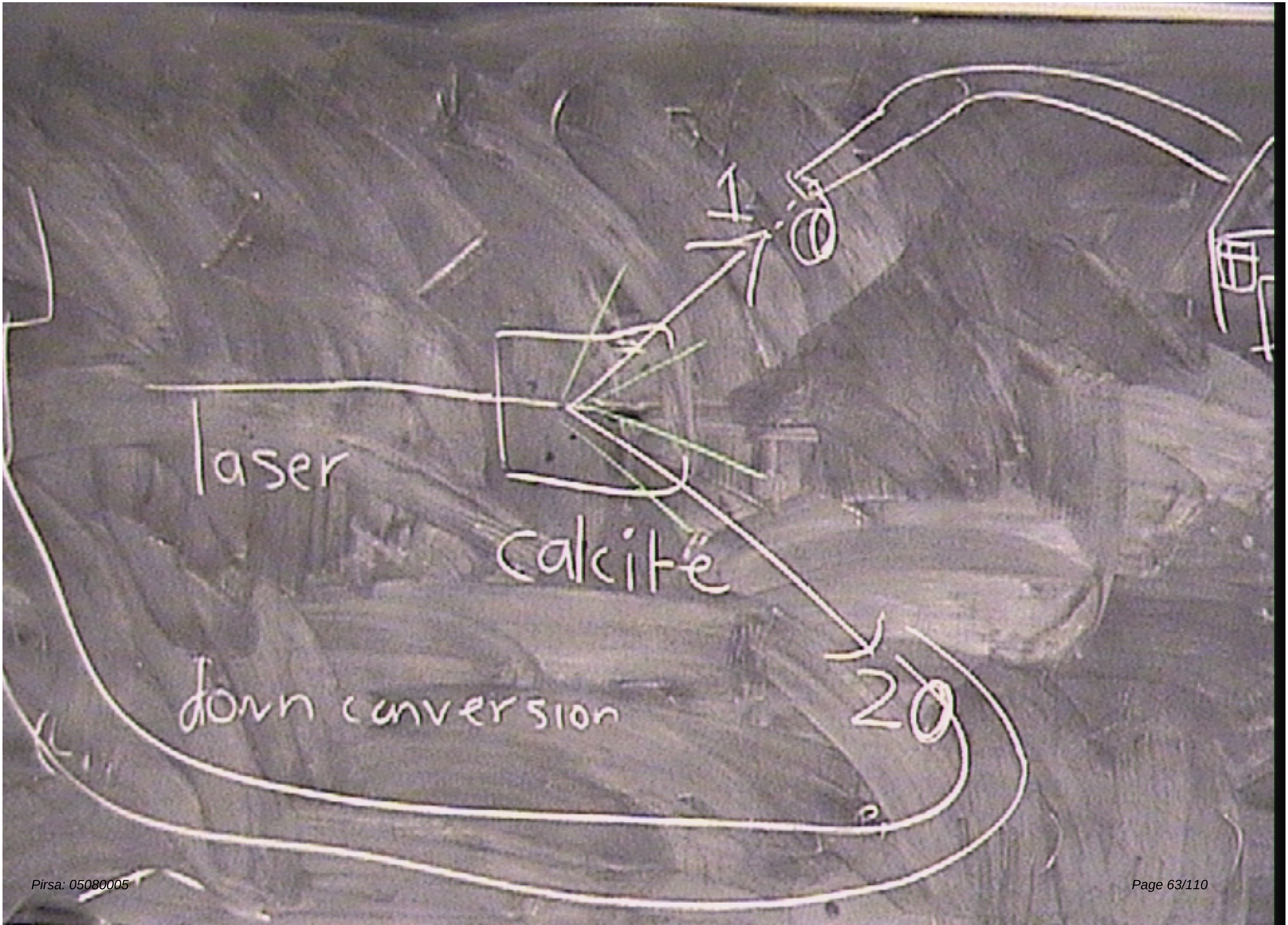

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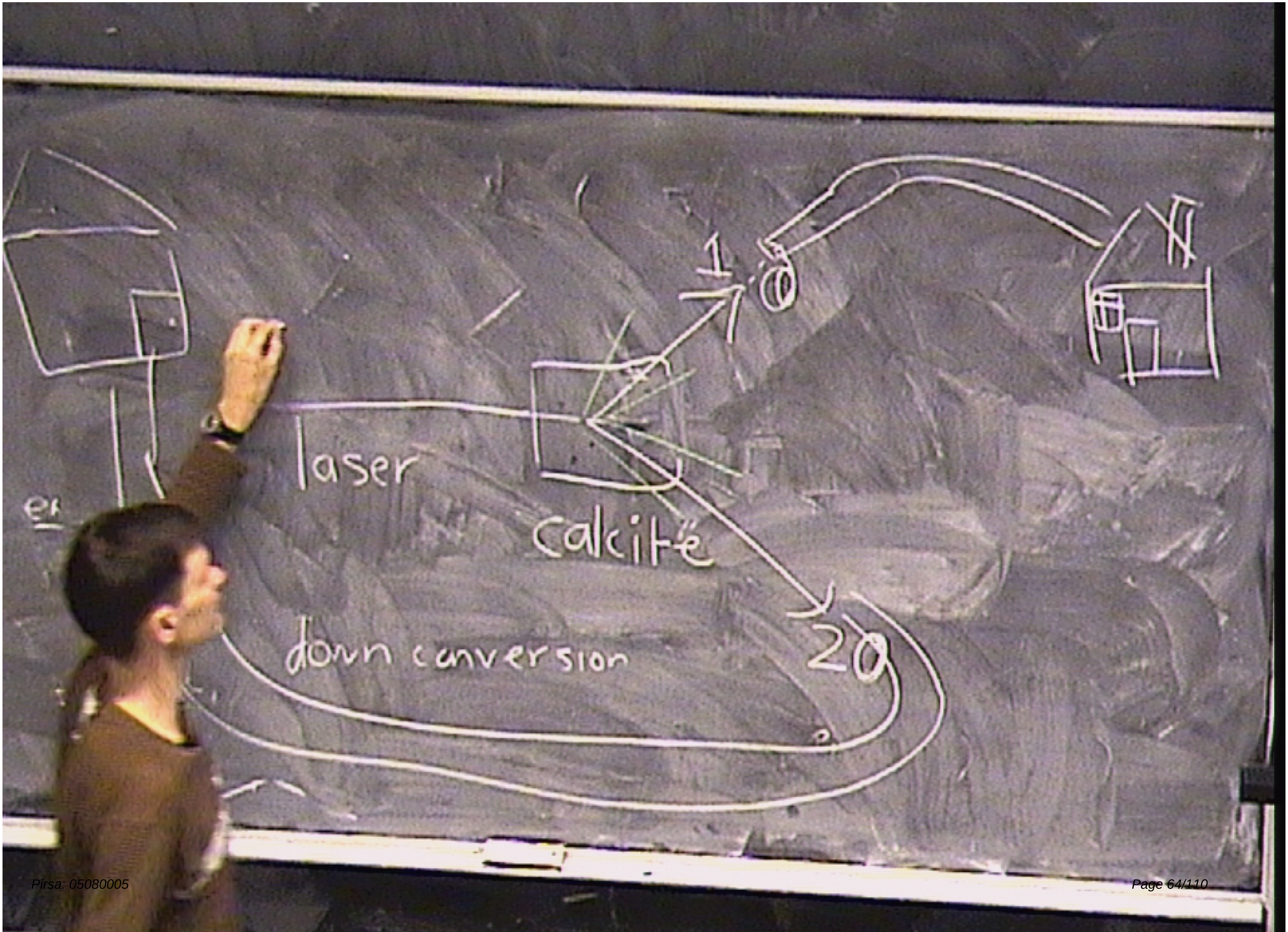
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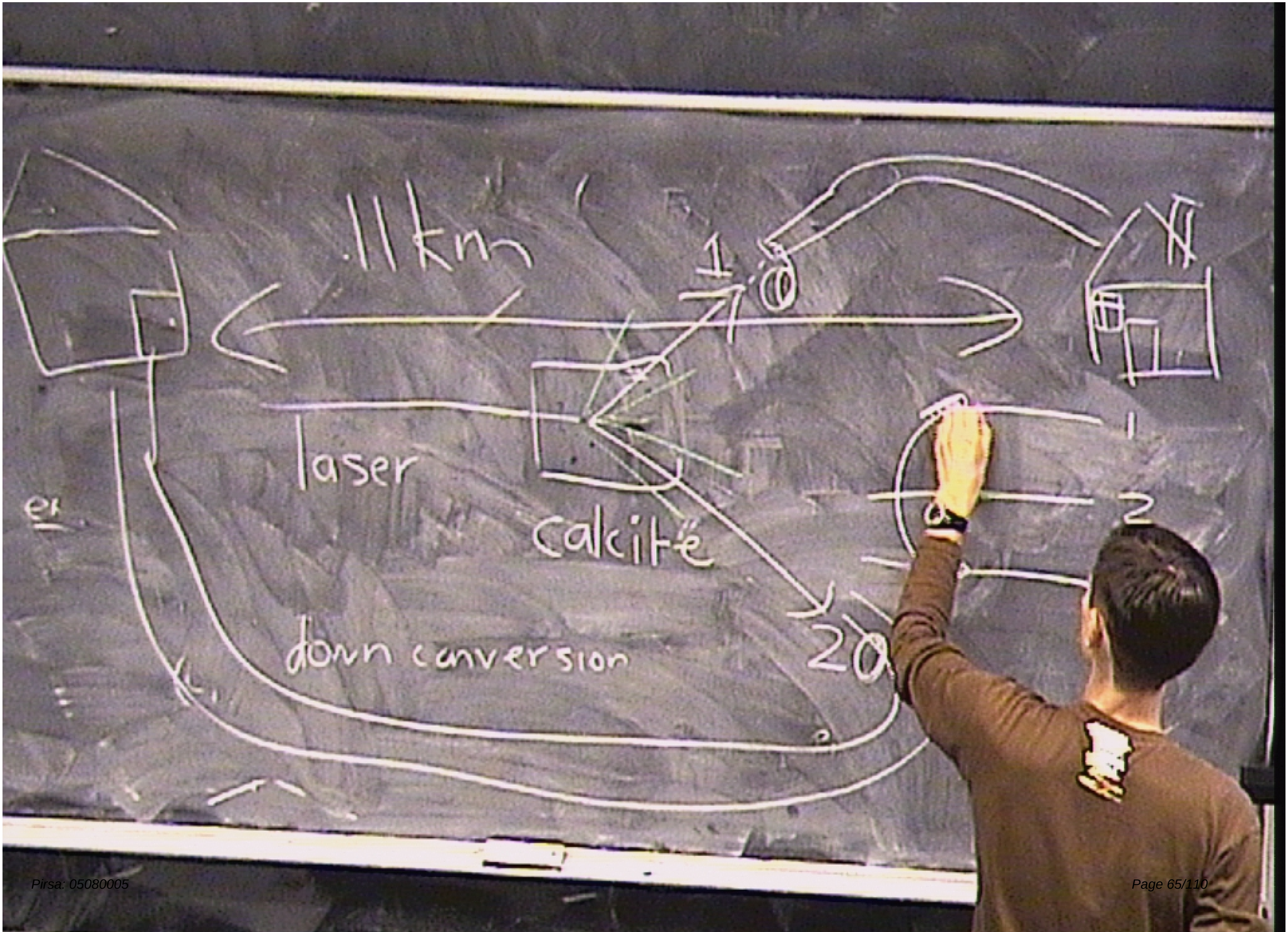
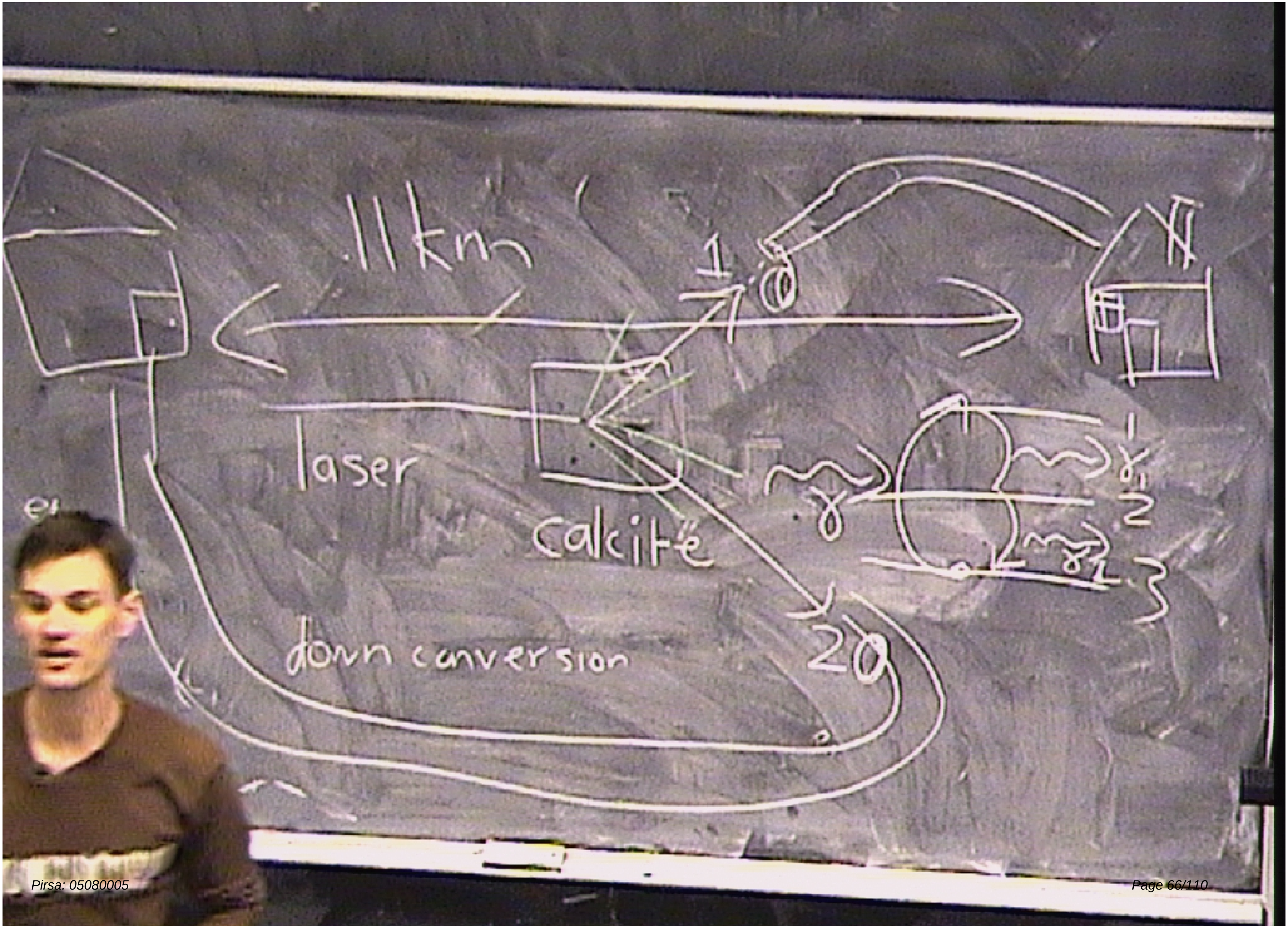
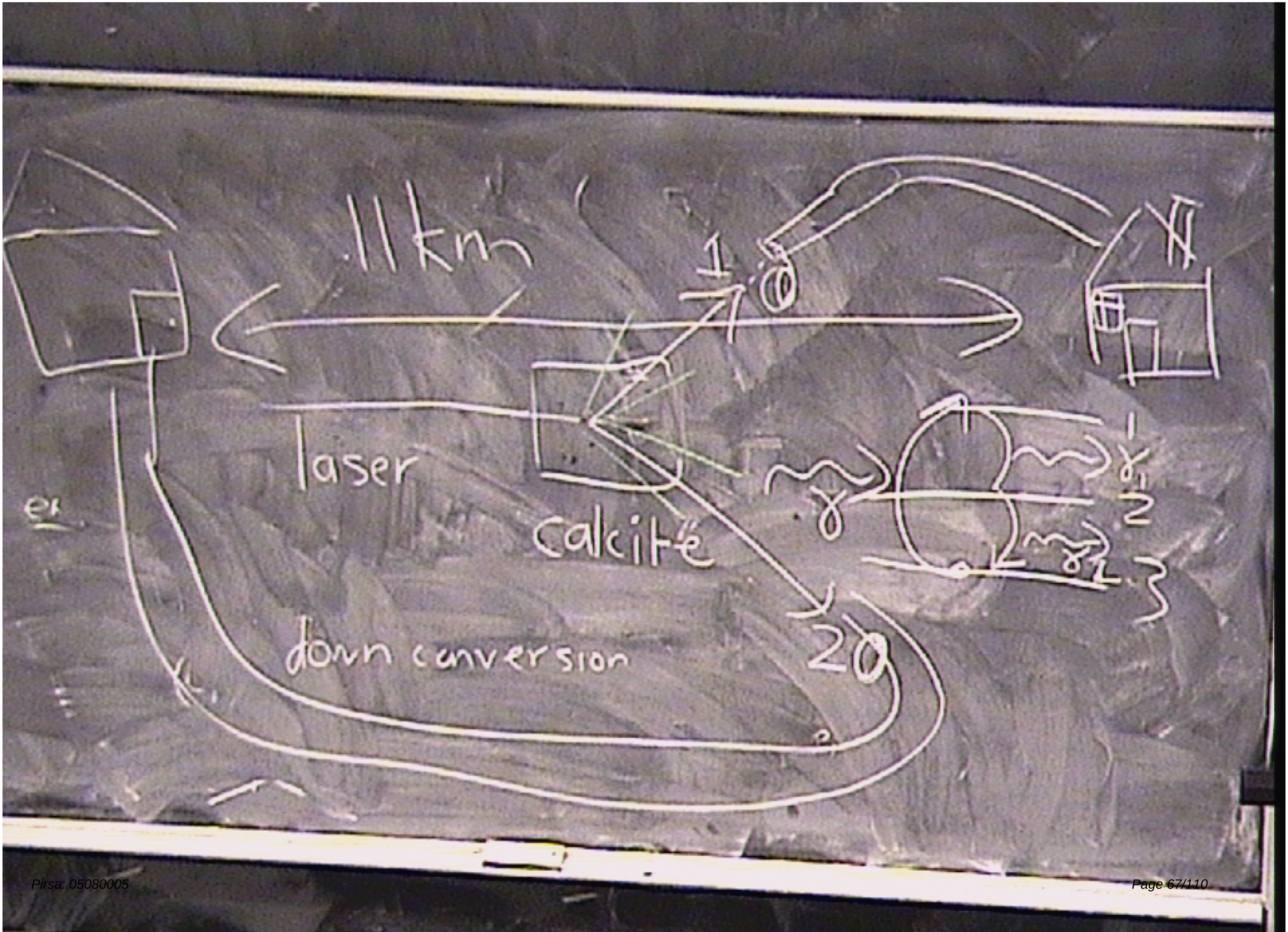
down conversion
2


11km
1
laser
calcite
2
down conversion
1
2


11 km
1
2
3
1
2
laser
calcite
down conversion


11km
laser
calcite
down conversion
1
2
3
10
20

$\epsilon_{123} = +1$ (2 3 1) even

$\epsilon_{223} = 0$

$\{L_a, L_b\} = \sum_c \epsilon_{abc} L_c$

exp(...) $\vec{q} = \ldots = \exp(a \underline{J_i}) \vec{q}$

$\hbar = \frac{h}{2\pi}$

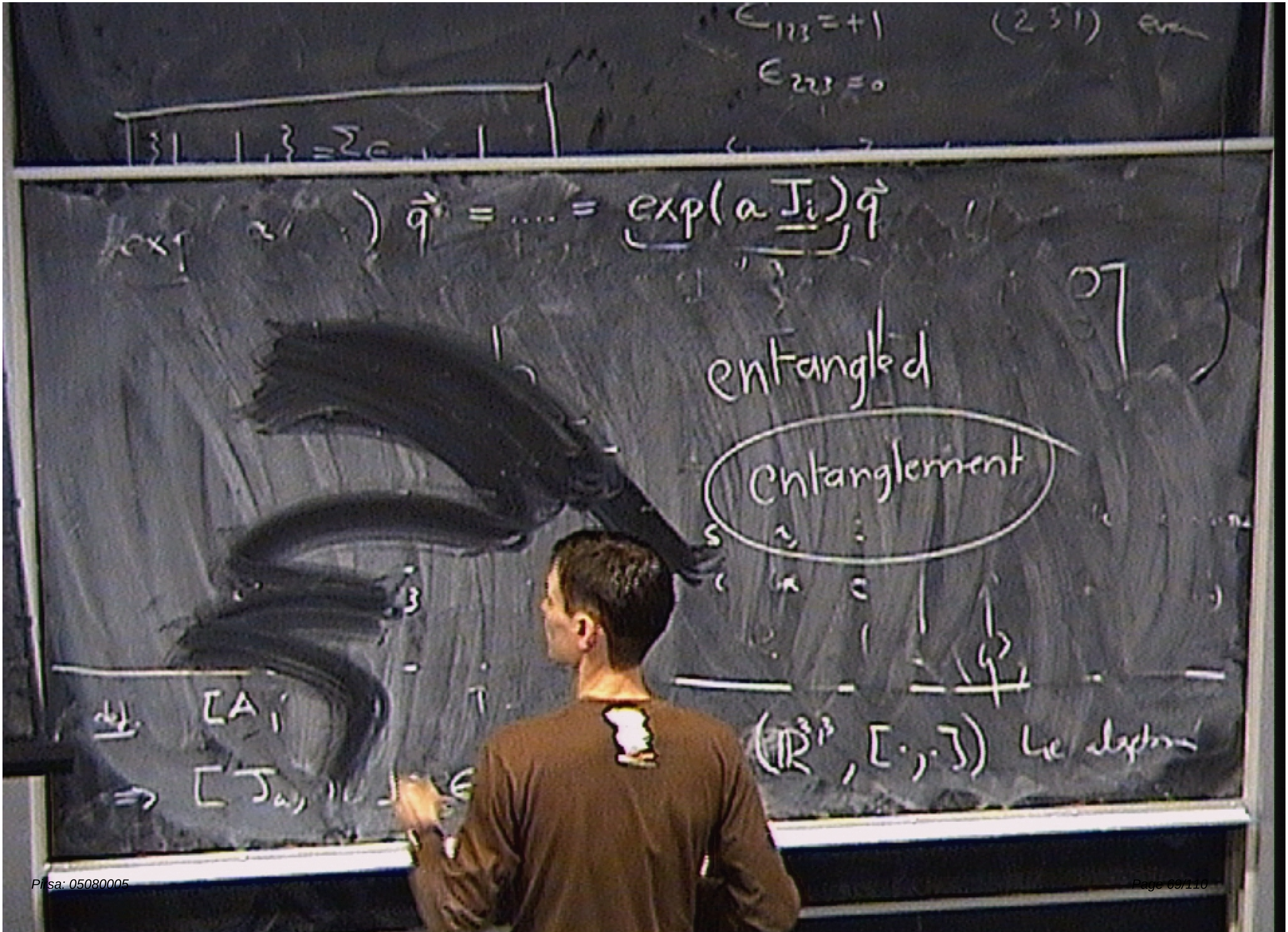
entangled

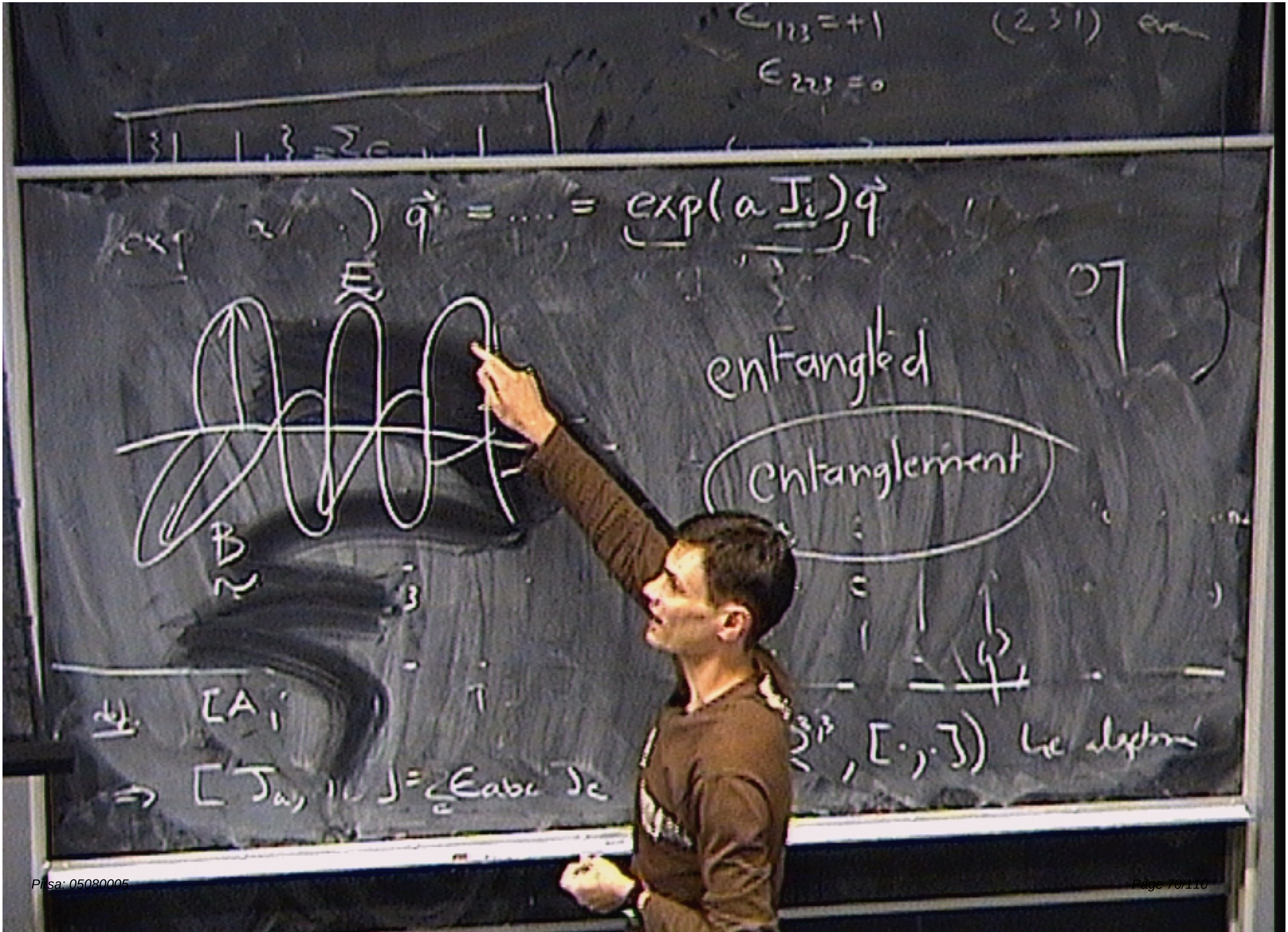
entanglement

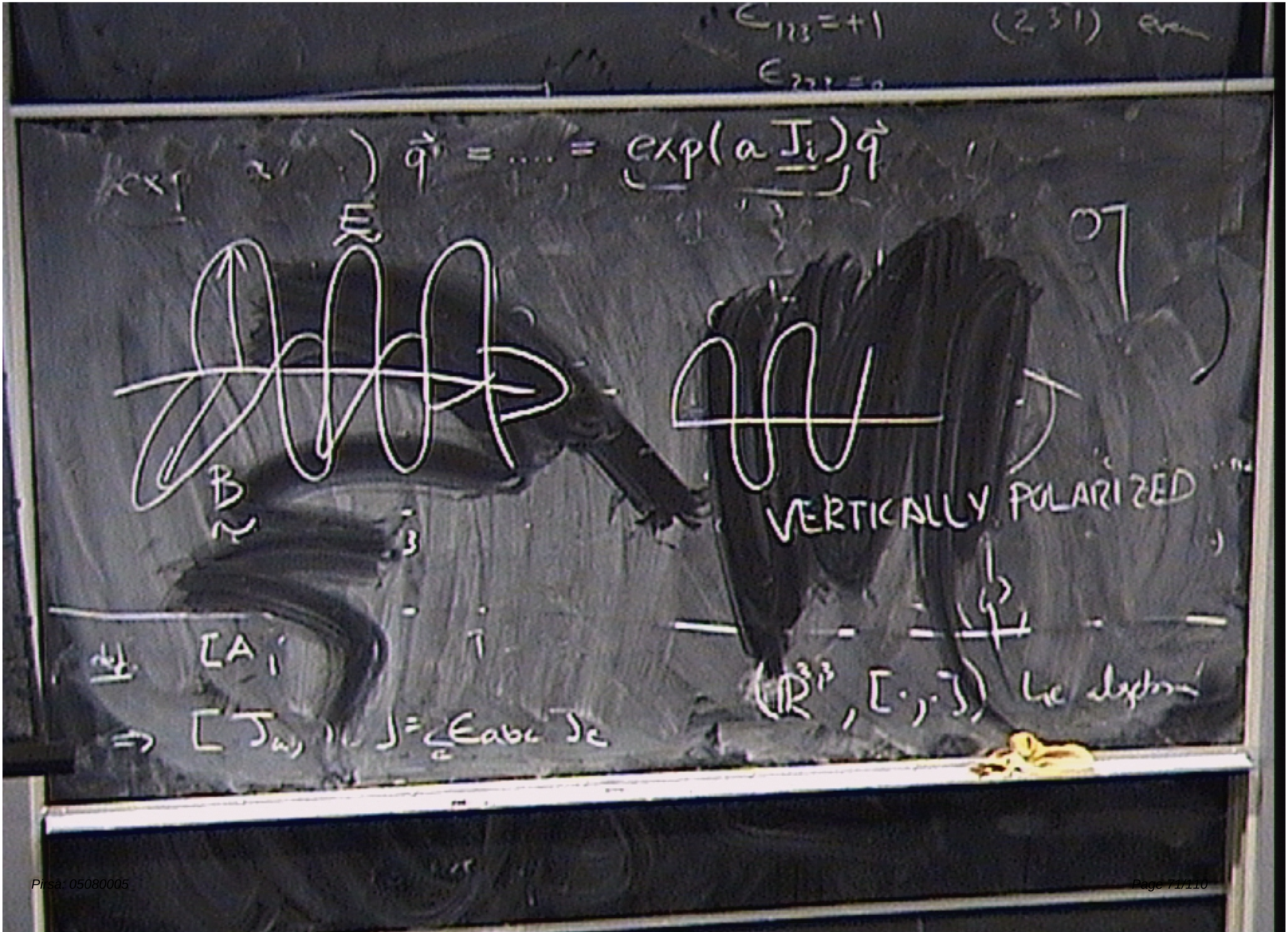
$(\mathbb{R}^{3}, [\cdot,\cdot])$ Lie algebra

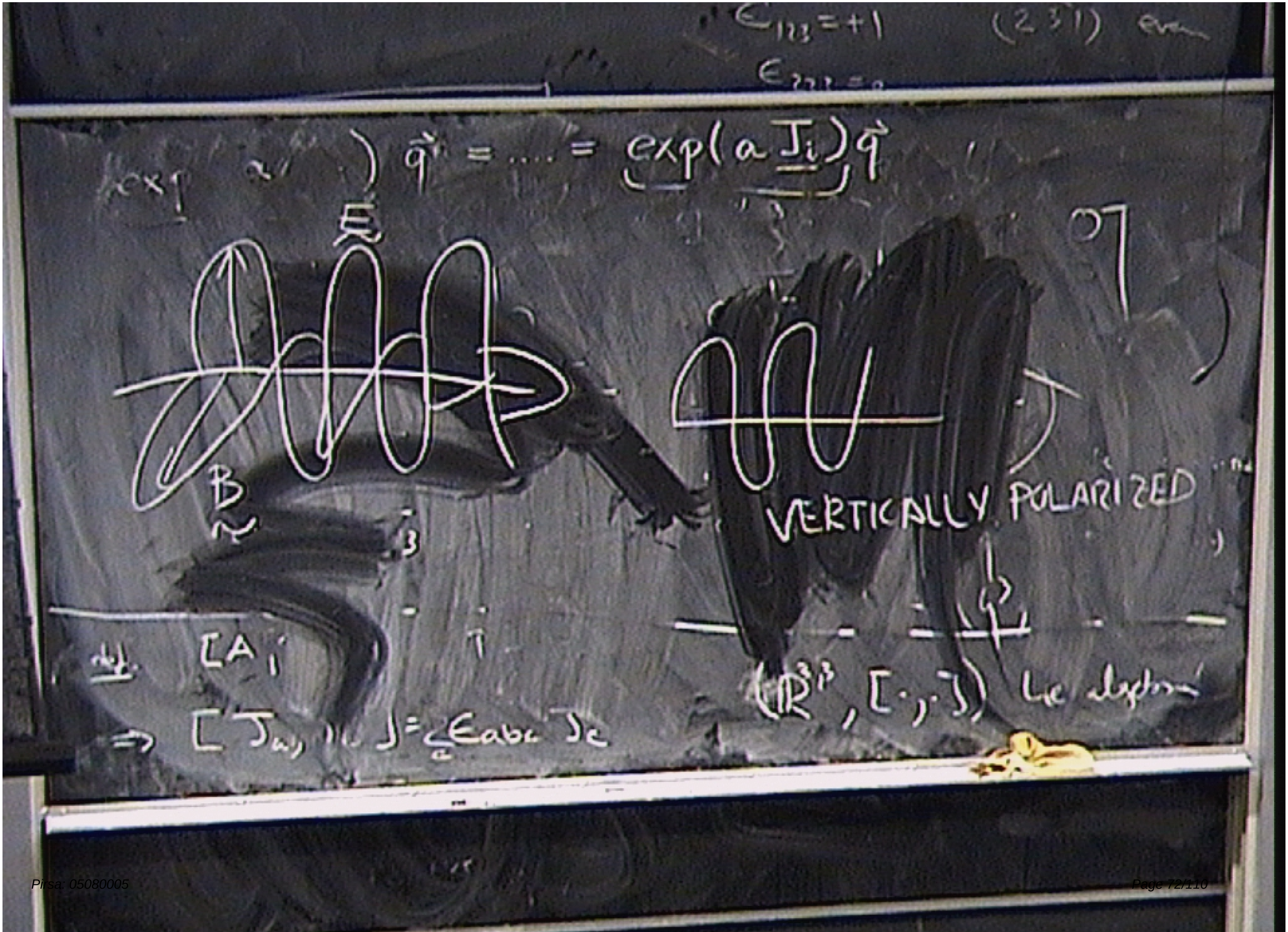
def. [A, ...

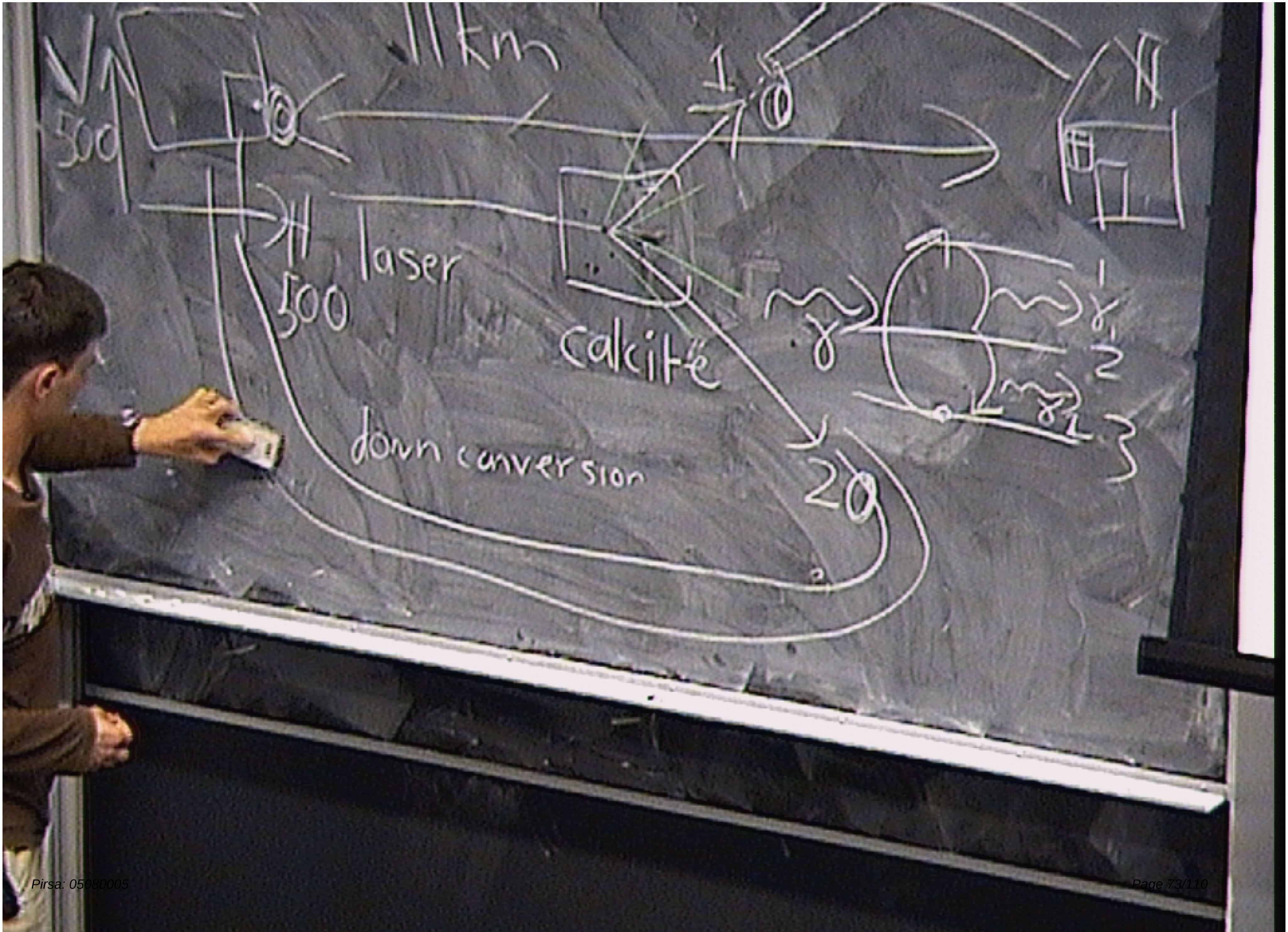
$\Rightarrow [J_a, J_b] = \sum_c \epsilon_{abc} J_c$


ε123 = +1
(2 3 1) even
ε223 = 0
= exp(a Ji) q
entangled
entanglement

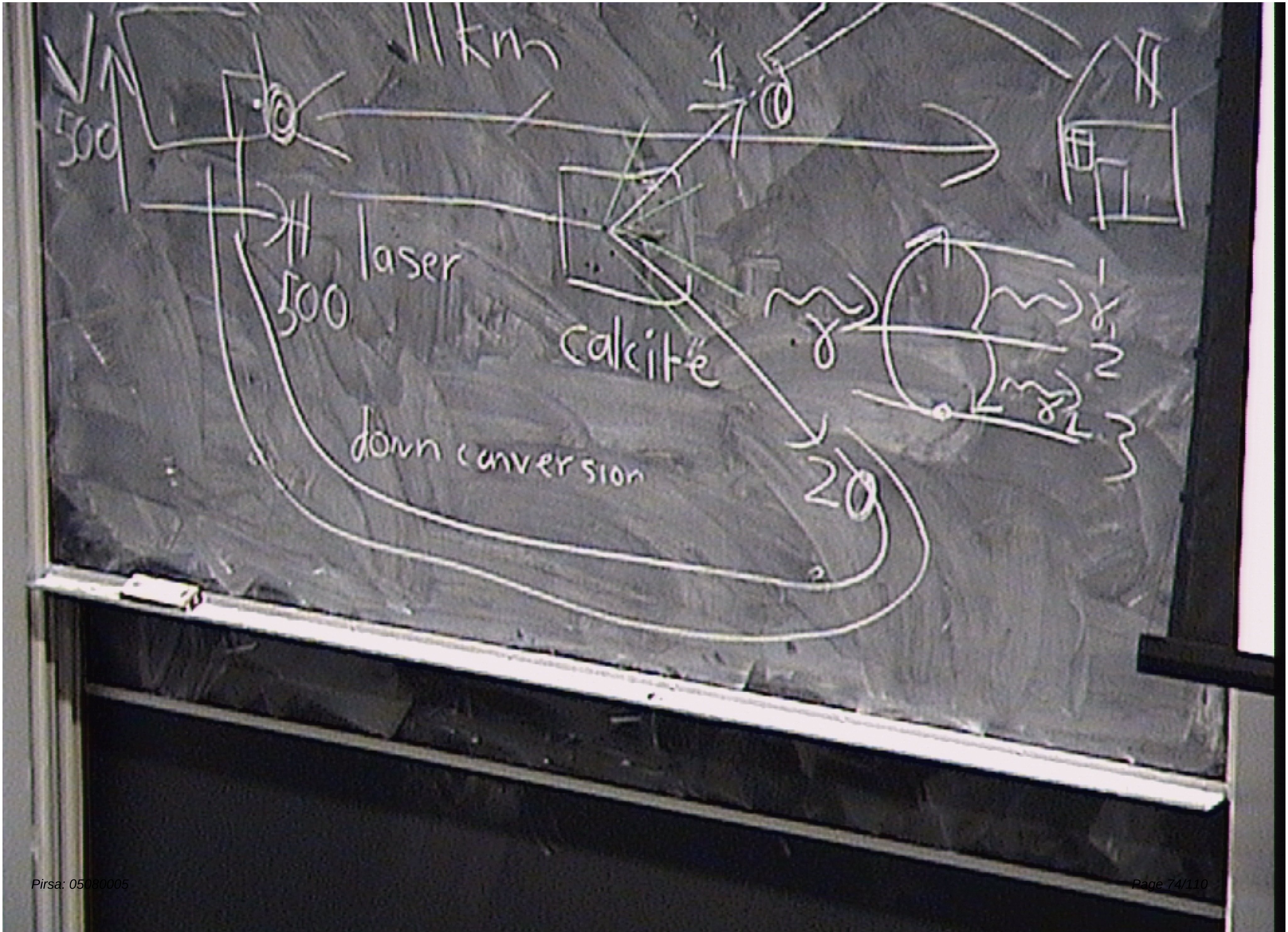

ε123=+1
(2 3 1) even
ε223=0
entangled
entanglement

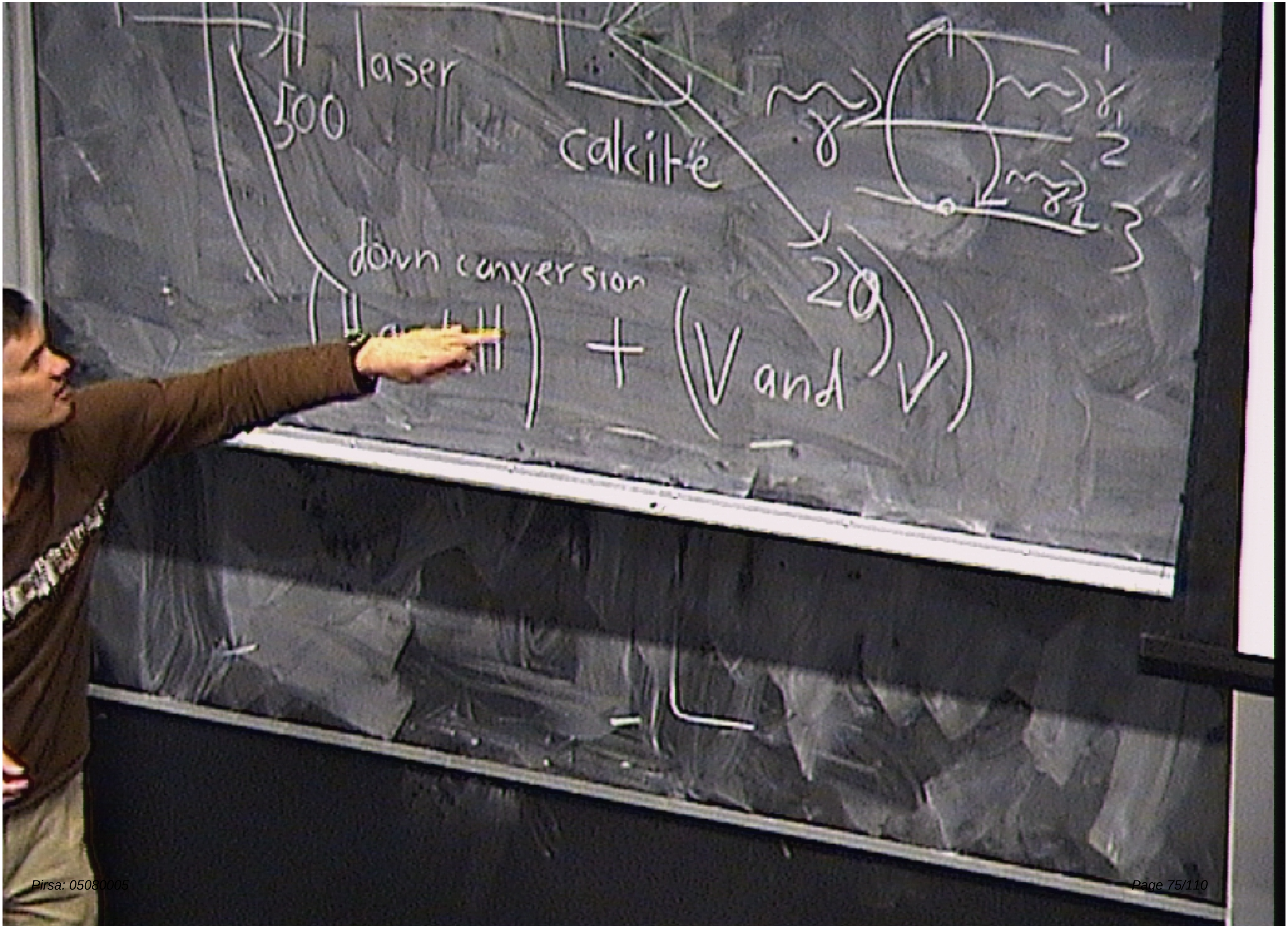

ε₁₂₃ = +1
(2 3 1) even
= exp(a J_i) q
VERTICALLY POLARIZED
B
E
[A
[J_a , J_b] = Σ_c ε_abc J_c

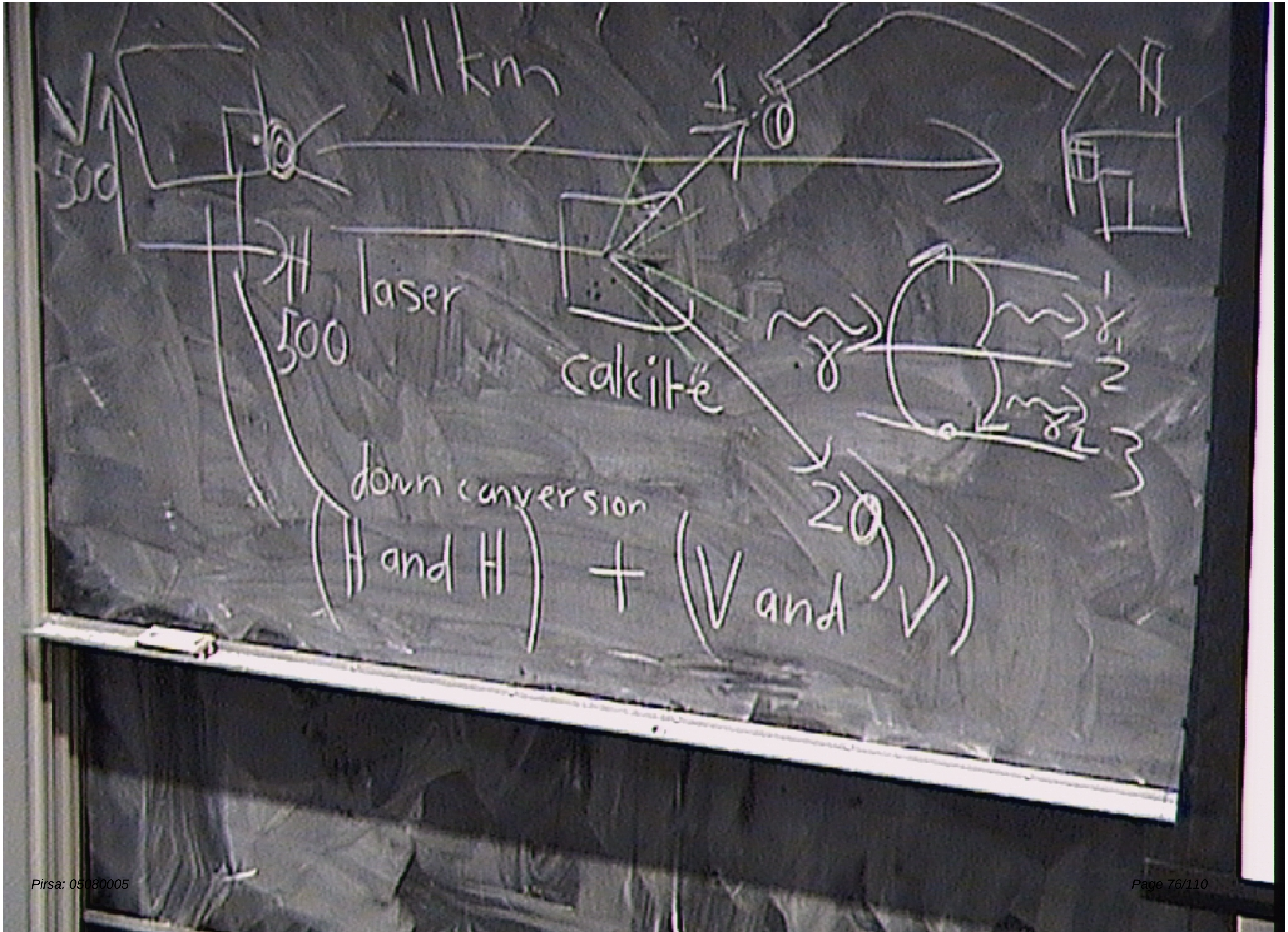

ε123 = +1
(2 3 1) even
exp(a Ji) q
VERTICALLY POLARIZED

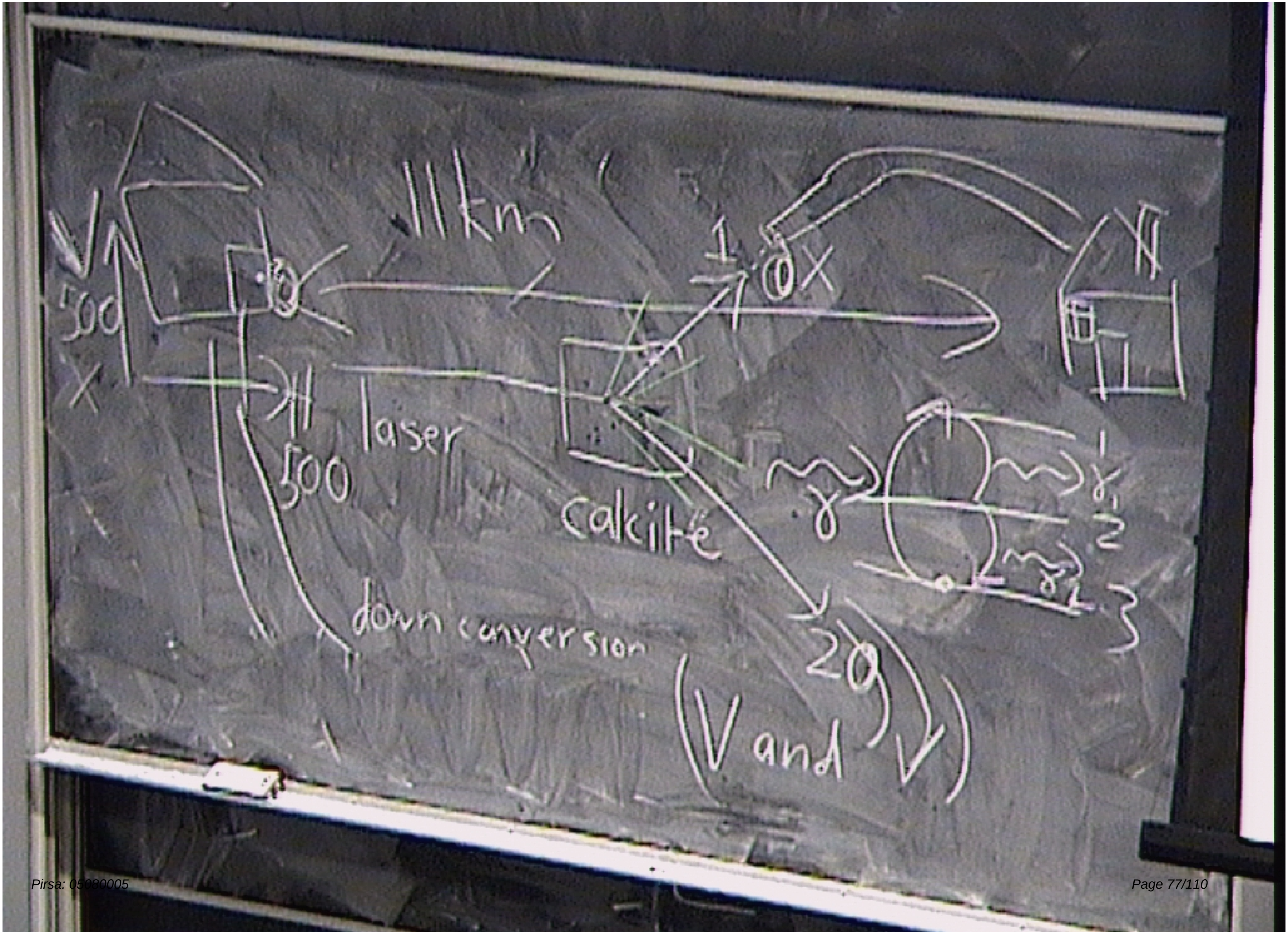


11 km
500
1
laser
500
calcite
down conversion
20
2
3

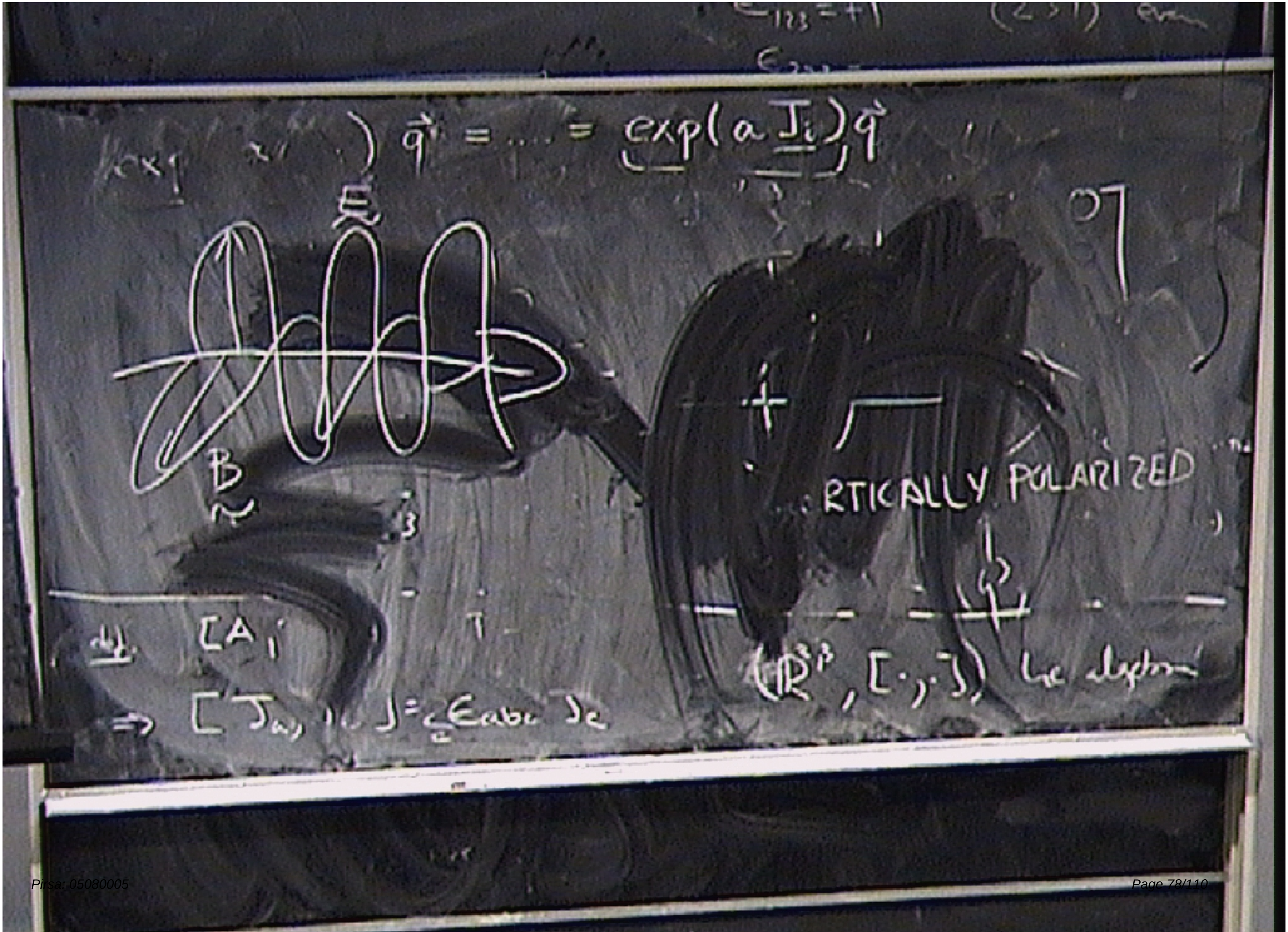

11 km
500
1
laser
500
calcite
γ
1
2
3
20
down conversion

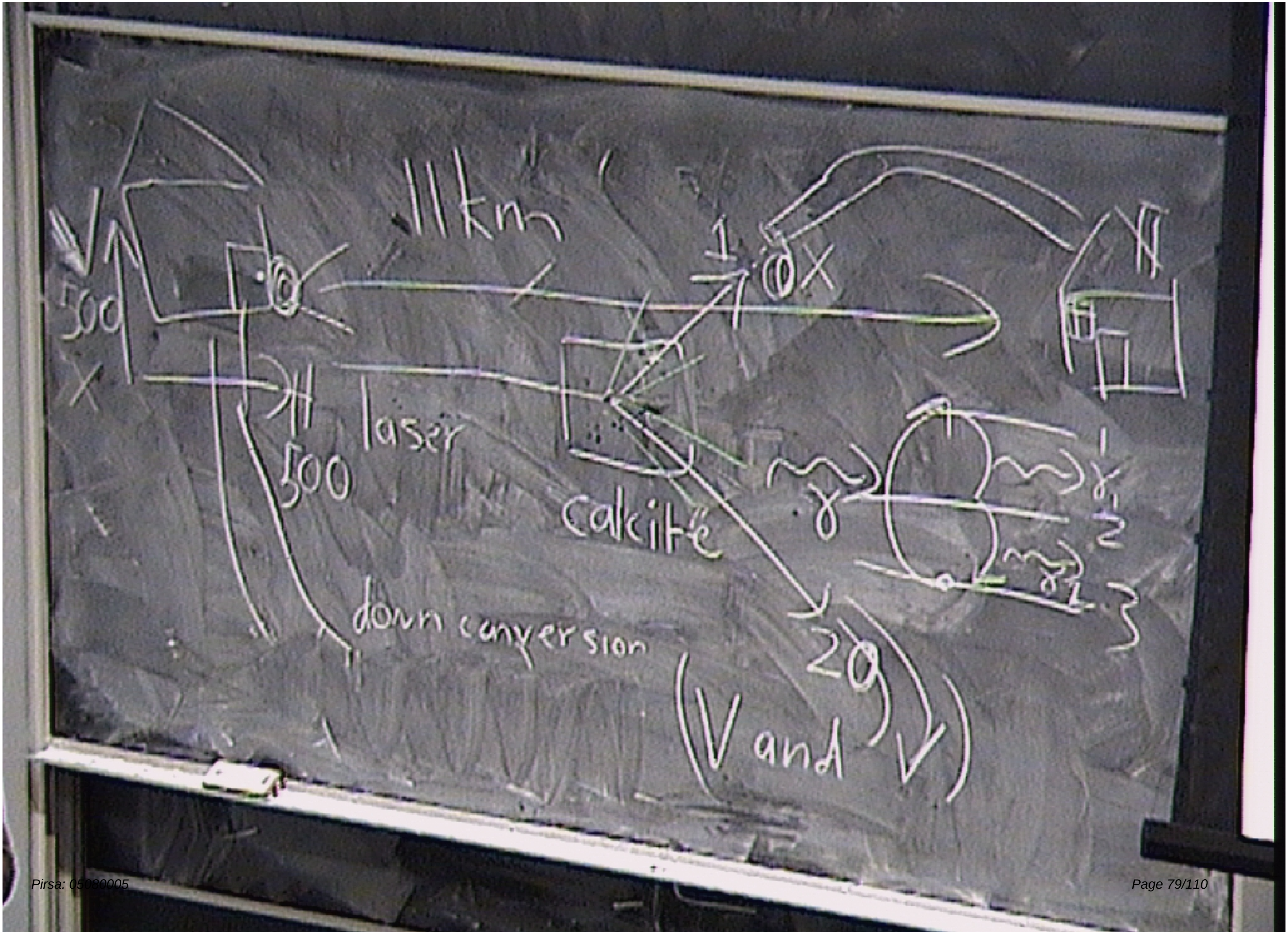

laser
500
calcite
down conversion
(H and H) + (V and V)
20

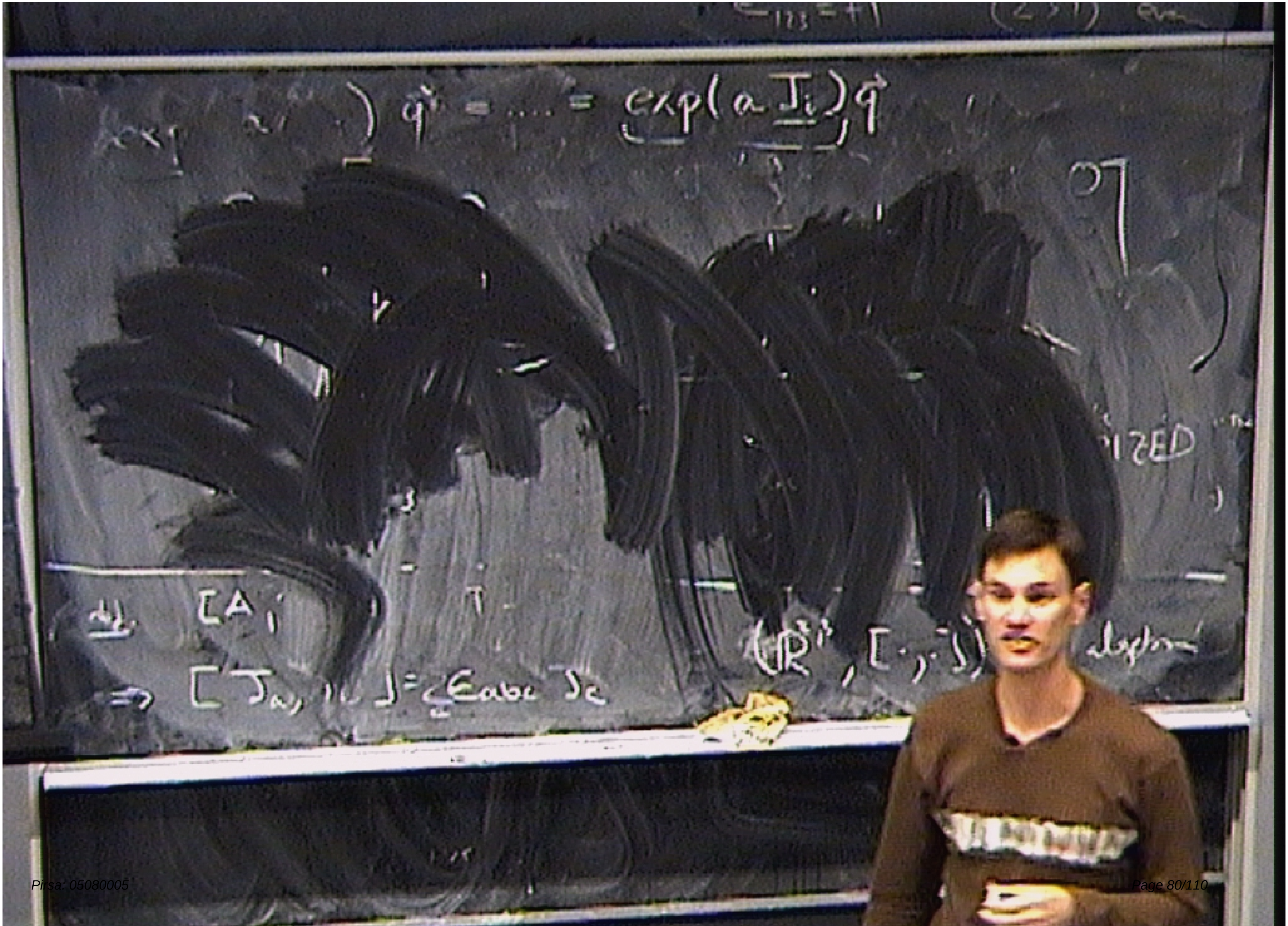

11 km
500
1
H
laser
500
calcite
down conversion
γ
20
(H and H) + (V and V)

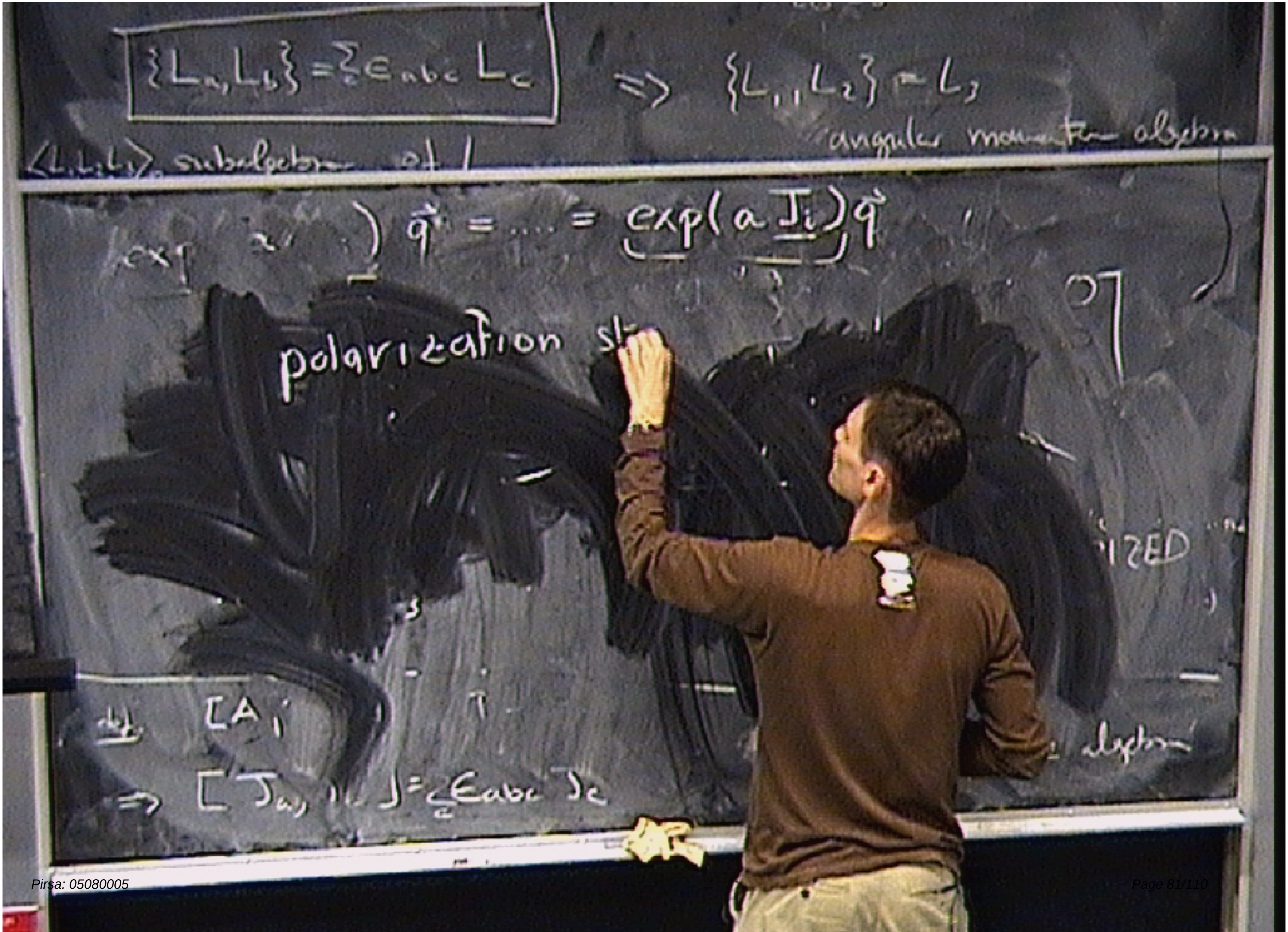


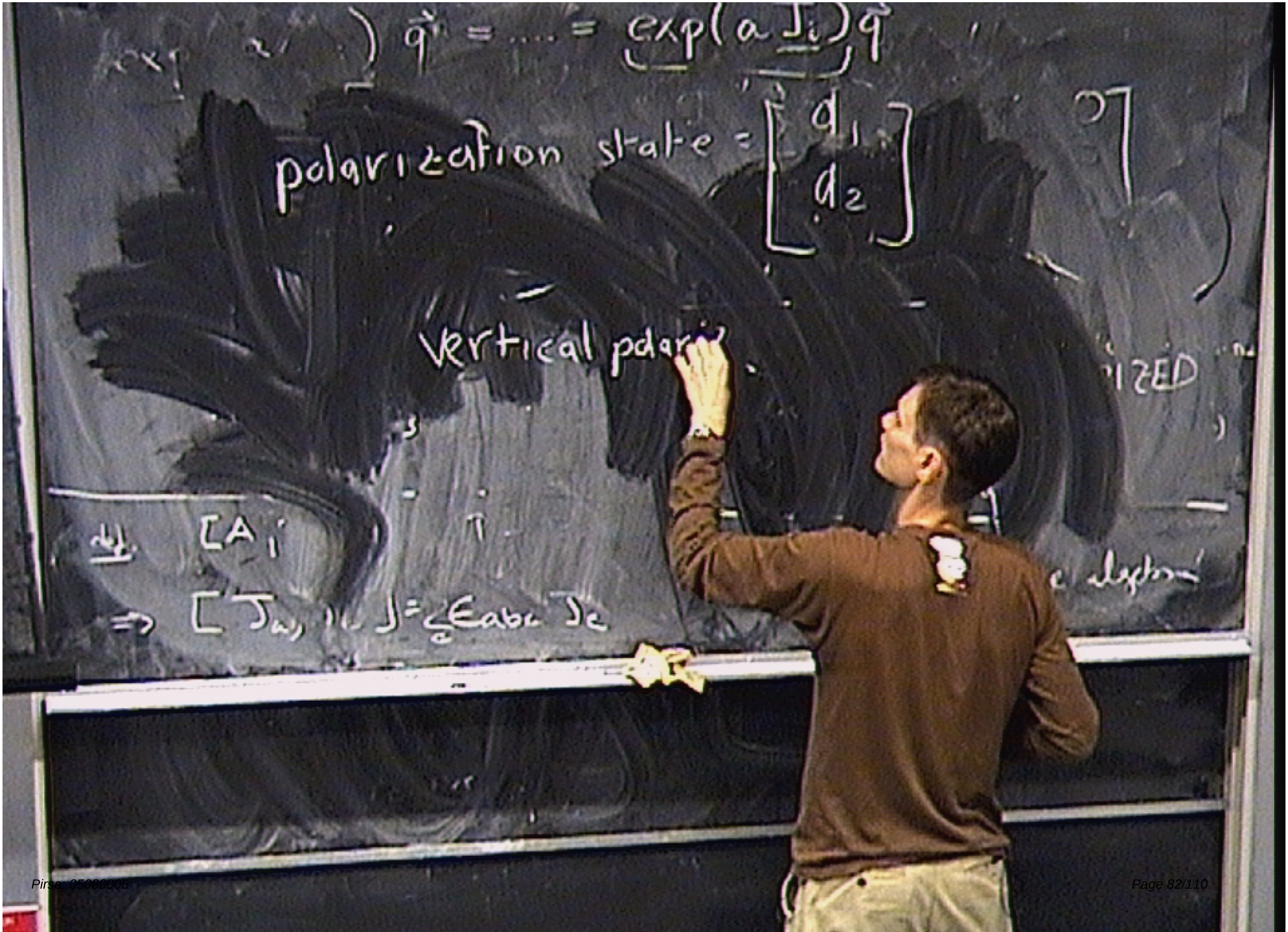
11 km
500
laser
500
calcite
down conversion
1
20
(V and V)
1
2
3

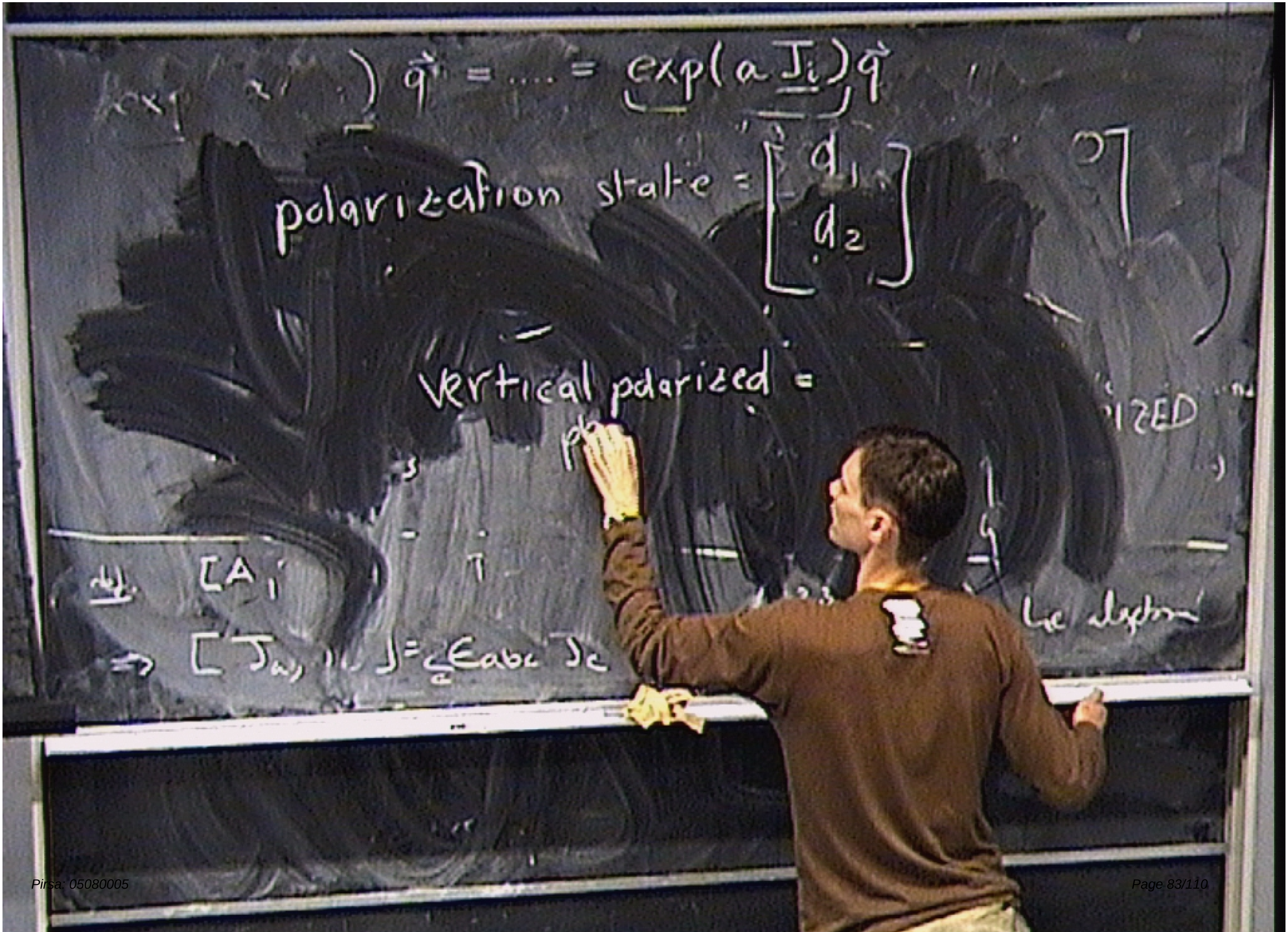

exp(a J_i) q
RTICALLY POLARIZED
B

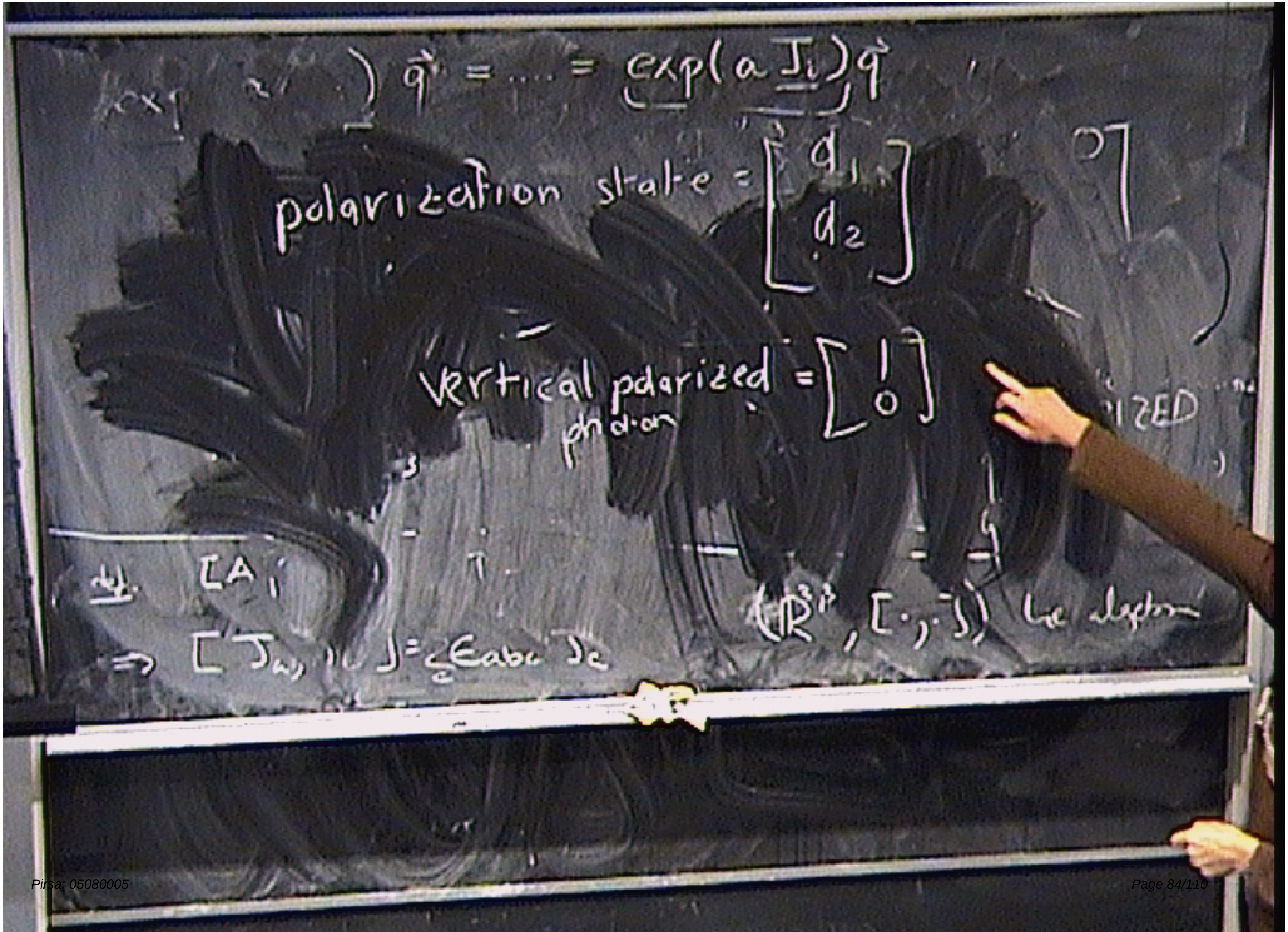

11 km
500
laser
500
calcite
down conversion
20
(V and V)




{L_a, L_b} = Σ ε_abc L_c
=> {L_1, L_2} = L_3
angular momentum algebra
q = ... = exp(a J_i) q
polarization st
=> [J_a, J_b] = Σ ε_abc J_c


polarization state =
Vertical

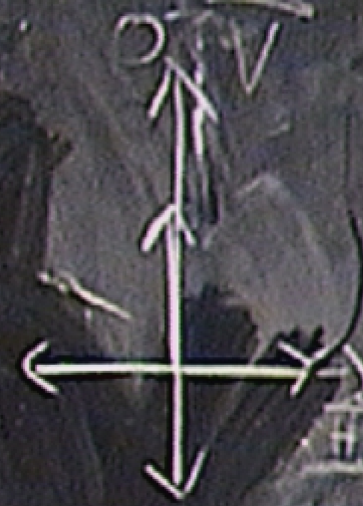

exp(a J_i) q
polarization state =
q_1
q_2
vertical polarized =


$\exp(a J_i)\vec{q} = \ldots = \exp(a \underline{J_i})\vec{q}$
polarization state $= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$
vertical polarized photon $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\Rightarrow [J_a, J_b] = \sum_c \epsilon_{abc} J_c$
$(\mathbb{R}^3, [\cdot,\cdot])$ Lie algebra

$\exp(\ldots)\,\vec{q} = \ldots = \exp(a\,\underline{J_i})\,\vec{q}$

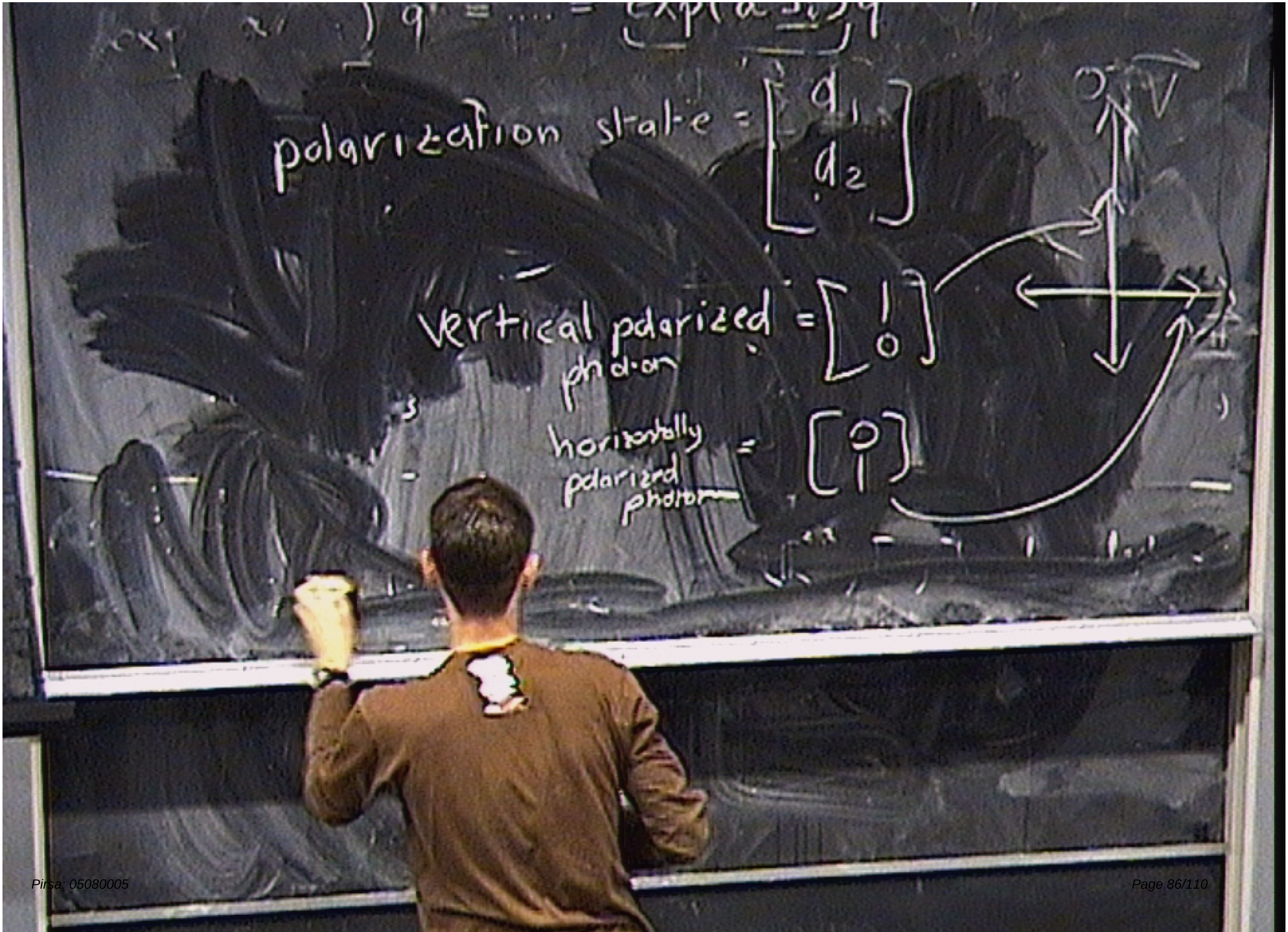
polarization state $= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

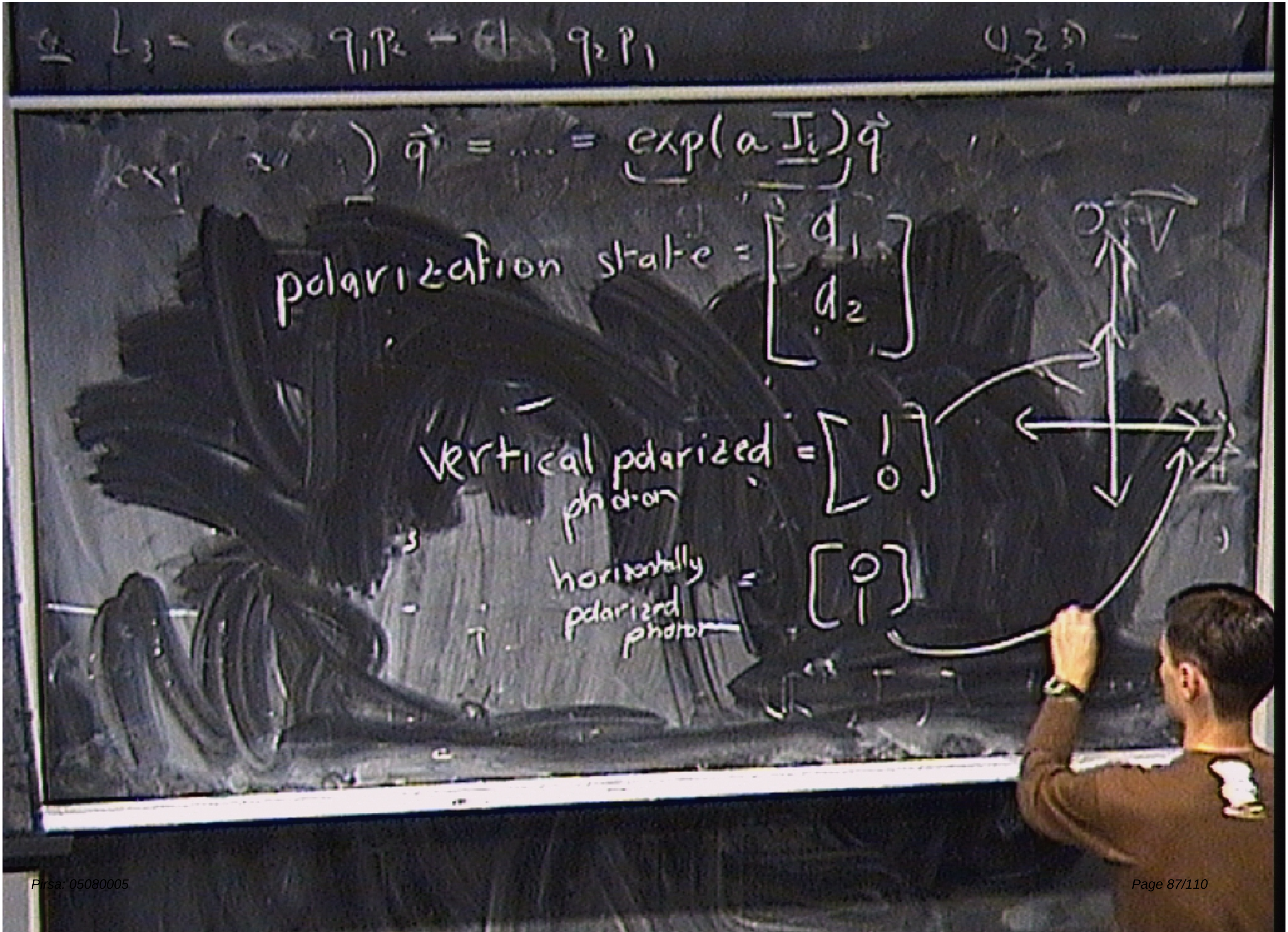
vertical polarized photon $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

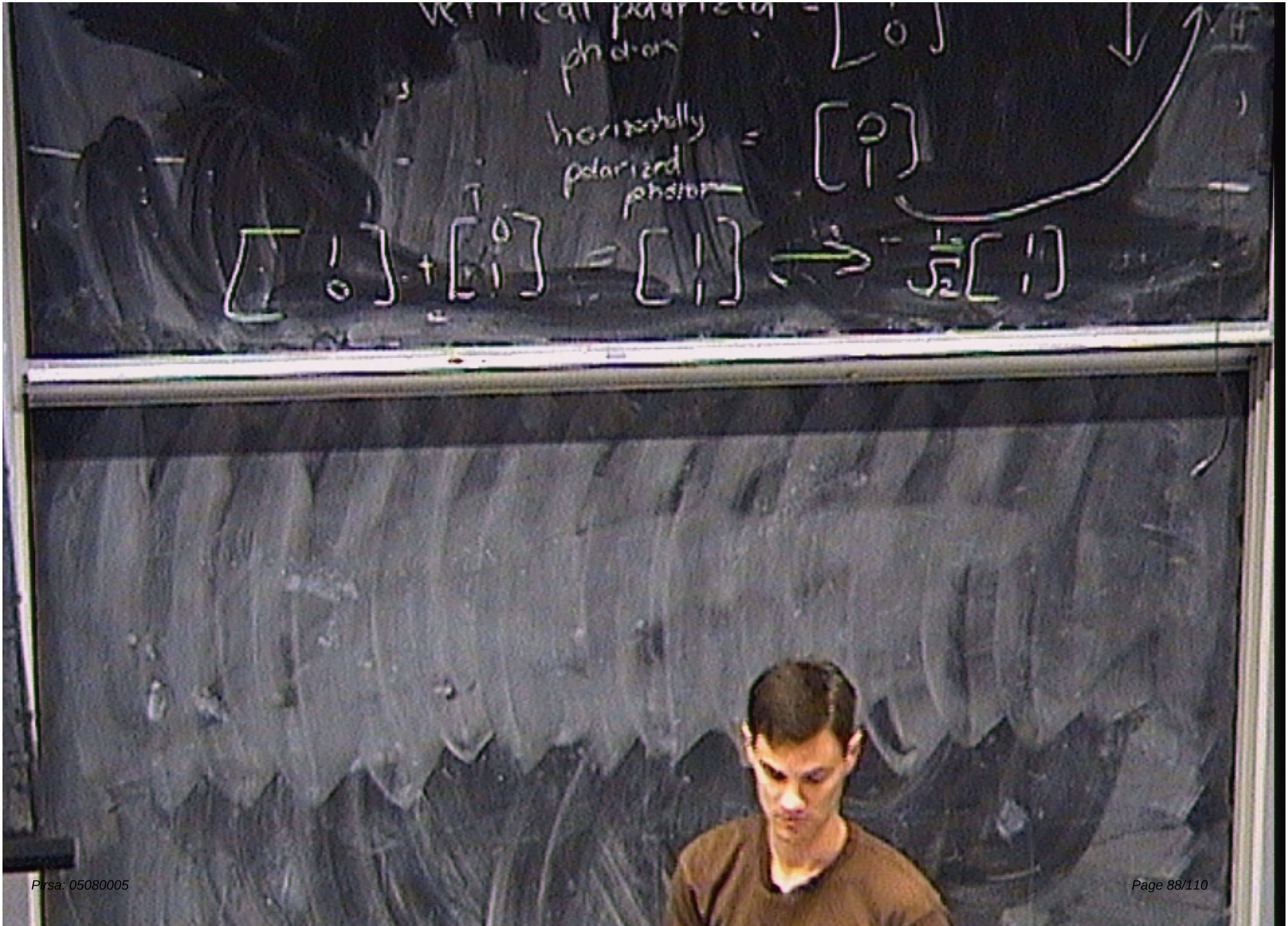


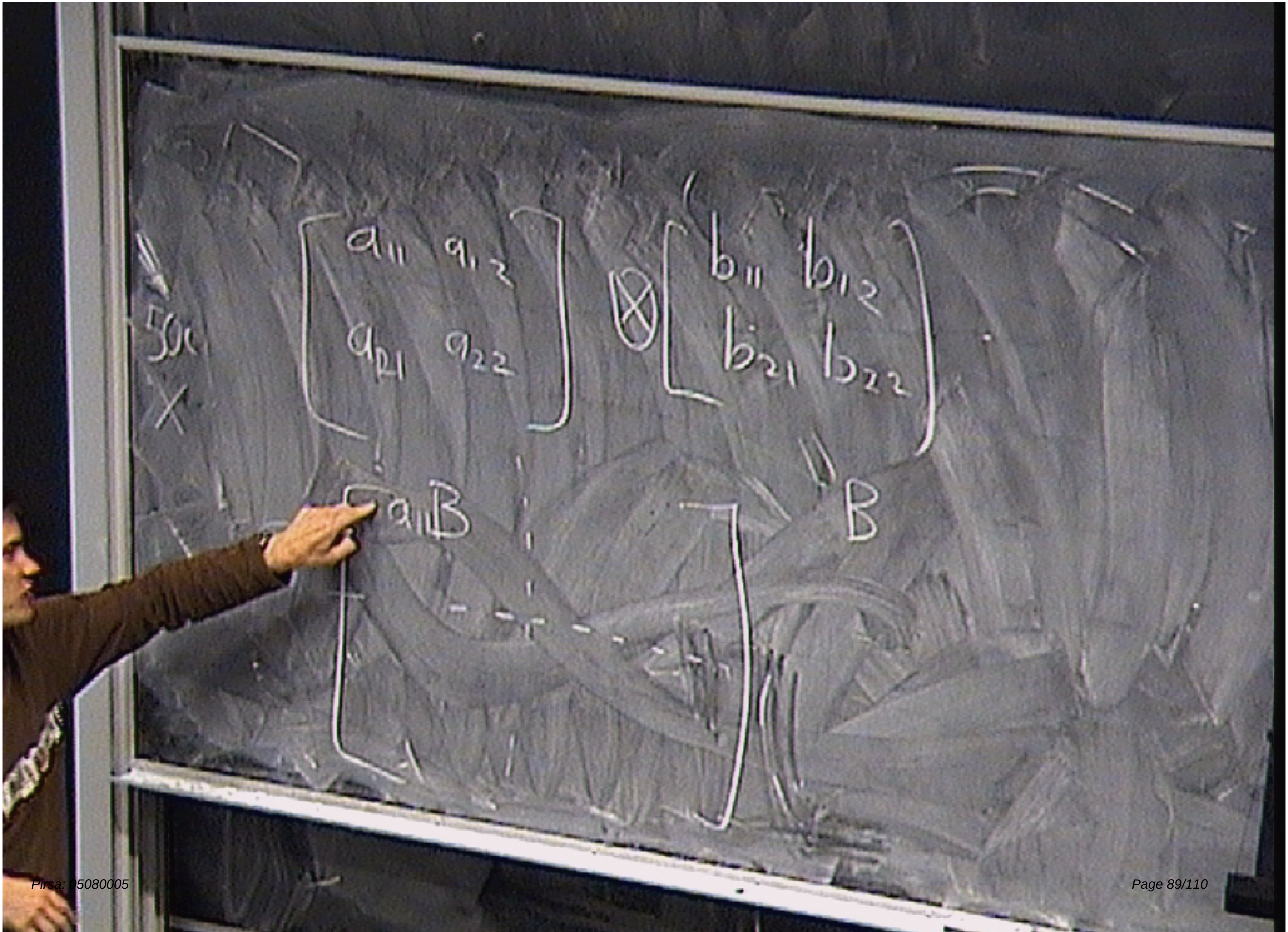
def. $[A_i$

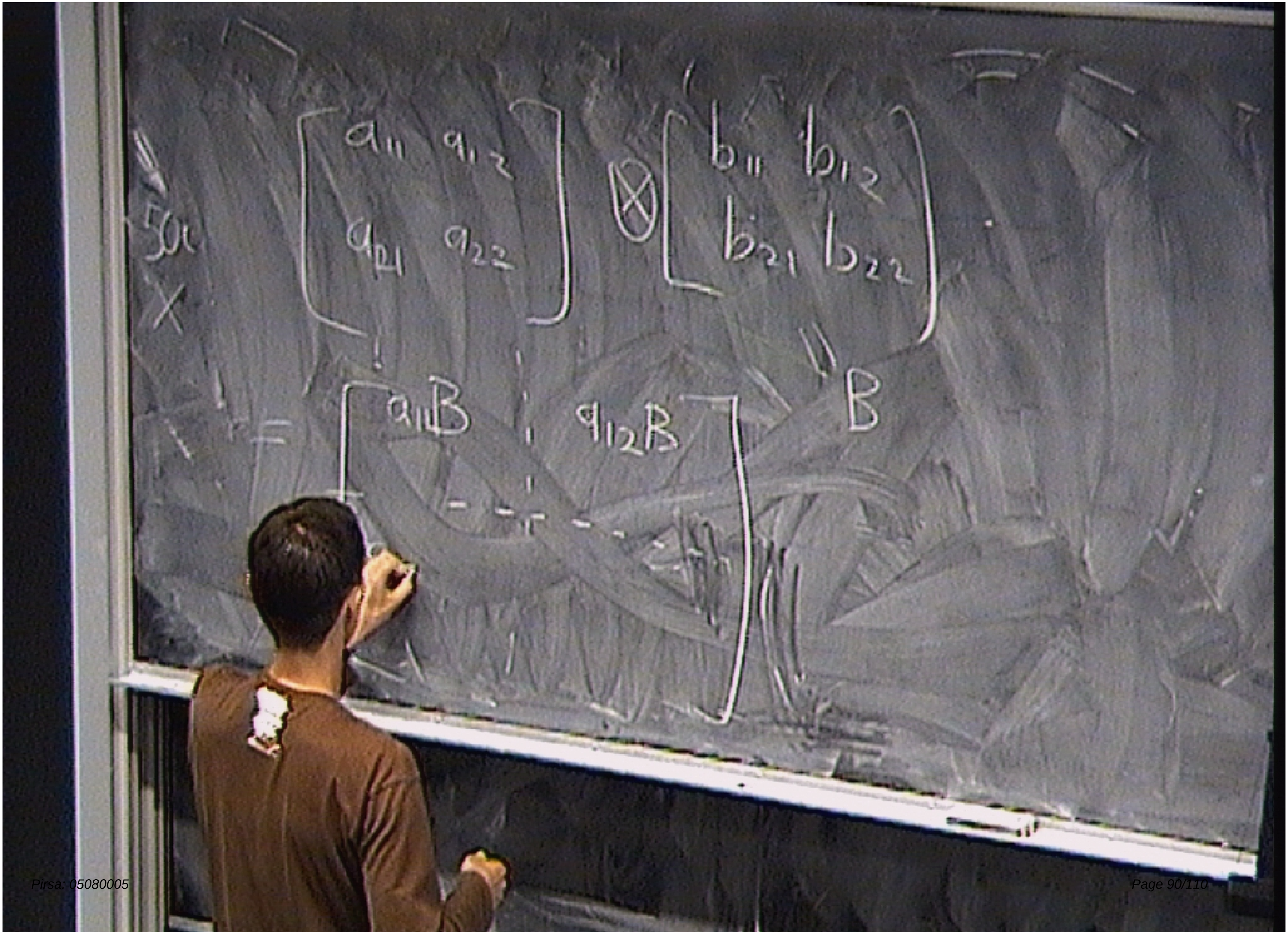
$\Rightarrow [J_a, J_b] = \sum_c \epsilon_{abc} J_c$

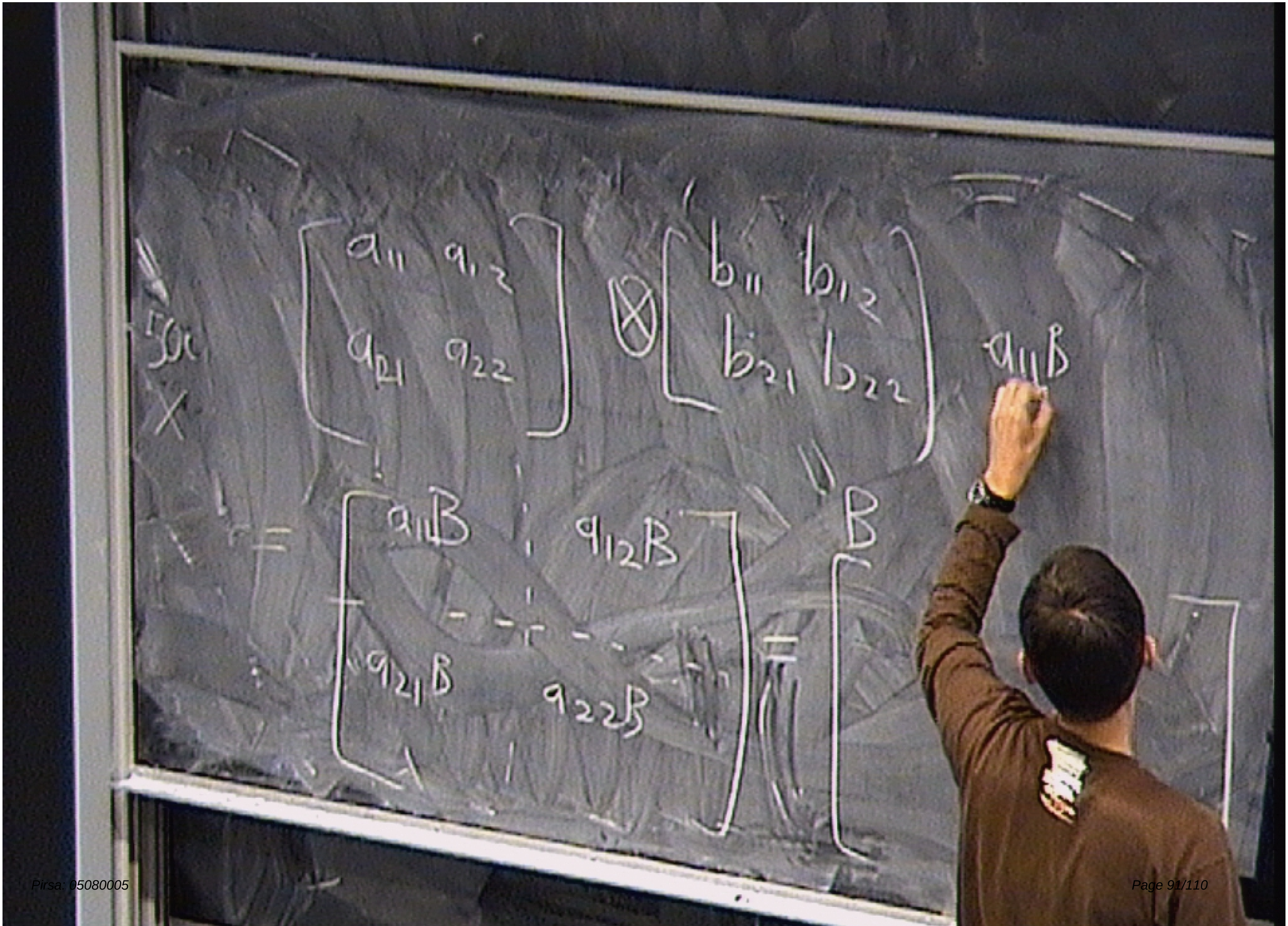

polarization state = [q1 q2]
vertical polarized photon = [1 0]
horizontally polarized photon = [0 1]


polarization state = [q1 q2]
vertical polarized photon = [1 0]
horizontally polarized photon = [0 1]
exp(a J_i) q

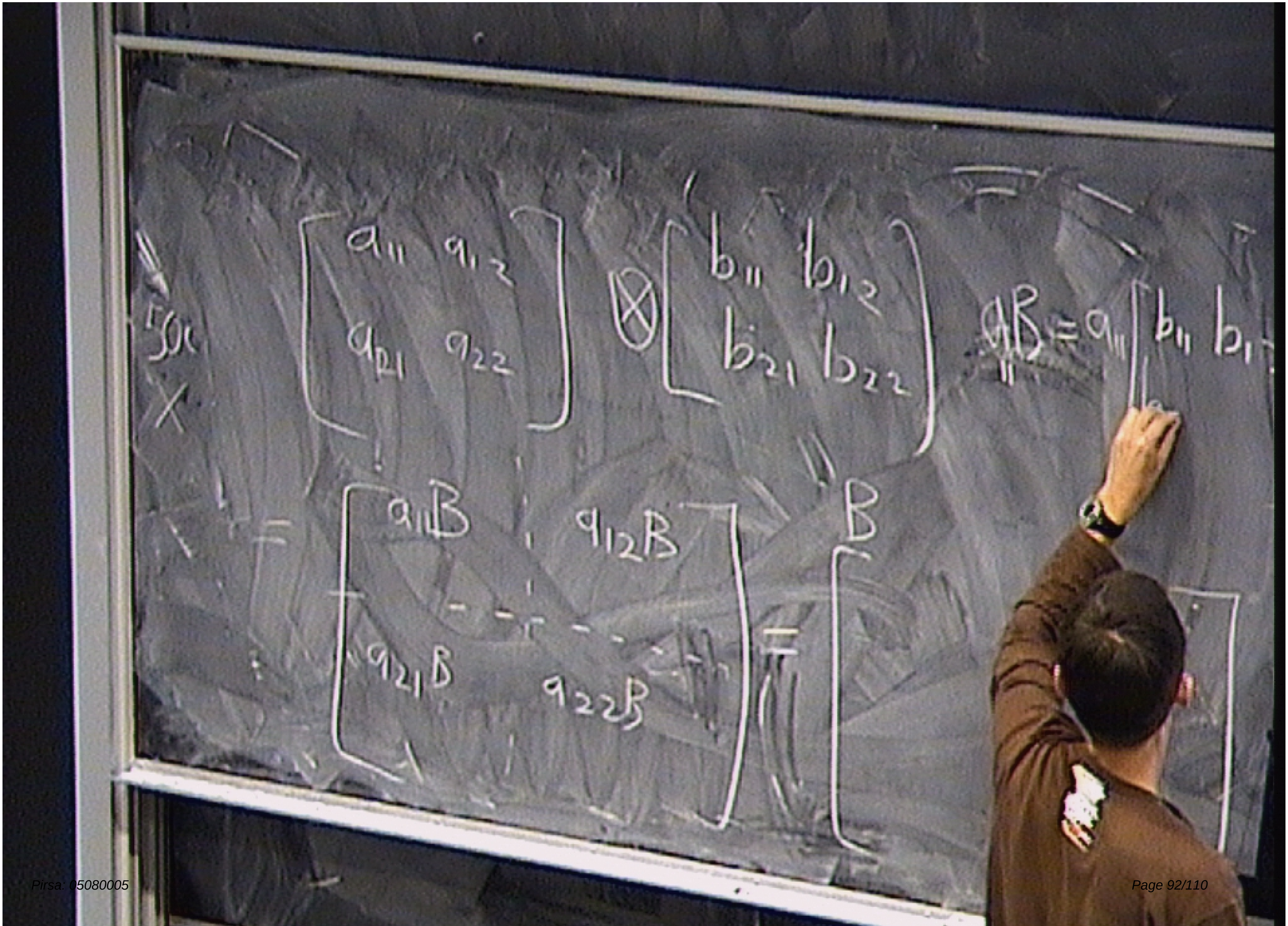

vertical polarized photon
horizontally polarized photon

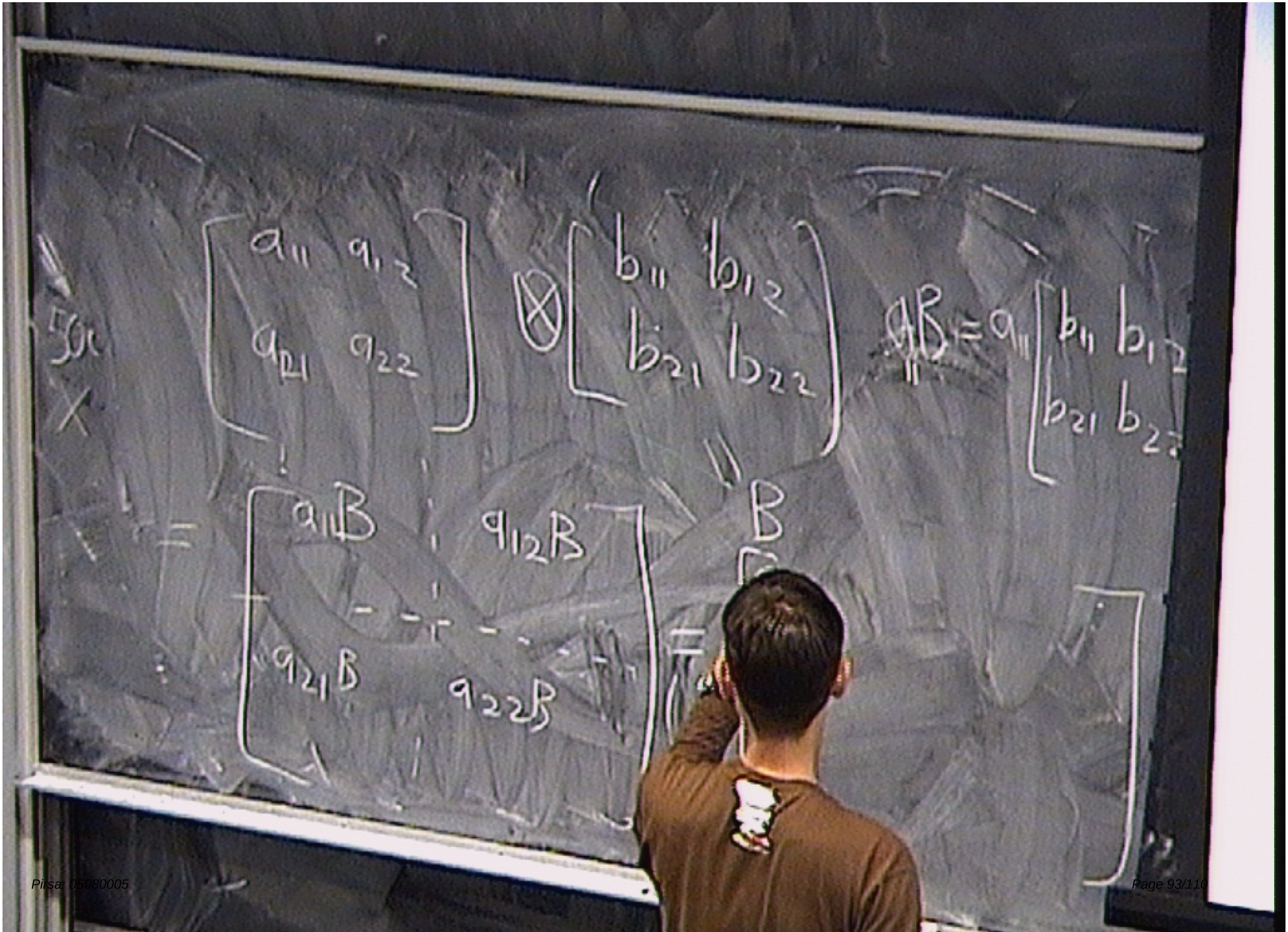

a11 a12
a21 a22
⊗
b11 b12
b21 b22
a11B
B


a11 a12 ⊗ b11 b12
a21 a22 b21 b22
= a11B a12B
B

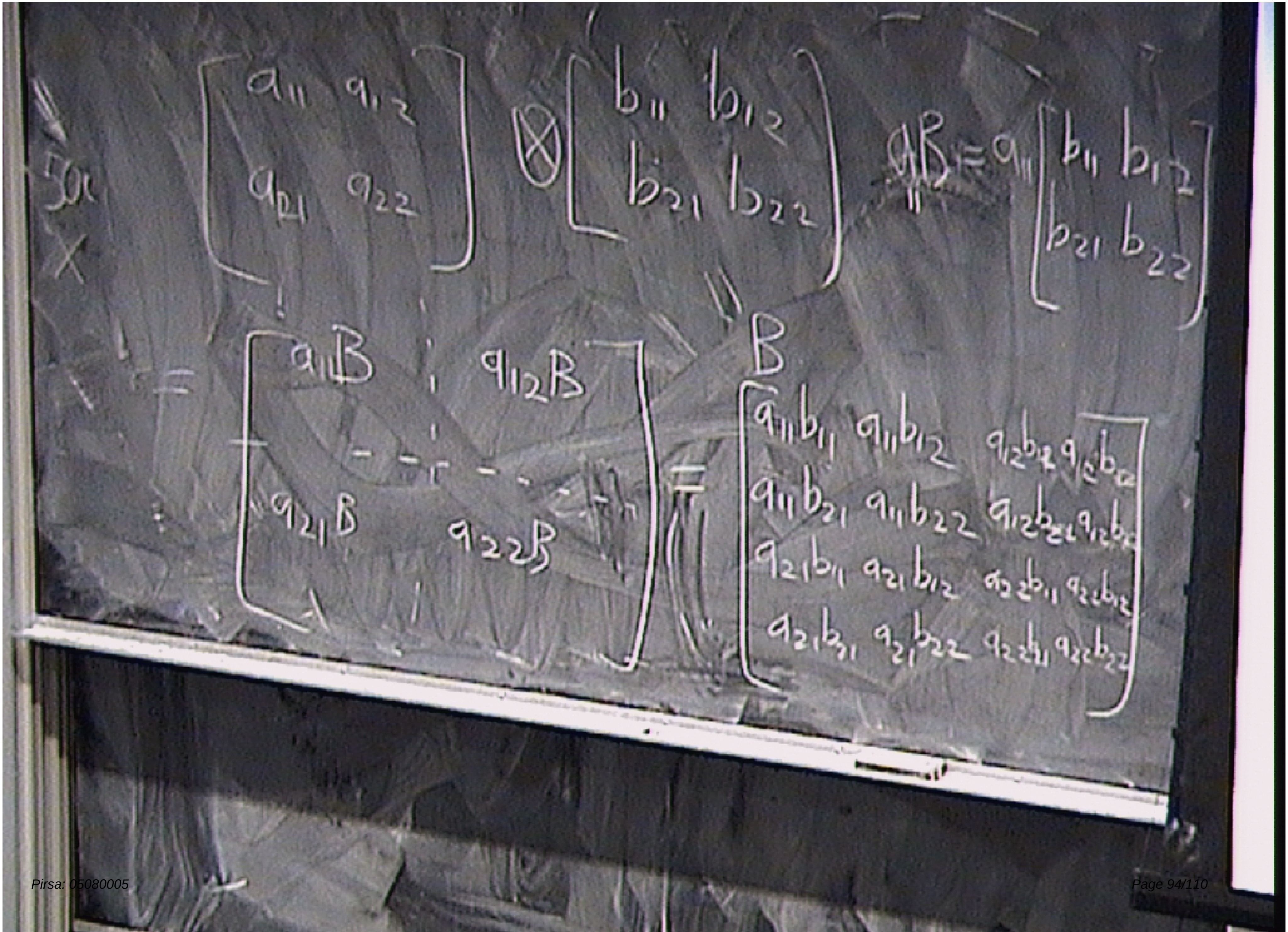


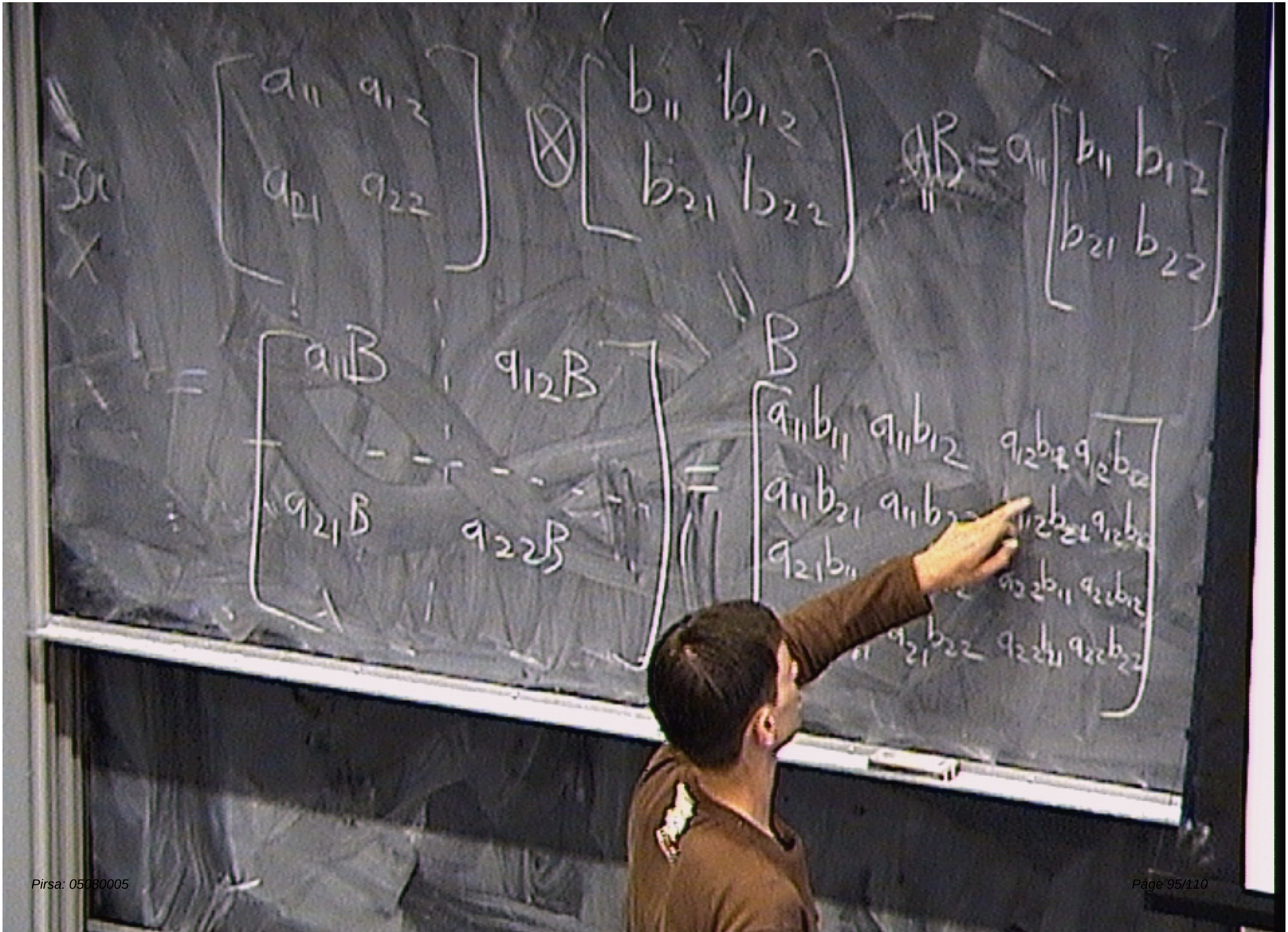
a11 a12
a21 a22
⊗
b11 b12
b21 b22
=
a11B a12B
a21B a22B
B
=
a11B

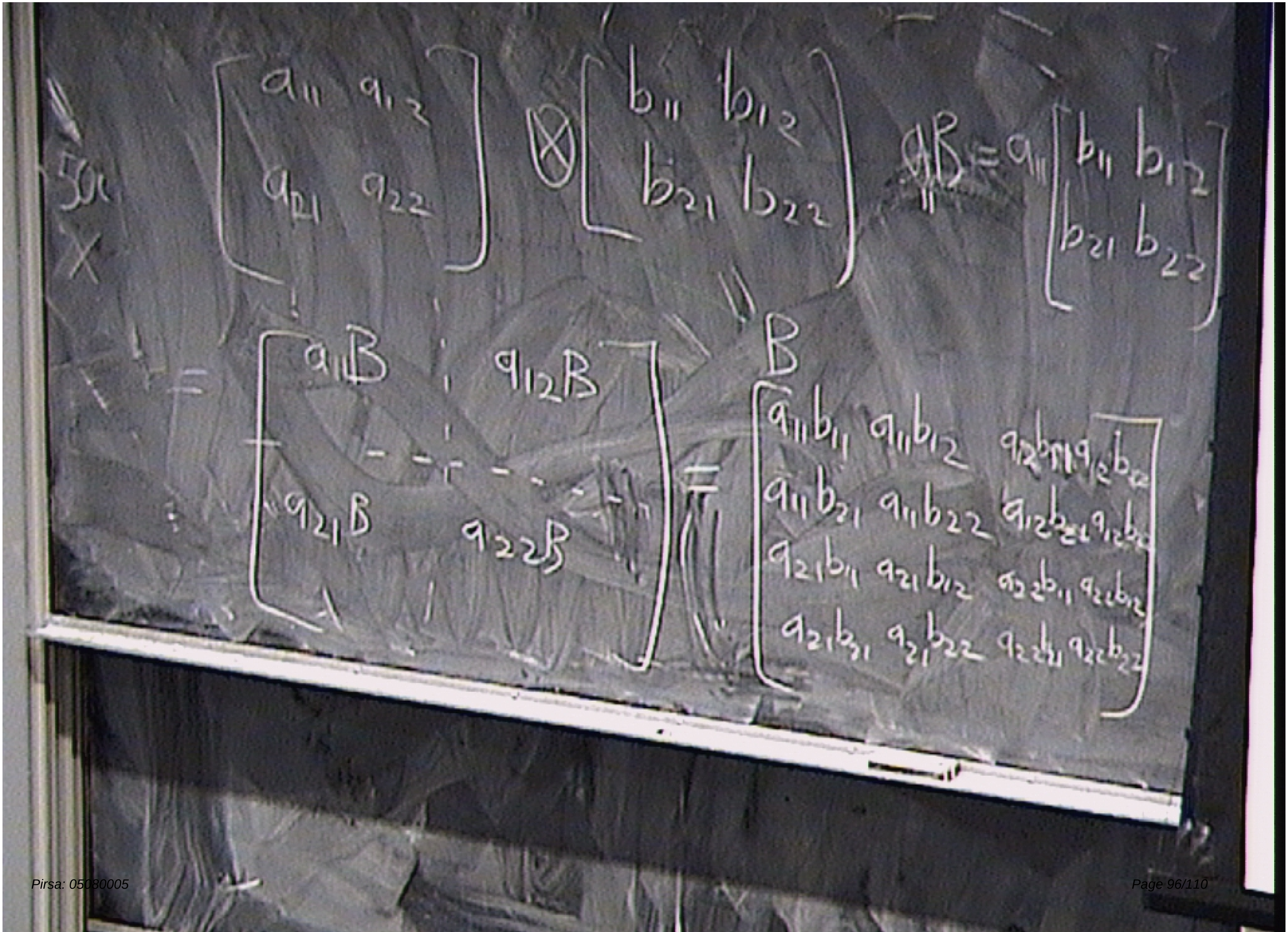

a11 a12
a21 a22
⊗
b11 b12
b21 b22
=
a11B a12B
a21B a22B
B
aB = a11 b11 b1

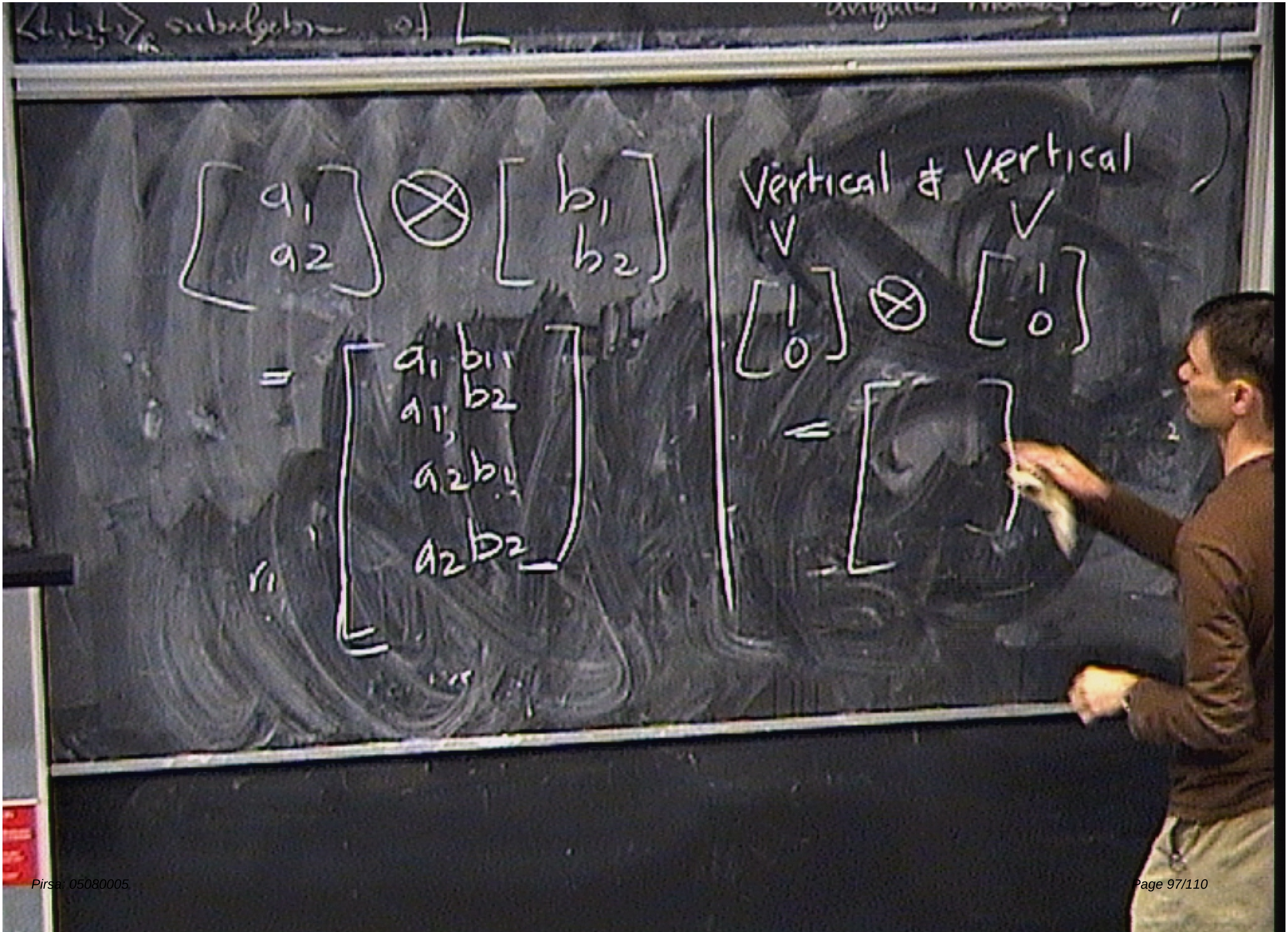


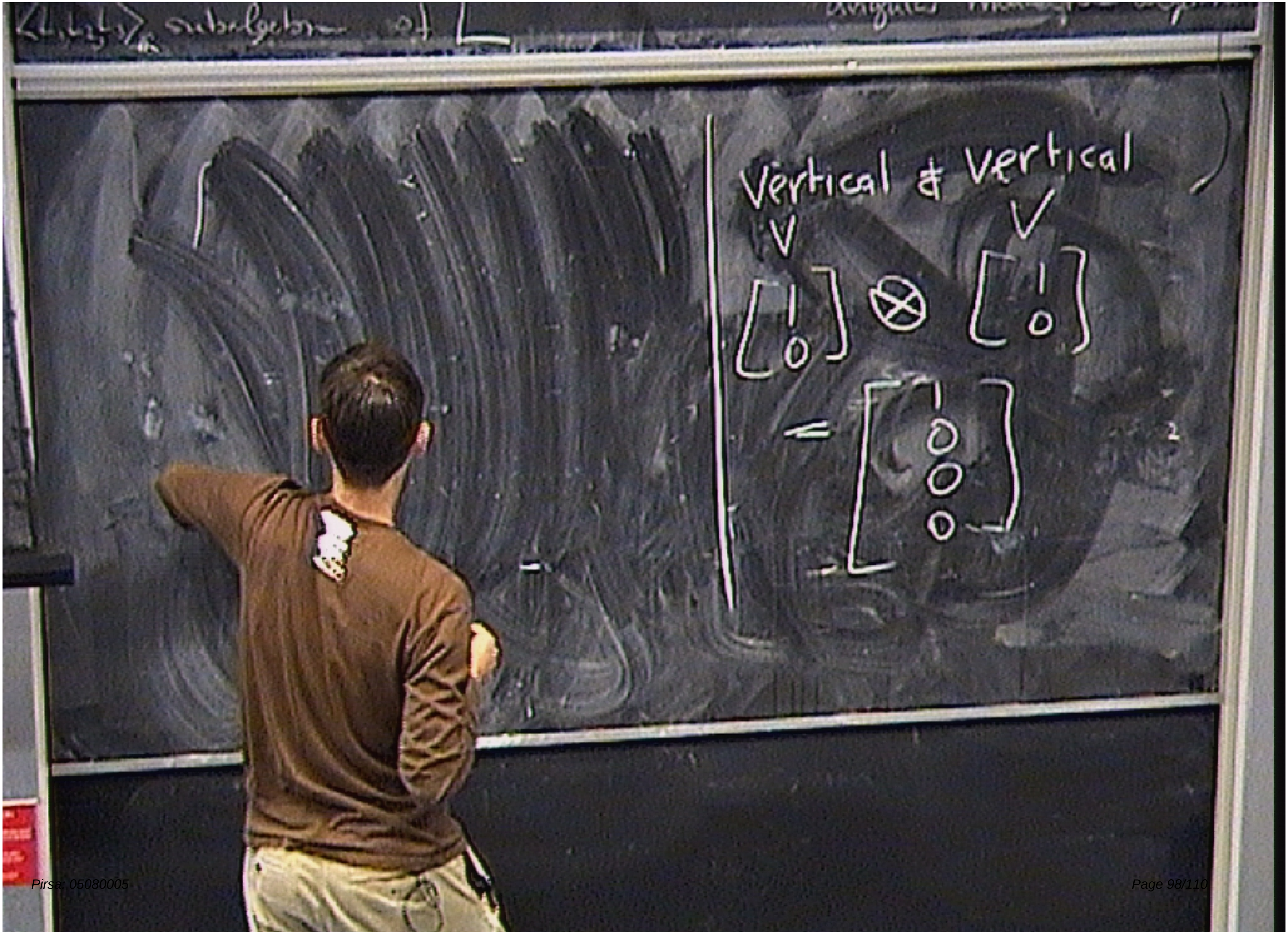
a11 a12
a21 a22
b11 b12
b21 b22
a11B a12B
a21B a22B
B

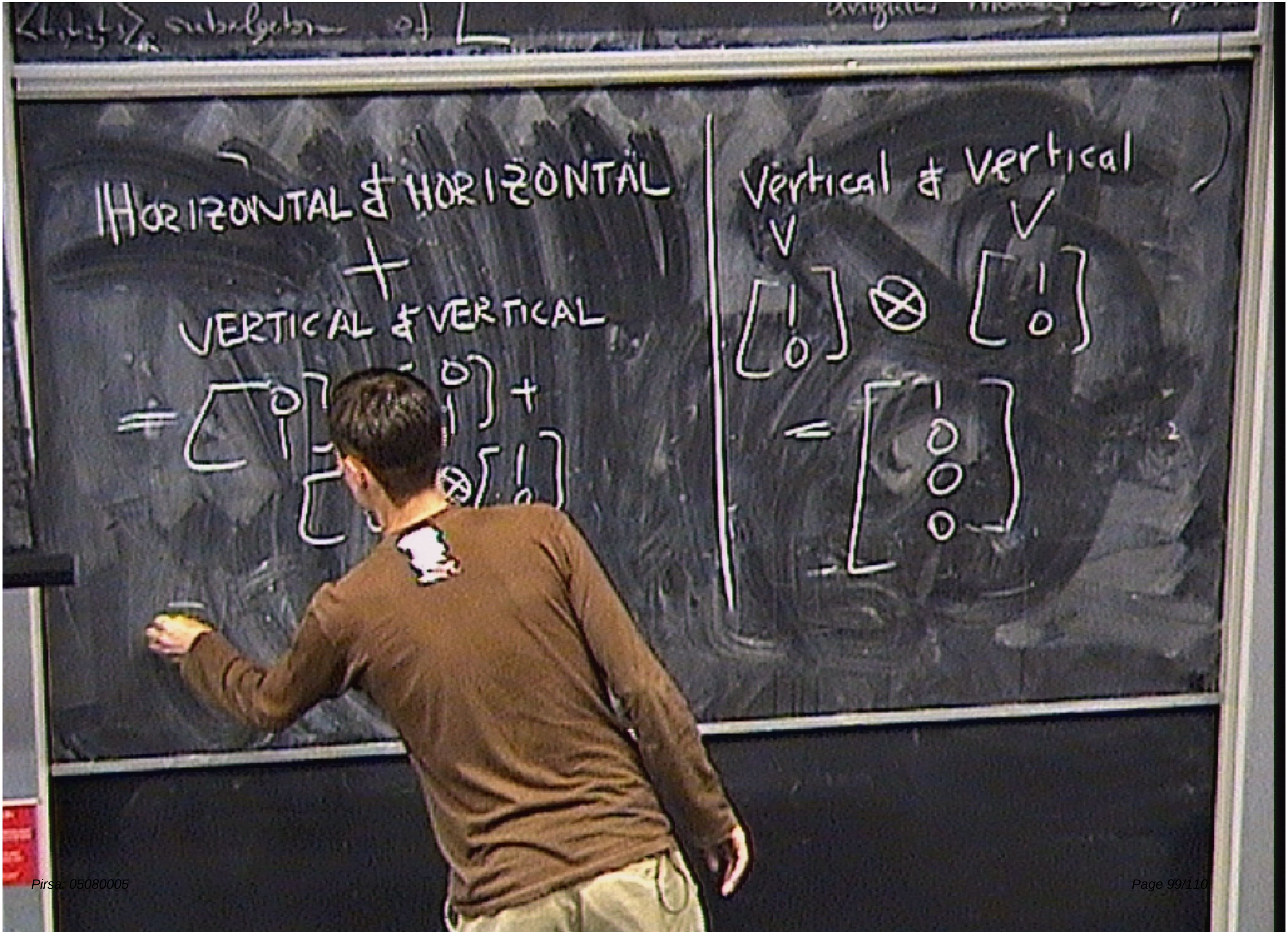

a11 a12
a21 a22
⊗
b11 b12
b21 b22
a11 B = a11 [b11 b12; b21 b22]
= [a11 B a12 B; a21 B a22 B]
B
a11b11 a11b12 a12b11 a12b12
a11b21 a11b22 a12b21 a12b22
a21b11 a21b12 a22b11 a22b12
a21b21 a21b22 a22b21 a22b22

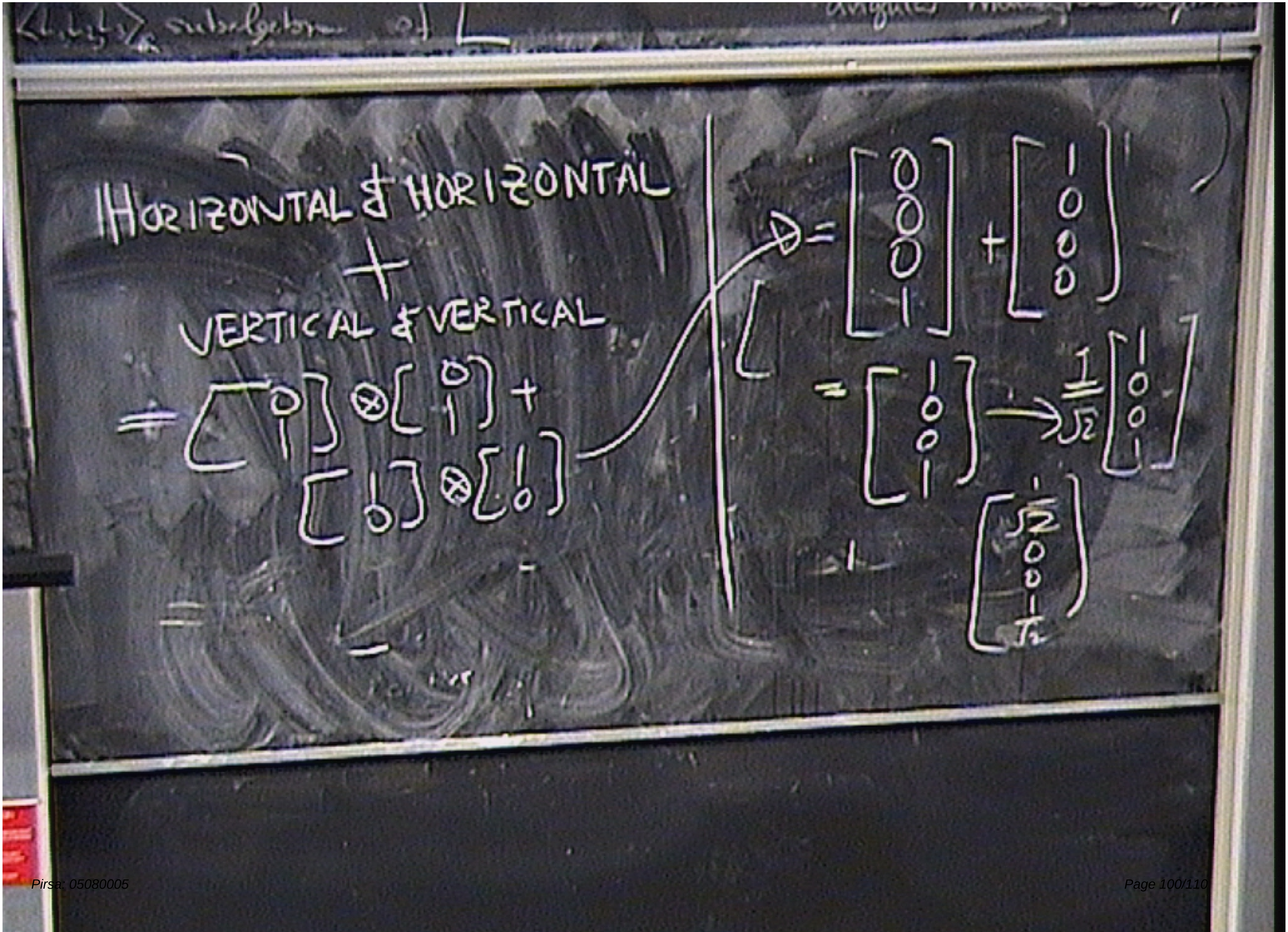


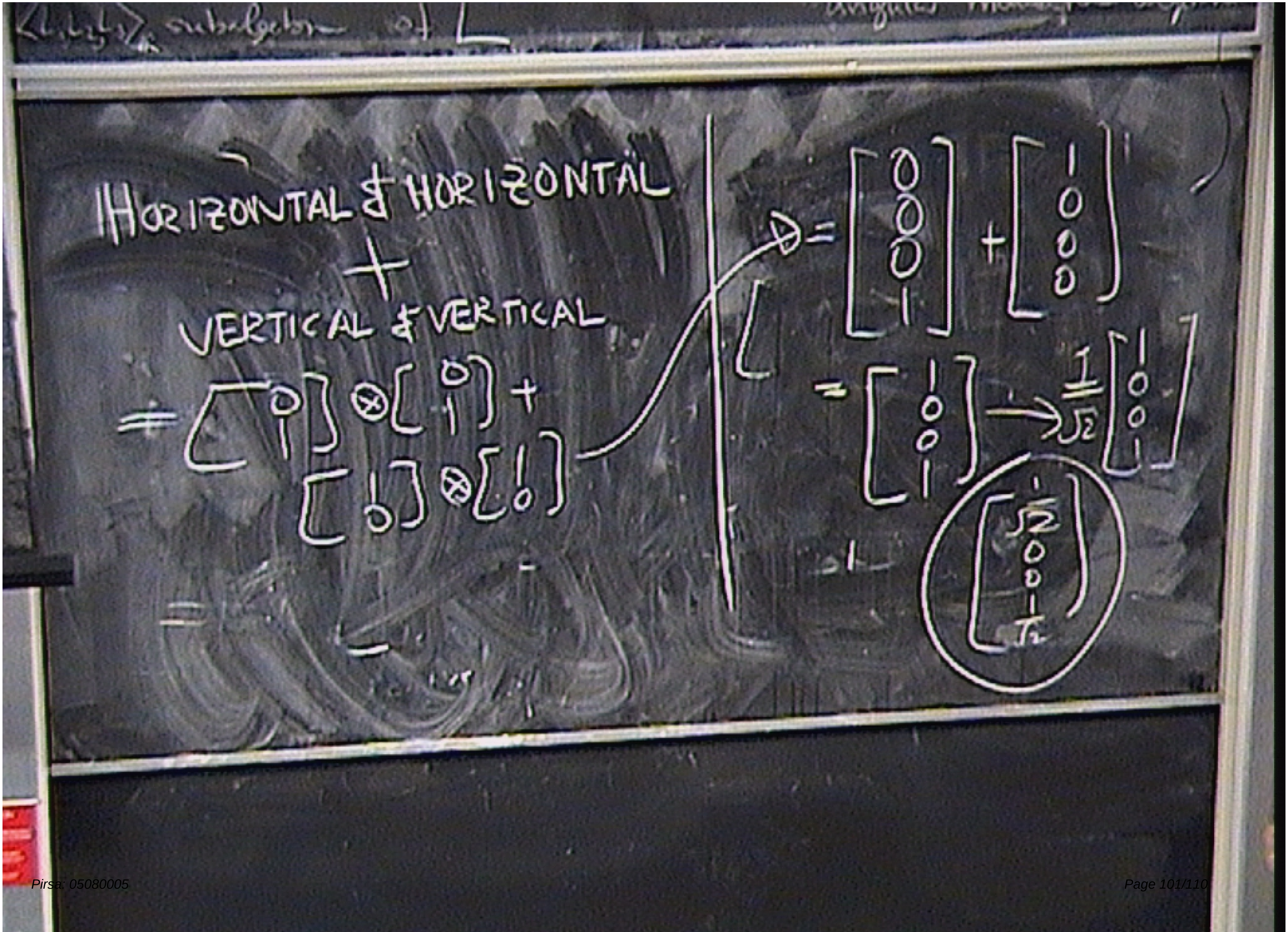

a11 a12 a21 a22 ⊗ b11 b12 b21 b22
a11B = a11 [b11 b12 b21 b22]
a11B a12B a21B a22B
B
a11b11 a11b12 a12b11 a12b12
a11b21 a11b22 a12b21 a12b22
a21b11 a21b12 a22b11 a22b12
a21b21 a21b22 a22b21 a22b22

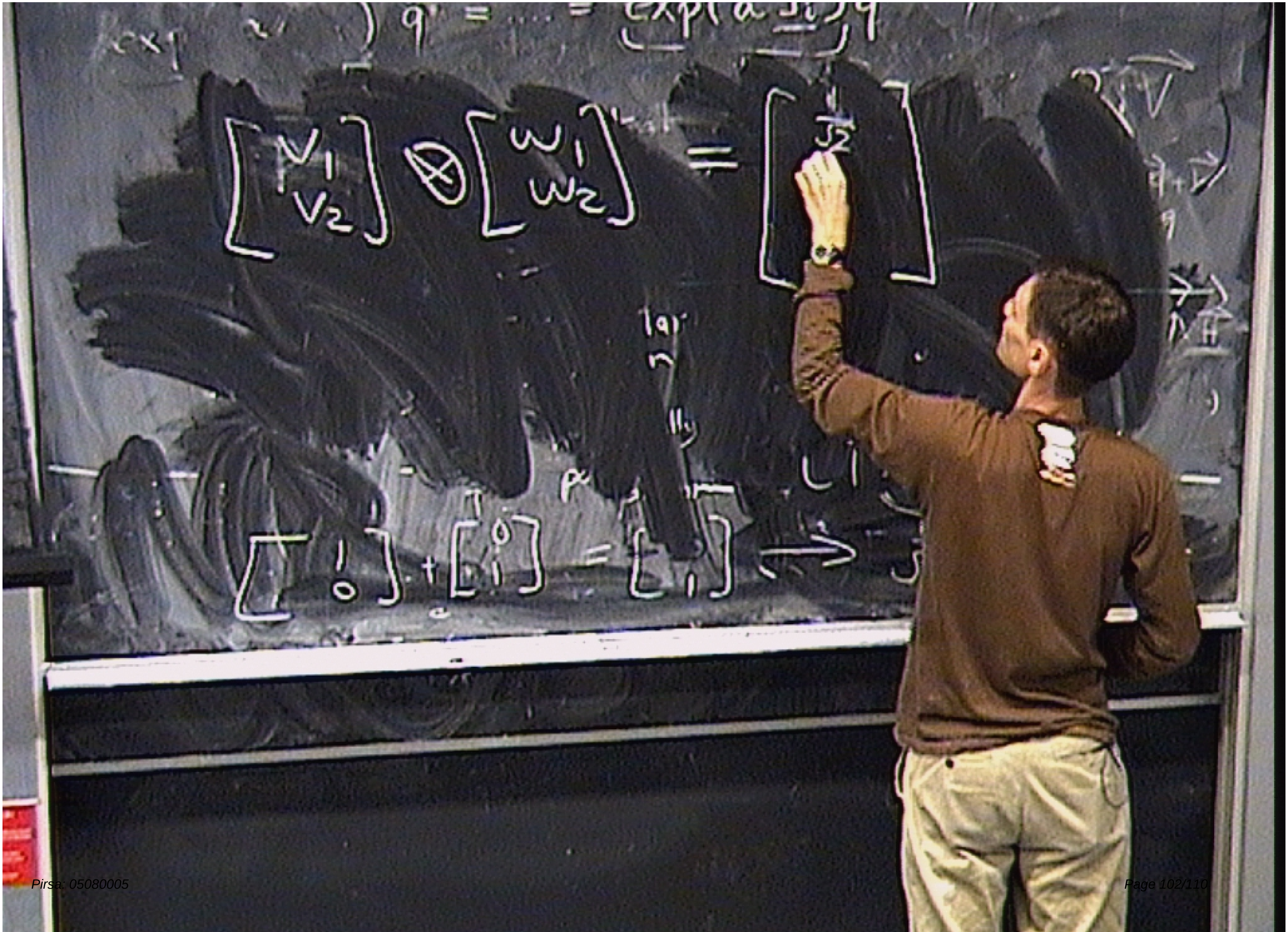

Vertical & Vertical
V
V
a1
a2
b1
b2
a1 b1
a1 b2
a2 b1
a2 b2
1
0

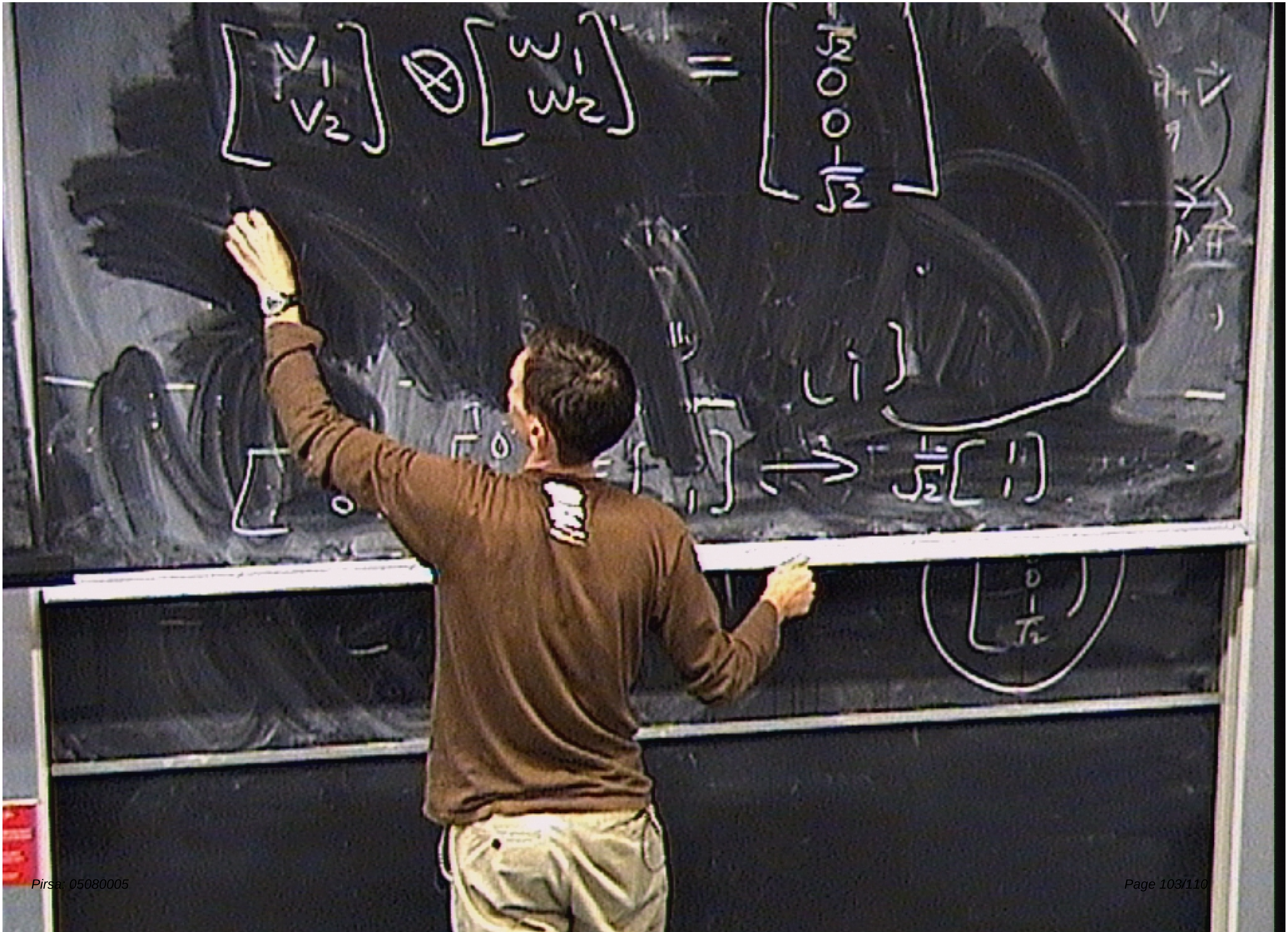

Vertical & Vertical
V
V

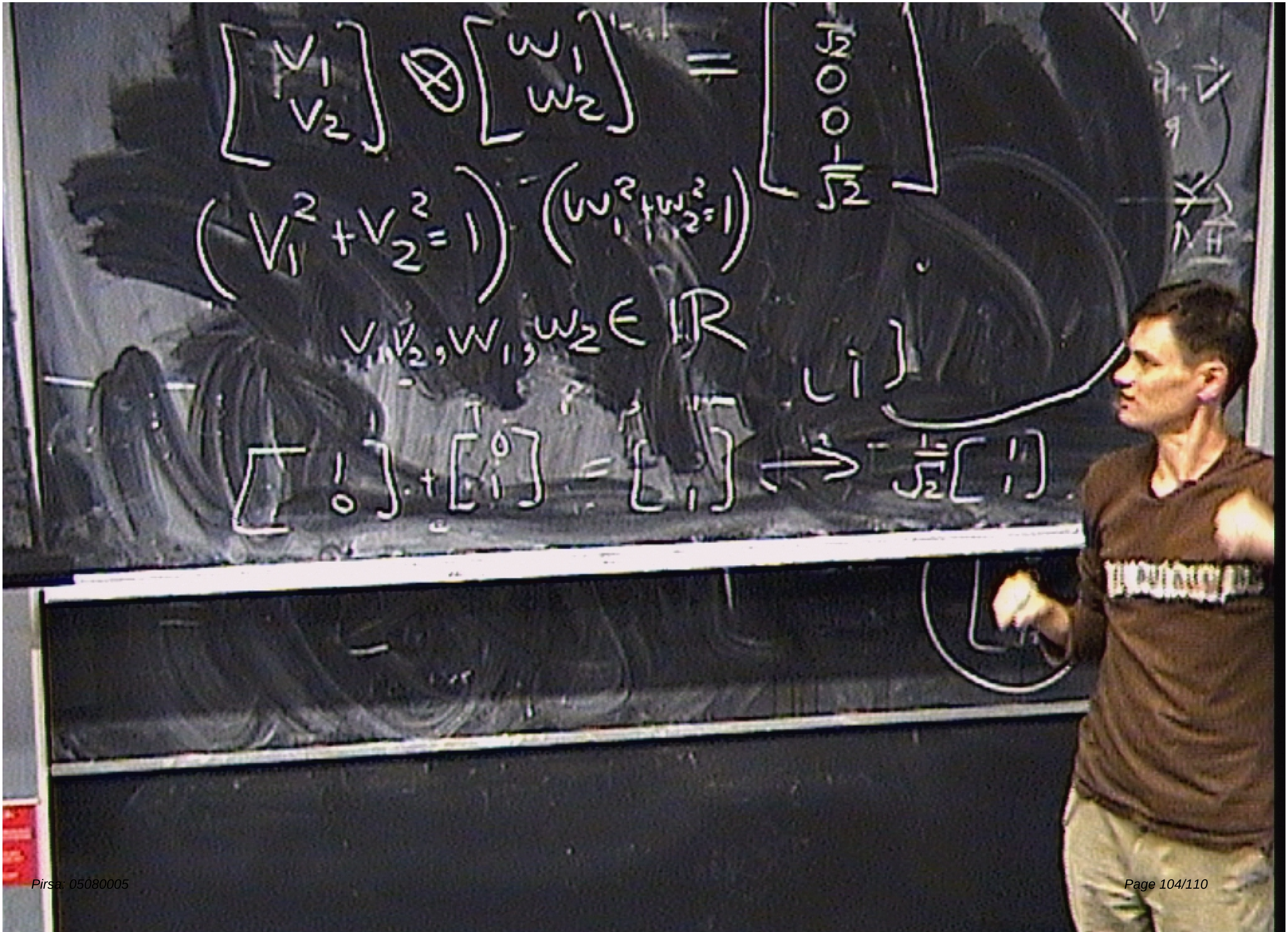

HORIZONTAL & HORIZONTAL
+
VERTICAL & VERTICAL
Vertical & Vertical

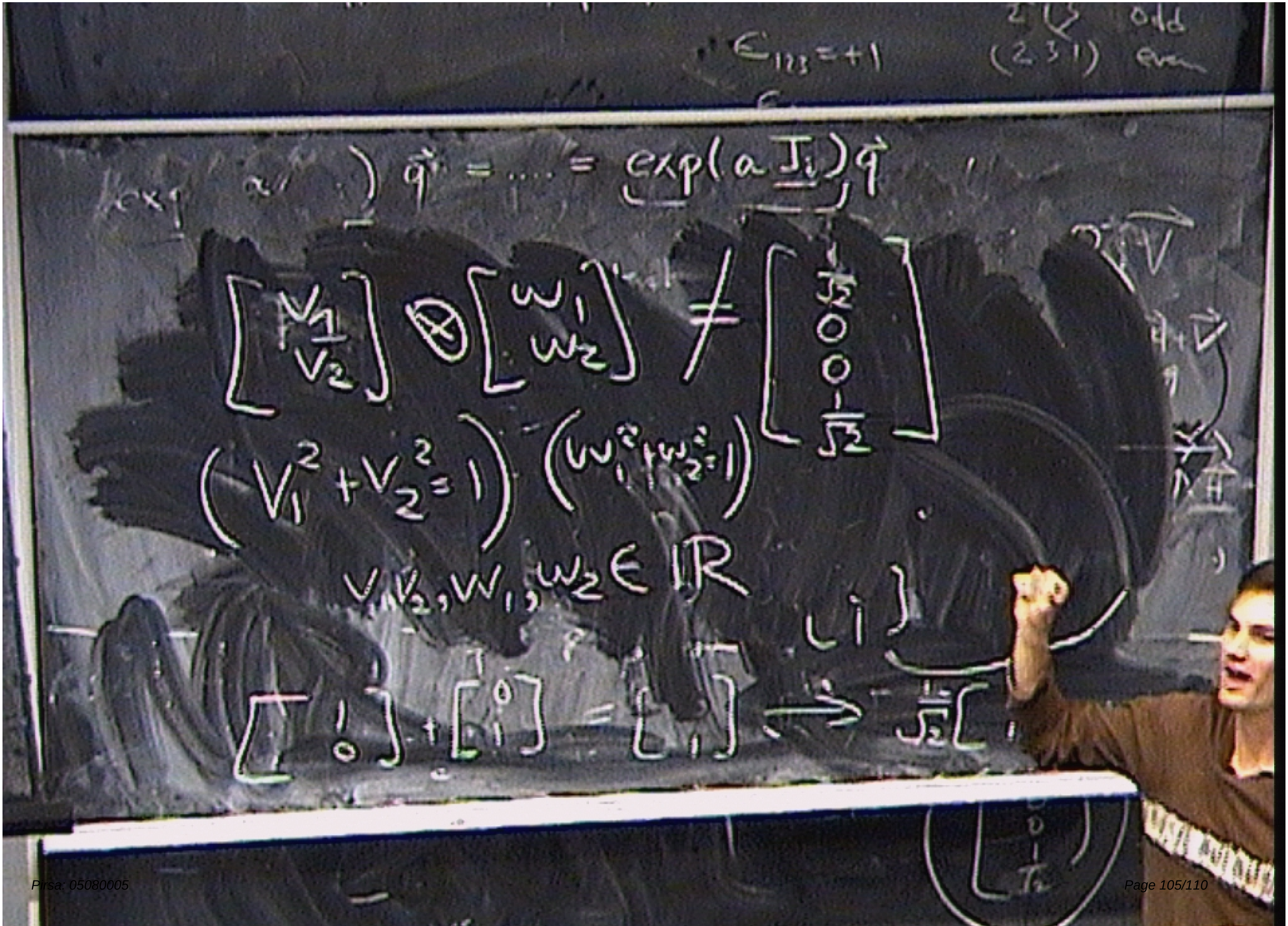

HORIZONTAL & HORIZONTAL
+
VERTICAL & VERTICAL


HORIZONTAL & HORIZONTAL
+
VERTICAL & VERTICAL






[v1 v2] ⊗ [w1 w2] = [1/√2 0 0 1/√2]
(v1² + v2² = 1) (w1² + w2² = 1)
v1, v2, w1, w2 ∈ ℝ
[1 0] + [0 1] = [1 1] ⟶ 1/√2 [1 1]


ε123 = +1
(2 3 1) even
) q = ... = exp(a J_i) q
[v1 v2] ⊗ [w1 w2] ≠ [1/√2 0 0 1/√2]
(v1² + v2² = 1) (w1² + w2² = 1)
v1, v2, w1, w2 ∈ ℝ

$\epsilon_{123} = +1$ $(2\,3\,1)$ even

odd

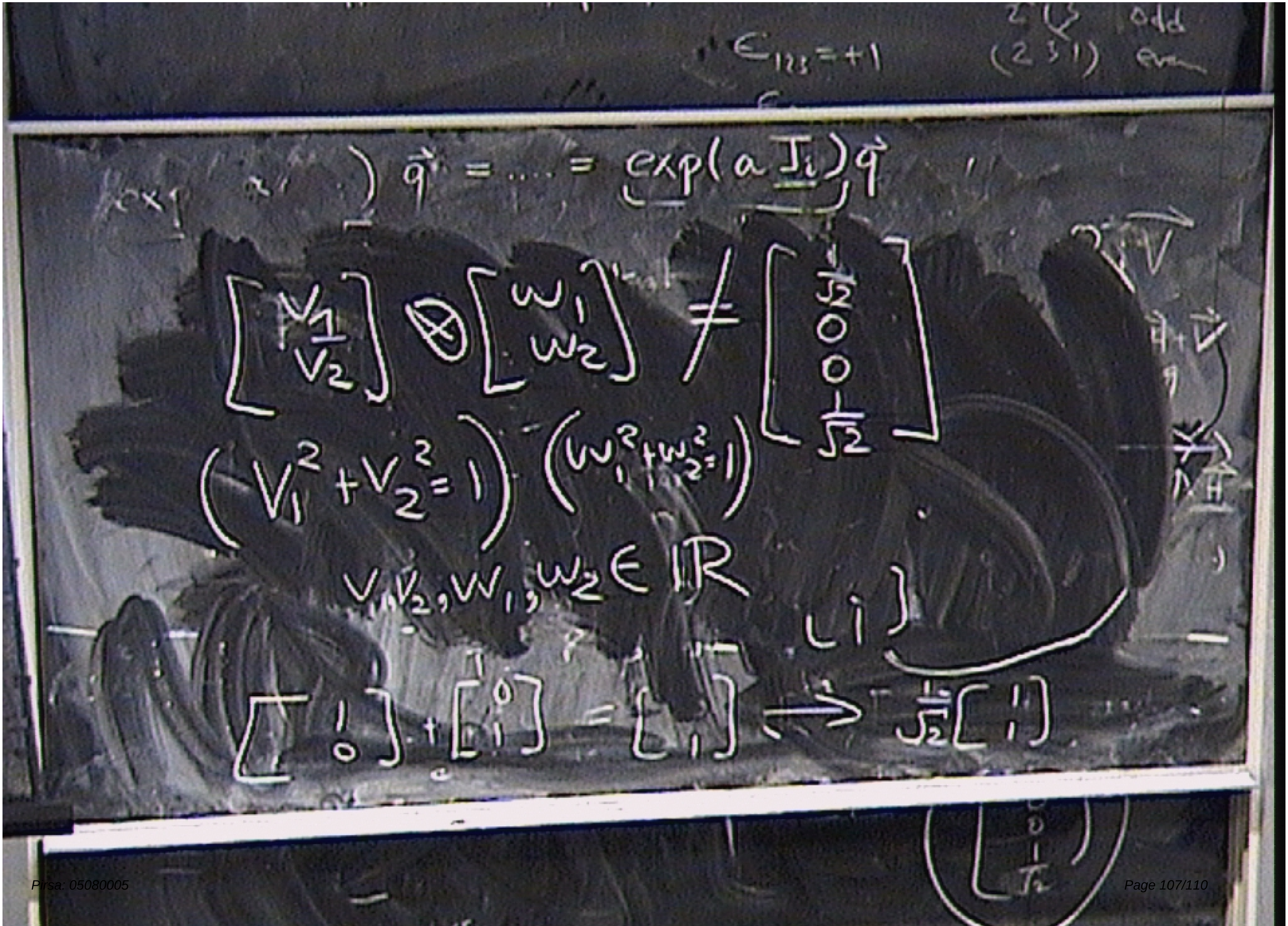
$\ldots) \vec{q} = \ldots = \exp(a \underline{J_i}) \vec{q}$

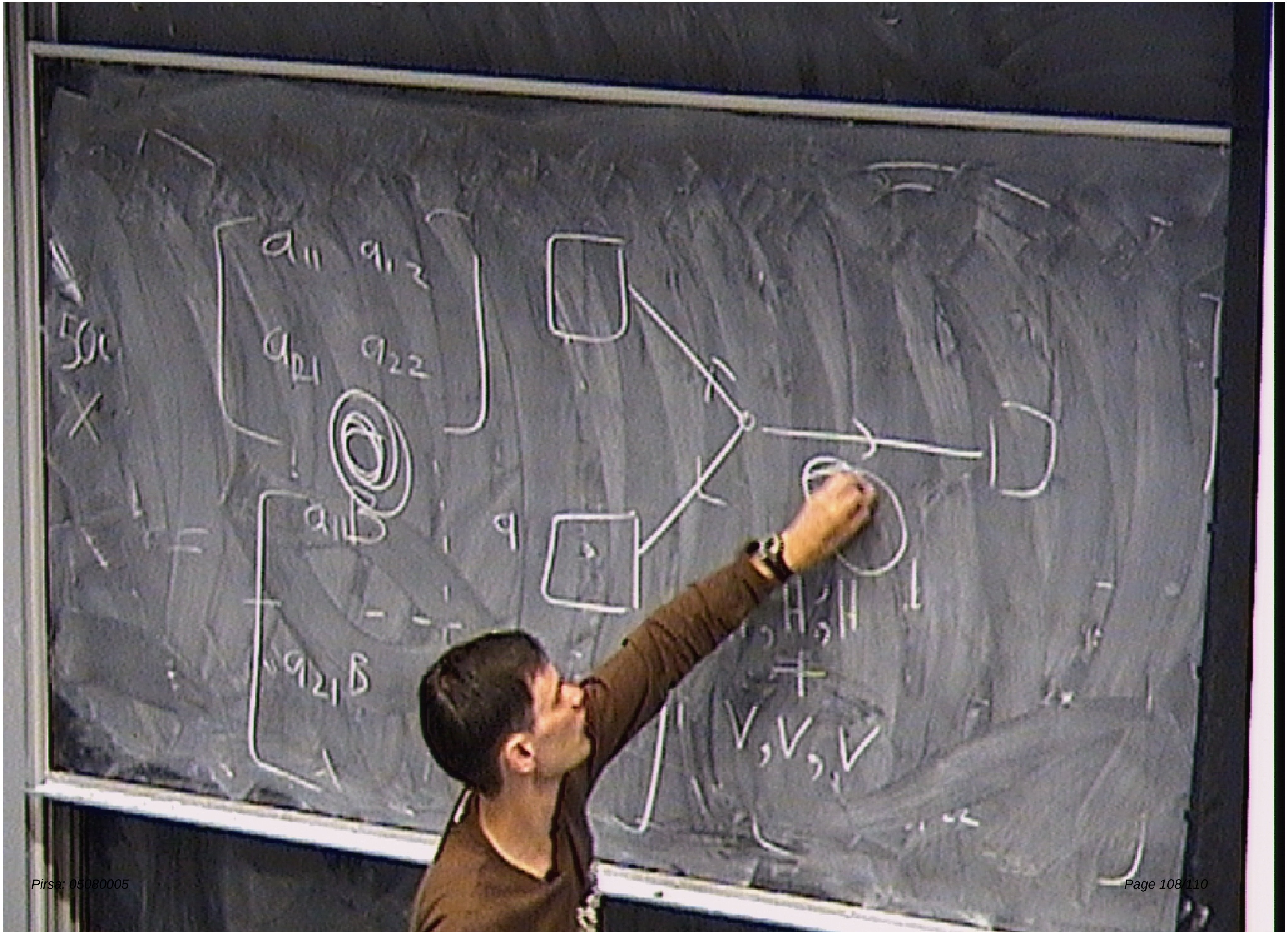
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(v_1^2 + v_2^2 = 1) \quad (w_1^2 + w_2^2 = 1)$$

$$v_1, v_2, w_1, w_2 \in \mathbb{R}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$


odd
$\epsilon_{123} = +1$
(2 3 1) even
$) \vec{q} = \ldots = \exp(a I_i) \vec{q}$
$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$(v_1^2 + v_2^2 = 1)$ $(w_1^2 + w_2^2 = 1)$
$v_1, v_2, w_1, w_2 \in \mathbb{R}$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longleftrightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$




H, H, H
+
V, V, V


a11 a12
a21 a22
a11 B
a21 B
H, H, H
V, V, V